25.08.2020 Introduction to the course

(I) Graphics Pipeline

- Modeling (geometric model)
- Transforms
- Visibility
- Illumination and Shading
- Color, Texture

+ Perception and "Interaction"

(II) Curves and surfaces : Interpolation, Approximation

Bezier approx, B-Splines

(III) Rendering

Ray-tracing

+ BLENDER [Python]
(UI)

(IV) Multimedia :

- Text
- Image / Graphics
- Sound
- Video
- Animation

"INTERACTION"

+ Storage
+ Retrieval

} Compression, Databases, Streaming

+ Processing } Editing

Evaluation Policy (Feedback)

- Mid Sem Exam - 30%
- End Sem Exam - 40%
- Assignments/Quizzes - 30%

Prerequisites and Expectations :

- Linear Algebra
- Willingness to learn and experiment
- I will write "codes"

"Everybody knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions."
Felix Klein

Question : Can I draw a straight line passing through given three points?

Science/Math ~ Only if colinear

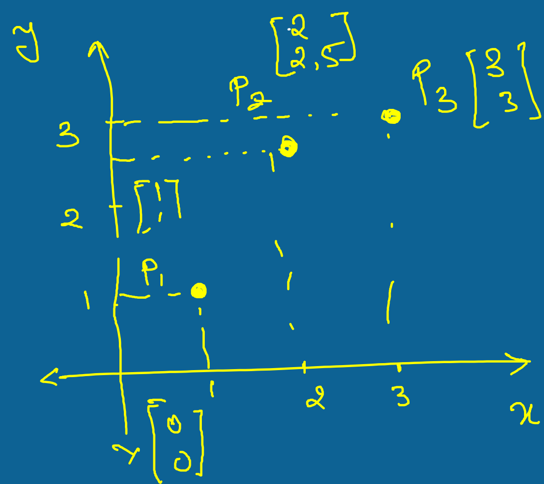
Engineer ~ why not ? "Best possible straight line"

Square matrices $\longrightarrow$ Non square matrices

$$[A]_{n\times n}\,[x]_{n\times 1} = [b]_{n\times 1} \qquad [A]_{m\times n}\,[x]_{n\times 1} = [b]_{m\times 1}$$

$$x = A^{-1} b \qquad ?$$

Find (m, c) $\quad \tilde{y}_i = \bar{m} x_i + c$



$$\boxed{y = mx + c}$$ and

P_1, P_2, P_3 are "on the line"

$$y_1 = \boxed{m} x_1 + \boxed{c}$$
$$y_2 = m x_2 + c$$
$$y_3 = m x_3 + c$$

$$A p = b$$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{3\times 1}$$

$$\sum_{i=1}^{3} \left(y_i - (m x_i + c) \right)^2 \sim E(m, c)$$

$$\underset{m,\, c \,\in\, \mathbb{R}}{\operatorname{argmin}} \; E(m, c)$$

- Calculus way
- Linear Algebra

$=1$

Solve:

$$A \cdot p = b \qquad \left\{ \arg\min_{p} \| b - A p \|^2 \right\}$$

$$A^T A p = A^T b \Longrightarrow \boxed{\tilde{p} = \underbrace{(A^T A)^{-1} A^T}_{\text{pseudoinverse}} b}$$

$$\boxed{y = \underline{a_0} + \underline{a_1} x + \underline{a_2} x^2 + \dots + a_{n-1} x^{n-1}}$$ polynomial of deg n-1

27.08.2020

- Geometric Model
- Introduction to Blender
- Sounds (Talking Tom)

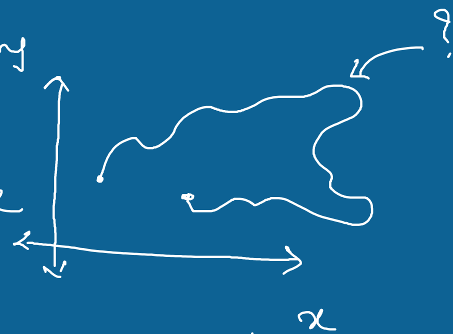
Point Cloud — structured representation

- volume
- shape
- curve

$c : (a, b) \to \mathbb{R}^n$

$n = 2$, planar curve

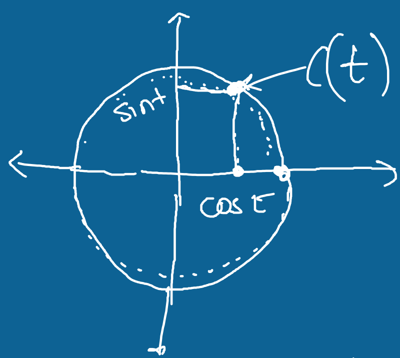
$n = 3$, space curve



"Graph"

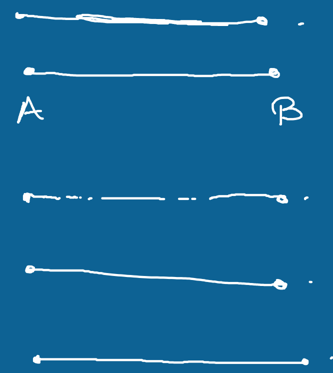
$c : (0, 2\pi] \to \mathbb{R}^2$ $\qquad c(t) = (\cos t, \sin t)$

$t \in (0, 2\pi]$

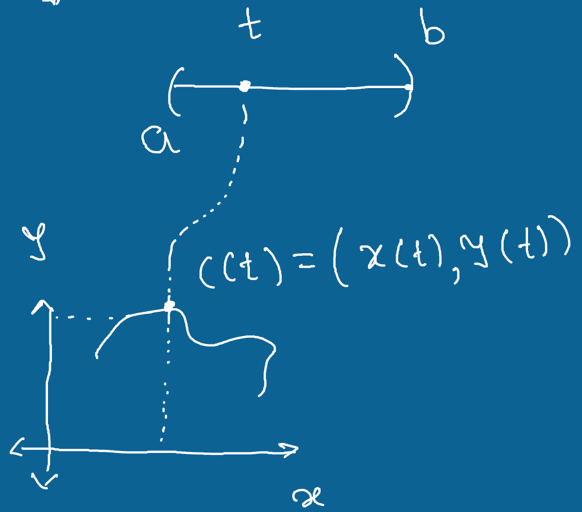


"parametric curves"

Q:



$\boxed{(a,b) \subset \mathbb{R}}$



$c(t) = (t, \cos t, \sin t)$

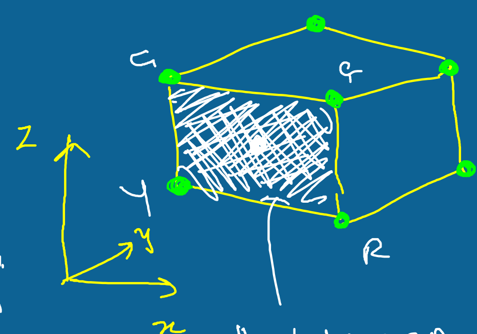
$c : (a,b) \longrightarrow \mathbb{R}^3$

- "Interaction"

- Shape : "surface"

MESH

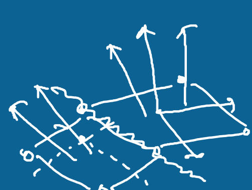
"Triangulated Mesh"



- vertices
— edges
▨ faces

"INTERPOLATION"

- Surface Patches — B-Spline, Bezier



- Derivatives
- Integrate
- Curvature of the surface at a point

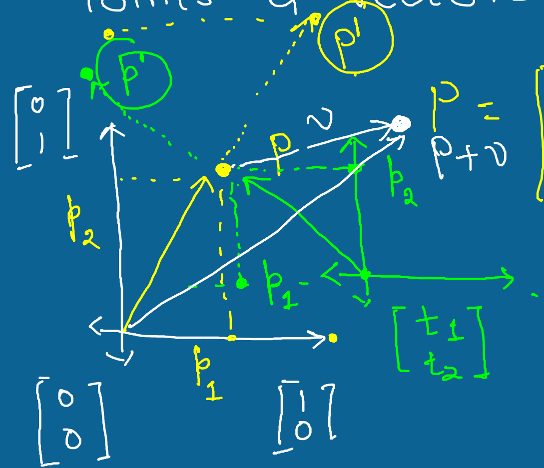
Patched Surface.

- Continuity

01.09.2020 • Curve, Surface

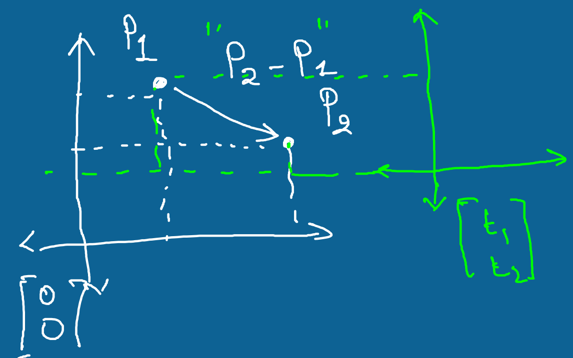
• Points & Vectors

"standard basis"



$$p = \begin{bmatrix}p_1\\p_2\end{bmatrix} = p_1\begin{bmatrix}1\\0\end{bmatrix} + p_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}$$

$$p + v = p$$



$P_0, P_1, P_2, \ldots, P_N \in \mathbb{R}^n$ Exercise:

$$\sum_{i=0}^{N} \omega_i P_i = P_0 + \sum_{i=1}^{N} \omega_i \underbrace{(P_i - P_0)}_{\text{vectors}} \sim \text{Point}, \quad \sum_{i=0}^{N}\omega_i = 1$$

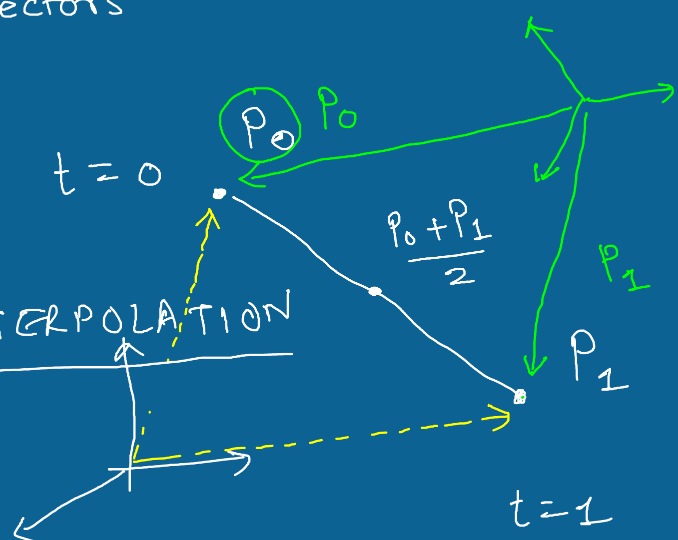
"Barycentric Sum"

+ $P_0, P_1 \in \mathbb{R}^3$

LINEAR INTERPOLATION

$$P(t) = (1-t)\,P_0 + t\,P_1$$

$$t \in [0,1]$$



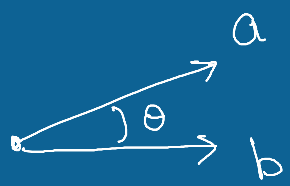
$P(t) = (1-t)P_0 + tP_1$

• Parametric Blending:

$$:= \omega_0 P_0 + \sum_{i=1}^{N} \omega_i P_i$$

$$:= 1\,P_0 - (1-\omega_0)P_0 + \sum_{i=1}^{N} \omega_i P_i$$

$$:= P_0 + \sum_{i=1}^{N} \omega_i P_i - \underbrace{(1-\omega_0)}_{(\omega_1+\omega_2+\ldots+\omega_N)}P_0$$



Inner Product

$$\langle a \mid b \rangle = \|a\|\,\|b\| \cos\theta$$

$a, b \in \mathbb{R}^N$

$$\left\langle \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \middle| \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \right\rangle = \sum_{i=1}^{N} a_i b_i$$

Blending

$P_0, P_1 \in \mathbb{R}^N$

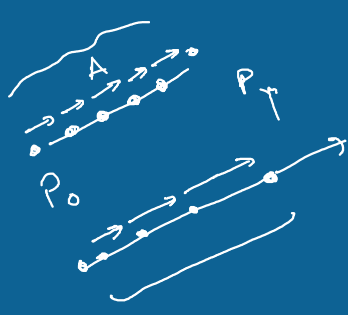
$(1-t)P_0 + tP_1$ — A

$(1-t)^2 P_0 + t^2 P_1$ — $\cancel{B}$

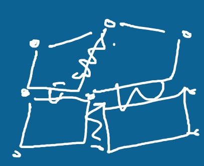
$\boxed{(1-t^2)P_0 + t^2 P_1}$ — C

$$(1-t)^2 + t^2 = 1 - 2t + t^2 + t^2 = 1 - 2t + 2t^2 \neq 1 \quad \forall t \in [0,1]$$

$$P(t) \sim \frac{dP}{dt} = \begin{cases} (-P_0 + P_1) & \text{— A} \\ -2(1-t)P_0 + 2tP_1 & \text{— } \cancel{B} \\ -2tP_0 + 2tP_1 & \text{— C} \end{cases}$$

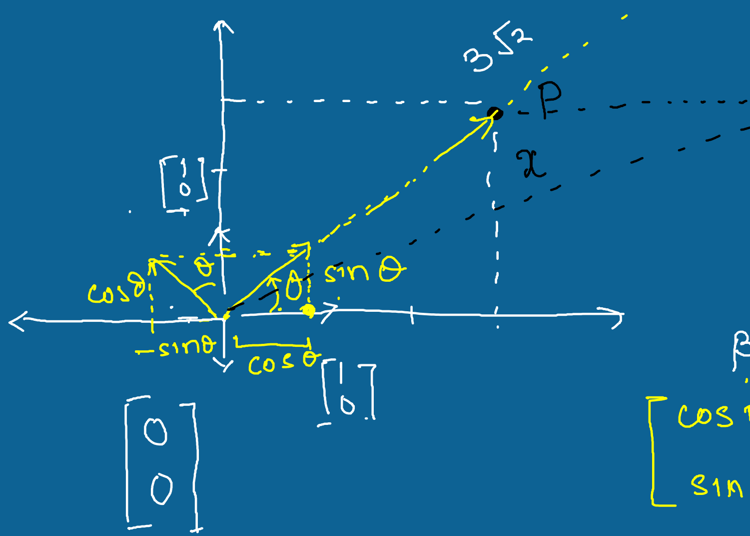


$2t(P_1 - P_0)$



03.09.2020 Change of basis

Standard Basis S matrix



$$b = 3\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = \overbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}^{s_1 \quad s_1}\begin{bmatrix}3\\3\end{bmatrix}$$

$$b \sim \begin{bmatrix}3\\3\end{bmatrix}$$

If R is rotation

$$\boxed{\begin{matrix} R^T R = I \\ R R^T = I \end{matrix}}$$

$$\beta_1 = \begin{bmatrix}\cos \pi/4\\ \sin \pi/4\end{bmatrix} \qquad \beta_2 = \begin{bmatrix}-\sin \pi/4\\ \cos \pi/4\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix} = \overbrace{\begin{bmatrix}\cos \pi/4 & -\sin \pi/4\\ \sin \pi/4 & \cos \pi/4\end{bmatrix}}^{B}\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}\cos \pi/4 & \sin \pi/4\\ -\sin \pi/4 & \cos \pi/4\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}3\sqrt{2}\\0\end{bmatrix}$$

$$\overset{\checkmark}{S\,[b]_S} = B\,\overset{?}{[b]}_B$$

$$\Rightarrow \boxed{[b]_B = B^{-1} S\, [b]_S}$$

Change of Basis

$S = I_{2\times 2}$

Ex. $H = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$ $\quad x = \begin{bmatrix}3\\3\end{bmatrix}$ $\quad B = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

$y = Hx$ $\qquad$ (1) $\bar{y}$? $\qquad$ (2) $\bar{H}$?

$$y = \begin{bmatrix}6\\3\end{bmatrix} \qquad \bar{y} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}6\\3\end{bmatrix} = \begin{bmatrix}9/\sqrt{2}\\ -3/\sqrt{2}\end{bmatrix} \qquad B^{-1} H B$$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
S \quad x : \mathbb{R}^N \arrow[r, "H \in \mathbb{R}^{N\times N}"] \arrow[d, "B^{-1}"'] & \mathbb{R}^N \quad S \arrow[d, "B^{-1}"] \\
B \quad \mathbb{R}^N \arrow[r, "\bar{H}"'] & \mathbb{R}^N \quad B
\end{tikzcd}
$$

$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ $\qquad y = Hx$

$\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_N \end{bmatrix}$ $\qquad \bar{y} = B^{-1} y$

$\bar{x} = B^{-1} x$

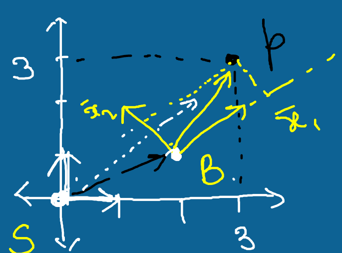
(I) $Sx = B\bar{x} \Rightarrow B^{-1} S x = \bar{x}$

(II) $B^{-1} H = \bar{H} B^{-1}$

$$\boxed{\bar{H} = B^{-1} H B}$$

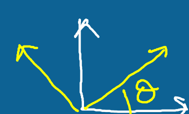
"COMMUTATIVE DIAGRAM"

body centric frame



world coordinate frame

${}^S p = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\qquad {}^B p = ?$



1. Coordinate Frame has origin & orientation

- $t = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad \underline{O} = \underline{O} + t$

- $\underline{R}_\theta \cdot \underline{S} = B$ $\qquad x = B \cdot \bar{x} + t$

$$\boxed{x = R_\theta \cdot \bar{x} + t}$$

$\Rightarrow R_\theta \bar{x} = x - t \Rightarrow \boxed{\bar{x} = R_\theta^{-1}(x - t)}$

Ex $\theta = \pi/4 \quad x = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad t = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \bar{x} = ?$

Homogeneous Coordinate System:

$$x = R_\theta \bar{x} + t \qquad x, \bar{x} \in \mathbb{R}^2 \qquad \bar{x} = R_\theta^{-1}(x - t)$$

$$\begin{bmatrix} \overset{2\times1}{x} \\ \underset{1\times1}{1} \end{bmatrix} = \begin{bmatrix} \overset{2\times2}{R_\theta} & \overset{2\times1}{t} \\ \underset{1\times2}{0} & \underset{1\times1}{1} \end{bmatrix} \begin{bmatrix} \overset{2\times1}{\bar{x}} \\ \underset{1\times1}{1} \end{bmatrix} = \begin{bmatrix} R_\theta \bar{x} + t \\ 1 \end{bmatrix}$$

A ↑

$$\boxed{\tilde{x} = A \tilde{\bar{x}}} \qquad \boxed{\tilde{\bar{x}} = A^{-1} \tilde{x}}$$

$$\boxed{A^{-1} = \begin{bmatrix} R_\theta^{-1} & -R_\theta^{-1} t \\ 0 & 1 \end{bmatrix}} \longleftarrow \bar{x} = R_\theta^{-1} x - R_\theta^{-1} t$$

05.09.2020

Circular Convolution as Matrix vector multiplication

$x = x_0 \quad x_1 \quad x_2 \quad x_3$

$h = h_0 \quad h_1 \quad h_2 \quad h_3$

$$h : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$\left\{ \begin{aligned} & h(x^1 + x^2) = h(x^1) + h(x^2) \\ & h(ax) = a\,h(x) \end{aligned} \right.$$

LINEARITY OF 'h'.

										k	y
			h_0	h_1	h_2	h_3					
x_3	x_2	x_1	x_0	x_3	x_2	x_1	$x_0 \ldots$			$\boxed{0}$	$x_0h_0 + x_3h_1 + x_2h_2 + x_1h_3$
$\ldots$	x_3	x_2	x_1	x_0	x_3	x_2	$x_1 \ldots$			1	$x_1h_0 + x_0h_1 + x_3h_2 + x_2h_3$
	$\ldots$	x_3	x_2	x_1	x_0	x_3	$x_2 \ldots$			2	$x_2h_0 + x_1h_1 + x_0h_2 + x_3h_3$
		x_0	x_3	x_2	x_1	x_0	$x_3 \ldots$			3	$x_3h_0 + x_2h_1 + x_1h_2 + x_0h_3$
x_3	x_2	x_1	x_0	x_3	x_2	x_1	x_0			$\boxed{4}$	repeating!!

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} h_0 & h_3 & h_2 & h_1 \\ h_1 & h_0 & h_3 & h_2 \\ h_2 & h_1 & h_0 & h_3 \\ h_3 & h_2 & h_1 & h_0 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

$$\boxed{y = H x}$$

Circulant matrix

$$h = \frac{1}{\sqrt{30}} \begin{array}{c} \begin{array}{ccccc} 0 & & 29 & & 3999 \end{array} \\ \left[\overbrace{1 \;\ldots\; 1}\;\; 0 \ldots\ldots 0\right] \end{array}$$

$$x = x_0 \; x_1 \; x_2 \ldots x_{3999}$$

(my voice $F_s = 4k$)

$$H = \begin{array}{c} \begin{array}{ccc} 0 & & 3999 \end{array} \\ \begin{bmatrix} \underset{(h_0)}{1/\sqrt{30}} & \underset{(h_{3999})}{0} & 0 & \ldots\ldots & \underset{(h_{30})}{0} & \underset{(h_{29})}{1/\sqrt{30}} & \ldots & \underset{(h_1)}{1/\sqrt{30}} \\ \text{circshift right} & & & & & & & \\ & & & & & & & \vdots \end{bmatrix} \end{array}$$

$\boxed{3999}$ circ right shifts

4000 ROWS

$y = Hx$ ← circulant matrix

$H : \mathbb{R}^N \to \mathbb{R}^N$

$[D, \Lambda] = \mathrm{eig}(H);$

D — columns are eigenvectors

Λ — eigenvalues

stubborn

$A v = \lambda . v$

$D = \begin{bmatrix} | & | & \cdots & | \end{bmatrix}$

$\Lambda = \begin{bmatrix} \square & & 0 \\ & \square & \\ 0 & & \ddots \end{bmatrix}$ eigenvalues

$H D_1 = \Lambda_{1,1} . D_1$

$H \begin{bmatrix} | & | & | & & | \\ D_1 & D_2 & D_3 & \cdots & D_N \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \Lambda_{1,1} D_1 & \Lambda_{22} D_2 & \cdots & \Lambda_{NN} D_N \\ | & | & & | \end{bmatrix}$

$\boxed{HD = \cancel{\Lambda D}}$ → $D\Lambda$

$H D = \underline{\underline{D}} \begin{bmatrix} \Lambda_{11} & & & \\ & \Lambda_{22} & & \\ & & \ddots & \\ & & & \Lambda_{NN} \end{bmatrix} \cancel{D}$

$\boxed{D^{-1} H D = \Lambda}$ ← diagonal matrix!

standard Basis

x myvoice + noise

$$\begin{array}{ccc} \mathbb{R}^N & \xrightarrow{\quad H \quad} & \mathbb{R}^N \\ \Big\downarrow D^{-1} & & \Big\downarrow D^{-1} \quad \Big\uparrow D \\ \mathbb{R}^N & \xrightarrow{\; D^{-1} H D = \Lambda \;} & \mathbb{R}^N \end{array}$$

H ← (convolution) circulant matrix

filtered voice

D^{-1} FOURIER TRANSFORM

D INVERSE FOURIER TRANSFORM

Basis {eigenvectors of H}

$x = x_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$I . x = D X$ $\boxed{X = D^{-1} x}$

$x = X_0 [e_1] + X_1 [e_2] + X_2 [e_3] + X_3 [e_4]$

$$x \longrightarrow X = D^{-1} x$$

$$y \longrightarrow Y = D^{-1} y$$

$$H \longrightarrow \bar{H} = D^{-1} H D$$

diagonal

Λ

$$\boxed{y = Hx} \qquad Y = \bar{H} X$$

$$\boxed{Y = \Lambda \underline{X}}$$

circulant

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underset{4\times 4}{\begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

diagonal

$$\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_2 & \\ 0 & & & \lambda_3 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$*$ $4\,[(4*) + (3+)]$

$16 *$ $\qquad$ $12 +$

$Y_0 = \lambda_0 X_0 \qquad Y_2 = \lambda_2 X_2$

$Y_1 = \lambda_1 X_1 \qquad Y_3 = \lambda_3 X_3$

DFT / FFT

$4 *$

$$Hx = H\left(x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3\right)$$

$$= x_0 H e_0 + x_1 H e_1 + x_2 H e_2 + x_3 H e_3$$

$$= \lambda_0 x_0 e_0 + \lambda_1 x_1 e_1 + \lambda_2 x_2 e_2 + \lambda_3 x_3 e_3$$

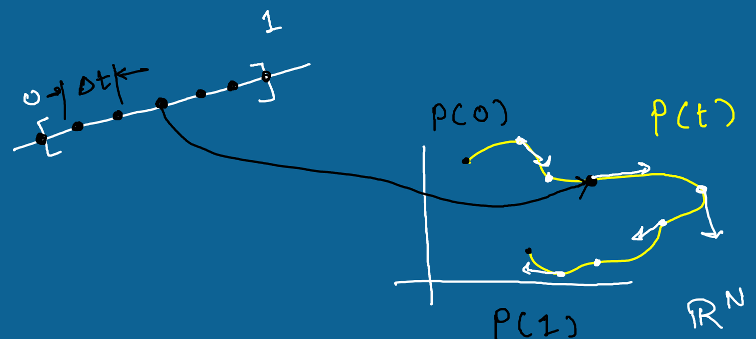
08.09.2020

Curve $P(t)$ $\qquad \boxed{s} : [0,1] \longrightarrow [a,b] \qquad s(t)$

$P : [0,1] \to \mathbb{R}^N$

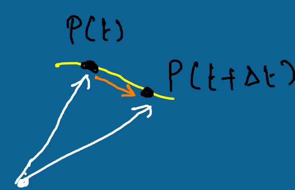
$\uparrow\downarrow$ reparametrization

$[a,b]$



Tangent Vector: $\dfrac{dP(t)}{dt} = P'(t) = \lim\limits_{\Delta t \to 0} \dfrac{[P(t+\Delta t) - P(t)]}{\Delta t}$

(velocity)



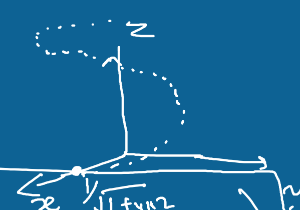
+ $P(t) = (\cos 2\pi t, \sin 2\pi t, t) \sim (x(t), y(t), z(t))$

$P'(t) = (x'(t), y'(t), z'(t)) = (-2\pi \sin(2\pi t), 2\pi \cos(2\pi t), 1)$

$\|P'(t)\| = \sqrt{4\pi^2(\sin^2 + \cos^2) + 1} = \sqrt{1+4\pi^2} \sim \text{constant}$

Def

+ $\|P'(t)\| = 1 \quad \forall t$ — unit speed curve.



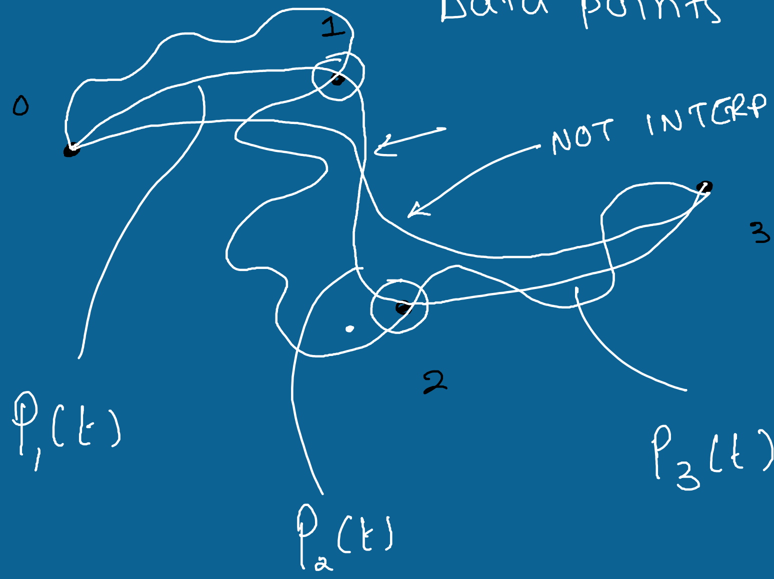
+ $$\boxed{P(s) = \left(\frac{1}{\sqrt{1+4\pi^2}} \cos 2\pi s, \frac{1}{\sqrt{1+4\pi^2}} \sin 2\pi s, s \right)}$$

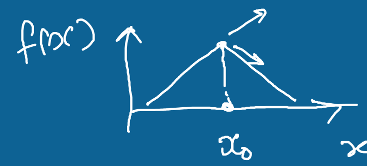
$$P'(s) = \left(-\frac{2\pi}{\sqrt{1+4\pi^2}} \sin 2\pi s, \frac{2\pi}{\sqrt{1+4\pi^2}} \cos 2\pi s, 1 \right)$$

$\|P'(s)\| = 1$ — unit speed curve

+ $P(t)$ parametric curve $\quad P(s(t))$ — unit speed curve

Data points — Interpolation / Approximations





$f'(x_0)$?

$$P(t) = \begin{cases} P_1(t), & [0, a] \\ P_2(t) & [a, b] \\ P_3(t) & [b, 1] \end{cases}$$

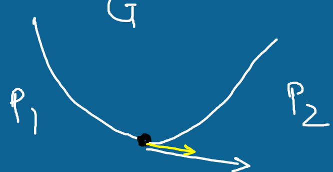
$$\begin{cases} P_1(t=a) = P_2(t=a) \\ P_1'(t=a) = P_2'(t=a) \\ \vdots \\ P_1^n(t=a) = P_2^n(t=a) \end{cases}$$

Geometric Continuity (G^n) — (C^n) Parametric Continuity

G^0

$P_1(end) = P_2(start)$

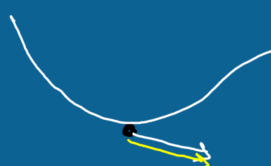
G^1



✓ $P_1(end) = P_2(start)$

✓ $$\frac{P_1'(end)}{\|P_1'(end)\|} = \frac{P_2'(start)}{\|P_2'(start)\|}$$

✓ $$\frac{P_1^n(end)}{\|P_1^n(end)\|} = \frac{P_2^n(start)}{\|P_2^n(start)\|}$$



C^1

$P_1(end) = P_2(start)$

$P_1'(end) = P_2'(start)$

$\vdots$

$P_1^n(end) = P_2^n(start)$

$P(t) = At + B$ Polynomial in '$\underline{t}$'

$\uparrow \quad \uparrow$

Geometric entities

Points, Vectors

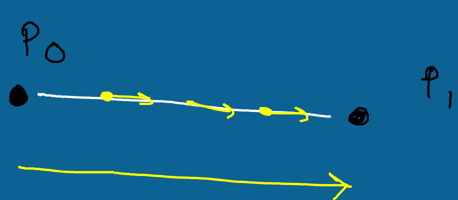
$P(t) = A\underline{t}^3 + B\underline{t}^2 + C\underline{t} + D\underline{1}$ Polyn. deg-3

"$\underline{\text{Basis}}$" Points, tangents

Scheme Geometry

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}$$

$$\boxed{P(t) = T(t)\, M\, G} \longrightarrow \boxed{P'(t) = T'(t)\, M\, G}$$



$P(t) = (1-t)P_0 + tP_1$ $\qquad P(t=0) = P_0$

$= t(P_1 - P_0) + P_0$ $\qquad P(t=1) = P_1$

$$\boxed{P(t) = T(t) \quad M \quad G}$$

$$P(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} P_1 - P_0 \\ P_0 \end{bmatrix} = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

$$P'(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = P_1 - P_0$$

$\forall t$

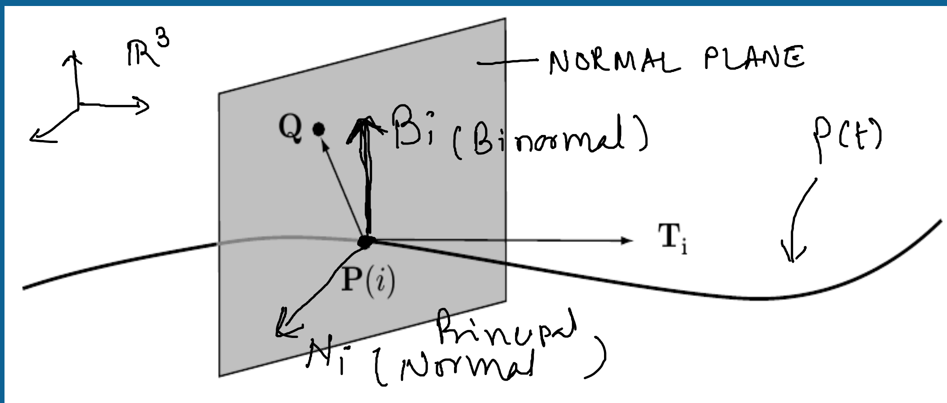
$B_1(t) + B_0(t) = 1$

$$\boxed{P(t) = B_1(t)\, P_1 + B_0(t)\, P_0}$$

Blending functions

10.09.2020 Parametric curve $P(t) = T(t)\,M\,G$

- Unit tangent $T(t) = P'(t) / \|P'(t)\|$ (Intrinsic Prop)



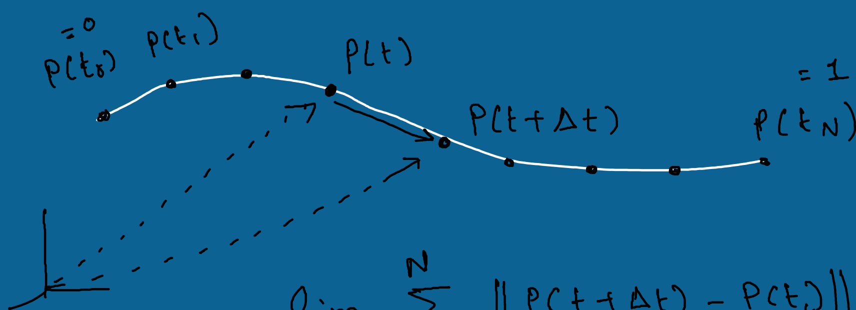
$\langle Q - P(i) \mid T_i \rangle = 0$

$\langle Q \mid T_i \rangle - \langle P(i) \mid T_i \rangle = 0$

$Ax + By + Cz + D = 0$

- Principal Normal Vector

N(t)



$p(t) = (\cos 2\pi t, \sin 2\pi t, t)$

$$\lim_{\Delta t \to 0} \sum_{k=1}^{N} \| P(t_k + \Delta t) - P(t_k) \|$$

$P: [0,1] \to \mathbb{R}^3$

$=$ Length of $P(t)$

$$L = \int_0^1 \sqrt{(-2\pi \sin 2\pi t)^2 + (2\pi \cos 2\pi t)^2 + 1}\; dt$$

$P: [0,1] \to \mathbb{R}^3$

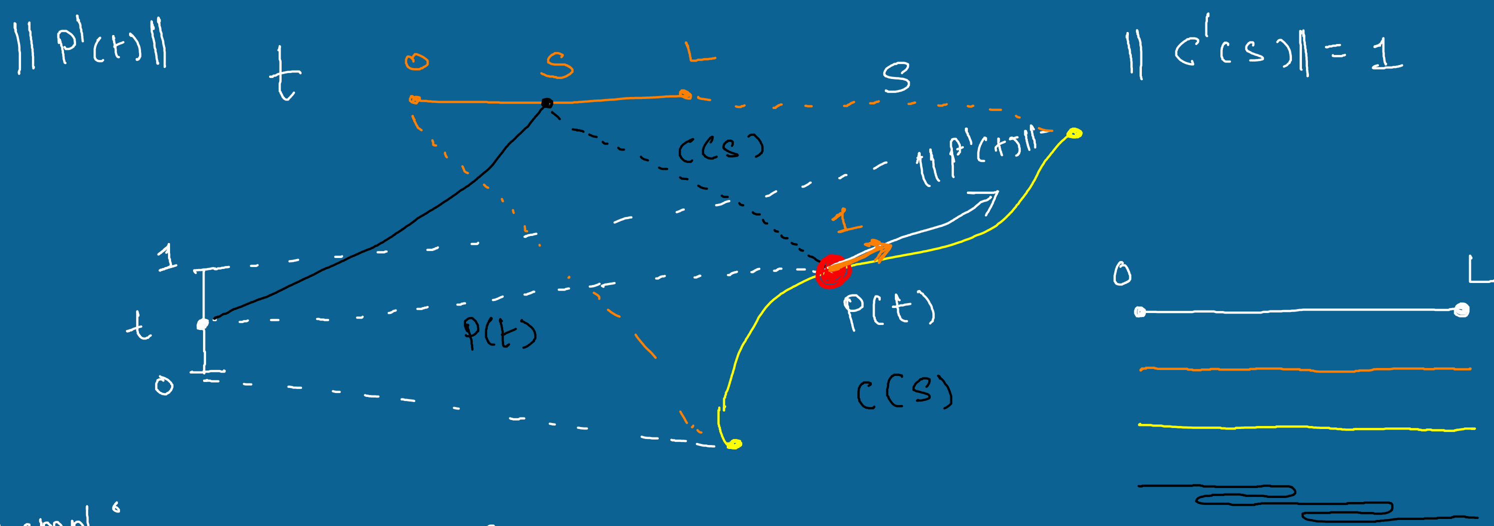
$$= \int_0^1 \sqrt{1 + 4\pi^2}\; dt = \sqrt{1+4\pi^2}$$

$$\sum_{k=1}^{N} \left\{ \lim_{\Delta t \to 0} \left(\frac{\| P(t_k + \Delta t) - P(t_k) \|}{\Delta t} \right) \right\} \Delta t$$

$$\boxed{\int_0^1 \| p'(t) \|\, dt = L}$$

Reparametrization: Unit Speed parametrization.

$$P : [0,1] \to \mathbb{R}^3 \quad \rightsquigarrow \quad s : [0,1] \to [0,L]$$



__Example:__ $p(t) = (1-t)\,P_0 + t\,P_1$

$t \in [0,1]$, Length of $P(t) = \displaystyle\int_0^1 \|P'(t)\|\,dt = \|P_1 - P_0\|$

$\downarrow$

$s \in [0, \|P_1 - P_0\|]$ $\qquad \boxed{s = \|P_1 - P_0\| . t} \qquad t = \dfrac{s}{\|P_1 - P_0\|}$

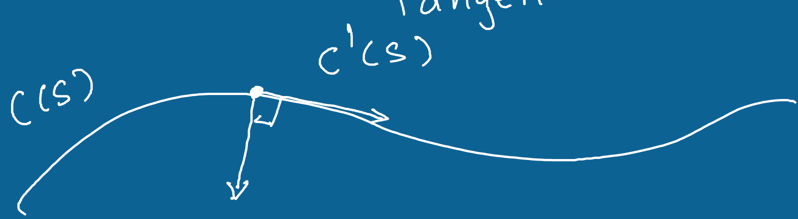
$$P(t) = (1-t)P_0 + t\,P_1 \Rightarrow C(s) = \left(1 - \frac{s}{\|P_1 - P_0\|}\right) P_0 + \frac{s}{\|P_1 - P_0\|} . P_1$$

$t \in [0,1]$

$$C(s) = \frac{1}{L}\left[\left(L - s\right) P_0 + s\,P_1\right] \quad s \in [0, L]$$

$$\frac{dP(t)}{dt} = P_1 - P_0 \qquad \frac{dC(s)}{ds} = \frac{P_1 - P_0}{L}, \quad \frac{d^2 c}{ds^2} = \text{"}0\text{"}$$

$\|p'(t)\| = L$ units/sec $\qquad \|c'(s)\| = 1$ unit/sec



$c'(s) =$ velocity

$c''(s) = \; ?$

$\dfrac{c''(s)}{\|c''(s)\|} \leftarrow$ principal normal

Constant

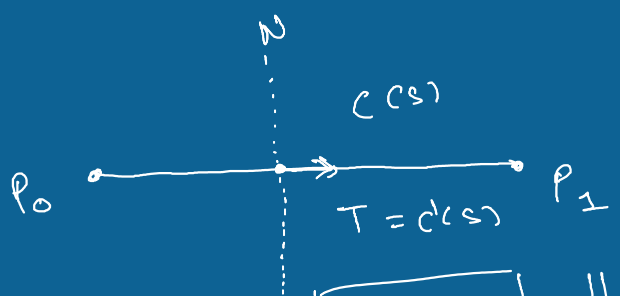
$\|c'(s)\|^2 = \langle c'(s) | c'(s) \rangle = \underline{\underline{1}}$

$c''(s) \perp c'(s)$

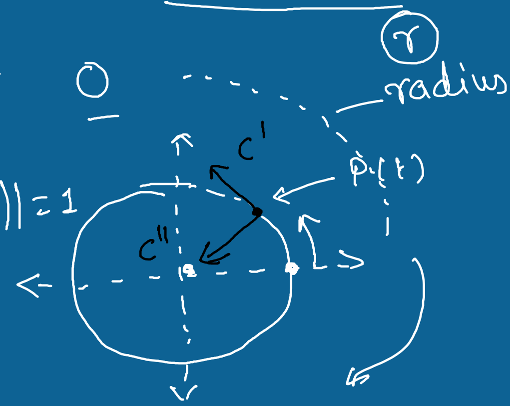
$$\frac{d}{ds} \langle c'(s) | c'(s) \rangle = \langle c''(s) | c'(s) \rangle + \langle c'(s) | c''(s) \rangle$$

$$= 2 \underline{\langle c''(s) | c'(s) \rangle}$$

$$= \underline{0}$$



$\boxed{c''(s) = 0}$ $\qquad \|c''(s)\| = 0$ $\qquad \|c''(s)\| = 1$



$\|c''(s)\| = \dfrac{1}{r}$

$t \in [0, 1]$

$s \in [0, 2\pi]$

$c(s) = (\cos s, \sin s)$ $\qquad p(t) = (\cos 2\pi t, \sin 2\pi t)$

$\|c'(s)\| = 1 \qquad c'(s) = (-\sin s, \cos s) \qquad p'(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)$

$\|p'(t)\| = 2\pi$

$c''(s) = (-\cos s, -\sin s)$

$s = 2\pi t \Rightarrow t = s/2\pi$

$\langle c''(s) | c'(s) \rangle = 0 \qquad \boxed{\|c''(s)\| = 1}$

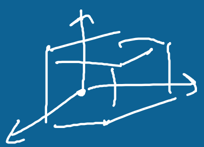
$c'(s) \; (T) \longrightarrow$ length

$c''(s) \; (N) \longrightarrow$ Curvature

$$\boxed{K(t) = p''(t) - \frac{\langle p''(t) | p'(t) \rangle \, p'(t)}{\|p'(t)\|^2}}$$

- Frenet - Serre Frames : $\{T, N, B\}$

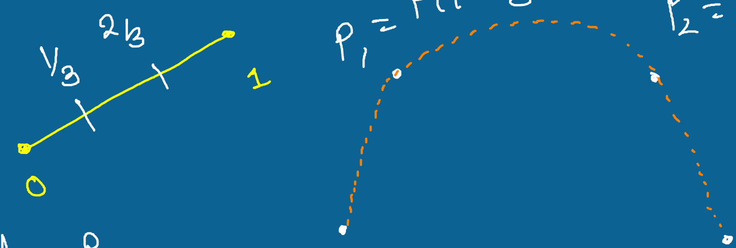
Tut - 3



12.09.2020

Ex. 1 $P(t)$, $P_0 \ldots P_3$ $t \in [0,1]$

$P(t) = at^3 + bt^2 + ct + d$



$P(0) = a.0 + b.0 + c.0 + d = P_0$

$P(1/3) = a.(1/3)^3 + b.(1/3)^2 + c.(1/3) + d = P_1$

$P(2/3) = a(2/3)^3 + b(2/3)^2 + c(2/3) + d = P_2$

$P(1) = a.1 + b.1 + c.1 + d = P_3$

$P_i \in \mathbb{R}^2$ or $\mathbb{R}^3$

$\mathbb{R}^N$

Invertible ✓

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/3^3 & 1/3^2 & 1/3 & 1 \\ 2/3^3 & 2/3^2 & 2/3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

— system of Linear equations

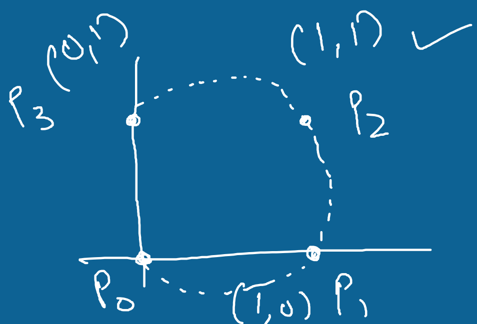
4 - unknowns, 4 - eqs

$$a \begin{bmatrix} 0 \\ 1/3^3 \\ 2/3^3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1/3^2 \\ 2/3^2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$G = \begin{bmatrix} -4.5 & 13.5 & -13.5 & 4.5 \\ 9 & -22.5 & 18 & -4.5 \\ -5.5 & 9 & -4.5 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ ~~~ Cubic interp

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix}^T \left\{ \begin{array}{l} P(t) \\ T(t)\, G\, P \end{array} \right.$$

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} G \boxed{\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}}$$ ← Interpolating Polynomial



— Plot $P(t)$

$T(t).G$ ← Blending functions

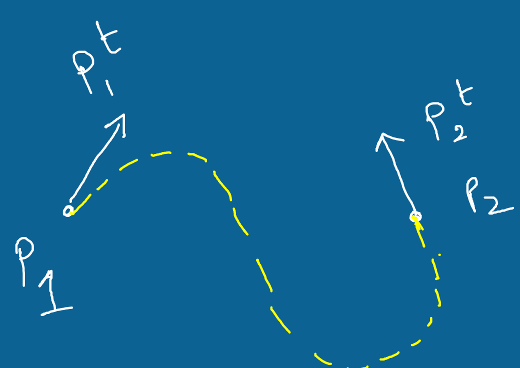
$$\left\{ \begin{array}{l} G_0(t) = (-4.5t^3 + 9t^2 - 5.5t + 1), \quad G_1(t) = (13.5t^3 - 22.5t^2 + 9t) \\ G_2(t) = (-13.5t^3 + 18t^2 - 4.5t), \quad G_3(t) = (4.5t^3 - 4.5t^2 + t) \end{array} \right\}$$

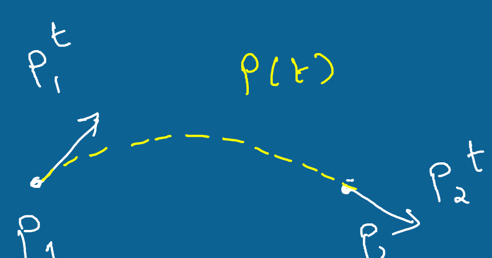
$$P(t) = \sum_{i=0}^{3} G_i(t) P_i$$

$\sum G_i(t) = 1$

$\forall t \in [0,1]$

Ex 2





$P(t=0) = P_1$ $\qquad$ $P(t=1) = P_2$

$P'(t=0) = P_1^t$ $\qquad$ $P'(t=1) = P_2^t$

$P(t) = at^3 + bt^2 + ct + d$ $\qquad P'(t) = 3at^2 + 2bt + c$

$P(0) = a.0 + b.0 + c.0 + d = P_1$

$P(1) = a.1 + b.1 + c.1 + d = P_2$

$P'(0) = 3a.0 + 2b.0 + c = P_1^t$

$P'(1) = 3a.1 + 2b.1 + c = P_2^t$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$ — 4 eq / 4 unknowns

Basis matrix

$$P(t) = [t^3 \quad t^2 \quad t \quad 1] \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{H} \begin{bmatrix} P_1 \\ P_2 \\ P_1^t \\ P_2^t \end{bmatrix}$$

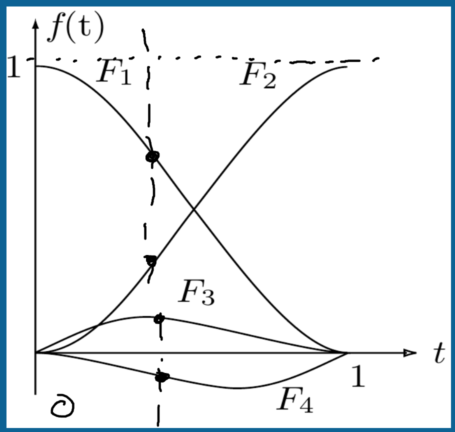
$P'(t) = [3t^2 \quad 2t \quad 1 \quad 0]$

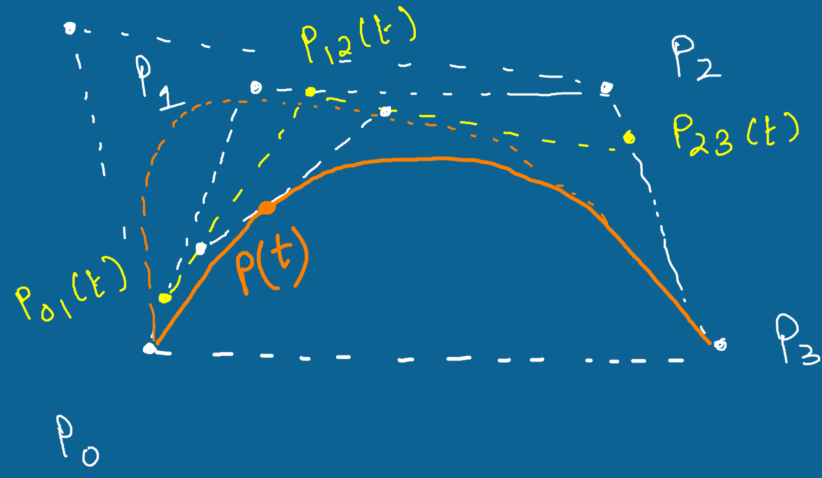
$F_1(t) = 2t^3 - 3t^2 + 1$, $F_2(t) = -2t^3 + 3t^2$, $F_3(t) = t^3 - 2t^2 + t$

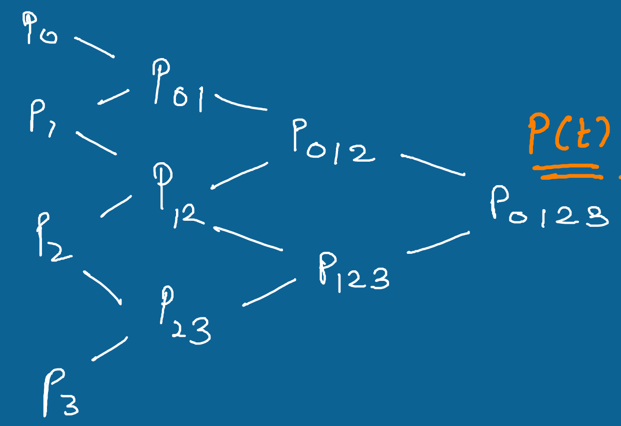
$F_4(t) = t^3 - t^2$

$$\boxed{\sum F_i(t) = 1}$$

$\forall t \in [0, 1]$

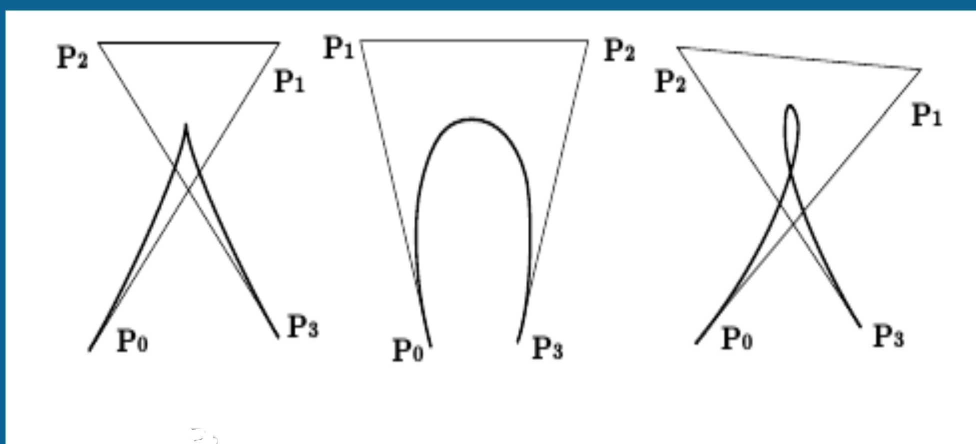


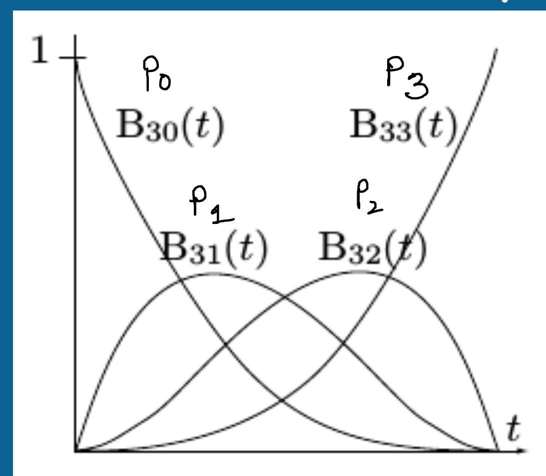




$$P_{i,i+1}(t) = (1-t)P_i + t P_{i+1}$$

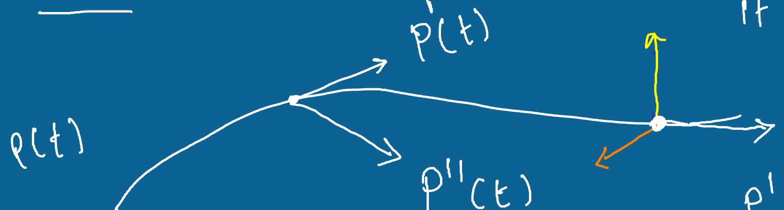
Bezier - Hermite





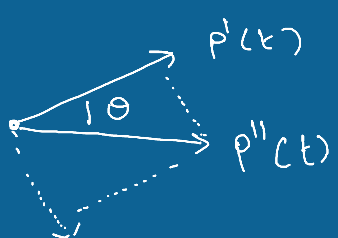
15.09.2020

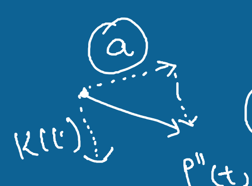
if t is not arc length parameter



not true
$p'(t) \not\perp p''(t)$

$s:[0,1] \to [0,L]$
to
$s(t) = \int_0^t \|p'(t)\|\,dt$





(a) $\langle p''(t) \mid p'(t) \rangle \dfrac{p'(t)}{\|p'(t)\|}$

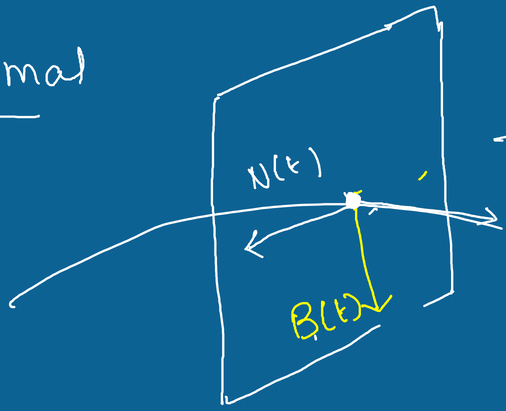
(b) $K(t) = p''(t) - \langle p''(t) \mid p'(t) \rangle \dfrac{p'(t)}{\|p'(t)\|}$

$t(s)$, $P(t)$
$\downarrow$
$\tilde{P}(s)$

$$\boxed{N(t) = \frac{K(t)}{\|K(t)\|}}$$ unit Normal / Principal Normal.

$\approx p''(s) / \|p''(s)\|$

Binormal

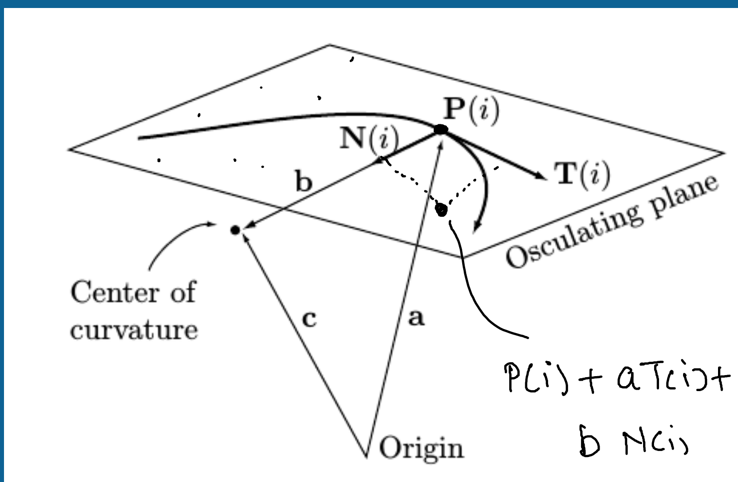


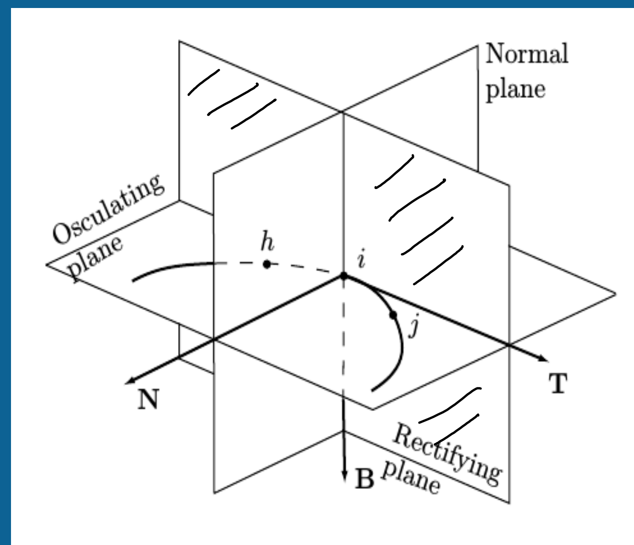
$T(t) = p'(t) / \|p'(t)\|$

$p'(t)$

$B(t) = T(t) \times N(t)$

Osculating Plane:



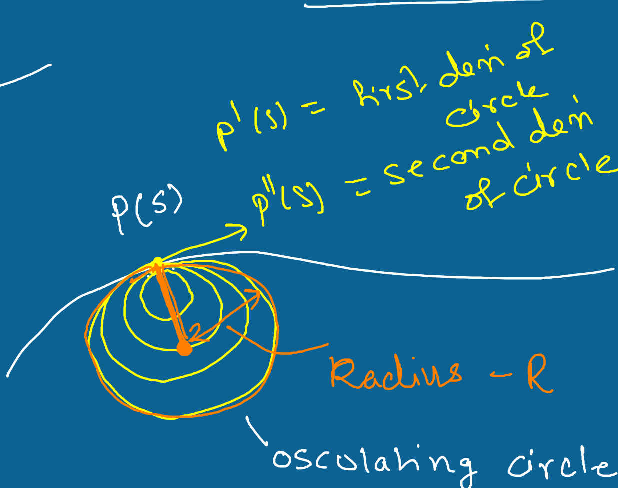


Curvature

$$\boxed{p''(s) = k \, . \, N(s)}$$

curvature

$k = 1/R$



JUST TOUCHES AND AGREES ON FIRST TWO DERIVATIVES

$$\boxed{k(t) = \frac{p'(t) \times (p''(t) \times p'(t))}{\|p'(t)\|^4}}$$

$$\boxed{k(t) = k \, . \, N(t)}$$

Exercise: $P(t)$

$P : [0,1] \to \mathbb{R}^3$

$s : [0,1] \to [0, L]$

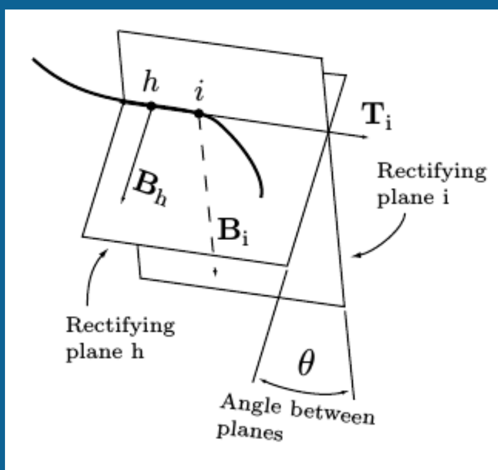
arc-length parameter

$s(t) \searrow$

$P(t)$ $t(s)$

$$\frac{d^2 P(t(s))}{ds^2} = \frac{P'(t) \times (P''(t) \times P'(t))}{\|P'(t)\|^4} \sim \frac{d^2}{ds^2} P(s) = \kappa N(s)$$

TORSION: deviation from planarity!



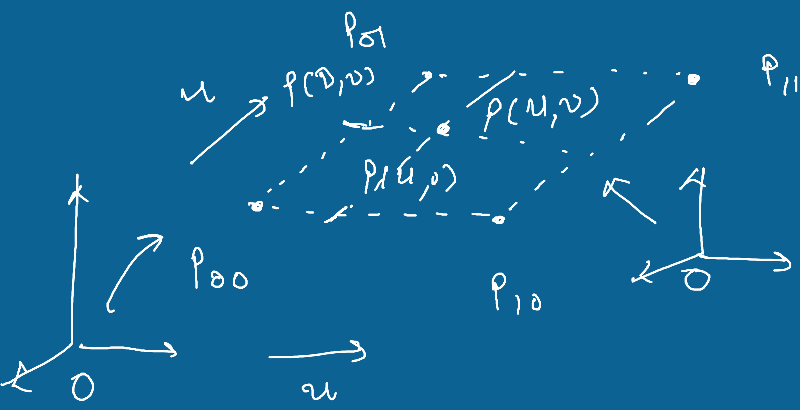
Surfaces:

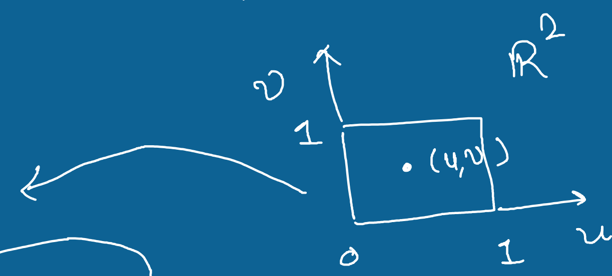
$P(u,v)$

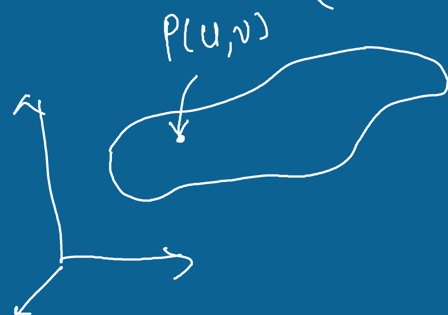
$P(u=0, v=0) = P_{00}$

$\vdots$

$P(u=1, v=1) = P_{11}$







$P: [0,1]\times[0,1] \longrightarrow \mathbb{R}^3$

$$P(u,v) = P_{00}(1-u)(1-v) + P_{01}(1-u)v + P_{10}\,u(1-v) + P_{11}\,uv$$

$P_{ij} \in \mathbb{R}^3$

$$P(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

Bilinear surface Patch.

$$= \begin{bmatrix} (1-u)P_{00} + uP_{10} & (1-u)P_{01} + uP_{11} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

$$\boxed{P(u,v) = (1-u)(1-v)P_{00} + u(1-v)P_{10} + (1-u)vP_{01} + uvP_{11}}$$

$$P(u,v) = \begin{bmatrix} u & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix}$$

$$\boxed{P(u,v) = \underbrace{U(u)}\,\underbrace{N}\,\underbrace{P}\,\underbrace{N^T}\,\underbrace{V^T(v)}}$$

Bi-(linear / quadratic / cubic) surface patch

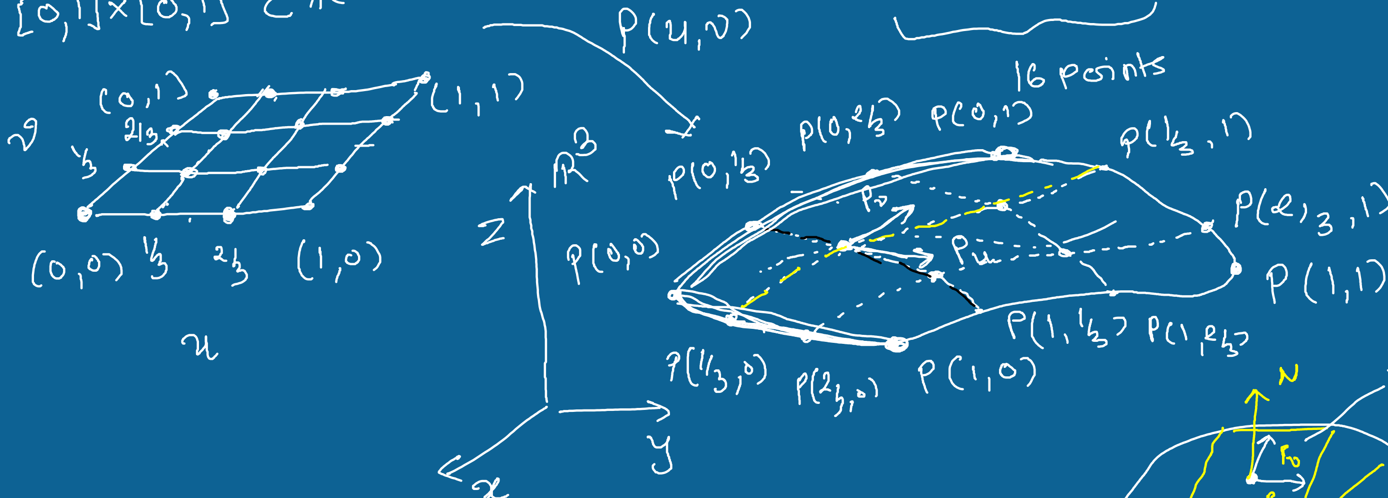
17.09.2020

Bi-cubic surface Patch:

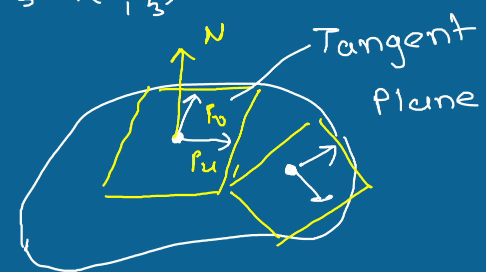
$$P(u,v) = U(u).G.P.G^T.V^T(v)$$

$$= [u^3\ u^2\ u\ 1]\ G \underbrace{\begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ \vdots & & & \\ \vdots & & & \\ P_{30} & & & P_{33} \end{bmatrix}}_{\text{16 points}} G^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

$(u,v) \in [0,1]\times[0,1] \subset \mathbb{R}^2$

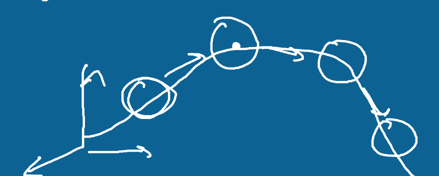


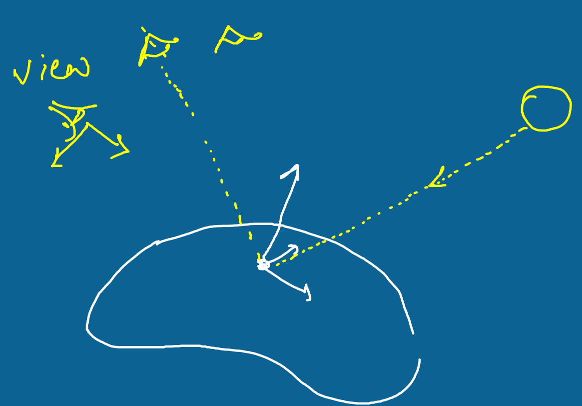
$N = P_u \times P_v$



Tangents: Two coordinate curves

$$\frac{\partial P(u,v)}{\partial u} = P_u \qquad \frac{\partial}{\partial v} P(u,v) = P_v$$





$P(\underline{u}) = U.G.P \quad , \quad P(\underline{v}) = VGP$

$$P(u) \otimes P(v) = (U.G.P)(VG\underline{P})^T$$

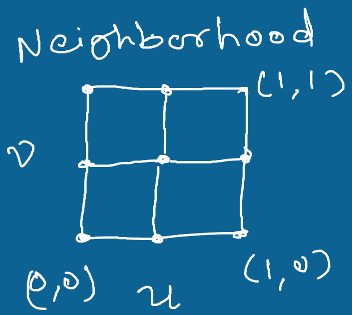
$$= UG\,PP^T\,G^T V^T$$

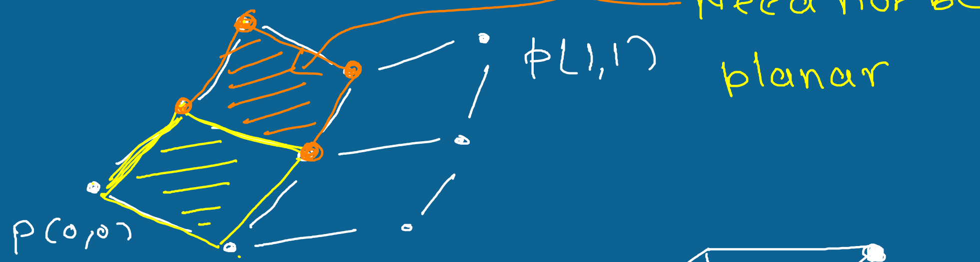
$$\boxed{P(u,v) = UGP\,G^T V^T}$$

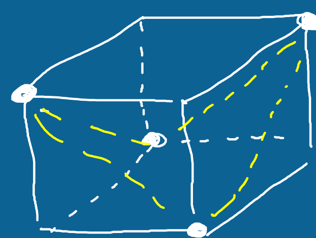
Plot Surface: Point Cloud $P(u,v)$

Connectivity / Edges & Faces

Neighborhood







Exercise

$P(u,v) \in \mathbb{R}^3$

R

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\pi/4 & \sin\pi/4 & 0 \\ -\sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotate w.r.t z-axis

$$R(t) = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$t \in [0, 2\pi]$

$t \sim \text{linspace}(0, 2\pi, 100)$

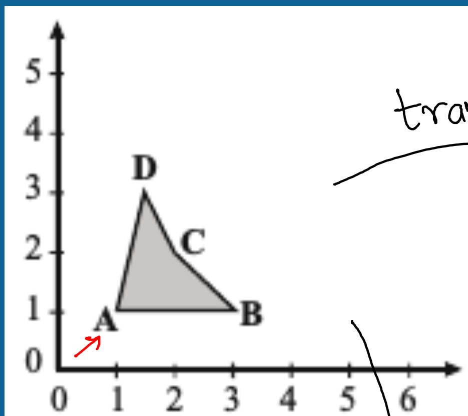
20 seconds animation

$\boxed{P(u,v,t)}$

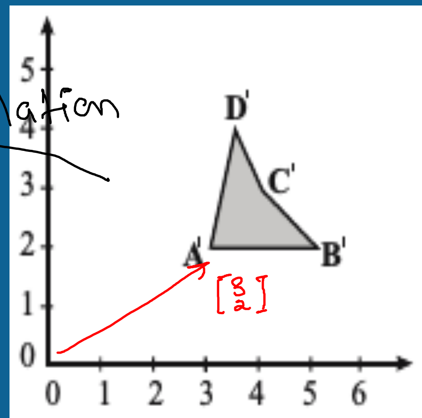
5 frames / second

19.09.2020

Transformations



translation



$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

translation

$p' = p + t$

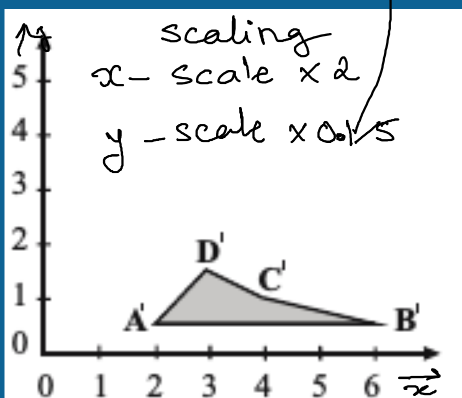
$A' = A + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$B' = B + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

scaling

x- scale x 2

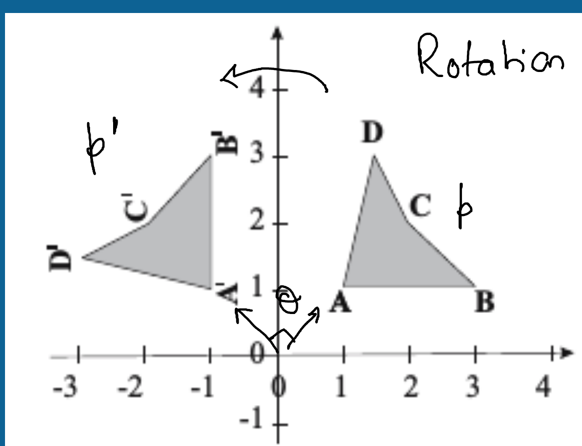
y-scale x 0.5



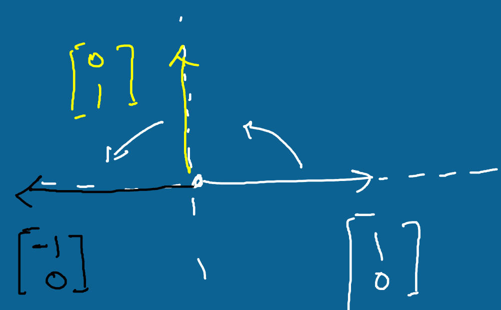
$p' = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} p$

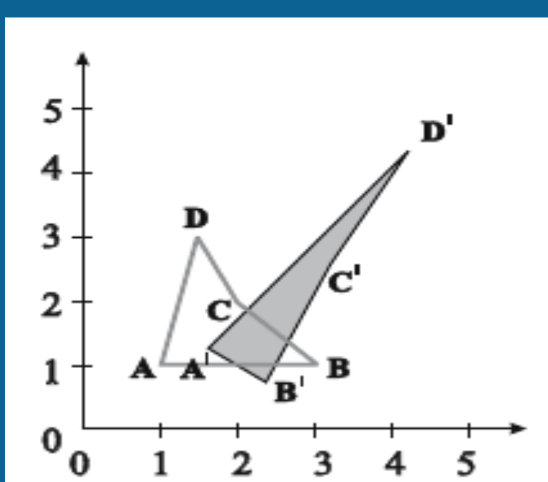
$p = \begin{bmatrix} x \\ y \end{bmatrix}$

Rotation



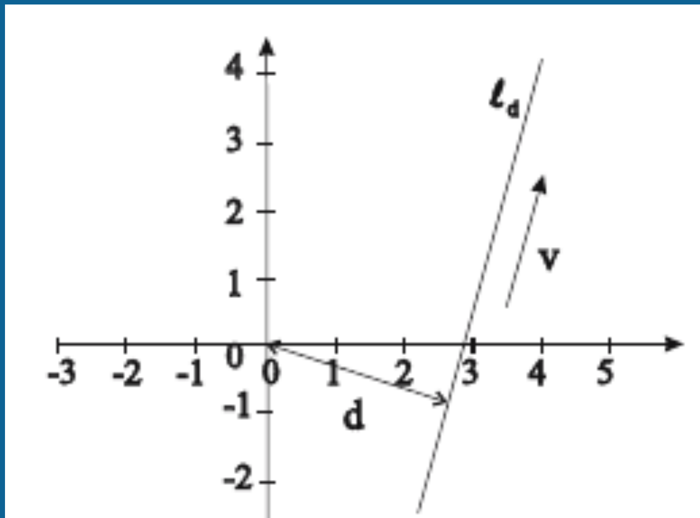
$p' = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$





Shear transformation

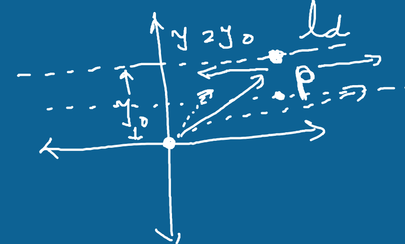
$\begin{bmatrix} 0.4 & 1.2 \\ 0.3 & 1.6 \end{bmatrix}$



$$p' = p + \overset{\text{factor}}{r}\, d\, \underline{v}$$

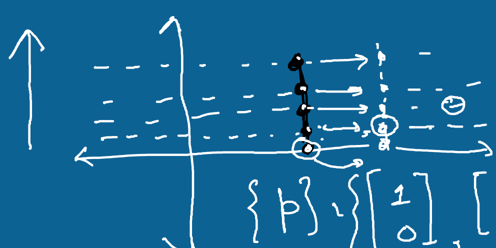
Ex. Shear in the dir of x-axis

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



"Shear about origin of factor r in direction v"

$$\boxed{p' = p + r y_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$



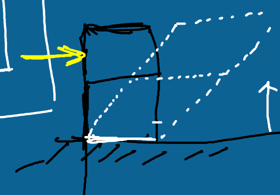
$$p' = \begin{bmatrix} x \\ y_0 \end{bmatrix} + \begin{bmatrix} r y_0 \\ 0 \end{bmatrix} = \begin{bmatrix} x + r y_0 \\ y_0 \end{bmatrix}$$

$$\{p\},\ \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$$

Shear transformation

$$\boxed{p' = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y_0 \end{bmatrix}}$$

Shear



$$\{p'\} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \{p\}$$

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \{p\}$$

Composition of Transformations :

$$\boxed{R_1 R_2 = R_2 R_1}$$

$$\{R, t, s, r\}$$

$$t(R(p)) \ ?\ R(t(p))$$

$$\underline{s}(\underline{R}(p)) \ ?\ R(s(p))$$

$R : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad s : \mathbb{R}^2 \to \mathbb{R}^2$

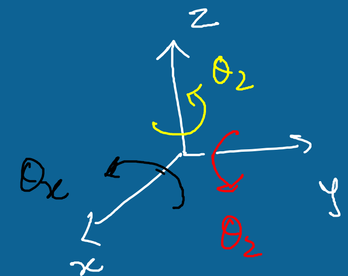
$t : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad r : \mathbb{R}^2 \to \mathbb{R}^2$

"Matrix multiplication"

Rotation in $\mathbb{R}^3$

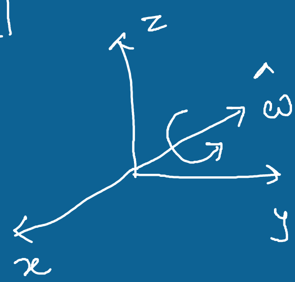
$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}, \quad R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta_x \\ 0 & 1 & 0 \\ \sin\theta_x & 0 & \cos\theta_x \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

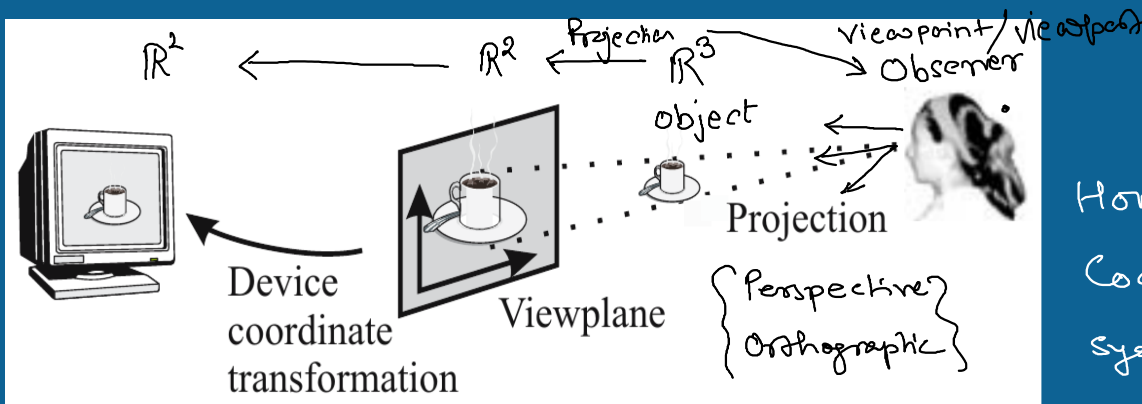


Rotate w.r.t. ω

axis of rot $\hat{\omega}$



"Rodrigue's Formula"



Homogeneous Coordinate system