2

Continuous Dynamics — Exercises

1. A **tuning fork**, shown in Figure 2.1, consists of a metal finger (called a **tine**) that is displaced by striking it with a hammer. After being displaced, it vibrates. If the tine has no friction, it will vibrate forever. We can denote the displacement of the tine after being struck at time zero as a function $y \colon \mathbb{R}_+ \to \mathbb{R}$. If we assume that the initial displacement introduced by the hammer is one unit, then using our knowledge of physics we can determine that for all $t \in \mathbb{R}_+$, the displacement satisfies the differential equation

$$\ddot{y}(t) = -\omega_0^2 y(t)$$

where ω_0^2 is a constant that depends on the mass and stiffness of the tine, and where $\ddot{y}(t)$ denotes the second derivative with respect to time of y. It is easy to verify that y given by

$$\forall\, t \in \mathbb{R}_+, \quad y(t) = \cos(\omega_0 t)$$

is a solution to the differential equation (just take its second derivative). Thus, the displacement of the tuning fork is sinusoidal. If we choose materials for the tuning fork so that $\omega_0 = 2\pi \times 440$ radians/second, then the tuning fork will produce the tone of A-440 on the musical scale.

(a) Is $y(t) = \cos(\omega_0 t)$ the only solution? If not, give some others.

Solution: The following is a solution for any constant α:

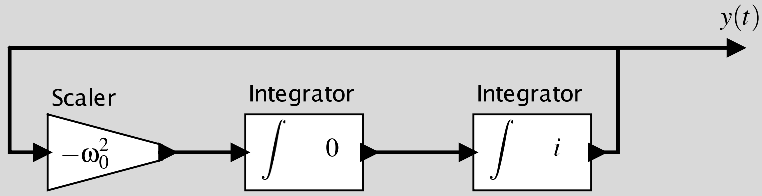
$$y(t) = \alpha \cos(\omega_0 t)\ .$$

(b) Assuming the solution is $y(t) = \cos(\omega_0 t)$, what is the initial displacement?

Solution: $y(0) = \cos(\omega_0 \times 0) = 1$.

(c) Construct a model of the tuning fork that produces y as an output using generic actors like Integrator, adder, scaler, or similarly simple actors. Treat the initial displacement as a parameter. Carefully label your diagram.

Solution: The following model will do the job:



Here, i is the initial displacement.

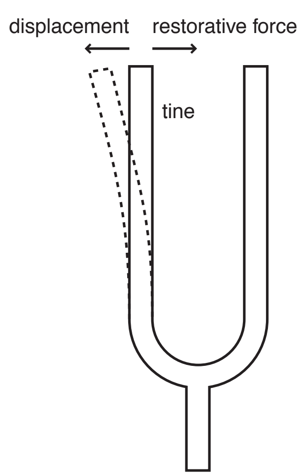


Figure 2.1: A tuning fork.

3. This exercise studies linearity.

   (a) Show that the helicopter model defined in Example 2.1 is linear if and only if the initial angular velocity $\dot{\theta}_y(0) = 0$.

Solution: The input T_y and output $\dot{\theta}_y$ are related by

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau.$$

First, we need to show that if $\dot{\theta}_y(0) = 0$, then superposition applies. Then we need to show if $\dot{\theta}_y(0) \neq 0$, superposition does not apply. For the first problem, if $\dot{\theta}_y(0) = 0$ then we have

$$\dot{\theta}_y(t) = \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau.$$

Suppose the input is given by

$$T_y = aT_1 + bT_2$$

where a and b are real numbers and T_1 and T_2 are signals. Then the output is

$$\begin{aligned}
\dot{\theta}_y(t) &= \frac{1}{I_{yy}} \int_0^t (aT_1(\tau) + bT_2(\tau)) d\tau \\
&= \frac{a}{I_{yy}} \int_0^t T_1(\tau) d\tau + \frac{b}{I_{yy}} \int_0^t T_1(\tau) d\tau.
\end{aligned}$$

It is easy to see that the first term is a times what the output would be if the input were only T_1, and the second term is b times what the output would be if the input were only T_2. That is, if the system function is S, the output is

$$\dot{\theta}_y(t) = a(S(T_1))(t) + b(S(T_2))(t).$$

Next, assume that $\dot{\theta}_y(0) \neq 0$. With the same input as above, we get the output

$$\begin{aligned}
\dot{\theta}_y(t) &= \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t (aT_1(\tau) + bT_2(\tau)) d\tau \\
&= \dot{\theta}_y(0) + \frac{a}{I_{yy}} \int_0^t T_1(\tau) d\tau + \frac{b}{I_{yy}} \int_0^t T_1(\tau) d\tau.
\end{aligned}$$

We can now see that the output is

$$\dot{\theta}_y(t) = a(S(T_1))(t) + b(S(T_2))(t) - \dot{\theta}_y(0),$$

so superposition does not apply.

   (b) Show that the cascade of any two linear actors is linear.

Solution: Given an actor with function S_1 and another with function S_2, the cascade composition is an actor with function $S_1 \circ S_2$, the composition of the two functions. If S_1 and S_2 both satisfy the superposition property, then

$$\begin{aligned} S_2(S_1(ax_1 + bx_2)) &= S_2(aS_1(x_1) + bS_1(x_2)) \\ &= aS_2(S_1(x_1)) + bS_2(S_1(x_2)). \end{aligned}$$

Hence, the composition also satisfies superposition.

(c) Augment the definition of linearity so that it applies to actors with two input signals and one output signal. Show that the adder actor is linear.

Solution: A system model $S\colon X_1 \times X_2 \to Y$, where X_1, X_2, and Y are sets of signals, is linear if it satisfies the superposition property:

$$\forall\, x_1, x_1' \in X_1 \text{ and } \forall\, x_2, x_2' \in X_2 \text{ and } \forall\, a, b \in \mathbb{R},$$
$$S(ax_1 + bx_1', ax_2 + bx_2') = aS(x_1, x_2) + bS(x_1', x_2').$$

The adder component is given by

$$S(x_1, x_2) = x_1 + x_2.$$

Hence

$$\begin{aligned} S(ax_1 + bx_1', ax_2 + bx_2') &= ax_1 + bx_1' + ax_2 + bx_2' \\ &= a(x_1 + x_2) + b(x_1' + x_2') \\ &= aS(x_1, x_2) + bS(x_1', x_2'). \end{aligned}$$

4. Consider the helicopter of Example 2.1, but with a slightly different definition of the input and output. Suppose that, as in the example, the input is $T_y \colon \mathbb{R} \to \mathbb{R}$, as in the example, but the output is the position of the tail relative to the main rotor shaft. Specifically, let the x-y plane be the plane orthogonal to the rotor shaft, and let the position of the tail at time t be given by a tuple $((x(t), y(t))$. Is this model LTI? Is it BIBO stable?

Solution: In this case, the system can be modeled as a function with two output signals,

$$S\colon (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})^2,$$

where

$$(S(T_y))(t) = (x(t), y(t)),$$

where $(x(t), y(t))$ is the position of the tail in the x-y plane. This model is clearly not linear. If the input torque doubles, for example, the output values will not double. In fact, the output values are constrained to lie on a circle centered at the origin, regardless of the input. For this reason, the model is BIBO stable. The output is always bounded. Thus, while our previous model was linear and unstable, this one is nonlinear and stable. Which model is more useful?

6. A DC motor produces a torque that is proportional to the current through the windings of the motor. Neglecting friction, the net torque on the motor, therefore, is this torque minus the torque applied by whatever load is connected to the motor. Newton's second law (the rotational version) gives

$$k_T i(t) - x(t) = I \frac{d}{dt}\omega(t), \tag{2.1}$$

where k_T is the motor torque constant, $i(t)$ is the current at time t, $x(t)$ is the torque applied by the load at time t, I is the moment of inertia of the motor, and $\omega(t)$ is the angular velocity of the motor.

(a) Assuming the motor is initially at rest, rewrite (2.1) as an integral equation.

Solution: Integrating both sides, we get

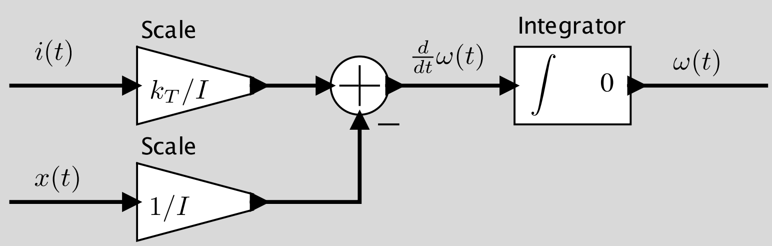
$$\int_0^t (k_T i(\tau) - x(\tau))d\tau = I\omega(t).$$

Solving for $\omega(t)$ we get

$$\omega(t) = \frac{1}{I}\int_0^t (k_T i(\tau) - x(\tau))d\tau.$$

(b) Assuming that both x and i are inputs and ω is an output, construct an actor model (a block diagram) that models this motor. You should use only primitive actors such as integrators and basic arithmetic actors such as scale and adder.

Solution: A solution is shown below:



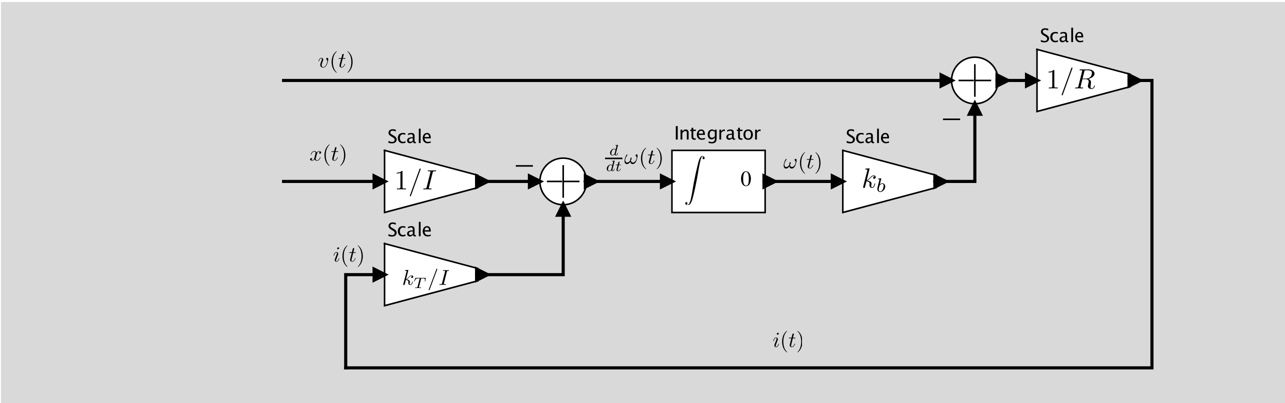
(c) In reality, the input to a DC motor is not a current, but is rather a voltage. If we assume that the inductance of the motor windings is negligible, then the relationship between voltage and current is given by

$$v(t) = Ri(t) + k_b\omega(t),$$

where R is the resistance of the motor windings and k_b is a constant called the motor back electromagnetic force constant. The second term appears because a rotating motor also functions as an electrical generator, where the voltage generated is proportional to the angular velocity.

Modify your actor model so that the inputs are v and x rather than i and x.

Solution: A solution is shown below:



Scale
$v(t)$
$1/R$
$-$
$x(t)$
Scale
$1/I$
$-$
$\frac{d}{dt}\omega(t)$
Integrator
$\int$ 0
$\omega(t)$
Scale
k_b
Scale
$i(t)$
k_T/I
$i(t)$

7 Sensors and Actuators — Exercises

1. Show that the composition $f \circ g$ of two affine functions f and g is affine.

Solution: Assume

$$f(x) = a_1 x + b_1$$

and

$$g(x) = a_2 x + b_2.$$

Then

$$(f \circ g)(x) = a_1(a_2 x + b_2) + b_1 = (a_1 a_2)x + (a_1 b_2 + b_1),$$

which is an affine function

$$(f \circ g)(x) = a_3 x + b_3,$$

where $a_3 = a_1 a_2$ and $b_3 = a_1 b_2 + b_1$.

2. The dynamic range of human hearing is approximately 100 decibels. Assume that the smallest difference in sound levels that humans can effectively discern is a sound pressure of about 20 μPa (micropascals).

   (a) Assuming a dynamic range of 100 decibels, what is the sound pressure of the loudest sound that humans can effectively discriminate?

Solution: We have

$$100 = 20\log_{10}\left(\frac{H-L}{p}\right)$$

where p is 20 μPa. Assume $L = 0$ (the lowest sound pressure represents no sound at all) and solve for H to get

$$H = p10^{100/20} = 2Pa.$$

   (b) Assume a perfect microphone with a range that matches the human hearing range. What is the minimum number of bits that an ADC should have to match the dynamic range of human hearing?

Solution: At 6 dB per bit, to match 100dB, we need at least 100/6 = 16.7 bits. Since fractional bits are not possible, we need at least 17 bits.

3. The following questions are about how to determine the function

$$f\colon (L,H) \to \{0,\ldots,2^B-1\},$$

for an accelerometer, which given a proper acceleration x yields a digital number $f(x)$. We will assume that x has units of "g's," where 1g is the acceleration of gravity, approximately $g = 9.8\text{meters}/\text{second}^2$.

(a) Let the bias $b \in \{0,\ldots,2^B-1\}$ be the output of the ADC when the accelerometer measures no proper acceleration. How can you measure b?

Solution: Place the accelerometer horizontally so that there is no component of gravity along the axis being measured. In theory, you could also put the accelerometer in free fall in a vacuum, but this would require a rather complicated experimental setup, and it would also require that the accelerometer not be twisting while it falls.

(b) Let $a \in \{0,\ldots,2^B-1\}$ be the *difference* in output of the ADC when the accelerometer measures 0g and 1g of acceleration. This is the ADC conversion of the sensitivity of the accelerometer. How can you measure a?

Solution: Place the accelerometer at rest so that gravity is along the axis being measured, then subtract b.

(c) Suppose you have measurements of a and b from parts (3b) and (3a). Give an affine function model for the accelerometer, assuming the proper acceleration is x in units of g's. Discuss how accurate this model is.

Solution: The affine function model is

$$f(x) = ax + b.$$

This function has two sources of inaccuracy. First, $f(x)$ can only take on integer values in the set $\{0,\ldots,2^B-1\}$, so there will be quantization errors. Second, any proper acceleration outside the measurable range will be saturated at either 0 or 2^B-1.

(d) Given a measurement $f(x)$ (under the affine model), find x, the proper acceleration in g's.

Solution:

$$x = \frac{f(x)-b}{a}.$$

(e) The process of determining a and b by measurement is called **calibration** of the sensor. Discuss why it might be useful to individually calibrate each particular accelerometer, rather than assume fixed calibration parameters a and b for a collection of accelerometers.

Solution: Sensors vary from device to device due to manufacturing variability, so even accelerometers with identical designs may exhibit different calibration parameters.

(f) Suppose you have an ideal 8-bit digital accelerometer that produces the value $f(x) = 128$ when the proper acceleration is 0g, value $f(x) = 1$ when the proper acceleration is 3g to the right, and value

$f(x) = 255$ when the proper acceleration is 3g to the left. Find the sensitivity a and bias b. What is the dynamic range (in decibels) of this accelerometer? Assume the accelerometer never yields $f(x) = 0$.

Solution: The sensitivity is $a = 127/3$ and the bias is $b = 128$. The precision is $p = 3/127 \approx$ 0.024g. The range is given by $H = 3$g and $L = -3$g. The dynamic range is therefore

$$D_{dB} = 20\log_{10}(6/0.024) = 48\text{dB}.$$