Filters

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Electric filters

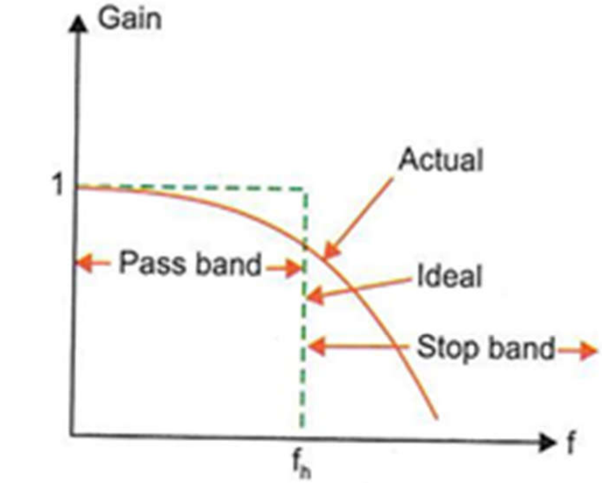
Filter: Which allows/pass the input signals to the output in desired range of frequencies

- Passive filters (R, L, C)
  - Input signal is not amplified
- Active filters (transistors, op-amps)
  - Input signal is amplified

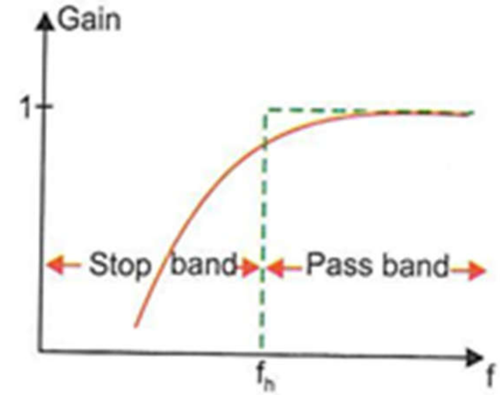
Filter classification

- Low pass filter (LPF)
- High pass filter (HPF)
- Band pass filter (BPF)
- Band reject filter/Band stop filter (BSF)

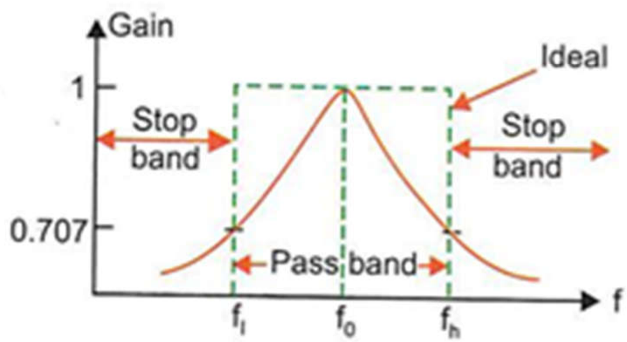
Frequency response of various filters



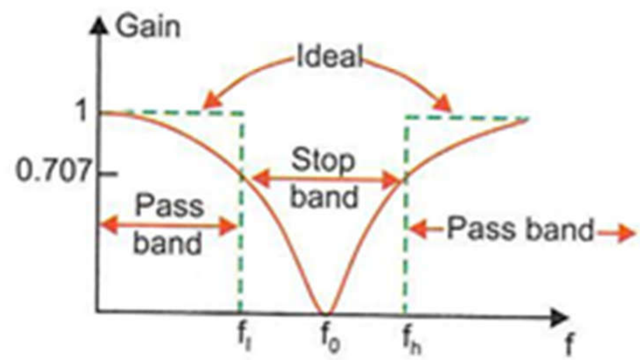
(a) **LPF**



(b) **HPF**

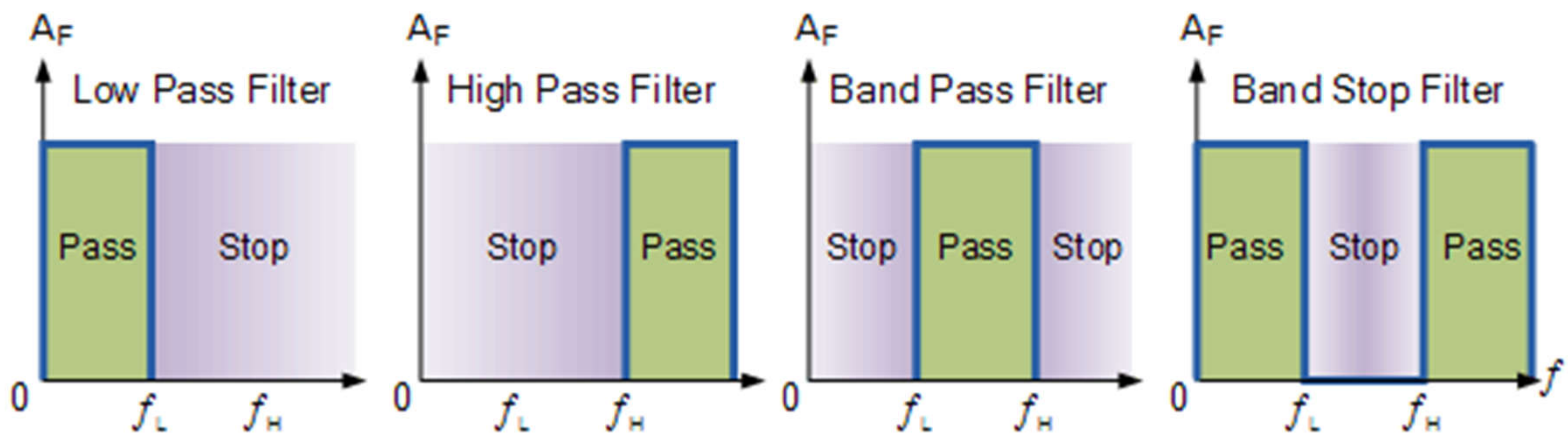


(c) **BPF**



(d) **BSF**

Ideal Filter Response Curves

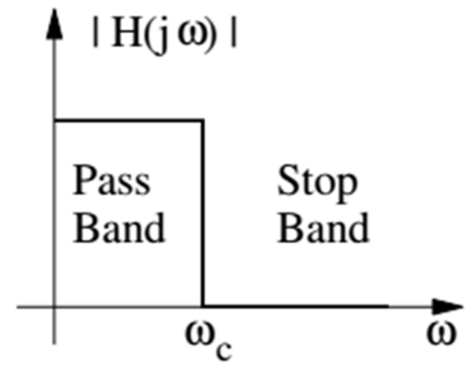


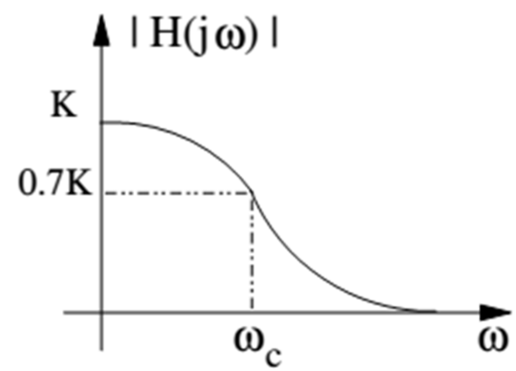
Passive Low-Pass RL filter

- Transfer function of RL LPF

$$V_o = \frac{R}{R + j\omega L} V_i \quad \rightarrow \quad H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R + j\omega L} = \frac{1}{1 + j(\omega L/R)}$$

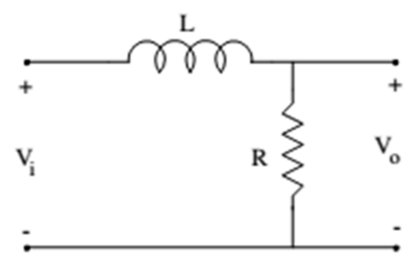
$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega L/R)^2}}$$





Cut-off frequency

- Ideal filter: The frequency between the pass- and-stop bands is called the cut-off frequency (ω_c)
- Practical filter: The frequency at which the magnitude $|H(j\omega)|$ is reduced to $1/\sqrt{2}$ (=0.7) times the maximum magnitude



Cut-off frequency of RL LPF

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}}$$

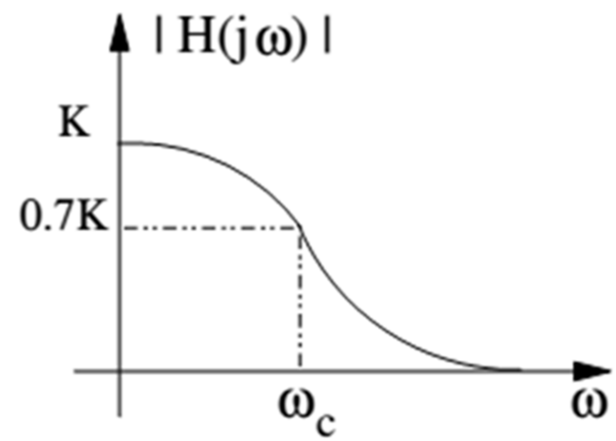
$$At\ f = 0\,Hz,$$

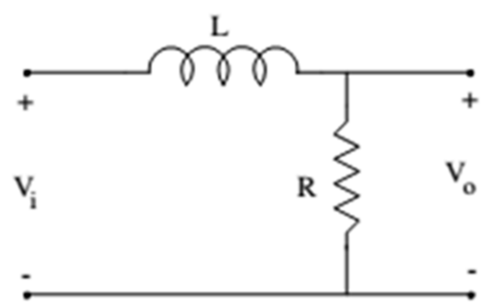
$$\left|H(j\omega)\right| = 1$$

$$At\ \omega = \omega_c, \left|H(j\omega)\right| = 1/\sqrt{2}$$

$$\frac{1}{\sqrt{1+\left(\frac{\omega_c L}{R}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\omega_c = \frac{R}{L}\ and\ H(j\omega) = \frac{1}{1+j\left(\frac{\omega}{\omega_c}\right)}$$





Bode Plots and Decibel

The voltage transfer function of a two-port network (and/or the ratio of output to input powers) is usually expressed in Bel:

$$\text{Number of Bels} = \log_{10}\left(\frac{P_o}{P_i}\right) \quad \text{or} \quad \text{Number of Bels} = 2\log_{10}\left|\frac{V_o}{V_i}\right|$$

because $P \propto V^2$. Bel is a large unit and decibel (dB) is usually used:

$$\text{Number of decibels} = 20\log_{10}\left|\frac{V_o}{V_i}\right| \quad \text{or} \quad \left|\frac{V_o}{V_i}\right|_{dB} = 20\log_{10}\left|\frac{V_o}{V_i}\right|$$

If several two-port network are placed in a cascade (output of one is attached to the input of the next), the overall transfer function, H, is equal to the product of all transfer functions

$$|H(j\omega)| = |H_1(j\omega)| \times |H_2(j\omega)| \times \ldots$$

$$20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega)| + 20\log_{10}|H_2(j\omega)| + \ldots$$

$$|H(j\omega)|_{dB} = |H_1(j\omega)|_{dB} + |H_2(j\omega)|_{dB} + \ldots$$

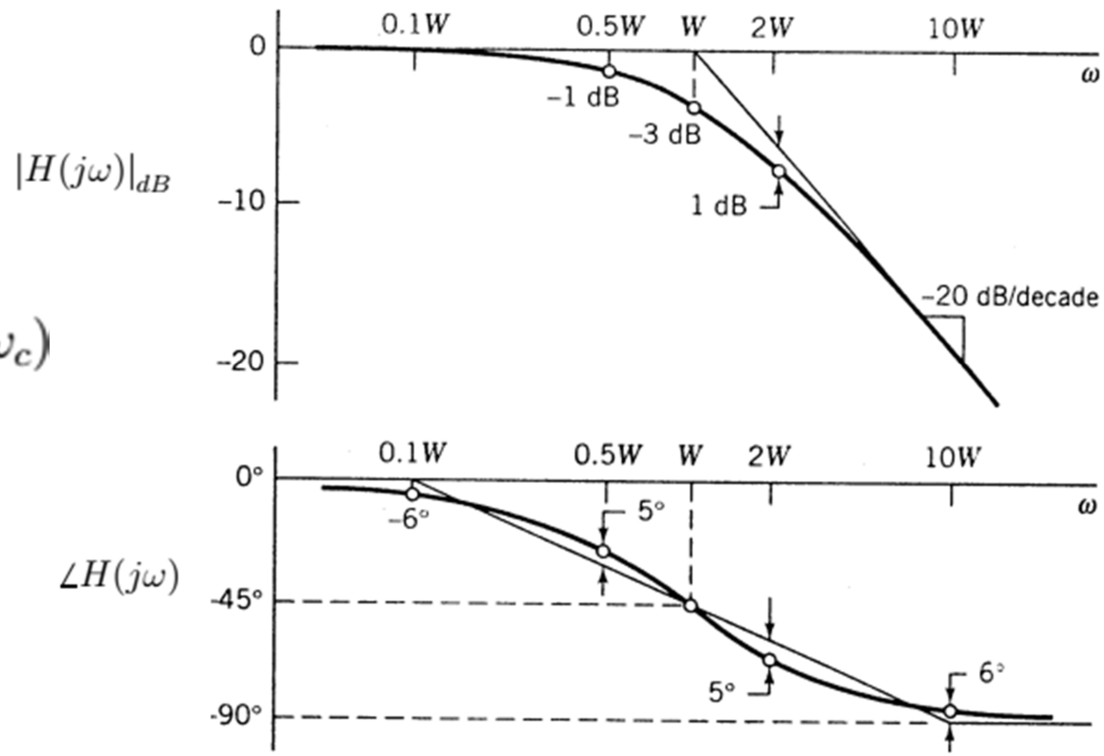
Bode plot

Magnitude vs. frequency and phase vs. frequency in a semi-log format

$$Magnitude = \left|H(j\omega)\right|_{dB}$$

$$Phase = \angle H(j\omega)$$

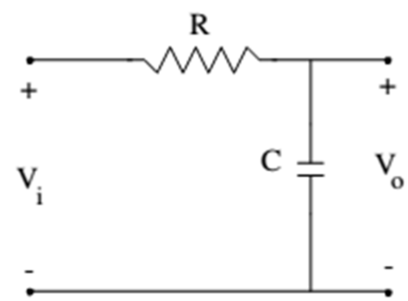
$$\angle H(j\omega) = -\tan^{-1}(\omega/\omega_c)$$



RC Low Pass Filter Circuit

$$V_o = \frac{1/(j\omega C)}{R + 1/(j\omega C)} V_i = \frac{1}{1 + j(\omega RC)} V_i$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$



Low Pass Filter Example

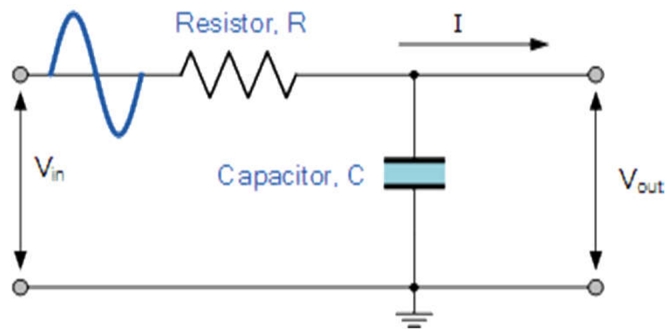
A Low Pass Filter circuit consisting of a resistor of 47 kΩ in series with a capacitor of 47 nF is connected across a 10 V sinusoidal supply. Calculate the output voltage (V_{out}) at a frequency of 100 Hz, 200 Hz and again at frequency of 10kHz, 11 kHz.

$$X_c = \frac{1}{2\pi fC}$$

$$Z = R - jX_c$$

$$|Z| = \sqrt{R^2 + X_c^2}$$

$$V_{out} = V_{in}\frac{X_c}{\sqrt{R^2 + X_c^2}} = V_{in}\frac{X_c}{Z}$$



Low Pass Filter Example

Voltage Output at a Frequency of 100Hz.

$$Xc = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 100 \times 47 \times 10^{-9}} = 33{,}863\Omega$$

$$V_{OUT} = V_{IN} \times \frac{Xc}{\sqrt{R^2 + X_C^2}} = 10 \times \frac{33863}{\sqrt{4700^2 + 33863^2}} = 9.9v$$

Voltage Output at a Frequency of 10,000Hz (10kHz).

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 10{,}000 \times 47 \times 10^{-9}} = 338.6\Omega$$

$$V_{OUT} = V_{IN} \times \frac{X_C}{\sqrt{R^2 + X_C^2}} = 10 \times \frac{338.6}{\sqrt{4700^2 + 338.6^2}} = 0.718v$$

Low Pass Filter Example

Cut-off Frequency and Phase Shift

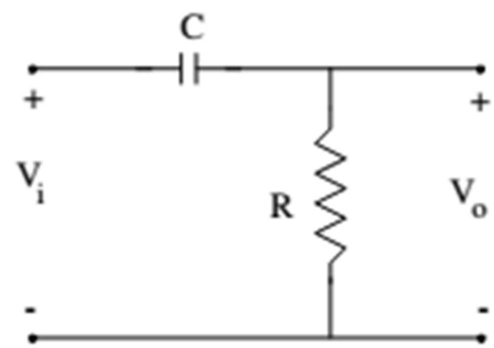
$$fc = \frac{1}{2\pi RC} = \frac{1}{2\pi \times 4700 \times 47 \times 10^{-9}} = 720\text{Hz}$$

$$\text{Phase Shift } \varphi = \text{-arctan } (2\pi f RC)$$

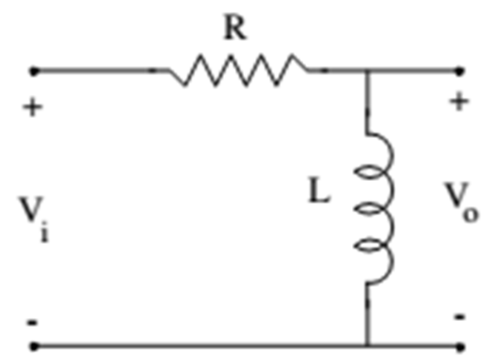
the cut-off frequency (ƒc) is given as 720Hz with an output voltage of 70.7% of the input voltage value and a phase shift angle of -45^{o}

High-pass RC filter

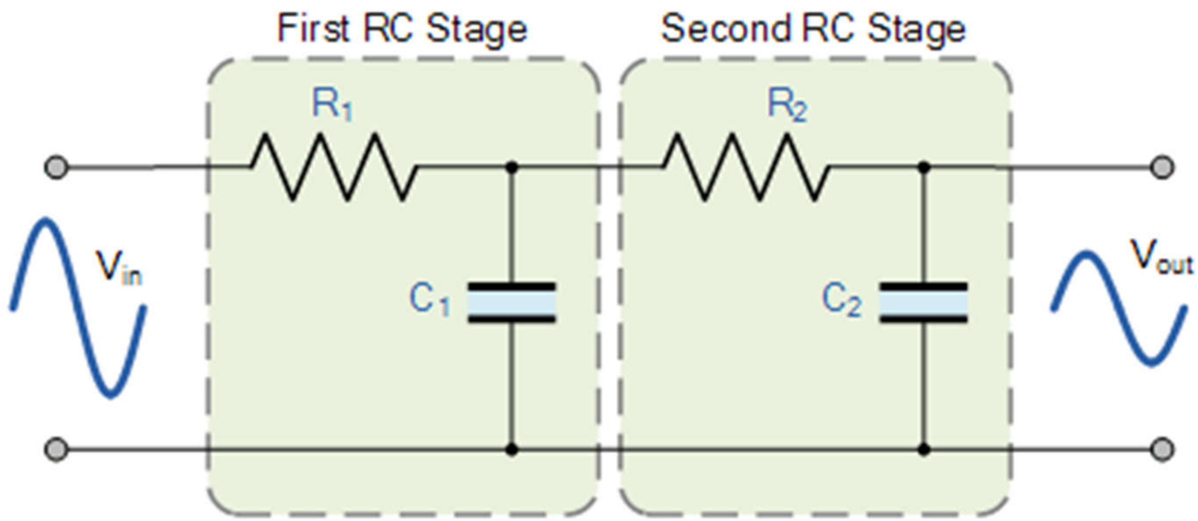
$$H(j\omega) = \frac{V_o}{V_i} = \frac{R}{R + 1/(j\omega C)} = \frac{1}{1 - j(1/\omega RC)}$$



High-pass RL filters

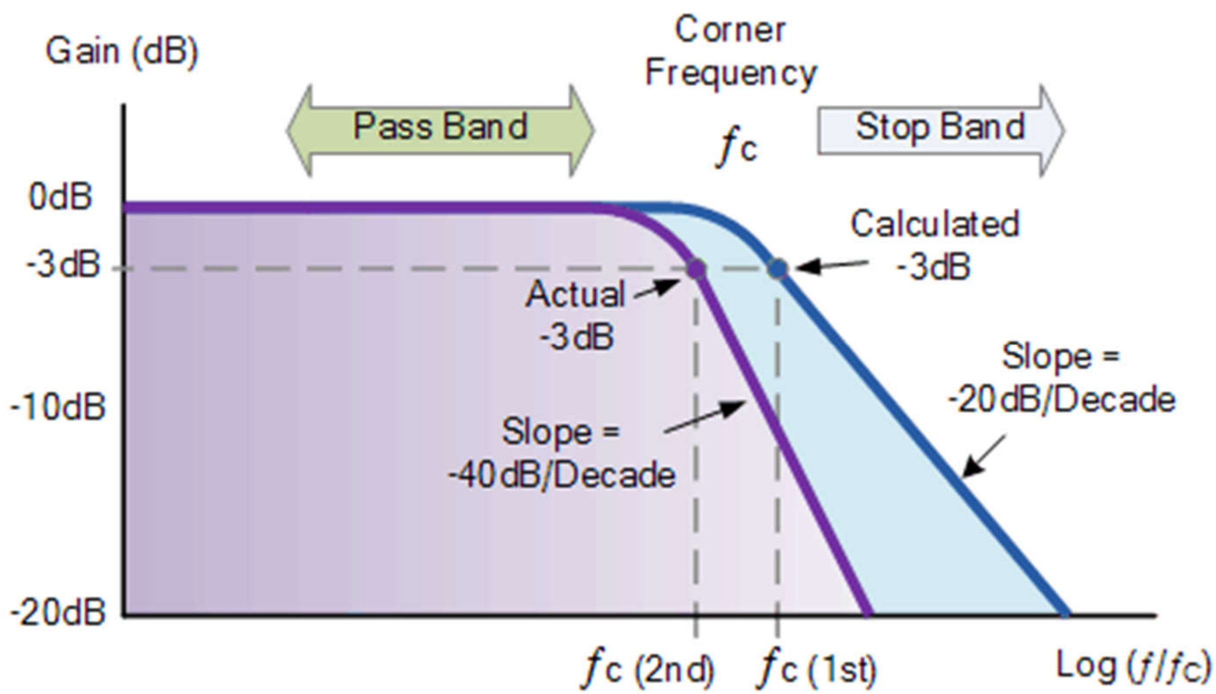


Second-order Low Pass Filter

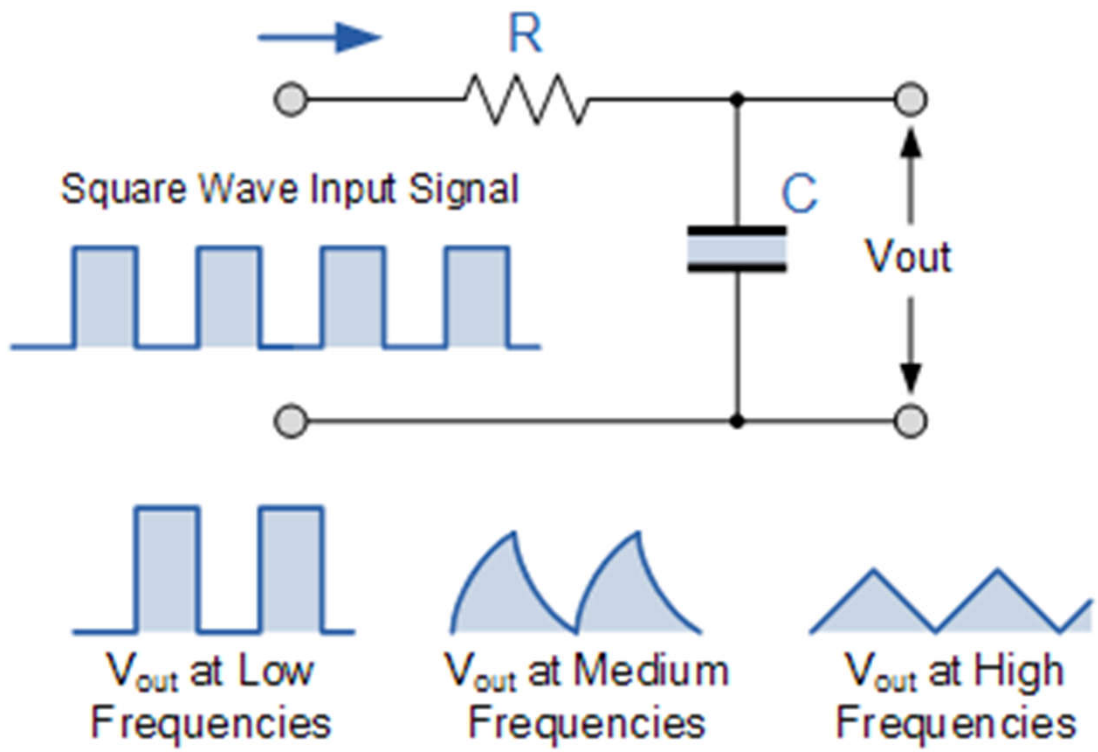


If -20dB/decade angle of the slope is not enough to remove an unwanted signal, then two stages of filtering can be used

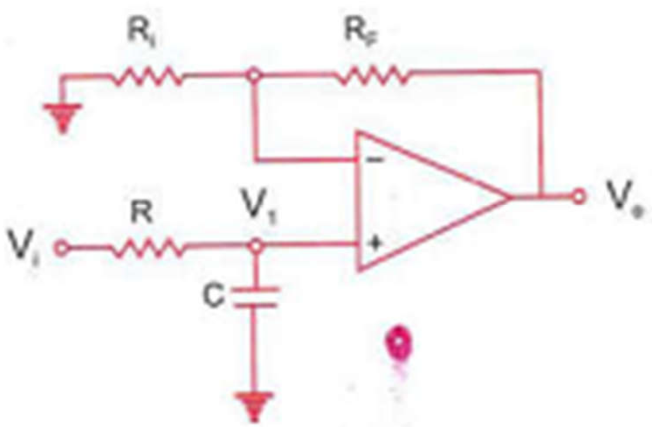
Frequency Response of a 2nd-order Low Pass Filter



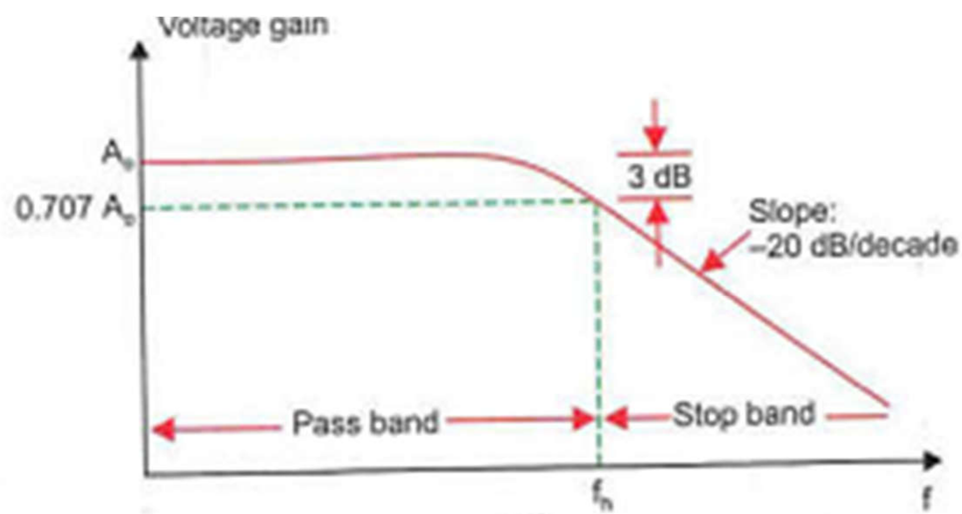
The RC Integrator Circuit



First order low pass active filter



(a)



(b)

First order low pass active filter

The voltage V_1 across the capacitor C in the s-domain is

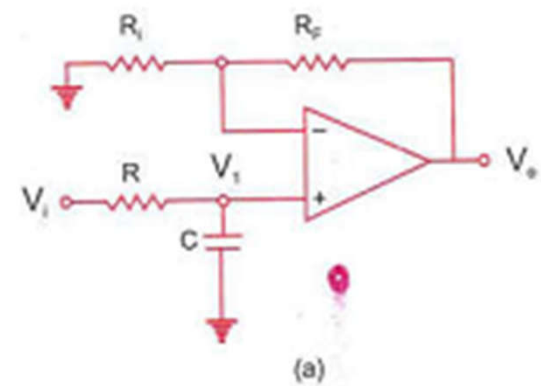
$$V_1(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_i(s)$$

So, $$\frac{V_1(s)}{V_i(s)} = \frac{1}{RCs+1}$$

where $V(s)$ is the Laplace transform of v in time domain.

The closed loop gain A_o of the op-amp is,

$$A_o = \frac{V_o(s)}{V_1(s)} = \left(1 + \frac{R_F}{R_i}\right)$$



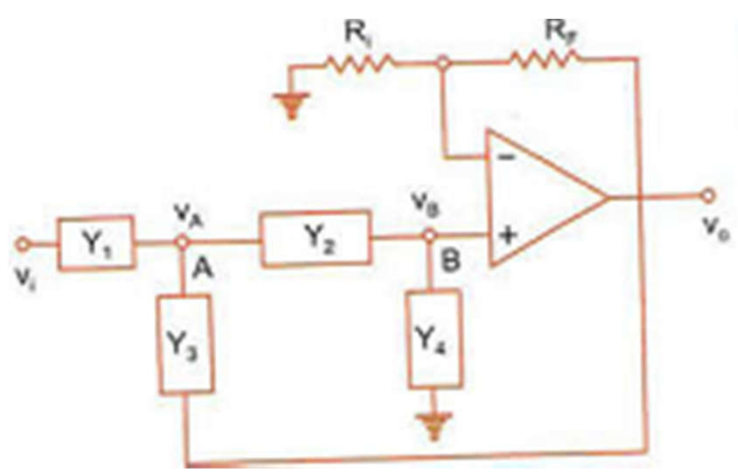
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{V_o(s)}{V_1(s)} \cdot \frac{V_1(s)}{V_i(s)} = \frac{A_o}{RCs+1}$$

Let $\omega_h = \frac{1}{RC}$

Therefore, $$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{A_o}{\frac{s}{\omega_h}+1} = \frac{A_o \omega_h}{s+\omega_h}$$

Second order active filter

- Sallen-key filter
- Two RC pairs
- -40 dB/decade



$$v_o = \left(1 + \frac{R_F}{R_i}\right) v_B = A_o v_B$$

KCL at node A

$$v_i Y_1 = v_A (Y_1 + Y_2 + Y_3) - v_o Y_3 - v_B Y_2$$

$$v_i Y_1 = v_A (Y_1 + Y_2 + Y_3) - v_o Y_3 - \frac{v_o}{A_o} Y_2$$

KCL at node B

$$v_A Y_2 = v_B (Y_2 + Y_4) = \frac{v_o}{A_o} (Y_2 + Y_4)$$

$$v_A = \frac{v_o}{A_o Y_2} (Y_2 + Y_4)$$

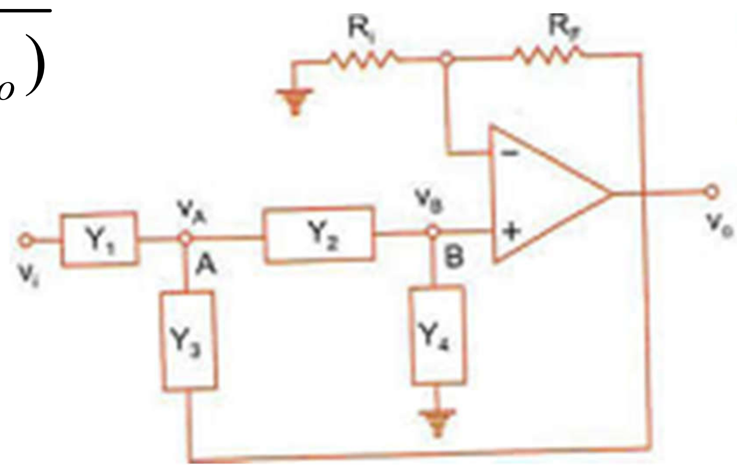
Second order active filter

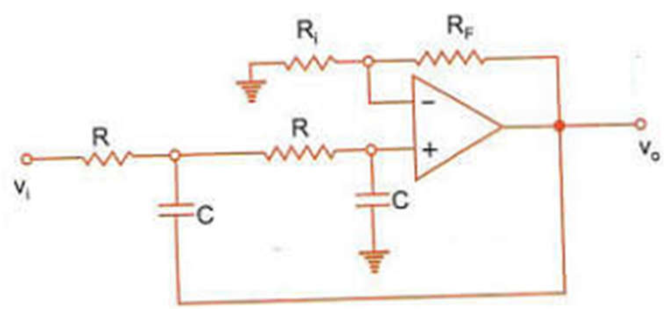
$$\frac{v_o}{v_i} = \frac{A_o Y_1 Y_2}{Y_1 Y_2 + Y_4 (Y_1 + Y_2 + Y_3) + Y_2 Y_3 (1 - A_o)}$$

$$Y_1 = Y_2 = 1/R, Y_3 = Y_4 = sC$$

$$Transfer\ \ function\, H(s)$$

$$H(s) = \frac{A_o}{s^2 C^2 R^2 + sCR(3 - A_o) + 1}$$





Transfer function of low pass second order filter

$$H(s) = \frac{A_o}{s^2C^2R^2 + sCR(3 - A_o) + 1}$$

$$H(s) = \frac{A_o\omega_h^2}{s^2 + \alpha\omega_h s + \omega_h^2}$$

$A_o = Gain$

ω_h = Upper cut-off frequency in radians/second

α = damping coefficient

$$\omega_h = \frac{1}{RC}$$

$$\alpha = (3 - A_o)$$

Transfer function of low pass second order filter

$$Substitute\; s = j\omega$$

$$H(j\omega) = \frac{A_o}{(j\omega/\omega_h)^2 + j\alpha(\omega/\omega_h) + 1}$$

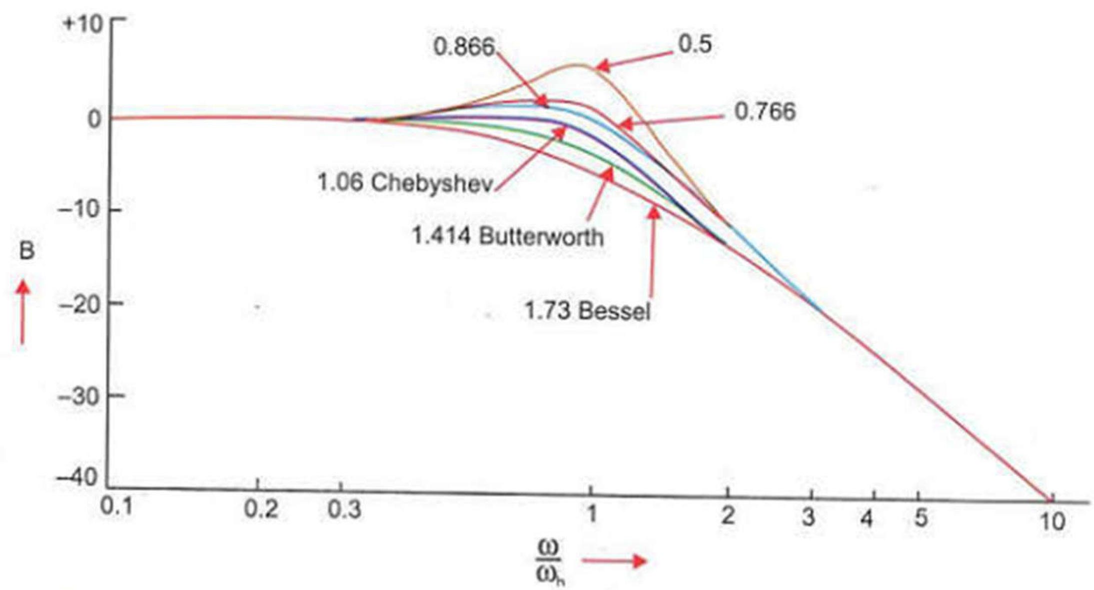
$$Normalized\; \exp ression\;\; for\; LPF$$

$$H(j\omega) = \frac{A_o}{s_n{}^2 + \alpha s_n + 1}$$

$$Normalized\;\; frequency$$

$$s_n = j\left(\frac{\omega}{\omega_h}\right)$$

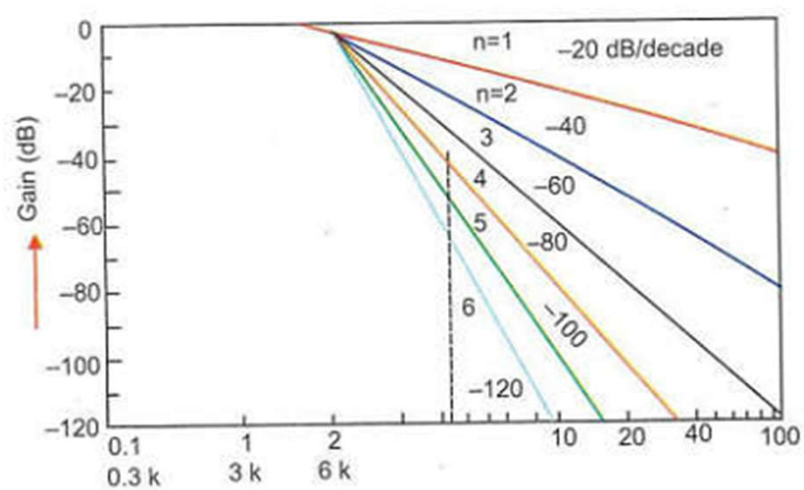
Frequency response for different values of α



- α, small oscillatory
- α = 1.414, Butterworth filter
- Audio amplifier uses Butterworth filter

Higher order filter

$$H(s) = \underbrace{\frac{A_{o1}}{s_n^2 + \alpha_1 s_n + 1}}_{\text{second order section}} \cdot \underbrace{\frac{A_{o2}}{s_n^2 + \alpha_2 s_n + 1}}_{\text{another second order section}} \cdot \underbrace{\frac{A_o}{s_n + 1}}_{\text{first order section}}$$



Design a second order Butterworth low-pass filter having upper cut-off frequency 1 kHz. Then determine its frequency response.

$$\frac{1.586}{s_n^2 + 1.414\, s_n + 1}$$

Now $A_o = 1 + R_F/R_i = 1.586 = 1 + 0.586$. Let $R_F = 5.86\ \text{k}\Omega$ and $R_i = 10\ \text{k}\Omega$.

Design a fourth order Butterworth low-pass filter having upper cut-off frequency 1 kHz.

$$A_{o1} = 3 - \alpha_1 = 3 - 0.765 = 2.235$$
$$A_{o2} = 3 - \alpha_2 = 3 - 1.848 = 1.152$$

The transfer function of fourth order low-pass Butterworth filter is

$$\frac{2.235}{s_n^2 + 0.765 s_n + 1} \cdot \frac{1.152}{s_n^2 + 1.848 s_n + 1}$$