Approximately Optimal Arm Identification

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Group 10

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The Setting

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Intuitively, ϵ is the tolerance in returning sub-optimal arms

Problem Statement

Given a bandit instance, the objective is to return a set of m, (ϵ, m)-optimal arms.

Considering the PAC framework, we want an algorithm that, with probability atleast $1 - \delta$, terminates and returns m such arms.

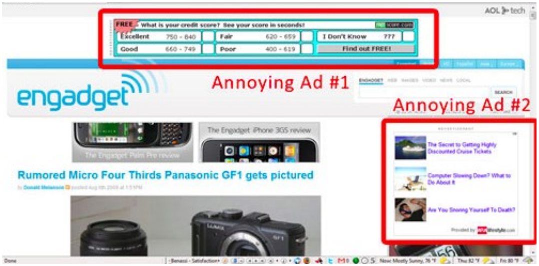
Applications

Webpage Advertisement Problem

A website owner has n options of advertisements but can only put up an m sized subset. His objective is to identify the m most likely to bring in more revenue.

Assumption: Viewership is roughly constant.

$\Rightarrow$ In this scenario, we model the quantized up-time as pulls and clicks as reward 1.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Annoying Ad #2
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Excellent 750 - 840
Fair 620 - 659
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Preliminary Approaches

Kalyanakrishnan, et. al. in [2] suggest some approaches towards algorithms for this problem.

The intuitive idea is that if an arm is sampled sufficiently large number of times, the confidence bounds on that arm is sufficiently tight and hence, with large probability (ϵ, m)-optimal arms will be returned.

We will briefly look at these.

Preliminary Approaches

Direct Algorithm

Each arm is sampled a sufficiently large number of times - $O(\frac{2}{\epsilon^2}\log(\frac{n}{\delta}))$ times. Then the empirically the best m arms are selected.

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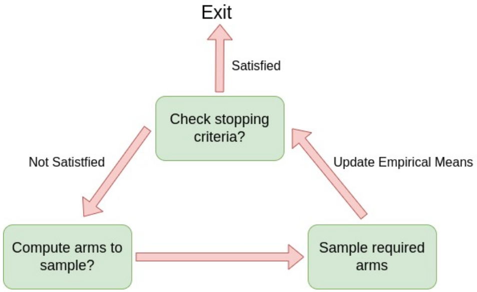
Halving Algorithm

The main idea of Halving is to perform binary search on the sequence of arms. This brings down sample to $O(\frac{m}{\epsilon^2}\log(\frac{m}{\delta}))$.

General Algorithmic Framework

Algorithm 1: A general algorithm

```
Result: $m$ $(\epsilon, m)$-optimal arms
initialize $history$;
while !shouldStop($history$) do
    $selectedArms$ = selectArms(history) ;
    $result$ = sample($selectedArms$) ;
    update($history$, $result$) ;
end
return topK($history$)
```



LUCB1

Sampling Strategy

At every pull, we maintain the sets of 'top' (m) and 'bottom' ($n - m$) arms. The paper discusses a greedy sampling strategy which is quite intuitive. According to this, we sample the arms which are more likely to have been misclassified.

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$$h_*^t = \operatorname*{argmin}_{h \in Top^t} \{\hat{p}_h^t - \beta(u_h, t)\}$$
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Here $\beta(u, t)$ is a function of the number of pulls u and total pulls t.

$$\beta(u, t) = \sqrt{\frac{1}{2u} \log(\frac{k_1 n t^4}{\delta})}, \text{ where } k_1 = \frac{5}{4}$$

LUCB1

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The sample complexity of this algorithm is $\mathcal{O}(H^{\epsilon/2} log(\frac{H^{\epsilon/2}}{\delta}))$, where $H^{\epsilon/2} = \sum_{a \in arms} \frac{1}{[\Delta_a \vee \frac{\epsilon}{2}]^2}$.

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- We replace δ in the expression of $\beta(u, t)$ by $\tilde{\delta}$.

$\tilde{\delta}$ should thus be the largest possible value that still would give the PAC guarantees. Intuitively, this will reduce $\beta(u, t)$ and hence number of the runs.

LUCB2 (Modified LUCB1)

The conditions which the variables should satisfy have been described below.

- $4 \cdot \tilde{\delta} \leq \cdot (146 \cdot n \cdot \log(\frac{n}{\delta}))^2$
- $\sum_{i=1}^{\infty} \frac{1}{i^3} \geq (\sum_{i=t_0}^{\infty} \frac{1}{i^3}) \cdot \frac{\tilde{\delta}}{\delta}$

We choose the maximum $\tilde{\delta}$ and minimum t_0 satisfying the above inequalities.
We can argue theoretically[1] that on the modifications mentioned above we can achieve the required confidence as required for the PAC algorithm with much smaller number of runs.

[1]Detailed reasoning and mathematical argument for the same would be provided in the report

Results

We ran LUCB1 (original) and LUCB2 (modified) for several bandit instances. Two of those instances are given below:

```
b1 = [0.9, 0.8, 0.7, 0.6, 0.595, 0.592, 0.3, 0.2, 0.1]
b5 = [0.9, 0.8, 0.795, 0.655, 0.6224, 0.61, 0.6, 0.5, 0.4,
    0.3, 0.2, 0.1]
```

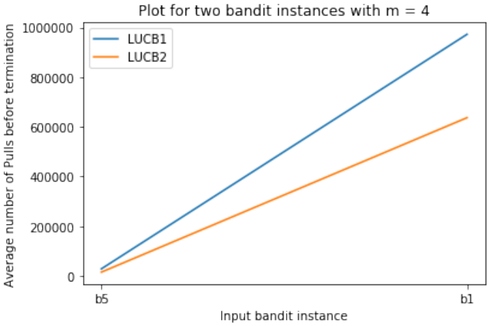

Figura: Sample complexity for $m = 4$, $\delta = 0{,}1$ over b1 and b5

Results

Here are some more instances with varied means and number of arms. In every case $m = 4$, $\delta = 0{,}1$.

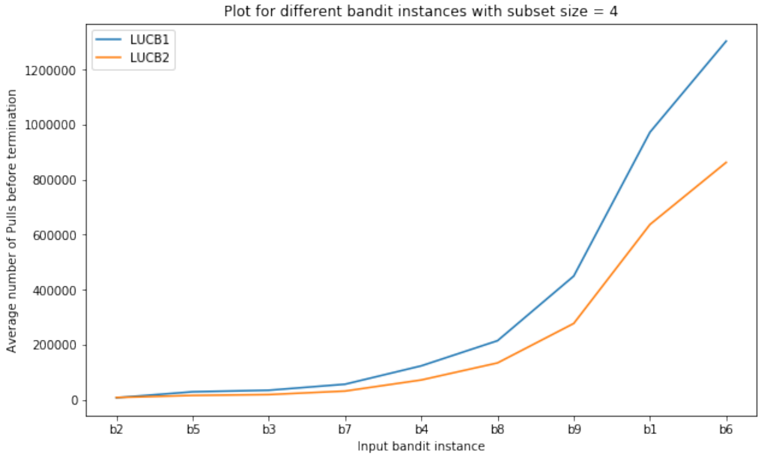

Figura: Sample complexity

An alternative notion of ϵ-closeness

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Objective

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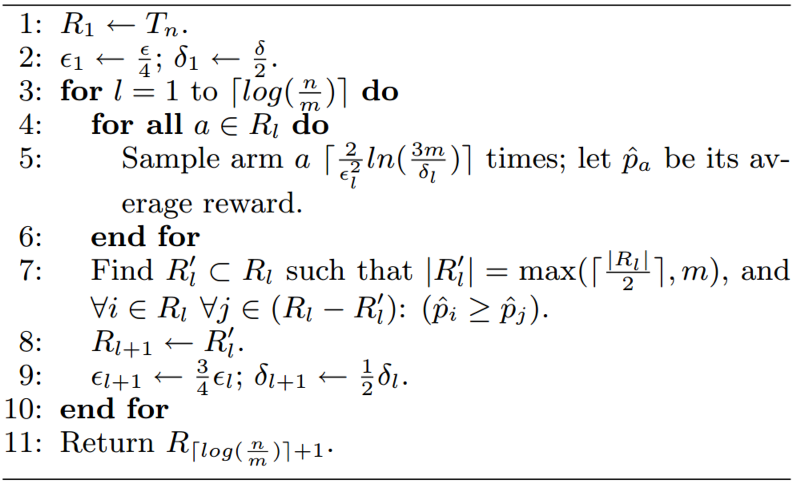
The main distinction is that here the tolerance is relative (proportional to true reward probability) rather than absolute.

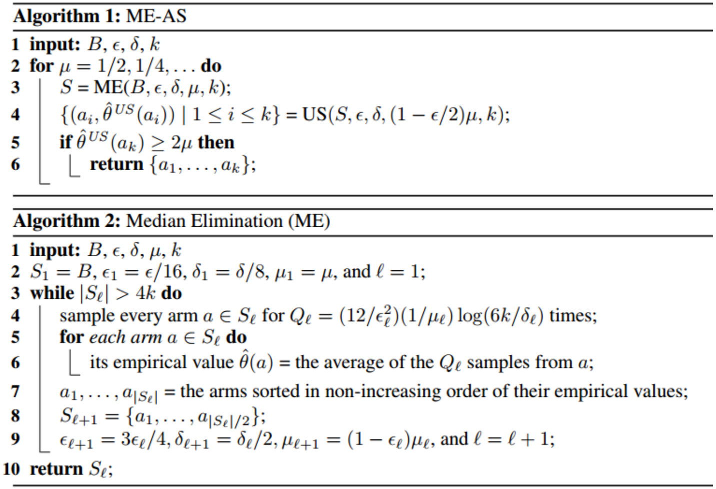
Second Look at Halving [2]



```
1:  $R_1 \leftarrow T_n$.
2:  $\epsilon_1 \leftarrow \frac{\epsilon}{4}$; $\delta_1 \leftarrow \frac{\delta}{2}$.
3:  **for** $l = 1$ to $\lceil log(\frac{n}{m}) \rceil$ **do**
4:      **for all** $a \in R_l$ **do**
5:          Sample arm $a$ $\lceil \frac{2}{\epsilon_l^2} ln(\frac{3m}{\delta_l}) \rceil$ times; let $\hat{p}_a$ be its average reward.
6:      **end for**
7:      Find $R'_l \subset R_l$ such that $|R'_l| = \max(\lceil \frac{|R_l|}{2} \rceil, m)$, and
        $\forall i \in R_l \; \forall j \in (R_l - R'_l)$: $(\hat{p}_i \geq \hat{p}_j)$.
8:      $R_{l+1} \leftarrow R'_l$.
9:      $\epsilon_{l+1} \leftarrow \frac{3}{4}\epsilon_l$; $\delta_{l+1} \leftarrow \frac{1}{2}\delta_l$.
10: **end for**
11: Return $R_{\lceil log(\frac{n}{m}) \rceil + 1}$.
```

Figura: Halving Algorithm

Adapting for Multiplicative Tolerance



Algorithm 1: ME-AS

```
1  input: $B$, $\epsilon$, $\delta$, $k$
2  for $\mu = 1/2, 1/4, \ldots$ do
3      $S = \mathrm{ME}(B, \epsilon, \delta, \mu, k)$;
4      $\{(a_i, \hat{\theta}^{US}(a_i)) \mid 1 \leq i \leq k\} = \mathrm{US}(S, \epsilon, \delta, (1 - \epsilon/2)\mu, k)$;
5      if $\hat{\theta}^{US}(a_k) \geq 2\mu$ then
6          return $\{a_1, \ldots, a_k\}$;
```

Algorithm 2: Median Elimination (ME)

```
1  input: $B$, $\epsilon$, $\delta$, $\mu$, $k$
2  $S_1 = B$, $\epsilon_1 = \epsilon/16$, $\delta_1 = \delta/8$, $\mu_1 = \mu$, and $\ell = 1$;
3  while $|S_\ell| > 4k$ do
4      sample every arm $a \in S_\ell$ for $Q_\ell = (12/\epsilon_\ell^2)(1/\mu_\ell)\log(6k/\delta_\ell)$ times;
5      for each arm $a \in S_\ell$ do
6          its empirical value $\hat{\theta}(a)$ = the average of the $Q_\ell$ samples from $a$;
7      $a_1, \ldots, a_{|S_\ell|}$ = the arms sorted in non-increasing order of their empirical values;
8      $S_{\ell+1} = \{a_1, \ldots, a_{|S_\ell|/2}\}$;
9      $\epsilon_{\ell+1} = 3\epsilon_\ell/4$, $\delta_{\ell+1} = \delta_\ell/2$, $\mu_{\ell+1} = (1 - \epsilon_\ell)\mu_\ell$, and $\ell = \ell + 1$;
10 return $S_\ell$;
```

Figura: Guess and Validate Mechanism

More ideas

For this variant, they propose a halving based algorithm with sample complexity $O(\frac{n}{\epsilon^2}\frac{1}{p_k^{true}}\log\frac{m}{\delta})$

Can we have an improvement over this by proposing a greedy strategy similar to LUCB?

References

[1] Cao, W., Li, J., Tao, Y., and Li, Z. On top-k selection in multi-armed bandits and hidden bipartite graphs. In *Advances in Neural Information Processing Systems* (2015), pp. 1036–1044.

[2] Kalyanakrishnan, S., and Stone, P. Efficient selection of multiple bandit arms: Theory and practice. In *Proceedings of the Twenty-seventh International Conference on Machine Learning (ICML 2010)* (2010), J. Fürnkranz and T. Joachims, Eds., Omnipress, pp. 511–518.

[3] Kalyanakrishnan, S., Tewari, A., Auer, P., and Stone, P. PAC subset selection in stochastic multi-armed bandits. In *Proceedings of the Twenty-ninth International Conference on Machine Learning (ICML 2012)* (New York, NY, USA, 2012), J. Langford and J. Pineau, Eds., Omnipress, pp. 655–662.

Thank You

THANK YOU
GRACIAS
ARIGATO
SHUKURIA
DANKSCHEEN
TASHAKKUR ATU
YAQHANYELAY
SUKSAMA
EKHMET
BIYAN
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