Sampling Distributions

The most important distinction in statistics

sample
vs
population

When analysing data (or reading the literature), think about whether you want to discuss the sample that you observed or want to make statements that are more generally true

Statistics is the only field that gives us the correct framework to generalize from our sample to the population

The most important distinction in statistics

Example: T cell counts from 40 women with triple negative breast cancer were observed. What can we do with this information?

Option 1: Discuss the 40 women. What was the mean T cell count? What was its variation?

Option 2: Generalise the information about the 40 women to make statements about all women with triple negative breast cancer

These are 2 different approaches to using the same information

Language for making these distinctions

Population

- Size N (usually ∞)
- Mean = μ

$$\mu = \sum p_j X_j \quad or \quad \int \ldots$$

- Variance = σ^2

$$\sigma^2 = \sum p_j \left(X_j - \mu \right)^2 \quad or \quad \int \ldots$$

Sample

- Size n
- Sample Mean = $\overline{X}$

$$\overline{X} = \frac{1}{n} \sum_{j=1}^{n} X_j$$

- Sample variance = s^2

$$s^2 = \frac{1}{n-1} \sum_{j=1}^{n} \left(X_j - \overline{X} \right)^2$$

Generalizing the sample to the population

Issue: We can calculate the sample mean and sample variance from our data, but the true mean and true variance are generally unknown

Fortunately, statisticians have learnt some things about how to recover* the true mean and true variance based only on sample means and sample variances

* with high probability

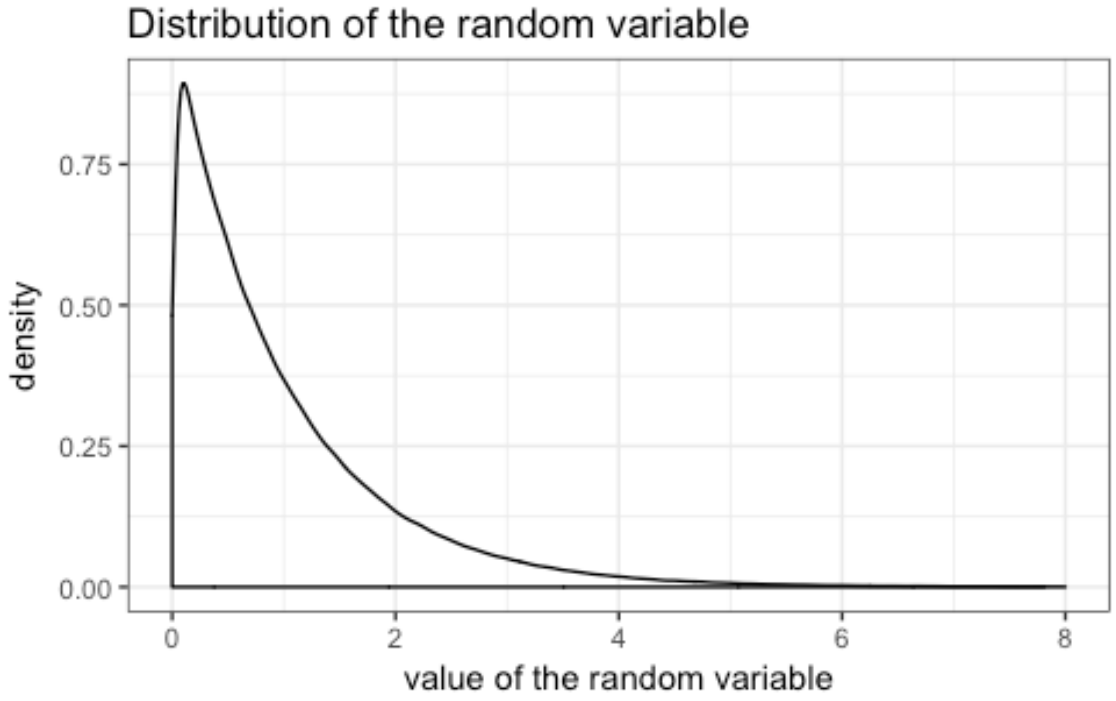
How do sample means behave?

Suppose we observe data $X_1, X_2, ..., X_n$. We can calculate $\bar{X}$ exactly, but what can we say about μ?

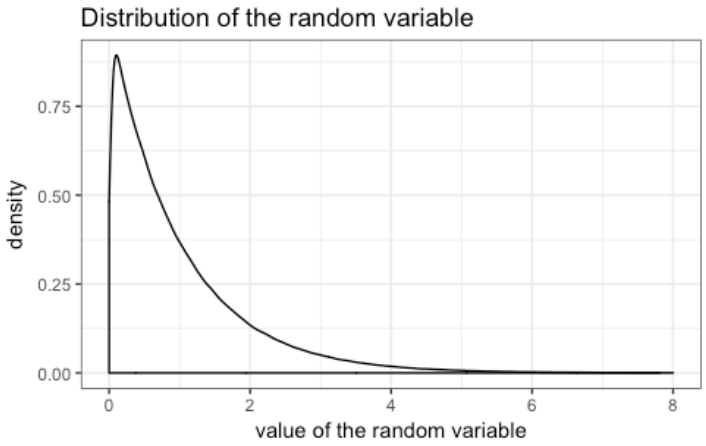
Idea: μ is probably close to $\bar{X}$

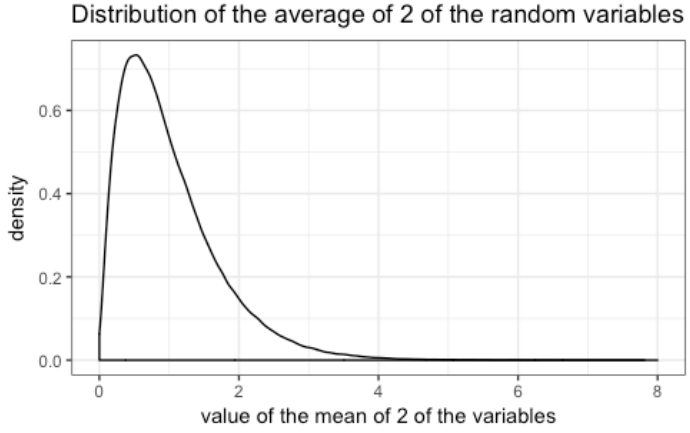
Goal: Make this more rigorous

Sums of Normal Random Variables



Sums of Normal Random Variables





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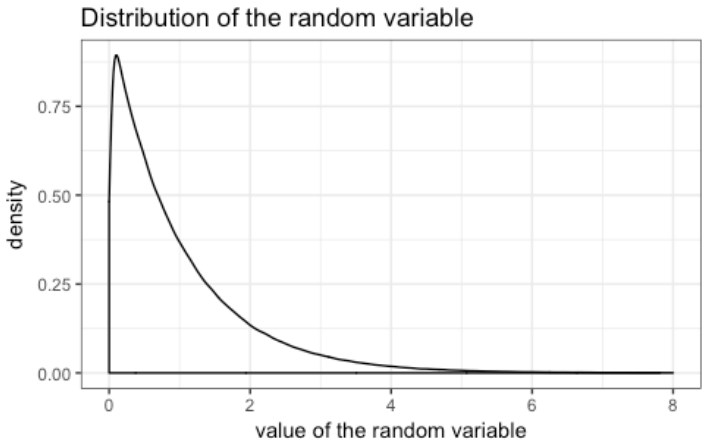

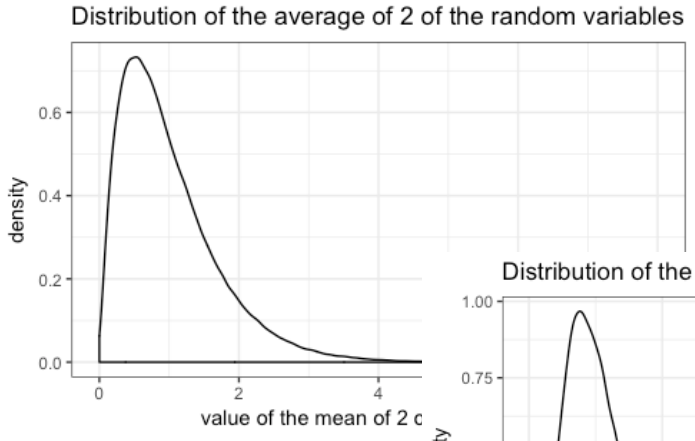

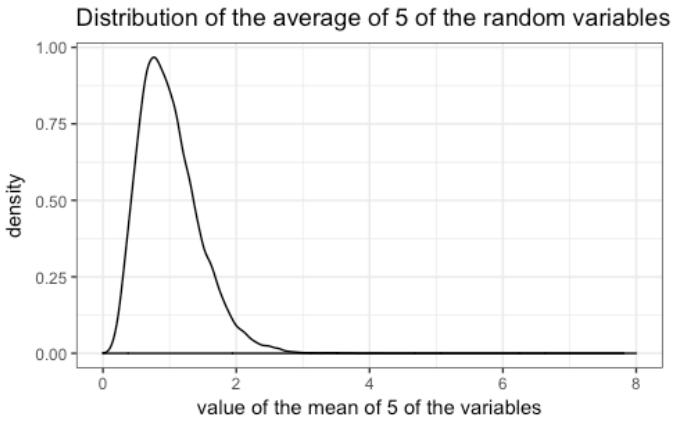

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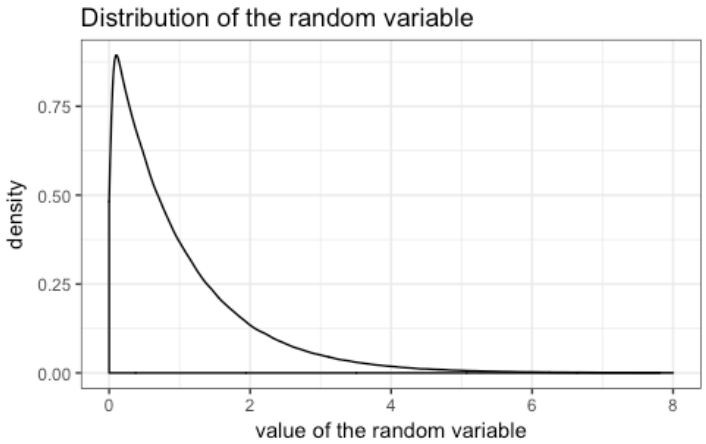
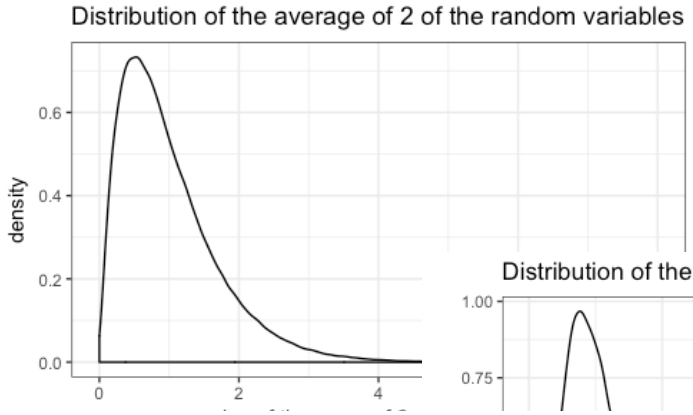
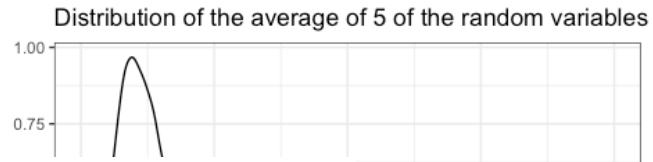
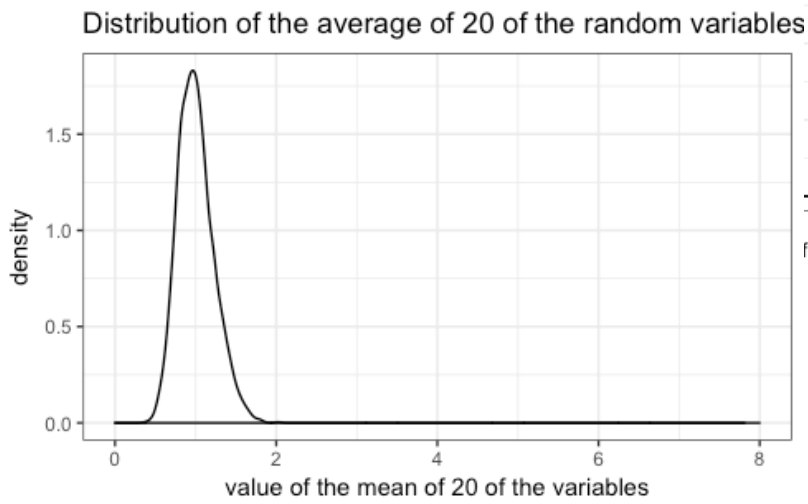
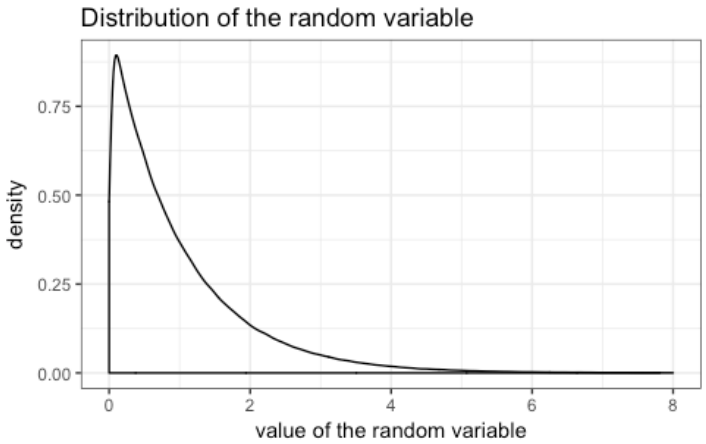


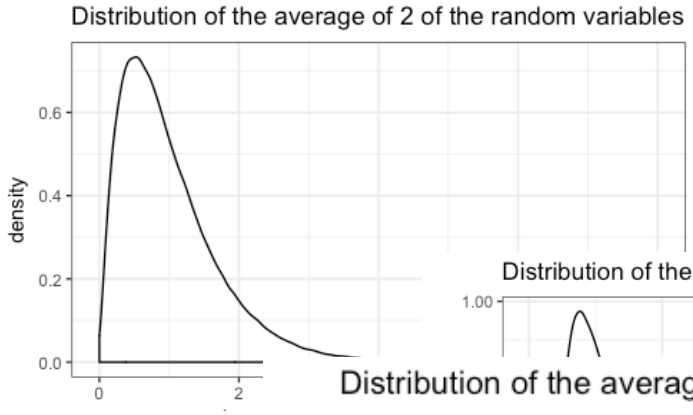


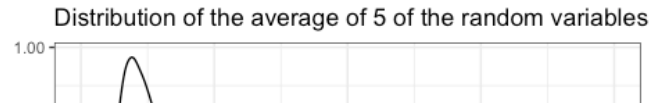


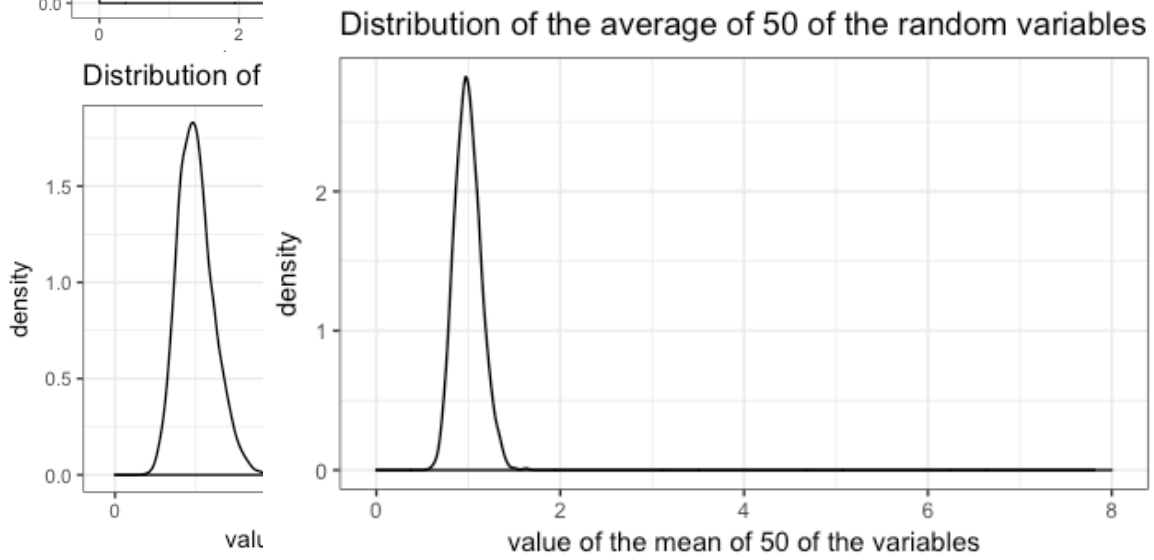


Sums of Normal Random Variables









Central limit theorem

This is the central limit theorem!

Central limit theorem:

If X_1, X_2, ..., X_n are independent and have the same distribution, and the variance of that distribution is σ^2, then if n is large,

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

approximately and under relatively weak conditions.

- In general, this applies for $n \geq 30$.
- As n increases, the normal approximation improves.

Central limit theorem

The central limit theorem allows us to use the sample ($X_1 \ldots X_n$) to discuss the population (μ)

We do not need to know the distribution of the data to make statements about the true mean of the population!

Distribution of the Sample Mean

EXAMPLE:

Suppose that for Seattle sixth grade students the mean number of missed school days is 5.4 days with a standard deviation of 2.8 days. What is the probability that a random sample of size 49 (say Ridgecrest's 6th graders) will have a mean number of missed days greater than 6 days?

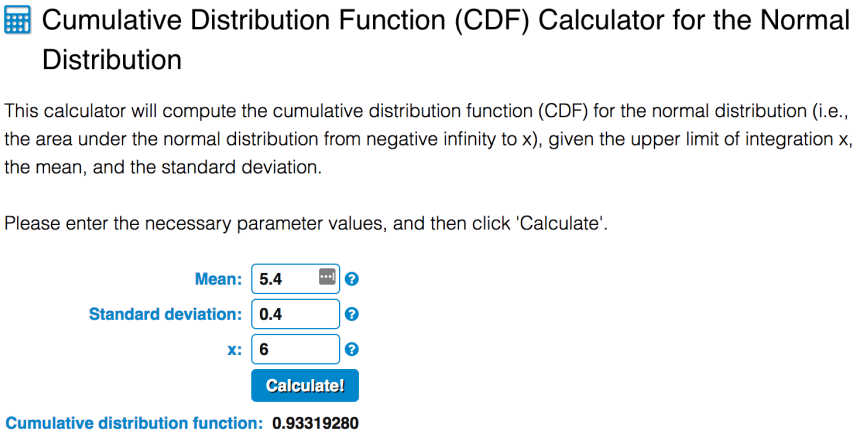
Find the probability that a random sample of size 49 from this population will have a mean greater than 6 days.

$\mu = 5.4$ days

$\sigma = 2.8$ days

$n = 49$

$$\text{So } \bar{X} \sim N\left(5.4, \frac{2.8}{\sqrt{49}}\right)$$



$$Pr(\bar{X} > 6) = 1 - 0.9332 = 0.0668$$

Quiz

What is the probability that a random sample (size 49) from this population has a mean between 4 and 6 days?

Recall: $\mu = 5.4$ days, $\sigma = 2.8$ days, $n = 49$

Quiz Solution

What is the probability that a random sample (size 49) from this population has a mean between 4 and 6 days?

Recall: $\mu = 5.4$ days, $\sigma = 2.8$ days, $n = 49$

$$\begin{aligned} Pr(4 \leq \bar{X} \leq 6) &= Pr(\bar{X} \leq 6) - Pr(\bar{X} \leq 4) \\ &= 0.9332 - 0.0002 \\ &= 0.9330 \end{aligned}$$

Confidence Intervals

Confidence Intervals

Confidence intervals are not just intervals!

"(L, U) is a 100p% confidence interval for a parameter θ"

means that

"For any possible correct value parameter θ, the interval (L, U) contains θ with probability at least p."

Since we don't know θ, we need to find a way to determine L and U from our data.

Note: Confidence intervals only concern parameters. Prediction intervals (different!) are intervals about random variables.

Confidence Intervals for the mean

Because

$$\overline{\mathrm{X}} \sim N\left(\mu, \frac{\sigma^2}{n}\right),$$

we know that

$$P\left[-1.96 \leq \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \leq +1.96\right] = 0.95.$$

Rearranging gives us that

$$\left(\overline{X} - 1.96 \times \frac{\sigma}{\sqrt{n}}, \overline{X} + 1.96 \times \frac{\sigma}{\sqrt{n}}\right)$$

is a 95% confidence interval for the true mean μ

Confidence Intervals

σ known

If we desire a (1 - α) confidence interval we can derive it based on the statement

$$P\left[Q_Z^{\left(\frac{\alpha}{2}\right)} < \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} < Q_Z^{\left(1-\frac{\alpha}{2}\right)} \right] = 1 - \alpha$$

That is, we find constants $Q_Z^{\left(\frac{\alpha}{2}\right)}$ and $Q_Z^{\left(1-\frac{\alpha}{2}\right)}$ that have exactly (1 - α) probability between them.

A (1 - α) Confidence Interval for the Population Mean

$$\left(\overline{X} + Q_Z^{\left(\frac{\alpha}{2}\right)} \times \frac{\sigma}{\sqrt{n}}, \overline{X} + Q_Z^{\left(1-\frac{\alpha}{2}\right)} \times \frac{\sigma}{\sqrt{n}} \right)$$

Confidence Intervals

σ known - EXAMPLE

Suppose the NIH reports that gestational times have a standard deviation of 6 days. A sample of 30 second time mothers yield a mean pregnancy length of 279.5 days. Construct a 90% confidence interval for the mean length of second pregnancies based on this sample.

$$\text{If } Z \sim N(0,1),$$

$$\text{then } Pr(Z < 1.645) = 0.95$$

$$\text{so } Pr(-1.645 < Z < 1.645) = 0.90$$

and our confidence estimate is

$$(279.5 - 1.645 \times \frac{6}{\sqrt{30}},$$

$$279.5 + 1.645 \times \frac{6}{\sqrt{30}})$$

Confidence Intervals
σ^2 unknown
t Distribution

When σ is unknown we replace it with the estimate, s, and use the t-distribution. The statistic

$$\frac{\overline{X} - \mu}{s / \sqrt{n}}$$

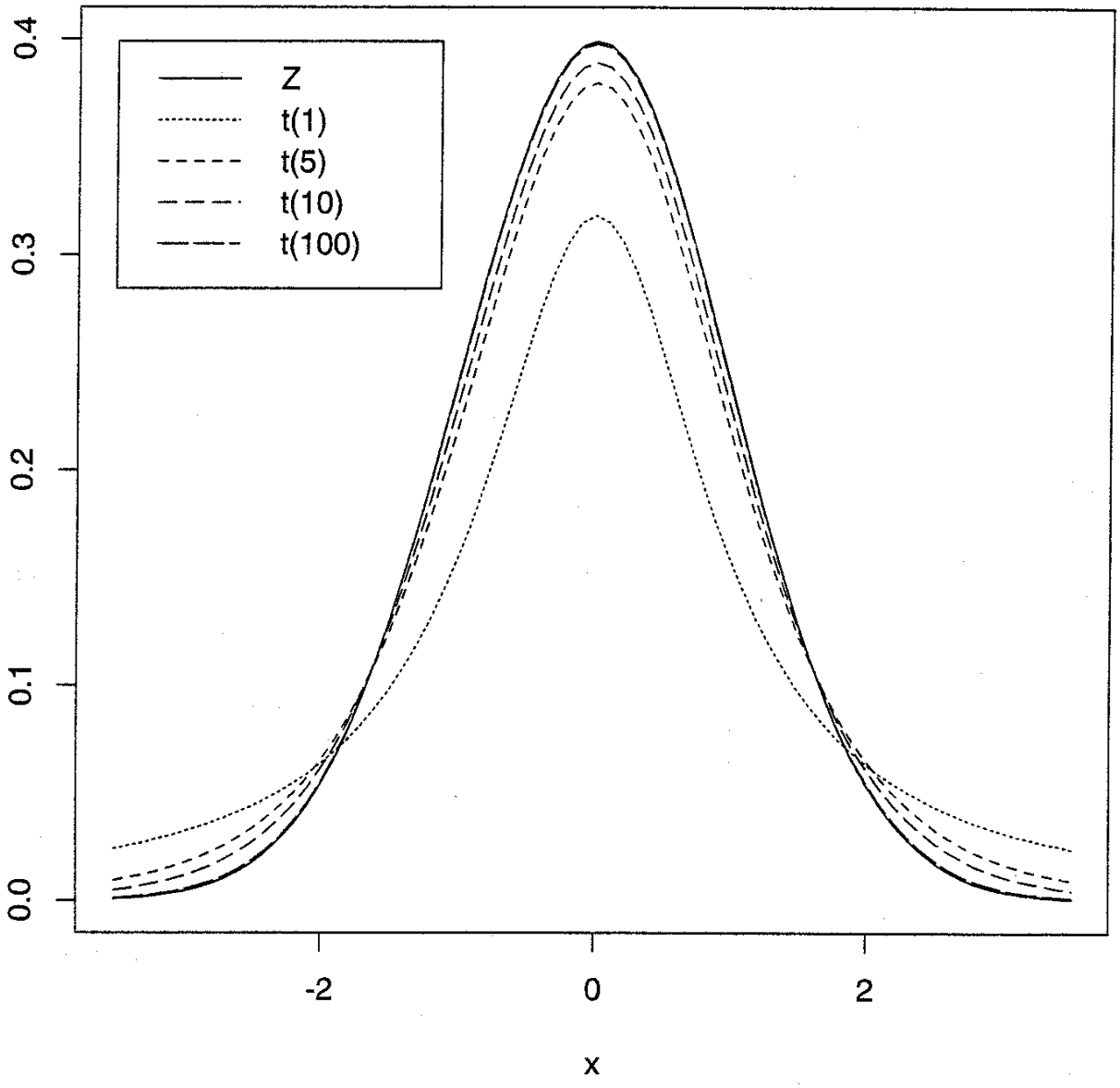
has a t-distribution with n-1 *degrees of freedom*.

We can use this distribution to obtain a confidence interval for μ even when σ is not known.

A (1-α) Confidence Interval for the Population Mean when σ is unknown

$$\left(\overline{X} + t_{(n-1)}^{\left(\frac{\alpha}{2}\right)} \times s / \sqrt{n},\ \ \overline{X} + t_{(n-1)}^{\left(1-\frac{\alpha}{2}\right)} \times s / \sqrt{n} \right)$$

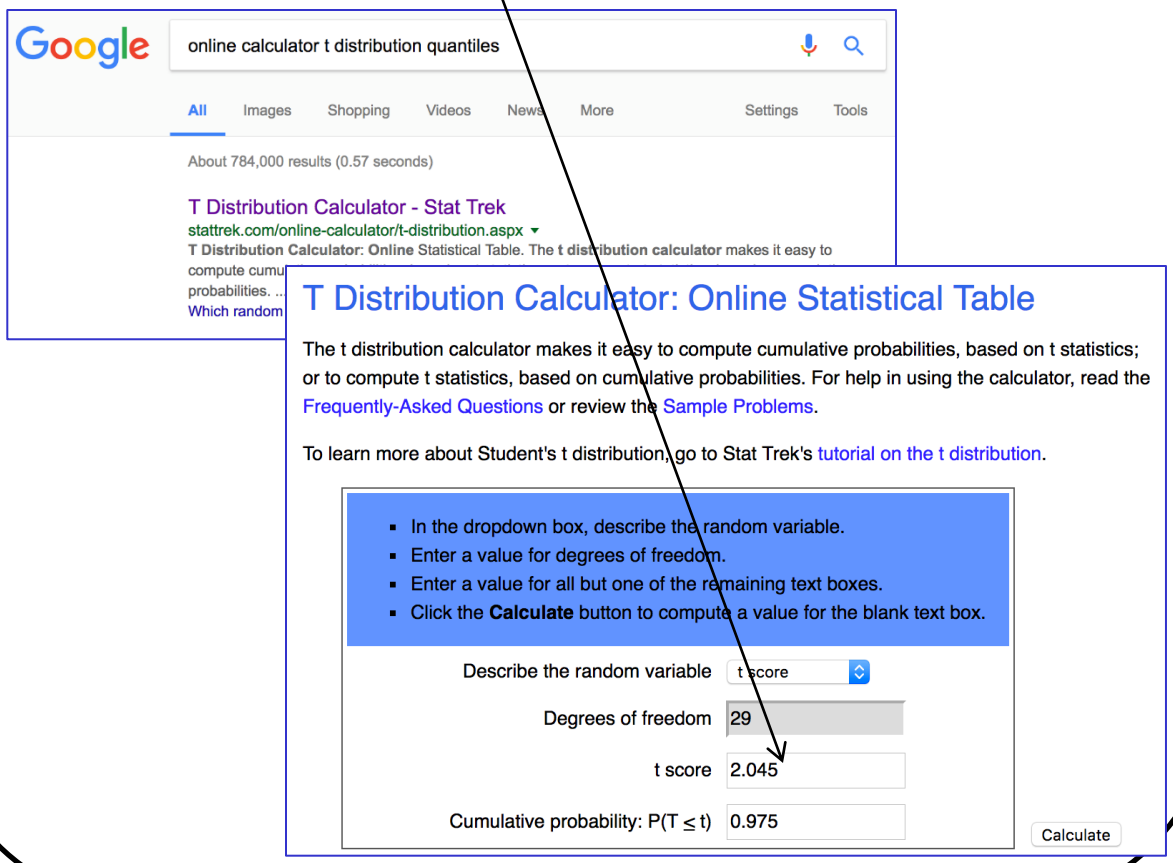
Normal and t distributions



Confidence Intervals - σ^2 unknown
t Distribution - EXAMPLE

Given our 30 moms with a mean gestation of 279.5 days and a variance of 28.3 days^2, we can now compute a 95% confidence interval for the mean length of pregnancies for second time mothers:

$$279.5 \pm t_{29}^{0.975} \times \frac{\sqrt{28.3}}{\sqrt{30}}$$



Additional material

(not covered in 2018)

Confidence Intervals - sample variance

Q: Can we derive a confidence interval for the sample variance?

A: Yes. We'll need the **Chi-square distribution**

Definition: The sum of squared independent standard normal random is a random variable with a **Chi-square** distribution with n degrees of freedom.

Let Z_i be standard normals, N(0,1). Let

$$X = Z_1^2 + Z_2^2 + \ldots + Z_n^2 = \sum_{i=1}^{n} Z_i^2$$

X has a χ^2(n) distribution

Chi-square Distribution

Properties of χ^2 (n): Let $X \sim \chi^2(n)$.

1. $X \geq 0$
2. $E[X] = n$
3. $V[X] = 2n$
4. **n**, the parameter of the distribution is called *the degrees of freedom*.

Chi-square Distribution
Sample Variance

The Chi-square distribution describes the distribution of the **sample variance**. Recall

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

and

$$(n-1)\frac{s^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2$$

Now the right side almost looks like

$$\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$$

which would be $\chi^2(n)$.

Since μ is estimated by $\overline{X}$ one degree of freedom is lost leading to …

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2 \quad \text{with n-1 degrees of freedom}$$

Chi-square Distribution
Confidence Interval for σ^2

We can use the Chi-square distribution to obtain a (1 - α) confidence interval for the **population variance**.

$$P\left[Q_{\chi^2(n-1)}^{\left(\frac{\alpha}{2}\right)} < (n-1)\frac{s^2}{\sigma^2} < Q_{\chi^2(n-1)}^{\left(1-\frac{\alpha}{2}\right)} \right] = 1-\alpha$$

Now, inverting this statement yields:

$$P\left[s^2 \times (n-1) / Q_{\chi^2(n-1)}^{\left(1-\frac{\alpha}{2}\right)} < \sigma^2 < s^2 \times (n-1) / Q_{\chi^2(n-1)}^{\left(\frac{\alpha}{2}\right)} \right] = 1-\alpha$$

Therefore,

A (1 - α) **Confidence Interval for the Population Variance**

$$\left(s^2 \times (n-1) / Q_{\chi^2(n-1)}^{\left(1-\frac{\alpha}{2}\right)},\ s^2 \times (n-1) / Q_{\chi^2(n-1)}^{\frac{\alpha}{2}} \right)$$

Chi-square Distribution
Confidence Interval for σ^2 - example

Suppose for the second time mothers were not happy using the standard deviation of 6 days since it was based on the population of all mothers regardless of parity. The sample variance was 28.3 days^2. What is a 95% confidence interval for the variance of the length of second pregnancies?

Chi-square Distribution
Confidence Interval for σ^2 - Solution

Suppose for the second time mothers were not happy using the standard deviation of 6 days since it was based on the population of all mothers regardless of parity. The sample variance was 28.3 days^2. What is a 95% confidence interval for the variance of the length of second pregnancies?

$$n = 30, s^2 = 28.3$$

$$Q_{\chi^2_{29}}^{0.025} = 16.1$$

$$Q_{\chi^2_{29}}^{0.975} = 45.7$$

$$(28.3 \times \frac{29}{45.7}, 28.3 \times \frac{29}{16.1})$$

$$\implies (17.96, 50.98) \text{ is a } 95\% \text{ CI for } \sigma^2$$

$$(4.24, 7.14) \text{ is a } 95\% \text{ CI for } \sigma$$

Summary

- General $(1 - \alpha)$ Confidence Intervals.
  - Confidence intervals are only for parameters!
- CI for μ, σ assumed known $\rightarrow$ Z.
- CI for μ, σ unknown $\rightarrow$ T.
- CI for $\sigma^2 \rightarrow \chi^2$

- $\uparrow$confidence $\rightarrow$ wider interval
- $\uparrow$sample size $\rightarrow$ narrower interval