An example showing the matrix version of the simple linear model

```
> library(faraway)
> #Matrix approach to Simple Linear Model
> #Experiment: 4 rats were injected with a dose(mcg) of a drug.
> #The time (minute) to run a maze was recorded for each rat.
> # x=dose y=time
```

Model: $\text{time}_i = \beta_0 \times 1 + \beta_1 \times \text{dose}_i + \text{error}_i$

```
> dose=c(0, 2, 4, 6)
```

← vector of doses $\boxed{n=4}$

```
> y=c(7, 5, 4, 4)
```

← vector of responses (time)

```
> X=as.matrix(cbind(1,dose))
```

← creates X-matrix

```
> X   #this is what the X-matrix looks like

       dose
[1,] 1    0
[2,] 1    2
[3,] 1    4
[4,] 1    6
```

$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}$ $\quad \underset{\sim}{Y} = \begin{bmatrix} 7 \\ 5 \\ 4 \\ 4 \end{bmatrix}$

```
> y  #this is what the vector of y responses looks like

[1] 7 5 4 4

> #create a data frame named "example" for later use
> example <- data.frame(dose,y)
> example
```

the four observations had the format (x_i, y_i) $i=1,2,3,4$

```
  dose y
1    0 7
2    2 5
3    4 4
4    6 4
```

$y_1 = \beta_0 \times 1 + \beta_1 \text{dose}_1 + \text{Error}_1$

$\vdots$

$y_4 = \beta_0 \times 1 + \beta_1 \text{dose}_4 + \text{Error}_4$

or $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\text{Error}}$

```
> XtX <- t(X) %*% X
> XtX #show XtX - should be a 2x2 symmetric matrix about the


diagonal
        dose
      4   12
dose 12   56
```

← the (1,2) element = the (2,1) element

$X'X = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix}$ and is $p \times p$

X' is $p \times n$ $\quad X$ is $n \times p$

$X'X$ is $p \times p$

$$(X'X) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix} = \begin{bmatrix} \overset{\text{sample size}}{n} & \Sigma x_i \\ \Sigma x_i & \Sigma (x_i)^2 \end{bmatrix}$$

```
> XtXi <- solve(t(X) %*% X)  #compute XtX inverse
> XtXi #show XtX inverse

            dose
      0.70 -0.15
dose -0.15  0.05
```

$\leftarrow (X'X)^{-1}$

```
> I2 <- XtX %*% XtXi
> I2 #should get back the Identity (2x2) matrix

        dose
      1    0
dose  0    1
```

$$(X'X)(X'X)^{-1} = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix} \begin{bmatrix} 0.70 & -0.15 \\ -0.15 & 0.05 \end{bmatrix} = \begin{bmatrix} 2.8-1.8 & -.6+.6 \\ 8.4-8.4 & -1.8+2.8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

```
> Xty <- t(X) %*% y
> Xty

      [,1]
        20
dose    50
```

$X'\underset{\sim}{Y} \longrightarrow$ 20

$$X'\underset{\sim}{Y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 50 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

```
> Bhat <- XtXi %*% t(X) %*% y   #Beta hat is equal to
(X'X)inverse * X'y

> Bhat

      [,1]
       6.5
dose  -0.5
```

$$\hat{\underset{\sim}{B}} = (X'X)^{-1} X'\underset{\sim}{Y} = \begin{bmatrix} 0.70 & -0.15 \\ -0.15 & 0.05 \end{bmatrix} \begin{bmatrix} 20 \\ 50 \end{bmatrix} = \begin{bmatrix} 14-7.5 \\ -3+2.5 \end{bmatrix} = \begin{bmatrix} 6.5 \\ -0.5 \end{bmatrix}$$

$$\hat{\underset{\sim}{B}} = \begin{bmatrix} \hat{B}_0 \\ \hat{B}_1 \end{bmatrix} = \begin{bmatrix} 6.5 \\ -0.5 \end{bmatrix}$$

```
> #compute the standard errors of Beta hat which can be found by
> #taking the square root of the diagonal elements of the
      matrix XtXi * sigma_hat
```

```
> mod <- lm(y ~ dose, data=example)
> modsum <- summary(mod)
> modsum #check to be sure the matrix computations agree with
the output from R

Call:
lm(formula = y ~ dose, data = example)

Residuals:
   1    2    3    4
 0.5 -0.5 -0.5  0.5
```

$SSE = (0.5)^2 + (-0.5)^2 + (-0.5)^2 + (0.5)^2 = 1.0$

$SSE = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \sum e_i^2 = \underline{1.0}$

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   6.5000     0.5916  10.987  0.00818 **
dose         -0.5000     0.1581  -3.162  0.08713 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.7071 on 2 degrees of freedom
Multiple R-squared:  0.8333,    Adjusted R-squared:   0.75
F-statistic:    10 on 1 and 2 DF,  p-value: 0.08713
```

n-2 or 4-2

```
> sigma_hat=sqrt(deviance(mod)/df.residual(mod)) #compute the
estimate of sigma

> sigma_hat
[1] 0.7071068
```

$\sqrt{MSE} = \sqrt{\frac{1}{2}} = \sqrt{.5} = \underline{0.7071}$

```
> sigma_hat_of_Beta_hat=sqrt(diag(XtXi))*sigma_hat
> sigma_hat_of_Beta_hat

                    dose
0.5916080 0.1581139
```

Known as the variance-covariance matrix of $\hat{B}$

$cov(\hat{B}_0, \hat{B}_1)$

$\boxed{\hat{\sigma}^2 \times (X'X)^{-1}} = \begin{bmatrix} (.5)(.70) & (.5)(-0.15) \\ (.5)(-0.15) & (.5)(0.05) \end{bmatrix} = \begin{bmatrix} 0.35 & -0.075 \\ -0.075 & 0.025 \end{bmatrix}$

```
> #done
```

$SQRT\left[\text{diagonal}\left[\hat{\sigma}^2 \times (X'X)^{-1}\right]\right] = \begin{bmatrix} SQRT(0.35) & \sim \\ \sim & SQRT(0.025) \end{bmatrix}$

std error of $\hat{B}_0 = \sqrt{.35} = \underline{\underline{.5916}}$ and std error of $\hat{B}_1 = \sqrt{0.025} = \underline{\underline{.1581}}$