HA NOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

Introduction to Cryptography and Security

Elliptic Curve Cryptosystems

Outline

1. Introduction to Elliptic Curve (EC)
2. EC Discrete Logarithm Problem
3. EC Diffie Hellman Key Exchange (ECDH)

Motivation:

Find public key family with shorter operands

Symmetric key size (bits)	Keysize for Elliptic Curve based scheme	Keysize for RSA or Diffie-Hellman
80	160	1024
112	224	2048
128	256	3072
192	384	7680
256	512	**15360**

Table: Bit lengths of public-key algorithms for different security levels

Definition

The **elliptic curve over** K is the set of all pairs $(x, y) \in K$ which fulfill

$$y^2 = x^3 + a \cdot x + b$$

together with an imaginary point of infinity $\mathcal{O}$, where

$$a, b \in K$$

and the condition

$$4 \cdot a^3 + 27 \cdot b^2 \neq 0.$$

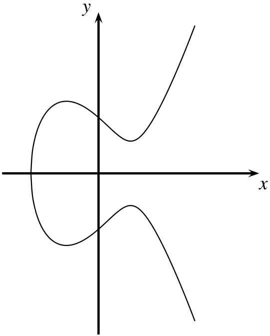
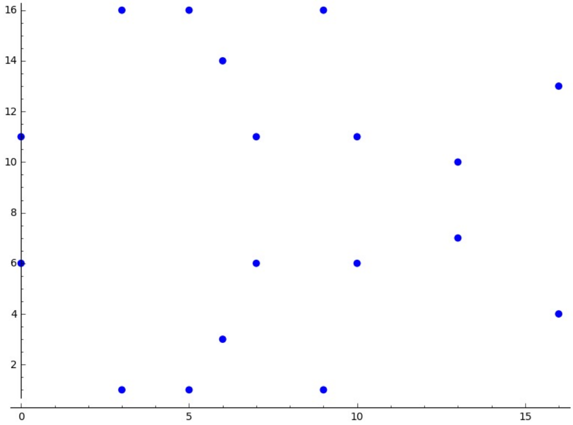


Figure: Elliptic curve $y^2 = x^3 - 3x + 3$ over $\mathbb{R}$

Elliptic curve $y^2 = x^3 + 2x + 2$ over $\mathbb{Z}_{17}$



$$
\begin{aligned}
&\mathcal{O}, \\
&(0,6),\ (0,11), \\
&(3,1),\ (3,16), \\
&(5,1),\ (5,16), \\
&(6,3),\ (6,14), \\
&(7,6),\ (7,11), \\
&(9,1),\ (9,16), \\
&(10,6),\ (10,11), \\
&(13,7),\ (13,10), \\
&(16,4),\ (16,13)
\end{aligned}
$$

Group Operations on EC

- Denote the group operation with addition symbol "+".
- Given two points and their coordinates

$$P = (x_1, y_1) \text{ and } Q = (x_2, y_2)$$

we have to compute coordinates of a third point R such that:

$$P + Q = R$$
$$(x_1,\ y_1) + (x_2, y_2) = (x_3, y_3)$$

- **Point Addition** $P + Q$: This is the case where we compute

$$R = P + Q \text{ and } P \neq Q.$$

- **Point Doubling** $P + P$: This is the case where we compute

$$P + Q \quad \text{but} \quad P = Q.$$

Group operations

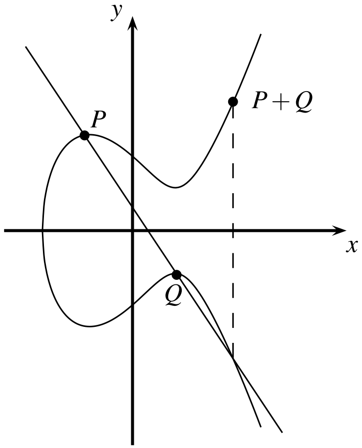


Figure: Point Addition

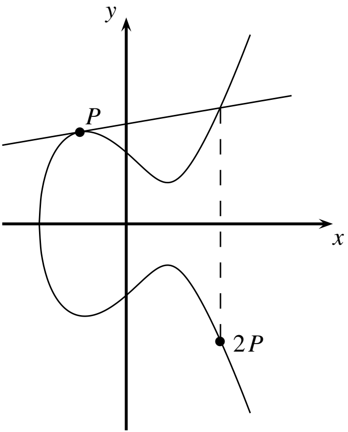


Figure: Point Doubling

Point Addition and Point Doubling

$$x_3 = s^2 - x_1 - x_2 \mod p$$
$$y_3 = s(x_1 - x_3) - y_1 \mod p$$

where

$$s = \begin{cases} (y_2 - y_1)/(x_2 - x_1) \mod p & \text{if } P \neq Q \\ (3{x_1}^2 + a)/(2y_1) \mod p & \text{if } P = Q. \end{cases}$$

Example

Consider the curve

$$E: \qquad y^2 = x^3 + 2x + 2 \mod 17$$

We want to double the point $P = (5, 1)$.

$$\begin{aligned}
2P &= P + P = (5,1) + (5,1) = (x_3, y_3) \\
s &= (3{x_1}^2 + a)/(2y_1) = (3 \cdot 5^2 + 2) \cdot (2 \cdot 1)^{-1} \mod 17 \\
&= 9 \cdot 2^{-1} = 13 \mod 17 \\
x_3 &= s^2 - x_1 - x_2 = 13^2 - 5 - 5 = 6 \mod 17 \\
y_3 &= s(x_1 - x_3) - y_1 = 13 \cdot (5 - 6) - 1 = -14 = 3 \mod 17 \\
2P &= (5,1) + (5,1) = (6,3)
\end{aligned}$$

Play with Sagemath

```
sage: E = EllipticCurve(GF(17),[2,2])
sage: E
Elliptic Curve defined by y^2 = x^3 + 2*x + 2 over
    Finite Field of size 17
sage: P = E(5,1)
sage: Q = P + P
sage: print Q
(6 : 3 : 1)
sage: E.is_on_curve(6,3)
True
```

Complete addition laws for elliptic curves

1. $\mathcal{O} + \mathcal{O} = \mathcal{O}$
2. $\mathcal{O} + (x_2, y_2) = (x_2, y_2)$
3. $(x_1, y_1) + \mathcal{O} = (x_1, y_1)$
4. $(x_1, y_1) + (x_1, -y_1) = \mathcal{O}$
5. for $y_1 \neq 0$,

$$(x_1, y_1) + (x_1, y_1) = (s^2 - 2x_1, s(x_1 - x_3) - y_1)$$

where $s = (3{x_1}^2 + a)/(2y_1)$

6. for $x_1 \neq x_2$,

$$(x_1, y_1) + (x_1, y_1) = (s^2 - x_1 - x_2, s(x_1 - x_3) - y_1)$$

where $s = (y_2 - y_1)/(x_2 - x_1)$

Properties

1. $\mathcal{O} + \mathcal{O} = \mathcal{O}$
2. $\mathcal{O} + (x_2, y_2) = (x_2, y_2)$
3. $(x_1, y_1) + \mathcal{O} = (x_1, y_1)$
4. $\underbrace{(x_1, y_1)}_{P} + \underbrace{(x_1, -y_1)}_{-P} = \mathcal{O}$

$\mathcal{O}$ is the identity element of the group.

```
sage: O = P + -P
sage: O
(0 : 1 : 0)
sage: O + O == O
True
sage: P + O
(5 : 1 : 1)
sage: P + O == P
True
sage: O + P == P
True
```

Projective coordinates

- The project point

$$(X : Y : Z),\ \ Z \neq 0$$

corresponds to the affine point $(X/Z, Y/Z)$.

- The project equation of elliptic curve is

$$Y^2 Z = X^3 + aXZ^2 + bZ^3$$

- The point at infinite $\mathcal{O}$ corresponds to$(0 : 1 : 0)$, while the negative of$(X : Y : Z)$ is $(X : -Y : Z)$.

Benefit of projective cordinates

- The explicit formulas for computing $+$ become much faster, by avoiding field inversions
- Thus the fundamental ECC operation kP becomes much faster

$(x', y') = 2(x, y)$	$(X' : Y' : Z') = 2(X : Y : Z)$
$s = \dfrac{3x^2 + a}{2y}$ $x' = s^2 - 2x$ $y' = s(x - x') - y$	$X' = 2XY\left(3X^2 + aZ^2\right)^2 - 8Y^2XZ$ $Y' = (3X^2 + aZ^2)(12Y^2XZ - (3X^2 + aZ^2)^2) - 8Y^4Z^2$ $Z' = 8Y^3Z^3$

Play with Sagemath

```
sage: E = EllipticCurve(GF(17),[2,2])
sage: E
Elliptic Curve defined by y^2 = x^3 + 2*x + 2 over
    Finite Field of size 17
sage: for P in E:
....:       print P
....:
```

```
(0 : 1 : 0)       (6 : 3 : 1)       (10 : 11 : 1)
(0 : 6 : 1)       (6 : 14 : 1)      (13 : 7 : 1)
(0 : 11 : 1)      (7 : 6 : 1)       (13 : 10 : 1)
(3 : 1 : 1)       (7 : 11 : 1)      (16 : 4 : 1)
(3 : 16 : 1)      (9 : 1 : 1)       (16 : 13 : 1)
(5 : 1 : 1)       (9 : 16 : 1)
(5 : 16 : 1)      (10 : 6 : 1)
```

Outline

1. Introduction to Elliptic Curve (EC)
2. EC Discrete Logarithm Problem
3. EC Diffie Hellman Key Exchange (ECDH)

Cyclic subgroups

Theorem

The points on an Elliptic curve together with $\mathcal{O}$ have a cyclic subgroups. Under certain conditions all points on an elliptic curve form a cyclic group.

Example

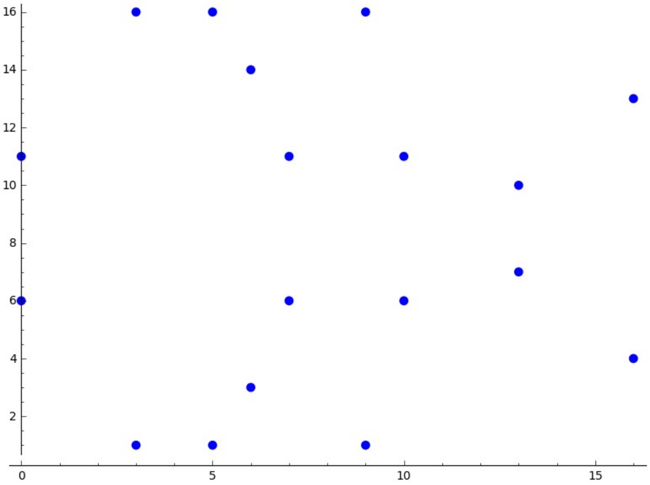
The points on EC $y^2 = x^3 + 2x + 2 \mod 17$ are

```
P  = (5,1)    6P = (16,13)  11P = (13,10) 16P = (10,11)
2P = (6,3)    7P = (0,6)    12P = (0,11)  17P = (6,14)
3P = (10,6)   8P = (13,7)   13P = (16,4)  18P = (5,16)
4P = (3,1)    9P = (7,6)    14P = (9,1)   19P = 0
5P = (9,16)   10P = (7,11)  15P = (3,16)
```

Example

$$y^2 = x^3 + 2x + 2 \mod 17$$

```
P   = (5,1)
2P  = (6,3)
3P  = (10,6)
4P  = (3,1)
5P  = (9,16)
6P  = (16,13)
7P  = (0,6)
8P  = (13,7)
9P  = (7,6)
10P = (7,11)
11P = (13,10)
12P = (0,11)
13P = (16,4)
14P = (9,1)
15P = (3,16)
16P = (10,11)
17P = (6,14)
18P = (5,16)
19P = 0
```



Elliptic Curved Discrete Logarithm Problem (ECDLP)

Definition

Given is an elliptic curve E. We consider an element P and another element T.

The DL problem is finding the integer d such that

$$\underbrace{P + P + \cdots + P}_{d \text{ times}} = dP = T$$

Exercise

Consider the curve

$$E: \quad y^2 = x^3 + 2x + 2 \mod 17$$

We already computed all "powers" of P.

P = (5,1)	6P = (16,13)	11P = (13,10)	16P = (10,11)
2P = (6,3)	7P = (0,6)	12P = (0,11)	17P = (6,14)
3P = (10,6)	8P = (13,7)	13P = (16,4)	18P = (5,16)
4P = (3,1)	9P = (7,6)	14P = (9,1)	19P = 0
5P = (9,16)	10P = (7,11)	15P = (3,16)	

For $P = (5,1)$ and $T = (16,4)$, find the integer d such that $dP = T$.

Group cardinality

Theorem (Hass)

Given an elliptic curve E modulo p, the number of points on the curve is denoted by $\#E$ and is bounded by:

$$p + 1 - \sqrt{p} \leq \#E \leq p + 1 + 2\sqrt{p}$$

- $\#E \approx p$
- If we need an elliptic curve with 2^{256} points, we have to use a prime with 256 bits.

Security

- All EC protocols rely on the hardness of ECDLP
- If EC is chosen carefully, the best algorithm for computing the ECDLP requires $\sqrt{p}$ steps
- Ex: $p \approx 2^{256}$
  Attack requires $\approx \sqrt{2^{256}} = 2^{128}$ steps.

> It is believed in general that the best algorithm to solve ECDLP takes time
> $$O(\sqrt{L})$$
> where L is order of P.

Outline

1. Introduction to Elliptic Curve (EC)
2. EC Discrete Logarithm Problem
3. EC Diffie Hellman Key Exchange (ECDH)

Phase I: ECDH Domain Parameters

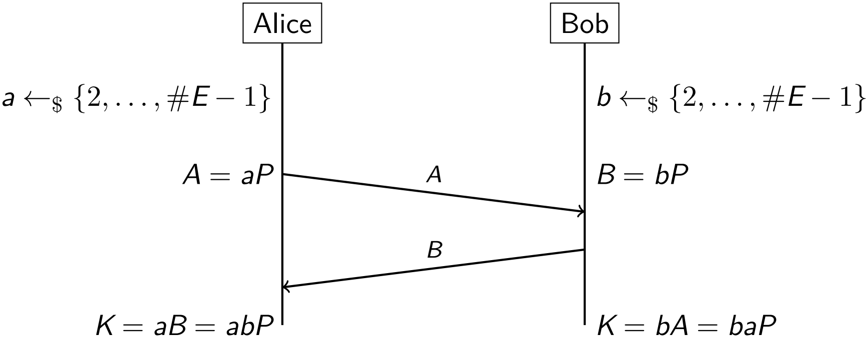
1. Choose a prime p and the elliptic curve

$$E: \qquad y^2 = x^3 + ax + b \mod p$$

2. Choose a point $P = (x_P, y_P)$

Phase II: Key Exchange

Public parameter: P, E



Single-scalar multiplication

```
def scalarmult(n,P):
    if n == 0: return 0
    if n == 1: return P
    R = scalarmult(n//2,P)
    R=R+R
    if n % 2: R = R + P
    return R
```

- CPU time is dominated by time to compute $\log_2(n)$ point doublings
- Example of worst case: 4 doublings; 4 more additions.

$$31P = 2(2(2(2P + P) + P) + P) + P.$$

- Average case is better: 5 doublings; 2 additions.

$$35P = 2(2(2(2(2P))) + P) + P.$$

The security of the DH key exchange

- An eavesdropper sees the values aP and bP
- It has to compute the value $K_{ab} = abP$
- The hardness of the computation is expressed via two problems believed to be difficult

Decision Diffie Hellman (DDH)

Given(P, aP, bP, cP), to decide if $ab = c$.

Computational Diffie Hellman (CDH)

Given(P, aP, bP), to compute abP.

Computational Diffie-Hellman Assumption

Computation DH assumption holds in E if: $P, aP, bP \nRightarrow abP$.

for all efficient algorithm A:

$$\Pr[A(P, aP, bP) = abP] < \text{ negligible}$$

where $P \leftarrow_{\$} \{ \text{ generators of } E \}, \; a, b \leftarrow_{\$} \mathbb{Z}_n$

The curve P256

The curve has the form

$$y^2 = x^3 - 3x + b \mod p$$

where

- the prime $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$
- and b in hexadecimal is:

  `5ac635d8 aa3a93e7 b3ebbd55 769886bc 651d06b0 cc53b0f6 3bce3c3e`

The curve P256

- The prime *p* is close to 2^{256}, the number of points is also close to 2^{256}.
- Then, computing discrete log take approximately 2^{128}.
- How was the odd looking parameter *b* in P256 selected? We don't know!
- P256 is widely used in practice.

Quiz 1

Consider the curve

$$E: \quad y^2 = x^3 + 2x + 2 \mod 17$$

and two points $P = (5, 1)$ and $Q = (10, 6)$ on E.

What is $R = P + Q$?

1. $R = (15, 7)$
2. $R = (3, 1)$
3. $R = \mathcal{O}$

Quiz 2

Consider the curve

$$E: \quad y^2 = x^3 + 2x + 2 \mod 17$$

and two points $P = (5, 1)$ and $Q = (10, 6)$ on E.

What is the integer d where $1 \leq d \leq \#E$, such that: $dQ = P$?

1. $d = 1$
2. $d = 13$
3. $d = 17$
4. There is no d.

VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG
SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

Thank you!