Deterministic Finite Automaton and Non-deterministic Finite Automaton

DFA and NFA

(Finite State Machines)

Notation and Definitions

- Alphabet
- String
- Language
- Operations on languages

Strings

- An alphabet is any finite set of distinct symbols
  - $\{0, 1\}$, $\{0,1,2,\ldots,9\}$, $\{a,b,c\}$
  - We denote a generic alphabet by Σ
- A string is any finite-length sequence of elements of Σ.
- e.g., if $\Sigma = \{a, b\}$ then *a*, *aba*, *aaaa*, ..., *abababbaab* are some strings over the alphabet Σ

Strings

- The length of a string ω is the number of symbols in ω. We denote it by $|\omega|$. $|aba| = 3$.
- The symbol ϵ denotes a special string called the empty string
  - ϵ has length 0
- String concatenation
  - If $\omega = a_1, \ldots, a_n$ and $\nu = b_1, \ldots, b_m$ then $\omega \cdot \nu$ (or $\omega\nu$) $= a_1, \ldots, a_n b_1, \ldots, b_m$
  - Concatenation is associative with ϵ as the identity element.
- If $a \in \Sigma$, we use a^n to denote a string of n a's concatenated
  - $\Sigma = \{0, 1\}, 0^5 = 00000$
  - $a^0 =_{def} \epsilon$
  - $a^{n+1} =_{def} a^n a$

Strings

- The reverse of a string ω is denoted by ω^R.
  - $\omega^R = a_n, \ldots, a_1$

- A substring y of a string ω is a string such that $\omega = xyz$ with $|x|, |y|, |z| \geq 0$ and $|x| + |y| + |z| = |\omega|$

- If $\omega = xy$ with $|x|, |y| \geq 0$ and $|x| + |y| = |\omega|$, then x is prefix of ω and y is a suffix of ω.
  - For $\omega = abaab$,
    - ϵ, a, aba, and $abaab$ are some prefixes
    - ϵ, $abaab$, aab, and $baab$ are some suffixes.

Strings

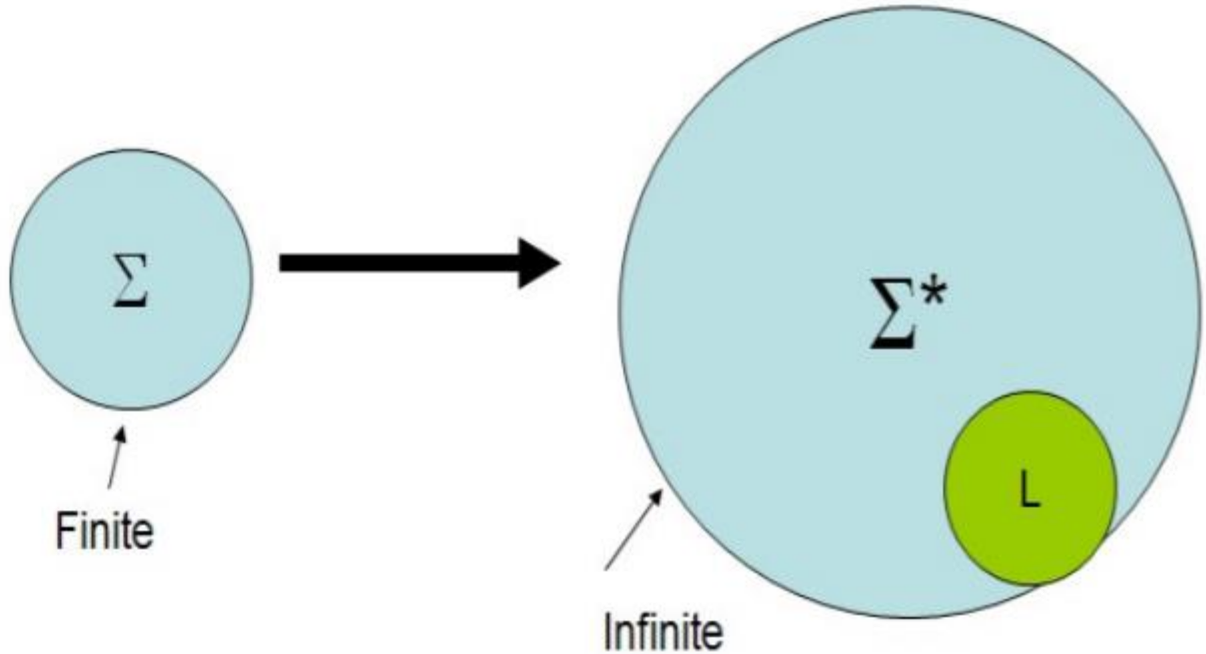
- The set of all possible strings over Σ is denoted by Σ^*.
- We define $\Sigma^0 = \{\epsilon\}$ and $\Sigma^n = \Sigma^{n-1} \cdot \Sigma$
  - with some abuse of the concatenation notation applying to sets of strings now
- So $\Sigma^n = \{\omega | \omega = xy \text{ and } x \in \Sigma^{n-1} \text{ and } y \in \Sigma\}$
- $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots \Sigma^n \cup \cdots = \bigcup_0^\infty \Sigma^i$
  - Alternatively, $\Sigma^* = \{x_1 x_2 \ldots\ x_n | n \geq 0 \text{ and } x_i \in \Sigma \text{ for all } i\}$
- Φ denotes the empty set of strings $\Phi = \{\}$,
  - but $\Phi^* = \{\epsilon\}$

Strings

- Σ^* is a countably infinite set of finite length strings
- If x is a string, we write x^n for the string obtained by concatenating n copies of x.
  - $(aab)^3 = aabaabaab$
  - $(aab)^0 = \epsilon$

Languages

- A language L over Σ is any subset of Σ^*



- L can be finite or (countably) infinite

Some Languages

- $L = \Sigma^*$ – The mother of all languages!
- $L = \{a, ab, aab\}$ – A fine finite language.
  - Description by enumeration
- $L = \{a^n b^n : n \geq 0\} = \{\epsilon,\ \mathrm{ab},\ \mathrm{aabb},\ \mathrm{aaabbb}, \ldots\}$
- $L = \{\omega | n_a(\omega) \text{ is even}\}$
  - $n_x(\omega)$ denotes the number of occurrences of x in ω
  - all strings with even number of a's.
- $L = \{\omega | \omega = \omega^R\}$
  - All strings which are the same as their reverses – palindromes.
- $L = \{\omega | \omega = xx\}$
  - All strings formed by duplicating some string once.
- $L = \{\omega | \omega$ is a syntactically correct Java program $\}$

Languages

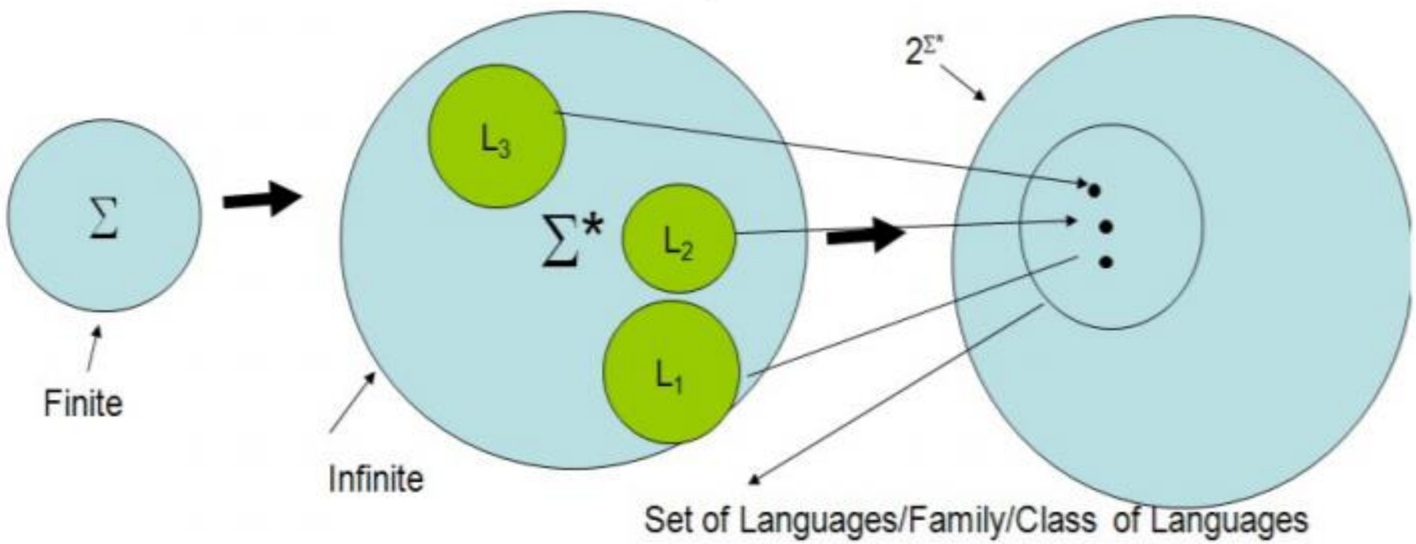
- Since languages are sets, all usual set operations such as intersection and union, etc. are defined.
- Complementation is defined with respect to the universe Σ^* : $\overline{L} = \Sigma^* - L$

Languages

- If L, L_1 and L_2 are languages:
  - $L_1 \cdot L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
  - $L^0 = \{\epsilon\}$ and $L^n = L^{n-1} \cdot L$
  - $L^* = \bigcup_0^\infty L^i$
  - $L^+ = \bigcup_1^\infty L^i$

Sets of Languages

- The power set of Σ^*, the set of all its subsets, is denoted as 2^{Σ^*}



AUTOMATA

- The control unit has some finite memory and it keeps track of what step to execute next.
- Additional memory (if any) is infinite - we never run out of memory!
    - Infinite but like a stack - only the top item is accessible at a given time.
    - Infinite but like a tape, any cell is (sequentially) accessible.

Finite State Automata

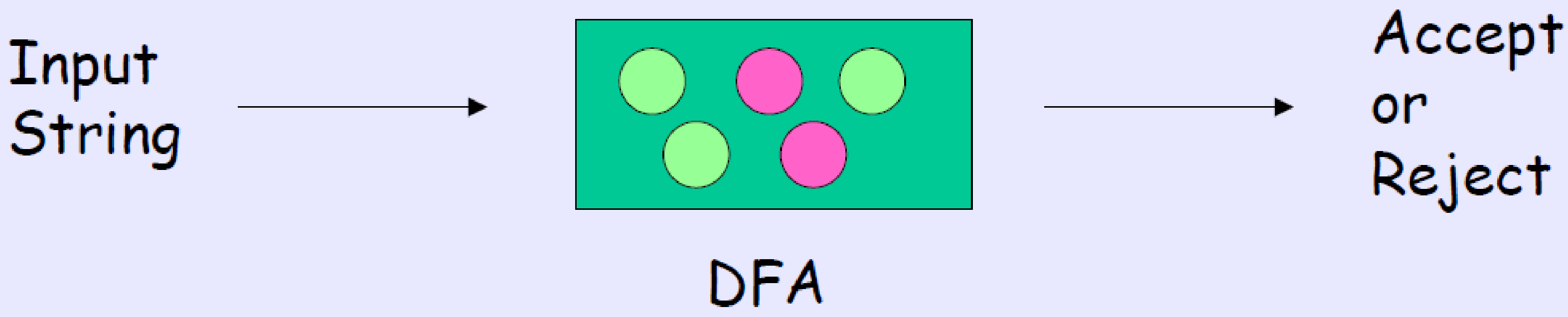
- Finite State Automata (FSA) are the simplest automata.
- Only the finite memory in the control unit is available.
- The memory can be in one of finite states at a given time – hence the name.
  - One can remember only a (fixed) finite number of properties of the past input.
  - Since input strings can be of arbitrary length, it is not possible to remember unbounded portions of the input string.
- It comes in Deterministic and Nondeterministic flavors.

Deterministic Finite State Automata (DFA)

- A DFA starts in a start state and is presented with an input string.
- It moves from state to state, reading the input string one symbol at a time.
- What state the DFA moves next depends on
  - the current state,
  - current input symbol
- When the last input symbol is read, the DFA decides whether it should accept the input string

Finite State Machines

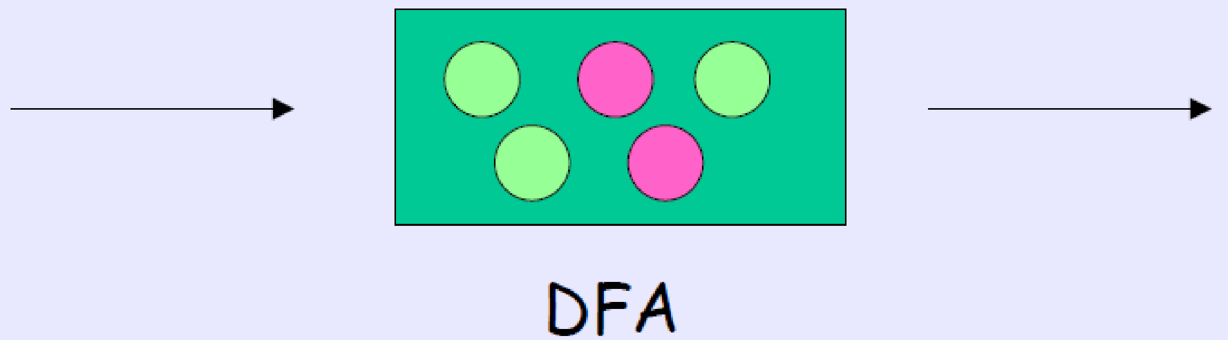
DFA



- A machine with finite number of states, some states are accepting states, others are rejecting states
- At any time, it is in one of the states
- It reads an input string, one character at a time

DFA

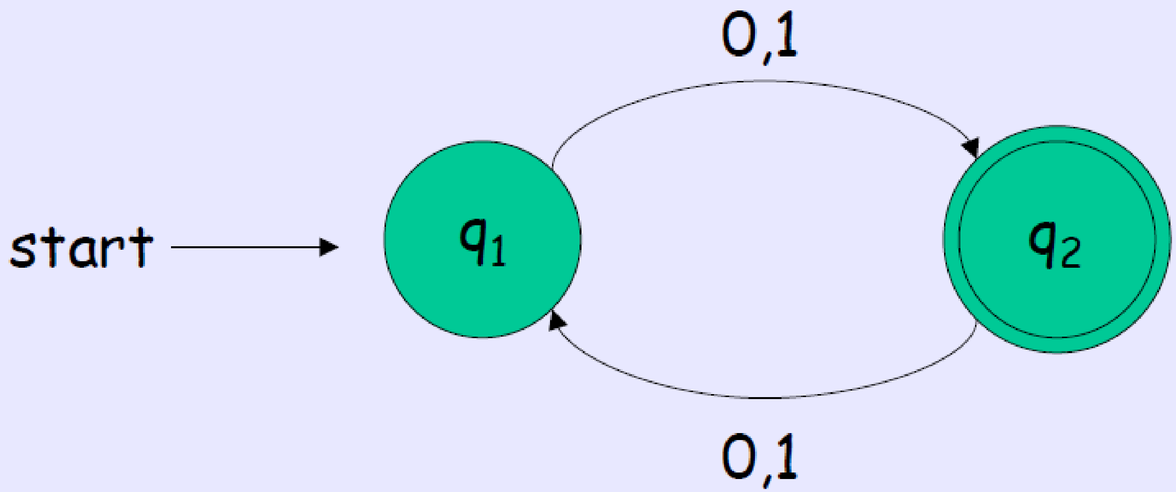
Input String



Accept or Reject

- After reading each character, it moves to another state depending on what is read and what is the current state
- If reading all characters, the DFA is in an accepting state, the input string is accepted.
- Otherwise, the input string is rejected.

Example of DFA



- The circles indicates the states
- If accepting state is marked with double circle
- The arrows pointing from a state q indicates how to move on reading a character when current state is q

DFA – Formal Definition

- A Deterministic Finite State Acceptor (DFA) is defined as the 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where
  - Q is a finite set of states
  - Σ is a finite set of symbols – the alphabet
  - $\delta : Q \times \Sigma \to Q$ is the next-state function
  - $q_0 \in Q$ is the (label of the) start state
  - $F \subseteq Q$ is the set of final (accepting) states

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Note, there must be exactly one start state.
Final states can be many or even empty !

Some Terminology

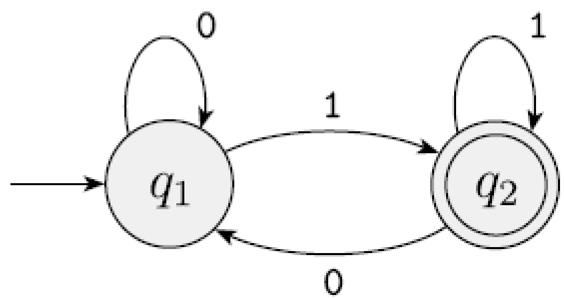
Let M be a DFA

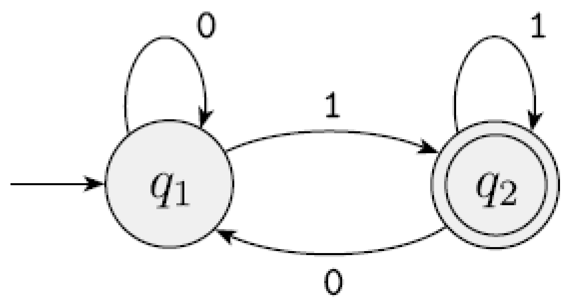
- Among all possible strings, M will accept some of them, and M will reject the remaining
- The set of strings which M accepts is called the language recognized by M
- That is, M recognizes A if

$$A = \{ w \mid M \text{ accepts } w \}$$

$L(M)$

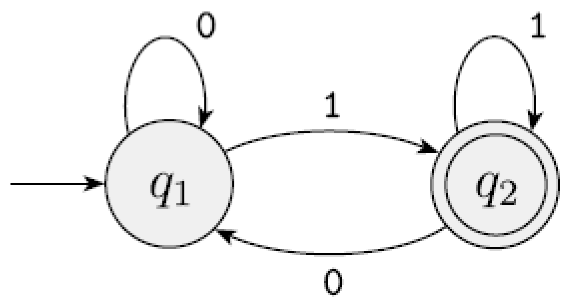
If A is the set of all strings that machine M accepts, we say that A is the ***language of machine M*** and write $L(M) = A$. We say that ***M recognizes A*** or that ***M accepts A***.


0
1
1
q1
q2
0



In the formal description, M_2 is $\big(\{q_1, q_2\}, \{\mathtt{0},\mathtt{1}\}, \delta, q_1, \{q_2\}\big)$. The transition function δ is

	0	1
q_1	q_1	q_2
q_2	q_1	q_2.



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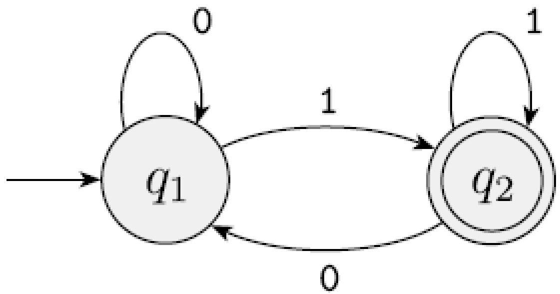
	0	1
q_1	q_1	q_2
q_2	q_1	q_2.

Remember that the state diagram of M_2 and the formal description of M_2 contain the same information, only in different forms. You can always go from one to the other if necessary.



M_2

$$L(M_2) = \{w \mid w \text{ ends in a } 1\}.$$

Consider the finite automaton M_3.

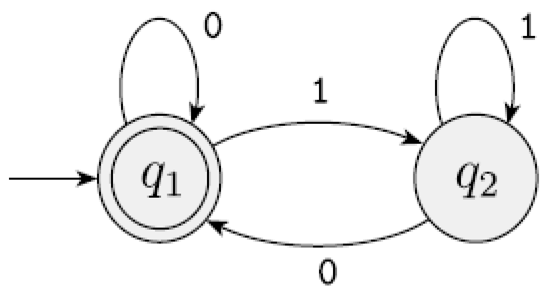


FIGURE 1.10
State diagram of the two-state finite automaton M_3

Can you describe this in the 5 tuple form?
In particular, can you write down the transition table?

Consider the finite automaton M_3.

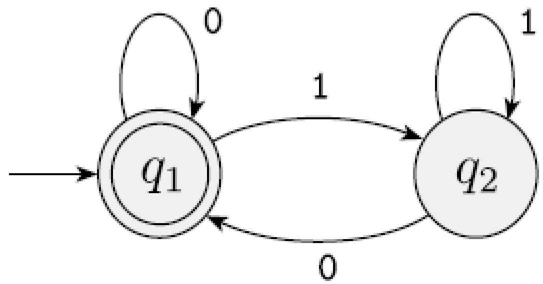


FIGURE **1.10**
State diagram of the two-state finite automaton M_3

What language M_3 recognizes?

Consider the finite automaton M_3.

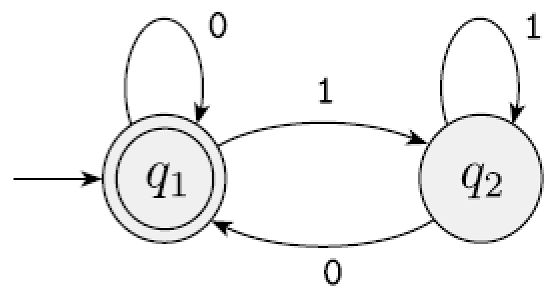


FIGURE 1.10
State diagram of the two-state finite automaton M_3

What language M_3 recognizes?

$L(M_3) = \{w |\ w \text{ is the empty string } \varepsilon \text{ or ends in a } 0\}.$

EXAMPLE 1.11

The following figure shows a five-state machine M_4.

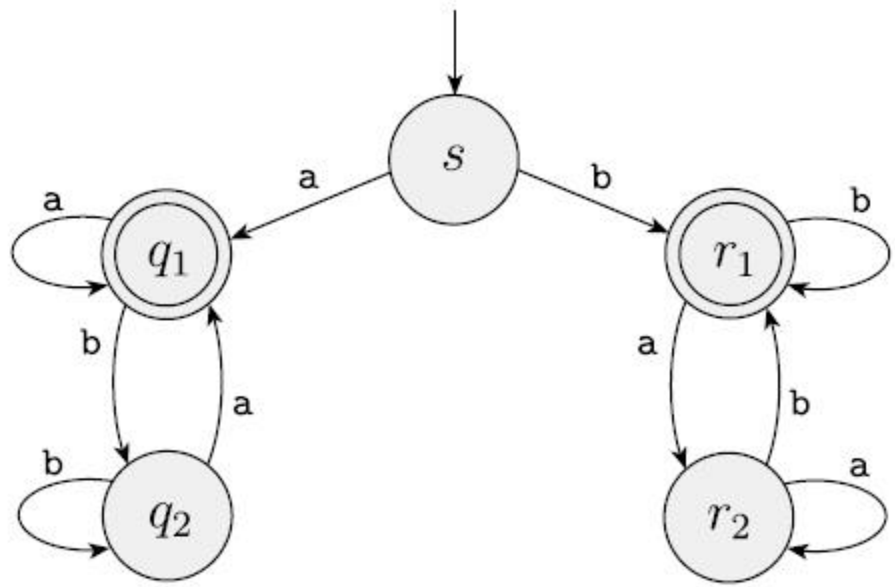


FIGURE 1.12
Finite automaton M_4

EXAMPLE 1.11

The following figure shows a five-state machine M_4.

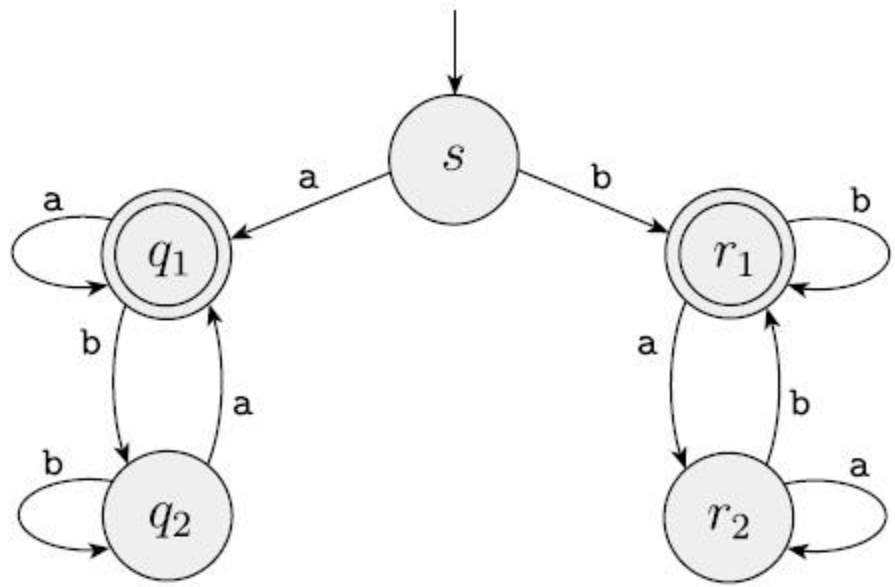


FIGURE 1.12
Finite automaton M_4

$L(M_4) =$ all strings that begin and end with the same character.

DFA for complement of a language

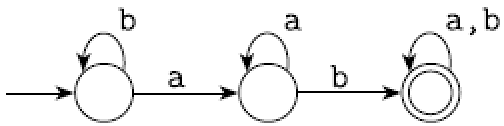
- Flip final and non-final states.

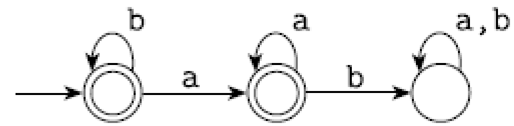
1.5 Each of the following languages is the complement of a simpler language. In each part, construct a DFA for the simpler language, then use it to give the state diagram of a DFA for the language given. In all parts, $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

[A]**a.** $\{w \mid w$ does not contain the substring $\mathtt{ab}\}$

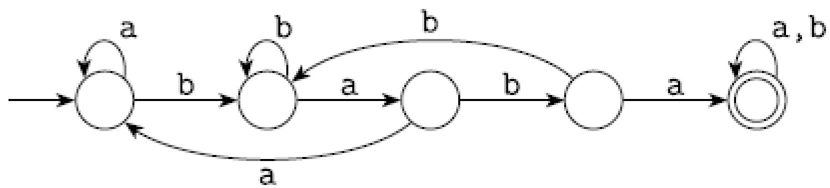
[A]**b.** $\{w \mid w$ does not contain the substring $\mathtt{baba}\}$

1.5 (a) The left-hand DFA recognizes $\{w|\ w \text{ contains } \texttt{ab}\}$. The right-hand DFA recognizes its complement, $\{w|\ w \text{ doesn't contain } \texttt{ab}\}$.

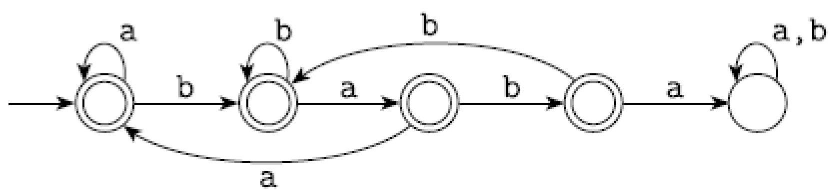




(b) This DFA recognizes $\{w|\ w \text{ contains } \texttt{baba}\}$.



This DFA recognizes $\{w|\ w \text{ does not contain } \texttt{baba}\}$.



Designing a DFA (Quick Quiz)

- How to design a DFA that accepts all binary strings representing a multiple of 5? (E.g., 101, 1111, 11001, ...)

Formally

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and let $w = w_1 w_2 \cdots w_n$ be a string where each w_i is a member of the alphabet Σ. Then M ***accepts*** w if a sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with three conditions:

1. $r_0 = q_0$,

2. $\delta(r_i, w_{i+1}) = r_{i+1}$, for $i = 0, \ldots, n-1$, and

3. $r_n \in F$.

Regular language [Ref: Sipser Book]

DEFINITION 1.16

A language is called a ***regular language*** if some finite automaton recognizes it.

The regular operations

DEFINITION 1.23

Let A and B be languages. We define the regular operations ***union***, ***concatenation***, and ***star*** as follows:

- **Union**: $A \cup B = \{x|\ x \in A \text{ or } x \in B\}$.
- **Concatenation**: $A \circ B = \{xy|\ x \in A \text{ and } y \in B\}$.
- **Star**: $A^* = \{x_1 x_2 \ldots x_k|\ k \geq 0 \text{ and each } x_i \in A\}$.

- These are similar to arithmetic operations.
- Note, * is a unary operator.

THEOREM 1.25

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

- The proof is by construction.
- We build a DFA for the union from the individual DFAs.

- The idea is simple: While reading the input simultaneously follow both machines.
  - Put a finger on current state. You need two fingers. You can move these two fingers as per the respective transition function.

PROOF

Let M_1 recognize A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, and
M_2 recognize A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct M to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$.

1. $Q = \{(r_1, r_2) | \; r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$.
   This set is the ***Cartesian product*** of sets Q_1 and Q_2 and is written $Q_1 \times Q_2$. It is the set of all pairs of states, the first from Q_1 and the second from Q_2.

2. Σ, the alphabet, is the same as in M_1 and M_2. In this theorem and in all subsequent similar theorems, we assume for simplicity that both M_1 and M_2 have the same input alphabet Σ. The theorem remains true if they have different alphabets, Σ_1 and Σ_2. We would then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$.

3. δ, the transition function, is defined as follows. For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let

$$\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big).$$

Hence δ gets a state of M (which actually is a pair of states from M_1 and M_2), together with an input symbol, and returns M's next state.

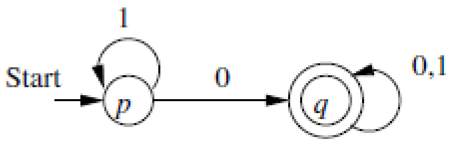
4. q_0 is the pair (q_1, q_2).

5. F is the set of pairs in which either member is an accept state of M_1 or M_2. We can write it as

$$F = \{(r_1, r_2) |\, r_1 \in F_1 \text{ or } r_2 \in F_2\}.$$

This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$. (Note that it is *not* the same as $F = F_1 \times F_2$. What would that give us instead?[3])

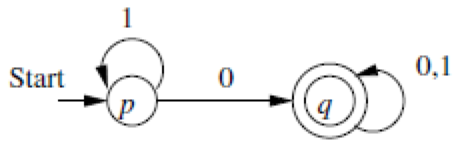
Union Example



(a)

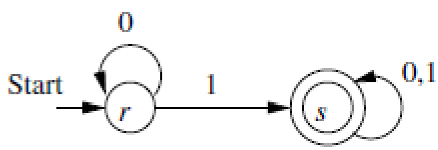
What is the language recognized by this DFA?

Union Example



(a)

What is the language recognized by this DFA?

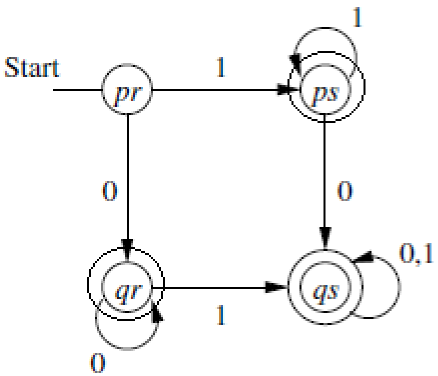


(b)

What is the language recognized by this DFA?

Find DFA for the union

Find DFA for the union

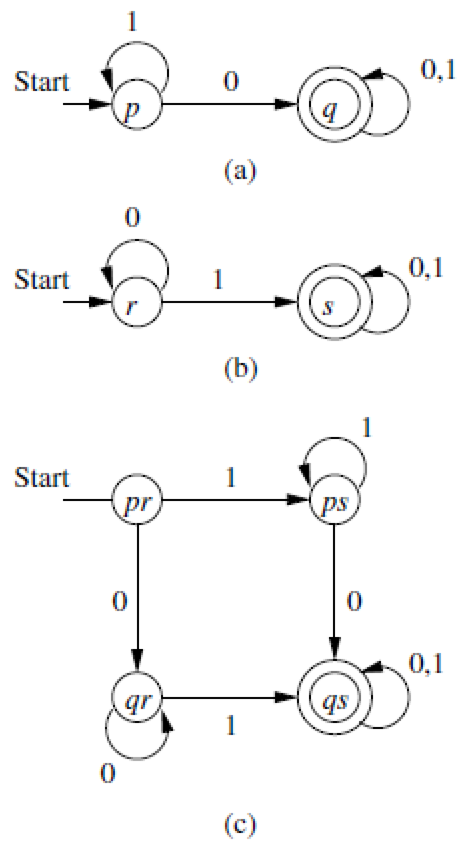


(c)

What about intersection?

- Intersection of two regular languages is also regular.
- Proof: by construction. Similar. Only final states will change.

Intersection



What else we can do with product principle?

- Set difference.
  - How?

1.4 Each of the following languages is the intersection of two simpler languages. In each part, construct DFAs for the simpler languages, then combine them using the construction discussed in footnote 3 (page 46) to give the state diagram of a DFA for the language given. In all parts, $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

a. $\{w \mid w$ has at least three a's and at least two b's$\}$

[A]b. $\{w \mid w$ has exactly two a's and at least two b's$\}$

c. $\{w \mid w$ has an even number of a's and one or two b's$\}$

[A]d. $\{w \mid w$ has an even number of a's and each a is followed by at least one b$\}$

e. $\{w \mid w$ starts with an a and has at most one b$\}$

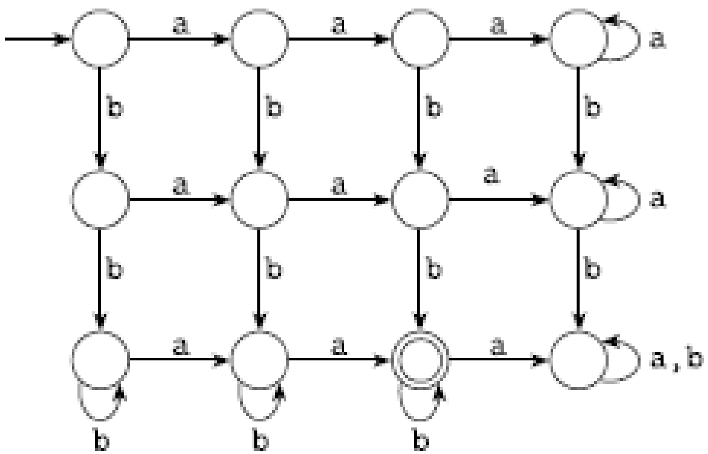
f. $\{w \mid w$ has an odd number of a's and ends with a b$\}$

g. $\{w \mid w$ has even length and an odd number of a's$\}$

1.4 (b) The following are DFAs for the two languages $\{w \mid w \text{ has exactly two a's}\}$ and $\{w \mid w \text{ has at least two b's}\}$.

- Now find product machine.

Combining them using the intersection construction gives the following DFA.



- This can be minimized. {Some states are redundant}.

NONDETERMINISM

- Useful concept, has great impact on ToC/algorithms.
- DFA is deterministic: every step of a computation follows in a unique way from the preceding step.
  - When the machine is in a given state, and upon reading the next input symbol, we know deterministically what would be the next state.
  - Only one next state.
  - No choice !!

NONDETERMINISM

- In a nondeterministic machine, several choices may exist for the next state at any point.
- Nondeterminism is a generalization of determinism.

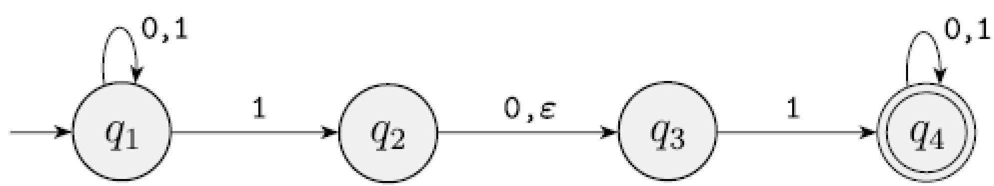


FIGURE 1.27
The nondeterministic finite automaton N_1

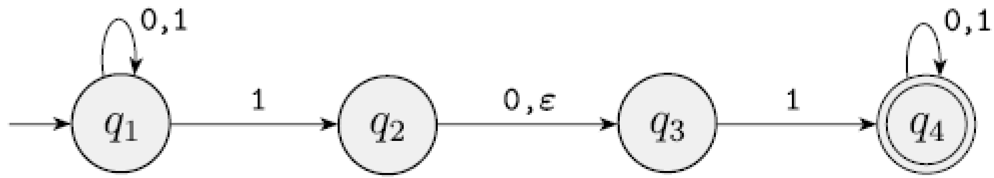


FIGURE 1.27
The nondeterministic finite automaton N_1

- More than one arrow from from q_1 on symbol 1.
- No arrow at all from q_3 on 0.
- There is Ɛ over an arrow !

How does an NFA compute?

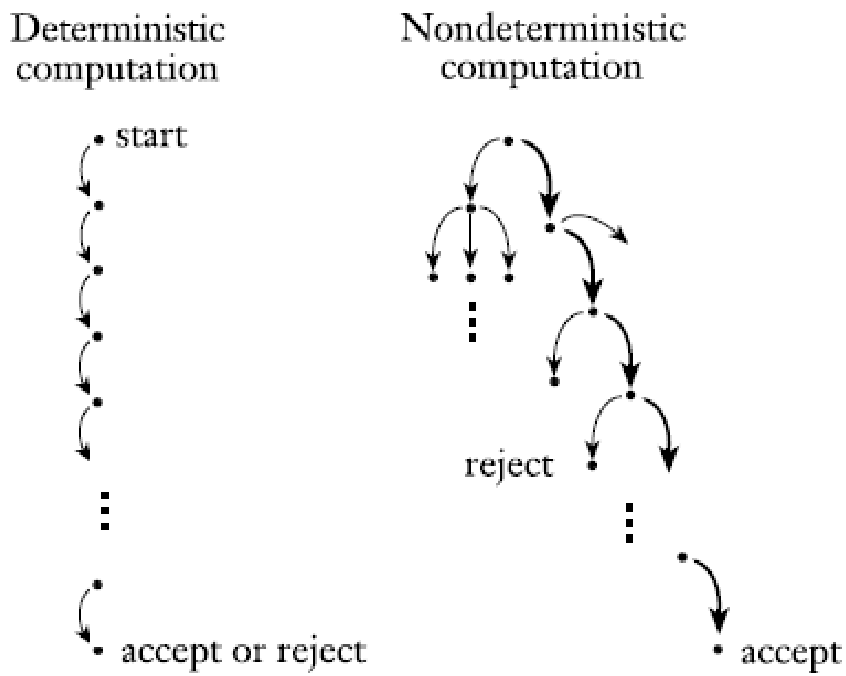


FIGURE 1.28
Deterministic and nondeterministic computations with an accepting branch

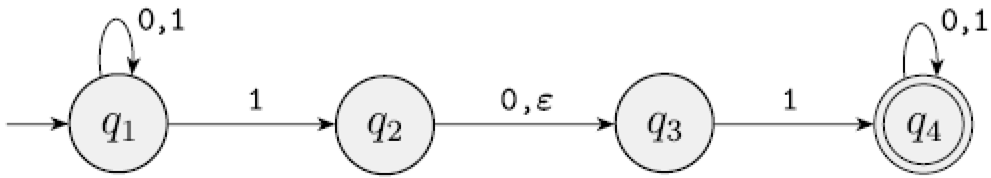


FIGURE 1.27
The nondeterministic finite automaton N_1

On input **010110**

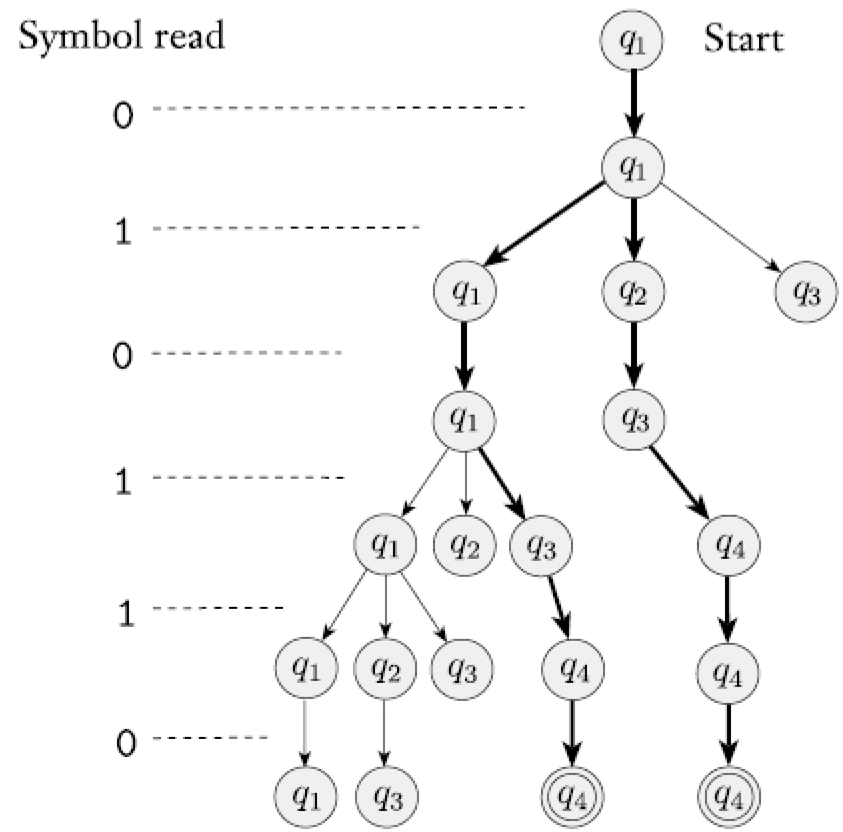


FIGURE 1.29
The computation of N_1 on input 010110

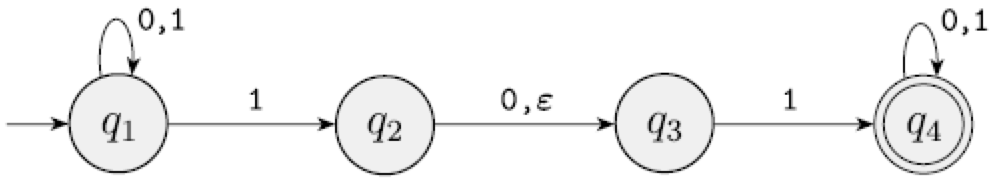


FIGURE 1.27
The nondeterministic finite automaton N_1

- What is the language accepted by this NFA?

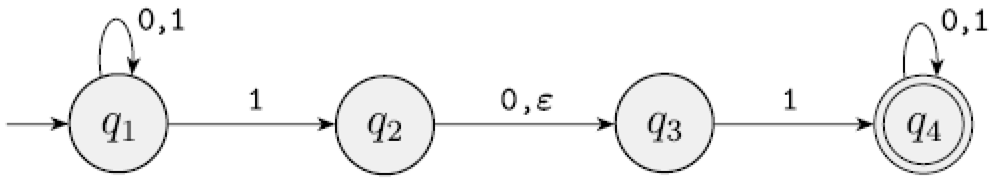


FIGURE **1.27**
The nondeterministic finite automaton N_1

- It accepts all strings that contain either 101 or 11 as a substring.

- Constructing NFAs is sometimes easier than constructing DFAs.
  - Later we see that every NFA can be converted into an equivalent DFA.

EXAMPLE 1.30

Let A be the language consisting of all strings over $\{0,1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

- Building DFA for this is possible, but difficult.
- Try this.

But NFA is easy to build.

EXAMPLE 1.30

Let A be the language consisting of all strings over $\{0,1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

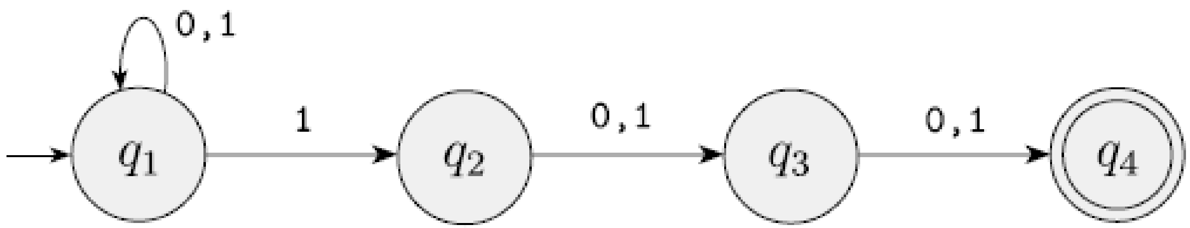


FIGURE 1.31
The NFA N_2 recognizing A

DFA for A

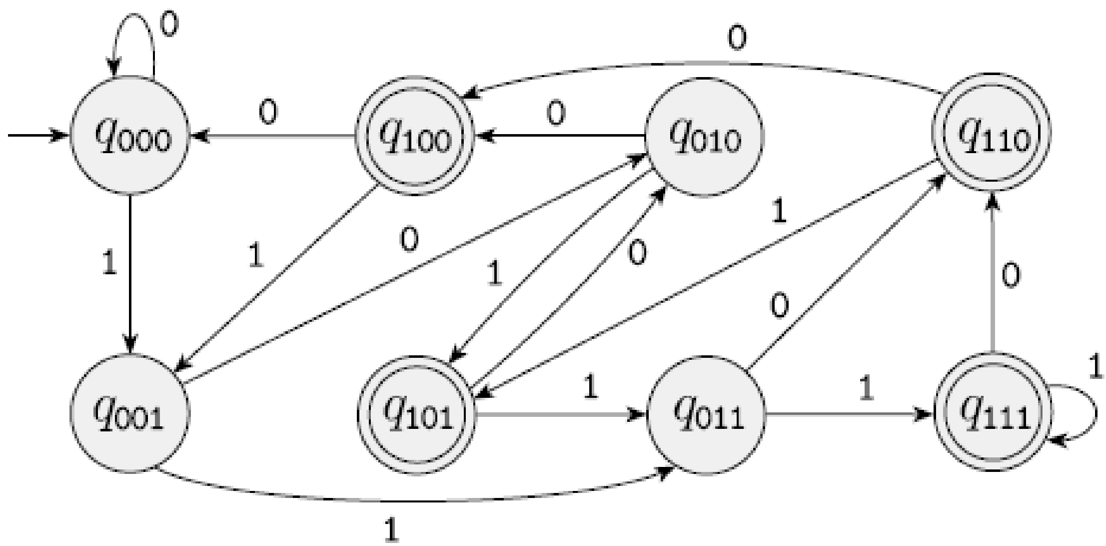


FIGURE 1.32
A DFA recognizing A

- See number of states and complexity !

Formal definition of NFA

We use Σ_{ε} to mean $\Sigma \cup \{\varepsilon\}$

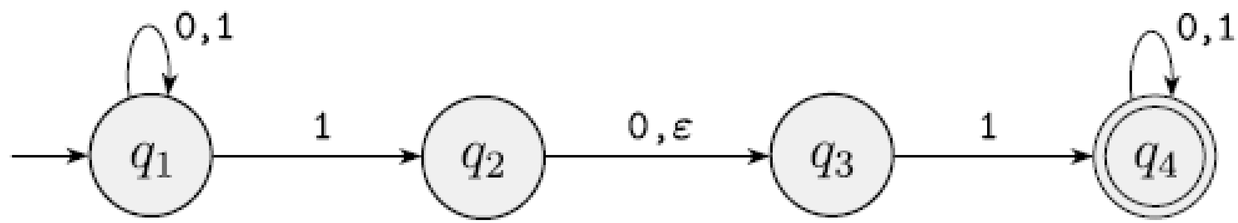
DEFINITION 1.37

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_{\varepsilon} \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

EXAMPLE 1.38

Recall the NFA N_1:



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

4. q_1 is the start state, and
5. $F = \{q_4\}$.

The formal definition of computation for an NFA is similar to that for a DFA. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over the alphabet Σ. Then we say that N ***accepts*** w if we can write w as $w = y_1 y_2 \cdots y_m$, where each y_i is a member of Σ_ε and a sequence of states $r_0, r_1, \ldots, r_m$ exists in Q with three conditions:

1. $r_0 = q_0$,

2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m-1$, and

3. $r_m \in F$.

Equivalence of NFAs and DFAs

- We say two machines are equivalent if they recognize the same language.

THEOREM 1.39

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Proof

- Proof by construction.
  - We build a equal DFA for the given NFA

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A.
We construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing A.

- First, for understanding purpose, we assume that there are no edges with Ɛ transitions.

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A.

We construct a DFA $M = (Q', \Sigma, \delta', {q_0}', F')$ recognizing A.

1. $Q' = \mathcal{P}(Q)$.
   Every state of M is a set of states of N. Recall that $\mathcal{P}(Q)$ is the set of subsets of Q.

2. For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$.

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$

3. ${q_0}' = \{q_0\}$.
   M starts in the state corresponding to the collection containing just the start state of N.

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$.
   The machine M accepts if one of the possible states that N could be in at this point is an accept state.

Can you convert the following

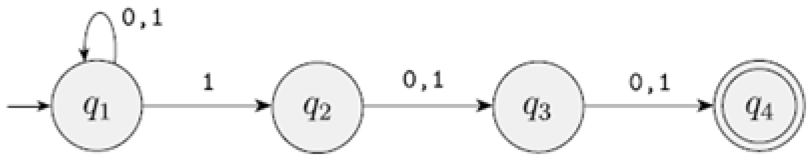


FIGURE 1.31
The NFA N_2

- What is the language accepted by this?

Now, considering Ɛ arrows

- For this purpose, we define Ɛ-CLOSURE of a set of states *R*.

Formally, for $R \subseteq Q$ let

$$E(R) = \{q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon \text{ arrows}\}.$$

- *E(R)* is Ɛ-CLOSURE of *R*.

- Then the transition is defined as,

  $\delta'(R, a) = \{q \in Q |\, q \in E(\delta(r, a))$ for some $r \in R\}$.

- Now the start state of the DFA should be

  $q_0{}' = E(\{q_0\})$

Example

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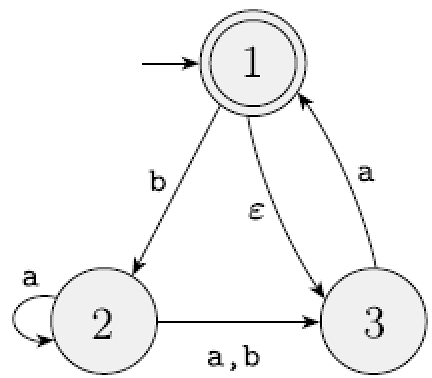
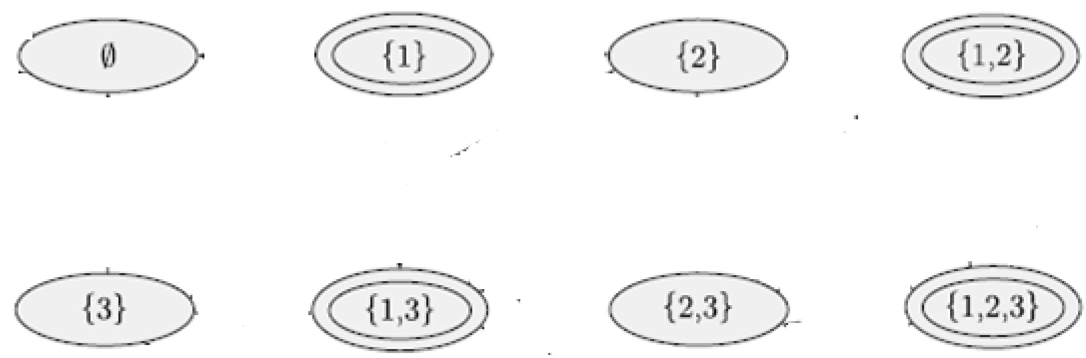


FIGURE 1.42
The NFA N_4



All possible states of the DFA.
(to be constructed; Final states are shown)

- Now we need to add edges, and
- identify the initial state.

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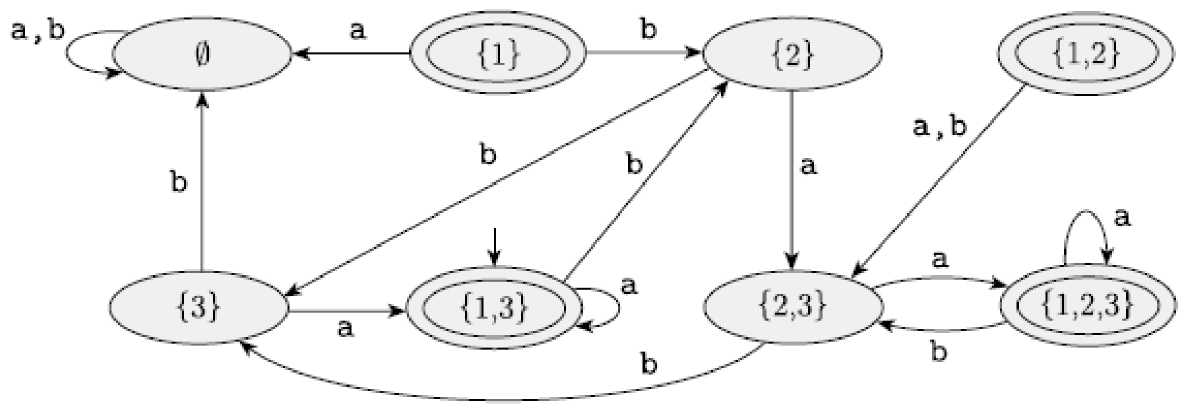


FIGURE 1.43
A DFA D that is equivalent to the NFA N_4

- But, some states are not reachable !
- Simplification can remove this.

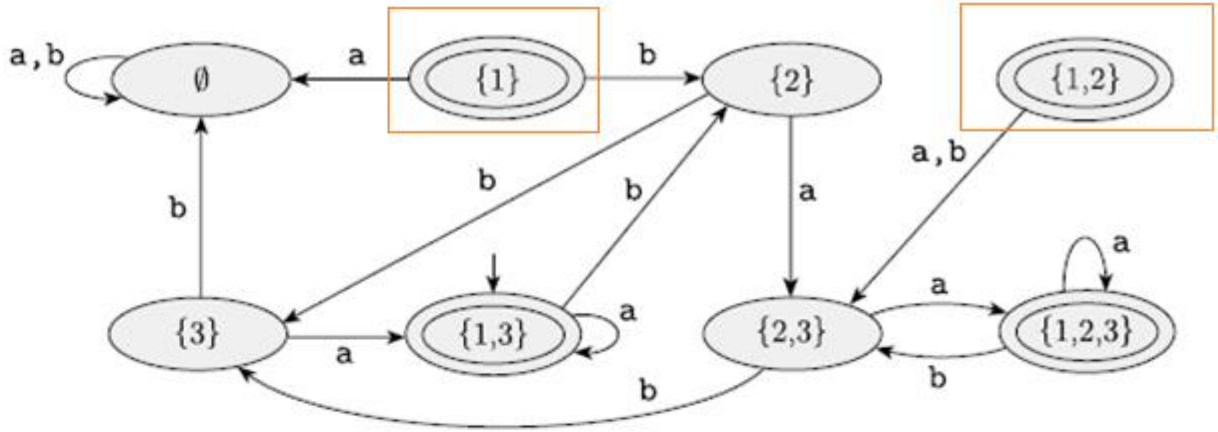
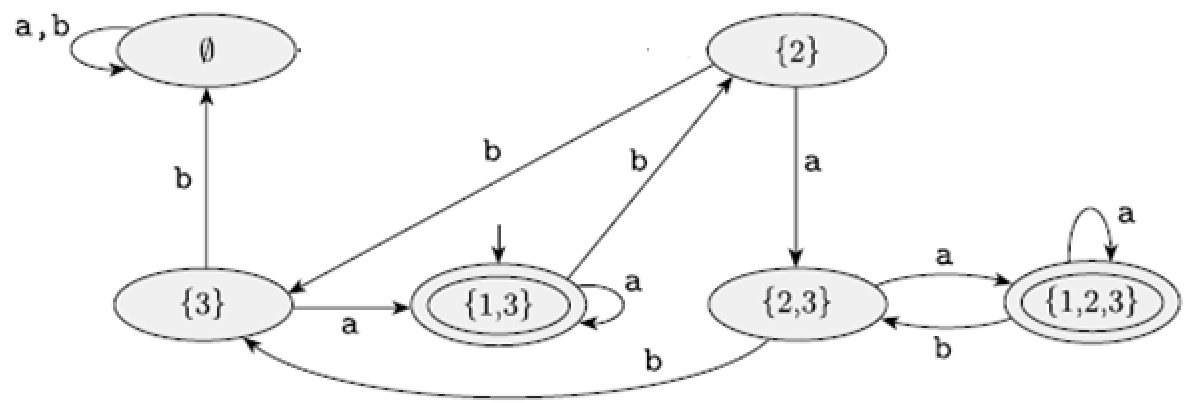
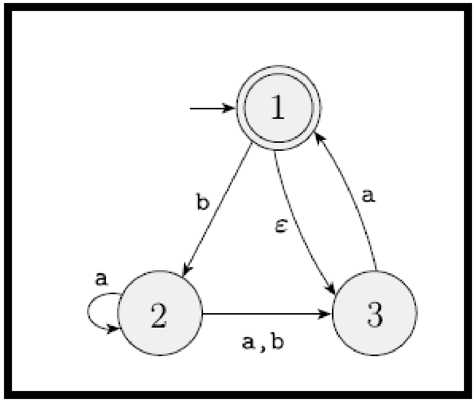


FIGURE 1.43
A DFA D that is equivalent to the NFA N_4

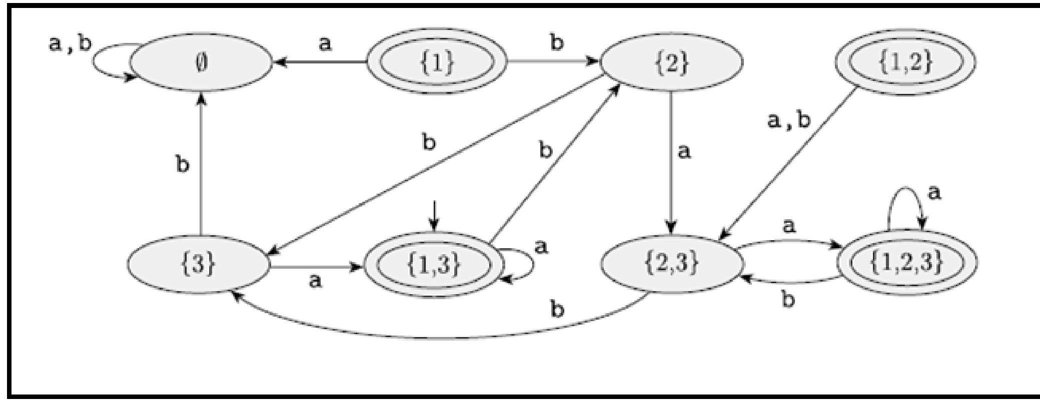


DFA D' which is equivalent to D.

Note: D' and D are different machines; but, they are equivalent.



N_4

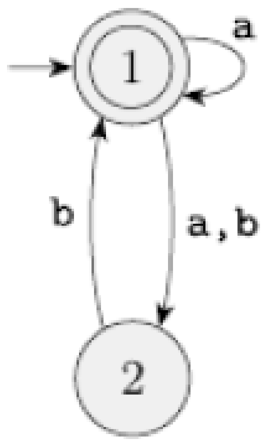


A DFA D that is equivalent to the NFA N_4

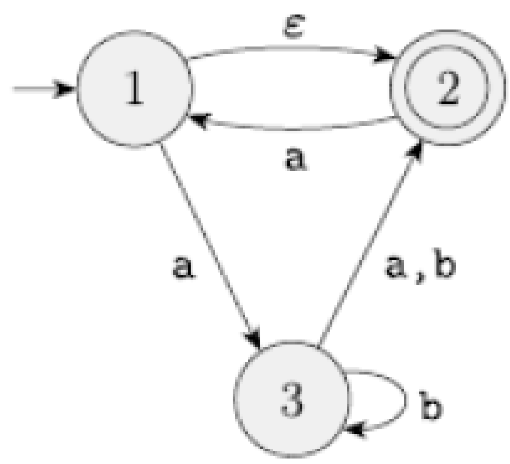
- Being in state 1 of N_4 upon reading input a the machine N_4 can be in state 1.
- Convince yourself that in the DFA D there are no mistakes.
  – From state $\{1\}$ with input a the DFA D goes to state ϕ.

Exercise

Convert the following NFAs to equivalent DFAs.



(a)



(b)

(Problem Source: Sipser's book exercise problem 1.16)

Exercise

- **1.7** Give state diagrams of NFAs with the specified number of states recognizing each of the following languages. In all parts, the alphabet is $\{0,1\}$.

  [A]**a.** The language $\{w \mid w \text{ ends with } 00\}$ with three states

  d. The language $\{0\}$ with two states

  g. The language $\{\varepsilon\}$ with one state

  h. The language 0^* with one state

- Can you convert each of above NFAs into a corresponding DFA.