

1. $\{E, A \rightarrow D, D \rightarrow (E \rightarrow B)\} \vdash A \rightarrow B$

1. A IP R_3

2. $A \rightarrow D$ P R_1

3. D 1,2, MP

4. $D \rightarrow (E \rightarrow B)$ P R_1

5. $E \rightarrow B$ 3,4, MP

6. E P R_1

7. \boxed{B} 5,6, MP

2. $\{\neg(B \rightarrow G) \rightarrow \neg A, A \rightarrow B\} \vdash A \rightarrow G$

1. A IP R_3

2. $A \rightarrow B$ P R_1

3. B 1,2, MP

4. $\neg(B \rightarrow G) \rightarrow \neg A$ P R_1

5. $A \rightarrow (B \rightarrow G)$ 4, E2, R_2

6. $B \rightarrow G$ 1,5, MP

7. \boxed{G} 3,6 MP

3. $\{B \wedge A\} \vdash A \vee B$

1. $B \wedge A$ P R_1

2. A 1, I4, R_2

3. $\boxed{A \vee B}$ 2, I1, R_2

4. $\{A \rightarrow (B \vee C), B \rightarrow D, \neg C \vee E, (D \vee E) \rightarrow \neg A\} \vdash \neg A$

1. $A \rightarrow (B \vee C)$ P R_1

2. A IP R_1

3. $B \vee C$ 1,2, I1, R_2

4. $(D \vee E) \rightarrow \neg A$ P R_1

5. $\neg(D \vee E)$ 2,4, I2, R_2

1. $\neg D \wedge \neg E$ 5, E1, R_2

2. $\neg D$ 6, I3, R_2

3. $\neg E$ 6, I4, R_2

4. $\neg C \vee E$ P R_1

5. $\neg C$ 8,9, I6, R_2

6. $B \rightarrow D$ P R_1

7. $\neg B$ 7,11, I12, R_2

8. C 12,3, I6, R_2

9. $\neg C \wedge C$ 10,13, I5, R_2

10. \boxed{F} 14, E2, R_2

11. $\neg A$

①. $\{z \rightarrow [y \rightarrow (r \vee s)], \neg(r \wedge s), r \leftrightarrow s\} \vdash (y \rightarrow z) \rightarrow \neg y$

1. $y \rightarrow z$ Ip R₀
2. $z \rightarrow [y \rightarrow (r \vee s)]$ p R₁
3. $\neg(r \wedge s)$ p R₁
4. $r \leftrightarrow s$ p R₁
5. $(r \rightarrow s) \wedge (s \rightarrow r)$ 4, E₂₂, R₂
6. $r \rightarrow s$ 5, I₃, R₂
7. $s \rightarrow r$ 5, I₄, R₂
8. $y \rightarrow [y \rightarrow (r \vee s)]$ 1, 2, I₃, R₂
9. $(y \wedge y) \rightarrow (r \vee s)$ 8, E₂₃, R₂
10. $y \rightarrow (r \vee s)$ 9, E₆, R₂
11. $\neg r \vee s$ 10, E₂₀, R₂
12. $\neg s \vee r$ 7, E₂₀, R₂
13. $\neg r \vee \neg s$ 3, E₁₆, R₂
14. $(\neg r \vee s) \wedge (\neg r \vee \neg s)$ 11, 13, I₅, R₂
15. $\neg r \vee (s \wedge \neg s)$ 14, E₁₄, R₂
16. $\neg r \vee F$ 15, E₈, R₂
17. $\neg r$ 16, E₂, R₂
18. $(\neg r \vee s) \wedge (s \vee r)$ 12, 13, I₅, R₂
19. $\neg s \vee (\neg r \wedge r)$ 18, E₁₄, R₂
 $\neg s \vee F$
20. $\neg s$ 19, E₂, R₂
21. $\neg r \wedge \neg s$ 17, 20, I₅, R₂
22. $\neg(r \vee s)$ 21, E₁₇, R₂
23. $\boxed{\neg y}$ 10, I₁₂, R₂

$$2. \textcircled{1} \vdash \neg x \rightarrow x$$

1. $\neg x$
2. $(\neg x) \rightarrow (\neg x \rightarrow \neg x)$
3. $\neg x \rightarrow \neg x$
4. $(\neg x \rightarrow \neg x) \rightarrow ((\neg x \rightarrow \neg x) \rightarrow x)$
5. $(\neg x \rightarrow \neg x) \rightarrow x$
6. $\neg x \rightarrow \neg x$
7. \boxed{x}

- $$\vdash (*) \vdash A \rightarrow A$$
- IP R₃ | 1. A IP R₃
 - , A₁
 - , 1, 2, MP
 - , A₃
 - , 3, 4, MP
 - (*)
 - , 5, 6, MP

$$3. \textcircled{2} \vdash (\neg \alpha) \rightarrow (\alpha \rightarrow \beta)$$

1. $\neg \alpha$
2. α
3. $\alpha \rightarrow (\neg \beta \rightarrow \alpha)$
4. $\neg \alpha \rightarrow (\neg \beta \rightarrow \alpha)$
5. $\neg \beta \rightarrow \alpha$
6. $\neg \beta \rightarrow \neg \alpha$
7. $(\neg \beta \rightarrow \neg \alpha) \rightarrow ((\neg \beta \rightarrow \neg \alpha) \rightarrow \beta)$
8. $(\neg \beta \rightarrow \neg \alpha) \rightarrow \beta$
9. $\boxed{\beta}$

- , IP R₃
- , IP R₃
- , A₁
- , A₁
- , 2, 3, MP
- , 1, 4, MP
- , A₃
- , 6, 7, MP
- , 5, 8, MP

$$\textcircled{3} \vdash ((\neg \beta) \rightarrow (\neg \alpha)) \rightarrow (\alpha \rightarrow \beta)$$

1. $\neg \beta \rightarrow \neg \alpha$
2. α
3. $(\neg \beta \rightarrow \neg \alpha) \rightarrow ((\neg \beta \rightarrow \neg \alpha) \rightarrow \beta)$
4. $(\neg \beta \rightarrow \neg \alpha) \rightarrow \beta$
5. $\alpha \rightarrow (\neg \beta \rightarrow \alpha)$
6. $\neg \beta \rightarrow \alpha$
7. $\boxed{\beta}$

- , IP R₃
- , IP R₃
- , A₃
- , 1, 3, MP
- , A₁
- , 2, 5, MP
- , 4, 6, MP

4. A₁: $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$

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|---------------------|-------------------|
| 1. α | IP R ₃ |
| 2. β | IP R ₃ |
| 3. $\boxed{\alpha}$ | 1-2 |

A₂: $\vdash (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

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|-----------------------------------------------------|----------------------------------------|
| 1. $\alpha \rightarrow (\beta \rightarrow \gamma)$ | IP R ₃ |
| 2. $\alpha \rightarrow \beta$ | IP R ₃ |
| 3. $(\alpha \wedge \beta) \rightarrow \gamma$ | 1, E ₂₃ , R ₂ |
| 4. $(\beta \wedge \alpha) \rightarrow \gamma$ | 3, E ₁₁ , R ₂ |
| 5. $\beta \rightarrow (\alpha \rightarrow \gamma)$ | 4, E ₂₃ , R ₂ |
| 6. $\alpha \rightarrow (\alpha \rightarrow \gamma)$ | 2, 5, I ₁₃ , R ₂ |
| 7. $(\alpha \wedge \alpha) \rightarrow \gamma$ | 6, E ₂₃ , R ₂ |
| 8. $\boxed{\alpha \rightarrow \gamma}$ | 7, E ₆ , R ₂ |

A₃: $\vdash (\neg \beta \rightarrow \neg \alpha) \rightarrow ((\neg \beta \rightarrow \alpha) \rightarrow \beta)$

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|----------------------------------------------------------|------------------------------------------------------|
| 1. $\neg \beta \rightarrow \neg \alpha$ | IP R ₃ |
| 2. $\neg \beta \rightarrow \alpha$ | IP R ₃ |
| 3. $\beta \vee \neg \alpha$ | 1, E ₂₀ , E ₉ , R ₂ |
| 4. $\beta \vee \alpha$ | 2, E ₂₀ , E ₉ , R ₂ |
| 5. $(\beta \vee \neg \alpha) \wedge (\beta \vee \alpha)$ | 3, 4, I ₅ , R ₂ |
| 6. $\beta \vee (\neg \alpha \wedge \alpha)$ | 5, E ₁₄ , R ₂ |
| 7. $\beta \vee F$ | 6, E ₈ , R ₂ |
| 8. $\boxed{\beta}$ | 7, E ₂ , R ₂ |