Problems on PAMs and PFAs

December 2, 2019

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Abstract

In this report we will explore two systems - piecewise affine maps (PAMs) and probabilistic finite automata (PFAs) - by viewing them as transformers of state vectors. We will study properties of these transition systems such as mortality, convergence and escape. Additionally, we will uncover some connections between these problems.

Contents

1 Preliminaries

We will view probabilistic finite automata as transformers of state distribution vectors. We first formally define PFAs.

Definition 1 (PFA)**.** *A probabilistic finite automaton is a tuple $\mathcal{A} = (S, \Sigma, (M_\alpha)_{\alpha\in\Sigma})$ where S is a finite set of states, Σ is the alphabet of actions and $(M_\alpha)_{\alpha\in\Sigma}$ is a set of stochastic matrices which specify how the probability distribution in each state gets transformed for each choice of action α.*

We recall the definition of one-step strategies for a PFA or MDP from [1].

Definition 2 (Strategy for an MDP)**.** *A one step strategy τ for an MDP is a map $\tau : \Sigma \times S \to [0, 1]$ satisfying $\sum_{\alpha\in\Sigma} \tau(\alpha, s) = 1$ for all $s \in S$. This strategy is associated with the stochastic matrix $M_\tau = \sum_{\alpha\in\Sigma} \tau(\alpha, s)M_{(\alpha,i)}$.*

Definition 3 (Strategy for a PFA)**.** *A one-step strategy τ for a PFA is a map $\tau : \Sigma \to [0,1]$ satisfying $\sum_{\alpha\in\Sigma} \tau(\alpha) = 1$. This strategy is associated with the stochastic matrix $M_\tau = \sum_{\alpha\in\Sigma} \tau(\alpha) M_\alpha$.*

A strategy σ then is simply a concatenation of one step strategies $\tau_1\tau_2\cdots$. The state probability distribution reached after m-steps starting from $\mathbf{v}$ is then given by $\mathbf{v} \cdot M_{\tau_1} M_{\tau_2} \cdots M_{\tau_m}$

Equivalence of Views We note that this definition of strategy of an MDP is equivalent to the usual definition of a fully observable strategy (i.e. that the state of the MDP is known each time an action is taken).

This is easy to see as follows. Let the conventional strategy be $\rho = \tau_1'(s_1)\tau_2'(s_2)\cdots$ where each $\tau_i'(s_i)$ is a stochastic one step (PFA) strategy chosen after observing state s_i. Now we can combine the strategies taken after observing each $s \in S$ to get an (MDP) strategy $\sigma = \tau_1\tau_2\cdots$, where τ_i is such that $\tau_i(s,\cdot) = \tau_i'(s)$ for all $s \in S$.

Equivalence of MDPs and Row-closed PFAs We first define row-closed PFAs as follows.

Definition 4 (Row-closed PFAs)**.** *An n-state PFA is said to be row closed if for two actions, α and β and every row index $i \in [n]$, there exists an action γ such that $M_\alpha[j] = M_\gamma[j]$ for each $j \neq i$ and $M_\beta[i] = M_\gamma[i]$, where $M[j]$ represents the j^{th} row of matrix M.*

Intuitively, we obtain a row closed PFA by exchanging corresponding rows across the transition probability matrices of any general PFA. The class of MDPs are in fact equivalent to the class of row-closed PFAs.

This is seen as follows. For any (MDP) strategy, we get a set of linear equations whose variables given the corresponding row-closed PFA strategy. Since, the number of variables ($|\Sigma|^{|S|}$) is always greater than the number of equations ($|S||\Sigma|$), we always find a solution. On the other hand, given a row-closed PFA, it is trivially equivalent to its MDP counter-part. Hence the classes are equivalent.

2 Safety and Escape Problems for PFAs

We first survey some results from [1], in which they consider the safety problem for PFAs w.r.t. a polytope. This problem is formalized as follows.

Safety w.r.t a polytope Let $\mathcal{A}$ be a PFA over n states and let H be a convex polytope over $\mathbb{R}^n$. This polytope is generated by the intersections of half spaces and is made stochastic, i.e. it is further intersected with the spaces $\sum_i x_i \leq 1$ and $\sum_i x_i \geq 1$ and $0 \leq x_i \leq 1$ for all i.

Now, a stategy $\sigma = \tau_1\cdots$ is said to be H-safe starting from $\mathbf{v} \in H$ if for all $m \in \mathbb{N}$, $\mathbf{v} \cdot M_{\tau_1} \cdot M_{\tau_2} \cdots M_{\tau_m} \in H$. This essentially means that there is a strategy such that the state probability distribution vector always remains inside the safe region H. Now dependending upon the initial points considered there are two variants to this problem. The existential variant asks whether there is a $\mathbf{v}_{init} \in H$ such that, there exists an H-safe strategy starting from $\mathbf{v}_{init}$. The universal variant asks whether $\forall \mathbf{v}_{init}$ there is a H-safe strategy starting from $\mathbf{v}_{init}$. In, [1], they show that the existential variant is undecidable while the universal variant is in EXPTIME.

Escape problem They also remark that there is a related problem called the escape problem. This is defined conversely to the safety problem. More formally, we can ask, whether there exists a strategy σ, such that from the starting vector $\mathbf{v}$, there exists an $m \in \mathbb{N}$, such that, $\mathbf{v} \cdot M_{\tau_1} \cdot M_{\tau_2} \cdots M_{\tau_m} \notin H$. That is eventually the trajectory escapes the polytope H. Once again we can define existential and universal variants for this problem.

Existential escape The existential version can be approached as follows. Firstly, we note that, it is sufficient to check that that there is a single step strategy to escape the polytope. Now, we also have that

$$\mathbf{v} = \sum_{\mathbf{u}_s \in V} \alpha_s \mathbf{u}_s$$

where V represents the vertex set of the polytope and $\alpha_s \in [0,1]$ with $\sum_s \alpha_s = 1$. Hence, we note that if the internal point eventually escapes with single step strategy τ, then one of the corner points must also escape with this same strategy. Also, we search over the (infinite) set of single strategies by simply considering a single action at a time. Due to linearity, if we have a mixed strategy that escapes, we must also have a pure strategy that does the same. We can hence decide existential escape in PTIME, given the V-represenation of the polytope.

Universal escape Universal escape is harder to check since the the individual points might have distinct strategies to escape. Moreover we cannot uniformize these strategies since the complement (region outside) of the polytope is non-convex and hence, we cannot use typical linear algebraic arguments. Still we note some observations.

Proof Ideas. For any initial point, $\mathbf{v}_{init}$, if there is a mixed strategy σ that escapes the polytope then there is also a pure strategy σ' that escapes the polytope. Hence, we can restrict our attention only to pure strategies. Now, in the same spirit as [1] let H_{win} be define as the set of initial distributions which cannot escape for H any strategy σ. We note that H_{win} is a convex closed polytope. Now, the problem reduces to the deciding the emptiness of H_{win}. We want the corners of H_{win} to map inside it for all choices of actions. If we had a bound on the number of corner points in H_{win}, we could write this as a linear program and check for satisfiability. But, a priori this seems to be difficult. For this consider the following example (in two dimensions) of a linear transformation that rotates the vector by an irrational multiple of π. Clearly then, the safe region for a point, would in fact be the circle containing that radius vector. The number of 'corner' points in this case is not even finite. However, we note that we have not made use of the original polytope. It is still unclear how we can make use of this information to obtain an effective characterization $\square$

3 Decision problems on PAMs

We consider the mortality problem as discussed in [2] for piece-wise affine maps. In particular we note that their proof of undecidability relies on the makes use of a variable number of partitions (see Remark 2 in [2]). It is remarked that the decidability of the mortality problem with a fixed number of partitions is open.

Consider a partitioning of a metric space X, generated by a set of half-planes. For each such partition created we define an affine transform operating on that partition. Formally, we define a piecewise affine map as follows.

Definition 5 (Piece-wise Affine Map)**.** *Consider a partitioning of $\mathbb{R}^k$ generated by intersections of finitely many half spaces. To this end consider the matrix $C \in \mathbb{Q}^{m\times n}$ and the vector $d \in \mathbb{Q}^m$. Let $sgn(x)$ be the vector extension of the sign function. Then, $sgn(Cx + d) \in \{0, 1\}^m$ for each vector $x \in \mathbb{R}^n$. Hence, we essentially get 2^m partitions, denoted by H_i; $i = \{1, 2, \cdots, 2^m\}$, generated by C and d. We have $\mathbb{R}^n = \bigcup_i H_i$. Consider a map $g : \mathbb{R}^n \to \mathbb{R}^n : x \to A_i x + b$ where $x \in H_i$, for some $A_i \in \mathbb{Q}^{n\times n}$ and $b_i \in \mathbb{Q}^n$. Then g is said to be a piecewise affine function (or piecewise affine map).*

Definition 6 (Decision Problems for Piecewise Affine Maps)**.** *Given a map $f : X \to X$ on a metric space X, with some arbitrary point (chosen as origin) at 0, we define the following problems related to the behaviour of f:*

1. *f is said to be mortal, if for all points $x_0 \in X$, the trajectory $x_{t+1} = f(x_t)$, reaches 0, i.e. $\exists t' \geq 0$ such that $x_{t'} = 0$.*

2. *f is globally convergent if for every initial point $x_0 \in X$, the trajectory $x_{t+1} = f(x_t)$ converges to 0.*

The PAM (global convergence) mortality problem asks, given a piecewise affine map f, whether f is (globally convergent) mortal.

These problems were considered by V. D. Blondel et. al. in [2] and shown to be undecidable. Their proof uses a reduction from the universal halting problem for 2-counter machines. Essentially for each program counter value, they define a unique partition and define the function operating in that partition so that it follows the control flow of the 2CM. Indeed the tricky part is having two possible next counter values for when the current counter has the 'zero check or decrement instruction'. Even then the number of partitions used by the the reduction increases with the number of counter values in the 2CM instance.

4 Undecidability for mortality and convergence with fixed number of partitions

In this section, we wish to give a proof of undecidability of the mortality problem with a fixed number of partitions, solving the above mentioned problem. More precisely we show the following.

Theorem 1. *Let g be a piece-wise affine function, with a fixed-number of partitions. Then the mortality problem for g is undecidable.*

The rest of this section is devoted to the proof of this theorem. We reduce from the well known Post's Correspondence Problem (PCP) [3]. More precisely, for a given PCP instance we will provide a piece-wise affine map f such that the PCP instance admits a solution if and only if f is mortal.

We note that PCP is undecidable even for $k = 7$, where k is the size of the instance. We crucially use this to ensure that we have a fixed number of partitions. Keeping this in mind, henceforth, we will assume that the the PCP instance has k number of partitions, where k is fixed and can be as small as 7. Moreover for each instance of PCP we note that there is a reverse instance obtained by reversing each string in α and β. It is easy to see that this reverse instance has a solution if and only if the original instance does too.

We reduce from fixed sized instances of PCP. Assume that the sized of the instance is k, and has been specified as $\alpha[i]$, $\beta[i]$, $i \in \{1, \ldots, 7\}$. We will define a function $g : \mathbb{R}^9 \to \mathbb{R}^9$.

We consider the following as a running example.

Example 1. *A simple PCP instance which has a solution*

Consider the following PCP instance $\alpha = [12, 22, 111]$ and $\beta = [121, 122, 1]$. It is easy to see that this has a solution given by the sequence of indices, $1, 3, 2$; which gives the string 1211122. The reverse instance corresponding to this is given by $\alpha_{rev} = [21, 22, 111]$ and $\beta_{rev} = [121, 221, 1]$. Henceforth, we maintain this convention and denote the reverse instance by α_{rev} and β_{rev}.

We start by fixing some predicates. Let $\mathbf{z}(x)$ denote `true` if x equals zero else `false`. Let $\mathbf{s}(x, y, z) = \neg(\mathbf{z}(x) \wedge \mathbf{z}(y) \wedge \mathbf{z}(z)) \wedge (\mathbf{z}(x) \vee \mathbf{z}(y) \vee \mathbf{z}(z))$ (i.e., some but not all variables are zero). Let $\mathbf{a}(x, y, z) = (\mathbf{z}(x) \wedge \mathbf{z}(y) \wedge \mathbf{z}(z))$ (i.e., all the variables are zero). Moreover we define the predicate $\mathbf{c}(x)$ as true when $0 \leq x < 1$ and false otherwise. We define g as follows; where are g_1 and g_2 are macros defined later, after providing some intuition.

$\mathbf{g(f_i, f_a, f_b, r_i, r_a, r_b, s, c_f, c_r)}$	*partition*
0	any variable < 0
0	any variable amongst fs or rs ≥ 1
0	$s \neq 1, 2$
0	$\mathbf{s}(f_i, f_a, f_b) \vee (\mathbf{a}(f_i, f_a, f_b) \wedge \neg\mathbf{c}(c_f))$
0	$\mathbf{s}(r_i, r_a, r_b) \vee (\mathbf{a}(r_i, r_a, r_b) \wedge \neg\mathbf{c}(c_r))$
0	$\mathbf{a}(f_i, f_a, f_b) \wedge \mathbf{a}(r_i, r_a, r_b)$
$g(f_i, f_a, f_b, r_i, r_a, r_b, 3 - s, c_f, c_r)$	$((\mathbf{a}(f_i, f_a, f_b) \wedge \mathbf{c}(c_f)) \vee (\mathbf{a}(r_i, r_a, r_b) \wedge \mathbf{c}(c_r)))$ $\wedge\, f_a = f_b \wedge r_a = r_b \wedge (\neg(\mathbf{a}(f_i, f_a, f_b) \wedge \mathbf{a}(r_i, r_a, r_b)))$
$g_1(f_i, f_a, f_b, r_i, r_a, r_b, 1, c_f, c_r)$	$\neg(\mathbf{a}(f_i, f_a, f_b) \vee \mathbf{a}(r_i, r_a, r_b)) \wedge s = 1$
$g_2(f_i, f_a, f_b, r_i, r_a, r_b, 2, c_f, c_r)$	$\neg(\mathbf{a}(f_i, f_a, f_b) \vee \mathbf{a}(r_i, r_a, r_b)) \wedge s = 2$

The intuition is that, $f_i = 0.d_1 d_2 d_3 \cdots$ will represent a solution to the PCP instance where each digit d_i corresponds to an index in the PCP instance provided. Even then, crucially we cannot lead such a solution to a known immortal point, else the function will be trivially immortal. All immortal points must therefore be implicit in the sense that each such point must reveal the solution to the PCP instance. Taking forward this idea we note that we can, using this PCP instance, oscillate between points in $\mathbb{R}^7$, without hitting zero. The main idea is that f_i and r_i will represent the sequences of indices for the PCP instance solution. We assume that all the number are represented in base 10 and that $k = 7$. We first introduce some notation, let $\mathsf{digit}(x)$ denote the first digit after the decimal place of x.

g_1 is defined as follows (the primed variables signify the new values - after applying the function).

$$\begin{aligned} g_1(f_i, f_a, f_b, r_i, r_a, r_b, 1, c_f, c_r) \;=\; \Big\langle & f_i \cdot 10 - \mathsf{digit}(f_i), \\ & f_a \cdot 10^{|\alpha[\mathsf{digit}(f_i)]|} - \alpha[\mathsf{digit}(f_i)], \\ & f_b \cdot 10^{|\beta[\mathsf{digit}(f_i)]|} - \beta[\mathsf{digit}(f_i)], \\ & (r_i + \mathsf{digit}(f_i))/10, \\ & (r_a + \alpha_{rev}[\mathsf{digit}(f_i)])/10^{|\alpha[\mathsf{digit}(f_i)]|}, \\ & (r_b + \beta_{rev}[\mathsf{digit}(f_i)])/10^{|\beta[\mathsf{digit}(f_i)]|}, 1, c_f - 1, c_r + 1 \Big\rangle \end{aligned}$$

In case $\mathsf{digit}(f_i) \geq 8$, we once again set $g(x)$ to zero. Essentially we check the first digit of f_i following the decimal point, to see which PCP index to match next. g_2 is defined next, similar to g_1 but reading from r and writing back to f.

$$\begin{aligned} g_2(f_i, f_a, f_b, r_i, r_a, r_b, 1, c_f, c_r) = \quad & \Big\langle (f_i + \mathsf{digit}(r_i))/10, \\ & (f_a + \alpha_{rev}[\mathsf{digit}(r_i)])/10^{|\alpha[\mathsf{digit}(r_i)]|}, \\ & (f_b + \beta_{rev}[\mathsf{digit}(r_i)])/10^{|\beta[\mathsf{digit}(r_i)]|}, \\ & r_i \cdot 10 - \mathsf{digit}(r_i), \\ & r_a \cdot 10^{|\alpha[\mathsf{digit}(r_i)]|} - \alpha[\mathsf{digit}(r_i)], \\ & r_b \cdot 10^{|\beta[\mathsf{digit}(r_i)]|} - \beta[\mathsf{digit}(r_i)], 1, c_f + 1, c_r - 1 \Big\rangle \end{aligned}$$

We now claim that g so defined is immortal if and only if the PCP instance has a solution.

Considering our running example [1] we can see that the starting point $x = \langle 0.132, 0.12211122, 0.12211122, 0, 0, 0, 1, 3, 0\rangle$ is indeed immortal. Since $s = 1$ initially, the elements of the PCP instance are transferred from f_i, f_a and f_b to r_i, r_a and r_b respectively albeit in reverse direction. Eventually then we end up with the configuration $\langle 0, 0, 0, 0.231,\ 0.22111221,\ 0.22111221, 2, 0, 3\rangle$. Now, since $s = 2$, the reverse process happens and the elements are transferred to f_i, f_a and f_b from r_i, r_a and r_b respectively. This process continues ad-infinitum and since all the points encountered are non-zero, they are all immortal and consequently, g is immortal.

Lemma 1. *The PCP instance has a solution if and only if g as defined above has an immortal point.*

Proof. One direction is to easy to see. If the PCP instance has a solution given by the sequence of indices $I = i_1, i_2, \cdots i_l$, then, we may begin with the point $x = (0.i_1 i_2 \cdot\cdot, 0.\alpha[i_1]\alpha[i_2]\cdot\cdot\alpha[i_l], 0.\beta[i_1]\beta[i_2]\cdot\cdot\beta[i_l], 0, 0, 0, 1, l, 0)$ where $|I|$ denotes the length of the sequence of indices. Clearly we read off from f_i until it becomes zero. Note that when f_i becomes zero so do f_a, f_b and c_f. Moreover once this happens, we have $c_r = l$, $r_i = 0.\overline{rev(i_1 i_2 \cdot\cdot i_l)}$ $r_a = 0.\overline{rev(\alpha[i_1]\alpha[i_2]\cdot\cdot\alpha[i_l])}$, $r_b = 0.\overline{rev(\beta[i_1]\beta[i_2]\cdot\cdot\beta[i_l])}$, where $rev()$ of a sequence denotes the reverse sequence. Note that we have $r_a = r_b$ (and trivially $f_a = f_b = 0$). Once this happens (with $s = 1$), we set $s \leftarrow 3 - s$ and the same process repeats in the reverse direction as sequences are read off from r and written back into into f. This cycle repeats ad infinitum and hence all these points are immortal (clearly none of them is 0).

For the converse, we assume that the function g is immortal, more specifically for some point x. Clearly we must have the s element of x to be either of 1 or 2. WLOG let us assume it is 1. Then following the definition of g, we read off symbols from f_i until one of f_i, f_a or f_b hits zero. Moreover this process continues for at-most $\lfloor c_f \rfloor$ many steps. Hence exactly after c_f steps either, (a) all of f_i, f_a, f_b, c_f reach zero or (b) only some reach zero. In the latter case, at the next iteration the function goes to zero and hence x is mortal. In the former case, we begin the reverse direction after setting $s = 2$ if $r_a = r_b$. In this case if we are able to return r_i, r_a, r_b, c_r to zero simultaneously (only way in which we can have immortality), then the indices in r_i constitute a solution to the PCP instance. If we are not able to reach zero simultaneously for r_i, r_a, r_b, c_r then we set the value of g to zero once again making the initial point x mortal. Moreover, the phase in which we read from r is also bounded by $\lfloor c_r \rfloor$ and hence must terminate. This shows that any immortal point x leads us to a solution of the PCP instance and hence the claim is proved. $\square$

We note that $\mathbf{0}$ is a fixed point of the function defined above. Hence, in this case mortality is in fact equivalent to global convergence. Hence we have the following corollary.

Corollary 1. *The global convergence problem for PAM is undecidable with even with a fixed (but suitably large) number of partitions*

5 Convergence and Mortality on restricted instances

5.1 One partition

Lemma 2. *The PAM instance with a single map is globally convergent iff if for all eigenvalues λ_i, we have $|\lambda_i| < 1$.*

Proof. We take cases on the eigenvalues of the transition matrix corresponding to the single map.

- If all eigenvalues are real, we consider the dominant eigenvalue. If $\max |\lambda_i| < 1$ in magnitude then we are done and the sequence converges. Otherwise, x_n may diverge away from 0. This is true since the vector obtained after the k^{th} iteration from vector $\mathbf{u}$ can be written as $\sum_i \alpha_i \lambda_i^k \mathbf{v}_i$ where $\mathbf{v}_i$ denote the eigenvectors of the matrix and α_i denotes the components of $\mathbf{u}$ along the $\mathbf{v}_i$s.

- If a matrix has a complex eignevalue λ and eigenvector $\mathbf{v}$ then we have the following:

$$A\overline{\mathbf{v}} = \overline{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$$

  Hence, in fact, even $\overline{\lambda}$ and $\overline{\mathbf{v}}$ is an eigen value and eigen vector pair for this matrix. We than consider $\mathbf{v} + \overline{\mathbf{v}}$ as the initial vector. Clearly then, we have $A^t(\mathbf{v} + \overline{\mathbf{v}}) = 2 \cdot Re(\lambda^t \mathbf{v})$. This quantity goes to zero $\iff$ $|\lambda| < 1$.

Since the eigenvectors form a basis for $\mathbb{C}^d$, we have that $|\lambda_i| < 1 \iff$ the PAM converges to the origin. □

Mortality with one partition We can take a similar approach for the mortality problem with one partition. If the matrix A has any non-zero eigenvalue λ, then the system is not mortal. This is easy to see, since we can take the eigenvector $\mathbf{v}$ corresponding to λ. Then we have, for all t,

$$A^t\mathbf{v} = \lambda^t\mathbf{v} \neq 0$$

Hence the initial point corresponding to $\mathbf{v}$ is immortal.

5.2 Two partitions

Now let us consider the system with two partitions.

In the case of two partitions, we try to apply an idea similar to the one for the one partition case. We have two cases, depending on whether the dividing hyper-plane passes through the origin.

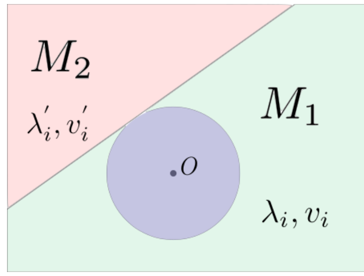

Figure 1: Case 1

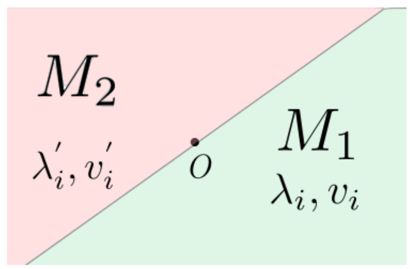

Figure 2: Case 2

Case 1: Hyperplane does not pass through the origin In this case, we basically have a ball $\mathcal{B}$ of non-zero radius centred at O and that completely lies inside the first (green) partition.

In this, if we have $\lambda_i \geq -1$ for some i, then $\mathbf{v}_i$ is an immortal point. Additionally, if we have complex eigenvalue then we reason similarly to the single partition case. More formally, we have that $A^t\mathbf{v} = 2 \cdot Re(\lambda^t\mathbf{v})$. Hence if $|\lambda| < 1$, we can find a point inside the ball, from where we have a sequence of iterates that always belong to the $\mathcal{B}$ and never hit zero. The cases that remain are when each eigenvalue is such that either - (1) $\lambda < -1$ or (2) λ is complex and $|\lambda| > 1$. In both of these cases, we have a diverging sequence of iterates and hence, we will eventually hit the second (red) partition. Hence, for this case, we need to reason about the second partition as well.

Case 2: Hyperplane does pass through the origin In this case, we have that the origin lies on the line separating the two hyperplanes.

The case when there is atleast one positive eigenvalue λ_i in either of the partitions immediately gives us immortality since, we can choose $\mathbf{v}_i$ as the initial vector and $\lambda_i^t \cdot \mathbf{v}_i \neq \mathbf{0}$ for all t.

Open Cases We hence note that following cases of the two partition problem are still unsolved:

1. Hyperplane not passing through the origin with each eigenvalue λ being such that (a) $\lambda < -1$ or (b) λ is complex with $|\lambda| > 1$

2. Hyperplane passes through the origin with each eigenvalue being such that $\lambda \leq 0$ or λ is complex

6 PFA-PAM Bisimulation

6.1 Bounded PAM

We note that the above proof of undecidability in section 4 works even when the partition space is bounded. Indeed the only place where the current version poses a problem is that the counter may get unbounded. But we can resolve this by using an encoding such as *counter* $\sim \frac{1}{2^c}$ for the counter. Then the counter values will be bounded within the interval $[0, 2)$.

Now, using the fact that the partitions and the piecewise defined functions are all affine we can consider a PAM problem defined on $[0, 1]^d$ without losing generality. We call the function a bounded such as one mentioned above as the bounded PAM.

Corollary 2. *The mortality and global convergence problems for bounded PAM are undecidable even for a fixed number of partitions*

6.2 PAM to PFA

We wish to develop a bisimulation between the sequence generated of states reached in the PAM problem and distributions reached in the case of a PFA (probabilistic finite automaton). We recall the definition of PFA as presented earlier 1.

Challenges in the bisimulation In the PAM case, we are constrained to use a specific affine function depending upon the partition. In the PFA case however, the agent any select an arbitrary (possibly randomized) strategy. Hence we need to somehow need to enforce the choice of actions by making use of the fact the the distribution needs to be constrained within a polytope.

In order to generate this bisimulation, we will proceed in two steps. We will show that for any bounded PAM f we have a bisimulation $f \leftrightarrow \mathcal{M}$ for some newly defined transition system $\mathcal{M}$. Additionally we will demonstrate the bisimulation $\mathcal{M} \leftrightarrow \mathcal{A}$ to prove our claim.

We will first define the intermediate transition system we consider.

Definition 7 (Single Partition Multi Map)**.** *Consider a bounded metric space $\mathcal{X}$. Let for $i \in [m]$, let $A_i \in \mathbb{Q}^{n\times n}$ and $b_i \in \mathbb{Q}^n$ generating the following maps $g_i : \mathbb{R}^n \to \mathbb{R}^n : a \to A_i x + b_i$. Now the transition is defined as choosing any of the m, actions from the current point inside $\mathcal{X}$.*

We will first start with an example illustrating both of these steps.

Example 2. *Consider the bounded PAM f, defined on $[-1, 1]^2$. We consider the case with just four partitions generated by the lines $x \geq 0$ and $y \geq 0$. Let the maps be M_1, M_2, M_3 and M_4 for the four partitions corresponding to $[+, +], [-, +], [-, -], [+, -]$.*

We first demonstrate how we can have a bisimulation between this PAM and a suitably designed SPMM system. This part is in fact easy to see. The idea is to introduce new dimensions, one for each half-plane constraint in the original PAM instance. For the running example then, we consider a 4-dimensional space with (after adding 2 extra dimensions). Now each of these new dimensions are bounded below by 0 and above by a sufficiently large (finite) constant t. The space $\mathcal{X}$ then becomes $[-1, 1]^2 \times [0, t]^2$.

Note the main challenge in the SPMM system is that the transition may be selected arbitrarily from the set of available ones. However we still want to enforce the constraint from PAM, that being the choice of linear transformation respects the current partition. The additional dimensions are used to achieve this.

More concretely, we define the actions as follows.

$$M_1' = \begin{bmatrix} M_1 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \multicolumn{2}{c}{\mathbf{0}_{2\times 4}} \end{bmatrix} \qquad M_2' = \begin{bmatrix} M_2 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \\ \multicolumn{2}{c}{\mathbf{0}_{2\times 4}} \end{bmatrix}$$

$$M_3' = \begin{bmatrix} M_3 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \\ \multicolumn{2}{c}{\mathbf{0}_{2\times 4}} \end{bmatrix} \qquad M_4' = \begin{bmatrix} M_4 & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \\ \multicolumn{2}{c}{\mathbf{0}_{2\times 4}} \end{bmatrix}$$

Now, noting that the values in the last two dimensions must be non-negative, we essentially have enforced that each M_i' may only be taken in its respective quadrant. More, generally we incorporate the equations corresponding to the hyper-planes in the linear transforms, and this along with the constraint on the added dimensions enforces the necessary condition.

Now we move to the second part of the bisimulation which is to show that a SPMM can be transformed into a PFA on a polytope.

First, we note that since the space of interest is bounded in all dimensions, we can make the system stochastic by applying an affine transformation to the entire state space. In our running example, this can be seen as follows.

For instance in the above example we first shift the space by the vector $\langle 1,1,0,0\rangle$ to the metric space $\mathcal{X} = [0,2]^2 \times [0,t]^2$. Now we scale this space by $1/2$ and $1/t$ in both directions to get, the space $[0,1]^4$. This now can be made stochastic by shrinking the current states even further and adding a fresh 'trash' state which will hold all the probability mass not in the current states. We note that the polytope that the original constraints genrated by the SPMM are still maintained by the current polytope that we will consider.

Now what remains is defining the new set of actions that will be permitted in the polytope. In fact these actions are simply stochastic versions of the actions in the SPMM.

References

[1] Akshay, S., Genest, B., and Vyas, N. Distribution-based objectives for markov decision processes. *CoRR abs/1804.09341* (2018).

[2] Blondel, V. D., Bournez, O., Koiran, P., Papadimitriou, C. H., and Tsitsiklis, J. N. Deciding stability and mortality of piecewise affine dynamical systems. *Theoretical Computer Science 255*, 1-2 (2001), 687–696.

[3] Post, E. L. A variant of a recursively unsolvable problem. *Bulletin of the American Mathematical Society 52*, 4 (1946), 264–268.