German University in Cairo
Media Engineering and Technology
DMET 502 Computer Graphics, Winter 2023
Assoc. Prof. Dr. Rimon Elias

Practice assignment 5 solution
3D Transformations

Q 1: A tetrahedron is to be rotated through an angle of $45°$ about a line passing through the points $[2, 1, 0]^T$ and $[6, 5, 0]^T$. Derive the required 3D transformation matrix.

Solution: The line extending from $[2, 1, 0]^T$ to $[6, 5, 0]^T$ does not coincide with any of the 3 axes, which means that some transformations are needed before being able to rotate the tetrahedron about it. Since the line does not pass through the origin two transformations are needed; the first transformation translates the line to pass through the origin, and the second rotates this line to coincide with the x-axis, which is a rotation about the z-axis.

$$M1 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M2 = \begin{bmatrix} cos(-45°) & -sin(-45°) & 0 & 0 \\ sin(-45°) & cos(-45°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rotation about the line is performed in the third transformation matrix $M3$; this rotation is about the x-axis since the line now coincides with this axis.

$$M3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(45°) & -sin(45°) & 0 \\ 0 & sin(45°) & cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After the rotation about the line, the first two transformations need to be undone. Matrices $M4$ and $M5$ undo matrices $M2$ and $M1$ respectively.

$$M4 = \begin{bmatrix} cos(45°) & -sin(45°) & 0 & 0 \\ sin(45°) & cos(45°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M5 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = M5 * M4 * M3 * M2 * M1 = \begin{bmatrix} 0.853 & 0.146 & 0.5 & 0.148 \\ 0.146 & 0.853 & -0.5 & -0.145 \\ -0.5 & 0.5 & 0.707 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q 2: A unit cube is centered at $[2, 5, 3]^T$. This cube is to be rotated through an angle of $45°$ about a line passing through its center and parallel to the y-axis. Derive the required 3D transformation matrix.

Solution: In order to rotate the cube about the line, this line needs to coincide with either the x, y, or z axes. Since the line to be rotated about is already parallel to the y-axis, then this line could simply be translated to coincide with this axis.

$$M1 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After the translation, the unit cube could be rotated about the y-axis.

$$M2 = \begin{bmatrix} cos(45°) & 0 & sin(45°) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(45°) & 0 & cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, after rotating, the cube is translated back.

$$M3 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The three previously calculated matrices are used to derive the final transformation matrix M.

$$M = M3 * M2 * M1 = \begin{bmatrix} 0.707 & 0 & 0.707 & -1.535 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 2.293 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q 3: A unit cube is centered at $[2, 5, 3]^T$. Rotate this cube through an angle of $45°$ about a line passing through the origin and having a direction vector $[7, 7, 7]^T$. Derive the required 3D transformation matrix.

Solution: The line to be rotated about does not coincide with any of the three axes, nor does it lie on any of the xy, yz, or zx planes. Therefore, two rotation transformations are needed in order to be able to rotate the cube about this line; the first rotation places the line on a plane, while the second rotation makes the line coincide with an axis.

The line is first rotated to lie on the xy plane; this rotation is about the y-axis and in a counterclockwise direction. Since this line splits the angle between the x-axis and z-axis in half, the rotation is by $45°$.

For better visualization, follow this link `https://www.geogebra.org/3d/kyuxfjuv`

$$M1 = \begin{bmatrix} cos(45°) & 0 & sin(45°) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(45°) & 0 & cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The next step would be to rotate the line to coincide with the x-axis, yet before being able to do that, the angle between the line and the x-axis is needed.

The diagonal between the origin and $[7, 0, 7]^T$ is the projection of the line between $[0, 0, 0]^T$ and $[7, 7, 7]^T$ onto the zx plane, it is also the hypotenuse of an isosceles right-angled triangle. The lengths of the other two sides of this triangle are determined using the point $[7, 0, 7]^T$, which indicates that both sides are of length 7. Using the Pythagoras theorem for right-angled triangles, the length of the diagonal is $\sqrt{7^2 + 7^2}$, which is equal to $7\sqrt{2}$.

After the first rotation, which was performed using $M1$, the diagonal calculated earlier was rotated to coincide with the x-axis, and form one side of a right-angled triangle. The second side of this triangle is perpendicular to the first one, lies on the xy plane, and ranges from 0 to 7 in the y direction having the length 7. The line between the origin and $[7, 7, 7]^T$ is the hypotenuse of the right-angled triangle in question, where the other two sides have the lengths 7 and $7\sqrt{2}$. The angle needed for the rotation is calculated as follows:

$$tan\,\theta = \frac{opposite\ side}{adjacent\ side} = \frac{7}{7\sqrt{2}} \rightarrow \theta = 35.26°$$

To coincide with the x-axis, a rotation of $-35.26°$ about the z-axis is performed.

$$M2 = \begin{bmatrix} cos(-35.26°) & -sin(-35.26°) & 0 & 0 \\ sin(-35.26°) & cos(-35.26°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The unit cube can now be rotated by $45°$ about the line that coincides with the x-axis.

$$M3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(45°) & -sin(45°) & 0 \\ 0 & sin(45°) & cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rotations in $M2$ and $M1$ are now undone in $M4$ and $M5$ respectively.

$$M4 = \begin{bmatrix} cos(35.26°) & -sin(35.26°) & 0 & 0 \\ sin(35.26°) & cos(35.26°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M5 = \begin{bmatrix} cos(-45°) & 0 & sin(-45°) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(-45°) & 0 & cos(-45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final transformation matrix M is computed as follows:

$$M = M5 * M4 * M3 * M2 * M1 = \begin{bmatrix} 0.8047 & -0.3106 & 0.5058 & 0 \\ 0.5059 & 0.8047 & -0.3106 & 0 \\ -0.3106 & 0.5059 & 0.8047 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q 4: Derive the transformation matrix that applies the following series of 3D transformations to a 3D object:

1. a translation by the vector $[2, 4, 6]^T$,
2. a shearing transformation in the x and y directions with the shearing factors 5 and 3 respectively,
3. a scaling of the object using factors 5 and 3 in the x and y directions respectively,
4. a rotation of the object through an angle of $70°$ about the z-axis.

Solution:

1. The translation matrix: $M1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
2. The shearing matrix: $M2 = \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
3. The scaling matrix: $M3 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
4. The rotation matrix: $M4 = \begin{bmatrix} cos(70°) & -sin(70°) & 0 & 0 \\ sin(70°) & cos(70°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   $M = M4 * M3 * M2 * M1 = \begin{bmatrix} 1.71 & -2.817 & 0.099 & -7.254 \\ 4.695 & 1.026 & 26.553 & 172.812 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$