ENPM 667 Project 2 submission

Topic:

FINAL PROJECT

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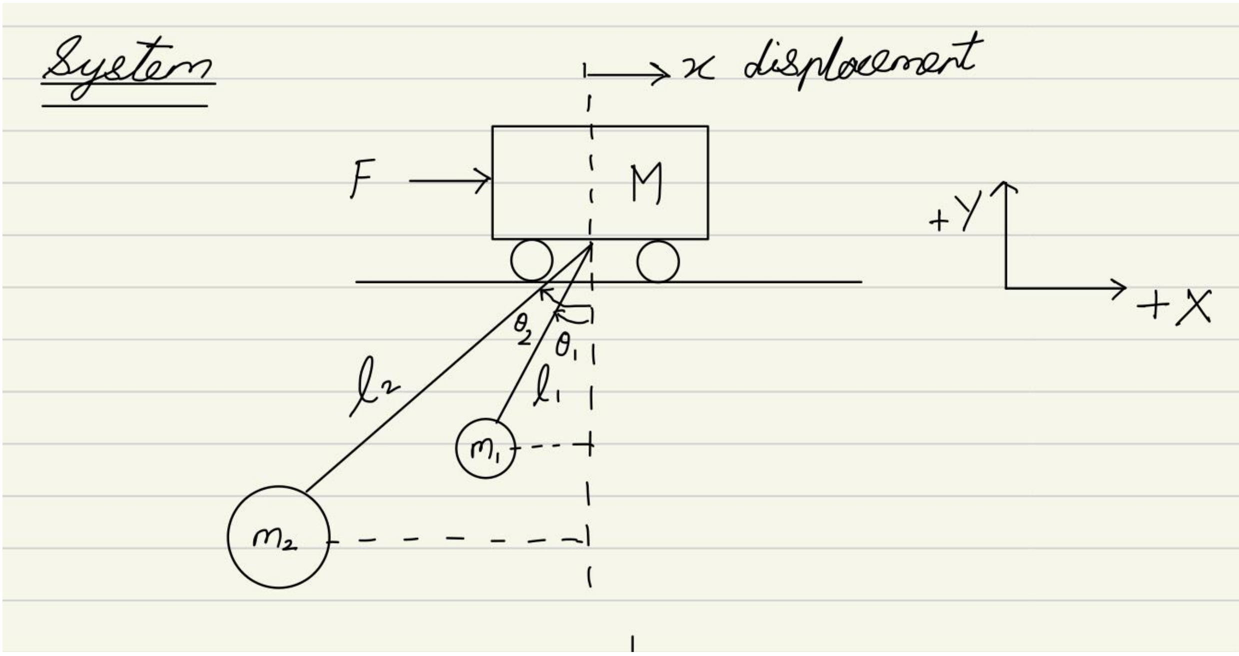
Contents

DYNAMIC MODEL OF THE SYSTEM:



M_{crane}: mass of crane

m_1: mass of ball 1

m_2: mass of ball 2

θ_1: angle made by rod of ball 1 with normal.

θ_2: angle made by rod of ball 2 with normal.

$\dot{\theta}_1$: angular velocity of rod 1.

$\dot{\theta}_2$: angular velocity of rod 2.

$\ddot{\theta}_1$: angular acceleration of rod 1.

$\ddot{\theta}_2$: angular acceleration of rod 2.

$\dot{x}$: velocity of cart.

$\ddot{x}$: acceleration of cart.

PART A: EQUATIONS OF MOTION AND NON-LINEAR STATE REPRESENTATION

The dynamic model of the system will be obtained using the Euler-Lagrange method. The states of the system are (x, θ_1, θ_2)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$

The Lagrangian(L) of the system describes the difference between the Kinetic Energy and the Potential Energy of the system.

Initial Conditions of the system are given by:

Let us start by defining the position of the objects in the system.

1. Position of the cart:

$$X_{crane} = x$$
$$Y_{crane} = o$$

2. Position of Mass_1:

$$\text{r} = x_1 i + y_1 j$$
$$x_1 = x - l_1 sin\theta_1$$
$$y_1 = -l_1 cos\theta_1$$

3. Position of Mass_2:

$$\text{r} = x_2 i + y_2 j$$
$$x_2 = x - l_2 sin\theta_2$$
$$y_2 = -l_2 cos\theta_2$$

4. Velocity of crane:

$$\dot{x}_{crane} = \dot{x}$$
$$\dot{y}_{crane} = 0$$

5. Velocity of Mass_1:

$$\dot{x}_1 = \dot{x} - l_1 \dot{\theta}_1 cos\theta_1$$
$$\dot{y}_1 = l_1 \dot{\theta}_1 sin\theta_1$$

6. Velocity of Mass_2:

$$\dot{x}_2 = \dot{x} - l_2 \dot{\theta}_2 cos\theta_2$$
$$\dot{y}_2 = l_2 \dot{\theta}_2 sin\theta_2$$

Calculation of the Lagrangian:

Difference between the kinetic energy and potential energy of the system

Kinetic Energy = Kinetic Energy of the cart + Kinetic Energy of the ball1 + Kinetic Energy of ball2

$$Kinetic\ Energy(KE) = \frac{1}{2}M_{crane}(\dot{x})^2 + \frac{1}{2}m_1\left(\dot{x_1}^2 + \dot{y_1}^2\right) + \frac{1}{2}m_2\left(\dot{x_2}^2 + \dot{y_2}^2\right)$$

$$KE = \frac{1}{2}M_{crane}(\dot{x})^2 + \frac{1}{2}m_1\left(\dot{x}^2 + {l_1}^2\dot{\theta_1}^2 - 2\dot{x}l_1\dot{\theta}_1 cos\theta_1\right) + \frac{1}{2}m_2\left(\dot{x}^2 + {l_2}^2\dot{\theta_2}^2 - 2\dot{x}l_2\dot{\theta}_2 cos\theta_2\right)$$

$$KE = \frac{1}{2}(M_{crane} + m_1 + m_2)\dot{x} - m_1 l_1\dot{x}\dot{\theta}_1 cos\theta_1 + \frac{1}{2}m_1{l_1}^2\dot{\theta_1}^2 - m_2 l_2\dot{x}\dot{\theta}_2 cos\theta_2 + \frac{1}{2}m_2{l_2}^2\dot{\theta_2}^2$$

The Potential Energy of the system is negative because we have considered the reference as the center of the cart and the balls are below the center of the cart.

$Potential\ Energy(PE) = -m_1 g l_1 cos\theta_1 - m_2 g l_2 cos\theta_2$

The Lagrangian is given by,

$$L = Kinetic\ Energy - Potential\ Energy$$

$$L = \frac{1}{2}(M_{crane} + m_1 + m_2)\dot{x} - m_1 l_1\dot{x}\dot{\theta}_1 cos\theta_1 + \frac{1}{2}m_1{l_1}^2\dot{\theta_1}^2 - m_2 l_2\dot{x}\dot{\theta}_2 cos\theta_2 + \frac{1}{2}m_2{l_2}^2\dot{\theta_2}^2 - m_1 g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2$$

Now we compute the equation of motion using this Lagrangian,

$$\frac{\partial L}{\partial \dot{x}} = M_{crane}\dot{x} - m_1 l_1\dot{\theta}_1 cos\theta_1 - m_2 l_2\dot{\theta}_2 cos\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M_{crane} + m_1 + m_2)\ddot{x} - m_1 l_1\left(\ddot{\theta}_1 cos\theta_1 - \dot{\theta_1}^2 sin\theta_1\right) - m_2 l_2\left(\ddot{\theta}_2 cos\theta_2 - \dot{\theta_2}^2 sin\theta_2\right)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1\left(2{l_1}^2\dot{\theta}_1 - 2\dot{x}l_1\dot{\theta}_1 cos\theta_1\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1{l_1}^2\ddot{\theta}_1 + m_1\dot{x}\dot{\theta}_1 l_1 sin\theta_1 - m_1\ddot{x}l_1 cos\theta_1$$

$$\frac{\partial L}{\partial \theta_1} = m_1\dot{x}l_1\dot{\theta}_1 sin\theta_1 - m_1 g l_1 sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2\left(2{l_2}^2\dot{\theta}_2 - 2\dot{x}l_2\dot{\theta}_2 cos\theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 {l_2}^2 \ddot{\theta}_2 + m_2 \dot{x} \dot{\theta}_2 l_2 sin\theta_2 - m_2 \ddot{x} l_2 cos\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = m_2 \dot{x} l_2 \dot{\theta}_2 sin\theta_2 - m_2 g l_2 sin\theta_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

$$F = (M_{crane} + m_1 + m_2)\ddot{x} - m_1 l_1 \left(\ddot{\theta}_1 cos\theta_1 - {\dot{\theta}_1}^2 sin\theta_1\right) - m_2 l_2 \left(\ddot{\theta}_2 cos\theta_2 - {\dot{\theta}_2}^2 sin\theta_2\right)$$

$$\ddot{x} = \frac{m_1 l_1 \left(\ddot{\theta}_1 cos\theta_1 - {\dot{\theta}_1}^2 sin\theta_1\right) + m_2 l_2 \left(\ddot{\theta}_2 cos\theta_2 - {\dot{\theta}_2}^2 sin\theta_2\right) + F}{M_{crane} + m_1 + m_2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0$$

$$m_1 {l_1}^2 \ddot{\theta}_1 + m_1 \dot{x} \dot{\theta}_1 l_1 sin\theta_1 - m_1 \ddot{x} l_1 cos\theta_1 - \left(m_1 \dot{x} l_1 \dot{\theta}_1 sin\theta_1 - m_1 g l_1 sin\theta_1\right) = 0$$

$${l_1}^2 \ddot{\theta}_1 - m_1 l_1 cos\theta_1 \ddot{x} + m_1 l_1 g.sin\theta_1 = 0$$

Rearranging the equation,

$$\ddot{\theta}_1 = \frac{m_1 l_1 cos\theta_1 \ddot{x} - m_1 l_1 g.sin\theta_1}{{l_1}^2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$

$$m_2 {l_2}^2 \ddot{\theta}_2 + m_2 \dot{x} \dot{\theta}_2 l_2 sin\theta_2 - m_2 \ddot{x} l_2 cos\theta_2 - \left(m_2 \dot{x} l_2 \dot{\theta}_2 sin\theta_2 - m_2 g l_2 sin\theta_2\right) = 0$$

$${l_2}^2 \ddot{\theta}_2 - m_2 l_2 cos\theta_2 \ddot{x} + m_2 l_2 g.sin\theta_2 = 0$$

Rearranging the equation,

$$\ddot{\theta}_2 = \frac{m_2 l_2 cos\theta_2 \ddot{x} - m_2 l_2 g.sin\theta_2}{{l_2}^2}$$

The equations of motion of the system are

$$\ddot{x} = \frac{F - \left(\left(\frac{g}{2}\right)\left(m_1 sin(2\theta_1) + m_2 sin(2\theta_2)\right) - m_1 l_1 {\dot{\theta}_1}^2 sin\theta_1 - m_2 l_2 {\dot{\theta}_2}^2 sin\theta_2\right)}{\left(M_{crane} + m_1 sin^2\theta_1 + m_2 sin^2\theta_2\right)}$$

$$\ddot{\theta}_1 = \frac{1}{l_1}(\ddot{x}cos\theta_1 - gsin\theta_1)$$

$$\ddot{\theta}_2 = \frac{1}{l_2}(\ddot{x}cos\theta_2 - gsin\theta_2)$$

PART B: LINEARIZING THE SYSTEM AND STATE SPACE REPRESENTATION OF THE SYSTEM

To get the state representation of the system we require A and B matrix which can be found by linearizing the system. First, we calculate the Jacobian of the of state space matrix. The state and state variables can be defined as,

The state representation is

$$X = AX + BU$$

$$\dot{x}(t) = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

To linearize the system, we substitute $\dot{x}(t) = 0$ and we substitute the equilibrium points in the Jacobian matrix. The equilibrium points are given as $x = 0,\ \theta_1 = 0,\ \theta_2 = 0$

Hence the matrix A of the system is a 6x6 matrix which is represented as

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \theta_1} & \frac{\partial \dot{x}}{\partial \dot{\theta}_1} & \frac{\partial \dot{x}}{\partial \theta_2} & \frac{\partial \dot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \theta_1} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} & \frac{\partial \ddot{x}}{\partial \theta_2} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_1}{\partial x} & \frac{\partial \dot{\theta}_1}{\partial \dot{x}} & \frac{\partial \dot{\theta}_1}{\partial \theta_1} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_1}{\partial \theta_2} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_1}{\partial x} & \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_1}{\partial \theta_1} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_1}{\partial \theta_2} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_2}{\partial x} & \frac{\partial \dot{\theta}_2}{\partial \dot{x}} & \frac{\partial \dot{\theta}_2}{\partial \theta_1} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_2}{\partial \theta_2} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_2}{\partial x} & \frac{\partial \ddot{\theta}_2}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_2}{\partial \theta_1} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_2}{\partial \theta_2} & \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2} \end{bmatrix}_{(0,0,0)}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{-m_1 g}{M_{crane}} & 0 & \dfrac{-m_2 g}{M_{crane}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{(M_{crane} + m_1)g}{M_{crane} l_1} & 0 & \dfrac{-m_2 g}{M_{crane} l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-m_1 g}{M_{crane} l_2} 0 & 0 & \dfrac{(M_{crane} + m_2)g}{M_{crane} l_2} & 0 \end{bmatrix}$$

The matrix B is the represented as

$$B = \begin{bmatrix} \dfrac{\partial \dot{x}}{\partial u} \\ \dfrac{\partial \ddot{x}}{\partial u} \\ \dfrac{\partial \dot{\theta}_1}{\partial u} \\ \dfrac{\partial \ddot{\theta}_1}{\partial u} \\ \dfrac{\partial \dot{\theta}_2}{\partial u} \\ \dfrac{\partial \ddot{\theta}_2}{\partial u} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ {}^{1}/_{M_{crane}} \\ 0 \\ {}^{1}/_{M_{crane} l_1} \\ 0 \\ {}^{1}/_{M_{crane} l_2} \end{bmatrix}$$

PART C: CONDITIONS OF THE CONTROLLABILITY

The controllability of the system can be found using the controllability matrix C

$$\text{C} = [B\ AB\ A^2B\ A^3B\ A^4B\ A^5B]$$

Here A and B matrix are calculated from the linearized system. For the system to be controllable the rank of the controllability matrix should be equal to the number of state variables of the system. In this case there are 6 states hence the if the rank of the controllability matrix is 6 then the system is controllable. We have used the rank function in MATLAB on the controllability matrix to get the rank of the matrix. The rank of the matrix is 6 which is equal to the number of states of the system is controllable. The controllability matrix should not be invertible for the system to be controllable. The determinant of the system is

$$\det(C) = \frac{g^{12} * (l_1 - l_2)^2}{(M * l_1 * l_2)^6}$$

Here

1. If $l_1 = l_2$ then the matrix becomes invertible hence the system is not controllable.
2. If the mass of the crane is very large, then determinant becomes almost zero and the system is no more controllable.

PART D: 1 LQR CONTROLLER DESIGN FOR LINEAR SYSTEM

Mass of Crane: 1000kg

Mass m1 = 100kg

Mass m2 = 100kg

Length of the ball1: 20m

Length of the ball2: 10m

g = 10m/s^2

The cost function of LQR controller is given by the following equation

$$J(k, \vec{x}) = \int_0^{\infty} X^T(t)\, QX(t) + U_k^T(t) R U_k(t) dt$$

We try to find the k for which the cost function is minimum. In our project we have set the Q and R matrix as follows,

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 250 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2000 \end{bmatrix}$$

$$R = 0.0001$$

The feedback gain k is given by equation

$$k = R^{-1} B_k^T P$$

Where P is symmetric positive definite solution of the Ricatti Equation given by

$$A^T P + PA - PBR^{-1}B^T P = -Q$$

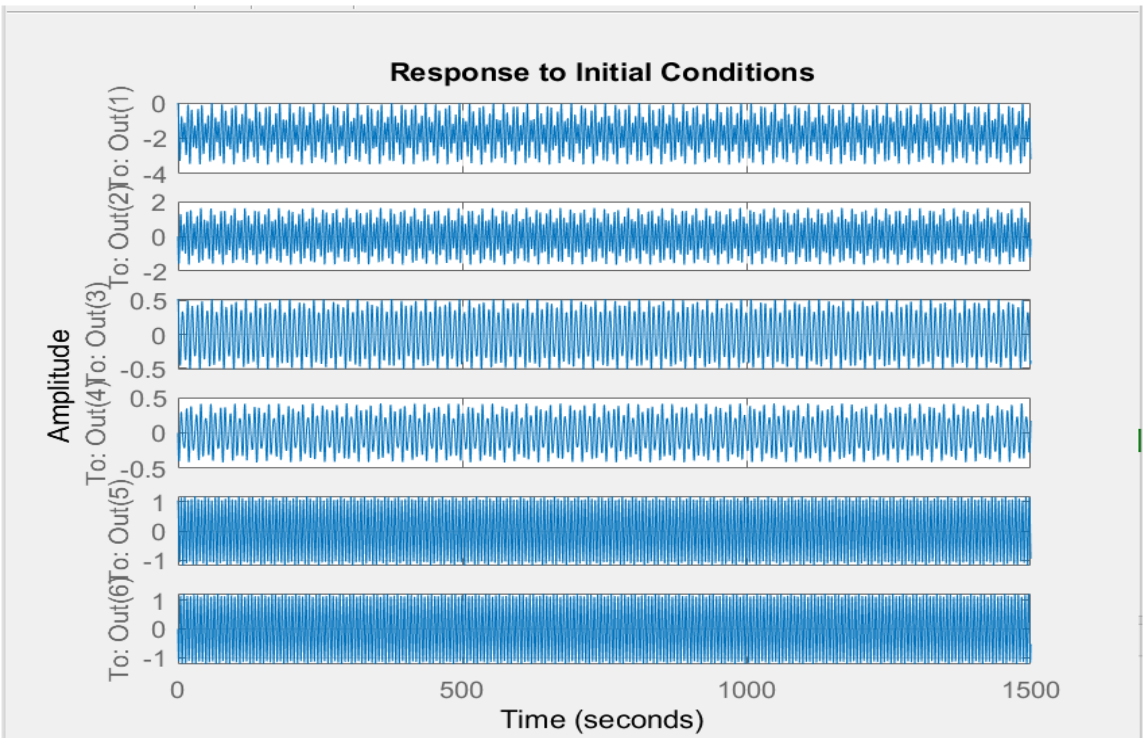
We have set the initial values of the states as follows

$$x = [0, 0, 30, 0, 60, 0]$$

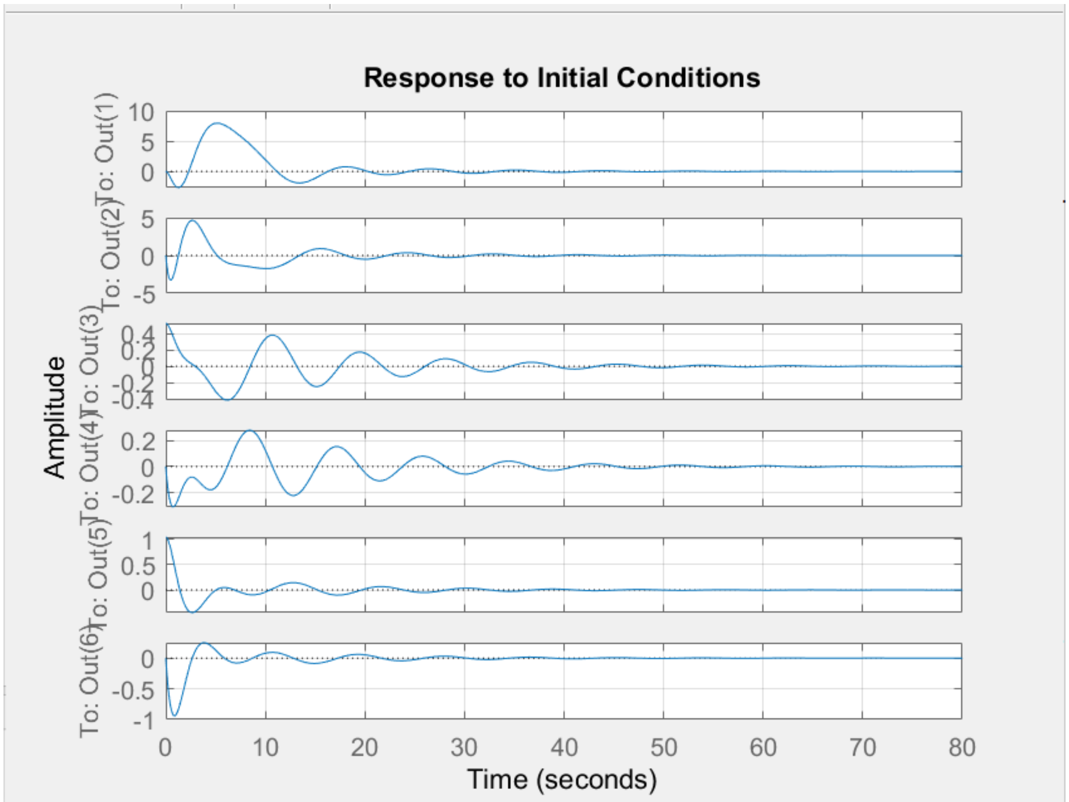
In this project we have used the `lqr` from MATLAB to find the feedback gain. This function returns poles, P and k. We used this feedback to derive the closed loop matrix A_k

$$A_k = A - Bk$$

We substituted this closed loop matrix with the feedback and created new state space representation. The following are the graphs before applying LQR controller to the system.



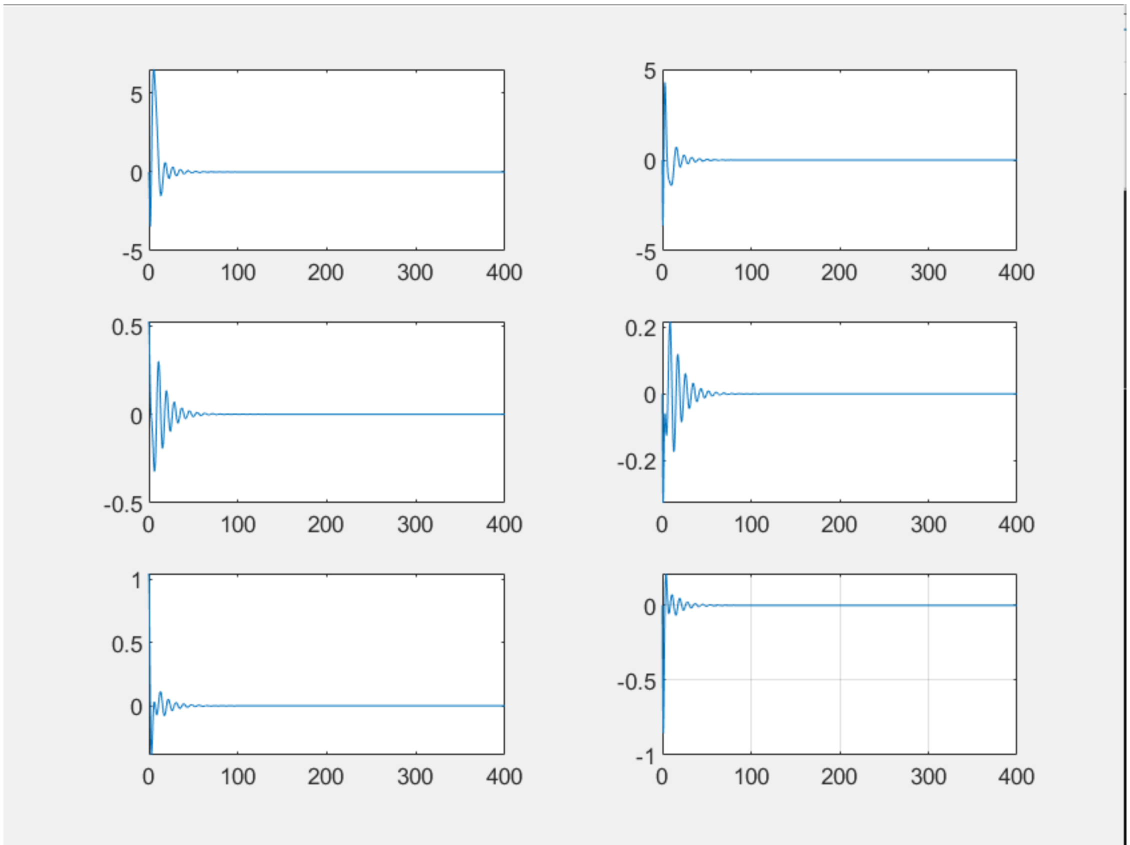
Graphs of the state response after applying LQR to the linearized system.



In the Non-Linear system we cannot assume that $x = 0,\ \theta_1 = 0,\ \theta_2 = 0$ here the input to the system will also be changing due to the feedback given by

$$F = F - kX$$

Due to which the state response will also vary for each time step. We have considered total time of 400 seconds with a time step of 0.001 second. The state response graphs in nonlinear system are represented as follows



Stability check using Lyapunov indirect method

In this method we check the eigen values of the system if the eigen values lie in the left half plane i.e., their real part is negative we infer that the system is stable. The eigen values are as follows

$$\lambda_1 = -0.7761 + 0.7555i$$

$$\lambda_2 = -0.7761 - 0.7555i$$

$$\lambda_3 = -0.3219 + 0.3843i$$

$$\lambda_3 = -0.3219 - 0.3843i$$

$$\lambda_5 = -0.0751 + 0.7285i$$

$$\lambda_6 = -0.0751 - 0.7285i$$

All eigenvalues real part is negative hence we conclude that the system is stabilized.

PART E: OBSERVERVABILITY FOR FOUR OUTPUT STATES COMBINATION

The condition to check if a system is observable for the selected states is to find the rank of the observability matrix. This can be done in MATLAB either by calculating the rank of the observability matrix using the

`Ob = obsv(A,C)` and `rank(Ob)` or by calculating the Observability Matrix and finding its rank. The observability condition is given by:

$$rank\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix} = 6$$

If this condition fails, the selected output states render the equation unobservable. Below are the selected state vectors and their observability analysis.

A) $x(t)$

The C matrix for this state is a *1*x*6* matrix given as,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

After calculating the rank of the observability matrix for the above condition, we get a rank of 6. Hence, the system is observable for the given combination of state variables.

```
Rank of matrix st1
     6

The output vector for state x is observable
```

B) $(\theta_1(t),\ \theta_2(t))$

The C matrix for this state is a *2*x*6* matrix given as,

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

After calculating the rank of the observability matrix for the above condition, we get a rank of 4. Hence, the system is NOT observable for the given combination of state variables.

```
Rank of matrix st2
     4

The output vector for state theta1 and theta2 is not observable
```

C) $(x(t),\ \theta_2(t))$

The C matrix for this state is a *2*x*6* matrix given as,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

After calculating the rank of the observability matrix for the above condition, we get a rank of 6. Hence, the system is observable for the given combination of state variables.

```
Rank of matrix st3
     6

The output vector for state x and theta2 is observable
```

D) $(x(t),\ \theta_1(t),\ \theta_2(t))$
The C matrix for this state is a $3x6$ matrix given as,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

After calculating the rank of the observability matrix for the above condition, we get a rank of 6. Hence, the system is observable for the given combination of state variables.

```
Rank of matrix st4
     6

The output vector for state x, theta1 and theta2 is observable
```

Hence, we conclude that the systems A), C) and D) are observable.

PART F: LUENBERGER OBSERVER

The basic task of an observer is to estimate states for a system which the controller cannot directly observe using a state output(s) which it can observe. We make use of the Luenberger observer on the three observable combination of the output states concluded in Part E. The observer is applied on both the linearized state space equation and the original non-linear state space equations. The methods are shown in the individual sections below.

Linear systems

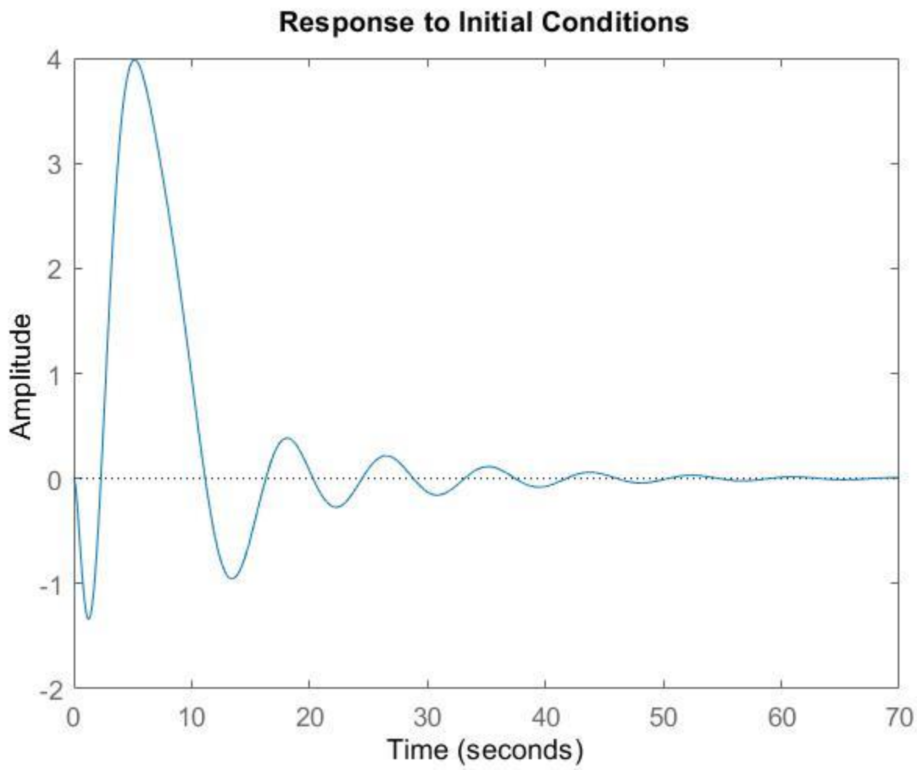
In order to obtain estimated states for Linear systems, the following state space closed loop system is solved:

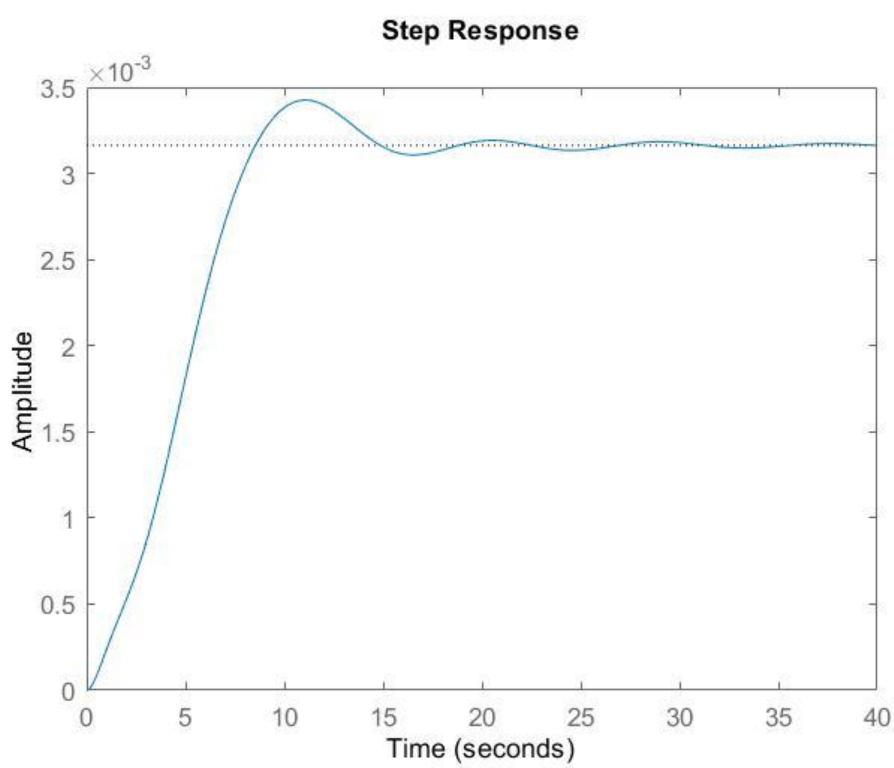
$$\begin{bmatrix} \dot{X}(t) \\ \dot{\hat{X}}(t) \end{bmatrix} = \begin{bmatrix} A - B_K K & B_K K \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} X(t) \\ \hat{X}(t) \end{bmatrix} + \begin{bmatrix} B_D \\ 0 \end{bmatrix} U_D(t)$$

Here,
$\hat{X}(t)$ are the estimated states
L is the observability gain

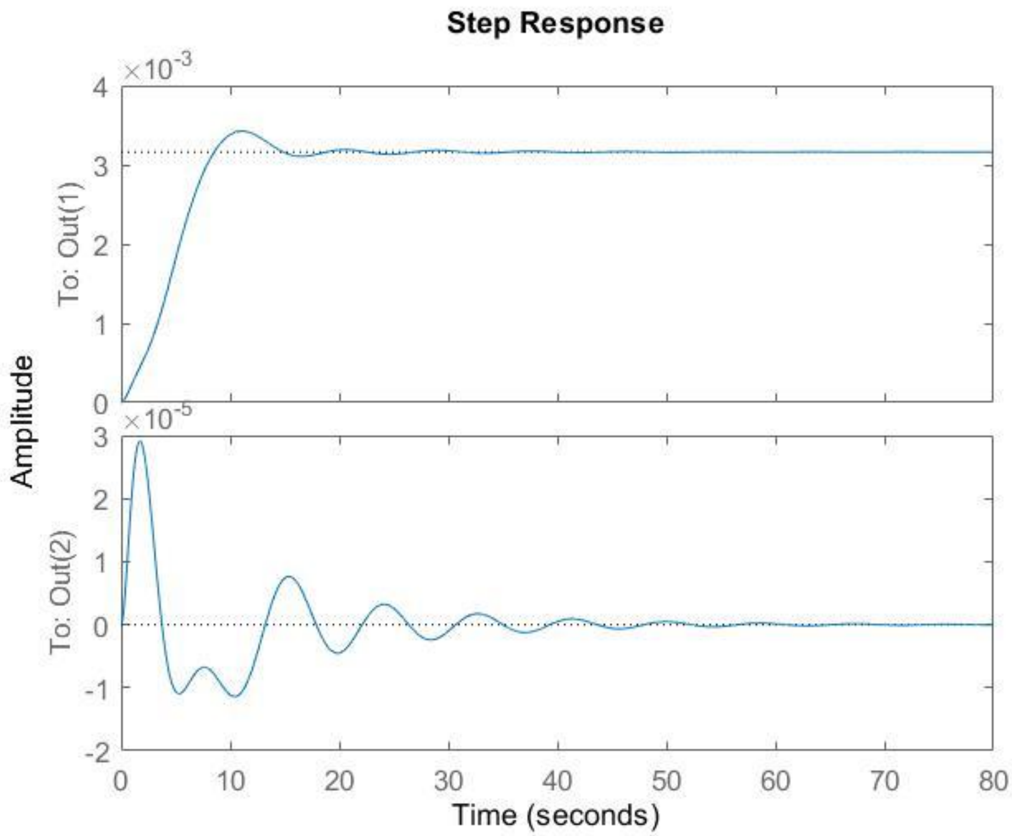
The observability gain matrix is calculated by pole-placement of the original linearized state space equation.

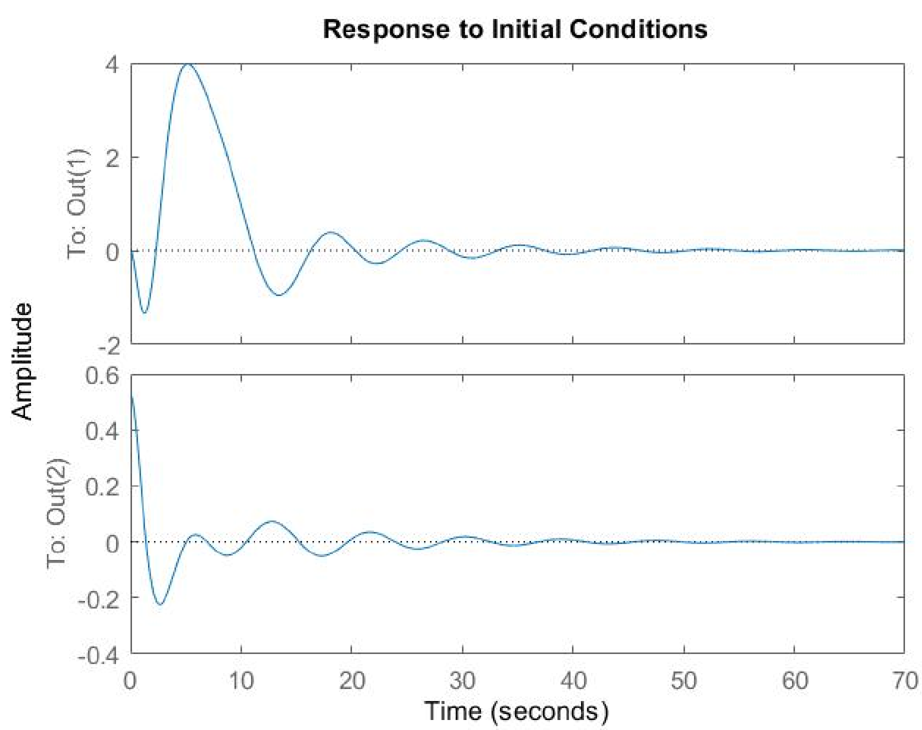
- States 1


Response to Initial Conditions
Amplitude
Time (seconds)
4
3
2
1
0
-1
-2
0
10
20
30
40
50
60
70

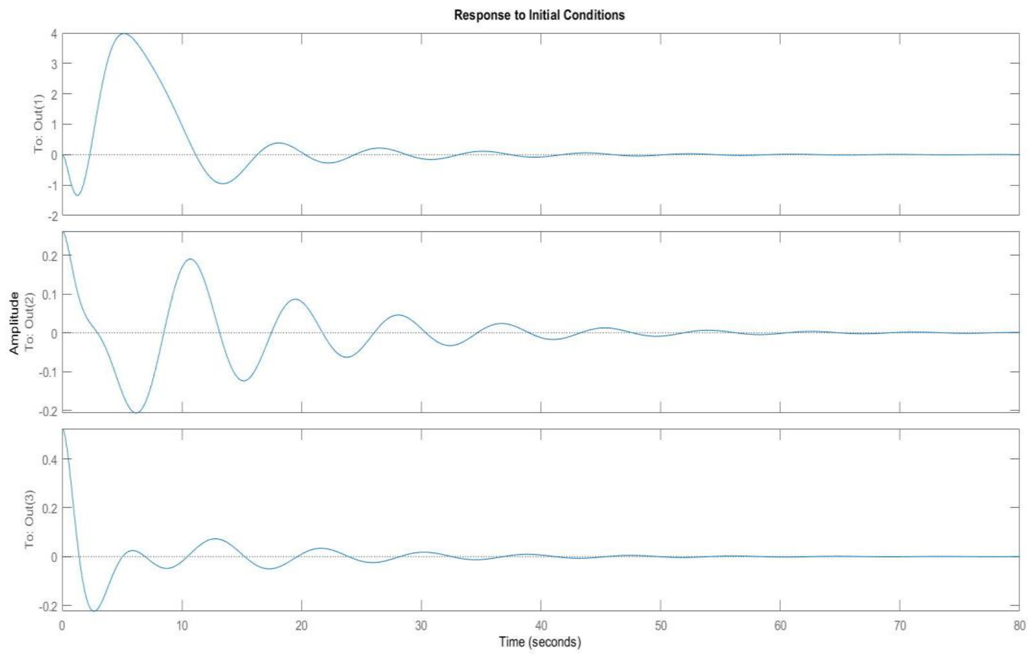

Step Response
×10^-3
Amplitude
Time (seconds)
3.5
3
2.5
2
1.5
1
0.5
0
0
5
10
15
20
25
30
35
40

- States 2

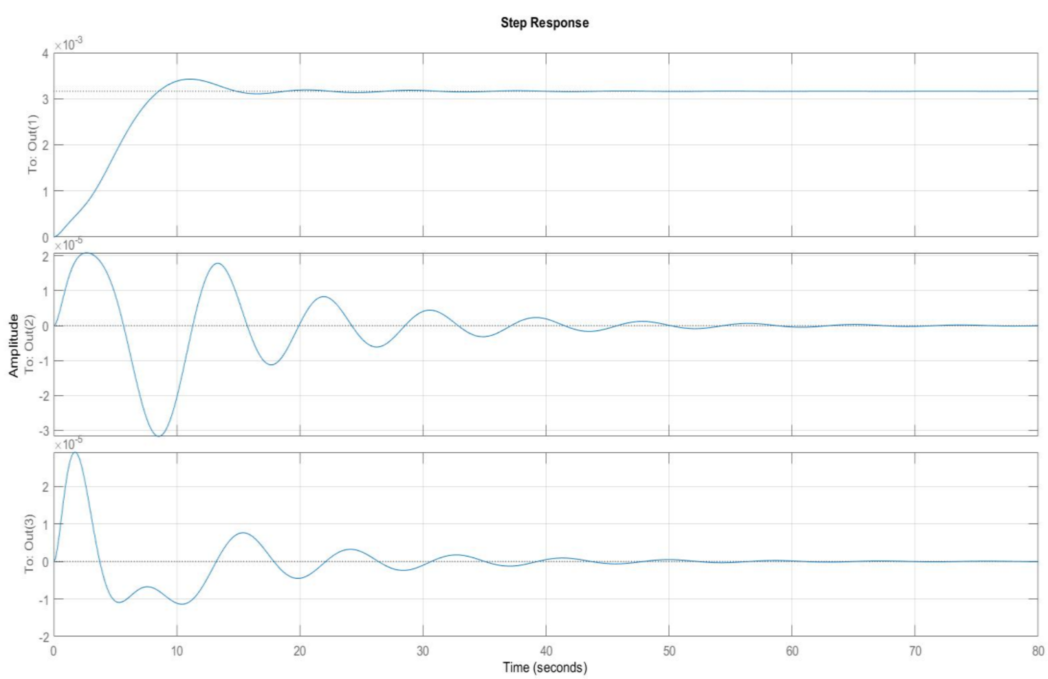

Step Response
×10^-3
4
3
2
1
0
To: Out(1)
×10^-5
3
2
1
0
-1
-2
To: Out(2)
Amplitude
0
10
20
30
40
50
60
70
80
Time (seconds)


Response to Initial Conditions
4
2
0
-2
To: Out(1)
0.6
0.4
0.2
0
-0.2
-0.4
To: Out(2)
Amplitude
0
10
20
30
40
50
60
70
Time (seconds)

- States 3



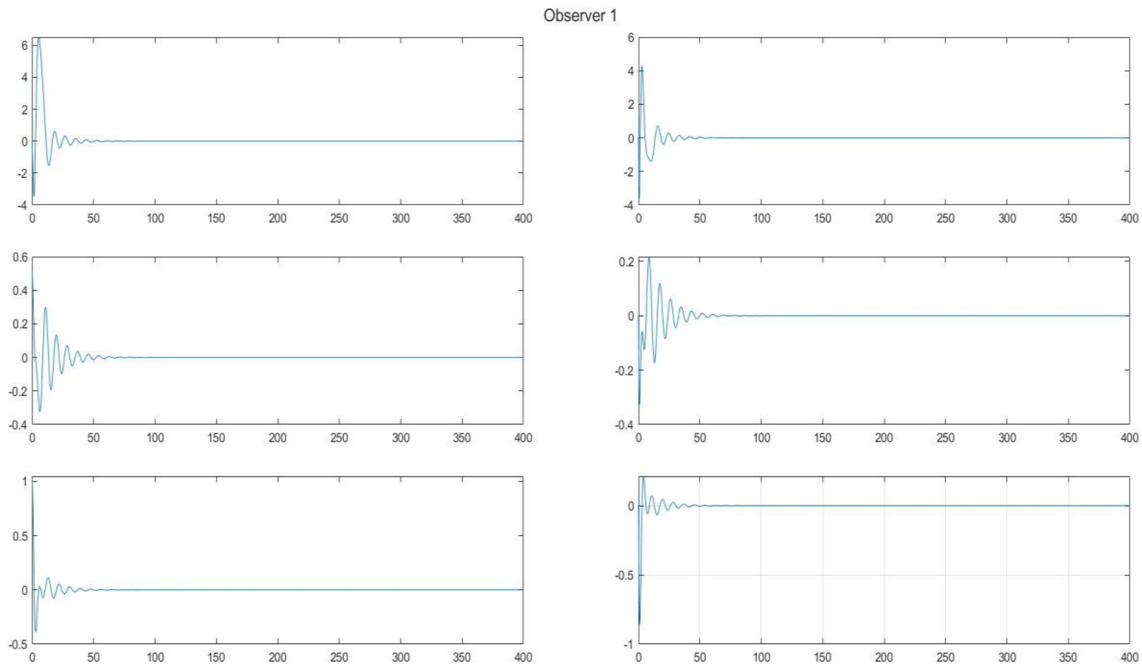
Response to Initial Conditions
To: Out(1)
Amplitude
To: Out(2)
To: Out(3)
Time (seconds)



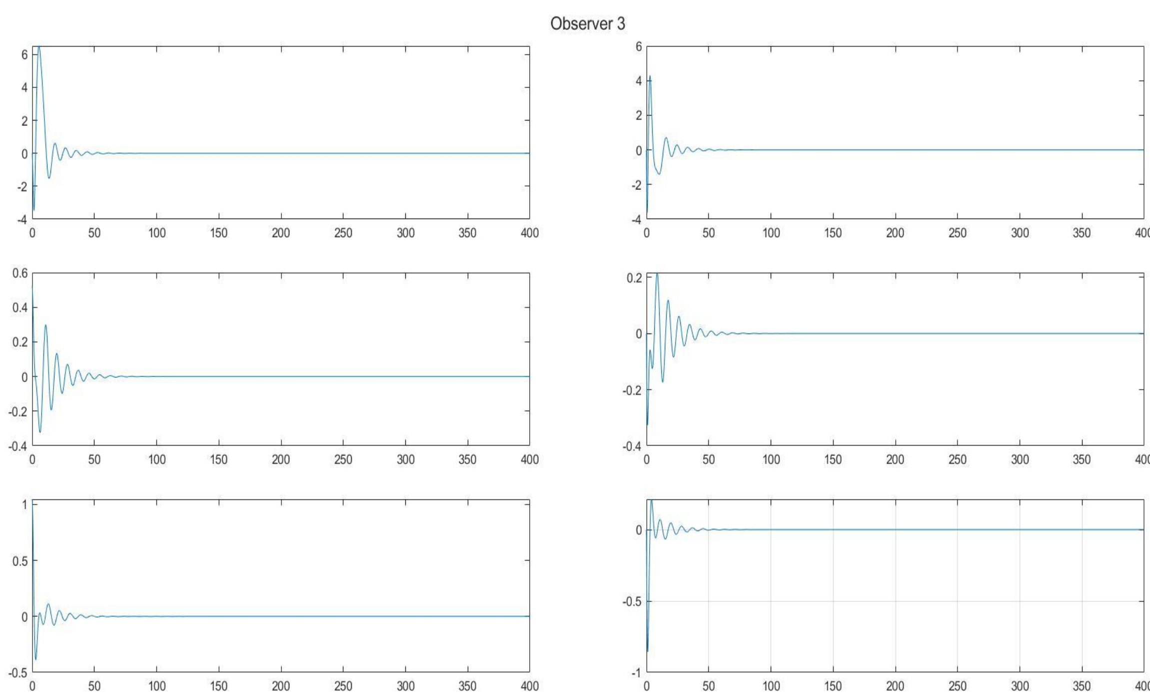
Step Response
To: Out(1)
Amplitude
To: Out(2)
To: Out(3)
Time (seconds)

Nonlinear systems

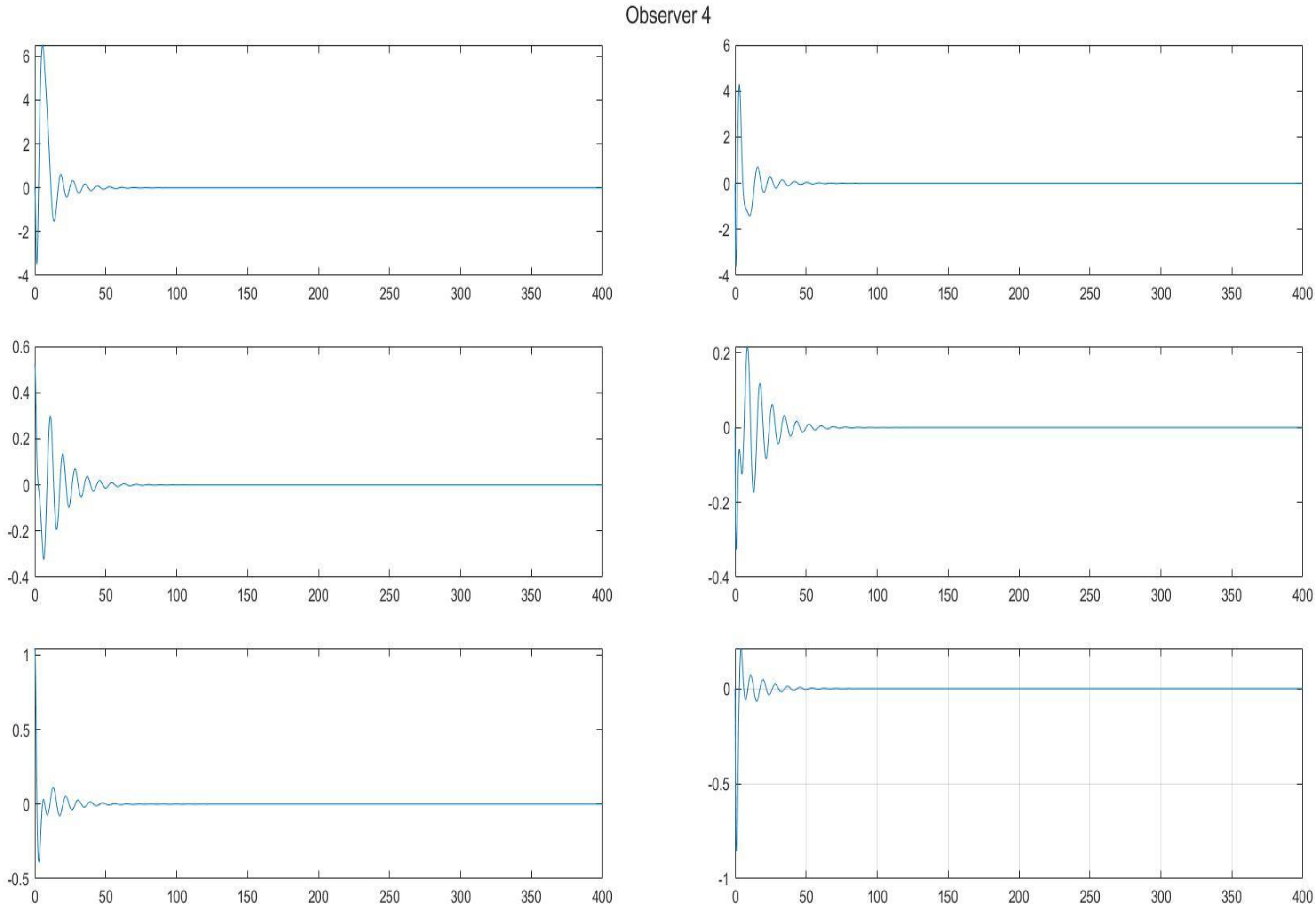
- States 1



- States 3



- States 4



PART G: LQG controller

The Linear Quadratic Gaussian Controller is an extended version of the LQR controller. The LQG controller is applied on nonlinear systems considering disturbances in the system. It uses feedback from a certain output observable state and computes the estimated states using the Kalman filter.

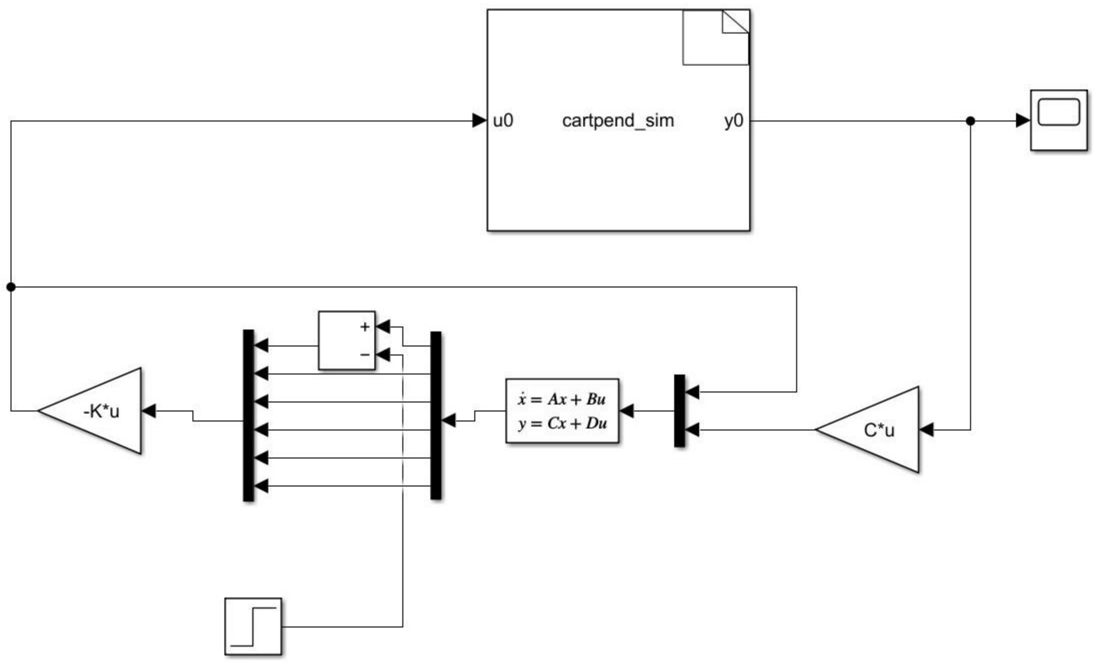


Figure 1: LQG Simulink model

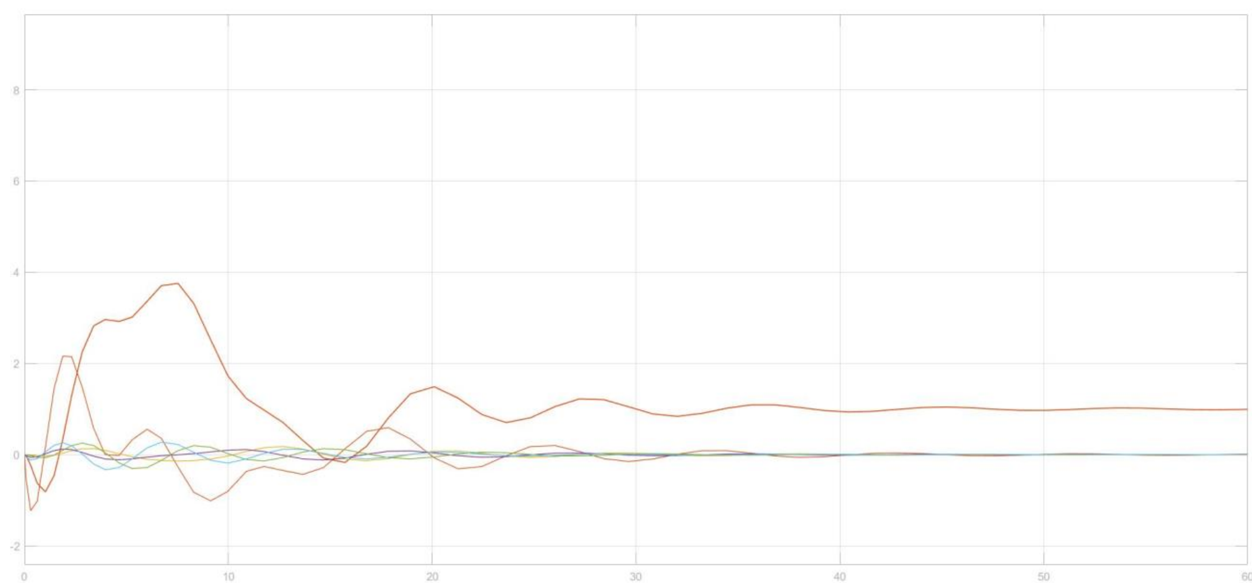


Figure 2: LQG output