Ramanujan τ-Functions

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Definition 1. *For any $A \in GL(2,\mathbb{R})$, the slash operator defined on $f : \mathcal{H} \to \mathbb{C}$ is*

$$f|_k A(z) = (det(A))^{k/2} j_A(z)^{-k} f(Az),$$

we have $j_A(z) = cz + d$, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 2.

$$\Delta_n := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | ad = n, 0 \leq b < d \right\}.$$

Lemma 3. *There is a one to one correspondence*

$$\Delta_n \times SL(2,\mathbb{Z}) \leftrightarrow SL(2,\mathbb{Z}) \times \Delta_n.$$

That is for any $\rho \in \Delta_n$, $\tau \in SL(2,\mathbb{Z})$, there exist unique $\tau' \in \Gamma, \rho' \in \Delta_n$, such that $\rho\tau = \tau'\rho'$.

Proof. If $\rho = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n$ and $\tau = \begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$, we want to find $\tau' = \begin{pmatrix} \alpha' & * \\ \gamma' & \delta' \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\rho' \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \Delta_n$ such that $\rho\tau = \tau'\rho'$. This is we want to solve:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha' & * \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}.$$

This is

$$\begin{pmatrix} a\alpha + b\gamma & * \\ d\gamma & d\delta \end{pmatrix} = \begin{pmatrix} a'\alpha' & * \\ a'\gamma' & b'\gamma' + d'\delta' \end{pmatrix},$$

and this is equal to

$$\begin{cases} a'\alpha' & = a\alpha + b\gamma; \\ a'\gamma' & = d\gamma; \\ b'\gamma' + d'\delta' & = d\delta. \end{cases}$$

For $\gamma \neq 0$, We can see that $a' \mid a\alpha + b\gamma$ and $a' \mid d\gamma$, so

$$(\frac{a\alpha + b\gamma}{a'}, \frac{d\gamma}{a'}) = (\alpha', \beta').$$

We can see that $(\alpha', \beta') = 1$, so $a' = (a\alpha + b\gamma, d\gamma)$. In addition, we have

$$\begin{cases} a' &= (a\alpha + b\gamma, d\gamma); \\ d' &= n/(a\alpha + b\gamma, d\gamma); \\ \alpha' &= (a\alpha + b\gamma)/(a\alpha + b\gamma, d\gamma); \\ \gamma' &= d\gamma/(a\alpha + b\gamma, d\gamma). \end{cases} \tag{1}$$

We can see that a', α', γ' are positive integers. For d', we have

$$(a, \gamma) = (a\alpha, \gamma) = (a\alpha + b\gamma, \gamma),$$

is a divisor of a due to $(\alpha, \gamma) = 1$, so we have $(a\alpha + b\gamma, d\gamma) \mid ad$ and d is an integer. What we need is prove

$$(\gamma', d') \mid d\delta, \tag{2}$$

and we will get a unique $0 \leq b' < d'$ and δ' from Bezout's theorem. From equation 1, we have $(\gamma', d') = (ad, d\gamma)/(a\alpha + b\gamma, d\gamma)$, and proving 2 is equal to prove

$$(a, \gamma) \mid \delta(a\alpha + b\gamma, d\gamma).$$

But this is from $(a, \gamma) \mid (a\alpha + b\gamma)$ and $(a, \gamma) \mid \gamma$.

If $\gamma = 0$, then we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix},$$

so we take $b' \equiv au + b \pmod{d}$ and $v = d^{-1}(au + b - dv)$, and v is an integer for $d \mid (au + b - dv)$.

For surjection, we need some matrices transform. If we have $\tau' \in \Gamma$, $\rho' \in \Delta_n$, we want to find τ and γ. We can assume that $\tau'\rho' = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, where $xw - yz = n$. Then we have

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} w/(z,w) & * \\ -z/(z,w) & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

With a matrix like $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ we will get the surjection. $\square$