Lecture 13a: Markov Decision Processes

CSCI 360

Introduction to Artificial Intelligence

USC

Here is where we are…

	3/1		Project 2 Out	
9	3/4	3/5	Quantifying Uncertainty	[Ch 13.1-13.6]
	3/6	3/7	Bayesian Networks	[Ch 14.1-14.2]
10	3/11	3/12	*(spring break, no class)*	
	3/13	3/14	*(spring break, no class)*	
11	3/18	3/19	Inference in Bayesian Networks	[Ch 14.3-14.4]
	3/20	3/21	Decision Theory	[Ch 16.1-16.3 and 16.5]
	3/23		Project 2 Due	
12	3/25	3/26	***Advanced topics*** *(Chao traveling to National Science Foundation)*	
	3/27	3/28	***Advanced topics*** *(Chao traveling to National Science Foundation)*	
	3/29		Homework 2 Out	
13	4/1	4/2	Markov Decision Processes	[Ch 17.1-17.2]
	4/3	4/4	Decision Tree Learning	[Ch 18.1-18.3]
	4/5		Homework 2 Due	
	4/5		Project 3 Out	
14	4/8	4/9	Perceptron Learning	[Ch 18.7.1-18.7.2]
	4/10	4/11	Neural Network Learning	[Ch 18.7.3-18.7.4]
15	4/15	4/16	Statistical Learning	[Ch 20.2.1-20.2.2]
	4/17	4/18	Reinforcement Learning	[Ch 21.1-21.2]
16	4/22	4/23	Artificial Intelligence Ethics	
	4/24	4/25	Wrap-Up and Final Review	
	4/26		Project 3 Due	
	5/3	5/2	**Final Exam** (2pm-4pm)	

Outline

- What is AI?
- Part I: Search
- Part II: Logical reasoning
- **Part III: Probabilistic reasoning**
  - Quantifying Uncertainty
  - Bayesian Networks
  - Inference in Bayesian Networks
  - Decision Theory
  - **Markov Decision Processes**
- Part IV: Machine learning

Recap: *Making a rational decision*

Rational decision depends on

(1) The relative importance of various goals and

(2) likelihood that (and degree to which) they will be reached

Decision theory = Utility theory + Probability theory

Choose an action that yields the ***maximum expected utility (MEU)***, averaged over all the possible outcomes of the action, weighted by the probability

Recap: *Maximum expected utility (MEU)*

- Choosing an action that maximizes the expected utility

$$\boxed{action = \operatorname*{argmax}_{a} EU(a|\mathbf{e})}$$

- This principle defines all of AI

Given the evidence (e) and a set of actions, pick the action (a) that has the MEU

Recap: *Utility function (U)*

- Utility function, denoted ***U(s')***, expresses the desirability of the state ***s'***

- Example:

$$s' = \{\ getting\ A,\ getting\ C\ \}$$

$U(\ getting\ A\ ) = 4.0$

$U(\ getting\ C\ ) = 2.0$

Recap: *Expected utility (EU)*

- Expected utility, ***EU(a|e)***, is the weighted average

$$EU(a|\mathbf{e}) = \sum_{s'} P(\textsc{Result}(a) = s' \mid a, \mathbf{e})\, U(s')$$

*a: with professor **Y***   *e: taking the course*   ***s'*** = { *getting A*, *getting C* }

Example:

*P(**RESULT**(a) = getting A | a, e) * U(getting A) = 75% * 4.0*

*P(**RESULT**(a) = getting C | a, e) * U(getting C) = 25% * 2.0*

*EU(a|e) = 75%*4.0 + 25%*2.0*

Recap: *Example*

- *Taking a course with* ***Professor X***

getting B(3)

- *Taking a course with* ***Professor Y***

getting A(4) getting C(2)

✓ *Choose Professor Y (seeking the best case – 4.0)*
✓ *Choose Professor X (avoiding the worst case – 2.0)*

Which one do **you** choose?

Recap: *Example*

- *Taking a course with* ***Professor X***

getting B(3)

(100%)

3*100% = 3

- *Taking a course with* ***Professor Y***

getting A(4) getting C(2)

(75%) (25%)

4*75% + 2*25% = 3.5

SHOULD

Which one ~~do~~ you choose?

Recap: *Example*

- *Taking a course with* ***Professor X***

getting B(3)

(100%)

3*100% = 3

- *Taking a course with* ***Professor Y***

getting A(4) getting C(2)

(25%) (75%)

4*25% + 2*75% = 2.5

SHOULD

Which one ~~do~~ you choose?

Recap: *Maximum expected utility (MEU)*

- Choosing an action that maximizes the expected utility

$$\boxed{action = \underset{a}{\operatorname{argmax}}\, EU(a|\mathbf{e})}$$

- This principle defines all of AI

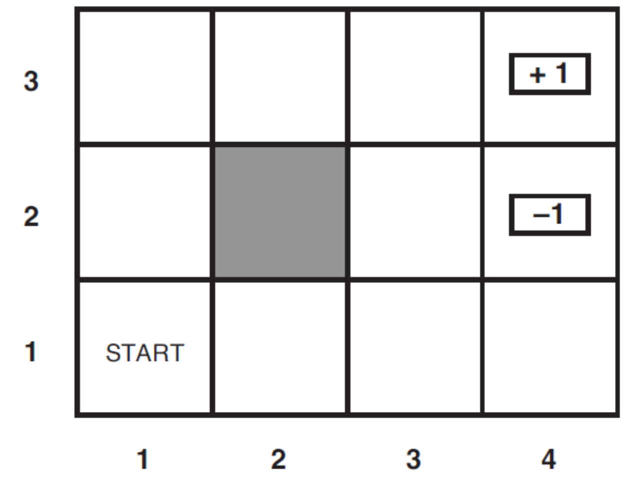
Given the evidence (e) and a set of actions, pick the action (a) that has the MEU

Outline of today's lecture

- Sequential Decision
  - A sequence of decisions (versus *one-shot* decision)
- Value Iteration
  - Algorithm for solving the sequential decision problem
- Policy Iteration
  - An alternative algorithm

Example problem

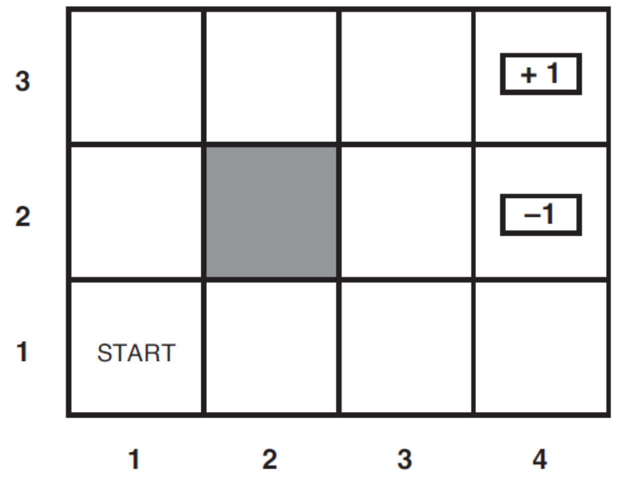
- Initial state: (1,1)
- Goal state: (4,1) utility +1
  (4,2) utility -1
- Actions:
  – Up, Down, Left, Right utility -0.04
  – *(won't move it running into the wall)*
- Transition model:
  – If RESULT(s, a) is **deterministic**, then it's a **search** problem

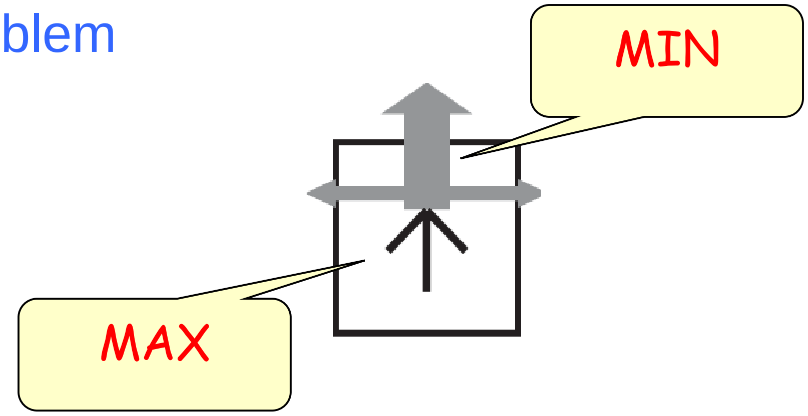


What's the **branching factor**?

Example problem (2)

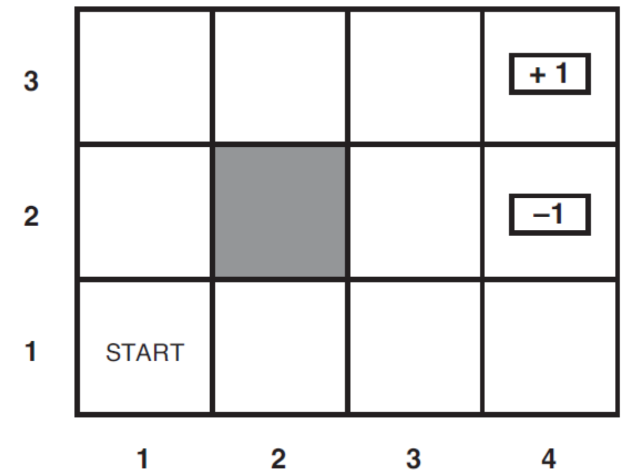
- Initial state: (1,1)
- Goal state: (4,1) utility +1
  (4,2) utility -1
- Actions:
  - Up, Down, Left, Right utility -0.04
  - *(won't move it running into the wall)*
- Transition model:
  - If RESULT(s, a) is **non-deterministic**, then it's an **adversarial search** problem

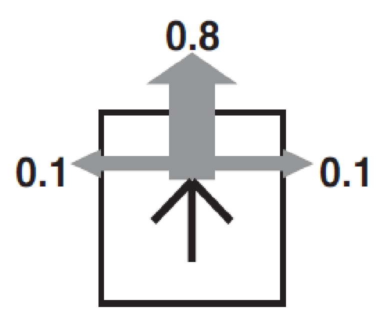




Example problem (3)

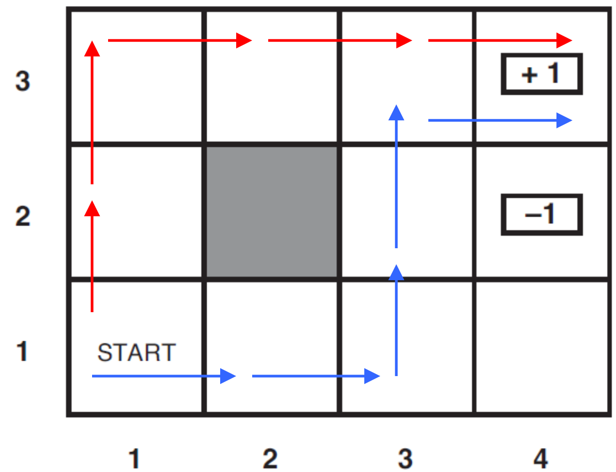
- Initial state: (1,1)
- Goal state: (4,1) utility +1
  (4,2) utility -1
- Actions:
  - Up, Down, Left, Right utility -0.04
  - *(won't move it running into the wall)*
- Transition model:
  - If RESULT(s, a) is **non-deterministic**, and **probabilistic**, then it's called MDP (Markov decision process)

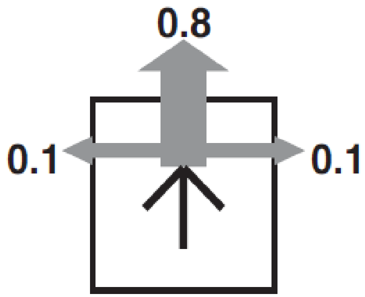




Example action sequence

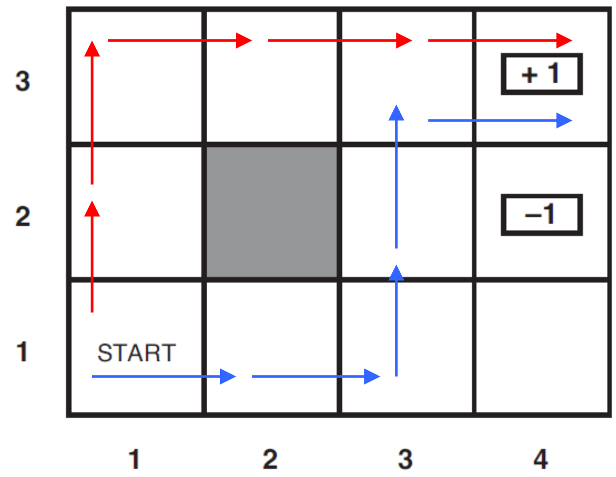
- Consider *[Up, Up, Right, Right, Right]*
- What’s the probability that it will reach the goal state (4,1)?
  - 0.8 * 0.8 * 0.8 * 0.8 * 0.8 = **0.32768**
- The other way for the given action sequence to reach (4,1)
  - 0.1 * 0.1 * 0.1 * 0.1 * 0.8 = **0.00008**

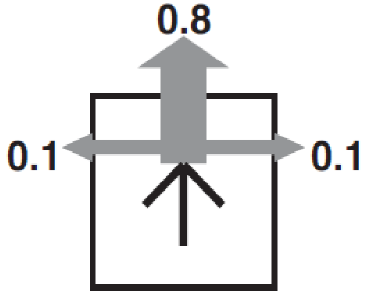




Example action sequence

- Consider *[Up, Up, Right, Right, Right]*
- What's the probability that it will reach the goal state (4,1)?
  - 0.8 * 0.8 * 0.8 * 0.8 * 0.8 = **0.32768**
  - *Reward: 4 * (-0.04) + 1 = 0.84*
- The other way for the given action sequence to reach (4,1)
  - 0.1 * 0.1 * 0.1 * 0.1 * 0.8 = **0.00008**
  - *Reward: 4 * (-0.04) + 1 = 0.84*

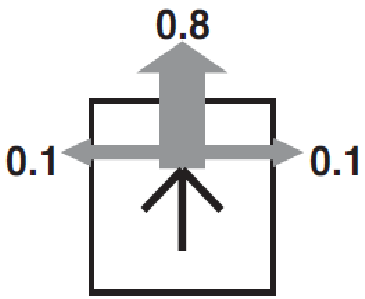




Markov decision process (MDP)

- MDP is a sequential decision problem that has
  - a **fully observable, stochastic** environment
  - a **Markovian** transition model, and
  - the **additive** rewards
- State space formalism
  - Initial state (s0)
  - Actions(s) = {a1, a2, … } for each state
  - Transition model P(s' | s, a) for each state and each action
  - Reward function R(s)

Solution to a MDP



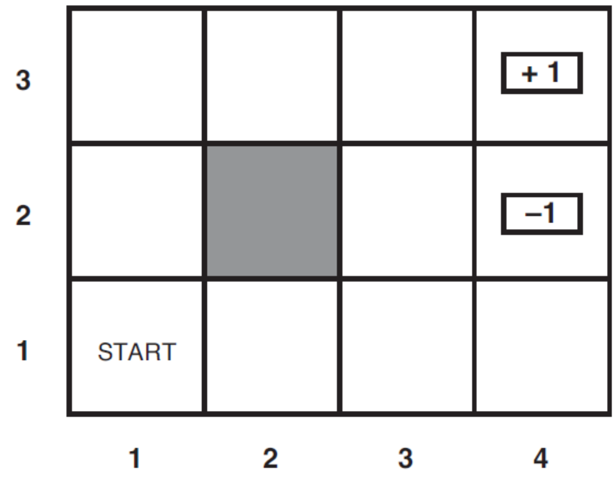
- It <u>should not</u> be any fixed action sequence
  - Example: *[Up, Up, Right, Right, Right]*
  - Because of non-deterministic outcomes of an action, there must be contingency plans

- It should be a **policy**, denoted *$\pi(s)$*, which specifies what the agent should do for any state *(s)* that it might reach

Question:

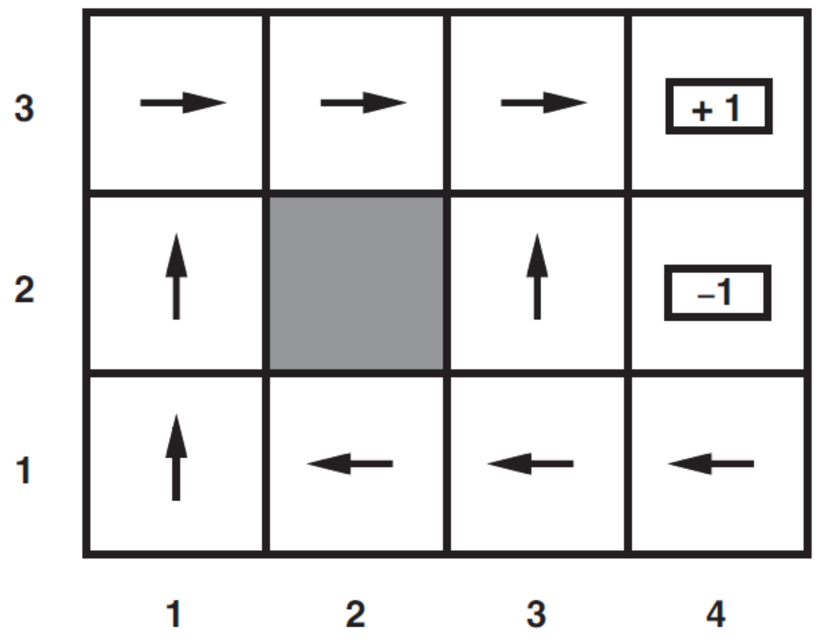
How many policies are there?

$4*4*...*4 = 4^9$



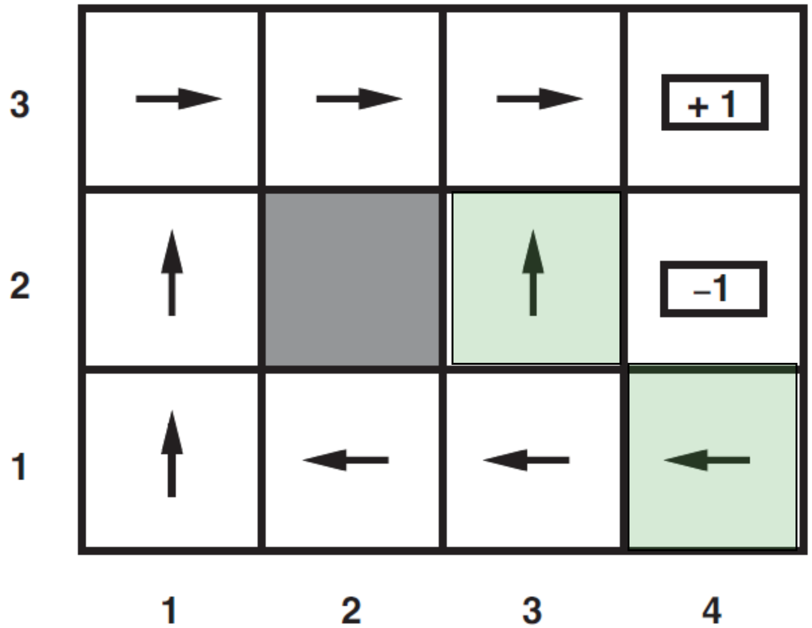
Optimal solution for *R(s) = -0.04*

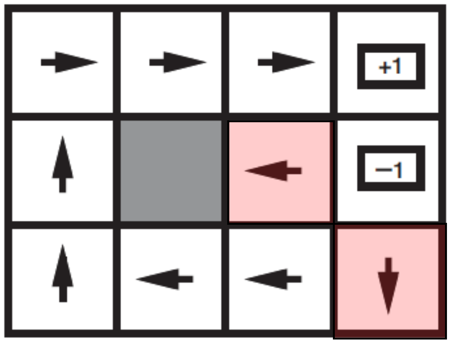
- When the cost of taking a step is fairly small compared to the penalty for ending up in (4,2) by accident
  - The policy recommends taking the long way round



Optimal solution for *-0.0221 ≤ R(s) ≤ 0*

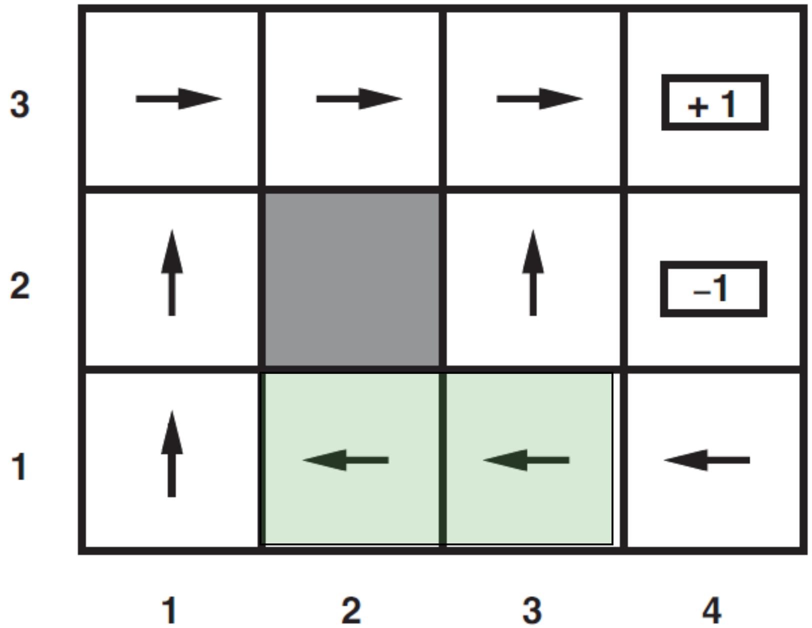
- When life is only slightly dreary…
  - Agent takes no risks at all

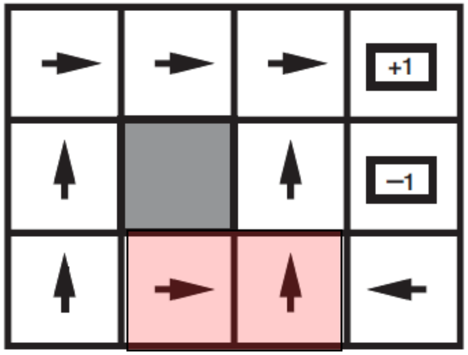




Optimal solution for *-0.4278 ≤ R(s) ≤ 0.0850*

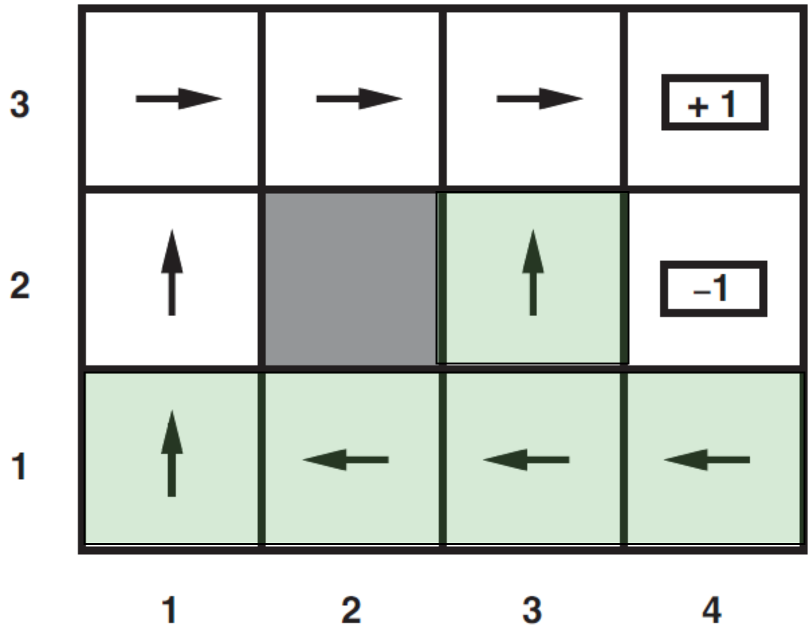
- When life is quite unpleasant..
  - Agent takes the shortest route to +1 state, and is willing to risk failing into the -1 state by accident

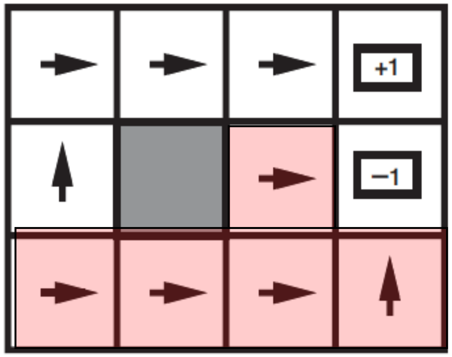




Optimal solution for *R(s) ≤ -1.6284*

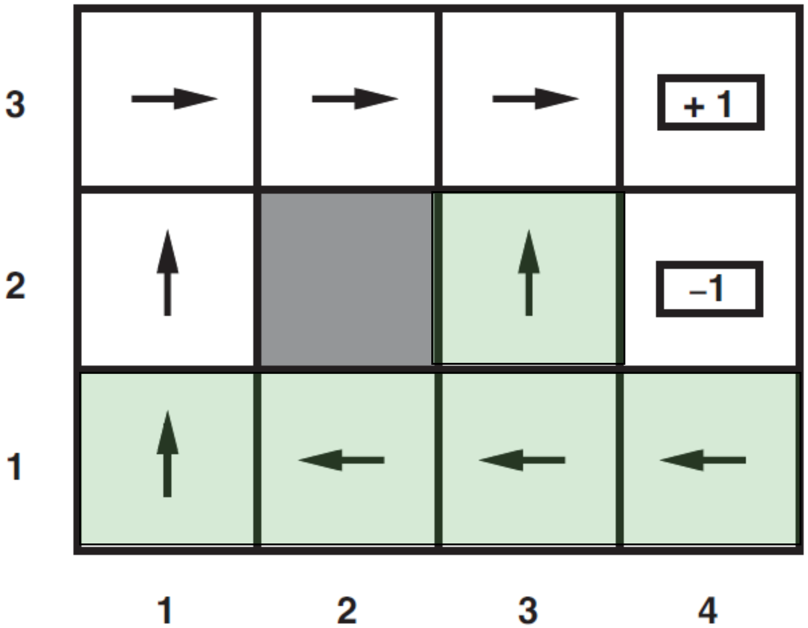
- Life is so painful…
  - Agent heads to the nearest exit, even if the exit is the -1 state

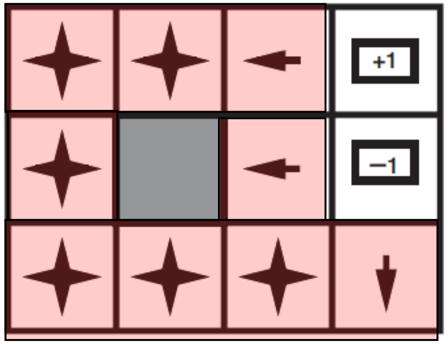




Optimal solution for *R(s) > 0*

- Life is so enjoyable…
  - Agent avoids both exit





Characteristic of MDPs

- Careful balancing of **risk** and **reward**, which is required by many real-world decision problems
  - AI
  - Operations research
  - Economics
  - Control theory

Utilities over time: *two options*

- Performance of agent is measured by a **sum** of rewards for the states visited
  - **Finite horizon**: a fixed time (N) after which nothing matters
    - This is actually complicated, since the optimal action in a given state could change over time
  - **Infinite horizon**: without a fixed time limit
    - This is simpler, since the optimal action depends only on the current state

We focus only on the "**infinite horizon**"

Sum of rewards: *two options*

- Additive rewards

$$U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \cdots$$

  - **Problem**: the sum is always infinite – hard to compare

- Discount rewards

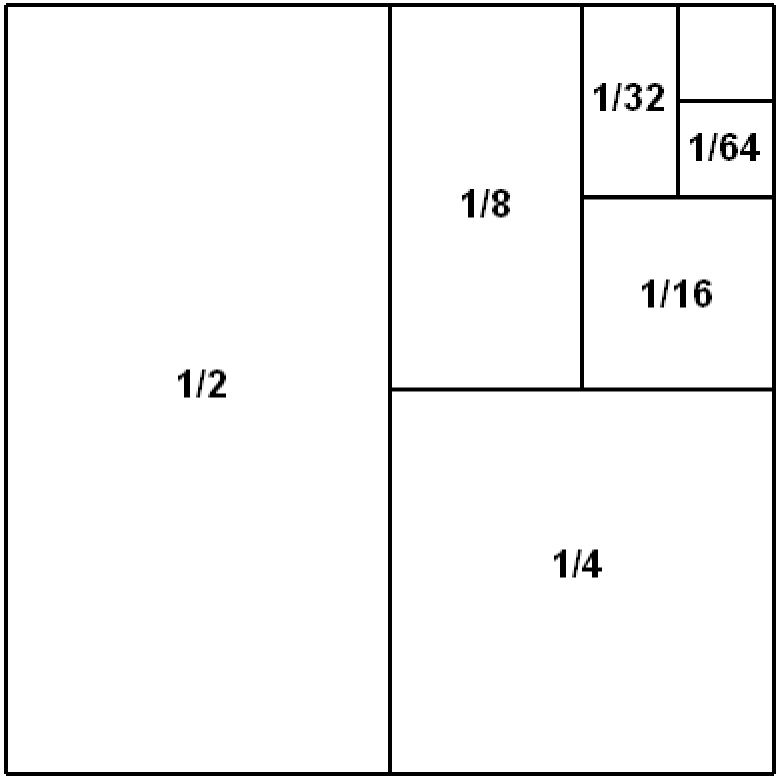
$$U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots$$

  - Discount factor (γ) is a number between 0 and 1 *(e.g., γ=0.95)*
  - Equivalent to an interest rate of *(1/γ) – 1 = 0.0526*

  - We focus only on "**discount rewards**"

Recap: *Geometric series*

- Question: $1/2 + 1/4 + 1/8 + 1/16 + \cdots \;=\; ?$



Discount factor: example

America's Got Talent Winner is Not an Instant Millionaire

Last night, NBC's *America's Got Talent* announced the winner of its sixth season. Landau Eugene Murphy, Jr., a 36 year old car wash detailer from West Virginia, was overcome with emotion as he was told of the $1 million prize and the opportunity to headline a show at Caesar's Palace in Las Vegas.

. . .

But if you read the fine print on the screen at the end of the finale last night, the million dollar prize is actually a 40-year long annuity. In reality, Murphy, whose impressive singing voice resembles that of Frank Sinatra, can expect an annual payout of only $25,000—before taxes, that is.

$25,000 * 40 = $1,000,000

Source: Forbes, September 15, 2011

Discount factor: example

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But if you read the fine print on the screen at the end of the finale last night, the million dollar prize is actually a 40-year long annuity. In reality, Murphy, whose impressive singing voice resembles that of Frank Sinatra, can expect an annual payout of only $25,000—before taxes, that is. Murphy will be offered a lump cash payment in lieu of the annuity, but this will likely be in the $300,000 range (again, before taxes).

$25,000 * 40 = $1,000,000

Source: Forbes, September 15, 2011

Which one is better?

Discount factor: simplified example

- A similar example with fewer payouts (to fit on the slide):

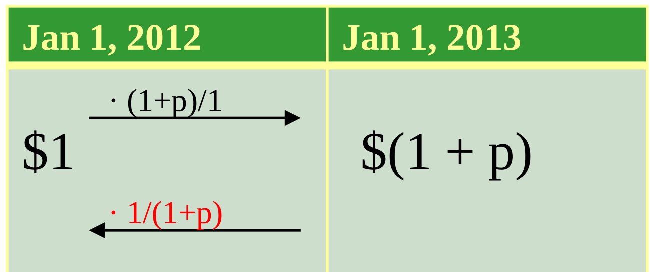
Jan 1, 2012	Jan 1, 2013	Jan 1, 2014	Jan 1, 2015
$25,000	$25,000	$25,000	$25,000

the interest rate is 5% per year

- Equivalent to a lump sum of $93,081
  - **How do we know this?**

Discount factor: simplified example

- If we put $1 into a savings account with **interest rate** p, after one year, we will have $(1 + p) in the account



- The $(1 + p) after one year is equivalent to $1 now

- We call 0 < 1/(1+p) ≤ 1 the **discount factor** γ (gamma).

Discount factor: simplified example

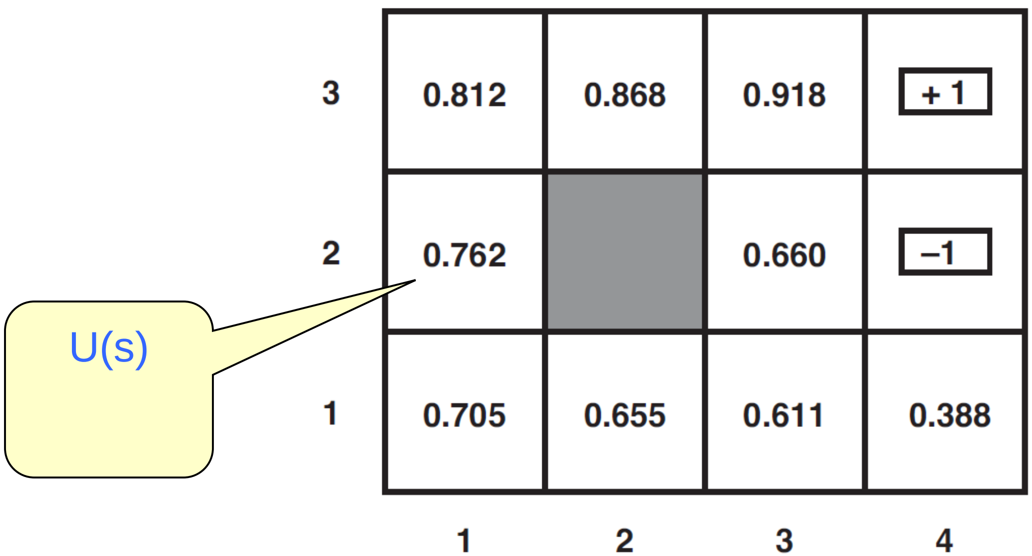
- For an interest rate of 5% (i.e. discount factor of $\gamma \approx 0.952$), what is the lump-sum payoff?

Jan 1, 2012	Jan 1, 2013	Jan 1, 2014	Jan 1, 2015
$25,000	$25,000	$25,000	$25,000

- $(1+ \gamma+ \gamma^2+\gamma^3) * \$25,000 \approx \$$**93,081**.20
  - The amount in 2015
    - $\$93,081.20 * (1.05)^3 = \$107,753$
    - $(1.05^3 + 1.05^2 + 1.05 + 1) * \$25,000 = \$107,753$

Difference between R(s) and U(s)

- ***R(s)*** – the "**short-term**" reward for being in state (s)
- ***U(s)*** – the "**long-term**" **total** reward from (s) onward



$$R(s) = \begin{cases} +1 \\ -0.04 \\ -1 \end{cases}$$

Problem – *need to compute both of them*

- Expected utility of executing policy (*π*) starting in state (*s*)

$$\boxed{U^{\pi}(s) = E\left[\sum_{t=0}^{\infty} \gamma^{t} R(S_t)\right]}$$

- Out of all policies (*π*) that the agent could choose, the one with the highest utilities is

$$\boxed{\pi_s^* = \underset{\pi}{\operatorname{argmax}}\, U^{\pi}(s)}$$

Do we have to enumerate all policies?

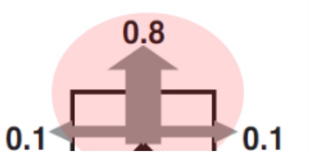
Outline of today's lecture

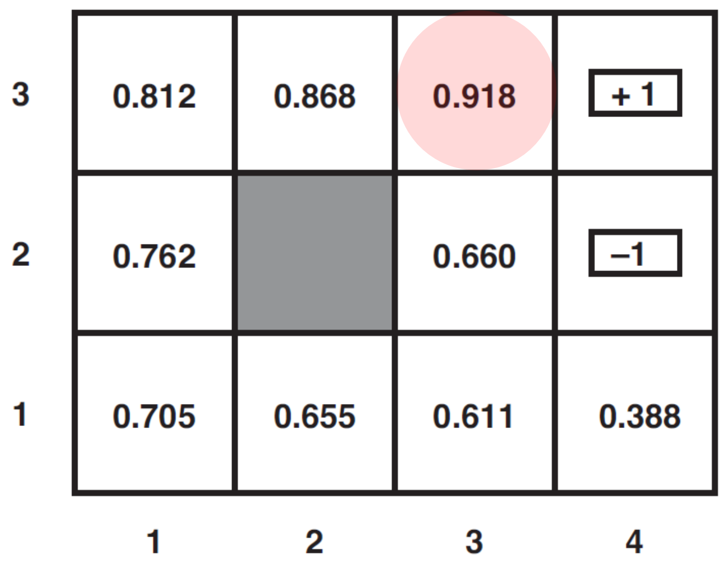
- Sequential Decision (MDP)
  - A sequence of decisions (versus *one-shot* decision)
- **Value Iteration**
  - Algorithm for computing optimal policy for MDP
- Policy Iteration
  - An alternative algorithm

Given U(s), compute optimal policy π*(s)

- Pick the action with the maximal expected utility (MEU)

$$\pi^*(s) = \operatorname*{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$$





Up:

Down:

Left:

Right:

Given U(s), compute optimal policy π*(s)

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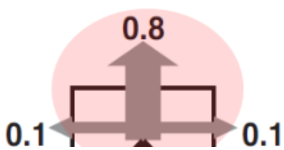
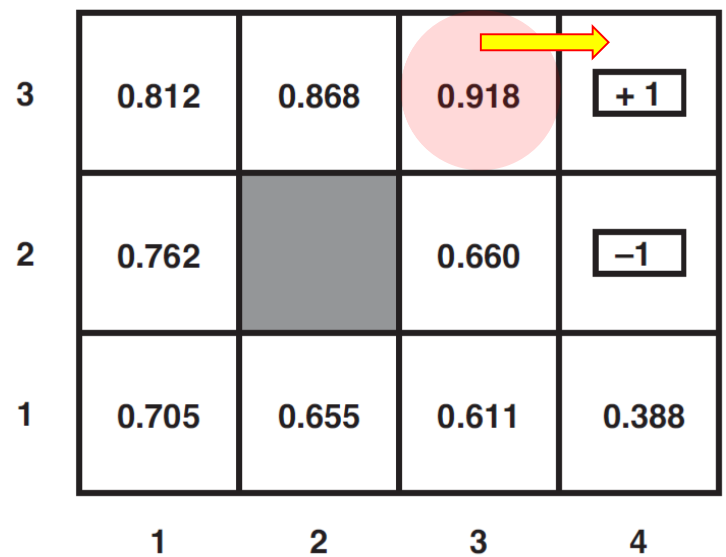
$$\pi^*(s) = \underset{a \in A(s)}{\operatorname{argmax}} \sum_{s'} P(s' \mid s, a) U(s')$$





Up:

$0.8*0.918 + 0.1*0.868 + 0.1*1 = 0.9212$

Down:

$0.8*0.660 + 0.1*0.868 + 0.1*1 = 0.7148$

Left:

$0.8*0.868 + 0.1*0.660 + 0.1*0.918 = 0.8522$

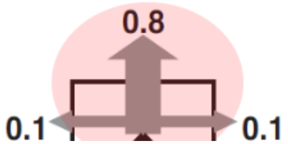
Right:

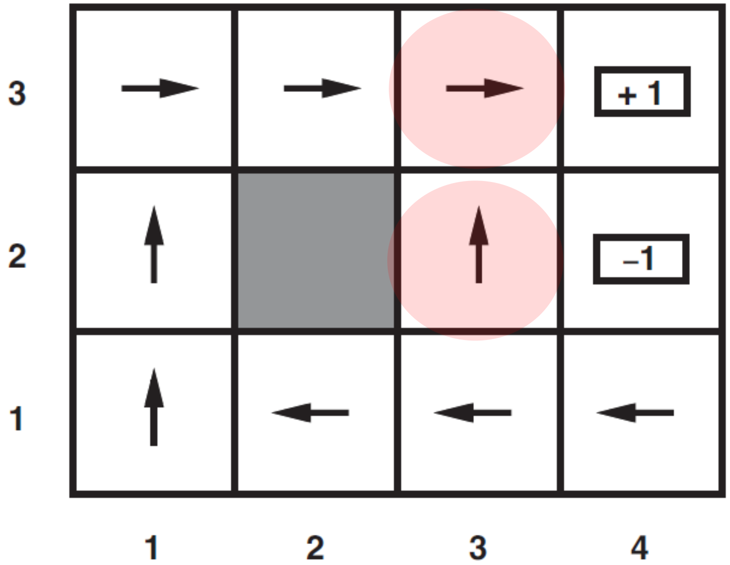
$0.8*1 + 0.1*0.918 + 0.1*0.660 = \boxed{0.9578}$

Given optimal policy π*(s), compute U(s)

- State utility is maximized when following the optimal policy

$$\pi^*_s = \operatorname*{argmax}_{\pi} U^{\pi}(s)$$





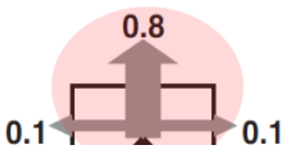
(3,3) : Right

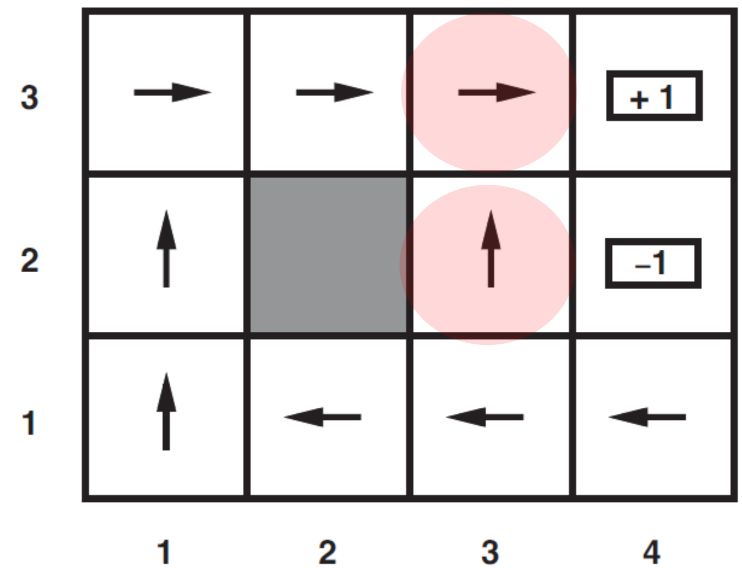
(3,2) : Up

Given optimal policy π*(s), compute U(s)

- State utility is maximized when following the optimal policy

$$\pi_s^* = \operatorname*{argmax}_{\pi} U^{\pi}(s)$$





(3,3) : Right

$0.8*1 + 0.1*U(S_{3,3}) + 0.1*U(S_{3,2}) - 0.04 = U(S_{3,3})$

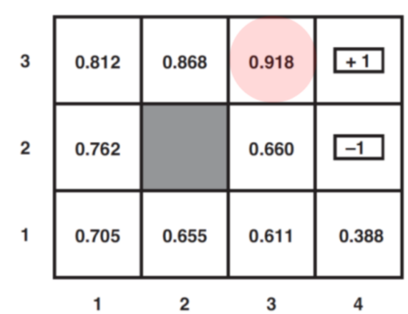
(3,2) : Up

$0.8*U(S_{3,3}) + 0.1*U(S_{3,2}) + 0.1*(-1) - 0.04 = U(S_{3,2})$

$0.76 + 0.1*U(S_{3,2}) = 0.9*U(S_{3,3})$

$0.8*U(S_{3,3}) - 0.14 = 0.9*U(S_{3,2})$

$U(S_{3,3}) = 0.9179$



A “chicken and egg” situation

- Which comes first?
  - Given U(s), we can easily compute the optimal policy π*(s)
  - Given the optimal policy π*(s), we can easily compute U(s)

3	→	→	→	+1
2	↑		↑	–1
1	↑	←	←	←
	1	2	3	4

3	0.812	0.868	0.918	+1
2	0.762		0.660	–1
1	0.705	0.655	0.611	0.388
	1	2	3	4

It’s the maximum number among all policies - enumerate all policies to get it?

Basic idea for solving MDP

- (1) Calculate the utility of each state
- (2) Use state utilities to select optimal action in each state

Richard Bellman
USC professor
(from 1965 to 1984)

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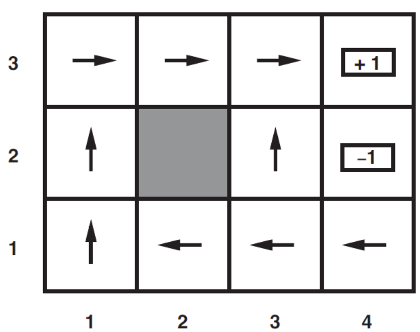
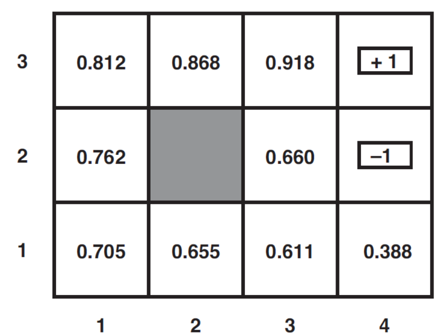


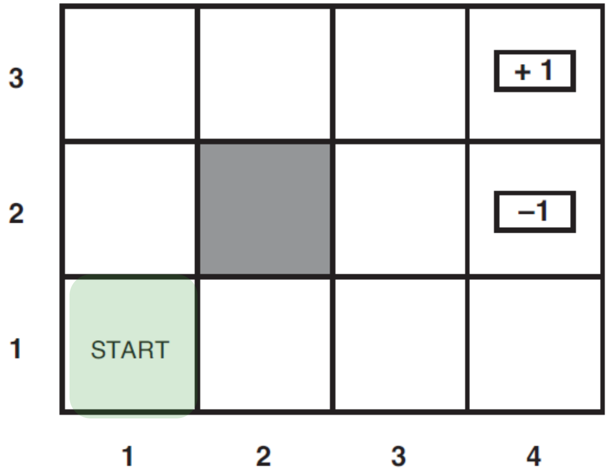


Bellman equation (1957)

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$$

Example:

$$\begin{aligned} U(1,1) = -0.04 + \gamma \max[\ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), && (Up) \\ & 0.9U(1,1) + 0.1U(1,2), && (Left) \\ & 0.9U(1,1) + 0.1U(2,1), && (Down) \\ & 0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1)\ ]. && (Right) \end{aligned}$$



Similarly, write down U(1,2), U(1,3), U(1,4),
U(2,1), U(2,2), U(2,3), U(2,4),
U(3,1), U(3,2), U(3,3), U(3,4)

Solutions to these equations are unique!

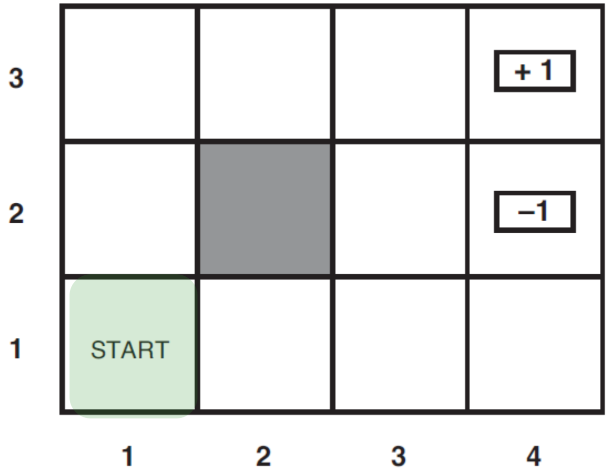
Bellman equation (1957)

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$$

You can't directly solve these equations due to the non-linear (**max**) operation

Example:

$$\begin{aligned} U(1,1) = -0.04 + \gamma \max[\ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), && (Up) \\ & 0.9U(1,1) + 0.1U(1,2), && (Left) \\ & 0.9U(1,1) + 0.1U(2,1), && (Down) \\ & 0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1)\ ]. && (Right) \end{aligned}$$



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Solutions to these equations are unique!

Iterative approach

- Start with **arbitrary initial values** for U(s), compute right-hand side of the equation, and plug it into left-hand side
  - Updating U(s) based on U(s'), the utilities of neighbor states (s')

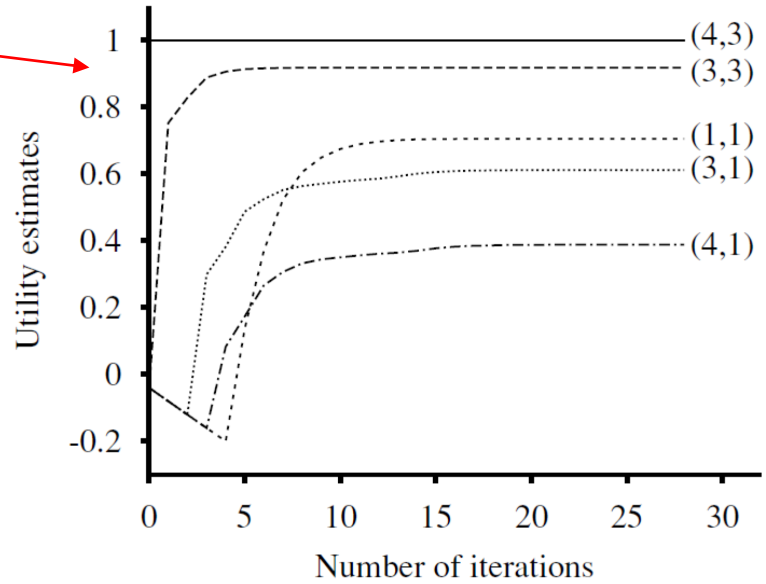
$$\boxed{U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U_i(s')}$$

Iterative approach: *how does it work in practice?*

- Start with **arbitrary initial values** for U(s), compute right-hand side of the equation, and plug it into left-hand side
  - Updating U(s) based on U(s'), the utilities of neighbor states (s')

$$\boxed{U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U_i(s')}$$

U($S_{3,3}$) = 0.9179



Value iteration algorithm

function VALUE-ITERATION(mdp, ϵ) **returns** a utility function
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
    rewards $R(s)$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', vectors of utilities for states in S, initially zero
    δ, the maximum change in the utility of any state in an iteration

  repeat
    $U \leftarrow U'$; $\delta \leftarrow 0$
    for each state s **in** S **do**
      $U'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\, U[s']$
      if $|U'[s] - U[s]| > \delta$ **then** $\delta \leftarrow |U'[s] - U[s]|$
  until $\delta < \epsilon(1-\gamma)/\gamma$
  return U

Value iteration algorithm

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Wikipedia: geometric progression

$$\sum_{k=m}^{\infty} ar^k = \frac{ar^m}{1-r}$$

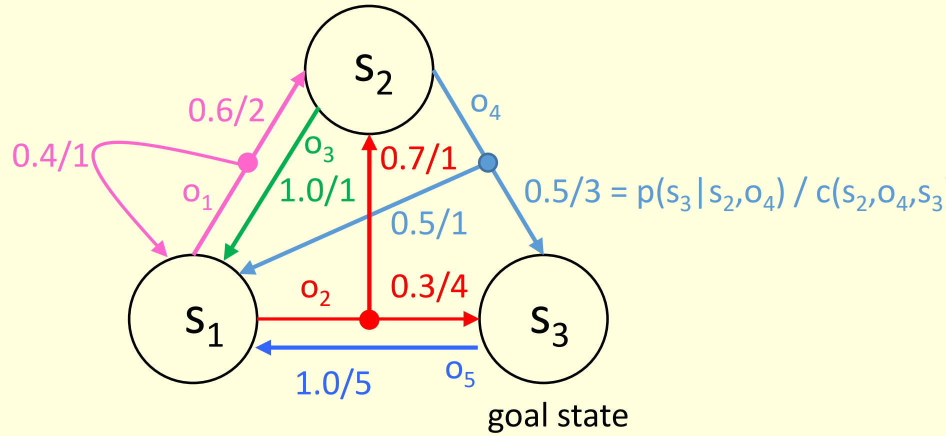
MDP solving example

```
#include <stdio.h>
#define min(x,y) ((x)<(y))?(x):(y);

main()
{
  float gamma=0.95;
  float s1=0.0, s2=0.0, s3=0.0;
  float o1,o2,o3,o4,o5;
  char *ostar1, *ostar2, *ostar3;
  int i;

  for (i=1; i<10000; ++i)
    {
      o1 = 0.4*(1.0+gamma*s1)+0.6*(2.0+gamma*s2);
      o2 = 0.7*(1.0+gamma*s2)+0.3*(4.0+gamma*s3);
      o3 = 1.0*(1.0+gamma*s1);
      o4 = 0.5*(1.0+gamma*s1)+0.5*(3.0+gamma*s3);
      o5 = 1.0*(5.0+gamma*s1);
      if (o1<o2)
        {
          s1 = o1;
          ostar1 = "o1";
        }
      else
        {
          s1 = o2;
          ostar1 = "o2";
        }
      if (o3<o4)
        {
          s2 = o3;
          ostar2 = "o3";
        }
      else
        {
          s2 = o4;
          ostar2 = "o4";
        }
      s3 = o5;
      ostar3 = "o5";
      printf("iteration %5d: s1 (execute %s) %5.2f (o1 %5.2f, o2 %5.2f), s2 (execute %s) %5.2f (o3 %5.2f, o4 %5.2f), s3 (execute %s) %5.2f\n",
             i, ostar1, s1, o1, o2, ostar2, s2, o3, o4, ostar3, s3);
    }
}
```

γ=0.95



MDP solving result

γ=0.95

$c_{s_1,iteration}$ $c_{s_2,iteration}$ $c_{s_3,iteration}$

```
iteration    1: s1 (execute o1)  1.60 (o1  1.60, o2  1.90), s2 (execute o3)  1.00 (o3  1.00, o4  2.00), s3 (execute o5)  5.00
iteration    2: s1 (execute o1)  2.78 (o1  2.78, o2  3.99), s2 (execute o3)  2.52 (o3  2.52, o4  5.14), s3 (execute o5)  6.52
iteration    3: s1 (execute o1)  4.09 (o1  4.09, o2  5.43), s2 (execute o3)  3.64 (o3  3.64, o4  6.42), s3 (execute o5)  7.64
iteration    4: s1 (execute o1)  5.23 (o1  5.23, o2  6.50), s2 (execute o3)  4.89 (o3  4.89, o4  7.57), s3 (execute o5)  8.89
iteration    5: s1 (execute o1)  6.37 (o1  6.37, o2  7.68), s2 (execute o3)  5.97 (o3  5.97, o4  8.71), s3 (execute o5)  9.97
iteration    6: s1 (execute o1)  7.42 (o1  7.42, o2  8.71), s2 (execute o3)  7.05 (o3  7.05, o4  9.76), s3 (execute o5) 11.05
iteration    7: s1 (execute o1)  8.44 (o1  8.44, o2  9.74), s2 (execute o3)  8.05 (o3  8.05, o4 10.78), s3 (execute o5) 12.05
iteration    8: s1 (execute o1)  9.40 (o1  9.40, o2 10.69), s2 (execute o3)  9.02 (o3  9.02, o4 11.73), s3 (execute o5) 13.02
iteration    9: s1 (execute o1) 10.31 (o1 10.31, o2 11.61), s2 (execute o3)  9.93 (o3  9.93, o4 12.65), s3 (execute o5) 13.93
iteration   10: s1 (execute o1) 11.18 (o1 11.18, o2 12.47), s2 (execute o3) 10.80 (o3 10.80, o4 13.51), s3 (execute o5) 14.80
...
iteration 9995: s1 (execute o1) 27.64 (o1 27.64, o2 28.94), s2 (execute o3) 27.26 (o3 27.26, o4 29.98), s3 (execute o5) 31.26
iteration 9996: s1 (execute o1) 27.64 (o1 27.64, o2 28.94), s2 (execute o3) 27.26 (o3 27.26, o4 29.98), s3 (execute o5) 31.26
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iteration 9998: s1 (execute o1) 27.64 (o1 27.64, o2 28.94), s2 (execute o3) 27.26 (o3 27.26, o4 29.98), s3 (execute o5) 31.26
iteration 9999: s1 (execute o1) 27.64 (o1 27.64, o2 28.94), s2 (execute o3) 27.26 (o3 27.26, o4 29.98), s3 (execute o5) 31.26
```
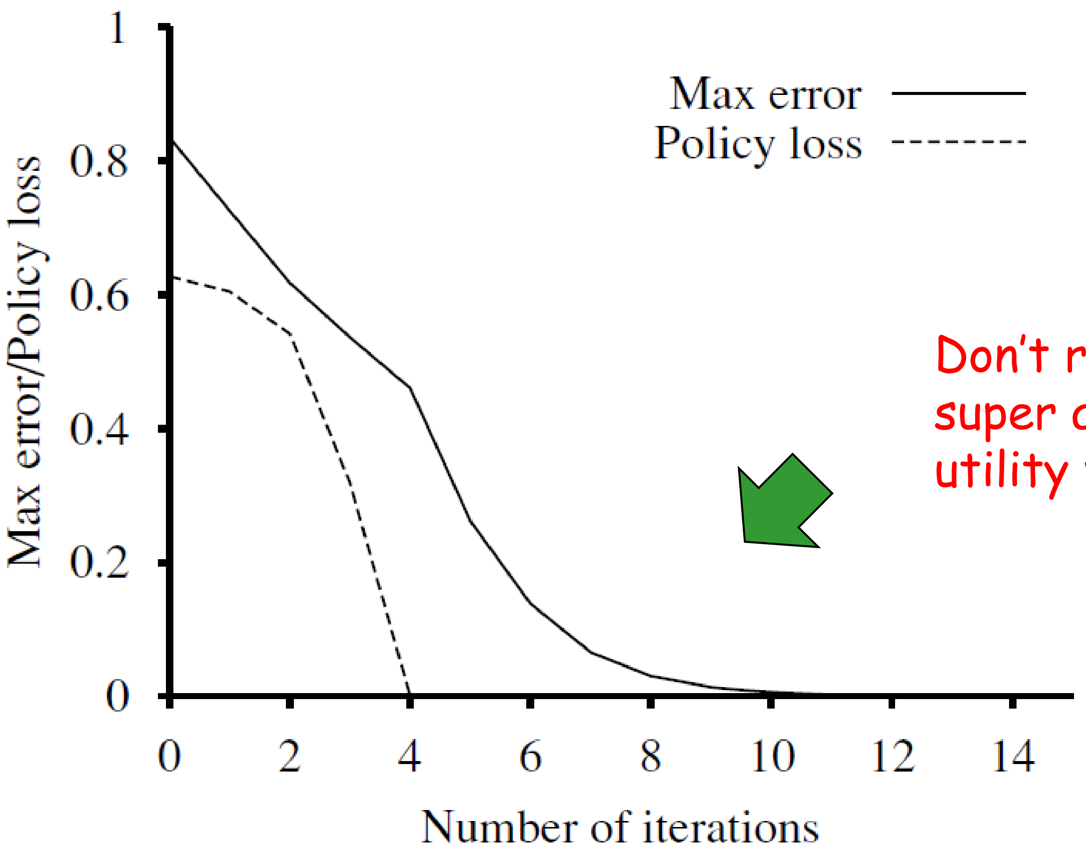

- If one can stop after executing a handful of iterations, then
  - execute o_1 in s_1, o_3 in s_2 and o_5 in s_3 (see iteration 9999)
  - …
  - execute o_1 in s_1, o_3 in s_2 and o_5 in s_3 (see iteration 2)
  - execute o_1 in s_1, o_3 in s_2 and o_5 in s_3 (see iteration 1)

Policy loss: *due to $U_i(s) - U(s)$*

- Difference between
  - The MEU policy based on $U_i(s)$ of the *i-th* value iteration
  - The MEU policy based on the actual (perfect) utility $U(s)$



Don't really need super accurate utility values…

Outline of today's lecture

- Sequential Decision (MDP)
  - A sequence of decisions (versus *one-shot* decision)
- Value Iteration
  - Algorithm for computing optimal policy for MDP
- **Policy Iteration**
  - An alternative algorithm

Alternative approach to solving MDP

- (1) **Policy evaluation**: given a policy ***π(s)***, compute ***U(s)***
- (2) **Policy improvement**: compute a new policy

$$\boxed{\pi^*(s) = \operatorname*{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')}$$

Alternative approach to solving MDP

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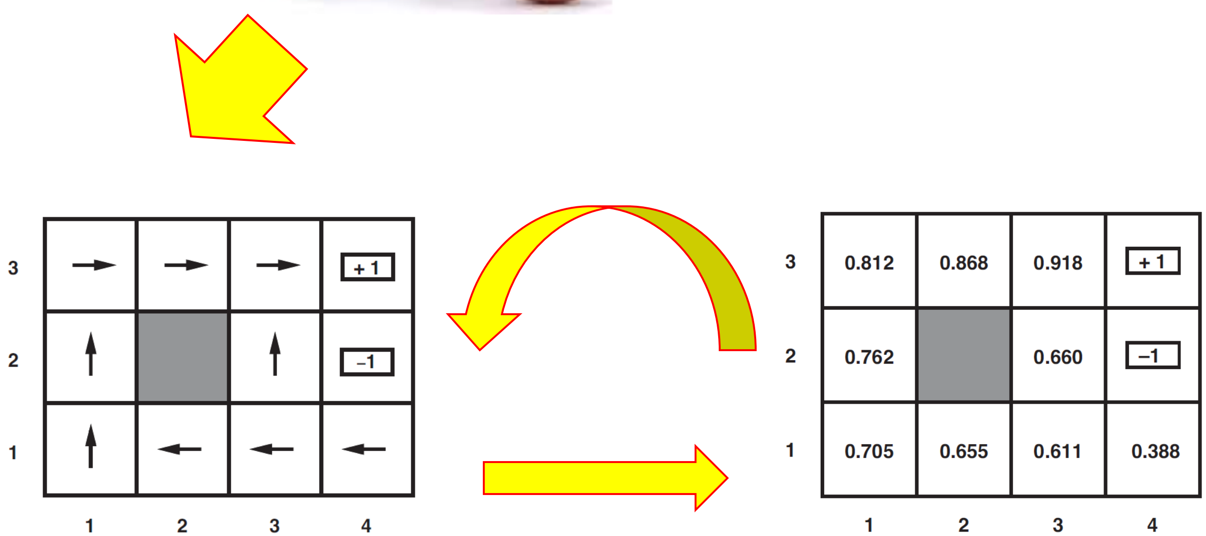


Policy evaluation

- Given a policy *π(s)*, compute *U(s)*

$$\boxed{U_i(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s')}$$

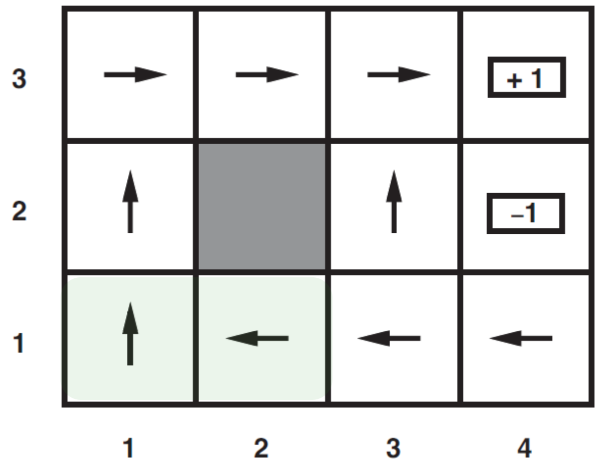
This is a set of *linear* equations, which is polynomial time solvable (by LP solver); in contrast, the Bellman equation $U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$ is nonlinear.

Policy evaluation: *example*

- Given a policy ***π(s)***, compute ***U(s)***

$$U_i(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s')$$

This is a set of ***linear*** equations, which is polynomial time solvable (by LP solver); in contrast, the Bellman equation $U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s')$ is nonlinear.



$$\begin{aligned} U_i(1,1) &= -0.04 + 0.8U_i(1,2) + 0.1U_i(1,1) + 0.1U_i(2,1)\ , \\ U_i(1,2) &= -0.04 + 0.8U_i(1,3) + 0.2U_i(1,2)\ , \\ &\vdots \end{aligned}$$

Policy improvement: *Given U(s), compute policy π*(s)*

- Pick the action with the maximal expected utility (MEU)

$$\pi^*(s) = \underset{a \in A(s)}{\operatorname{argmax}} \sum_{s'} P(s' \mid s, a) U(s')$$

- We already talked about this!

Policy improvement: *Given U(s), compute policy π*(s)*

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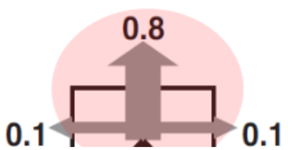
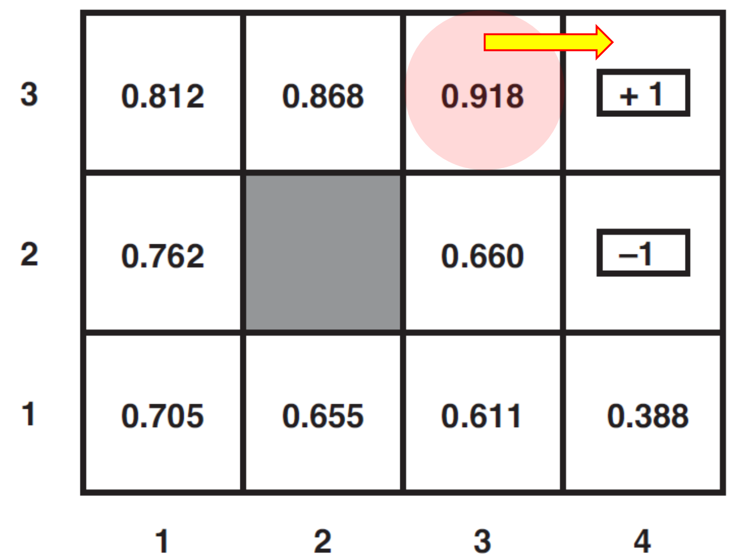
$$\pi^*(s) = \underset{a \in A(s)}{\operatorname{argmax}} \sum_{s'} P(s' \mid s, a) U(s')$$





Up:

$0.8*0.918 + 0.1*0.868 + 0.1*1 = 0.9212$

Down:

$0.8*0.660 + 0.1*0.868 + 0.1*1 = 0.7148$

Left:

$0.8*0.868 + 0.1*0.660 + 0.1*0.918 = 0.8522$

Right:

$0.8*1 + 0.1*0.918 + 0.1*0.660 = \boxed{0.9578}$

Overall algorithm

function POLICY-ITERATION(mdp) **returns** a policy
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$
  local variables: U, a vector of utilities for states in S, initially zero
    π, a policy vector indexed by state, initially random

  repeat
    $U \leftarrow$ POLICY-EVALUATION(π, U, mdp)
    $unchanged? \leftarrow$ true
    for each state s **in** S **do**
      if $\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\ U[s'] > \sum_{s'} P(s' \mid s, \pi[s])\ U[s']$ **then do**
        $\pi[s] \leftarrow \operatorname{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\ U[s']$
        $unchanged? \leftarrow$ false
  until $unchanged?$
  return π

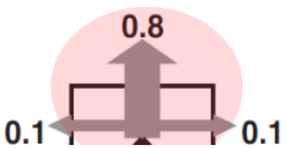
Summary of today's lecture

- Sequential Decision (MDP)
  - A sequence of decisions (versus *one-shot* decision)
- Value Iteration
  - Algorithm for computing optimal policy for MDP
- Policy Iteration
  - An alternative algorithm

Quiz 9

- Pick the action with the maximal expected utility (MEU)

$$\pi^*(s) = \underset{a \in A(s)}{\operatorname{argmax}} \sum_{s'} P(s' \mid s, a) U(s')$$



Up:

0.8*(???)+ 0.1*{???) + 0.1*(???) = ???

Down:

0.8*(???)+ 0.1*{???) + 0.1*(???) = ???

Left:

0.8*(???)+ 0.1*{???) + 0.1*(???) = ???

Right:

0.8*(???)+ 0.1*{???) + 0.1*(???) = ???

	1	2	3	4
3	0.812	0.868	0.918	+ 1
2	0.762		0.660	–1
1	0.705	0.655	0.611	0.388