COMM 401: Signals & Systems Theory

Lecture 4

- **Properties of LTI Systems**
- **(Cont.)**

Remember: Convolution Sum

The output of LTI system is given by:

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

Symbolic representation of convolution

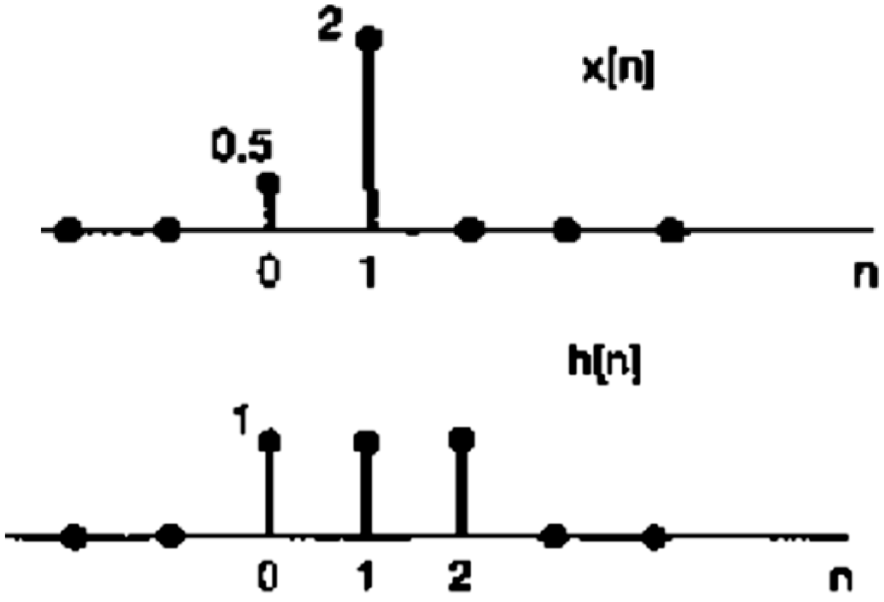
Impulse Response

$$\boxed{y[n] = x[k] * h[n]}$$

LTI systems are fully described by their impulse response

- **Example:**

A LTIS has impulse response h[n] shown in the Fig, find the output y[n] if the input to the system is given by x[n]



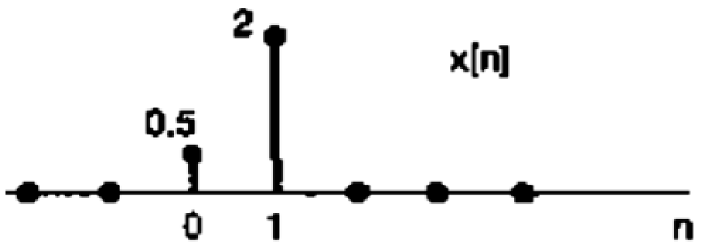
The output is given by:

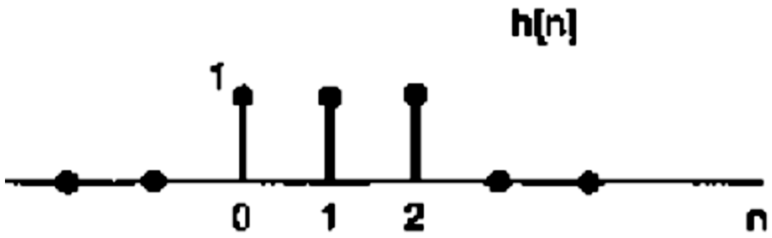
$$y[n]=x[n]*h[n] \Longleftrightarrow y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

We have different method to find the convolution:

First method:

Since x[n] has finite length

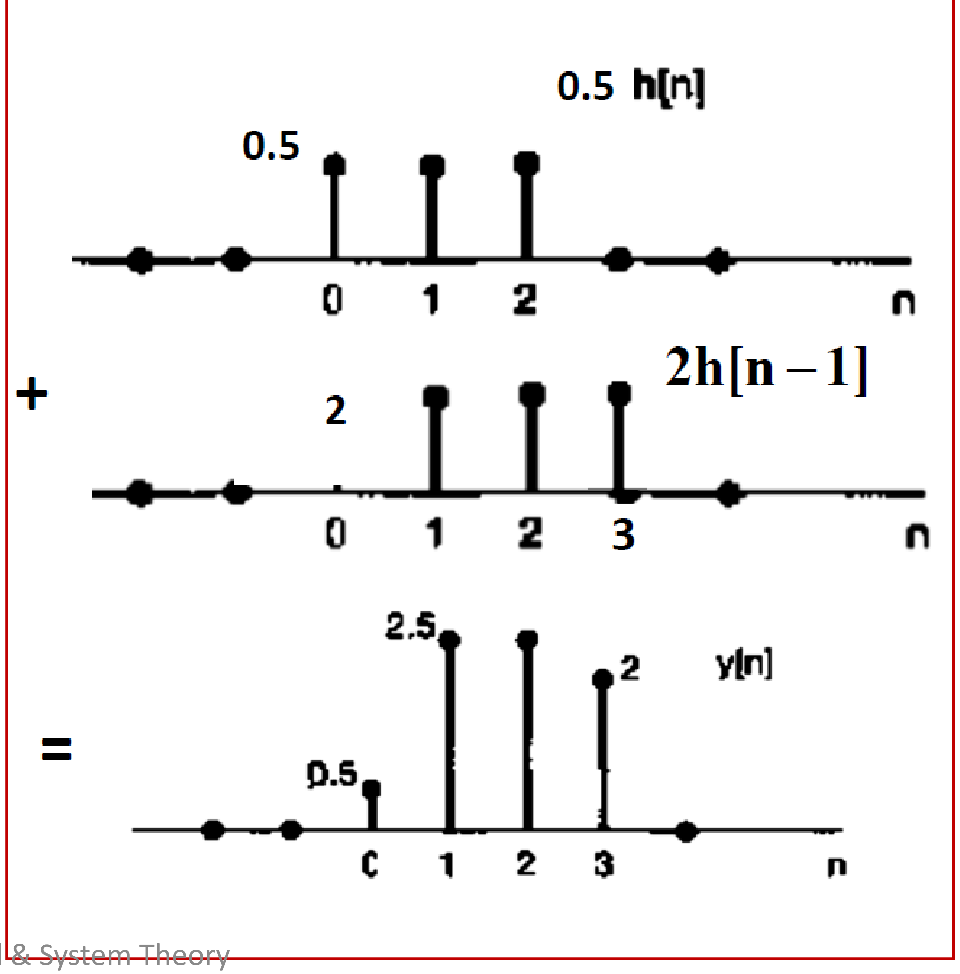




Rules:

$$x(t) * \delta(t) = x(t)$$
$$x(t) * \delta(t-a) = x(t-a)$$
$$x(t-b) * \delta(t-a) = x(t-a-b)$$

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= (0.5\delta[n] + 2\delta[n-1]) * h[n] \\
&= 0.5\,\delta[n] * h[n] + 2\delta[n-1] * h[n] \\
&= 0.5\,h[n] + 2h[n-1]
\end{aligned}
$$



Second method (similar to first method):

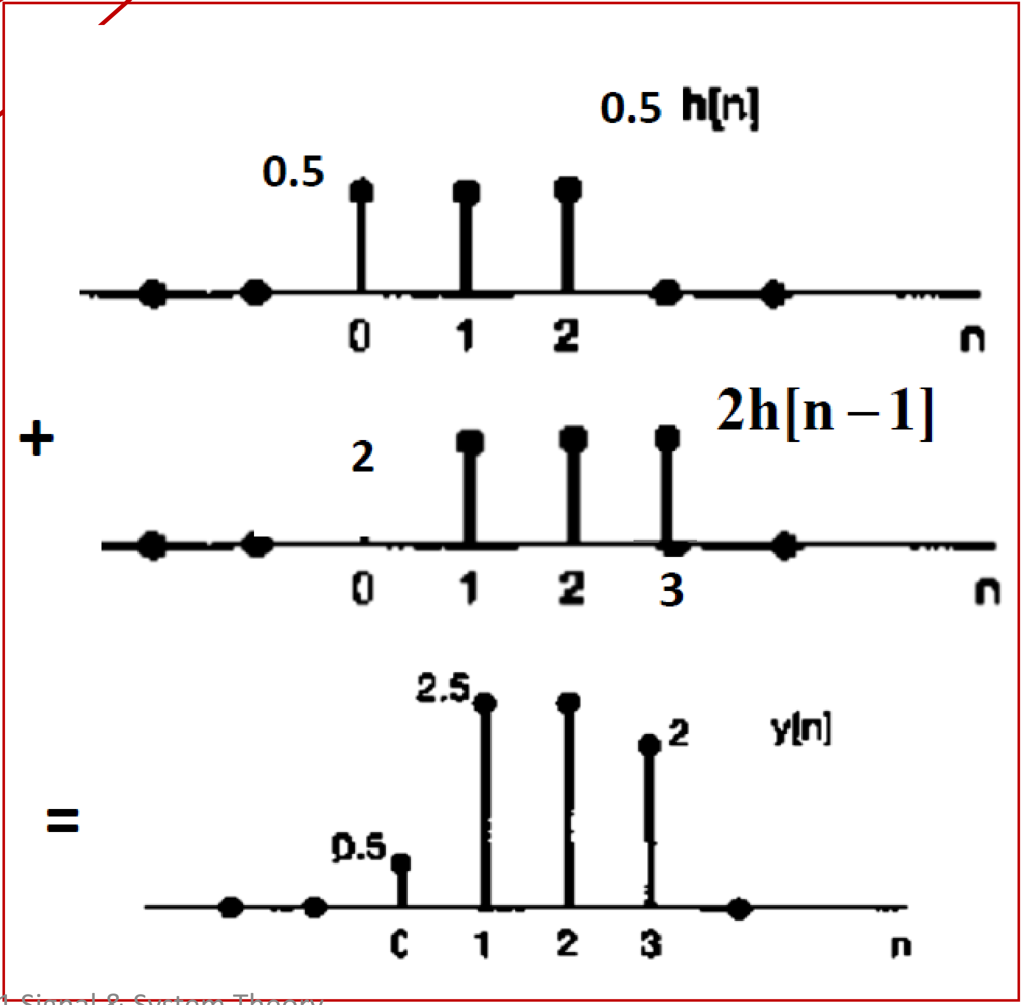
Since x[n] has finite length

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=0}^{1} x[k]h[n-k]$$

$$= x[0]h[n] + x[1]h[n-1]$$

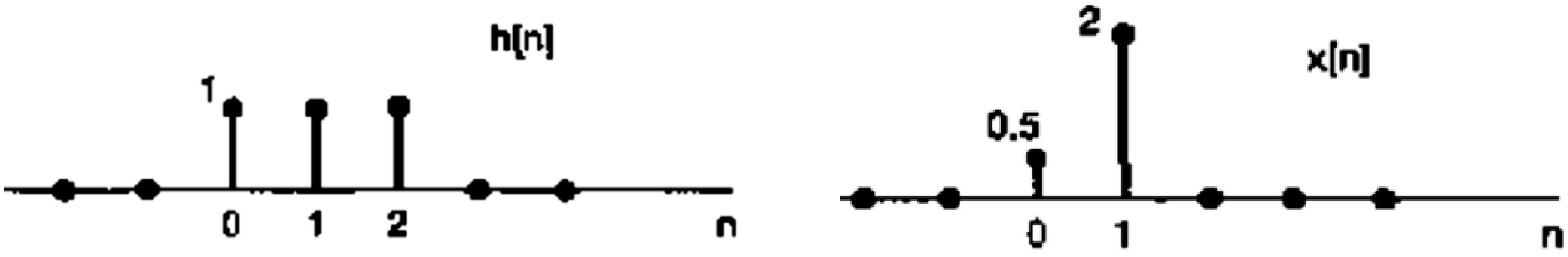
$$= 0.5h[n] + 2h[n-1]$$



3rd method: (General)

$$y[n] = x[n] * h[n] \iff y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Example: A LTIS has impulse response h[n] shown in the Fig, find the output y[n] if the input to the system is given by x[n]



Steps:

1- To get x[k], rename the axis to be k instead of n

2- To get h[n-k] do the following:

- Rename the axis to be k instead of n.
- Make a time shift and reversal to get h[n-k] .

3- The output y[n] is obtained by multiplying x[k] and h[n-k] and sum the result for all k

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

$$For\ n<0,\ \ y[n]=0$$

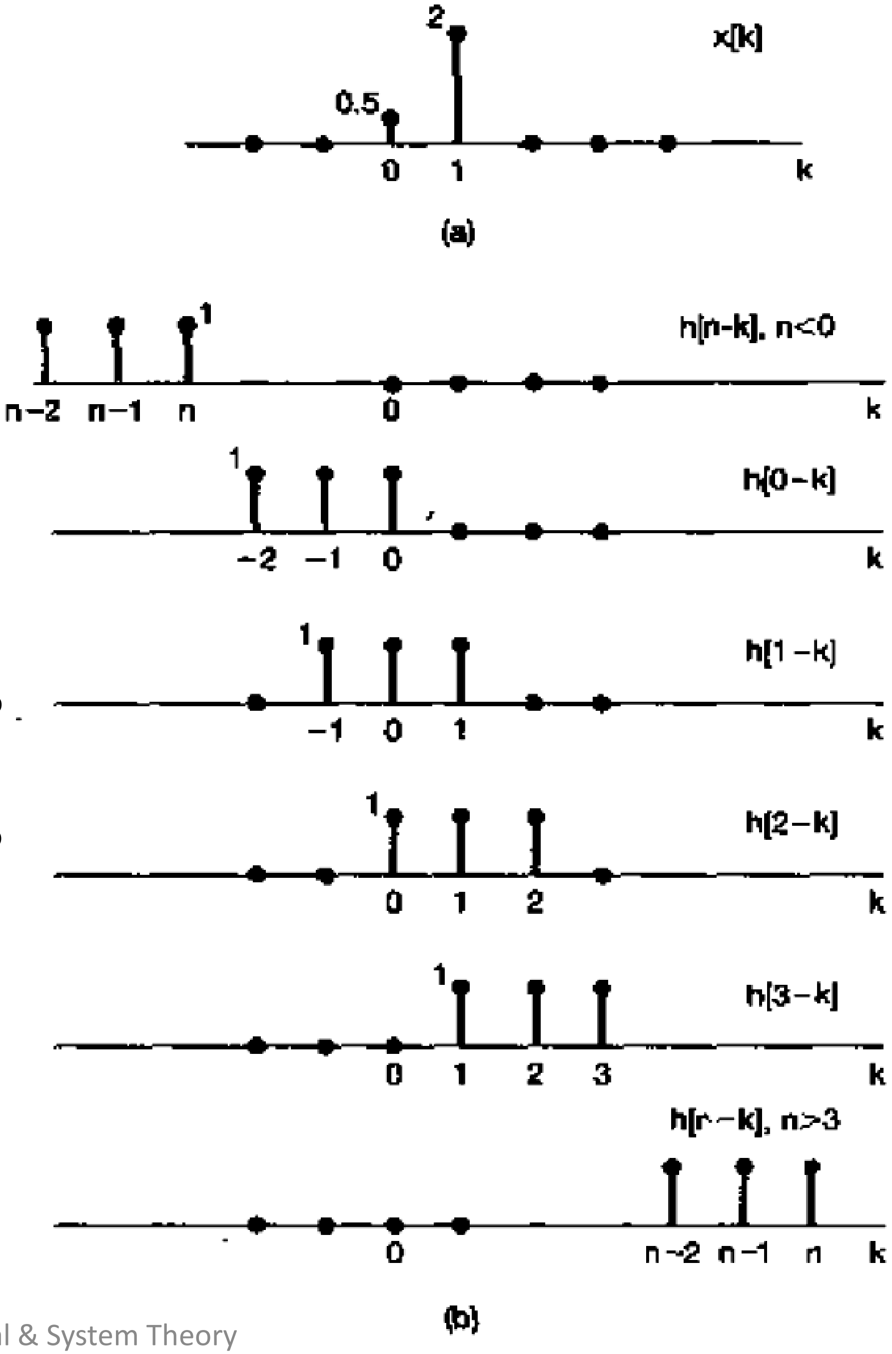
$$For\, n=0, \quad y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]=0.5$$

$$For\, n=1, \quad y[1]=\sum_{k=-\infty}^{\infty} x[k]h[1-k]=0.5+2=2.5$$

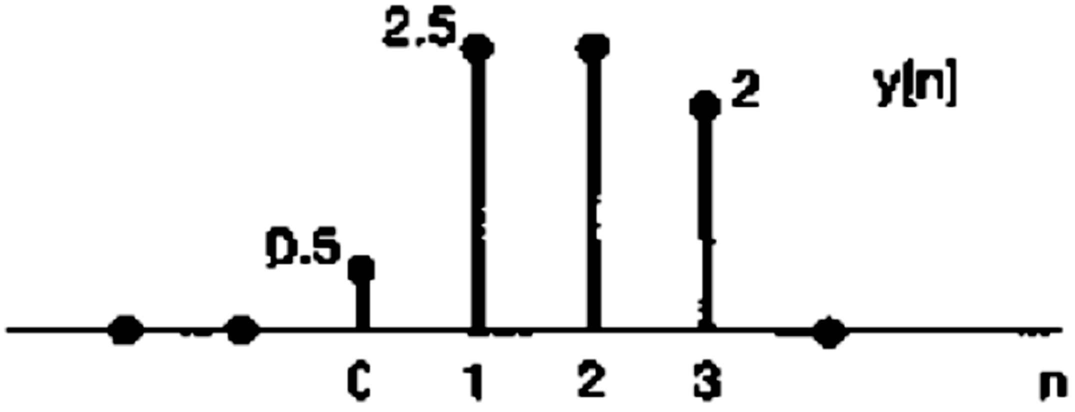
$$For\, n=2, \quad y[2]=\sum_{k=-\infty}^{\infty} x[k]h[2-k]=0.5+2=2.5$$

$$For\, n=3, \quad y[3]=\sum_{k=-\infty}^{\infty} x[k]h[3-k]=2$$

$$For\ n>3,\ \ y[n]=0$$

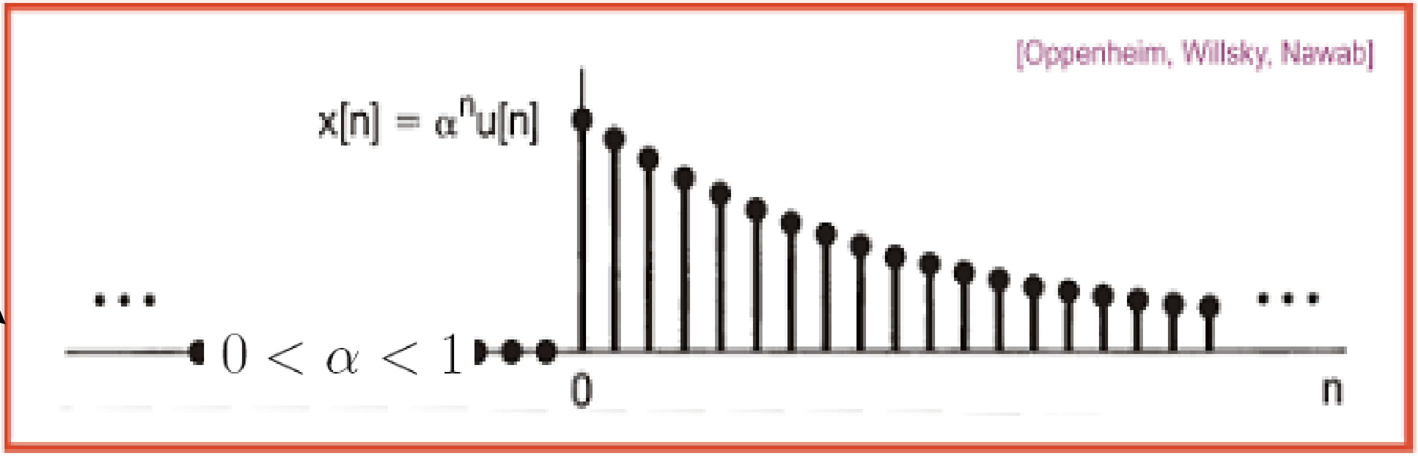


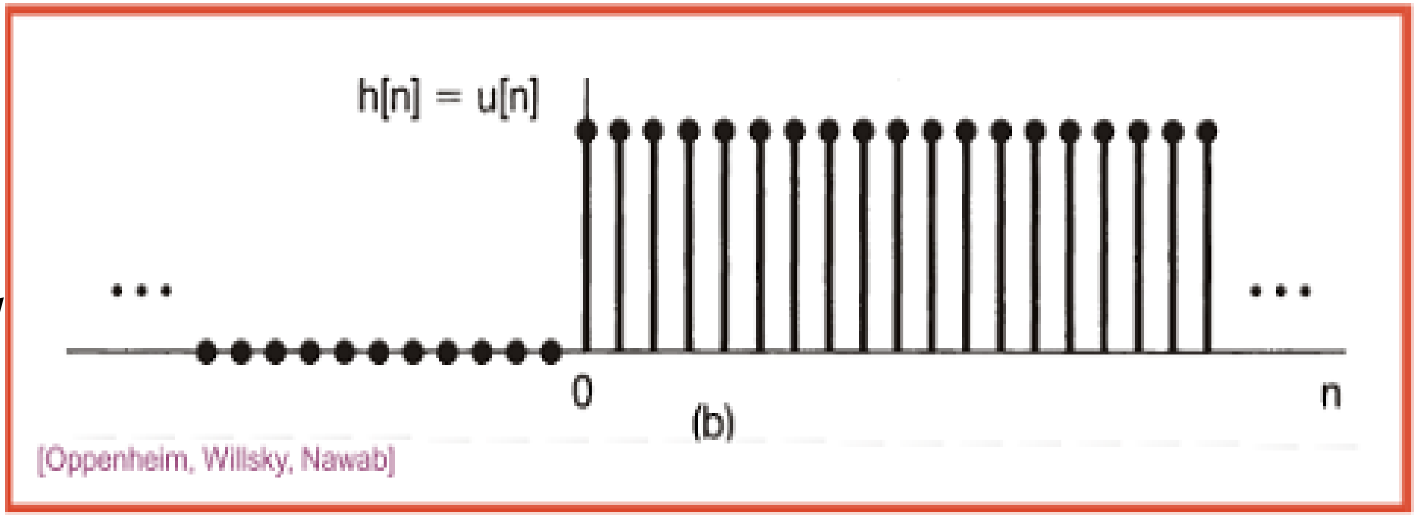
The output y[n]:



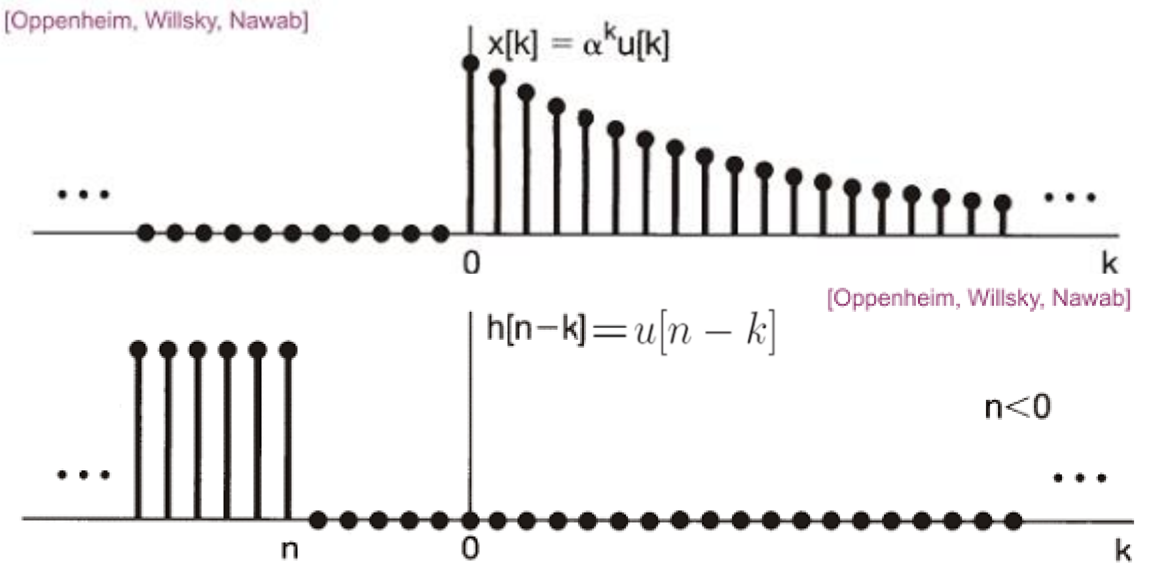
Example 2: Calculate $y[n] = x[n] * h[n]$

Given:





Example 2



Solution for $n < 0$

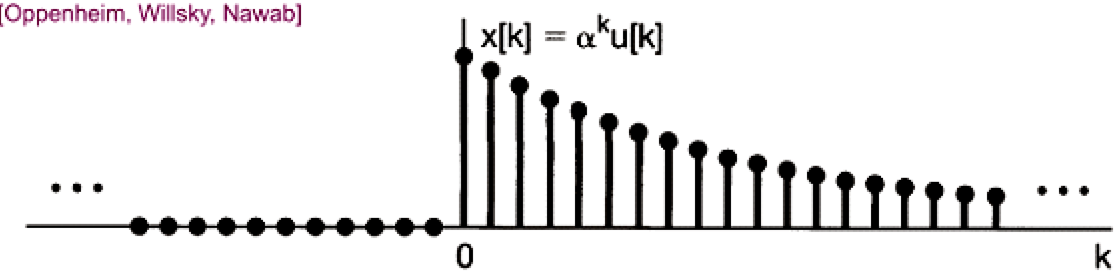
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] =$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k] u[n-k] = 0$$

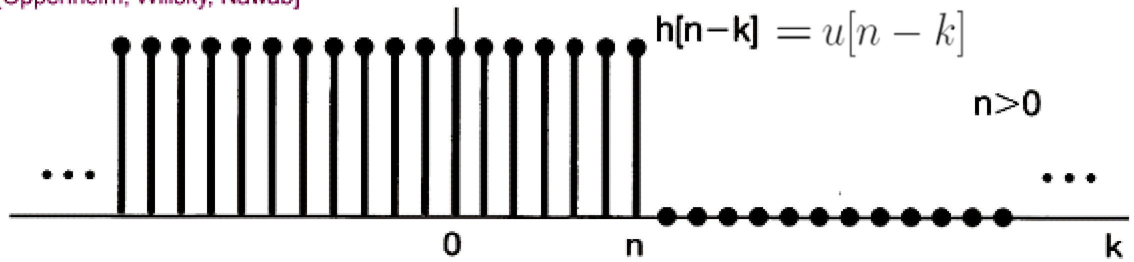
Example 2

Solution for $n \geq 0$

[Oppenheim, Willsky, Nawab]



[Oppenheim, Willsky, Nawab]



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]u[n-k]$$

$$= \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$\boxed{\sum_{k=0}^{n_1} a^k = \frac{1-a^{n_1+1}}{1-a}}$$

CT LTI Systems: The Convolution Integral

Remember:

Like the discrete time case, any continuous time signal can be expressed in terms of impulses;

This property is described by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

CT LTI Systems: The Convolution Integral

Convolution Integral

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Symbolic representation of convolution

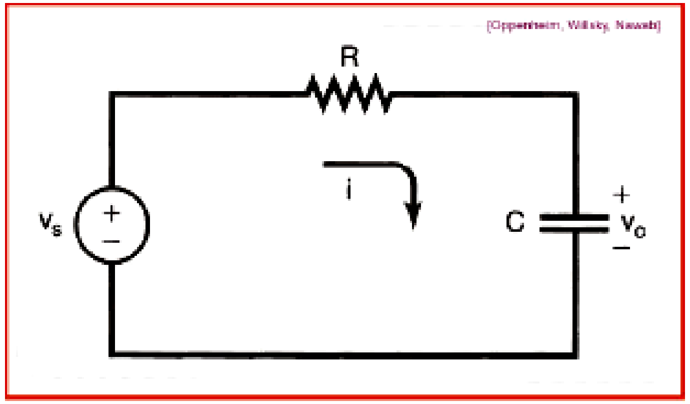
$$\boxed{y(t) = x(t) * h(t)}$$

Practical Example:

Given the impulse response of the low pass filter (LTIS) shown in Fig.:

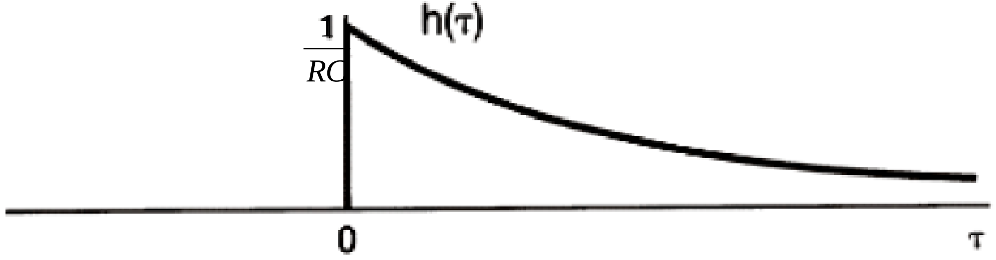
$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

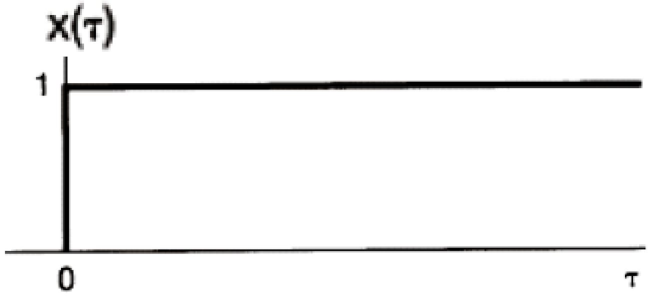
Find the output when a dc voltage (u(t) is its input.

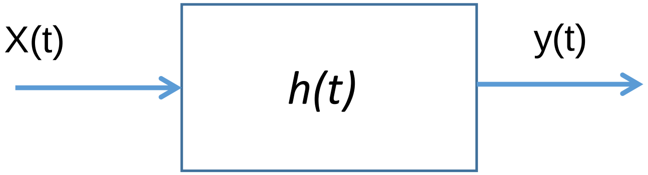


solution

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$







$$y(t) = x(t) * h(t)$$

transfer function $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + sRC}$

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$at\, t < 0$

$$y(t) = 0$$

$at\ t > 0$

$$y(t) = \frac{1}{RC}\int_{-\infty}^{\infty} e^{\frac{-\tau}{RC}}\ u(\tau)\, x(t-\tau)\, d\tau$$

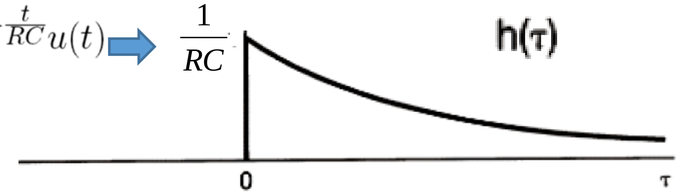
$$= \frac{1}{RC}\int_{0}^{t} e^{\frac{-\tau}{RC}}\ d\tau$$

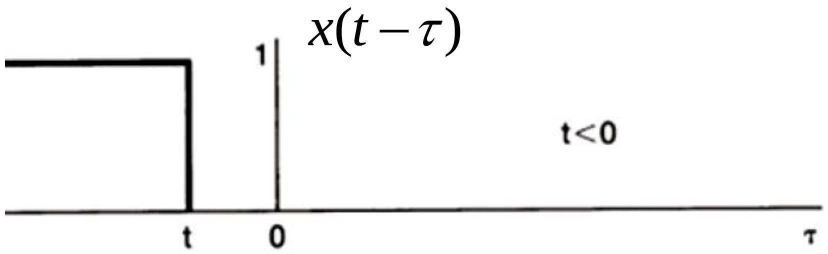
$$= \frac{-RC}{RC}\left(e^{\frac{-t}{RC}} - 1\right)$$

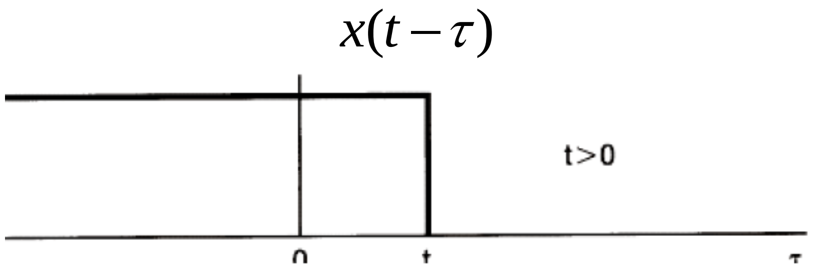
$$= \left(1 - e^{\frac{-t}{RC}}\right) u(t)$$

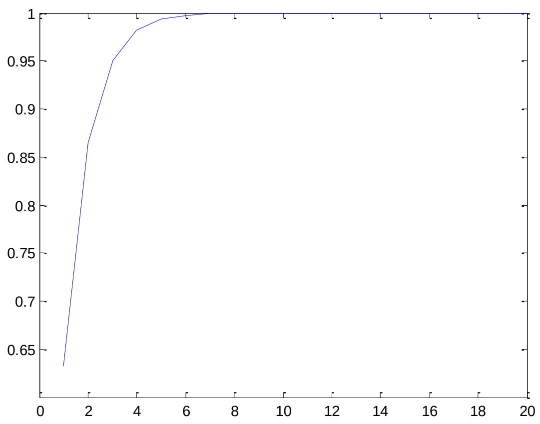
To start from t=zero

$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$









LPF

RC=1

Properties of LTI Systems

Proof of properties
In the Appendix at
the end of the
lecture

Commutative property in CT

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Commutative property in DT

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

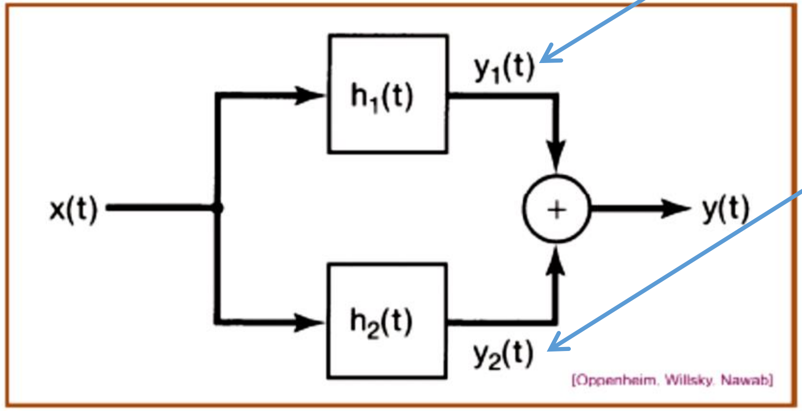
$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Properties of LTI Systems

Distributive property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



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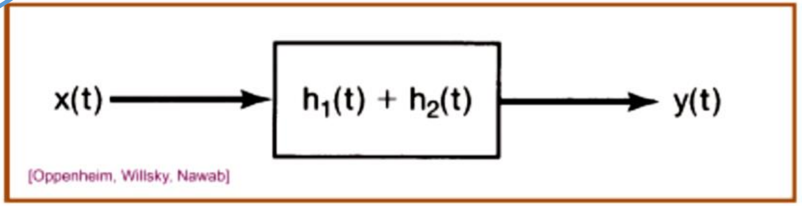


Proof of properties In the Appendix at the end of the lecture

Properties of LTI Systems

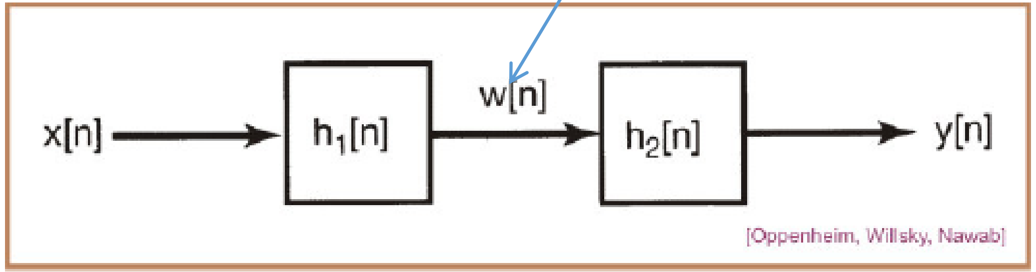
Proof of properties In the Appendix at the end of the lecture

Associative property

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

- Cascade connection of two LTI system represented by single LTI system



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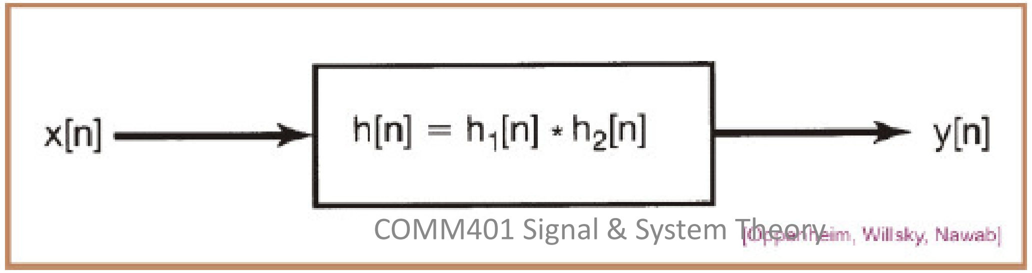
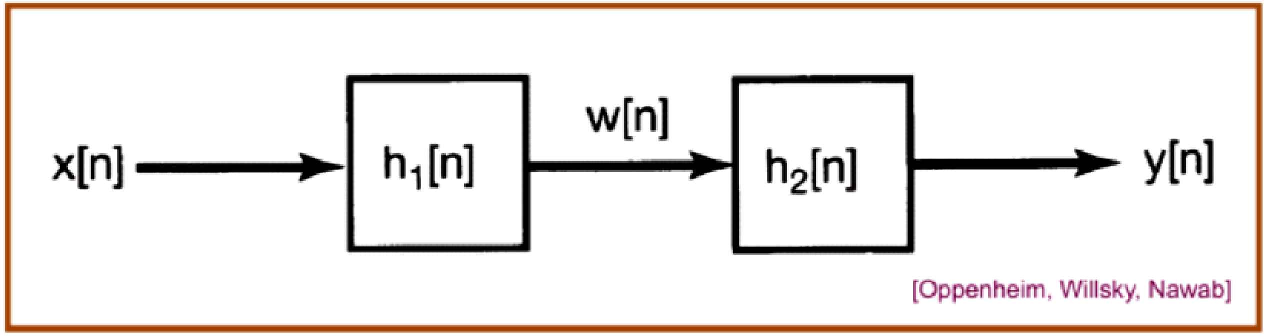


Properties of LTI Systems

order in a cascade of LTI systems irrelevant



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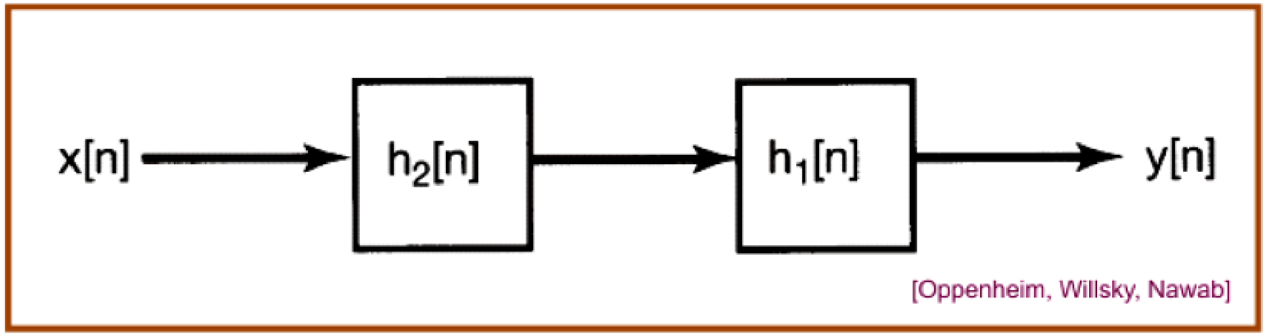
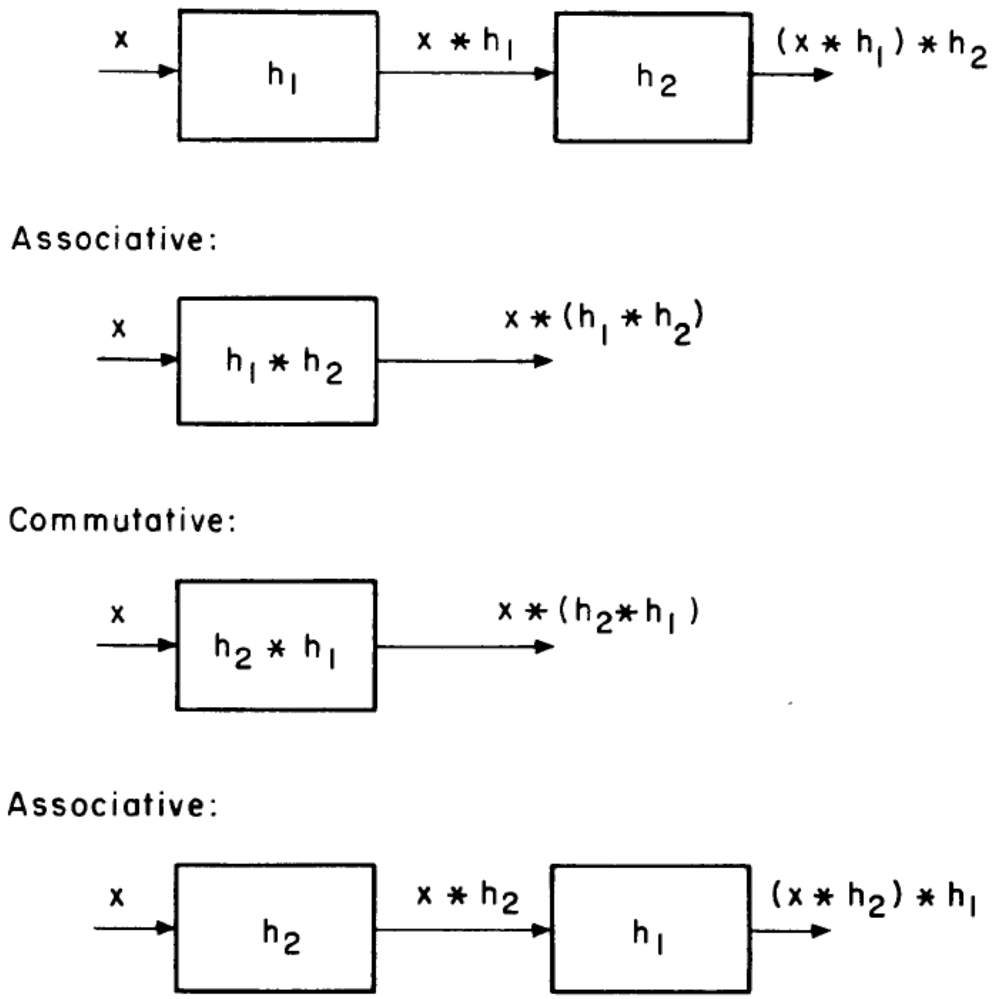


Summary of Properties of LTI Systems

- LTI system can be cascaded to **any order**:



Reading Materials

Appendix A

Proofs of Properties

Properties of LTI Systems

Distributive property: proof

$$\begin{aligned}
x(t) * (h_1(t) + h_2(t)) &= \int_{-\infty}^{+\infty} x(\tau)(h_1(t-\tau) + h_2(t-\tau))d\tau \\
&= \int_{-\infty}^{+\infty} x(\tau)h_1(t-\tau)d\tau \\
&+ \int_{-\infty}^{+\infty} x(\tau)h_2(t-\tau)d\tau \\
&= x(t) * h_1(t) + x(t) * h_2(t)
\end{aligned}$$

Similarly, the distributive property for LTI DT Systems can be proved.
sum instead of convolution integral.

Properties of LTI Systems

Commutative property in CT

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

Proof:

$$h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \overset{\tau'=t-\tau}{=} -\int_{+\infty}^{-\infty} h(t-\tau')x(\tau')d\tau'$$

$$= \int_{-\infty}^{+\infty} h(t-\tau')x(\tau')d\tau' = x(t) * h(t)$$

Properties of LTI Systems

Commutative property in DT

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

Proof:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \overset{r=n-k}{=} \sum_{r=-\infty}^{+\infty} x[n-r]h[r]$$

$$= h[n] * x[n]$$

Thank You