Ramanujan τ-Functions

Yu Xiong

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1 First Note

Definition 1. *For any $A \in GL(2,\mathbb{R})$, the slash operator defined on $f : \mathcal{H} \to \mathbb{C}$ is*

$$f|_k A(z) = (det(A))^{k/2} j_A(z)^{-k} f(Az),$$

we have $j_A(z) = cz + d$, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 2.

$$\Delta_n := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | ad = n, 0 \leq b < d \right\}.$$

Lemma 3. *There is a one to one correspondence*

$$\Delta_n \times SL(2,\mathbb{Z}) \leftrightarrow SL(2,\mathbb{Z}) \times \Delta_n.$$

That is for any $\rho \in \Delta_n$, $\tau \in SL(2,\mathbb{Z})$, there exist unique $\tau' \in \Gamma, \rho' \in \Delta_n$, such that $\rho\tau = \tau'\rho'$.

Proof. If $\rho = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n$ and $\tau = \begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$, we want to find $\tau' = \begin{pmatrix} \alpha' & * \\ \gamma' & \delta' \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\rho' \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \Delta_n$ such that $\rho\tau = \tau'\rho'$. This is we want to solve:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha' & * \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}.$$

This is

$$\begin{pmatrix} a\alpha + b\gamma & * \\ d\gamma & d\delta \end{pmatrix} = \begin{pmatrix} a'\alpha' & * \\ a'\gamma' & b'\gamma' + d'\delta' \end{pmatrix},$$

and this is equal to

$$\begin{cases} a'\alpha' & = a\alpha + b\gamma; \\ a'\gamma' & = d\gamma; \\ b'\gamma' + d'\delta' & = d\delta. \end{cases}$$

For $\gamma \neq 0$, we can see that $a' \mid (a\alpha + b\gamma)$ and $a' \mid d\gamma$, so

$$(\frac{a\alpha + b\gamma}{a'}, \frac{d\gamma}{a'}) = (\alpha', \gamma').$$

From $\alpha'\delta' - \beta'\gamma' = 1$, we can see that $(\alpha', \gamma') = 1$, so $a' = (a\alpha + b\gamma, d\gamma)$. In addition, we have

$$\begin{cases} a' &= (a\alpha + b\gamma, d\gamma); \\ d' &= n/(a\alpha + b\gamma, d\gamma); \\ \alpha' &= (a\alpha + b\gamma)/(a\alpha + b\gamma, d\gamma); \\ \gamma' &= d\gamma/(a\alpha + b\gamma, d\gamma). \end{cases} \tag{1}$$

We can see that a', α', γ' are positive integers. For d', we have

$$(a, \gamma) = (a\alpha, \gamma) = (a\alpha + b\gamma, \gamma),$$

is a divisor of a due to $(\alpha, \gamma) = 1$, so we have $(a\alpha + b\gamma, d\gamma) \mid ad$ and d' is an integer.

We have $\alpha'\delta' \equiv 1 \pmod{|\gamma'|}$ and $(\alpha', \gamma') = 1$, there is a $\delta'' \in \mathbb{Z}$, such that $\delta' \equiv \delta'' \pmod{|\gamma'|}$, so $\delta' = \delta'' + \gamma' m$. From $b'\gamma' + d'\delta' = d\gamma$, we have $(b' + md')\gamma' + \delta'' d' = d\delta$.

$$\begin{aligned} \alpha' d\delta - \alpha'\delta'' d' &\equiv \alpha' d\delta - d' && \pmod{|\gamma'|} \\ &\equiv \frac{ad\alpha\delta + bd\delta\gamma - n}{(a\alpha + b\gamma, d\gamma)} && \pmod{|\gamma'|} \\ &\equiv \frac{(\alpha\delta - 1)n}{(a\alpha + b\gamma, d\gamma)} + \gamma'\delta b && \pmod{|\gamma'|} \\ &\equiv 0 && \pmod{|\gamma'|}. \end{aligned}$$

So we find b' and δ'.

If $\gamma = 0$, then we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix},$$

so we take $b' \equiv au + b \pmod{d}$ and $v = d^{-1}(au + b - dv)$, and v is an integer for $d \mid (au + b - dv)$.

For surjection, we need some matrices transform. If we have $\tau' \in \Gamma$, $\rho' \in \Delta_n$, we want to find τ and γ. We can assume that $\tau'\rho' = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, where $xw - yz = n$. Then we have

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} w/(z,w) & * \\ -z/(z,w) & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

With a matrix like $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ we will get the surjection. $\square$

2 Second Note

Definition 4. *Let* $\chi : \mathbb{Z} \to \mathbb{C}$, *and* $\chi(ab) = \chi(a)\chi(b)$, *for any* $(a, b) \in \mathbb{Z}^2$. *If* $n \geq 1$, *we define Hecke operators* T_n *on* $f : \mathcal{H} \to \mathbb{C}$ *to be*

$$T_n f := \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{b \pmod d} f(\frac{a\tau + b}{d}).$$

Theorem 5. *For any* $m, n \geq 1$, *we have*

$$T_m T_n f = \sum_{d|(m,n)} \chi(d) d^{k-1} T_{\frac{mn}{d^2}} f$$

Proof. From Definition 4, we have

$$\begin{aligned}
&mnT_mT_nf(\tau) \\
&= \sum_{\substack{a_1d_1=m \\ a_2d_2=n}} \chi(a_1a_2)(a_1a_2)^k \sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{d_2}}} f\left(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \tau\right) \\
&= \sum_{d|(m,n)} \sum_{\substack{a_1d_1=m \\ a_2d_2=n \\ (a_1,d_2)=d}} \chi(a_1a_2)(a_1a_2)^k \sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{d_2}}} f\left(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \tau\right)
\end{aligned}$$

$$= \sum_{d|(m,n)} \chi(d)d^k \sum_{\substack{a_1'd_1=m/d \\ a_2d_2'=n/d \\ (a_1',d_2')=1}} \chi(a_1'a_2)(a_1'a_2)^k \sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{dd_2'}}} f\left(\begin{pmatrix} da_1'a_2 & d(a_1'b_2 + b_1d_2') \\ 0 & dd_1d_2' \end{pmatrix} \tau\right). \tag{2}$$

We can see that for any $a \in \mathbb{R}$ and $\gamma \in SL(2, \mathbb{Z})$, we have $f(a\gamma\tau) = f(\gamma\tau)$, so

$$f\left(\begin{pmatrix} da_1'a_2 & d(a_1'b_2 + b_1d_2') \\ 0 & dd_1d_2' \end{pmatrix} \tau\right) = f\left(\begin{pmatrix} a_1'a_2 & a_1'b_2 + b_1d_2' \\ 0 & d_1d_2' \end{pmatrix} \tau\right).$$

I claim that for fixed a_1', a_2, d_1, d_2' and $(a_1', d_2') = 1$, if b_1 runs over $\pmod{d_1}$ and b_2 runs over $\pmod{d_2'}$, then we have $a_1'b_2 + b_1d_2'$ runs over $\pmod{d_1d_2'}$ one times.

(b_1, b_2) has d_1d_2' ways to take, so we just need to show that taking different (b_1^*, b_2^*) we have $a_1'b_2 + b_1d_2' \not\equiv a_1'b_2^* + b_1^*d_2' \pmod{d_1d_2'}$. In fact, if they are equal, we have $a_1'(b_2 - b_2^*) \equiv d_2'(b_1^* - b_1) \pmod{d_1d_2'}$, so we have $d_2' \mid a_1'(b_2 - b_2^*)$. But $(a_1', d_2') = 1$, so $b_2 \equiv b_2^* \pmod{d_2'}$. So we prove the claim.

So we have

$$\sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{dd_2'}}} f\left(\begin{pmatrix} a_1'a_2 & a_1'b_2 + b_1d_2' \\ 0 & d_1d_2' \end{pmatrix} \tau\right) = d \sum_{b \pmod{d_1d_2'}} f\left(\begin{pmatrix} a_1'a_2 & b \\ 0 & d_1d_2' \end{pmatrix} \tau\right)$$

We have

$$
\begin{aligned}
&mnT_mT_n \\
&= \sum_{d|(m,n)} \chi(d)d^{k+1} \sum_{\substack{a_1'd_1=m/d \\ a_2d_2'=n/d \\ (a_1',d_2')=1}} \chi(a_1'a_2)(a_1'a_2)^k \sum_{b \pmod{d_1d_2'}} f\left(\begin{pmatrix} a_1'a_2 & b \\ 0 & d_1d_2' \end{pmatrix}\tau\right) \\
&= \sum_{d|(m,n)} \chi(d)d^{k+1} \sum_{ad=\frac{mn}{d^2}} \chi(a)a^k \sum_{b \pmod{d}} f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\tau\right) \\
&= \sum_{d|(m,n)} \chi(d)d^{k+1} T_{\frac{mn}{d^2}} f.
\end{aligned}
$$

□

Recall that we have

Definition 6.

$$\Delta_n := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} |ad = n, 0 \le b < d \right\}.$$

Lemma 7. *There is a one to one correspondence*

$$\Delta_n \times SL(2,\mathbb{Z}) \leftrightarrow SL(2,\mathbb{Z}) \times \Delta_n.$$

That is for any $\rho \in \Delta_n$, $\tau \in SL(2,\mathbb{Z})$, there exist unique $\tau' \in \Gamma, \rho' \in \Delta_n$, such that $\rho\tau = \tau'\rho'$.

If we take $\chi(a) = 1$ for all $a \in \mathbb{Z}$, we have

$$
\begin{aligned}
T_nf &= \frac{1}{n} \sum_{ad=n} \chi(a)a^k \sum_{b \pmod{d}} f(\frac{a\tau + b}{d}) \\
&= n^{k-1} \sum_{\rho \in \Delta_n} d^{-k} f(\rho\tau) \qquad\qquad \text{for ad=n} \\
&= n^{k/2-1} \sum_{\rho \in \Delta_n} f|_k\rho(\tau).
\end{aligned}
$$

So we have

Proposition 8. *$T_n : M_k(\Gamma) \to M_k(\Gamma)$, and if*

$$f = \sum_{m=0}^{\infty} a(m)e(mz),$$

then we have $T_nf(z) = \sum_{m=1}^{\infty} a_n(m)e(mz)$, where

$$a_n(m) = \sum_{d|(m,n)} d^{k-1}a(mnd^{-2}).$$

Proof. From Lemma 7, we have

$$\begin{aligned}
T_n f|_k \gamma &= n^{k/2-1} \sum_{\rho \in \Delta_n} f|_k \rho\gamma(\tau) \\
&= n^{k/2-1} \sum_{\rho' \in \Delta_n} f|_k \gamma' \rho'(\tau) \\
&= T_n f.
\end{aligned}$$

For the Fourier expansion, we have

$$\begin{aligned}
T_n f(z) &= \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \le b < d} f(\frac{az+b}{d}) \\
&= \frac{1}{n} \sum_{ad=n} a^k \sum_{0 \le b < d} \sum_{m=0}^{\infty} a(m) e(\frac{amz+bm}{d}) \\
&= \frac{1}{n} \sum_{ad=n} a^k \sum_{m=0}^{\infty} a(m) \sum_{0 \le b < d} e(\frac{amz+bm}{d}) \\
&= \frac{1}{n} \sum_{ad=n} a^k d \sum_{m=0}^{\infty} a(m) e(amz) \\
&= \sum_{N=0}^{\infty} \sum_{\substack{ad=n \\ am=N}} a^{k-1} a(md) e(Nz).
\end{aligned}$$

□

Corollary 9. *We have* $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$*, and*

$$\tau(n)\tau(m) = \sum_{d|(m,n)} d^{11} \tau(\frac{mn}{d^2}).$$

3 Third Note

In this part, I will introduce the following theorem, whose proof is from Math Stackexchange, and there are some corollaries giving some congruence of τ functions, which is from [1].

Theorem 10.

$$(1-n)\tau(n) = 24 \sum_{j=1}^{n-1} \sigma(j) \tau(n-j).$$

Definition 11. δ_k *is an operator from holomorphic functions to holomorphic functions.*

$$\delta_k(f) := 12\theta(f) - kE_2 f,$$

where $\theta(f) := q\frac{df}{dq}$, $q = e^{2\pi i\tau}$ *and*

$$E_2(\tau) = 1 - 2\sum_{n=1}^{\infty} \sigma q^n.$$

Lemma 12. $\delta_k : M_k(\Gamma) \to M_{k+2}(\Gamma)$ *and* $\delta_k(S_k(\Gamma)) \subset S_{k+2}(\Gamma)$.

Proof. To prove $\delta_k(M_k(\Gamma)) \subset M_{k+2}(\Gamma)$, we just need to show for any $f \in M_k(\Gamma)$ and $\alpha \in \Gamma$, we have

$$(\delta_k f)|_{k+2}\alpha = \delta_k f. \tag{3}$$

We can see that $(\delta_k f)|_{k+2}\alpha = 12(\theta_k f)|_{k+2}\alpha - kE_2|_2\alpha f|_k\alpha$. Firstly, we have $\theta(f) = q\frac{df}{dq} = \frac{1}{2\pi i}f'(\tau)$, so we have

$$\theta(f)|_{k+2}\alpha = \frac{1}{2\pi i}f'|_{k+2}\alpha = \frac{1}{2\pi i}(c\tau + d)^{-k-2}f'(\alpha\tau).$$

We have $f(\alpha\tau) = (c\tau + d)^k f(\tau)$, and taking the derivative of the equation we have $\frac{1}{(c\tau+d)^2}f'(\alpha\tau) = kc(c\tau + d)^{k-1}f(\tau) + (c\tau + d)^k f'(\tau)$. So we have

$$f'(\alpha\tau) = kc(c\tau + d)^{k+1}f(\tau) + (c\tau + d)^{k+2}f'(\tau).$$

Taking the $f'(\alpha\tau)$ to $(\theta_k f)|_{k+2}\alpha$, we have

$$\begin{aligned}
(\theta_k f)|_{k+2}\alpha &= \frac{1}{2\pi i}(c\tau + d)^{-k-2}f'(\alpha\tau) \\
&= \frac{1}{2\pi i}\left(\frac{kc}{c\tau + d}f(\tau) + f'(\tau)\right) \\
&= \frac{kcf(\tau)}{2\pi i(c\tau + d)} + \theta f.
\end{aligned}$$

So we have

$$\begin{aligned}
(\delta_k f)|_{k+2}\alpha - \delta_k f &= 12(\theta_k f)|_{k+2}\alpha - kE_2|_2\alpha f - (12\theta(f) - kE_2 f) \\
&= kf(\tau)\left(\frac{12c}{2\pi i(c\tau + d)} - E_2|_2\alpha + E_2\right) \\
&= 0.
\end{aligned}$$

Last equation is from $E_2|_2\alpha = E_2 + \frac{12c}{2\pi i(c\tau+d)}$. $\square$

Proof of Theorem 10. We have $\delta_k\Delta \in S_{14}(\Gamma) = 0$, so $\delta_k\Delta = 0$. So we have

$$\delta_k\Delta = 12\sum_{n=1}^{\infty} n\tau(n)q^n - 12E_2\Delta = 0.$$

Hence, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n\tau(n)q^n &= \left(1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(\sum_{n=1}^{\infty}\tau(n)q^n\right) \\
&= \sum_{n=1}^{\infty}\tau(n)q^n - 24\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\tau(m)\sigma(n-m)q^n\right).
\end{aligned}$$

□

Corollary 13. *We have*

(a)$\tau(2n) \equiv 0 \pmod 8$.

(b)$\tau(3n-1) \equiv \tau(3n) \equiv 0 \pmod 3$.

(c)$\tau(4n-1) \equiv 0 \pmod 4$.

Proof. We know that $\tau(n)$ are integers, so if we modulo 24 on the two sides of the equation of Theorem 10, we have

$$(1-n)\tau(n) \equiv 0 \pmod{24}.$$

If $n = 2k$, we have $(1-2k)\tau(2k) \equiv 0 \pmod 8$, so we have (a).
If $n = 3k-1$, we have $(2-3k)\tau(3k-1) \equiv 2\tau(n) \equiv 0 \pmod 3$, so we get (b).
If $n = 4k-1$, we have $(2-4k)\tau(4k-1) \equiv 2(1-2k)\tau(4k-1) \equiv 0 \pmod{24}$, so we have $(1-2k)\tau(4k-1) \equiv 0 \pmod 4$, and we have (c). □

Proposition 14. *$\tau(n)$ is odd if and only if n is the square of an odd number*

This proposition is proved by Hansraj Gupta, but I haven't check.

Proposition 15. *We have*

(i)$\tau(6n-1) \equiv 0 \pmod 6$.

(ii)$\tau(8n-1) \equiv 0 \pmod 8$.

Proof. For (i), we know that $\left(\frac{-1}{6}\right) = -1$, so from the corollary and proposition, we have (i).

For (ii), we need the observation that $\tau(8n-r)\sigma(r-1) \equiv 0 \pmod 2$, for $1 < r \leq 8n-1$. From the proposition, we can see that $\tau(8n-r)$ is odd if and only if $r \equiv 7 \pmod 8$. Then we can see that $r-1 \equiv 6 \pmod 8$, so we can see that there is a prime factor of $r-1$ has odd power, and from the multiplication of σ, we get $\sigma(r-1) \equiv 0 \pmod 2$. Hence, we get (ii). □

References

[1] R. P. Bambah, S. Chowla, H. Gupta, and D. B. Lahiri. Congruence properties of Ramanujan's function $\tau(n)$. *Quart. J. Math. Oxford Ser.*, 18:143–146, 1947.