Astro 212: Dynamical Astronomy
Problem Set 1
Due: Wednesday, January 18, 2017

In this problem set, you will become comfortable with orbital elements and with numerically integrating orbits in the Python package REBOUND. As an example, you will apply your work to radial velocity curves for the detection of extrasolar planets.

Note: In the below, I am assuming that you are working in Python. REBOUND is also available in C, and for the non-REBOUND problems, you can use whatever tools you like. However, you will need to be able to, for example, compare calculations on the same plot. I expect that using Python will be easiest.

Reading: Class Notes Week 1; Murray & Dermott Ch. 2.1–2.4,2.7–2.8

0. In this class you will be writing a set of utilities for doing dynamics calculations. Create a Python module called "dynamics" to organize them.

   (a) In a convenient place, make a directory called `dynamics`. For example:
   `/Users/ruth/pythonstuff/dynamics/`
   Add the directory hosting `dynamics` to your PYTHONPATH. For example, if you are using bash for your shell (the default on Macs), add this line to the `.bashrc` file in your home directory (`cd ~` gets you to your home directory):
   ```
   export PYTHONPATH=$PYTHONPATH:/Users/ruth/pythonstuff
   ```

   (b) Tell Python that `dynamics` is a module by creating an empty file called `__init__.py` in the `dynamics` directory. A simple way is to type:
   ```
   touch __init__.py
   ```
   while in the directory `dynamics`.

   (c) Add a file to the `dynamics` directory called `orbconvert.py`. In this file, you will define functions. For example:
   ```
   def funcname(x,y):
       z = x + y
       return z
   ```
   Import any packages that you need for your functions at the top of the file.

   (d) Try importing your function into another piece of python code:
   ```
   from dynamics.orbconvert import *
   print (funcname(2,3))  # my examples are using Python 3 syntax
   ```
   Once you have finalized your development of each function, add it to the file `orbconvert.py` or to another `.py` file in the `dynamics` directory. You may want to work on your functions in a jupyter notebook and add them to your module once you're happy with them.

1. An orbit may be specified by the cartesian coordinates (x, y, z, v_x, v_y, v_z) or by a set of six orbital elements. One standard example of a set of orbital elements is $(a, e, i, \Omega, \omega, f)$. Write your own code to convert between cartesian coordinates and orbital elements.

   (a) Write a function `carttoels` that takes cartesian coordinates (x, y, z, v_x, v_y, v_z) as inputs and returns orbital elements $(a, e, i, \omega, \Omega, f)$. You will need an additional input parameter. What is it? You can find a set of useful expressions in the Class Notes and in Murray & Dermott Ch. 2.8.

(b) Write the inverse function `elstocart` that takes orbital elements as input and returns cartesian coordinates.

(c) Write a function that converts from heliocentric cartesian coordinates to barycentric cartesian coordinates for the two-body problem. You will need yet another input parameter. What is it? Write the inverse function that converts from barycentric to heliocentric coordinates.

2. Write your own code to generate solutions to the two-body problem as a function of time.

(a) Using one of the iterative methods discussed in class (see the Class Notes Week 1) or in Murray & Dermott Ch. 2.4, write a numerical solver that solves Kepler's equation:

$$M = E - e \sin E \tag{1}$$

Your function should take M as an input and return E.

(b) Consider a Jupiter-mass planet orbiting the Sun with a semi-major axis $a = 1$ AU and an eccentricity $e = 0.1$. Use your knowledge about elliptical orbits and your solver from part (a) to plot the radial distance between the planet and the Sun as a function of time for at least one orbital period. Assume that at time 0, the planet is located at pericenter.

(c) Write the expressions that relate the vector pointing from the Sun to the planet and the vectors pointing from the barycenter to each body. Use these expressions to plot the radial distance of the planet from the system's barycenter and the radial distance of the Sun from the system's barycenter as a function of time. How does the distance from the Sun to the barycenter compare with the radius of the Sun?

(d) Check your converters from Question 1: For each point in time in part (b), convert from heliocentric orbital elements to heliocentric cartesian coordinates, then heliocentric cartesian to barycentric cartesian, then barycentric cartesian to barycentric orbital elements. Replot the distances from the barycenter to the Sun and to the planet as a function of time. Hopefully, you will get the same answer as you did in part (c)!

3. Set up REBOUND and compare an elliptical orbit with your analytic results from Question 2.

(a) Follow the instructions on the class webpage to install REBOUND and follow the links to the documentation. Here is a snippet of code to get you started by giving you a sense of what you can do:

```
import rebound as rb
sim = rb.Simulation()
sim.add(m=1.0)
sim.add(m=1.0e-3, a=1.0, e=0.1)
sim.move_to_com()  # moves all particles to the barycentric frame
for p in sim.particles:
    print(p.m, p.x, p.y, p.z, p.vx, p.vy, p.vz)
sim.integrate(1000.)
sim.status()
for p in sim.particles:
    print(p)
for o in sim.calculate_orbits(heliocentric=True): print(o)
%matplotlib inline
fig = rb.OrbitPlot(sim, unitlabel="[AU]", color=True, periastron=True)
```

(b) Use REBOUND to integrate the same orbit that you plotted in Question 2 for at least one orbital period. Recreate each of your plots from Question 2. This will require integrating to each time step that you want to record and figuring out how to get REBOUND to print out the quantities that you want. You should not need to do any complicated conversions yourself.

(c) Plot $x(t)$ and $v_x(t)$ using your analytic code and using REBOUND. Compare.

4. The first large population of planets was discovered using radial velocities (RV), and the RV method remains the best way to measure a planet's mass. Derive the radial velocity formula for exoplanet detection. Following convention, we will define our coordinate system as follows: The center of mass of the star/planet system is at the origin. The reference xy-plane is the plane of the sky, perpendicular to our line of sight. The z-axis is directed away from the observer. Let the star have mass m_1 and the planet have mass m_2. The position of the star with respect to the origin is $\vec{r_1}$ and the position of the planet is $\vec{r_2}$, so that the relative vector $\vec{r} = \vec{r_2} - \vec{r_1}$. The radial velocity of the star measured by the observer is $\dot{z}_1$.[1] Recall that coordinates in the planet's orbital plane may be transformed to an arbitrary coordinate system using a series of three rotations $P_z(\Omega)P_x(i)P_z(\omega)$ (see the Class Notes Week 1 or Murray & Dermott Section 2.8). The matrices giving rotation about the z-axis and x-axis are:

$$P_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{2}$$

For $\vec{r}$ in our RV reference frame and $\vec{r'}$ in the orbital plane of the planet,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P_z(\Omega)P_x(i)P_z(\omega) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{3}$$

(a) Write down an expression for z_1 as a function of the masses and (a subset of) the orbital elements $(a, e, i, \Omega, \omega, f)$.

(b) Use the expressions for $\dot{r}$ and $r\dot{f}$ along a two-body orbit to write $\dot{z}_1$ as a function of the masses and the orbital elements (but no time derivatives). Show that your answer is of the form

$$\dot{z}_1 = V_0 + K[e\cos\omega + \cos(\omega + f)] \tag{4}$$

where V_0 is a constant component of the radial velocity and K is the radial velocity amplitude.

(c) Explain why radial velocities can be used to measure $m_2 \sin i$ rather than m_2. Note that this degeneracy is present even for eccentric orbits, meaning that the radial velocity induced by a high-mass planet on an inclined elliptical orbit looks like the pattern induced by a lower-mass planet on an edge-on elliptical orbit.

[1] RV coordinates are confusing. The reference line (the x-axis) points from the origin in the direction of the north equatorial pole. Longitudes are then measured from that reference line in the "east of north" direction (counter-clockwise on the sky). To make this a right-handed system, the z-axis should point toward the observer, despite radial velocities being measured as positive away from the observer. Furthermore, for a binary, the argument of pericenter of the primary (which RV people call the longitude of pericenter) is measured from where the primary passes the sky plane from nearer the observer to farther from the observer. And the argument of the secondary (the fainter component) is measured from where it passes the sky plane from farther from the observer to near the observer. This is just to give them the same value.

(d) Show that for a circular orbit,

$$\frac{m_2 \sin i}{M_{Jup}} = \left(\frac{K}{28.4 \text{ m/s}}\right)\left(\frac{M_*}{M_\odot}\right)^{2/3}\left(\frac{T}{\text{yr}}\right)^{1/3} \tag{5}$$

5. Measure the mass of the first discovered exoplanet.

   (a) Using Figure 1 and your expression for the radial velocity amplitude, estimate the minimum mass, $m_2 \sin i$, of 51 Peg b. It orbits a solar-like star. Estimate the semi-major axis of the planet's orbit.

   (b) Using REBOUND, integrate a planet on a circular orbit with the mass and semi-major axis that you estimated. Plot the radial velocity signal. Does it look like the figure? The difficult part here will be getting the coordinate system right.

   (c) Check Mayor & Queloz (1995; full reference in the figure capion). How close did you get to the right $m_2 \sin i$?

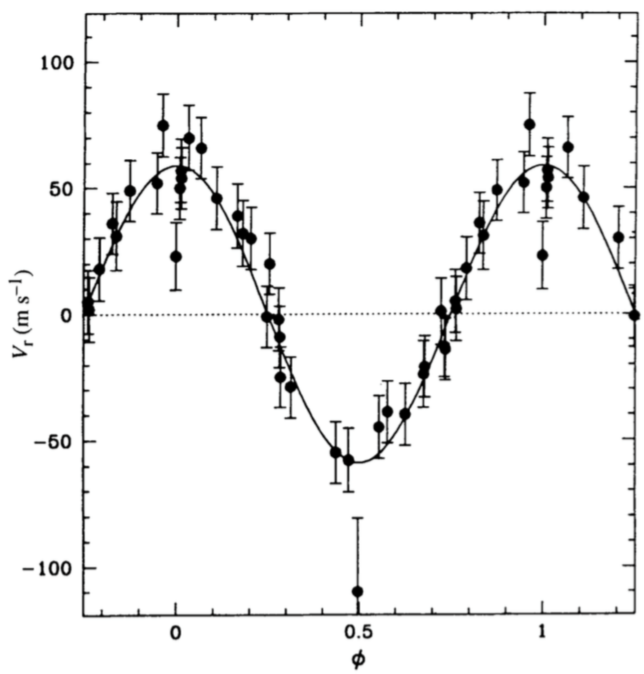

Figure 1: Observed radial velocities, V_r, plotted as a function of orbital phase, ϕ, for the first reported extrasolar planet, 51 Peg b. These data are folded with a period of 4.2 days, which corresponds to the time elapsed from $\phi = 0$ to $\phi = 1$. The planet is orbiting a solar-like star. This is Figure 4 from the discovery paper: Mayor, M. & Queloz, D. *Nature*, 378, 355 (1995). (As a point of historical interest, the exoplanet HD 114762 b was actually discovered earlier, but it wasn't recognized as a planet at the time of its discovery because it has minimum mass $M \sin i = 11 M_{Jup}$. Now we know that exoplanet masses reach this high value.)