An Optimization for Computing Transcendental Numbers using Zeckendorf's Theorem

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Introduction

The value of π was computed in 1987 to an accuracy of over 100 million decimal places[4]. Over the years, the ability to calculate π accurately and efficiently has been a large area of research[2]. Though we have enough digits of π to compute the volume of the universe accurately, finding new algorithms and series for the convergence of π will allow us to find ways to accelerate the convergence of other similar series and to increase computer efficiency.

Engineering Goal: To find a way to accelerate the computation of π using the Zeckendorf theorem and to find a pair of Lucas numbers that results in a counterexample of the normalcy of π.

Model

The arctan(x) series: $4 \cdot \sum_{i=0}^{\infty} \frac{(-1)^n}{2n+1} = 4(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}...)$ is accurate to only three decimal places after 10,000 terms for the computation of π. However, the current most commonly used series for π, the Bailey–Borwein–Plouffe formula (BBP formula) shown below, is accurate to 63 decimal places after 50 terms[3]:

$$\pi = \sum_{i=0}^{\infty} [\frac{1}{16^i}(\frac{4}{8i+1}) - (\frac{2}{8i+4}) - (\frac{1}{8i+5}) - (\frac{1}{8i+6})]$$

$$\pi^2 = \frac{9}{8}\sum_{i=0}^{\infty} \frac{1}{64^i}(\frac{16}{(6i+1)^2}) - (\frac{24}{(6i+2)^2}) - (\frac{8}{(6i+3)^2}) - (\frac{6}{(6i+4)^2}) - (\frac{1}{(6i+5)^2})$$

This formula allows you to find the nth digit of π in hexadecimal as well as the the value of π[1].

Proof of [1]:

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{2}}} \frac{x^{k-1}}{1-x^8}\,dx &= \int_0^{\frac{1}{\sqrt{2}}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx \quad (\text{for any } k<8) \\
&= \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)} \\
&= \sum_{i=0}^{\infty} \frac{1}{16^i}(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}) \\
&= \int_0^{\frac{1}{\sqrt{2}}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\,dx \\
y := \sqrt{2}x \\
&= \int_0^1 \frac{16y-16}{y^4-2y^3+4y-4}\,dy \\
&= \int_0^1 \frac{4y}{y^2-2}\,dy - \int_0^1 \frac{4y-8}{y^2-2y+2}\,dy \\
&= \pi \;\blacksquare
\end{aligned}$$

Zeckendorf Theorem

Theorem: every positive integer can be represented uniquely as the sum of one or more distinct and non consecutive Fibonacci numbers[6]. My unique idea of how to use the Zeckendorf theorem to compute a series:

$$\begin{aligned}
\text{Let } S(n) &= \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \\
S(13) &= \sum_{i=0}^{13} i = \sum_{i=0}^{8} i + \sum_{j=9}^{13} j \\
&= \frac{(8)(8+1)}{2} + \sum_{j=1}^{5}(j+8) \\
s'_8(5) = \sum_{j=1}^{5}(j+8) &= (9+10+11+12+13) \\
&= ((8+1)+(8+2)+(8+3)+(8+4)+(8+5)) \\
S(13) &= \frac{(8)(8+1)}{2} + 8\cdot 5 + 15 = 91
\end{aligned}$$

Lets shift the equation by $j = k+8$. We will define this new notation: $S'_8(5)$ which we will define as shifting S(n) forward by 8 and summing till 5.

Algorithm

By using the Zeckendorf theorem, we can derive the value of π, by breaking up any upper limit of the series as the sum of Fibonacci numbers. Let us continue on our last proof and derive the following algorithm: we will have a precomputed table of $S'_{F_{N+1}}(F_n)$ because these numbers will appear frequently. We have shown that you can use the Zeckendorf theorem to accelerate the series $\sum_{i=0}^{13} n$. We implemented this model as well for the BBP series. If you want to compute $\sum_{j=0}^{13}$ you can look up the value for $\sum_{j=0}^{8}$ which is stored in your table and add this to the value you compute for $\sum_{j=0}^{5}(j+8)$. This technique of precomputing values will drastically reduce run time. The 100th Fibonacci number is 354224848179261915075. If you store all the integer values of the series of π for all the Fibonacci numbers up till the 100th, you will be able to find any intermediate value of the nth digit of π between 0 and approximately 3 quintilian ($3\cdot 10^{18}$). According to our literature review, this the 1st application of the Zeckendorf theorem to compute the series for π.

$$\begin{aligned}
S'_m(n) &= \sum_{j=0}^{n}(j+m) \\
S(n) &= \sum_{i=0}^{n} i \;\text{ then} \\
S'_m(n) &= \frac{(N)(N+1)}{2} - \frac{(m)(m+1)}{2} \quad \text{where N = m+n} \\
S'_m(n) &= \frac{(m+n)(m+n+1)}{2} - \frac{(m)(m+1)}{2} \\
&= s(m+n) - s(m) \\
S'_m(n) &= mn + s(n) \;\blacksquare
\end{aligned}$$

$$\begin{aligned}
30 = 21+8+1 \Rightarrow \sum_{i=0}^{30} &= \sum_{j=0}^{21} + \sum_{j=21}^{29} + \sum_{j=29}^{30} \\
S(30) = S(21) + S_{21}(8)... &= \sum_{i=0}^{21} + \sum_{j=1}^{8}(j+21) + ... \\
&= S(21) + S'_{21}(8) + S'_{29}(1) \\
\text{Similarly } S(21) &= S(13) + S'_{13}(8)
\end{aligned}$$

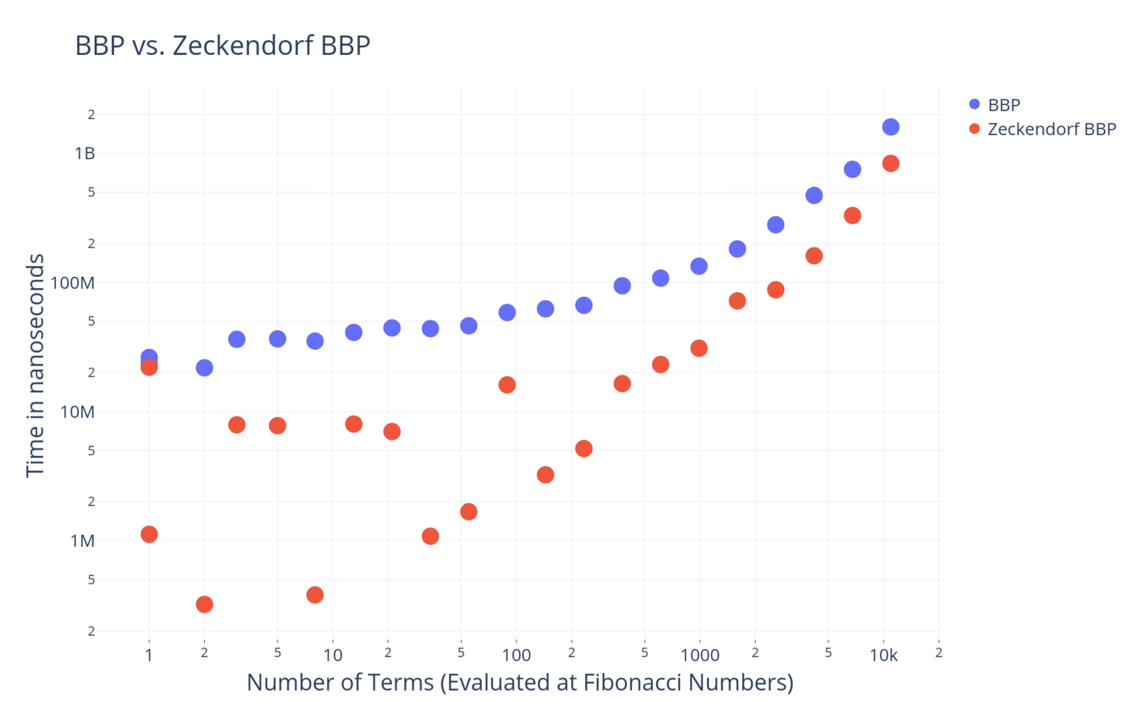

Fig. 1: Time trials were done using the standard BBP algorithm and were compared to the optimized algorithm using the Zeckendorf series (all images made by Advay Koranne)

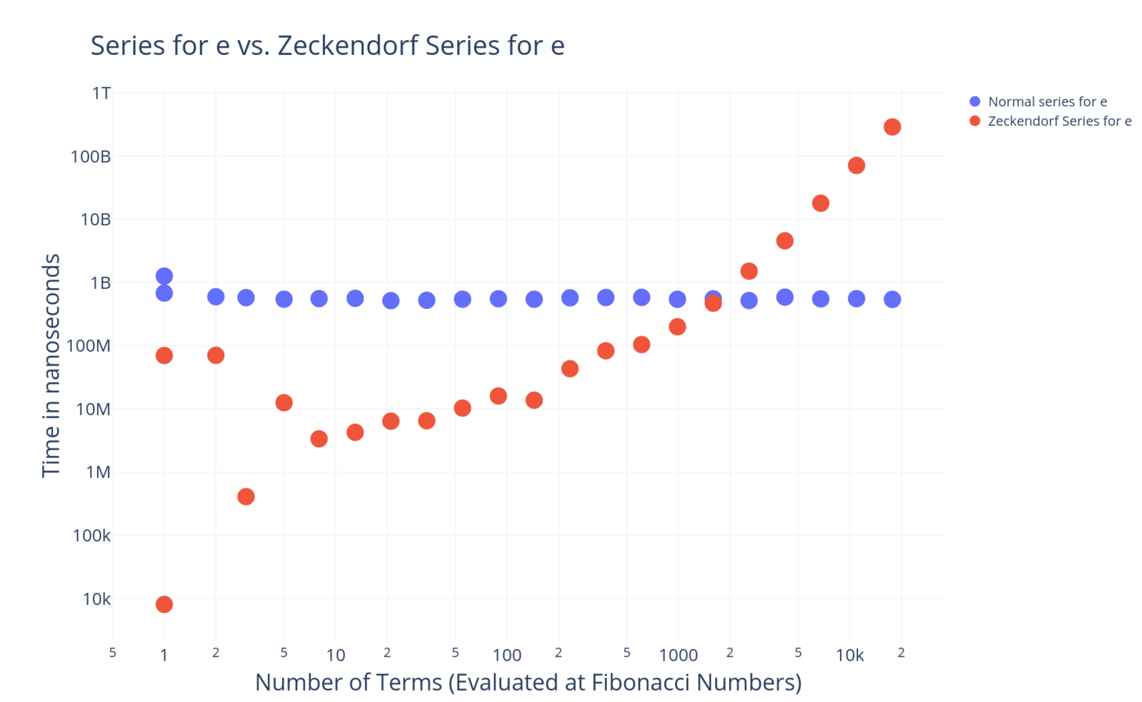

Fig. 2: Time trials were done using the standard series for e and were compared to the optimized series for e using the Zeckendorf series (all images made by Advay Koranne)

We implemented the proposed Zeckendorf algorithm in Java using the BigDecimal class. We invented and tested the **Zeckendorf-BBP-algorithm** for every Fibonacci number in between [0, 10946] and did the shift with the previous Fibonacci number. We also implemented this for the series of e: $\sum_{k=0}^{\infty}\frac{1}{k!}$ and did the same approach as we had done for the Zeckendorf algorithm for π. However, after a certain number due to the factorial part of the denominator, the computational time of the standard series seemed to become better. With an optimized factorial function, this can be further improved to continue out performing the standard series.

$$e = \sum_{k=0}^{10946}\frac{1}{k!} = \sum_{k=0}^{6765}\frac{1}{(k)!} + \sum_{k=1}^{4,181}\frac{1}{(k+6765)!}$$

The time complexity for the standard BBP algorithm is **O(nlog(n)M(log(n)))** where M(j) is the complexity of multiplying j bit integers[3]. The run-time complexity for the Zeckendorf Telescoping algorithm is approximately **O(log(n))**. The viability of computing these transcendental numbers by breaking up the series allows us to improve the time complexity significantly as evident in our computational results.

Lucas Pairs - Possible application of the Zeckendorf Algorithm

The question still remains whether there exists a pair of Lucas numbers (starting seeds for Fibonacci sequence) that can result in a non-normal sequence of π. The idea of π's normality was introduced by E. Borel in 1909 [7]. A normal number would mean that a digit occurs at the same frequency as the others. In loose terms this can be related to "randomness." A known example of a number being normal is Champernowne constant (0.123456789101112), which is normal in base 10 [2]. The standard Fibonacci sequence is the following: (0,1,1,2,3,5,13,21,...). **Lucas numbers are similar except the starting two numbers can be different eg.(2,1,3,4,7,11,18,29,47,...).**

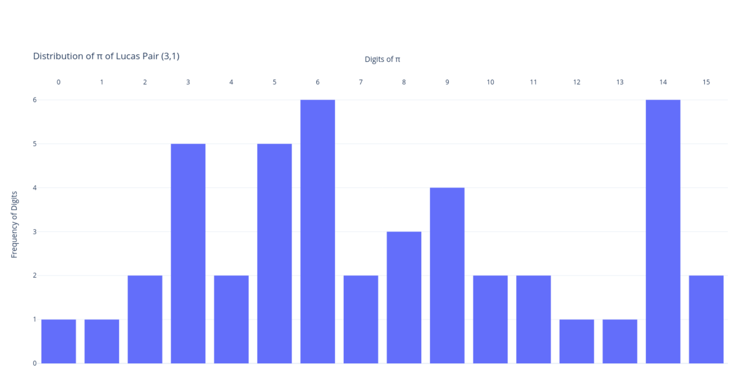

Fig. 3: 45 terms of the Lucas Series with the seed (3,1) were sampled and their nth digit of π was recorded (all images made by Advay Koranne)

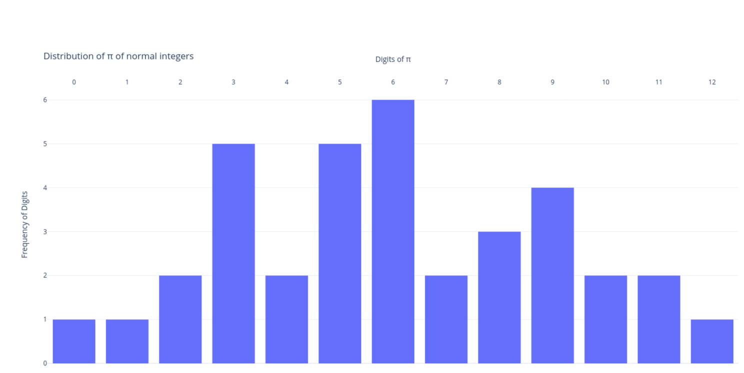

Fig. 4: The number [0,45] were sampled and their nth digit of π was recorded (all images made by Advay Koranne)

Results

Digit count until the 8 Billionth Lucas number (base-16) of π.

Lucas-Number	digit
0	1
1	1
2	2
3	5
4	2
5	5
6	6
7	2
8	3
9	4
A	2
B	2
C	1
D	1
E	6
F	2
Total	44

Digit count of π using Lucas Pairs.

Nth	Lucas Number	Digit
3	5	7
4	9	9
5	14	9
6	23	3
7	37	4
8	60	15
9	97	13
10	157	12
...	...	...
40	292072113	6
41	472582606	15
42	764654719	16
43	1237237325	10
44	2001892044	5
45	3239129369	5
46	5241021413	16
47	8480150782	6

To go up until the 8 billionth Lucas pair required a total of 1492246 seconds or approximately 17 days[5]. Our preliminary research using the standard Fibonacci sequence or Lucas Pair of (1,1) showed us that there exists Lucas numbers, which avoid the digit 'D' (decimal 13) up to the 45th term (which is already close to the 3 trillionth digit of Pi). Our graphs for the Lucas Pair (3,1) show us that with 45 samples, the distribution is very different than normal sampling on the interval [0,45]. With enough computational power and using the optimized Zeckendorf series, it seems likely to find a sequence of numbers that excludes certain digits or results in a non-normal series of π on a large closed interval.

Conclusion

This application of using the Zeckendorf algorithm to compute π can be applied to other transcendental numbers such as π, e, a^b where $a \neq (0,1)$ and b is an irrational number, seems like a viable approach to optimize the computation of such numbers. Other applications that require large series for expensive computations can use such a methodology to drastically decrease run time. This specific model can further be used to find more Lucas Pairs and build on the research that was done in the application of the Zeckendorf Algorithm and to determine whether or not there exists a sequence of Lucas Pairs that results in a non-normal sequence. The implementation of this algorithm can further be improved by doing more than just a two series break up. For example, rather than breaking 30 into $21+9$ you can break it into $21+8+1$, which will further increase the optimization of the algorithm. Further, research is being conducted on creating the structure to store the Fibonacci series for π and to continue finding Lucas pairs.

References

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