Ramanujan τ-Functions

Yu Xiong

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In this part, I will introduce the following theorem, whose proof is from Math Stackexchange, and there are some corollaries giving some congruence of τ functions, which is from [1].

Theorem 1.

$$(1-n)\tau(n) = 24\sum_{j=1}^{n-1}\sigma(j)\tau(n-j).$$

Definition 2. *δ_k is an operator from holomorphic functions to holomorphic functions.*

$$\delta_k(f) := 12\theta(f) - kE_2 f,$$

where $\theta(f) := q\frac{df}{dq}$, $q = e^{2\pi i\tau}$ and

$$E_2(\tau) = 1 - 2\sum_{n=1}^{\infty}\sigma q^n.$$

Lemma 3. *$\delta_k : M_k(\Gamma) \to M_{k+2}(\Gamma)$ and $\delta_k(S_k(\Gamma)) \subset S_{k+2}(\Gamma)$.*

Proof. To prove $\delta_k(M_k(\Gamma)) \subset M_{k+2}(\Gamma)$, we just need to show for any $f \in M_k(\Gamma)$ and $\alpha \in \Gamma$, we have

$$(\delta_k f)|_{k+2}\alpha = \delta_k f. \tag{1}$$

We can see that $(\delta_k f)|_{k+2}\alpha = 12(\theta_k f)|_{k+2}\alpha - kE_2|_2\alpha f|_k\alpha$. Firstly, we have $\theta(f) = q\frac{df}{dq} = \frac{1}{2\pi i}f'(\tau)$, so we have

$$\theta(f)|_{k+2}\alpha = \frac{1}{2\pi i}f'|_{k+2}\alpha = \frac{1}{2\pi i}(c\tau+d)^{-k-2}f'(\alpha\tau).$$

We have $f(\alpha\tau) = (c\tau+d)^k f(\tau)$, and taking the derivative of the equation we have $\frac{1}{(c\tau+d)^2}f'(\alpha\tau) = kc(c\tau+d)^{k-1}f(\tau) + (c\tau+d)^k f'(\tau)$. So we have

$$f'(\alpha\tau) = kc(c\tau+d)^{k+1}f(\tau) + (c\tau+d)^{k+2}f'(\tau).$$

Taking the $f'(\alpha\tau)$ to $(\theta_k f)|_{k+2}\alpha$, we have

$$\begin{aligned}(\theta_k f)|_{k+2}\alpha &= \frac{1}{2\pi i}(c\tau+d)^{-k-2}f'(\alpha\tau)\\ &= \frac{1}{2\pi i}\left(\frac{kc}{c\tau+d}f(\tau)+f'(\tau)\right)\\ &= \frac{kcf(\tau)}{2\pi i(c\tau+d)}+\theta f.\end{aligned}$$

So we have

$$\begin{aligned}(\delta_k f)|_{k+2}\alpha-\delta_k f &= 12(\theta_k f)|_{k+2}\alpha - kE_2|_2\alpha f-(12\theta(f)-kE_2 f)\\ &= kf(\tau)\left(\frac{12c}{2\pi i(c\tau+d)}-E_2|_2\alpha+E_2\right)\\ &= 0.\end{aligned}$$

Last equation is from $E_2|_2\alpha = E_2+\frac{12c}{2\pi i(c\tau+d)}$. □

Proof of Theorem 1. We have $\delta_k\Delta\in S_{14}(\Gamma)=0$, so $\delta_k\Delta=0$. So we have

$$\delta_k\Delta = 12\sum_{n=1}^{\infty} n\tau(n)q^n - 12E_2\Delta = 0.$$

Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n\tau(n)q^n &= \left(1-24\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(\sum_{n=1}^{\infty}\tau(n)q^n\right)\\ &= \sum_{n=1}^{\infty}\tau(n)q^n - 24\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\tau(m)\sigma(n-m)q^n\right).\end{aligned}$$

□

References

[1] R. P. Bambah, S. Chowla, H. Gupta, and D. B. Lahiri. Congruence properties of Ramanujan's function $\tau(n)$. *Quart. J. Math. Oxford Ser.*, 18:143–146, 1947.