Ramanujan τ-Functions

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Definition 1. *Let $\chi : \mathbb{Z} \to \mathbb{C}$, and $\chi(ab) = \chi(a)\chi(b)$, for any $(a,b) \in \mathbb{Z}^2$. If $n \geq 1$, we define Hecke operators T_n on $f : \mathcal{H} \to \mathbb{C}$ to be*

$$T_n f := \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{b \pmod d} f(\frac{a\tau + b}{d}).$$

Theorem 2. *For any $m, n \geq 1$, we have*

$$T_m T_n f = \sum_{d|(m,n)} \chi(d) d^{k-1} T_{\frac{mn}{d^2}} f$$

Proof. From Definition 1, we have

$$\begin{aligned}
&mnT_mT_nf(\tau)\\
&= \sum_{\substack{a_1d_1=m\\a_2d_2=n}} \chi(a_1a_2)(a_1a_2)^k \sum_{\substack{b_1 \pmod{d_1}\\b_2 \pmod{d_2}}} f\left(\begin{pmatrix} a_1 & b_1\\ 0 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ 0 & d_2\end{pmatrix}\tau\right)\\
&= \sum_{d|(m,n)} \sum_{\substack{a_1d_1=m\\a_2d_2=n\\(a_1,d_2)=d}} \chi(a_1a_2)(a_1a_2)^k \sum_{\substack{b_1 \pmod{d_1}\\b_2 \pmod{d_2}}} f\left(\begin{pmatrix} a_1 & b_1\\ 0 & d_1\end{pmatrix}\begin{pmatrix} a_2 & b_2\\ 0 & d_2\end{pmatrix}\tau\right)\\
= &\sum_{d|(m,n)} \chi(d)d^k \sum_{\substack{a_1'd_1=m/d\\a_2d_2'=n/d\\(a_1',d_2')=1}} \chi(a_1'a_2)(a_1'a_2)^k \sum_{\substack{b_1 \pmod{d_1}\\b_2 \pmod{dd_2'}}} f\left(\begin{pmatrix} da_1'a_2 & d(a_1'b_2+b_1d_2')\\ 0 & dd_1d_2'\end{pmatrix}\tau\right).
\end{aligned} \tag{1}$$

We can see that for any $a \in \mathbb{R}$ and $\gamma \in SL(2,\mathbb{Z})$, we have $f(a\gamma\tau) = f(\gamma\tau)$, so

$$f\left(\begin{pmatrix} da_1'a_2 & d(a_1'b_2+b_1d_2')\\ 0 & dd_1d_2'\end{pmatrix}\tau\right) = f\left(\begin{pmatrix} a_1'a_2 & a_1'b_2+b_1d_2'\\ 0 & d_1d_2'\end{pmatrix}\tau\right).$$

I claim that for fixed a_1', a_2, d_1, d_2' and $(a_1', d_2') = 1$, if b_1 runs over $\pmod{d_1}$ and b_2 runs over $\pmod{d_2'}$, then we have $a_1'b_2 + b_1d_2'$ runs over $\pmod{d_1d_2'}$ one times.

(b_1, b_2) has d_1d_2' ways to take, so we just need to show that taking different (b_1^*, b_2^*) we have $a_1'b_2 + b_1d_2' \not\equiv a_1'b_2^* + b_1^*d_2' \pmod{d_1d_2'}$. In fact, if they are equal, we have $a_1'(b_2-b_2^*) \equiv d_2'(b_1^*-b_1) \pmod{d_1d_2'}$, so we have $d_2' \mid a_1'(b_2-b_2^*)$. But $(a_1', d_2') = 1$, so $b_2 \equiv b_2^* \pmod{d_2'}$. So we prove the claim.

So we have

$$\sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{dd_2'}}} f\left(\begin{pmatrix} a_1'a_2 & a_1'b_2+b_1d_2' \\ 0 & d_1d_2' \end{pmatrix}\tau\right) = d \sum_{b \pmod{d_1d_2'}} f\left(\begin{pmatrix} a_1'a_2 & b \\ 0 & d_1d_2' \end{pmatrix}\tau\right)$$

We have

$$\begin{aligned}
&mnT_mT_n \\
&= \sum_{d|(m,n)} \chi(d)d^{k+1} \sum_{\substack{a_1'd_1=m/d \\ a_2d_2'=n/d \\ (a_1',d_2')=1}} \chi(a_1'a_2)(a_1'a_2)^k \sum_{b \pmod{d_1d_2'}} f\left(\begin{pmatrix} a_1'a_2 & b \\ 0 & d_1d_2' \end{pmatrix}\tau\right) \\
&= \sum_{d|(m,n)} \chi(d)d^{k+1} \sum_{ad=\frac{mn}{d^2}} \chi(a)a^k \sum_{b \pmod d} f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\tau\right) \\
&= \sum_{d|(m,n)} \chi(d)d^{k+1} T_{\frac{mn}{d^2}} f.
\end{aligned}$$

$\square$

Recall that we have

Definition 3.

$$\Delta_n := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad = n, 0 \le b < d \right\}.$$

Lemma 4. *There is a one to one correspondence*

$$\Delta_n \times SL(2,\mathbb{Z}) \leftrightarrow SL(2,\mathbb{Z}) \times \Delta_n.$$

That is for any $\rho \in \Delta_n$, $\tau \in SL(2,\mathbb{Z})$, there exist unique $\tau' \in \Gamma, \rho' \in \Delta_n$, such that $\rho\tau = \tau'\rho'$.

If we take $\chi(a) = 1$ for all $a \in \mathbb{Z}$, we have

$$\begin{aligned}
T_n f &= \frac{1}{n} \sum_{ad=n} \chi(a)a^k \sum_{b \pmod d} f\left(\frac{a\tau+b}{d}\right) \\
&= n^{k-1} \sum_{\rho\in\Delta_n} d^{-k} f(\rho\tau) \qquad\qquad \text{for ad=n} \\
&= n^{k/2-1} \sum_{\rho\in\Delta_n} f|_k\rho(\tau).
\end{aligned}$$

So we have

Proposition 5. *$T_n : M_k(\Gamma) \to M_k(\Gamma)$, and if*

$$f = \sum_{m=0}^{\infty} a(m)e(mz),$$

then we have $T_n f(z) = \sum_{m=1}^{\infty} a_n(m)e(mz)$, where

$$a_n(m) = \sum_{d|(m,n)} d^{k-1} a(mnd^{-2}).$$

Proof. From Lemma 4, we have

$$\begin{aligned} T_n f|_k \gamma &= n^{k/2-1} \sum_{\rho \in \Delta_n} f|_k \rho\gamma(\tau) \\ &= n^{k/2-1} \sum_{\rho' \in \Delta_n} f|_k \gamma' \rho'(\tau) \\ &= T_n f. \end{aligned}$$

For the Fourier expansion, we have

$$\begin{aligned} T_n f(z) &= \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \le b < d} f(\frac{az+b}{d}) \\ &= \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \le b < d} \sum_{m=0}^{\infty} a(m) e(\frac{amz+bm}{d}) \\ &= \frac{1}{n} \sum_{ad=n} a^k \sum_{m=0}^{\infty} a(m) \sum_{0 \le b < d} e(\frac{amz+bm}{d}) \\ &= \frac{1}{n} \sum_{ad=n} a^k d \sum_{m=0}^{\infty} a(m) e(amz) \\ &= \sum_{N=0}^{\infty} \sum_{\substack{ad=n \\ am=N}} a^{k-1} a(md) e(Nz). \end{aligned}$$

□

Corollary 6. *We have $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$, and*

$$\tau(n)\tau(m) = \sum_{d|(m,n)} d^{11} \tau(\frac{mn}{d^2}).$$