Hypothesis Testing

Hypothesis Testing
Motivation

1. Is the chance of getting a cold different when subjects take vitamin C than when they take placebo? (Pauling 1971 data).

2. Suppose that 6 out of 15 students in a grade-school class develop influenza, whereas 20% of grade-school children nationwide develop influenza. Is there evidence of an excessive number of cases in the class?

Hypothesis Testing
Motivation

3. In a study of 25 hypertensive people we find a mean serum-cholesterol level of 220 mg/ml. In the population the mean serum cholesterol is 211 mg/ml with standard deviation of 46 mg/ml.

- Is the data consistent with that model?
- What if $\overline{X}$ = 230 mg/ml?
- What if $\overline{X}$ = 250 mg/ml?
- What if the sample was of n=100 instead of 25?

Hypothesis Testing

Define:

μ = population mean serum cholesterol for male hypertensives

Hypothesis:

1. Null Hypothesis: Generally, the hypothesis that the unknown parameter equals a fixed value.

   H_0: $\mu = 211$ mg/ml

2. Alternative Hypothesis: contradicts the null hypothesis.

   H_A: $\mu \neq 211$ mg/ml

Hypothesis Testing

Decision / Action:

We assume that either H_0 or H_A is true. Based on the data we will choose one of these hypotheses.

	H_0 Correct	H_A Correct
Decide H_0	$1-\alpha$	β
Decide H_A	α	$1-\beta$

α = **"size"**
$1 - \beta$ = "**power**"

Hypothesis Testing

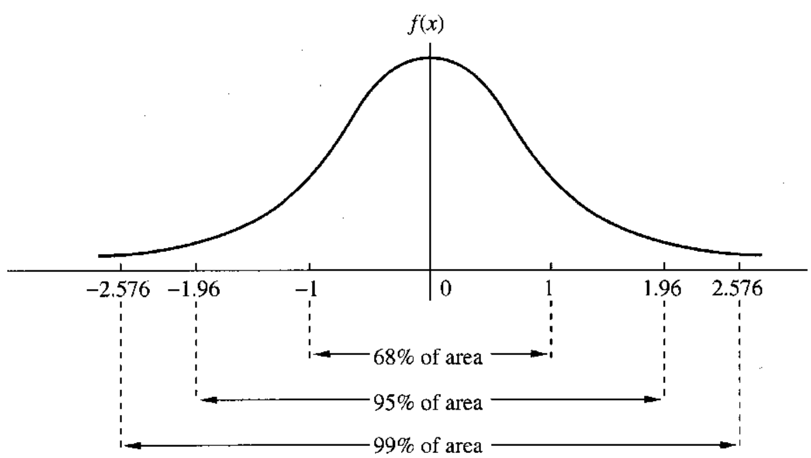
Let's fix α, for example, $\alpha = 0.05$.

$$0.05 = \alpha = P[\text{ choose } H_A \mid H_0 \text{ true }]$$

$$\alpha = P[\text{ reject } H_0 \mid H_0 \text{ true }]$$

Q: How to construct a procedure that makes this error with only 0.05 probability?

A: Suppose we assume H_0 is true and suppose that, using that assumption, the data should give us a standard normal, Z.



If $\mu = 0$ then $|Z|$ is rarely "large". A "large" $|Z|$ would make me question whether $\mu = 0$.

Hypothesis Testing

Therefore, **we reject** H_0 **if** $|Z| > 1.96$.

$\alpha = P[\text{reject } H_0 \mid H_0 \text{ true}] = 0.05$

Then if we do find a large value of $|Z|$ we can claim that:

- **Either** H_0 is true and something unusual happened (with probability α)…
- **or**, H_0 is not true.

Given α and H_0 we can construct a test of H_0 with a specified significance level. But remember, we start by assuming that H_0 is true - we haven't proved it is true. Therefore, we usually say

- $|Z| > 1.96$ then we **reject** H_0.
- $|Z| < 1.96$ then we **fail to reject** H_0.

Hypothesis Testing

Cholesterol Example:

Let μ be the mean serum cholesterol level for male hypertensives. We observe

$$\overline{X} = 220 \text{ mg/ml}$$

Also, we are told that for the general population...

μ_0 = mean serum cholesterol level for males = 211 mg/ml

σ = std. dev. of serum cholesterol for males = 46 mg/ml

NULL HYPOTHESIS: mean for male hypertensives is the same as the general male population.

ALTERNATIVE HYPOTHESIS: mean for male hypertensives is different than the mean for the general male population.

$$H_0 : \mu = \mu_0 = 211 \text{ mg/ml}$$

$$H_A : \mu \neq \mu_0 \;\; (\mu \neq 211 \text{ mg/ml})$$

Hypothesis Testing

Cholesterol Example:

Test H_0 with significance level α.

Under H_0 we know:

$$\frac{\overline{X}-\mu_o}{\sigma/\sqrt{n}} \sim N(0,1)$$

Therefore,

•**Reject** H_0 if $\left|\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\right| > 1.96$ gives an $\alpha = 0.05$ test.

•**Reject** H_0 if

$$\overline{X} > \mu_0 + 1.96\frac{\sigma}{\sqrt{n}} \quad \text{or}$$

$$\overline{X} < \mu_0 - 1.96\frac{\sigma}{\sqrt{n}}$$

Hypothesis Testing

Cholesterol Example:

TEST: **Reject** H_0 if

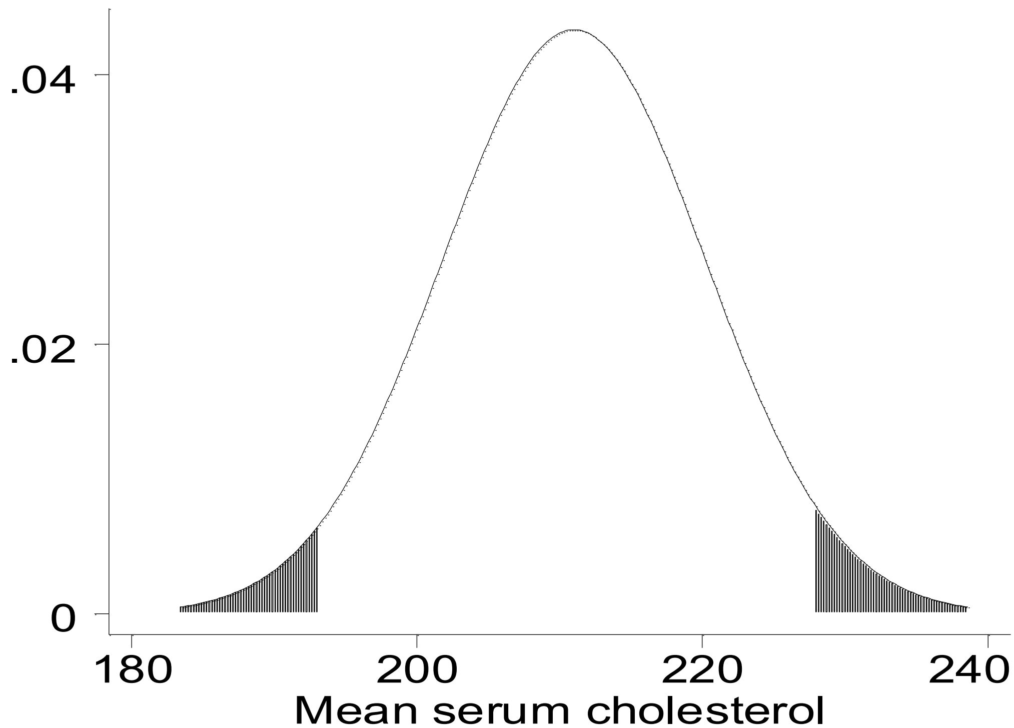
$$\overline{X} > 211 + 1.96\frac{46}{\sqrt{25}} \text{ or}$$
$$\overline{X} < 211 - 1.96\frac{46}{\sqrt{25}}$$

$$\overline{X} > 228.03 \text{ or}$$
$$\overline{X} < 192.97$$

In terms of Z ...

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$$

Reject H_0 if $Z<-1.96$ or $Z> 1.96$


.04
.02
0
180
200
220
240
Mean serum cholesterol

Hypothesis Testing

p-value:

- smallest possible α for which the observed sample would still reject H_0.
- probability of obtaining a result as extreme or more extreme than the actual sample (give H_0 true).

Hypothesis Testing

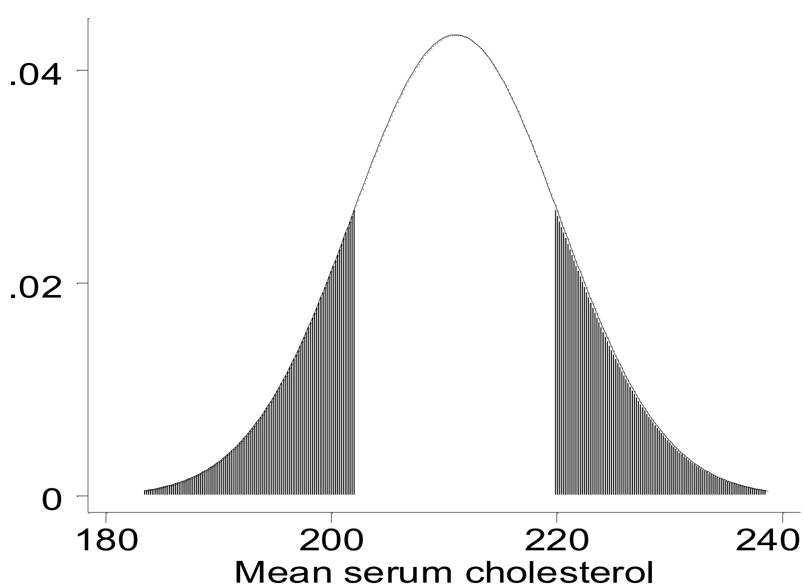
p-value: Cholesterol Example

$\overline{X}$ = 220 mg/ml n = 25 σ = 46 mg/ml

$$H_0 : \mu = 211 \text{ mg/ml}$$
$$H_A : \mu \neq 211 \text{ mg/ml}$$

p-value is given by:

$$2 * P[\overline{X} > 220] = .33$$



Determination of Statistical Significance for Results from Hypothesis Tests

Either of the following methods can be used to establish whether results from hypothesis tests are statistically significant:

(1) ~~The test statistic Z can be computed and compared with the critical value~~ $\cancel{Q_Z^{(1-\alpha/2)}}$ ~~at an~~ $\cancel{\alpha}$ ~~level of .05. Specifically, if~~ $\cancel{H_0: \mu = \mu_0}$ ~~versus~~ $\cancel{H_1: \mu \neq \mu_0}$ ~~are being tested and~~ $\cancel{|Z| > 1.96}$~~, then~~ $\cancel{H_0}$ ~~is rejected and the results are declared *statistically significant* (i.e., p < .05). Otherwise,~~ $\cancel{H_0}$ ~~is accepted and the results are declared *not statistically significant* (i.e., p ≥ .05). We refer to this approach as the **critical-value method**~~.

(2) The exact p-value can be computed, and if $p < .05$, then H_0 is rejected and the results are declared *statistically significant* . Otherwise, if $p \geq .05$ then H_0 is accepted and the results are declared *not statistically significant* . We will refer to this approach as the **p-value method** .

Guidelines for Judging the Significance of p-value

If $.05 \leq p < .10$, than the results are *marginally significant*.

If $.01 \leq p < .05$, then the results are *significant*.

If $.001 \leq p < .01$, then the results are *highly significant*.

If $p < .001$, then the results are *very highly significant*.

If $p > .1$, then the results are considered *not statistically significant* (sometimes denoted by NS).

Significance is not everything!

Hypothesis Testing and Confidence Intervals

Hypothesis Test: Fail to reject H_0 if

$$\overline{X} < \mu_0 + Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

$$\text{and} \quad \overline{X} > \mu_0 - Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

Confidence Interval: Plausible values for μ are given by

$$\mu < \overline{X} + Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

$$\text{and} \quad \mu > \overline{X} - Q_Z^{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

Hypothesis Testing
"how many sides?"

Depending on the alternative hypothesis a test may have a **one-sided alternative** or a **two-sided alternative**. Consider

$$H_0 \; : \; \mu = \mu_0$$

We can envision (at least) three possible alternatives

$$H_A \; : \; \mu \neq \mu_0 \qquad (1)$$

$$H_A \; : \; \mu < \mu_0 \qquad (2)$$

$$H_A \; : \; \mu > \mu_0 \qquad (3)$$

(1) is an example of a "two-sided alternative"

(2) and (3) are examples of "one-sided alternatives"

The distinction impacts

- Rejection regions
- p-value calculation

Hypothesis Testing
"how many sides?"

Cholesterol Example: Instead of the two-sided alternative considered earlier we may have only been interested in the alternative that hypertensives had a higher serum cholesterol.

$$H_0 : \mu = 211$$

$$H_A : \mu > 211$$

When do one vs two sided? If a small or large value would change the direction of your research?

Given this, an $\alpha = 0.05$ test would reject when

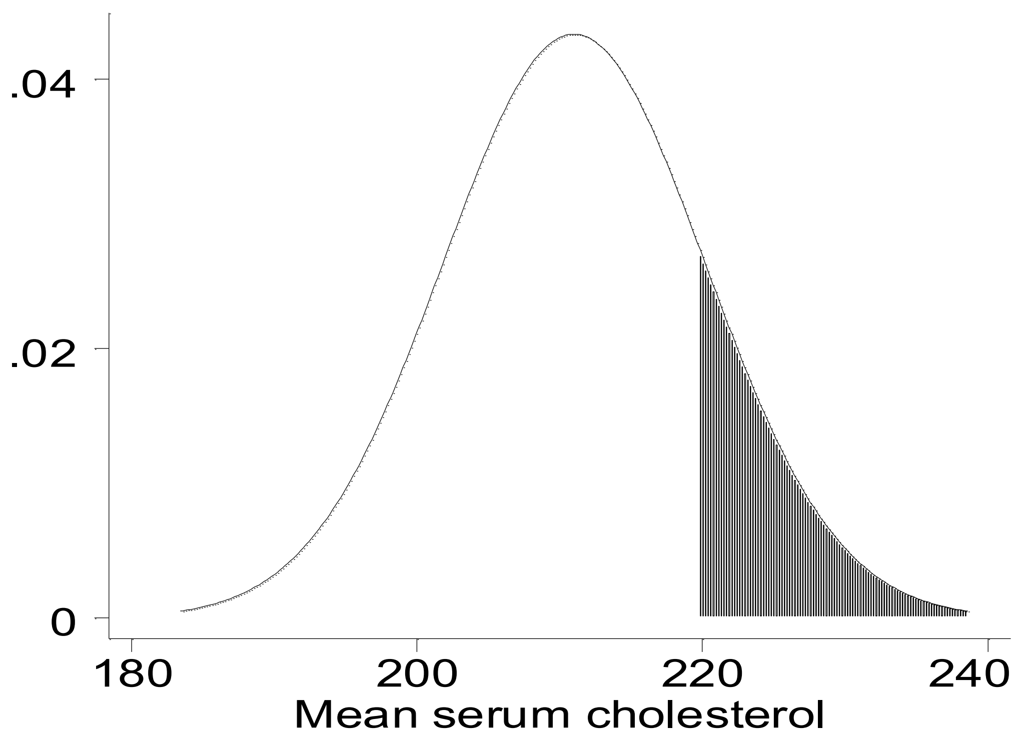
$$\frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} = Z > Q_Z^{(1-0.05)} = 1.65$$

We put all the probability on "one-side".

The p-value would be half of the previous,

$$\text{p-value} = P[\,\overline{X} > 220]$$

$$= .163$$


.04
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Mean serum cholesterol

Hypothesis Testing

Through this worked example we have seen the basic components to the statistical test of a scientific hypothesis.

Summary

1. Identify H_0 and H_A
2. Identify a test statistic
3. Determine a significance level, $\alpha = 0.05$, $\alpha = 0.01$
4. Critical value determines rejection / acceptance region
5. p-value
6. Interpret the result

Other things to note

- General idea (compare p-value of test statistic to cutoff) is same for all hypothesis tests
- - The test stat and distribution changes for different tests (regression parameters, proportions, etc.)
- - Bayesians take a different approach to hypothesis testing