GUC
German University in Cairo
الجامعة الألمانية بالقاهرة

Mathematics Department
Dr Rami Younes

Engineering Math 301
Final Exam

December 19^{th}**, 2019**

Barcode

Instructions

1. The **duration** of this exam is **180 minutes**.
2. Calculators are allowed.
3. There are **8 questions** in total.
4. There are **13 pages** of this exam in total. **Limit your answers** to pages **2-10**. Use pages **11-13** for **scratch**.
5. Find a **formula sheet** attached with this exam.
6. **For each question**, **read** the given **instructions** carefully before you start providing your answers.
7. **For full credit**, let your **answers** be **detailed** to the best of your knowledge. Straight **calculator answers** will result in significant **reduction** of your score.
8. Do not spend too much time on any particular question. **Tackle the questions you feel comforatble answering first.**
9. **Indicate** clearly **question numbers** that you are solving.

Question	1	2	3	4	5	6	7	8	Total
Max. score	12	14	12	12	12	16	10	12	100
Score									

1. Consider the function

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 4 \\ 8 - x & \text{if } 4 < x \le 8 \end{cases}$$

and denote by $\tilde{f}(x)$ its **odd periodic extension**.

(a) Find $\tilde{f}(-1)$ and $\tilde{f}(10)$.

(b) Show that for all $0 \le x \le 8$:

$$f(x) = \frac{32}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{8}.$$

(c) Deduce the value of the sum:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

(3+6+3 points)

Solution

(a) Notice that $\tilde{f}(x)$ is periodic of period $T = 2L$ with $L = 8$. Thus, $T = 16$ is a period of $\tilde{f}(x)$. Since $\tilde{f}(x)$ is odd and periodic it follows that

$$\begin{aligned} \tilde{f}(-1) &= -\tilde{f}(1) = -f(1) = -1 \\ \tilde{f}(10) &= \tilde{f}(10-16) = \tilde{f}(-6) = -\tilde{f}(6) = -f(6) = -(8-6) = -2. \end{aligned}$$

(b) $\tilde{f}(x)$ is periodic with period $T = 2L = 16$. Furthermore, $\tilde{f}(x)$ is continous and admits left and right derivatives at all x (a sketch of the graph of $\tilde{f}(x)$ shows these facts). Therefore, the representation by Fourier series theorem applies to $\tilde{f}(x)$ and it follows that for all x,

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{L}x + b_n \sin n\frac{\pi}{L}x,$$

where

i. for all $n \ge 0 : a_n = 0$ (since $\tilde{f}(x)$ is odd)

ii. for all $n \ge 1$:

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx \\ &= \frac{1}{4}\int_0^8 f(x)\sin\frac{n\pi}{8}x\,dx \\ &= \frac{1}{4}\left[\int_0^4 x\sin\frac{n\pi}{8}x\,dx + \int_4^8 (8-x)\sin\frac{n\pi}{8}x\,dx\right] \end{aligned}$$

Thus, integration by parts implies that for $n \geq 1$:

$$4b_n = -\frac{8}{n\pi}\left[x\cos\frac{n\pi}{8}x\right]_0^4 + \frac{8}{n\pi}\int_0^4 \cos\frac{n\pi}{8}x\,dx - \frac{8}{n\pi}\left[(8-x)\cos\frac{n\pi}{8}x\right]_4^8 - \frac{8}{n\pi}\int_4^8 \cos\frac{n\pi}{8}x\,dx$$
$$= \frac{8^2}{n^2\pi^2}\left[\sin\frac{n\pi}{8}x\right]_0^4 - \frac{8^2}{n^2\pi^2}\left[\sin\frac{n\pi}{8}\right]_4^8$$
$$= \frac{8^2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2}\right) = \frac{128}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Consequently,

$$b_{2n} = 0 \text{ (for } n \geq 1) \text{ and } \quad b_{2n+1} = 32\frac{(-1)^n}{\pi^2(2n+1)^2} \text{ (for } n \geq 0)$$

Hence, for all $-\infty < x < \infty$,

$$\begin{aligned}\tilde{f}(x) &= \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{8}x \\ &= \sum_{n=0}^{\infty} b_{2n+1}\sin\frac{(2n+1)\pi x}{8} \\ &= \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{8}\end{aligned}$$

In particular, for $0 \leq x \leq 8$, $\tilde{f}(x) = f(x)$ and it follows that:

$$f(x) = \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{8} \qquad (*)$$

(c) Let $x = 4$ in $(*)$. Thus,

$$\begin{aligned}f(4) &= \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi 4}{8} \\ \Longleftrightarrow 4 &= \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi}{2} \\ \Longleftrightarrow 4 &= \frac{32}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}(-1)^n \\ \Longleftrightarrow \frac{\pi^2}{8} &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.\end{aligned}$$

Therefore,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

2. (a) **Solve** the following ODE **according to the values of** ω:

$$y'' + \omega y = 0.$$

(b) **Solve** the following ODE for $\omega > \mathbf{1}$:

$$y'' + \omega y = \cos x - \sin x.$$

(7+7 points)

Solution

(a) The ODE is homogeneous, linear, of order 2. Its characteristic equation is

$$\lambda^2 + \omega = 0 \iff \lambda^2 = -\omega \qquad (E)$$

Distinguish 3 cases.

- $\omega = 0$:

$$\lambda = 0 \text{ is a double root of } (E) \Longrightarrow y = Ax + B,$$

where A and B are arbitrary constants.

- $\omega > 0$:

$$\lambda = \pm i\sqrt{\omega} \text{ are distinct imaginary roots of } (E) \Longrightarrow y = A\cos\sqrt{\omega}x + B\sin\sqrt{\omega}x,$$

where A and B are arbitrary constants.

- $\omega < 0$:

$$\lambda = \pm\sqrt{-\omega} \text{ are distinct real roots of } (E) \Longrightarrow y = Ae^{-\sqrt{-\omega}x} + Be^{\sqrt{-\omega}x},$$

where A and B are arbitrary constants.

(b) The ODE is linear non-homogeneous. It follows that its general solution is

$$y = y_H + y_p$$

where y_H is the general solution of its complementary equation and y_p any particular solution.

For $\omega > 1$, it follows that $y_H = A\cos\sqrt{\omega}x + B\sin\sqrt{\omega}x$. By the method of undetermined coefficients, a particular solution in the form

$$y_p = M\cos x + N\sin x$$

may be found.

$$y_p' = -M\sin x + N\cos x$$
$$y_p'' = -M\cos x - N\sin x$$

$$
\begin{aligned}
& y_p'' + \omega y_p = \cos x - \sin x \\
\Longleftrightarrow\ & -M\cos x - N\sin x + \omega(M\cos x + N\sin x) = \cos x - \sin x \\
\Longleftrightarrow\ & (-M + \omega M)\cos x + (-N + \omega N)\sin x = \cos x - \sin x \\
\Longleftrightarrow\ & M(\omega - 1) = 1 \quad \text{and} \quad N(\omega - 1) = 1 \\
\Longleftrightarrow\ & M = \frac{1}{\omega - 1} \quad \text{and} \quad N = -\frac{1}{\omega - 1}
\end{aligned}
$$

Therefore,

$$
y = A\cos\sqrt{\omega}x + B\sin\sqrt{\omega}x + \frac{1}{\omega - 1}\cos x - \frac{1}{\omega - 1}\sin x,
$$

where A and B are arbitrary constants..

3. (a) Solve the following initial value problem:

$$y''y = y'^2 + y', \qquad y(0) = 1, \qquad y'(0) = 1.$$

(b) Consider the two dimensional Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

i. Show that the function $u(x,y) = a\ln(x^2+y^2) + b$ is a solution of (1) for any a and b.

ii. Find a solution of (1) that satisfies the following boundary conditions

$$u \equiv 110 \text{ on the circle } x^2+y^2 = 1 \qquad \text{and} \qquad u \equiv 0 \text{ on the circle } x^2+y^2 = 100.$$

(6+6 points)

Solution

(a)

$$\begin{aligned}
y''y = y'^2 + y' &\iff y''y = y'(y'+1) \qquad (y' \neq -1 \text{ and } y \neq 0 \text{ for the solution of the IVP})\\
&\iff \frac{y''}{y'+1} = \frac{y'}{y}\\
&\iff \int \frac{y''}{y'+1}dx = \int \frac{y'}{y}dx\\
&\iff \ln|y'+1| = \ln k|y|, \qquad \text{for some constant } k > 0\\
&\iff y' - Cy = -1, \qquad \text{for some constant } C(= \pm k)
\end{aligned}$$

Notice that the initial consditions imply that

$$y'(0) - Cy(0) = -1 \Longrightarrow 1 - C = -1 \Longrightarrow C = 2.$$

Thus, y is a solution of the linear first order ODE: $y' - 2y = -1$. The solution formula for such ODEs then implies that

$$y = e^{2x}\left[-\int e^{-2x}dx + c\right] = ce^{2x} + \frac{1}{2}.$$

Again, the initial conditions imply that:

$$y(0) = c + \frac{1}{2} \iff c = \frac{1}{2}.$$

Therefore, $y = \dfrac{e^{2x}+1}{2}$.

(b) i. For any a and b,

$$\begin{aligned}
u_x &= \frac{2ax}{x^2+y^2} \\
u_{xx} &= 2a\frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2}\right] = 2a\frac{y^2-x^2}{(x^2+y^2)^2} \\
u_y &= \frac{2ay}{x^2+y^2} \\
u_{yy} &= 2a\frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right] = 2a\frac{x^2-y^2}{(x^2+y^2)^2}
\end{aligned}$$

Thus,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2a\frac{y^2-x^2}{(x^2+y^2)^2} + 2a\frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

which implies that $u(x,y)$ is a solution of the Laplace equation.

ii. Choose a and b so that $u(x,y) = a\ln(x^2+y^2)+b$ is the desired solution.

- On the circle $x^2+y^2=1$, $u(x,y) = a\ln 1 + b = b = 110$
- On the circle $x^2+y^2=100$,

$$u(x,y) = a\ln 100 + 110 = 0 \Longrightarrow a = -\frac{110}{2\ln 10} = -\frac{55}{\ln 10}$$

Therefore, the desired solution is

$$u(x,y) = -\frac{55}{\ln 10}\ln(x^2+y^2) + 110.$$

4. Consider the following ODE:

$$y'' - xy = 0 \tag{2}$$

(a) Show that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution of (2) if and only if

$$c_2 = 0 \quad \text{and} \quad c_n = \frac{1}{n(n-1)} c_{n-3} \quad \text{for } n \geq 3.$$

(b) Deduce two **linearly independent** power series solutions of (2) and give its general solution. For each of the solutions give c_n **explicitly** in terms of n.

(8+4 points)

Solution

(a) If $y = \sum_{n=0}^{\infty} c_n x^n$, the term-by-term differentiation property of power series implies that

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Thus,

$$y'' - xy = 0 \iff \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Reindexing both series to obtain only terms in x^n,

$$\begin{aligned}
&\iff \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0 \\
&\iff 2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0 \\
&\iff 2c_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} - c_{n-1}\right] x^n = 0 \\
&\iff c_2 = 0 \text{ and } (n+2)(n+1) c_{n+2} - c_{n-1} = 0, \text{ for all } n \geq 1
\end{aligned}$$

Therefore, the sequence of coefficients c_n is defined by the recurrence relation

$$c_2 = 0 \qquad \text{and} \qquad c_{n+2} = \frac{1}{(n+2)(n+1)} c_{n-1} \qquad n = 1, 2, \ldots$$

which can be rewritten as

$$c_2 = 0 \qquad \text{and} \qquad c_n = \frac{1}{n(n-1)} c_{n-3} \quad n = 3, 4, \ldots$$

(b) Since in each recurrence step we subtract 3 from n, we distinguish 3 cases

- $n = 3k, \quad (k \geq 1)$

$$\begin{aligned}
c_n &= \frac{1}{n(n-1)} c_{n-3} \\
&= \frac{1}{n(n-1)} \frac{1}{(n-3)(n-4)} c_{n-6} \\
&= \dots \\
&= \frac{1}{n(n-1)(n-3)(n-4)\dots 6.5.3.2} c_0
\end{aligned}$$

- $n = 3k+1, \quad (k \geq 1)$

$$\begin{aligned}
c_n &= \frac{1}{n(n-1)} c_{n-3} \\
&= \frac{1}{n(n-1)} \frac{1}{(n-3)(n-4)} c_{n-6} \\
&= \dots \\
&= \frac{1}{n(n-1)(n-3)(n-4)\dots 7.6.4.3} c_1
\end{aligned}$$

- $n = 3k+2, \quad (k \geq 1)$

$$\begin{aligned}
c_n &= \frac{1}{n(n-1)} c_{n-3} \\
&= \frac{1}{n(n-1)} \frac{1}{(n-3)(n-4)} c_{n-6} \\
&= \dots \\
&= \frac{1}{n(n-1)(n-3)(n-4)\dots 8.7.5.4} c_2 \\
&= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= \sum_{k=0}^{\infty} c_{3k} x^{3k} + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2} \\
&= \left(c_0 + \sum_{k=1}^{\infty} c_{3k} x^{3k} \right) + \left(c_1 + \sum_{k=1}^{\infty} c_{3k+1} x^{3k+1} \right) \\
&= c_0 \left(1 + \sum_{k=1}^{\infty} \frac{1}{3k(3k-1)\dots 6.5.3.2} x^{3k} \right) + c_1 \left(x + \sum_{k=1}^{\infty} \frac{1}{(3k+1)3k\dots 7.6.4.3} x^{3k+1} \right) \\
&= c_0 y_0 + c_1 y_1.
\end{aligned}$$

5. **A closed rectangular box is to have a volume** $V = 768$ cm^3. The cost of the material used in the box is 4 EGP/cm^2 for the sides and 6 EGP/cm^2 for top and bottom.

   (a) Give the cost of making such a box in terms of its dimensions.

   (b) How should the box be made in order to minimize the total cost of production?

   (c) What is the lowest cost of such a box?

(3+7+2 points)

Solution

(a) Let x and y be the length and width (in cm) of the top and bottom faces and let z be the height (in cm) of the box. The EGP cost of making such a box is

$$C(x, y, z) = 2(6xy) + 2(4xz) + 2(4yz) = 12xy + 8xz + 8yz.$$

(b) **First Approach**

The cost is to be minimized under the constraint of a fixed volume $V = 768$. Thus, we solve the constrained optimization problem using the Lagrange multiplier method.

The minimum cost is attained at a solution (x, y, z) of the system

$$\nabla C(x, y, z) = \lambda \nabla V(x, y, z) \text{ and } V(x, y, z) = xyz = 768.$$

That is,

$$\begin{cases} C_x(x,y,z) & = \lambda V_x(x,y,z) \\ C_y(x,y,z) & = \lambda V_y(x,y,z) \\ C_z(x,y,z) & = \lambda V_z(x,y,z) \\ V(x,y,z) & = 768 \end{cases} \iff \begin{cases} 12y + 8z & = \lambda yz & (1) \\ 12x + 8z & = \lambda xz & (2) \\ 8x + 8y & = \lambda xy & (3) \\ xyz & = 768 & (4) \end{cases}$$

Multiplying (1) by x, (2) by y and (3) by z yields the system:

$$\begin{cases} 12xy + 8xz & = 768\lambda & (5) \\ 12xy + 8zy & = 768\lambda & (6) \\ 8xz + 8yz & = 768\lambda & (7) \\ xyz & = 768 & (8) \end{cases}$$

Notice that

- $(8) \Longrightarrow x \neq 0, y \neq 0, \text{ and } z \neq 0$
- $(5) - (6) \Longrightarrow x = y$
- $(6) - (7) \Longrightarrow 3y = 2z$
- Subsitution into (8) implies that: $y(y)(3y/2) = 768 \Longrightarrow y = 8$. It follows that

$$x = y = 8 \text{ and } z = 12.$$

Another Appoach

$V(x, y, z) = xyz = 768 \Longrightarrow z = \dfrac{768}{xy}$ from which follows that

$$C = 12xy + \frac{6144(x+y)}{xy} = 12xy + \frac{6144}{x} + \frac{6144}{y}, \qquad (x, y) \neq (0, 0).$$

The minimum cost is attained at a critical point of C, that is, at a solution of

$$\begin{cases} C_x = 0 \\ C_y = 0 \end{cases} \iff \begin{cases} 12y - \dfrac{6144}{x^2} & = 0 \\ 12x - \dfrac{6144}{y^2} & = 0 \end{cases} \iff \begin{cases} y = \dfrac{512}{x^2} & (1) \\ x = \dfrac{512}{y^2} & (2) \end{cases}$$

Substitute (1) in (2) to obtain: $x^3 = 512 \iff x = 8$. Thus, $y = \dfrac{512}{8}$ and $(8, 8)$ is a unique critical point. Since a box that minimizes the cost must exist, it follows that the cost is minimum at the the dimensions:

$$x = y = 8 \text{ and } z = \frac{768}{8(8)} = 12.$$

Notice that the fact that C must attain a minimum at $(8, 8)$ may be verified using the second derivative test.

(c) The lowest cost is $C(8, 8, 12) = 12(8)(8) + 8(8)(12) + 8(8)(12) = 2304$.

6. (a) Let $r(t) = \begin{cases} 10\sin 2t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases}$. Show that the Laplace transform of $r(t)$ is $\mathcal{L}(r) = \dfrac{20(1-e^{-\pi s})}{s^2+4}$.

(b) **Knowing that**

$$\frac{1}{(s^2+4)(s^2+2s+2)} = -\frac{s+1}{10(s^2+4)} + \frac{s+3}{10(s^2+2s+2)},$$

find the inverse Laplace transform of **each** of the following functions:

i. $\dfrac{s-3}{s^2+2s+2}$ ii. $\dfrac{20}{(s^2+4)(s^2+2s+2)}$ iii. $\dfrac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}$.

(c) Solve the following initial value problem for a damped mass-spring system acted upon by a sinusoidal force for some time interval. **You may use the results you obtained in the above questions.**

$$y'' + 2y' + 2y = r(t), \qquad y(0) = 1, \qquad y'(0) = -5.$$

(4+8+4 points)

Solution

(a) Notice that $r(t) = 10[H(t) - H(t-\pi)]\sin 2t$. Thus,

$$\begin{aligned}
\mathcal{L}(r) &= 10\left[\mathcal{L}(H(t)\sin 2t) - \mathcal{L}(H(t-\pi)\sin 2t)\right] && \text{(linearity)} \\
&= 10\left[\mathcal{L}(\sin 2t) - e^{-\pi s}\mathcal{L}(\sin 2(t+\pi))\right] && \text{(t-shifting)} \\
&= 10\left[\mathcal{L}(\sin 2t) - e^{-\pi s}\mathcal{L}(\sin 2t)\right] \\
&= 10\left(1-e^{-\pi s}\right)\mathcal{L}(\sin 2t) \\
&= \frac{20(1-e^{-\pi s})}{s^2+4}
\end{aligned}$$

(b) i. Notice that

$$\frac{s-3}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} - \frac{4}{(s+1)^2+1}.$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{s-3}{s^2+2s+2}\right) &= \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - 4\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) \\
&= e^{-t}\cos t - 4e^{-t}\sin t.
\end{aligned}$$

ii. Notice that

$$\begin{aligned}
\frac{20}{(s^2+4)(s^2+2s+2)} &= -2\frac{s+1}{s^2+4} + 2\frac{s+3}{s^2+2s+2} \\
&= -2\frac{s}{s^2+4} - \frac{2}{s^2+4} + 2\frac{s+1}{(s+1)^2+1} + 4\frac{1}{(s+1)^2+1}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{20}{(s^2+4)(s^2+2s+2)}\right) &= -2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \mathcal{L}^{-2}\left(\frac{2}{s^2+4}\right) \\
&\quad + 2\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) + 4\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) \\
&= -2\cos 2t - \sin 2t + 2e^{-t}\cos t + 4e^{-t}\sin t.
\end{aligned}$$

iii. Since

$$\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} = e^{-\pi s}\mathcal{L}(-2\cos 2t - \sin 2t + 2e^{-t}\cos t + 4e^{-t}\sin t),$$

the $t-$shifting property implies that

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right) \\
&= H(t-\pi)\left[-2\cos 2(t-\pi) - \sin 2(t-\pi) + 2e^{-(t-\pi)}\cos(t-\pi) + 4e^{-(t-\pi)}\sin(t-\pi)\right] \\
&= H(t-\pi)\left[-2\cos 2t - \sin 2t - 2e^{-(t-\pi)}\cos t - 4e^{-(t-\pi)}\sin t\right]
\end{aligned}$$

(c) Let $\mathcal{L}(y) = Y$. Thus,

$$\begin{aligned}
\mathcal{L}(y') &= sY - y(0) = sY - 1 \\
\mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) = s^2Y - s + 5
\end{aligned}$$

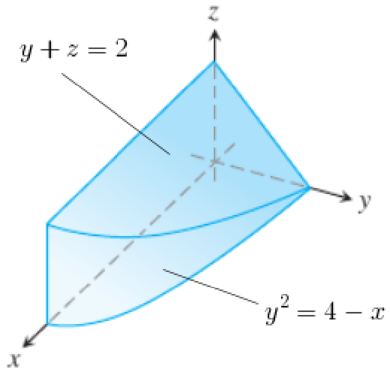
Taking the Laplace transform of both sides of the ODE and using the result of question (a) one obtains the subsidiary equation and its solution

$$\begin{aligned}
s^2Y - s + 5 + 2sY - 2 + 2Y &= \frac{20(1-e^{-\pi s})}{s^2+4} \\
\Longleftrightarrow Y &= \frac{s-3}{s^2+2s+2} + \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}
\end{aligned}$$

Taking the inverse transform of the last equation, one obtains the solution of the IVP

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left(\frac{s-3}{s^2+2s+2}\right) + \mathcal{L}^{-1}\left(\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right) - \mathcal{L}^{-1}\left(\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right) \\
&= e^{-t}\cos t - 4e^{-t}\sin t - 2\cos 2t - \sin 2t + 2e^{-t}\cos t + 4e^{-t}\sin t \\
&\qquad - H(t-\pi)\left[-2\cos 2t - \sin 2t - 2e^{-(t-\pi)}\cos t - 4e^{-(t-\pi)}\sin t\right] \\
&= \begin{cases} 3e^{-t}\cos t - 2\cos 2t - \sin 2t & \text{if } 0 < t < \pi \\ 3e^{-t}\cos t + e^{-(t-\pi)}(2\cos t + 4\sin t) & \text{if } t > \pi \end{cases}.
\end{aligned}$$

7. Consider the region D, shown in the figure, that is bounded by the coordinate planes, the plane $y+z=2$, and the parabolic cylinder $y^2=4-x$.



(a) Find a parametrization of S, where S is the piece of cylinder bounding D.

(b) Compute the surface area of S.

(4+6 points)

Solution

(a) S is a graph over the triangle $T=\{(y,z)|0\le y\le 1, 0\le z\le 2-y\}$. Thus, a parametrization of S is

$$\mathbf{r}(y,z)=(4-y^2,y,z),\,(y,z) \text{ in } T.$$

(b) Area $=\displaystyle\iint_T |\mathbf{r}_y\times\mathbf{r}_z|dA$

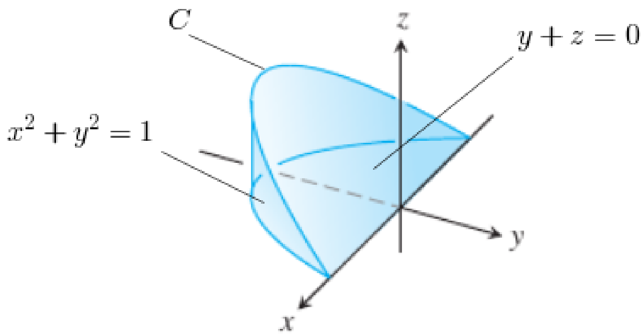
$$\mathbf{r}_y=(-2y,1,0) \qquad \mathbf{r}_z=(0,0,1)$$

$$\mathbf{r}_y\times\mathbf{r}_z=\mathbf{i}+2y\mathbf{j} \qquad |\mathbf{r}_x\times\mathbf{r}_y|=\sqrt{1+4y^2}$$

Thus,

$$\begin{aligned}
\text{Area} &= \int_0^2\int_0^{2-y}\sqrt{1+4y^2}dzdy\\
&= \int_0^2(2-y)\sqrt{1+4y^2}dy\\
&= 2\int_0^2\sqrt{1+4y^2}dy-\int_0^2 y\sqrt{1+4y^2}dy\\
&\underbrace{=}_{y=\frac{\tan t}{2}} \int_0^{\tan^{-1}4}\sec^3 t\,dt-\frac{1}{12}\left[(1+4y^2)^{3/2}\right]_0^2 \qquad \left(\text{compute }\int\sec^3 t\,dt\text{ aside}\right)\\
&= \frac{1}{2}\left[\sec x\tan x+\ln(\sec x+\tan x)\right]_0^{\tan^{-1}4}-\frac{1}{12}(17\sqrt{17}-1)\\
&= 2\sqrt{17}+\frac{1}{2}\ln(\sqrt{17}+4)-\frac{17\sqrt{17}-1}{12}\\
&= \frac{7\sqrt{17}+1+6\ln(\sqrt{17}+4)}{12}, \qquad \left(\text{since }\sec^2(\tan^{-1}4)=1+\tan^2(\tan^{-1}4)=17\right).
\end{aligned}$$

8. Consider the wedge W, shown in the figure, cut out from the cylinder $x^2+y^2=1$ by the planes $z=0$ and $z+y=0$.



(a) Compute $\int_C xdx + ydy + zdz$, where C is the boundary curve of W that is the intersection of the cylinder and the plane $y+z=0$. Assume that C is **positively oriented by the upward pointing unit normal** of the plane $y+z=0$.

(b) Consider the vector field

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2}+x)\mathbf{k}$$

i. Is $\mathbf{F}$ conservative? Justify your answer.

ii. Compute the net flux of $\mathbf{F}$ across the boundary of the wedge W.

(4+(3+5) points)

Solution

(a) **First Approach**

Notice that $xdx+ydy+zdz = d\left(\frac{x^2+y^2+z^2}{2}\right)$. Thus, the differential form $xdx+ydy+zdz$ is exact (with potential function $f(x,y,z) = \frac{x^2+y^2+z^2}{2}$) and

$$\begin{aligned}\int_C xdx+ydy+zdz &= \int_{(-1,0,0)}^{(1,0,0)} xdx+ydy+zdz \\ &= f(1,0,0) - f(-1,0,0) \\ &= 0.\end{aligned}$$

Another Approach

A parametrization of C is

$$\mathbf{r}(t) = (\cos t, \sin t, -\sin t), \qquad \pi \le t \le 2\pi.$$

Thus,

$$\begin{aligned}
\int_C xdx + ydy + zdz &= \int_\pi^{2\pi} (-\cos t \sin t + \sin t \cos t - \sin t \cos t)dt \\
&= \left[\frac{\cos^2 t}{2}\right]_\pi^{2\pi} \\
&= 0.
\end{aligned}$$

(b) Consider the vector field

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + x)\mathbf{k}$$

i. The domain of $\mathbf{F}$ is $\mathbb{R}^3$ which is simply connected. Thus, $\mathbf{F}$ is conservative if and only if curl $\mathbf{F} = \mathbf{0}$.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = (2xye^{xy^2} - x\cos xz)\mathbf{i} + \cdots \neq \mathbf{0}.$$

Therefore, $\mathbf{F}$ is not conservative (over $\mathbb{R}^3$).

ii. The components of $\mathbf{F}$ have continuous partial derivatives in W and the boundary of W is a closed piecewise smooth surface. Thus, the divergence theorem applies and

$$\begin{aligned}
\text{Flux} &= \iiint_W \text{div } \mathbf{F}\, dV && (\text{div } \mathbf{F} = 3y) \\
&= 3\iiint_W y\, dV \\
&= \int_\pi^{2\pi}\int_0^1\int_0^{-r\sin\theta} (r\sin\theta) r dz dr d\theta && \text{(change to cylindrical coordinates)} \\
&= -3\left(\int_\pi^{2\pi} \sin^2\theta\, d\theta\right)\left(\int_0^1 r^3 dr\right) \\
&= -\frac{3}{4}\left(\int_\pi^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta\right) \\
&= -\frac{3\pi}{8}.
\end{aligned}$$

End of Exam