The Method of Infinite Descent

> Berkeley Math Circle Intermediate II | Adwait Godbole | 13th Nov 2024

Warmup: the numbers.

0. What are integers, natural numbers, rational numbers, irrational numbers?

1. Prove that $\sqrt{2}$ is irrational.

   Proof (fill in the blanks).

   > We will do a proof by ________________. Suppose that $\sqrt{2}$ is rational. Then, there exist integers a and b such that $\sqrt{2} =$ ___________ (Eq. 1). We can assume that a **is the smallest such number**. Squaring both sides of Eq. 1, we get $2 =$ ___________, and hence, $a^2 = 2b^2$ (Eq. 2). This means that a^2 is even, and hence a is ________. Let $a = 2k$. Substituting this back in Eq. 2, we get ___________. This implies that b^2 is even, and hence b is even. Let $b = 2l$. Then $\sqrt{2} = \frac{a}{b} = \frac{2k}{2l} = \frac{k}{l}$. This contradicts our assumption that ________________ (why?).

Infinite Descent

0. **Going down the number line.** Which of these statements are true? Can you prove/disprove them?

   - **S1**: Suppose you have *integers* $a_1, a_2, a_3, \ldots, a_n$ for some n such that $a_i > a_{i+1}$. Then it cannot be that all the numbers are non-negative.
   - **S2**: Suppose you have *integers* $a_1, a_2, a_3, \ldots$ to infinity such that $a_i > a_{i+1}$. Then it cannot be that all the numbers are non-negative.
   - **S3**: Suppose you have *rational* numbers $a_1, a_2, a_3, \ldots$ to infinity such that $a_i > a_{i+1}$. Then it cannot be that all the numbers are non-negative.

1. **Well-ordering principle**: Every non-empty set S of non-negative integers has a smallest element.

   Proof (fill in the blanks).

   > Proof by contradiction. Suppose S has no least element, and let J is the set of numbers not in S. We will prove by induction that $J = \mathbb{N}$. Base case: Since ______ is the smallest non-negative integer, $0 \notin S$, and hence, ______________. Inductive case: Now suppose that $0, 1, \cdots, n \in J$. Then, we know that $n + 1 \notin$ ______ due to our assumption that ______________________________. Hence, ______________. This proves that $J = \mathbb{N}$. But this implies that ______________________, which is a contradiction. Hence, S has a least element.

The Method of Infinite Descent:

> Infinite descent is a proof technique that is used to prove that *a certain property does not hold* for a set of well-ordered numbers. We assume that the property holds for some number within that set, and then show that it also holds for a smaller number. Then, by the well ordering principle, we get a contradiction. This contradiction implies that the property does not hold for any number in the set.

More examples of Infinite Descent (resource1, resource2)

Diophantine Equations

0. Prove that the equation $x^3 + 3y^3 + 9z^3 = 0$ has no positive integer solutions.

 Proof (fill in the blanks).

 > Suppose that there is a positive integer solution. Consider (x, y, z) to be such a solution. Then, we know that 3 divides ____, let $x = 3k$. Then, we have, $27k^3 + 3y^3 + 9y^3 = 0$ and, dividing by 3, ________________________. Hence 3 divides ____. Let $y = 3l$. Substituting this back and dividing by 3, we get ________________________. This implies that 3 divides ____. Let $z = 3m$. Substituting this back and dividing by 3 *again*, we get ________________________, which means that (k, l, m) is also a solution. But, (k, l, m) is a __________ solution than (x, y, z). Hence, we get an infinite __________, which is a contradiction.

Side note: These are Diophantine equations (notes). A Diophantine equation is an equation where the unknowns are required to be integers. The name comes from the Greek mathematician Diophantus, who studied such equations. Can you write irrationality of $\sqrt{2}$ as a Diophantine equation with no (integer) solutions?

1. Prove that the equation $x^2 + y^2 = 3z^2$ has no *non-trivial* integer solutions.

2. **Fermat's Last Theorem** (Wikipedia article): There are no integer solutions to the equation $x^n + y^n = z^n$ for $n > 2$. *Warning*: This is a very very hard problem!.

3. **Fermat's Last Theorem Mini**: There are no non-trivial integer solutions to the equation $x^4 + y^4 = z^4$. This is an easy special case of Fermat's Last Theorem.

Non-algebraic examples of Infinite Descent (more problems)

4. (Sylvester's Problem) There are n points in the plane such that every pair of two points has another point that is collinear with them. Prove that all the points are collinear.

5. Consider triangle ABC. A perpendicular is drawn from A to BC at A_1. A perpendicular is drawn from A_1 to AC at B_1. A perpendicular is drawn from B_1 to AB at C_1. This process is repeated to get points A_2, B_2, C_2, and so on. Prove that all the points A_i, B_i, C_i are distinct.

Vieta Root Jumping (notes on Vieta Root Jumping/Flipping)

6. **IMO 1988 P6:** Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer.

Arthur Engel (Problem Solving Strategies: an awesome problem book by Engel) writes about the problem:

> Nobody of the six members of the Australian problem committee could solve it ... Since it was a number theoretic problem it was sent to the four most renowned Australian number theorists. They were asked to work on it for six hours. None of them could solve it in this time ... The problem committee submitted it to the jury of the 29th IMO marked with a double asterisk, which meant a superhard problem, possibly too hard to pose. After a long discussion, the jury finally had the courage to choose it as the last problem of the competition. Eleven students gave perfect solutions.