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Assignment - 05

Que. 1. If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$ prove that grad u, grad v and grad w are coplanar vectors.

$u = x+y+z$; $v = x^2+y^2+z^2$; $w = yz+zx+xy$

$\text{grad}\,u = \hat{i}+\hat{j}+\hat{k}$; $\text{grad}\,v = 2x\hat{i}+2y\hat{j}+2z\hat{k}$; $\text{grad}\,w = (y+z)\hat{i}+(x+z)\hat{j}+(x+y)\hat{k}$

$$[\text{grad}\,u\ \text{grad}\,v\ \text{grad}\,w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & x+z & x+y \end{vmatrix} \quad (R_2 \to R_2+R_3)$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 0 \qquad \{\because R_1 = R_2\}$$

$\because [\text{grad}\,u, \text{grad}\,v, \text{grad}\,w] = 0$, $\therefore$ grad u, grad v & grad w are coplanar vectors.

Que. 2. Find the values of λ and μ so that surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$ and $4x^2y+z^3 = 4$ intersect orthogonally at the point $(1,-1,2)$.

Let $\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x$; $\phi_2 = 4x^2y+z^3-4$

$\vec{n_1} = \vec{\nabla}\phi_1 = (2x\lambda-\lambda-2)\hat{i} - \mu z\hat{j} - \mu y\hat{k}$; $\vec{\nabla}\phi_2 = 8xy\hat{i}+4x^2\hat{j}+3z^2\hat{k}$

$\vec{n_1} = (\vec{\nabla}\phi_1)_{(1,-1,2)} = (\lambda-2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}$; $\vec{n_2} = (\vec{\nabla}\phi_2)_{(1,-1,2)} = -8\hat{i}+4\hat{j}+12\hat{k}$

$\because$ surfaces cuts orthogonally

$\therefore \vec{n}_1 \cdot \vec{n}_2 = 0$

$-8(\lambda-2) - 8\mu + 12\mu = 0$

$-8(\lambda-2) + 4\mu = 0$

$-2(\lambda-2) + \mu = 0$ —— (1)

$\because$ surfaces passes through point $(1,-1,2)$

$\therefore \lambda + 2\mu = \lambda + 2$

$\boxed{\mu = 1}$ Ans:

$-2\lambda + 4 + \mu = 0$

$\boxed{\lambda = \frac{5}{2}}$ Ans:

Que. 3. Find the directional derivative of $\phi = (x^2+y^2+z^2)^{-1/2}$ at the point $(3,1,2)$ in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$

We have $\phi = (x^2+y^2+z^2)^{-1/2}$

$$\vec{\nabla}\phi = \left\{\frac{-2x}{2(x^2+y^2+z^2)^{3/2}}\right\}\hat{i} + \left\{\frac{-2y}{2(x^2+y^2+z^2)^{3/2}}\right\}\hat{j} + \left\{\frac{-2z}{2(x^2+y^2+z^2)^{3/2}}\right\}\hat{k}$$

$$\vec{\nabla}\phi = \frac{-1}{(x^2+y^2+z^2)^{3/2}}\left\{x\hat{i} + y\hat{j} + z\hat{k}\right\}$$

$$(\vec{\nabla}\phi)_{(3,1,2)} = \frac{-1}{(14)^{3/2}}(3\hat{i} + \hat{j} + 2\hat{k})$$

let $\vec{n} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ $\quad |\vec{n}| = 7$

$(\vec{n})_{(3,1,2)} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

$$\therefore \text{Directional Derivative} = \frac{-1}{(14)^{3/2}}(3\hat{i} + \hat{j} + 2\hat{k}) \cdot \frac{(2\hat{i} + 6\hat{j} + 3\hat{k})}{7}$$

$$= \frac{-18}{(14)^{3/2}\,7}$$

$$= \frac{-18}{14\sqrt{14} \times 7}$$

$$= \frac{-9}{49\sqrt{14}} \quad \text{Ans:}$$

Que. 4. Find the directional derivative of $\nabla \cdot (\nabla\phi)$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$.

$$\vec{\nabla}\phi = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\vec{\nabla} \cdot (\vec{\nabla}\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Let $\vec{\nabla}\cdot(\vec{\nabla}\phi) = f(x,y,z)$

$\therefore\ f(x,y,z) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$$\vec{\nabla}f = (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz^4 + 48x^3yz^2)\hat{j} + (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

$$(\vec{\nabla}f)_{(1,-2,1)} = 348\hat{i} - 144\hat{j} + 400\hat{k}$$

Direction: let $\phi' = xy^2z - 3x - z^2$

$$\vec{\nabla}\phi' = (y^2z-3)\hat{i} + (2xyz)\hat{j} + (xy^2-2z)\hat{k}$$

$$\vec{n} = (\vec{\nabla}\phi')_{(1,-2,1)} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$|\vec{n}| = \sqrt{21}$$

$\therefore$ Directional Derivative $= (\vec{\nabla}f)_{(1,-2,1)} \cdot \hat{n}$

$$= (348\hat{i} - 144\hat{j} + 400\hat{k}) \cdot \frac{(\hat{i} - 4\hat{j} + 2\hat{k})}{\sqrt{21}}$$

$$= \frac{348 + 576 + 800}{\sqrt{21}}$$

$$= \frac{1724}{\sqrt{21}} \quad \text{Ans:}$$

Que. 5. The temprature at a point (x,y,z) in space is given by $T(x,y,z) = x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly.

Mosquito should fly in the direction of normal to the surface. since directional derivative maximum normal to the surface.

$$T(x,y,z) = x^2+y^2-z$$

$$\vec{\nabla}T = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\vec{n} = (\vec{\nabla}T)_{(1,1,2)} = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{n} = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$|\vec{n}| = 3$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$$

Que. 6. A vector field is given by $\vec{A} = (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}$. show that the field is irrotational and find the scalar potential.

$$\vec{A} = (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}$$

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+xy^2) & (y^2+x^2y) & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (2xy-2xy)\hat{k}$$

$$\text{curl}\,\vec{A} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$\because$ curl $\vec{A} = \vec{0}$

$\therefore$ $\vec{A}$ is irrotational.

Scalar potential: Let $\phi(x,y,z)$ be a scalar potential.

$$\vec{\nabla}\phi \cdot d\vec{r} = \vec{A} \cdot d\vec{r} \qquad : \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \{(x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}\} \cdot \{dx\hat{i} + dy\hat{j} + dz\hat{k}\}$$

$$\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = (x^2+xy^2)dx + (y^2+x^2y)dy$$

$$d\phi = x^2dx + xy^2dx + y^2dy + x^2y\,dy$$

$$d\phi = x^2dx + y^2dy + \frac{1}{2}(2xy^2dx + 2yx^2dy)$$

$$d\phi = x^2dx + y^2dy + \frac{1}{2}d\,x^2y^2$$

Integrating both side

$$\int d\phi = \int x^2dx + \int y^2dy + \frac{1}{2}\int d\,x^2y^2$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C$$

Que. 7. Show that the vector field $F = \frac{r}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential.

$$F = \frac{\vec{r}}{r^3}$$

$$\operatorname{div}\vec F = \operatorname{div}\frac{\vec r}{r^3} = \frac{1}{r^3}\operatorname{div}\vec r + \operatorname{grad}\frac{1}{r^3}\cdot\vec r \quad \left\{\because \operatorname{div} u\vec a = u\operatorname{div}\vec a + \nabla u\cdot\vec a\right\}$$

$$\operatorname{div}\vec F = \frac{1}{r^3}(3) + \left(-\frac{3}{r^4}\right)\frac{\vec r}{r}\cdot\vec r \quad \left\{\because \operatorname{grad} f(r) = f'(r)\frac{\vec r}{r}\right\}$$

$$\operatorname{div}\vec F = \frac{3}{r^3} - \frac{3}{r^3} \quad \left\{\because \vec r\cdot\vec r = r^2\right\}$$

$$\operatorname{div}\vec F = 0$$

$\because \operatorname{div}\vec F = 0$

$\therefore$ $\vec F$ is solenoidal.

$$\operatorname{curl}\vec F = \operatorname{curl}\frac{\vec r}{r^3} = \frac{1}{r^3}\operatorname{curl}\vec r + \operatorname{grad}\frac{1}{r^3}\times\vec r \quad \left\{\because \operatorname{curl} u\vec a = u\operatorname{curl}\vec a + \nabla u\times\vec a\right\}$$

$$\operatorname{curl}\vec F = \frac{1}{r^3}(\vec 0) - \left(\frac{3}{r^4}\right)\frac{\vec r}{r}\times\vec r \quad \left\{\because \operatorname{grad} f(r) = f'(r)\frac{\vec r}{r}\right\} \text{ and } \left\{\because \operatorname{curl}\vec r = \vec 0\right\}$$

$$\operatorname{curl}\vec F = \vec 0 - \vec 0 \quad \left\{\because \vec r\times\vec r = \vec 0\right\}$$

$$\operatorname{curl}\vec F = \vec 0$$

$\because \operatorname{curl}\vec F = \vec 0$

$\therefore$ $\vec F$ is irrotational.

Scalar potential   let $\phi(x,y,z)$ be a scalar potential

$$\vec\nabla\phi\cdot d\vec r = \vec F\cdot d\vec r$$

$$\left(\frac{\partial\phi}{\partial x}\hat i + \frac{\partial\phi}{\partial y}\hat j + \frac{\partial\phi}{\partial z}\hat k\right)\cdot\left(dx\hat i + dy\hat j + dz\hat k\right) = \frac{\vec r}{r^3}\left(dx\hat i + dy\hat j + dz\hat k\right)$$

$$\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \frac{(x\hat i + y\hat j + z\hat k)(dx\hat i + dy\hat j + dz\hat k)}{r^3}$$

$$d\phi = \frac{x\,dx + y\,dy + z\,dz}{r^3} = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

Integrating both side

$$\int d\phi = \int \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

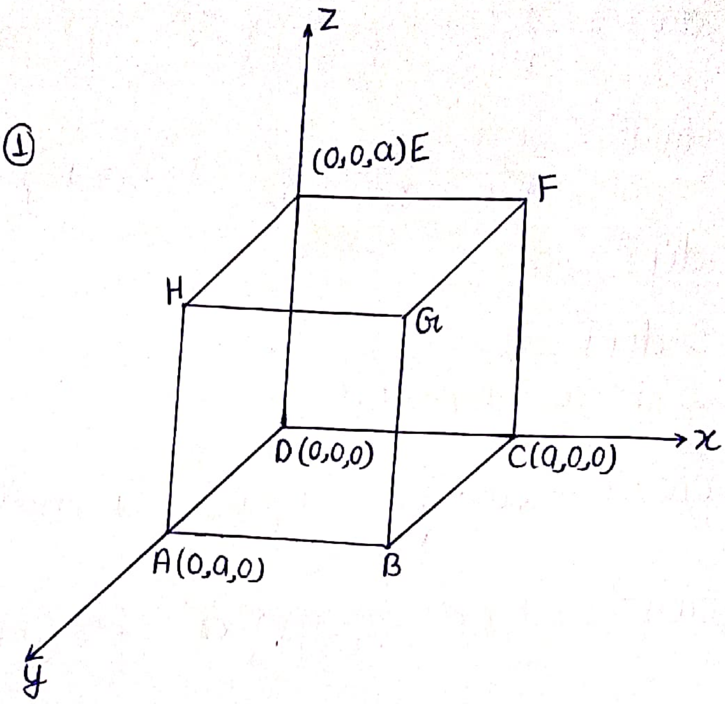
$$\int d\phi = \int d\left\{-(x^2+y^2+z^2)^{-1/2}\right\}$$

$$\phi = -(x^2+y^2+z^2)^{-1/2} = \frac{-1}{\sqrt{x^2+y^2+z^2}} + c \text{ Ans:}$$

Que. 9. Verify divergence theorem for $\vec{F} = (x^3 - yz)\hat{i} - 2x^2y\hat{j} + 2\hat{k}$, taken over the cube $0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a$.

Gauss's Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\vec{F}\,dv \quad \text{—— (1)}$$



$$\vec{F} = (x^3 - yz)\hat{i} - 2x^2y\hat{j} + 2\hat{k}$$

$$\text{div}\vec{F} = 3x^2 - 2x^2 = x^2$$

$$\therefore \text{R.H.S} = \iiint_V \text{div}\vec{F}\,dv$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} x^2\,dx\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} \frac{a^3}{3}\,dy\,dz$$

$$= \int_{z=0}^{a} \frac{a^4}{3}\,dz$$

$$= \frac{a^5}{3}$$

Plane	Equation	$\hat{n}$	ds	$\vec{F}\cdot\hat{n}$
$S_1 \to$ ABCD (xy)	$z=0$	$-\hat{k}$	$dx\,dy$	-2
$S_2 \to$ HGEF (xy)	$z=a$	$\hat{k}$	$dx\,dy$	2
$S_3 \to$ ADEH (yz)	$x=0$	$-\hat{i}$	$dy\,dz$	$yz - x^3$
$S_4 \to$ BCFG (yz)	$x=a$	$\hat{i}$	$dy\,dz$	$x^3 - yz$
$S_5 \to$ CDEF (xz)	$y=0$	$-\hat{j}$	$dx\,dz$	$2x^2y$
$S_6 \to$ ABHG (xz)	$y=a$	$\hat{j}$	$dx\,dz$	$-2x^2y$

$$\text{L.H.S} = \iint_S \vec{F}\cdot\hat{n}\,ds = \underbrace{\iint_{S_1}\vec{F}\cdot\hat{n}\,ds_1}_{I_1} + \underbrace{\iint_{S_2}\vec{F}\cdot\hat{n}\,ds_2}_{I_2} + \underbrace{\iint_{S_3}\vec{F}\cdot\hat{n}\,ds_3}_{I_3} + \underbrace{\iint_{S_4}\vec{F}\cdot\hat{n}\,ds_4}_{I_4} + \underbrace{\iint_{S_5}\vec{F}\cdot\hat{n}\,ds_5}_{I_5} + \underbrace{\iint_{S_6}\vec{F}\cdot\hat{n}\,ds_6}_{I_6}$$

$$I_1 = \int_{x=0}^{a}\int_{y=0}^{a} -2\,dy\,dx = -2a^2\ ; \quad I_2 = \int_{x=0}^{a}\int_{y=0}^{a} 2\,dy\,dx = 2a^2\ ; \quad I_3 = \int_{y=0}^{a}\int_{z=0}^{a} yz\,dz\,dy \quad \{\because x=0\}$$

$$I_4 = \int_{y=0}^{a}\int_{z=0}^{a} (a^3 - yz)\,dz\,dy = a^5 - \frac{a^4}{4}$$

$$I_3 = \frac{a^4}{4}$$

$$I_5 = 0 \quad (\because y=0)$$

$$I_6 = \int_{z=0}^{a}\int_{x=0}^{a} -2ax^2\,dx\,dz = -\frac{2a^5}{3}$$

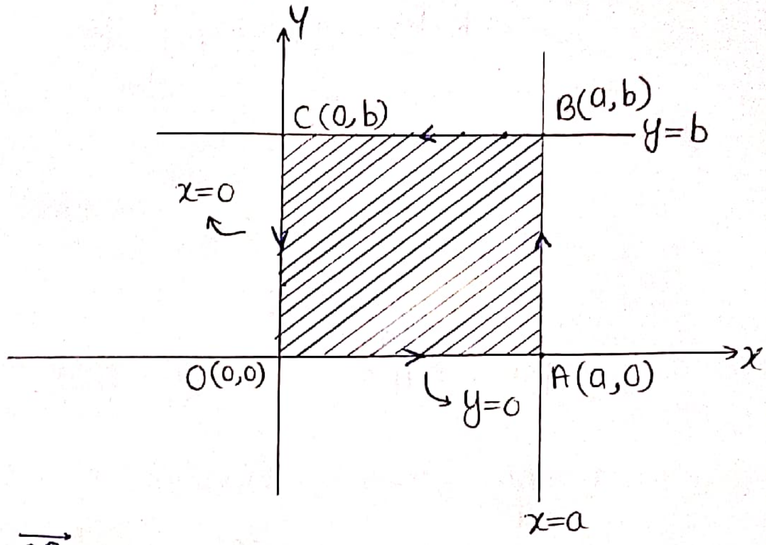
$$\therefore \text{L.H.S} = -2a^2 + 2a^2 + \frac{a^4}{4} + a^5 - \frac{a^4}{4} + 0 - \frac{2a^5}{3} = \frac{a^5}{3} \qquad \text{Ans:}$$

Q. 9. Verify stoke's theorem for $\vec{F}=(x^2-y^2)\hat{i}+2xy\hat{j}$ taken round the rectangle bounded by the lines $x=0, x=a, y=0, y=b$

Stoke's Theorem state that

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\vec{F}=(x^2-y^2)\hat{i}+2xy\hat{j}$$



$$L\cdot H\cdot S = \oint_C \vec{F}\cdot d\vec{r} = \oint_C (x^2-y^2)dx+2xydy$$

where $C: \vec{OA}+\vec{AB}+\vec{BC}+\vec{CO}$

$$= \int_{\vec{OA}}(x^2-y^2)dx+2xydy+\int_{\vec{AB}}(x^2-y^2)dx+2xydy+\int_{\vec{BC}}(x^2-y^2)dx+2xydy+\int_{\vec{CO}}(x^2-y^2)dx+2xydy$$

$\vec{OA}$: $y=0$, $dy=0$, $x=0$ to $x=a$

$\vec{AB}$: $x=a$, $dx=0$, $y=0$ to $y=b$

$\vec{BC}$: $y=b$, $dy=0$, $x=a$ to $x=0$

$\vec{CO}$: $x=0$, $dx=0$, $y=b$ to $y=0$

$$= \int_0^a x^2dx + \int_0^b 2aydy + \int_a^0 (x^2-b^2)dx + \int_b^0 0\,dy$$

$$= \frac{a^3}{3}+ab^2-\frac{a^3}{3}+ab^2+0$$

$$= 2ab^2$$

$$R\cdot H\cdot S = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2) & 2xy & 0 \end{vmatrix} = 0\hat{i}-0\hat{j}+4y\hat{k} = 4y\hat{k}$$

$\because$ surface S is in a xy plane

$\therefore\ \hat{n}=\hat{k}$

$\therefore\ \text{curl}\,\vec{F}\cdot\hat{n} = 4y$

$$ds = \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = dxdy$$

$$\therefore \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dy\,dx$$

$$R\cdot H\cdot S = \int_{x=0}^{a} 2\left[y^2\right]_{y=0}^{b} dx$$

$$= 2b^2\int_{x=0}^{a} dx$$

$$= 2ab^2$$

$L\cdot H\cdot S = R\cdot H\cdot S$ verified.

Que. 10. Verify green's theorem to evaluate $\int_C (2y^2dx + 3xdy)$, where c is the boundary of the closed region bounded by $y=x$ & $y=x^2$.

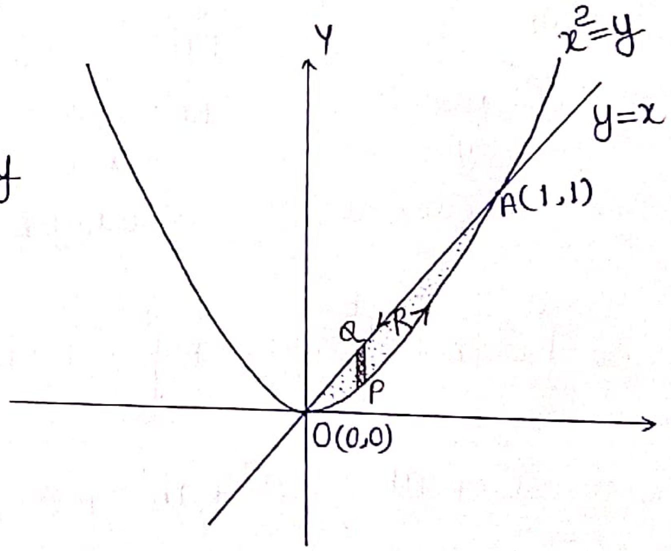
$$I = \int_C 2y^2dx + 3xdy \quad \text{where } c:\ y=x \ \&\ y=x^2$$

Green's theorem states that

$$\oint_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$\because F_1 = 2y^2 \ ; \quad F_2 = 3x$$

$$\frac{\partial F_1}{\partial y} = 4y \ ; \quad \frac{\partial F_2}{\partial x} = 3$$



$$\therefore I = \int_C 2y^2dx + 3xdy = \iint_R (3-4y)\,dxdy$$

limits $y=x^2$ to $y=x$
$x=0$ to $x=1$

$$I = \int_{x=0}^{1}\int_{y=x^2}^{x} (3-4y)\,dy\,dx = \int_{x=0}^{1}\left[\frac{(3-4y)^2}{-8}\right]_{y=x^2}^{x} dx = -\frac{1}{8}\int_{x=0}^{1}\left\{(3-4x)^2 - (3-4x^2)^2\right\} dx$$

$$I = -\frac{1}{8}\int_{x=0}^{1}(-16x^4 + 40x^2 - 24x)\,dx = \int_{x=0}^{1}(2x^4 - 5x^2 + 3x)\,dx$$

$$I = \left[\frac{2x^5}{5} - \frac{5x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \frac{2}{5} - \frac{5}{3} + \frac{3}{2} = \frac{12-50+45}{30} = \frac{7}{30} \quad \text{Ans:}$$

$$I = \frac{7}{30} \quad \text{Ans:}$$

$$L.H.S = \oint_C F_1 dx + F_2 dy$$

$$= \oint_C 2y^2 dx + 3x dy$$

where $C : \overrightarrow{OA} + \overrightarrow{AO}$

$$= \int_{\overrightarrow{OA}} 2y^2 dx + 3x dy \quad + \quad \int_{\overrightarrow{AO}} 2y^2 dx + 3x dy$$

$\overrightarrow{OA} : y = x^2$

$dy = 2x dx$

$\overrightarrow{AO} : y = x$

$dy = dx$

$$= \int_0^1 2x^4 dx + 6x^2 dx \quad + \quad \int_1^0 2x^2 dx + 3x dx$$

$$= \left[\frac{2x^5}{5} + \frac{6x^3}{3} \right]_0^1 - \left[\frac{2x^3}{3} + \frac{3x^2}{2} \right]_0^1$$

$$= \left(\frac{2}{5} + \frac{6}{3} \right) - \left(\frac{2}{3} + \frac{3}{2} \right)$$

$$= \frac{2}{5} + \frac{6}{3} - \frac{2}{3} - \frac{3}{2}$$

$$= \frac{2}{5} + \frac{4}{3} - \frac{3}{2}$$

$$= \frac{12 + 40 - 45}{30}$$

$$= \frac{7}{30}$$