$$\Rightarrow \bar{x}' \Rightarrow [\bar{x}_1' \;\; \bar{x}_2' \; \cdots\cdots \; \bar{x}_n'] = [\bar{x}_1 \;\; \bar{x}_2 \;\; \bar{x}_3 \; \cdots\cdots \; \bar{x}_n] \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}$$

β = transformed basis

α = earlier basis

$$\beta = \alpha B$$

Theorem (ii):

In the earlier case, if $\bar{x}' = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \cdots\cdots + \alpha_n \bar{x}_n$ and

$\bar{x} = \alpha_1' \bar{x}_1' + \alpha_2' \bar{x}_2' + \cdots\cdots + \alpha_n' \bar{x}_n'$

a vector in space V_n

then, $u' = B^{-1} u$

$$\left[\begin{array}{l} B \text{ is transformation matrix} \\ u = [\alpha_1 \; \alpha_2 \cdots\cdots \alpha_n] \\ u' = [\alpha_1' \; \alpha_2' \cdots\cdots \alpha_n'] \end{array}\right]$$

Matrix of a linear transformation:

→ 1) Given a linear transformation, find matrix of the linear transformation with respect to a basis.

→ 2) Given matrix of a L.T w.r.t a basis, find the transformation.

→ 3) Given matrix of a L.T w.r.t a basis, find the matrix wrt another basis.

$$\begin{array}{ccc} T: V & \to & W \\ \downarrow & & \downarrow \\ \bar{X} & & \bar{Y} \\ \bar{X}' & & \bar{Y}' \end{array} \qquad \begin{array}{c} \text{Matrix} \\ A \\ A' \end{array}$$

1) $T: V_3 \to V_2$; $T(\bar{x}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_2 + x_3 \end{pmatrix}$.

Basis of V_3 $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$ matrix of basis $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \bar{X}$

Basis of V_2 $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$

$$* \quad \boxed{T(\bar{X}) = \bar{Y} A}$$

Transformation of $\bar{X}$ = Basis of target vector space × Matrix of linear transformation.

$$T(\bar{x}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_2 + x_3 \end{pmatrix}$$

$T(\bar{X})$: $T\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. $\Rightarrow T(\bar{X}) = \begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix}$.

$T(\bar{x}) = \bar{Y} \cdot A$

$\begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot A$

$C = BA$

$B^{-1}C = B^{-1}BA$

$B^{-1}C = A$

$\mathrm{Adj}(A_{2\times 2}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\cancel{A = \begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}}$

$\cancel{A = \begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix} \frac{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}{1}}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix} = A$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & -1 & 1 \\ 5 & 2 & 1 \end{pmatrix}$

Matrix of linear transformation, $A = \begin{pmatrix} 0 & -1 & 1 \\ 5 & 2 & 1 \end{pmatrix}$.

2) $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$

$T : V_3 \to V_2$ $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

Basis

$\left\{ \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

$T(\bar{x}) = \bar{Y} \cdot A$

$T(\bar{x}) = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$

$T(\bar{x}) = \begin{pmatrix} 0 & 5 & -5 \\ 3 & 1 & -1 \end{pmatrix}$

$\Rightarrow T \begin{pmatrix} 5 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 5 & -5 \\ 3 & 1 & -1 \end{pmatrix}$

$\Rightarrow \quad T\begin{pmatrix}5\\1\\3\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}$

$T\begin{pmatrix}3\\2\\2\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}$

$T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-5\\-1\end{pmatrix}$

Find transformation definition

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \alpha_1\begin{pmatrix}5\\1\\3\end{pmatrix} + \alpha_2\begin{pmatrix}3\\2\\2\end{pmatrix} + \alpha_3\begin{pmatrix}1\\2\\1\end{pmatrix}$$

- $x_1 = 5\alpha_1 + 3\alpha_2 + \alpha_3$

$x_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3$

$x_3 = 3\alpha_1 + 2\alpha_2 + \alpha_3$

$x_1 - x_3 = 2\alpha_1 + \alpha_2$

$x_2 - 2x_3 = -5\alpha_1 - 2\alpha_2$

$2x_1 - 2x_3 = 4\alpha_1 + 2\alpha_2$

$x_2 - 2x_3 = -5\alpha_1 - \cancel{7}\alpha_2$

$2x_1 + x_2 - 4x_3 = -\alpha_1$

$\alpha_1 = -2x_1 - x_2 + 4x_3 \rightarrow (1)$

$x_1 - x_3 + 4x_1 + 2x_2 - 8x_3 = \alpha_2$

$5x_1 + 2x_2 - 9x_3 = \alpha_2 \rightarrow (2)$

$x_2 = -2x_1 - x_2 + 4x_3 + 10x_1 + 4x_2 - 18x_3 + 2\alpha_3$

$2x_1 + 2x_2 - 4x_3 - 10x_1 - 4x_2 + 18x_3 = 2\alpha_3$

$-8x_1 - 2x_2 + 14x_3 = 2\alpha_3$

$\alpha_3 = -4x_1 - x_2 + 7x_3 \rightarrow (3)$

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = T\left(\alpha_1\begin{pmatrix}5\\1\\3\end{pmatrix} + \alpha_2\begin{pmatrix}3\\2\\2\end{pmatrix} + \alpha_3\begin{pmatrix}1\\2\\1\end{pmatrix}\right)$$

$$= \alpha_1\begin{pmatrix}0\\3\end{pmatrix} + \alpha_2\begin{pmatrix}5\\1\end{pmatrix} + \alpha_3\begin{pmatrix}-5\\-1\end{pmatrix}$$

$$= \begin{pmatrix} 25x_1+10x_2-45x_3+20x_1+5x_2-35x_3 \\ -6x_1-3x_2+12x_3+5x_1+2x_2-9x_3+4x_1+x_2-7x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 45x_1+15x_2-80x_3 \\ 3x_1-4x_3 \end{pmatrix}$$

8)

Theorem: If A is the matrix of the L.T T wrt the basis $\bar{X}$ and $\bar{Y}$, and A' is the transformation matrix wrt the bases $\bar{X}'$ and $\bar{Y}'$. Then, there exists non-singular square matrices B and C such that

$$A' = C^{-1}AB.$$

If T is defined from V to V, then $A' = B^{-1}AB$.

Ex: $T: V_3 \to V_3$

$$\bar{X} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \longrightarrow A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\bar{X}' = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \longrightarrow A' = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ -4 & -4 & -5 \end{pmatrix}$$

$$A' = B^{-1}AB$$

Ans: $B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$

characteristic Equations:

$\rightarrow$ A is square matrix with real entries. characteristic equation of the given matrix is

$$|A - \lambda I| = 0, \quad [\det|A - \lambda I| = 0].$$

I => Identity matrix of same order as A.

Ex:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

characteristic eqⁿ:

$$\left|\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} = 0$$

The values of λ => characteristic root or Eigen values.

Ex: $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 1 \\ 3 & 1-\lambda & 2 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 2\lambda^2 + 10\lambda + 4 = 0.$$

$$(1-\lambda)(-\lambda+\lambda^2-2) - 2(-3\lambda-2) + 1(3-1+\lambda) = 0$$

$$(1-\lambda)(\lambda^2-\lambda-2) + 6\lambda+4+2+\lambda = 0.$$

$$\lambda^2-\lambda-2-\lambda^3+\lambda^2+2\lambda+7\lambda+6 = 0.$$

$$-\lambda^3+2\lambda^2+10\lambda+4 = 0$$

Eigen values and Eigen vectors of a square matrix:

$|A - \lambda I| = 0.$

$A \cdot \bar{x}_1 = \lambda_i \bar{x}_1$ [Eigen vectors satisfy this equation].

→ If λ is an eigen value of A, then λ^k is an eigen value of A^k

$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

$A_{n \times n} \Rightarrow$ atmost n eigen vectors

[for each value of λ, $\exists$ exists eigen vector]

ch eq^n

$= \left| \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0$

$= -\lambda + \lambda^2 - 2 = 0.$

$-\lambda^2 + \lambda + 2 = 0$

$-\lambda^2 + 2\lambda - \lambda + 2 = 0$

$-\lambda(\lambda - 2) - 1(\lambda - 2) = 0$

$\lambda = 2, -1. \Rightarrow$ Eigen values.

Eigen vectors?

$\underline{A} \cdot \bar{x}_i = \lambda \cdot \bar{x}_i$

$[A - \lambda I]\, \bar{x}_i = 0.$

$\begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} \bar{x}_i = 0.$

$\begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = 2$

$$\Rightarrow \begin{bmatrix} 1-2 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 = 0$$

$$x_1 - 2x_2 = 0$$

infinitely many soln.

Let $x_1 = 2 \Rightarrow x_2 = 1$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow$ Eigen vector

$\lambda = -1$

$$\begin{bmatrix} 2 & 2 \\ 1 & +1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 = 1 \Rightarrow x_2 = -1$$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$ Eigen vector.

$\rightarrow$ If λ is eigen value of A, then λ^k is an eigen value of A^K.

Proof: If λ is eigen value of A, then $\exists$ a vector $\bar{x}$ corresponding to λ; such that

$$A\bar{x} = \lambda\bar{x}.$$

$$A^k\bar{x} = A^{k-1}.\,A.\bar{x}$$

$$= A^{k-1}(\lambda\bar{x}) \quad [\because A\bar{x} = \lambda\bar{x}].$$

$$= \lambda.(A^{k-1}\bar{x})$$

$$= \lambda\,(A^{k-2}(A\bar{x})) \quad [\because A\bar{x} = \lambda\bar{x}]$$

$$= \lambda^2.\,A^{k-2}\bar{x}.\,\ldots\ldots = \lambda^k\bar{x}$$

$\Rightarrow A^k \bar{x} = \lambda^k \bar{x}$.

$\therefore \lambda^k$ is an eigen value of A^k.

* Two matrices A and A' are said to be similar if $\exists$ a non singular matrix B such that $A' = B^{-1}AB$.

<u>Theorem</u>: Two similar matrices A and $A' = B^{-1}AB$ have the same eigen values.

<u>Cayley Hamilton Theorem</u>:

A square matrix A satisfies it's own characteristic equation.

<u>Ex</u>: $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

characteristic eq^n

$\lambda^2 - \lambda - 2 = 0$.

Acc. to cayley Hamilton Theorem,

$$A^2 - A - \underset{(2I)}{2} = 0.$$

<u>Modal /spectral matrix</u>:

→ Modal matrix is the matrix formed by eigen vectors of the given matrix A.

→ Spectral matrix of a matrix A is a diagonal matrix where diagonal elements are eigen values of A.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

Modal matrix $= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

Spectral matrix = $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

→ A matrix 'A' is said to be diagonalizable, if $\exists$ a non-singular matrix P such that $D = P^{-1}AP$, where D is a diagonal matrix.

[A is said to be diagonalizable, if A is similar to a diagonal matrix D.]

for P = Modal Matrix,

D = Spectral Matrix.

for $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$;

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

→ A is not diagonalisable if some eigen values have ~~more than~~ multiplicity more than 1 (suppose m) and no. of eigen vectors formed corresponding to the eigen value is less than m.

$$D = P^{-1}AP$$

$$D^2 = P^{-1}APP^{-1}AP$$

$$D^2 = P^{-1}A^2P$$

$$D^3 = P^{-1}A^2P \cdot P^{-1}AP$$

$$D^3 = P^{-1}A^3P$$

$$\vdots$$

$$D^k = P^{-1}A^kP. \Rightarrow A^k = PD^kP^{-1}.$$

⇒ A linear transformation $T: V \to V$ is said to be diagonalizable if $\exists$ a basis X wrt which, matrix A of T is diagonalizable.

$AA^T = I$, $A^T = A^{-1}$ } orthogonal matrix

Singular Value Decompositor (SVD):

$A = USV^T$ $\qquad A_{m\times n} = U_{m\times m} S_{m\times n} V^T_{n\times n}$

S: Diagonal matrix

U, V: Orthogonal matrix.

How to find U, S and V?

$AA^T \Rightarrow$ sq. matrix

$A^TA \Rightarrow$ sq. matrix

AA^T & A^TA are similar $\Rightarrow$ same eigen values.

But $AA^T = m\times m$.

$A^TA = n \times n$.

$\therefore$ No. of eigen values = min(m, n).

If $m > n$;

n eigen values and remaining m-n eigen values for matrix AA^T are zeroes.

S => Diagonal matrix where diagonal elements are ~~diagonal~~ eigen values.

$U \Rightarrow m\times m$; $V \Rightarrow n \times n$

$\Rightarrow$ U is matrix of eigen vectors from AA^T.

V is matrix of eigen vectors from A^TA.

Ex:

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{4\times 2}$$

$U = 4\times 4$

$V = 2\times 2$

$S = 4 \times 2$

$$A\cdot A^T = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{bmatrix}_{2\times 4}$$

$$= \begin{bmatrix} 20 & 14 & 0 & 0 \\ 14 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T\cdot A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{bmatrix}_{2\times 4} \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2}$$

$$= \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

Eigen values of $A^TA \Rightarrow \begin{vmatrix} 5-\lambda & 11 \\ 11 & 25-\lambda \end{vmatrix} = 0.$

$(5-\lambda)(25-\lambda) = 121$

$125 - 30\lambda + \lambda^2 = 121$

$\lambda^2 - 30\lambda + 4 = 0.$

$$\lambda = \frac{30 \pm \sqrt{900-16}}{2}$$

$$\lambda = \frac{30 \pm \sqrt{884}}{2}$$

S = diagonal matrix of eigen values

$$\begin{bmatrix} \frac{30+\sqrt{884}}{2} & 0 \\ 0 & \frac{30-\sqrt{884}}{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 29.8 & 0 \\ 0 & 0.115 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

V from A^TA

$$\begin{bmatrix} 5-\lambda & 11 \\ 11 & 25-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\begin{bmatrix} (5-\lambda)x_1 + 11x_2 \\ 11x_1 + (25-\lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(5-\lambda)x_1 + 11x_2 = 0.$

$11x_1 + (25-\lambda)x_2 = 0.$

$\lambda = 29.8$

$-24.8x_1 + 11x_2 = 0.$

$11x_1 - 24.8x_2 = 0.$

$-(24.8)^2 x_2 + 11^2 x_2 = 0$

$11x_1 = 24.8x_2$

$x_1 = \frac{24.8}{11}x_2$

$11 \times (-24.8)x_1 + 121x_2 = 0$

$(24.8) \times 11x_1 - (14.8) \times (24.8)x_2 = 0$