10+2 PCM NOTES

BY

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(PDF version handwritten notes of Maths, Physics and Chemistry for 10+2 competitive exams like JEE Main, WBJEE, NEST, IISER Entrance Exam, CUCET, AIPMT, JIPMER, EAMCET etc.)

3D Geometry.

* **Section formulae:**

i) Division internally $m_1 : m_2$

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_2z_1+m_1z_2}{m_1+m_2}\right).$$

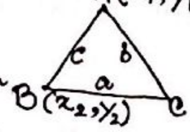
ii) Division externally $m_1 : m_2$

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right).$$

* Centroid of a triangle: $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

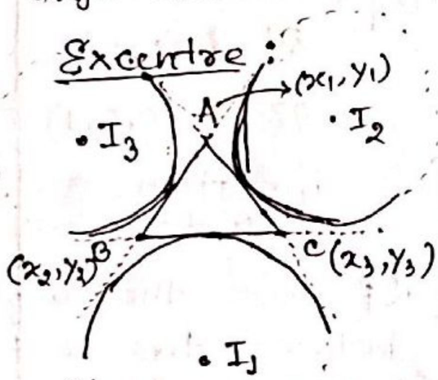
* Centroid of tetrahedron: $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$

* Incentre: pt. of intersection of internal angle bisectors



$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

* Excentre:

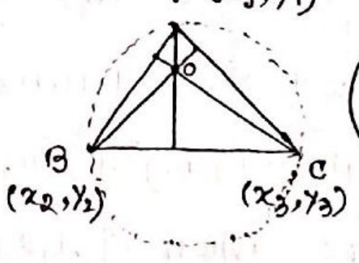


$$I_1 \equiv \left(\frac{ax_1-bx_2-cx_3}{a-b-c}, \frac{ay_1-by_2-cy_3}{a-b-c}\right).$$

$$I_2 \equiv \left(\frac{bx_2-ax_1-cx_3}{b-a-c}, \frac{by_2-ay_1-cy_3}{b-a-c}\right).$$

$$I_3 \equiv \left(\frac{cx_3-ax_1-bx_2}{c-a-b}, \frac{cy_3-by_2-ay_1}{c-b-a}\right)$$

* Circumcentre: Pt of intersection of perpendicular bisector of 3 sides.



Coordinates are:

$$\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+x_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right).$$

* Orthocentre: Pt of intersection of altitudes:

$$\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$$

* Direction cosines: (l, m, n) $(\cos\alpha, \cos\beta, \cos\gamma)$.

Direction ratios: (a, b, c) where a, b, c are three numbers proportional to direction cosines.

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* Useful results:

i) $x = l\,|\vec{r}|$, $y = m\,|\vec{r}|$, $z = n\,|\vec{r}|$.
(Projections on axes).

ii) $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k} = \dfrac{\vec{r}}{|\vec{r}|}$

iii) $l^2 + m^2 + n^2 = 1$. | $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ | $\sum \cos 2\alpha = -1$

iv) If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$, then (a, b, c) are the direction ratios of $\vec{r}$. Direction cosines of $\vec{r}$:

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

v) Direction ratios of line joining two points:
$P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ –

$$[(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)]$$

Direction cosines:

$$\left[\frac{x_2 - x_1}{|\vec{PQ}|}, \frac{y_2 - y_1}{|\vec{PQ}|}, \frac{z_2 - z_1}{|\vec{PQ}|}\right]$$

Direction cosines of

$\vec{OX} \rightarrow (1, 0, 0)$
$\vec{OY} \rightarrow (0, 1, 0)$
$\vec{OZ} \rightarrow (0, 0, 1)$

axes / Coordinate axes.

Direction ratios are infinite in number.

* Angle between two vectors: If θ is the angle between two vectors whose direction cosines are (l_1, m_1, n_1), (l_2, m_2, n_2),

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

$$\sin\theta = \left[(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2\right]^{1/2}$$

orthogonal vectors when $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

parallel vectors when $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

In terms of direction ratios,

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

orthogonal
$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$
parallel
$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

* If $P(\vec{r_1})$, $Q(\vec{r_2})$ are two pt's, then the length of projection of PQ on a line whose direction cosines are (l, m, n) : $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.

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* Straigt line:

Parametric form: passing through a pt with position vector $\vec{a}$ & parallel to a given vector $\vec{b}$ (dr → a, b, c) → (x_1, y_1, z_1).

$$\vec{r} = \vec{a} + \lambda\vec{b} \quad , \lambda \text{ is a scalar.}$$

Non-parametric form: | Cartesian form: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

$$(\vec{r} - \vec{a}) \times \vec{b} = 0 \quad \text{or} \quad \vec{r} \times \vec{b} = \vec{a} \times \vec{b}.$$

passing through two pt's having position vectors $\vec{a}$ & $\vec{b}$

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \text{or} \quad \vec{r} = \vec{b} + \lambda(\vec{b} - \vec{a}).$$

• non parametric: $(\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = 0$ or
$\vec{r} \times (\vec{b} - \vec{a}) = \vec{a} \times \vec{b}$

Cartesian form $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

* Bisector of angle: between the st. lines $\vec{r} = \vec{a} + \lambda\vec{b}$ & $\vec{r} = \vec{a} + \mu\vec{c}$.

$$\vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} \pm \frac{\vec{c}}{|\vec{c}|}\right\}, \text{ where } t \in \mathbb{R}$$

• between two vectors (non-collinear) $\vec{a}$ & $\vec{b}$

$$\vec{r} = \lambda(\hat{a} + \hat{b}).$$

* Perpendicular distance of ~~of~~ a point from a line: from a pt $P(\vec{a})$ on $\vec{r} = \vec{b} + \mu\vec{c}$

$$\sqrt{|\vec{a} - \vec{b}|^2 - \{(\vec{a} - \vec{b}) \cdot \hat{c}\}^2} \quad \text{or}$$

$$\left|\frac{(\vec{b} - \vec{a}) \times \vec{c}}{|\vec{c}|}\right|$$

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* Angle between two lines: between $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$
$\vec{r} = \vec{a_2} + \mu \vec{b_2}$

$$\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}| \cdot |\vec{b_2}|}$$

• cartesian form: between

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \ \& \ \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

* Condition of coplanarity of two lines:

$\vec{r} = \vec{a} + \lambda \vec{b}$ & $\vec{r} = \vec{c} + \mu \vec{d}$ are coplanar if

$$(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d}) = 0.$$

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{d}] = [\vec{c}\ \vec{b}\ \vec{d}]$$

* Shortest distance between skew lines: between.

$\vec{r} = \vec{a} + \lambda \vec{b}$ & $\vec{r} = \vec{c} + \mu \vec{d}$

$$\text{Distance} = \left| \frac{(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})}{|(\vec{b} \times \vec{d})|} \right|.$$

* Equ^n of Plane: General - $ax + by + cz + d = 0$.

Vectorical: passing through $P(\vec{a})$ & normal to the vector $\vec{n}$.

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}.$$

a pt on the plane (x_0, y_0, z_0) & normal to plane (a, b, c) then equ^n

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$\vec{r} \cdot \vec{n} = d$ | d is the distance of plane from origin.

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3D Geometry.

- **Intercept form:** Intercepts are a, b, c

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

- **Passing through three points:** $P(\vec{r_1})$, $\vec{r_2}$, $\vec{r_3}$

Vectorial: $[\vec{r}-\vec{r_1} \quad \vec{r}-\vec{r_2} \quad \vec{r}-\vec{r_3}] = 0$

Cartesian: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} = 0.$

- **Passing through a pt. & parallel to two given vectors:** $P(\vec{a})$ vector lines $\vec{b}$ & $\vec{c}$

$$\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c} \quad \text{(parametric)}$$

$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0.$ (non-parametric)

$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}].$

* **Distance of a pt. from a plane:**

$P(\vec{a})$ point and plane $\vec{r}\cdot\vec{n}=d$

$$\text{distance} = \frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$$

* **Angle between a line and a plane:**

Vectorial: $\vec{r}=\vec{a}+\lambda\vec{b}$ & $\vec{r}\cdot\vec{n}=d$

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}$$

Perpendicularity: $\vec{b}\times\vec{n}=0$ or $\vec{b}=\lambda\vec{n}$

Parallelism: $\vec{b}\cdot\vec{n}=0.$

* <u>Angle between two planes:</u>

$\vec{r}\cdot\vec{n_1} = d_1$ & $\vec{r}\cdot\vec{n_2} = d_2$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

perpendicularity : $\vec{n_1}\cdot\vec{n_2} = 0$

parallelism : $\vec{n_1}\times\vec{n_2} = 0$.

* Eqn of plane passing through intersection of two planes:

<u>Vector</u>: $(\vec{r}\cdot\vec{n_1} = d_1$ & $\vec{r}\cdot\vec{n_2} = d_2)$

$$(\vec{r}\cdot\vec{n_1} - d_1) + \lambda(\vec{r}\cdot\vec{n_2} - d_2) = 0$$

$$\Rightarrow \vec{r}\cdot(\vec{n_1} + \lambda\vec{n_2}) = d_1 + \lambda d_2.$$

[family of planes].

* Planes bisecting the angles between two planes:

Cartesian: $(a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0)$.

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{—(1)}$$

[writing the eqns of plane as their constant term positive, then

$a_1a_2 + b_1b_2 + c_1c_2 >$ or < 0 means

positive sign in the eqn (1) gives the bisector of the obtuse or acute angle.

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3D Geometry

* **Equⁿ of a sphere:** of radius a & centre $P(\vec{c})$.

$$|\vec{r}-\vec{c}| = a.$$

cartesians: centre (a,b,c) & radius K

$$(x-a)^2+(y-b)^2+(c-z)^2 = K^2.$$

• general equⁿ of sphere.

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

$$\left(\begin{array}{l}\text{centre } -u, -v, -w, \\ \text{radius } \sqrt{u^2+v^2+w^2-d}\end{array}\right).$$

• passing through 4 pt's.

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

• $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ as extremities of diameter, the equⁿ

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$

* • Condition of the sphere to touch a plane.

$$\vec{r}\cdot\vec{n} = d \quad ; \quad |\vec{r}-\vec{c}| = a$$

$$\frac{|\vec{c} * \vec{n} - d|}{|\vec{n}|} = a = \frac{|\vec{c}\cdot\vec{n} - d|}{|\vec{n}|}$$

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Extras.

* 3 pt's are collinear then $\frac{x_1-x_2}{x_2-x_3} = \frac{y_1-y_2}{y_2-y_3} = \frac{z_1-z_2}{z_2-z_3}$

* Planes in cartesian form:

• point-normal form – $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

• parallel to two lines with direction ratio/cosines (a_1, a_2, a_3) & (b_1, b_2, b_3) & passing through (x_0, y_0, z_0) – $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

• containing two pt's (x_1, y_1, z_1), (x_2, y_2, z_2) & a parallel line to plane with dr/dc (a,b,c) –

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ a & b & c \end{vmatrix} = 0$$

* Position of a point: Two pt's are on the same side,

$$(ax_1+by_1+cz_1+d)(ax_2+by_2+cz_2+d) > 0$$

opposite side $(ax_1+by_1+cz_1+d)(ax_2+by_2+cz_2+d) < 0.$

* Distance of a pt from the plane $ax+by+cz+d=0$

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

* Perpendicular foot of point (x_1, y_1, z_1) onto the plane $ax+by+cz+d=0$, is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = -\left(\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2}\right)$$

* Image of a pt. in a plane: of (x_1, y_1, z_1) in $ax+by+cz+d=0$.

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = -2\left(\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2}\right).$$

* Distance between planes –

parallel $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$ | $ax+by+cz+d_1=0$, $ax+by+cz+d_2=0$

* Projection of area: $A^2 = A_{xy}^2 + A_{yz}^2 + A_{zx}^2$

* Plane section ratio of a line joining two points: $\frac{m}{n} = -\left(\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

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3D Geometry

* Foot of perpendicular of (x_1, y_1, z_1) onto the line $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ is given by

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c} = \left(\frac{(ax_1+by_1+cz_1)-(a\alpha+b\beta+c\gamma)}{a^2+b^2+c^2}\right)$$

* Image of (x_1, y_1, z_1) from $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c} = 2\left(\frac{(ax_1+by_1+cz_1-(a\alpha+b\beta+c\gamma)}{a^2+b^2+c^2}\right)$$

* Shortest distance of skew lines:

$$\frac{[(x_1-x_2, y_1-y_2, z_1-z_2)\ (a_1, b_1, c_1)\ (a_2; b_2, c_2)]}{|(a_1, b_1, c_1) \times (a_2, b_2, c_2)|}.$$

* The eqn of plane passing through the intersection of the planes $\vec{r}\cdot\vec{n_1} = d_1$ & $\vec{r}\cdot\vec{n_2} = d_2$ is $\vec{r}\cdot(\vec{n_1} + \lambda\vec{n_2}) = d_1 + \lambda d_2$.

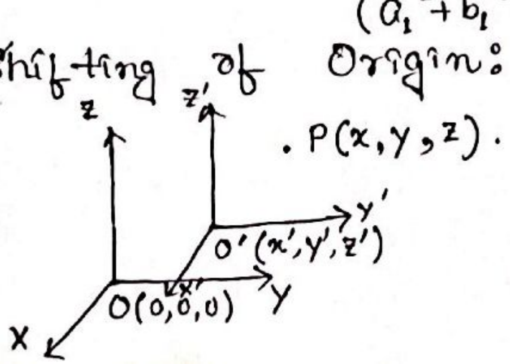
* Volume of tetrahedron: $\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

* Reflection of the plane $ax+by+cz+d=0$ on the place $a_1x+b_1y+c_1z+d_1=0$ is

$$2(aa_1+bb_1+cc_1)(a_1x+b_1y+c_1z+d_1) =$$

$$(a_1^2+b_1^2+c_1^2)(ax+by+cz+d).$$

* Shifting of Origin:



new coordinate. (x_1, y_1, z_1)

$x_1 = x - x'$

$y_1 = y - y'$

$z_1 = z - z'$

old coordinate (x, y, z).

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