CSE 599-I: Final Project

A Spectral Approach to Graph Coloring

Mike Shrieve & Alex Voet

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1 Introduction

Given a graph $G = (V, E)$, the chromatic number $\chi(G)$ is the least k for which there exists a k-coloring of the vertices. It is a NP-hard problem to determine the chromatic number, and even for a graph of known chromatic number, it is still NP-hard to find an optimal coloring. However, as discussed in class, there are non-optimal coloring algorithms for graphs of known chromatic number. For this reason (and certainly for others), it may be useful to put bounds on $\chi(G)$. Specifically, we investigated several chromatic bounds obtained from the spectrum of the graph and compared them experimentally to some less sophisticated bounds using the python package `graph-tool`. Our code is available at `github.com/advoet/spectral-coloring`.

2 Bounds on the Chromatic Number

Regardless of the structure of G, there exists a coloring with V colors: simply paint each vertex a different color. Better, we can *greedy color* our graph using $k = 1 + \max_{v \in V} \deg(\mathrm{v})$. There are several improvements on these bounds using the eigenvalue spectrum of a graph.

Definition 2.1. Let G be a graph and A its adjacency matrix. The spectrum of G is the set of eigenvalues of A, listed in decreasing order

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$$

Theorem. *The simplest bound is due to Wilf [Wil67]*

$$\chi(G) \leq \mathtt{Wilf}(G) = 1 + \mu_1.$$

Proof. The largest eigenvalue of the adjacency matrix is bounded below by the average vertex degree, so for any graph we can guarantee there is a vertex of degree $\lceil \mu_1 \rceil$. Additionally, the largest eigenvalue can only decrease when removing a vertex. These are Lemmas 4.2.1 and 4.2.2 in the referenced notes. Wilf's bound then follows by induction, removing a node of degree at most $\lceil \mu_1 \rceil$. This can only decrease the largest eigenvalue. The subgraph is inductively assumed to be $\lceil \mu_1' \rceil + 1$ colorable, and our restriction on the degree of the removed vertex guarantees we can extend the coloring.

ref: `cs.yale.edu/homes/spielman/561/2009/lect04-09.pdf` □

Theorem. *We have the following bound due to Hoffman and Howes [HH70]:*

$$\chi(G) \geq \mathtt{Hoff}(G) = 1 - \frac{\mu_1}{\mu_n}.$$

Proof. [EFP11]
The proof relies on the fact that a coloring is a partition of a graph into independent subsets. We present the referenced proof that an independent set $|I|$ has size bounded by

$$|I|/|V| \leq -\mu_n/(\mu_1 - \mu_n)$$

The Hoffman bound follows as we see at least $1 - \mu_1/\mu_n$ independent sets are required to cover all the vertices.

Assume we are on a d-regular graph and let $v_1, \ldots, v_n$ be an orthonormal system of eigenvectors with eigenvalues $\mu_1, \ldots, \mu_n$ respectively. The regularity implies that v_1 is the vector of all 1s. Now take I to be an indpendent set, and let $u = 1_I$ be the vector which is 1 for each vertex in I and 0s elsewhere. Since I is independent, notice

$$u^t A u = 0$$

If we write u as a linear combination of eigenvectors so that

$$u = \sum_{i=1}^{n} a_i v_i$$

then we have the following equalities:

$$|I|/|V| = \langle u, v_1 \rangle = a_1 = \sum_{i=1}^{n} a_i^2$$

where $\langle u, v \rangle := \frac{1}{n} u \cdot v$. Using these equalities, we find

$$0 = u^t A u = \sum_{i=1} a_i^2 \mu_i \geq a_1^2 \mu_1 + \sum_{i=2} a_i^2 \mu_n = a_1^2 \mu_1 + (a_1 - a_1^2)\mu_n$$

Then

$$0 \geq a_1 \mu_1 + (1 - a_1)\mu_n$$

and we are done, since $a_1 = |I|/|V|$. □

Theorem. *Wocjan and Elphick provide a generalized form of Hoffman's bound in [WE12]*

$$\chi(G) \geq \texttt{gHoff}(G) = 1 + \max_{m \in \{1,\ldots,n-1\}} \left\{ \frac{-\sum_{i=1}^{m} \mu_m}{\sum_{i=1}^{m} \mu_{n-i+1}} \right\}.$$

Conjecture. *Let S^+ be the sum of the squares of the positive eigenvalues, and S^- be the sum of the squares of the negative eigenvalues. Wocjan and Elphick conjecture the following improvement on the previous bounds:*

$$\chi(G) \geq \texttt{WocElp}(G) = 1 + \frac{S^+}{S^-}$$

They prove the weaker bound:

$$\chi(G) \geq \frac{S^+}{S^-}.$$

3 Generating the Graphs

The literature on generating graphs with a given chromatic number is surprisingly sparse. While it is relatively simple to randomly generate a graph with bounded chromatic number, it seems not much is known about generating graphs with known chromatic number beyond a single algorithm in [Lei79]

3.1 generate_n_colorable(n, k, p)

This simple algorithm was taken from [AK97].

As implemented, the algorithm generates an n colorable graph on $n * k$ vertices and provides a coloring where each color group has size k. There are no guarantees on the chromatic number beyond $\chi(G) \leq n$. Only in the case of $n = 3$ can we make any guarantees without excessive computational overhead, as it is quick to check if a graph is bipartite. The algorithm is as follows:

Algorithm:

1. Select vertices to form color groups (k vertices of each color, but this step generalizes easily)
2. For every pair of edges between vertices of different colors, add an edge with probability p.

3. (Optional, for $n = 3$) If graph is bipartite, throw it out and repeat.
4. (Optional) Introduce a clique of size n on vertices of distinct colors to guarantee $\chi(G) = n$

As it is impossible to add an edge between two vertices in the same color group, the resulting graph is guaranteed to be n colorable. The inclusion of optional step 3 is implemented in `generate_strict_three_colorable`. The inclusion of optional step 4 is implemented as `generate_strict_n_colorable`.

3.2 generate_test_graph(n, k, a, c, m, bs)

This algorithm was presented by Leighton in [Lei79] as a "proving ground" for graph coloring algorithms. While these graphs are not entirely random, they are reproducible, and Leighton makes the claim in his paper that they are "at least as hard to color" as randomly generated graphs. This claim seems dubious, but there is some data to back it up in his paper.

The algorithm is implemented exactly as stated in the paper.

Inputs:
n - Number of vertices
k - Desired chromatic number
a - Positive integer such that $p|m \Rightarrow p|(a-1)$ for all primes p, and for $p = 4$
c - Positive integer coprime to m
m - Positve integer much larger than n where $(n, m) = k$ and $(c, m) = 1$
bs - Nonnegative integers $(b_k, \ldots, b_2)$ with b_i equal to the number of cliques of size i to be added, $b_k \geq 1$

Algorithm:

1. Generate a sequence $\{X_i\}$ of integers in $[0, m)$ by the rule $X_{i+1} = aX_i + c \pmod{m}$
2. Let $Y_i = X_i \pmod{n}$
3. We require $b_k \geq 1$, so take $Y_1 \ldots Y_k$ and add edges to form a clique on those vertices and decrement b_k
4. Continue for each ℓ while $b_\ell > 0$ to take the next ℓ indices from $\{Y_i\}$ and introduce a clique on those vertices

A k-coloring is provided by $f(v_{Y_i}) = i \pmod{k}$.

3.3 random graphs

Additionally we tested the bounds for random graphs with two generation methods. One is generated by adding edges with uniform probability, and another by graph-tool's random graph method which outputs a random d regular graph for a given d in some range.

4 Data

To get a concrete upper bound on $\chi(G)$, we use the `graph-tool` sequential coloring algorithm. The sequential algorithm allows us to specify the order of vertices with respect to which it will attempt to find a coloring. We implemented two orderings: one totally random, and one which is ordered in decreasing orders of degrees; if multiple vertices have the same degree, their suborder is randomized. For a graph G, we run the first ordering 100 times, and the second ordering 50 times; out of the 150 colorings, we take the one with the fewest colors. This gives an upper bound on the chromatic number, as well as an explicit coloring that we can visualize.

Using randomly generated graphs, we investigate the following eigenvalue-based bounds: `Hoff`, `gHoff`, `Wilf`, and `WocElp`. We also compute an explicit coloring, and note the average and maximum degrees of the vertices. In the following, we fix an input to one of the graph generators, and run the experiment 15 times, we then average over all the bounds obtained and report the results in a table.

Table 1: Graph by Degree

Vertices	Method#1	Method#2	Method#3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

Table 2: Graph by Edge Probability

Vertices	Method#1	Method#2	Method#3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

Table 3: Leighton Graph

Vertices	Method#1	Method#2	Method#3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

Table 4: Strict n-colorable Graph

Vertices	Edge Probablity	Method#1	Method#2	Method#3
1	50	837	970	
2	47	877	230	
3	31	25	415	
4	35	144	2356	
5	45	300	556	

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References

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