# Carbon Taxes for Paris Agreement Targets

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After signing the Paris Agreement to limit the rise in global temperature within 2°C, countries are coming up with fiscal policies to limit their carbon emissions. Economic models should inform how to optimally implement these targets but so far have not because the Paris target is infeasible in the benchmark model. This paper challenges that infeasibility by incorporating a state-of-the art climate model into a neoclassical growth model; the climate model explicitly derives the nonlinear dynamics of the carbon cycle from physical laws, as opposed to the standard models with parameterized linear dynamics. My model predicts higher benefits of rapid mitigation policies than the standard carbon cycles used in the literature, which makes the Paris target feasible. After showing the Paris target is feasible, I calculate the optimal taxes to achieve the target. I solve the model's nonlinear feedback effects by employing a new numerical solution technique. I find that the carbon tax should be \$298 in 2020 and should grow at 1.56% per year till 2050, after which it should decline at an average annual rate of 0.5% per year. I also show that capital taxes would be required along with carbon taxes to implement the optimal allocation. Given the state of policies around the world, most countries have much lower carbon taxes and there is a need to drastically increase the tax base and level to achieve the Paris Agreement targets.

JEL-Classification: H21, H23, Q51, Q58

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# 1 Introduction

Time is a vital aspect of any policy to combat climate change. Changes today are systematically different than changes in the future. This is most obviously seen in situations with "tipping points" where the effects of climate change may be irreversible. Even without tipping points, time plays an important role because of the climate system. This paper seeks to better understand the importance of time within climate economics by incorporating a state of the art model of climate into a standard economic growth model. Time becomes especially important for immediate policy goals, like the Paris Agreement's target of limiting the rise in global atmospheric temperature to within 2 degrees Celsius increase from preindustrial levels.

However, economists have failed to address this point and there is a lack of policy prescriptions to achieve this target. Nordhaus (2016) claims that this target is unrealistic, even if industrial emissions are driven to zero today and stay there forever, because of the inertia of the climate system. Nordhaus concludes "The international target for climate change with a limit of 2 °C appears to be infeasible" (p. 3). But climatologists tell us that the Paris targets are technologically feasible, contrary to Nordhaus's claim, even if people disagree about whether it is a good policy objective.

First, I show the Paris agreement is technologically feasible. This is done by updating the standard climate model to a state of the art climate model from Glotter et al. (2014) that captures much of the nonlinear dynamics. Their "Bolin and Eriksson Adjusted Model" (BEAM) captures much of the known nonlinear chemistry of ocean carbon absorption. While this would require emission reduction by 13% annually, we can achieve the target without net negative global industrial emission levels. To solve this nonlinear dynamical system, I employ a new numerical solution technique in which the system's nonlinearities are preserved - a feature that is crucial while accurately modeling economic and environmental interconnections.

Using the updated carbon cycle, the paper theoretically characterizes and then quantifies the optimal taxes in an integrated climate-economy model. I combine a dynamic general equilibrium model of the world economy with the climate model and climate economy feedbacks based on the DICE framework (Nordhaus 2017). I choose the DICE model since it is widely applied in the literature, and is one of the three models used by the United States government to value the impacts of carbon dioxide emissions. The characterization and quantitative model allow me to show the following three results.

First, I quantify the optimal carbon tax for different scenarios to provide benchmark targets to policymakers. In all scenarios, the optimal policy suggests a significant ramping up of emission reduction efforts. To reach the Paris agreement target of remaining within 2°C rise in global atmospheric temperature, carbon taxes in 2020 should be as high as \$298.3 per ton of $CO_2$. The optimal taxes increase initially and then decline in all scenarios. While output is marginally lower by 2100 compared to a business as usual scenario, it is significantly higher by 2200.

Second, I show that a carbon tax alone is not enough to implement the optimal allocation. While this is true in multiple models, it has been largely ignored in the literature. I show that one needs a capital tax along with a carbon tax to correct two margins of distortions. Both of these taxes depend on the dynamics of the carbon cycle and the marginal effect of capital accumulation and mitigation expenditure on atmospheric carbon dioxide concentration. The expressions I derive show that the nonlinear dynamics play a crucial role in the time path and levels of these taxes.

Third, this paper shows the application of cutting edge numerical techniques to solve such models. The traditional technique of "first optimize, then discretize" involves finding the necessary continuous optimality conditions analytically and then optimize the equivalent discretized system. This makes it hard to add inequality bounds to the problem, like temperature constraints. Instead, I follow a "first discretize, then optimize" strategy. In this approach, I transform the original continuous problem into a discretized optimization problem which is typically large and sparse. I show how this problem can be reformulated as a nonlinear programming problem. This structure gives rise to a block sparse Jacobian and a block diagonal Hessian. I then show the implementation of multiple shooting algorithm (Bock and Plitt (1984)) through an open-source optimization framework called CasADi developed by Andersson, Åkesson, and Diehl (2012)[1]. This framework implements algorithmic differentiation to such large scale nonlinear problems and computes the Jacobian and Hessian matrices efficiently thereby significantly speeding up the iterations. This approach can be applied to other economic problems with large state spaces, nonlinear policy functions, and inequality path constraints.

The rest of the paper proceeds as follows. Because climate-economic models differ from standard neoclassical growth models, Section 2 briefly explains their structure and where I make updates to the standard models. Section 3 reviews the relevant literature. Section 4 lays out the model and explains the main modeling assumptions. Section 5 describes the computation strategy, functional forms, and parameter selection and the results. Section 6 concludes.

# 2 The Carbon Cycle in Integrated Assessment Models

The use of fuels produces carbon dioxide and its concentration in the atmosphere has been rising at alarming rates post the industrial revolution. This rise in atmospheric carbon dioxide concentration is a leading cause of climate change. Thus the use of carbon fuels gives rise to externalities. So an important question is how to price carbon to internalize the cost of this externality. I use an Integrated Assessment Models (IAMs), such

1. CasADi has been applied to solve Nordhaus' DICE model by Kellett, Faulwasser, and Weller (2016)

as that used in Nordhaus's Nobel Prize-winning work, which jointly models an economy and a climate system, particularly the carbon cycle. IAMs are then often used to estimate the impact of climate change. However, $CO_2$ does not simply accumulate in the atmosphere. Instead, a large part of it gets slowly absorbed by oceans globally. IAMs must model oceanic absorption to translate emissions into atmospheric $CO_2$ concentrations and finally into climate change. This absorption process is well-understood within the scientific literature.

However, cutting-edge climate models are too computationally difficult to be used in economic analysis. Therefore, IAMs use simplified representations of the climate system. Most use a linear absorption representation of the carbon cycle (example Nordhaus (2017), Golosov et al. (2014), Barrage (2014), Cai, Lenton, and Lontzek (2016)). In particular, in these models, the rate of flow of carbon dioxide from the atmosphere to the ocean happens at a constant rate.

The linear dynamic process of carbon absorption assumed in most IAMs does not match the known basic chemistry of ocean carbon absorption. In particular, if the oceanic concentration of $CO_2$ goes down as a result of rapid mitigation efforts, the rate of absorption of $CO_2$ by the oceans would go up. At the same time, if we continue on our business as usual path, the rate of absorption would go down. Without the incorporation of this mechanism, we would be either systematically underestimating or overestimating the benefits from current policies. This issue is highlighted in Hof et al. (2012), who claim that IAMs lead to much lower benefits in high mitigation scenarios because of their failure in capturing nonlinear climate dynamics. This also means that along a business as usual scenario, the damages from climate change would be much higher than predictions in standard models.

All of this becomes especially important if we focus on the Paris Agreement target of limiting the rise in global atmospheric temperature to within 2 degrees Celsius increase from preindustrial levels. Nordhaus (2016) claims that this target is unrealistic and given the inertia of the climate system, this will not be possible without large net negative global industrial emission levels. Given the nascent stage of development of carbon capture and storage technologies and their high costs, it will not be feasible to have large scale deployment of this technology needed to reach the level of decarbonization required to get to large negative net industrial emissions globally. Nordhaus (2016) thus concludes the Paris Agreement target is not technologically feasible. In this paper, I show that by incorporating the BEAM carbon cycle within a standard growth model, we can reach the Paris Agreement target. While this would require reducing emissions by 13% annually, we can achieve the target without net negative global industrial emission levels.

Climatologists have written about emission paths that would be consistent with different temperature targets, including Paris Agreement Targets. Walsh et al. (2017) shows that it is possible to reach the Paris agreement targets by 2100 with net positive anthropogenic emissions throughout this period. This is in stark contrast to Nordhaus' conclusions.

# 3 Literature Review

The present work builds on the seminal work by Nordhaus and brings in nonlinear dynamics in the carbon cycle. In doing so it builds on the papers in the economics field that talk about optimal climate policy like Nordhaus (2017), Golosov et al. (2014), Barrage (2014), Van Der Ploeg and Withagen (2014) and papers in the climate science field like Glotter et al. (2014), Cai, Lenton, and Lontzek (2016), Millar et al. (2017), Hartin et al. (2016). The reason for the inclusion of these nonlinearities is two-fold. One is to get better quantitative estimates, the other is to show reaching the Paris agreement target is feasible and what are the taxes required to do so. In doing so, it helps address a conflict between conclusions by economists and climatologists. Nordhaus (2016) claims achieving the Paris agreement target is not feasible while Hof et al. (2012) has argued integrated assessment models of the type used by Nordhaus systematically underestimate benefits of rapid mitigation. This paper helps to address this criticism. The results are consistent with Walsh et al. (2017) and other papers in the climate science literature.

On the theory side, the described carbon-capital tax link is novel and builds on the work by Barrage (2014). While a large number of papers have studied pollution pricing, theoretical work on this has primarily been in static settings. This paper jointly characterizes optimal carbon and capital tax in an infinite horizon growth model. Previous work has produced multiple insights on environmental policy with capital taxation, these have generally been in different settings. For example: several papers have modelled pollution and income taxes in endogenous growth models, Fullerton and Kim (2008), Mooij and Bovenberg (1997), Ligthart and Ploeg (1994) but they focus on long run outcomes along a balanced growth path. This paper studies the transition to the balanced growth path while taking long run growth rates as given. Second, the dynamics of capital and carbon taxation have been studied by many focusing on carbon as a rent generating resource rather than the climate externality that I focus on, for example, Franks, Edenhofer, and Lessmann (2017) and Groth and Schou (2007). Third, capital taxation has been studied as part of climate models to look at issues like time inconsistencies, Gerlagh and Liski (2017) or other considerations, like in Dao and Edenhofer (2018). Many papers have also looked at carbon taxation in general equilibrium models like Hassler and Krusell (2012), Golosov et al. (2014), Rezai and Van der Ploeg (2016), but they all consider simple carbon cycle models with linear depreciation.

On the quantitative side, we build on two branches. A rich and growing literature has developed integrated assessment models to quantify the effects of a variety of climate economy interactions like Manne and Richels (2005), Hope (2011), Acemoglu et al. (2012), Anthoff and Tol (2013), Lemoine and Traeger (2014), Cai, Lenton, and Lontzek (2016).

However, these papers assume linear dynamics in the carbon cycle. While a more detailed analysis of these nonlinear dynamics is common in natural sciences, it has not been included in economic analysis. This combination is rarely attempted because the computational problem becomes hard to solve. To address this aspect, my work shows the application of a multiple shooting algorithm from Bock and Plitt (1984) through an open-source optimization framework developed by Andersson et al. (2019) and Andersson, Åkesson, and Diehl (2012) using IPOPT solvers following Wächter and Biegler (2006) to solve optimal control problems in economics.

# 4 The General Model

I begin by describing the general setting in Section 4.1. I will then introduce a set of additional assumptions in Section 4.2 that are key in deriving the main results. In Section 4.3, we state the planning problem: how to optimally allocate resources over time, taking into account how the economy affects the climate. We then consider the decentralized economy in Section 4.4 and identify the optimal emission and capital taxes that implements the optimal allocation from the planner's problem. Section 5 goes over the algorithm, parameter choice and results from the computation exercises. Section 6 then concludes.

## 4.1 The Economy and Climate: A General Specification

The model is a standard deterministic neoclassical growth model with two types of capital. Physical capital and nature capital. The stock of $CO_2$ can be thought of as the amount of nature capital. A higher stock of nature capital is then bad for the economy. Resources can be used to reduce the stock of nature capital by investing it in abatement technology. Time is discrete and infinite. The economic module can be described in the following section.

**The Economy Module**

There is a standard representative household with preferences given by

$$\sum_{t=0}^{\infty} \beta^t U(c(t))$$

where $U$ is a standard concave period utility function, $c$ is consumption, and $\beta \in (0,1)$ is the discount factor. The production process consists of one final goods sector with output $Y(t)$. The resource constraint for this economy is given by

$$k(t+1) + c(t) + \theta_1(t)\mu(t)^{\theta_2} Y(t) = Y(t)\{1 - R(T_{AT}(t))\} + (1-\delta)K(t) \tag{1}$$

where the left-hand side is resource use: consumption, next period's capital stock and investment in abatement technology which is modeled as an aggregate economy wide level as a nonlinearfunction of the extent of the carbon cutback from a business as usual scenario. $\mu(t) \in [0,1]$ is the emission control rate which denotes the fraction of cutback, $\theta_1(t)$ denotes the cost of renewable (backstop) technology and is exogenously specified.

The first term on the right-hand side denotes the output net of damages. Net output is gross output reduced by damages. The second term is undepreciated capital. Here, $Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha}$, is gross output, denoted by a standard Cobb-Douglas technology. The damage fraction, $R(T_{AT}(t))$ denotes the loss in output or damages arising from the rise in atmospheric temperature. See section 4.2 for different functional forms of the damages function. $T_{AT}(t)$ denotes the rise in average global atmospheric temperature over its 1900 levels. Temperature is derived from the natural science module and will be described in section 4.1. Therefore, the climate part of the model has a feedback in the economy though these damages.

How the economy affects the climate system is the other crucial part. Output production generates emissions. The benefit of spending on abatement is that it reduces the amount of emission that is generated from output production.

$$\mathcal{E}(k, \mu, t) = \sigma(t)(1 - \mu(t))Y(t) + E^{\text{land}}(t) \tag{2}$$

where $\sigma(t)$ denotes the carbon intensity of output production and $E^{\text{land}}(t)$ denotes emissions from deforestation and land use. The 25 countries with the highest forest cover have all included forest-related mitigation measures in their Nationally Appropriate Mitigation Actions (NAMAs) and Nationally Determined Contributions (NDCs). Such measures include afforestation, reduced deforestation and degradation, enhancement of forest carbon stocks, forest conservation and agroforestry. [2] This is modeled as a gradual decline in $E^{\text{land}}(t)$ from about 2.93 $GtCO_2/year$ in 2015 to about 0.32 $GtCO_2/year$ in 2200.

Emissions are a flow that enters into the natural science module. The temperature at time $t$ depends on the history of emissions up to that time, $\mathcal{E}^t$

**The Natural Science Module**

Temperature is determined by the mass of carbon in the atmosphere ($M_{AT}$) which changes due to carbon emissions. Therefore, we need a system to model how the flow of emissions determines the mass of carbon in the atmosphere which in turn determines the rise in atmospheric temperatures ($T_{AT}$). Production generates a flow of emissions into the atmosphere, scaled down by the emission control rate. Based on a state of the art climate model with a three-stage storage capability (representing the storage of carbon and energy in the atmosphere, upper and lower oceans), we determine the mass of carbon in the atmosphere. Thus economic activity is coupled with the carbon concentration in the reservoirs and the resulting increase in radiative forcing and temperature.

2. The State of the World's Forests, 2018 by the Food and Agriculture Organization of the United Nations

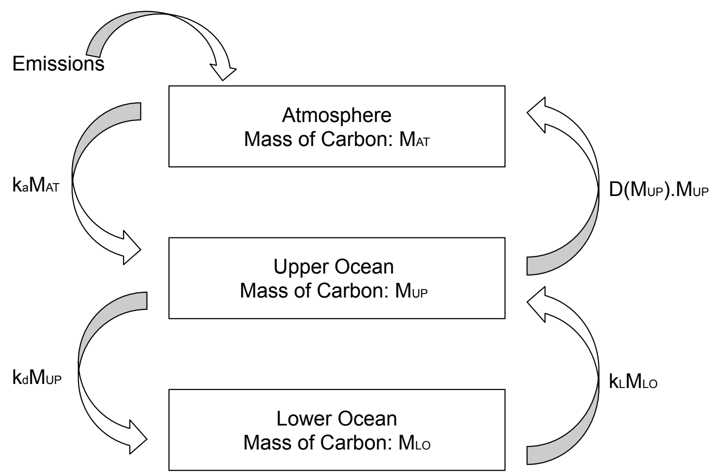


Figure 1: Carbon Cycle

$CO_2$ enters to global carbon cycle as emissions through the atmosphere. Over time, a part of it gets dissolved in the oceans. To model this, as depicted in Figure 1, we use a three-box carbon cycle model. The three boxes represent the Atmosphere, Upper Ocean (up to a depth of 50 meters from sea level) and the lower ocean. The mass of carbon in these three layers is represented by $M_{AT}, M_{UP}, M_{LO}$. The flow of carbon between these layers denoted by the arrows in Figure 1, is mathematically represented by a transition matrix. In the economics literature, this transition matrix is calibrated to match the amount of carbon in these three layers and the history of emissions up to that time. The transition rates thus estimated, are constants and give rise to linear transition dynamics. This is used to predict future carbon concentrations.

This calibration exercise implies that we are not taking into account the change in these transition rates that occur with time as the amount of carbon in these reservoirs changes. Based on multiple studies by climatologists we know that as the upper ocean absorbs $CO_2$, it becomes more acidic, thereby reducing its ability to absorb more $CO_2$. Thus ocean acidification results in slower uptake rates which lead to slower carbon sequestration in the global oceans.

This has resulted in a different estimate of the transition matrix for Nordhaus' model every few years. He admits this drawback in his papers.

> " The carbon cycle is limited because it cannot represent the complex interactions of ocean chemistry and carbon absorption. We have adjusted the carbon flow parameters to reflect carbon-cycle modeling for the 21st century, which show lower ocean absorption than for earlier periods. This implies that the (DICE) model overpredicts atmospheric absorption during historical periods."
> - Nordhaus and Sztorc (2013)

As a way of moving past this periodic re-calibration exercise, we use a self-updating representation of the carbon cycle. We use Glotter et al. (2014) to model the nonlinear climate dynamics that captures the effect of ocean acidification through a variable transition matrix.

The concentrations of $CO_2$ evolve according to:

$$\begin{bmatrix} M_{AT}(t) \\ M_{UP}(t) \\ M_{LO}(t) \end{bmatrix} = \Phi^{BEAM} \begin{bmatrix} M_{AT}(t-1) \\ M_{UP}(t-1) \\ M_{LO}(t-1) \end{bmatrix} + \begin{bmatrix} \mathcal{E}(k,\mu,t) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} M_{AT}(t-1) \\ M_{UP}(t-1) \\ M_{LO}(t-1) \end{bmatrix} \tag{3}$$

where

$$\Phi^{BEAM} = \begin{bmatrix} -k_a & D(M_{UP}(t-1)) & 0 \\ k_a & -D(M_{UP}(t-1)) - k_d & k_l \\ 0 & k_d & -k_l \end{bmatrix}$$

or, more compactly,

$$\mathbf{M}(t) = (1 + \Phi^{BEAM})\mathbf{M}(t-1) + [\mathcal{E}(k,\mu,t), 0, 0]^T$$

Where $\mathbf{M}(t) = [M_{AT}(t), M_{UP}(t), M_{LO}(t)]^T$. Here $D(M_{UP}) = A.B(M_{UP})$, where $A$ is the ratio of atmosphere to ocean $CO_2$ concentration at equilibrium. $B$ is the ratio of dissolved $CO_2$ to total ocean inorganic carbon at equilibrium and depends on the acidity of the upper ocean. More acidic sea water stores less carbon dioxide. Variation can alter the uptake rates dramatically. For example, in business as usual simulations, $D(M_{UP})$ raises by a factor of 5.2 as atmospheric $CO_2$ concentrations increase by 5 times their present level by 2200. This means the ocean's ability to hold inorganic carbon relative to the atmosphere would decrease nearly as fast as the increase in atmospheric $CO_2$ concentrations. This kind of a near cancellation would imply lowering of carbon uptake nearly 10 times their expected initial values.

$$A = k_H.\frac{AM}{OM/\Delta + 1};\ B = \frac{1}{1 + \frac{k_1}{[H^+]} + \frac{k_1 k_2}{[H^+]^2}}$$

where $[H^+]$ is obtained by solving the following quadratic

$$\frac{M_{UP}}{Alk} = \frac{1 + \frac{k_1}{[H^+]} + \frac{k_1 k_2}{[H^+]^2}}{\frac{k_1}{[H^+]} + \frac{2k_1 k_2}{[H^+]^2}}$$

Here, $[H^+]$ denotes the hydronium ion concentration.[3]. The model solves for $[H^+]$ at each timestep using the above equation and assuming constant alkalinity. Constant alkalinity is a reasonable assumption for several thousand years. Over significantly longer timescales, dissolution of calcium carbonate would help return pH to its original value, increasing sequestration of atmospheric $CO_2$. The model will therefore under predict $CO_2$

3. We can measure how acidic the upper ocean is by computing its $pH = -log_{10}([H^+])$. A pH of 7 is considered neutral while below 7 is considered acidic.

uptake on $\sim 10,000$ year timescales.

Note in figure 2 the variation in the transition coefficient $D(M_{UP})$. $CO_2$ like other gases is soluble in water, but unlike some gases like oxygen, it reacts with water to form several ionic and non-ionic species collectively called dissolved inorganic carbon (DIC). The relative amounts of these species determine the solubility of $CO_2$. The relative amount of these species in equilibrium depends on the level of acidity or pH of the ocean. Thus more acidic oceans imply lower sequestration of $CO_2$ which leads to a higher amount in the atmosphere. In all versions of Nordhaus' DICE models, the transition coefficients are fixed at some level, determined through a calibration exercise. Thus there is no change in the uptake rates in all versions of his models and those derived from it. The drawback of this method is that such a process is unable to predict future transition rates for carbon concentrations that we have not experienced in the recent past. This implies the calibrated number is instead capturing some average uptake rate based on past data which is not the best predictor for future absorption. Given this methodological flaw, DICE models have had to periodically update their carbon cycle coefficients to keep up with the changing reality. In general, they tend to underestimate the benefits of rapid mitigation policies and underestimate the harm from continuing with business as usual emission paths. Correcting both these margins is important to estimate optimal climate policy and the welfare gains of doing so.

Energy is continuously flowing into the atmosphere in the form of solar radiation. A part of this sunlight (about 30%) is reflected back to space and the rest is absorbed by the planet. And like any warm object with a cold surrounding, energy is continuously radiating back out into space as infrared radiation. Subtracting the energy flowing out from the energy flowing in gives us the net energy. If the net energy flow is positive, there will be warming (or cooling, if the number is negative) going on. Radiative forcing is a direct measure of the amount that the Earth's energy budget is out of balance.

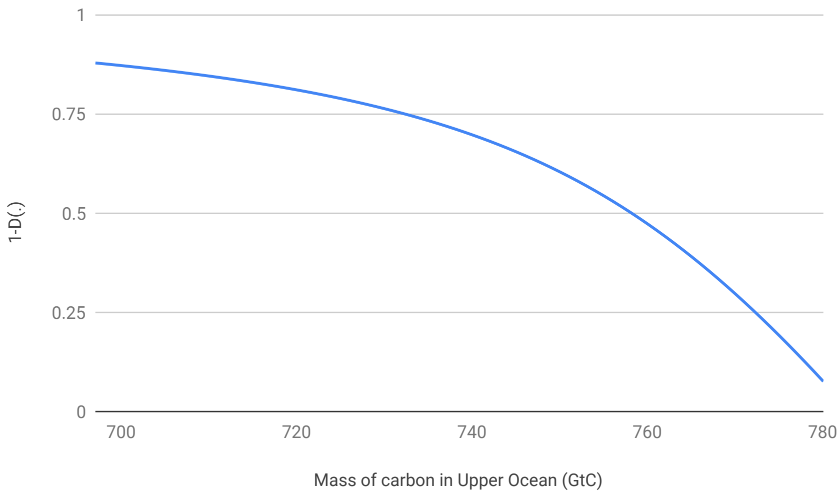


Figure 2: Fraction of carbon retained in the upper ocean

As the emissions accumulate in the atmosphere, it creates a blanket that prevents an increasing amount of incoming solar radiation to escape into space. This is modeled as an increase in radiative forcing due to the increased insulation remaining in the atmospheric layers. This increase in forcing, which leads to rise in atmospheric temperature is measured with respect to the level in 1750 and is defined to be the sum of the logarithmic increase of concentration of carbon with respect to its level in 1750, multiplied by the constant for equilibrium increase of forcing at a doubling of $CO_2$ and the exogenous forcing[4] resulting in:

$$\mathcal{F}(M_{AT}, t) = \eta \log_2 \left( \frac{M_{AT}(t)}{\overline{M_{AT}}} \right) + \mathcal{F}^{EX}(t) \tag{4}$$

where $\eta$ is the equilibrium increase of forcing due to a doubling of $CO_2$ concentration

4. Examples of sources of exogenous forcing include methane, aerosols, ozone, and chlorofluorocarbons (Nordhaus, 2008).

and $\overline{M_{AT}}$ denotes the amount of $CO_2$ in the atmosphere in pre-industrial times.

$$\mathcal{F}^{EX}(t) = \begin{cases} 0.25 + 0.025(t-1) & t \leq 2115 \\ 0.7 & \text{otherwise.} \end{cases} \tag{5}$$

While $CO_2$ is one of the largest contributors to radiative forcing, other gases also contribute to radiative forcing, like methane, nitrous oxide, cholofluorocarbons, hydrofluorocarbons, sulphur hexafluoride etc. All of this is clubbed together and considered as exogenous forcing in the model. Nordhaus (2016) uses the estimates in equation 5 and this is consistent with estimates from the IPCC (Myhre, Shindell, and Pongratz 2014).

The radiative forcing contributes to the rise in atmospheric temperature. As a result of the mixing reservoirs, the temperature of the lower oceans also increases, this is proportional to $\mu_{LO}$. There are two layers for temperature and they evolve according to the following equations:

$$\begin{aligned} T_{AT}(t) &= T_{AT}(t-1) + \mu_{AT}\Big[\mathcal{F}(M_{AT},t) - \kappa T_{AT}(t-1) - \gamma_{LO}\left(T_{AT}(t-1) - T_{LO}(t-1)\right)\Big] \\ T_{LO}(t) &= T_{LO}(t-1) + \mu_{LO}\gamma_{LO}\Big[T_{AT}(t-1) - T_{LO}(t-1)\Big] \end{aligned} \tag{6}$$

or more compactly,

$$\mathbf{T}(t) = (1 + \Phi^{TEMP})\mathbf{T}(t-1) + [\mu_{AT}\mathcal{F}(M_{AT}(t),t),0]^T$$

where $\mathbf{T}(t) = [T_{AT}(t), T_{LO}(t)]^T$, $\mu_{AT}$ is the climate response of the atmosphere, $\mu_{LO}$ is the climate response of the ocean, $\gamma$ relates atmosphere ocean heat transfer to temperature anomaly and $\kappa$ is the general climate sensitivity expressed in °C per doubling of $CO_2$. The sensitivity of average surface temperature to $CO_2$ levels in the atmosphere is benchmarked against the warming expected for a doubling of $CO_2$ levels from its preindustrial values. By comparing predictions of different climate models, the IPCC concluded that the likely range of this variable is 1.5° to 4.5°. Using this range of sensitivities, if $CO_2$ levels were stabilized at today's levels, global mean temperatures would eventually rise to around 1°C to 3°C. The increase in atmospheric temperature results in a higher damage fraction reducing output. It is this relation, that results in a trade-off between producing and abating, consuming today or at a later date.

#### Temperature Targets

Given the uncertainty regarding feedback effects in climate change, policymakers have often taken some temperature targets as the maximum permissible increase in global temperature that can be safely allowed without setting into motion dangerous feedback effects. The Paris agreement, for instance, sets this maximum temperature increase to 2°C. Under such circumstances, it is important to find the least cost or efficient way to achieve this target. I consider different scenarios with different temperature targets.

### 4.2 Specializing Some Assumptions

#### Preferences

Utility is given by an iso-elastic function of per capita consumption $(c(t))$ expressed in trillions of 2005 dollars per person. This ensures that we steer away from overconsumption, avoiding the possible negative influence it might have in the future. By setting the elasticity of marginal utility $\gamma$ as greater than one, we get a negative exponential shape of the utility function that results in risk-averse behaviour. The aggregate consumption level, $C(t)$ is obtained by multiplying the per capita consumption $c(t)$ with the population $(L(t))$.

$$U(C(t), L(t)) = L(t)\left(\frac{\left(\frac{C(t)}{L(t)}\right)^{1-\gamma} - 1}{1-\gamma} - 1\right)$$

Population grows following a projected growth rate upto 2050 and then asymptotically reaches a boundary $(L_a)$.

$$L(t+1) = L(t)\left(\frac{L_a}{L(t)}\right)^{l_g}$$

**Global GDP**

The global output concept is in terms of purchasing power parity (PPP) dollars as used by the International Monetary Fund (IMF). To compute this, we take the weighted growth rate of real gross domestic product (GDP) of different countries of the world, where the weights are the country shares of world nominal GDP using current international dollars. These estimates correspond closely to the estimates by the IMF of the growth of real output in constant international (PPP) dollars.

**Abatement Cost**

Nordhaus (2016) models the abatement cost at the aggregate economy wide level as a nonlinear function of the extent of the carbon cutback from a business as usual scenario.

$$\text{Abatement Cost} = \theta_1(t)\mu(t)^{\theta_2}Y(t),$$

where

$$\theta_1(t) = \sigma(t)\frac{p_b}{1000\theta_2}(1-\delta_{p_b})^{t-1}$$

$$\text{and } \sigma(t) = \sigma(t-1)e^{(-g_\sigma * (1-\delta_\sigma)^{5(t-1)} * 5)}$$

Here $\sigma(t)$ represents the $CO_2$ equivalent emission output ratio or the carbon intensity. It represents the amount of $CO_2$ emission generated from a unit of output produced at time $t$. It declines over time to reflect the widening set of technological alternatives, $\mu$ is the emission control rate or the proportion reduction in emission from the business as usual scenario to the policy target level. $\theta_2$ is the exponent showing the degree of nonlinearity in costs for emission cuts. $\theta_1(t)$ is the price of the backstop technology that is declining over time. $p_b$ is the price of backstop technology at initial time. The parameters are calibrated to match the average cost for emission cuts by different amounts from the business as usual scenario by different dates into the future. Nordhaus (2010) derives the abatement cost parameters based on cost estimates from different regions in the Fourth Assesment Report (IPCC 2007) and the Energy Modeling Forum(EMF) 22 report.

One can also think $\sigma(t)Y(t)$ as the amount of emission produced from output when $\mu = 0$. Following Golosov et al. (2014) one can think that this refers to the amount of energy used to produce $Y(t)$ level of output. $\mu$ is the fraction of energy produced from clean sources and $1 - \mu$ is the fraction of energy produced from fossil fuel. If firms use $\mu$ fraction of their energy input from clean energy then $(1 - \mu(t))\sigma(t)Y(t)$ is the level of emission. Then $\theta_1(t)\mu(t)^{\theta_2}Y(t)$ is the cost of producing this amount of clean energy.

### Damages

The damages function in Nordhaus (2016) takes the form of quadratic damages. This is estimated by assuming that damages are 1.8% of output at 2.5 °C (Nordhaus and Sztorc 2013). At high temperatures, it is assumed by default that the quadratic relationship of damages to temperature continues to apply.

$$R(T_{AT}) = a_2 T_{AT}^{a_3}$$

In a discussion of damage functions and catastrophic risks, Martin Weitzman argues that even if the Nordhaus estimate is appropriate for low-temperature damages, the increasingly ominous scientific evidence about climate risks implies much greater losses at higher temperatures (Weitzman 2010). He suggests that damages should be modeled at 50% of output at 6°C and 99 percent at 12°C as better representations of the current understanding of climate risks; the latter temperature can be taken as representing the end of modern economic life if not human life in general. In support of this disastrous projection for 12°C of warming, Weitzman cites recent research showing that at that temperature, areas where half the world's population now lives would experience conditions, at least once a year, that human physiology cannot tolerate - resulting in death from heatstroke within a few hours (Sherwood and Huber 2010). Weitzman creates a damage function that matches the Nordhaus estimate at low temperatures but rises to his suggested values at 6°C and 12°C. He modifies the above damages function by adding a higher power of $T_{AT}$ to the denominator:

$$R(T_{AT}) = \frac{1}{1 + \pi_1 T_{AT}^2 + \pi_2 T_{AT}^{6.76}}$$

When $T_{AT}$ is small, the quadratic term is more important, providing a close match to the original DICE damage function; when T is large, the higher-power term is more important, allowing the damage function to match Weitzman's values for higher temperatures.

### 4.3 The Planning Problem

I now return to the general formulation in Section 4.1, state the planning problem. The planner wants to maximize discounted sum of utilities subject to the resource constraint, the law of motion for the stock of carbon in the three reservoirs, the law of motion for tem-

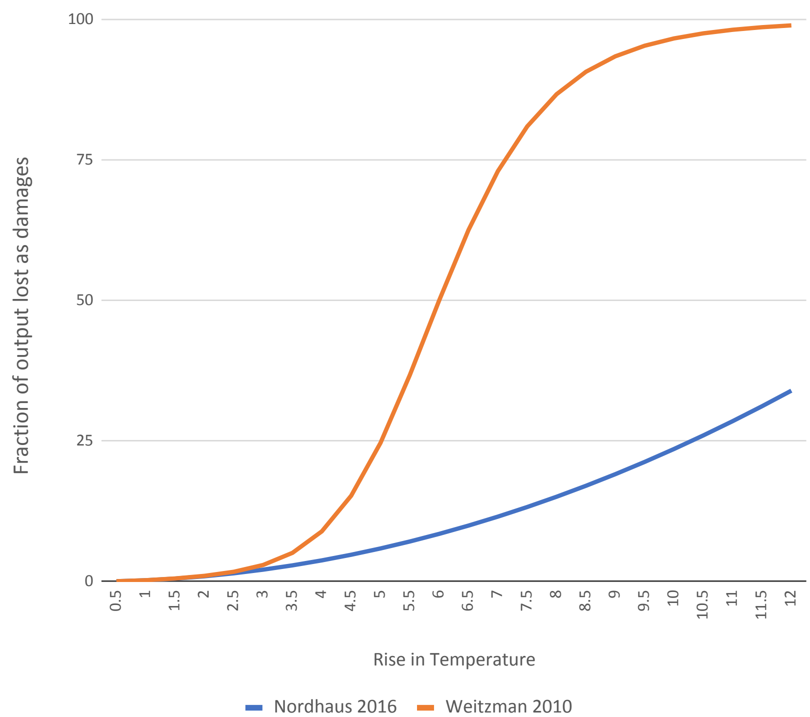


Figure 3: Damages Functions

perature and the temperature path constraints given some initial conditions. The planner realizes that by spending on abatement, he can affect the mass of carbon in the atmosphere in the next period. The general planning problem can be written as follows:

$$\max_{\{c(t),\mu(t),K(t+1)\}_{t=0}^{\infty}} \sum_{0}^{\infty} \beta^t U(c(t), n(t))$$

$$
\begin{aligned}
\text{subject to:} \quad & K(t+1) + c(t) + \theta_1(t)\mu(t)^{\theta_2}Y(t) = Y(t)\{1 - R(T_{AT}(t))\} + (1-\delta)K(t) \\
& \mathbf{M}(t) = (1 + \Phi^{BEAM})\mathbf{M}(t-1) + (\mathcal{E}(k,\mu,t), 0, 0)^T \\
& \mathbf{T}(t) = (1 + \Phi^{TEMP})\mathbf{T}(t-1) + [\mu_{AT}\mathcal{F}(M_{AT}(t)), 0]^T \\
& T_{AT}(t) \leq 2 \quad \forall t \\
& 0 \leq \mu(t) \leq 1 \quad \forall t \\
& K(0), \mathbf{M}(-1), \mathbf{T}(-1) \text{ given}
\end{aligned}
$$

where $\mathbf{M}(t) = [M_{AT}(t), M_{UP}(t), M_{LO}(t)]^T$, $\mathbf{T}(t) = [T_{AT}(t), T_{LO}(t)]^T$, $\mathcal{E}(k,\mu,t)$ is described in equation (2) and $\mathcal{F}(M_{AT}(t))$ is described in equation (4).

**Proposition 1.** *The solution to the planning problem implies there are two wedges. One represents the distortion in capital accumulation and the other represents the distortion in mitigation effort.*
*The capital wedge is given by:*

$$
\Lambda_K(t+1) = \sum_{i=1}^{\infty} \frac{\beta^{i-1}U_c(t+i)}{U_c(t+1)} F(t+i) R'\Big(T_{AT}(t+i)\Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \frac{\partial M_{AT}(t+i)}{\partial K(t+1)}
$$

*The mitigation wedge is given by:*

$$
\Lambda_\mu(t) = \sum_{i=0}^{\infty} \frac{\beta^{i}U_c(t+i)}{U_c(t)} F(t+i) R'\Big(T_{AT}(t+i)\Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \frac{\partial M_{AT}(t+i)}{\partial \mu(t)}
$$

*Proof.* The expressions come directly from the first order conditions. See A.1 for details. ■

In this problem, the history of emissions determines the amount of carbon in the atmosphere at any point in time. Thus an action that affects emissions at any given point in time has an immediate effect and an effect in every future point of time. Lets first consider the planner's capital accumulation decision. A higher capital stock in the future generates higher output, but this also means higher emissions. The planner's intertemporal Euler equation (7) shows the immediate returns through higher output in the first line. Apart from standard terms, this equation shows that the planner considers the effect higher output will have of the damages on the next period and the increased spending to keep the emission control rate at the same level.

The second line of the planner's Euler equation shows the externality that the higher next period output will have on subsequent periods. This term is the present value of marginal damages imposed in the future. A higher capital stock means higher output that generates higher damages. This effect through emissions affects every subsequent period, although its effect is dampened over time. Private agents do not consider how their actions have this effect on all subsequent periods and this gives rise to the first wedge. The planner wants to slow down capital accumulation to prevent high future damages.

The second decision the planner makes is the expenditure on abatement through the emission control rate $(\mu)$. The abatement cost function gives us the marginal cost spending on $\mu$. The right-hand side of equation (8) gives the present discounted value of benefits from increasing $\mu$. An increase in the emission control rate implies lower immediate emission. This means lower damages in the next period and lower stock of carbon in the atmosphere. This lowering of carbon stock at any one period has a benefit in every subsequent period and gives rise to a stream of benefits. Every future lower mass of carbon will mean lower damage that has to be converted into appropriate consumption units and then discounted back to compute the present discounted value of marginal benefits.

Given we are using the BEAM carbon cycle to model the climate side, where the transition rates depend on the mass of carbon, this has an interesting implication about the timing of policy. In particular, during times when oceans can absorb more carbon dioxide from the atmosphere, the benefit of investing in $\mu$ is larger. This effect is absent in standard linear models. In Section 5.3 we will see this leads to higher carbon taxes in the recent future and lower carbon taxes farther on.

Time-varying paths of exogenous parameters like $\theta_1$ and $\sigma$ have an important effect on these marginal decisions by the planner. Both of these are declining over time, which means its cheaper to abate in the future since prices are lower. Furthermore, it is less damaging to produce output in the future since it produces lower emissions due to declining carbon intensity. Changing the path of these parameters will change the level and time profile of both the mitigation wedge and the capital wedge.

Thus in the planner allocation, the objective is to balance both of these margins. The marginal benefit from having higher capital stock tomorrow must equal the private return plus the marginal externality damages. The marginal cost of spending on emission control must balance the present discounted value of marginal benefits from the same. In the next section, I talk about a possible decentralization to implement the planner's solution.

## 4.4 Decentralized Equilibrium

In the following section, I will outline one possible implementation of the solution to the planning problem.

### Consumers

An infinitely lived household has well behaved preferences over consumption. Labor is inelastically supplied by the households. Households own capital and rent it to the firms.

$$\begin{aligned}
&\max_{c,\hat{k}} && \sum_{0}^{T} \beta^t U(c(t)) \\
&\text{subject to} && \sum_{0}^{T} p(t)\left(c(t)+\hat{k}(t+1)\right) \leq \sum_{0}^{T} p(t)\left(w(t)n(t)+(1-\delta+r(t)-\tau_K(t))\hat{k}(t)+T(t)\right)
\end{aligned}$$

where $\tau_K(t)$ is the tax on capital, $r(t)$ is the rental rate of capital, $w(t)$ is the wage rate, $T(t)$ denotes lumpsum taxes from the government and $p(t)$ denotes Arrow-Debreu prices.

**Producers**

There is an arbitrary number of firms that employ labor and rent capital from households. They have access to the same neoclassical technology and produce a homogeneous good that they sell competitively to the households in the economy. They also choose the emission control rate given a particular profile of subsidies on $\mu(t)$. Firms take $T_{AT}$ as given and do not internalize how their actions affect it in the future. All input and output markets are assumed to be perfectly competitive. A representative firm solves the following static problem

$$
\begin{aligned}
\max_{k,\mu} \quad & \sum_{0}^{T} p(t)\Big[Y(t)\{1 - R(T_{AT}(t))\} - r(t)k(t) - w(t)n(t) \\
& + \tau(t)\mu(t) - \theta_1(t)\mu(t)^{\theta_2}Y(t)\Big] \\
\text{subject to} \quad & 0 \leq \mu(t) \leq 1 \quad \forall t \\
\text{where} \quad & R(T_{AT}(t)) \text{ is the appropriate damages function.} \\
\text{and} \quad & Y(t) = A(t)k(t)^{\alpha}n(t)^{1-\alpha}
\end{aligned}
$$

where $R(T_{AT}(t))$ denotes the fraction of output lost as damages due to rise in atmospheric temperature, $\mu(t)$ is the level of emission control rate chosen by the firms, $\sigma(t)$ is the carbon intensity of output, $\tau_\mu(t)$ denotes the subsidy on emission control. The presence of this subsidy implies a tax per unit of industrial emission. In 4.4 I show the implied carbon tax. This exposition is maintain an easy comparison with the planning problem. Note, if $\tau_m u = 0$, producers will choose an emission control rate of zero.

#### Government

The Government collects the capital taxes, pays the emission control subsidies and distributes the net revenue lumpsum to the consumers of the economy to balance the budget.

#### Carbon Tax

In the decentralization that was outlined, there was one tax on capital, $\tau_K(t)$ and a subsidy on the emission control rate, $\tau_\mu(t)$. In this section, I will show that the subsidy on the emission control rate, $\mu$, is equivalent to a tax on emission. This is what we call a carbon tax in the paper.

The total amount of dollars spent on the subsidy is given by $\tau_\mu(t)\mu(t)$. This subsidy helped reduce $\mu(t)\sigma(t)F(t)$ $GtCO_2$ emissions. Then the money spent on a unit reduction in emission of $CO_2$ is given by $\frac{\tau_\mu(t)}{\sigma(t)F(t)}$. From the planning problem, we know $\tau_\mu(t) = \theta_1(t)\theta_2\mu(t)^{\theta_2-1}F(t)$. Thus the money spent on a unit reduction of emission in time $t$ or the carbon tax is

$$\frac{\theta_1(t)\theta_2\mu(t)^{\theta_2-1}}{\sigma(t)} \ \$/GtCO_2$$

#### Competitive Equilibrium

A Competitive Equilibrium in this economy can now be formally defined as follows:

**Definition 1.** *A Competitive Equilibrium consists of an allocation for the consumer* $\{c(t), n(t), \hat{k(t)}\}$, *an allocation for the firm* $\{k(t), n(t), \mu(t)\}$, *prices* $\{p(t), w(t), r(t)\}$, *climate states* $\{M_{AT}(t), M_{UP}(t), M_{LO}(t), T_{AT}(t), T_{LO}(t)\}$ *and government policy* $\{\tau_K(t), \tau_\mu(t)\}$ *such that*

*(i) Consumers maximize utility subject to budget constraint given prices and policies.*

*(ii) Producers maximize profits given prices and policies.*

*(iii) Government budget balances*

*(iv) Markets clear*

*(v) Climate and Temperature evolves according to laws of motion*

**Proposition 2.** *Suppose the government sets policy according to the following rule:*

*The tax on capital is given by*

$$\tau_K(t+1) = \sum_{i=1}^{\infty} \frac{\beta^{i-1} U_c(t+i)}{U_c(t+1)} F(t+i) R'\Big(T_{AT}(t+i)\Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \frac{\partial M_{AT}(t+i)}{\partial K(t+1)}$$

*The subsidy for emission control rate is given by:*

$$\tau_\mu(t) = \sum_{i=0}^{\infty} \frac{\beta^{i} U_c(t+i)}{U_c(t)} F(t+i) R'\Big(T_{AT}(t+i)\Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \frac{\partial M_{AT}(t+i)}{\partial \mu(t)}$$

*and the net government revenue are rebated lump-sum to the representative consumer. Then the competitive equilibrium allocation coincides with the social planner's problem.*

*Proof.* The expressions come directly from the first order conditions. See A.2 for details. ∎

Two margins are distorted in the absence of the tax. Since firms do not pay for the negative externality that emissions create, they pay a higher rate of return on capital than what the planner would like. Faced with this, consumers over accumulate capital. Furthermore, firms would choose $\mu(t) = 0 \ \forall t$ as abatement is costly and they do not see any benefits to this investment. To correct these two margins, the planner needs at least two instruments. One possible decentralization is to put a tax on capital and a subsidy on emission control rate.

The intuition of this result is straightforward. Having a lower amount of carbon dioxide in the atmosphere helps in production analogous to physical capital. However, as the atmosphere is a public good, there is a free-rider problem. Private agents take the state of the climate as given and do not internalize the effect their actions have on the climate in subsequent periods. Private firm's incentives to invest in the upkeep of the atmosphere through emission reductions are distorted in the absence of a properly set subsidy. As firms use more capital, they pay a higher return on it, thereby making the household over accumulate capital. To address the twin distortions, the planner uses two instruments to align the first-order conditions in the decentralized problem with the first-order conditions in the planning problem. The subsidy on the emission control rate is isomorphic to a tax on carbon emission as outlined in Section 4.4. The literature has talked about this carbon tax but ignored the role of capital taxes to implement the optimal allocation.

# 5 Computation

In this section, I start by laying out the computational challenge in dealing with these types of models. Section 5.1 lays out how to convert the optimal control problem outlined in the previous section into a nonlinear programming problem. Section 5.2 lists the parameters used in the calibration exercises. Finally, section 5.3 shows the results obtained.

## 5.1 Strategy

Optimization of Integrated Assessment Models (IAMs) is generally computationally more complicated compared to optimizing growth models in economics and differential equation models, such as climate models. In growth models, the number of state variables is small and the policy functions are linear. This allows techniques like dynamic programming to be applied. In physical climate models, we can use recursive time-stepped algorithms as we do not perform any optimization in such models. However, as the current model combines a deterministic neoclassical growth model with a state of the art three-box climate model, the optimization problem becomes computationally more complex from a mathematical point of view. Now, we have to solve a set of equilibrium conditions, such as first-order conditions along with multiple equations of motion for each of the state variables. While optimization problems with fewer variables or linear solution paths can be solved quickly and efficiently, in general, the computational costs increase at an exponential rate depending on the number of variables and nonlinear solutions. For the current problem, a big computational challenge is that there are six continuous state variables, $M_{AT}, M_{UP}, M_{LO}, T^{AT}, T^{LO}, K$ and two control variables: $\mu, c$. Furthermore, the policy functions are nonlinear and would require a large number of grid points. Given the difficulty of using dynamic programming techniques for a nonlinear problem with a large state space, I employ a different technique more common in the engineering fields to solve this problem.

Numerical methods for solving optimal control problems emerged in the 1950s and were at the beginning typically based on either dynamic programming, which is limited to small state dimension, or methods based on the calculus of variations. Such approaches severely suffer from Bellman's "curse of dimensionality". There is also the difficulty of such algorithms to deal with inequality constraints, represented as bounds on the state and control paths. These methods are classified as indirect methods and use a "first optimize, then discretize" approach.

The emergence of powerful nonlinear programming (NLP) solvers, shifted the focus in the early 1980s to direct methods. Here, the original optimal control problem is transformed into a finite-dimensional nonlinear programming problem. This NLP is solved by cutting edge numerical optimization methods and the approach is often sketched as "first discretize, then optimize". A big advantage of this technique compared to indirect methods like dynamic programming is that now we can handle inequality path constraints. In our problem, the Paris agreement target of being within 2°C is essentially a path constraint on atmospheric temperature, which is a state variable. In the NLP formulation, structural changes in the active constraints during the optimization procedure can be treated by well-developed NLP methods that can deal with inequality constraints. (Diehl et al. (2006))

We implement the multiple shooting algorithm, following Bock and Plitt (1984), to reformulate the original problem as an NLP. At first, we discretize the state and control trajectories piecewise on a coarse grid. Nordhaus (2016) calibrates his model at 5 year time steps. To keep our estimates comparable, we also use a 5-year time step. Thus each time period in our model corresponds to five years. However, while discretizing the state and control variables, we use five times the number of periods as the number of intervals in the grid. Although the larger number of intervals slows down the optimization process, it ensures the solutions are smooth and numerical errors are small. Next, we formulate the NLP problem.

We start by guessing a sequence of values for each of the control and state variables for each interval. Let $S_t = [M_{AT}, M_{UP}, M_{LO}, T_{UP}, T_{LO}, K]$ denote our guess for the state variables and $u_t = [\mu, c]$ denote our guess for the control variables at time $t$. Let the entire sequence of state and controls be given by $\{S_t, u_t\}_t$. We solve the set of constraints, the six difference equations simultaneously given our guess of the control variables and the starting value of the state variables for each interval. Let $\Psi(.)$ denote the solution function for the states such that given $S_t$, $\Psi(S_t)$ is the value of the states in period $t+1$ according to the constraints of the program. According to our original guess for the state trajectories, $S_{t+1}$ was the value of the state at $t+1$. To constrain our guess for the state and control variables, we impose the following continuity constraint:

$$S_{t+1} - \Psi(S_t) = 0, \qquad t = 0, 1, ..., 5T$$

The solvers that solve the optimization problem now only need to iterate on the guess for trajectories of the state and control variables that make sure all the continuity constraints are satisfied. Although the NLP problem has a much higher number of variables that it has to solve for, the problem has a block sparse structure. The block sparse Joacobian contains the linearized dynamic equations of the model and the Hessian is block diagonal. To implement this algorithm, we use an open-source optimization framework called CasADi developed by Andersson, Åkesson, and Diehl (2012). CasADi can efficiently implement automatic differentiation (AD) algorithms to compute sparse Jacobian matrices quickly (Coleman and Verma (1998)) and helps converge to a solution much faster. Furthermore, this symbolic framework provides the building blocks to formulate such problems in a compact structure. It interfaces with different nonlinear programming solvers easily and can solve the problem efficiently. I use the IPOPT solver, following Wächter and Biegler (2006), since its appropriate for such large scale nonlinear programs. The framework can be implemented through MATLAB or Python and allows the user to easily include inequalities as path constraints and customize the solver to user specifications.

## 5.2 Parameter Selection

This section describes the calibration for the model outlined above. The calibration adopts many parameters and functional forms directly from Nordhaus (2016) to maintain close comparability with this benchmark. At the same time, the calibration presents novel features of deriving the carbon cycle parameters from the laws of natural science.

Table 1: Summary of Key Calibration Parameters

| Parameter | Value | Source and Notes |
|---|---|---|
| Preferences: $U(C(t), L(t)) = L(t)\left(\frac{\left(\frac{C(t)}{L(t)}\right)^{1-\gamma} - 1}{1-\gamma} - 1\right)$ | | |
| $\gamma$ | 1.45 | Nordhaus (2016), Elasticity of marginal utility |
| $\rho$ | 0.015 | Nordhaus (2016), Initial rate of social time preference per year |

| | | |
|---|---|---|
| Labor Supply: $L(t+1) = L(t)\left(\frac{L_a}{L(t)}\right)^{l_g}$ | | |
| $L_a$ | 11500 | Nordhaus (2016), Asymptotic population |
| $L_0$ | 7403 | Nordhaus (2016), Initial population (in millions) |
| $l_g$ | 0.134 | Nordhaus (2016), Population growth rate |
| Output: $Y(t) = A(t)K(t)^{\alpha}L(t)^{1-\alpha}$ | | |
| $\alpha$ | 0.3 | Nordhaus (2016), Capital elasticity in production function |
| $\delta$ | 0.1 | Nordhaus (2016), Capital depreciation (5 Years) |
| Productivity: $A(t+1) = \frac{A(t)}{1-g_a e^{-\delta_a * 5(t-1)}}$ | | |
| $A_0$ | 5.115 | Nordhaus (2016), Initial Total Factor Productivity (TFP) |
| $g_a$ | 0.076 | Nordhaus (2016), Initial TFP rate |
| $\delta_a$ | 0.005 | Nordhaus (2016), TFP increase rate |
| Nordhaus Damages: $R(T_{AT}) = a_2 T_{AT}^{a_3}$ | | |
| $a_2$ | 0.00236 | Nordhaus (2016), Damage multiplier |
| $a_3$ | 2 | Nordhaus (2016), Damage exponent |
| Weitzman Damages: $R(T_{AT}) = \frac{1}{1+\pi_1 T_{AT}^2 + \pi_2 T_{AT}^{6.76}}$ | | |
| $\pi_1$ | 0.00245 | Weitzman (2010) |
| $\pi_2$ | 0.00000502 | Weitzman (2010) |
| Abatement Costs: $\theta_1(t) = \sigma(t)\frac{p_b}{1000\theta_2}(1-\delta_{p_b})^{t-1}$ | | |
| $p_b$ | 550 | Nordhaus (2016), Initial backstop price |
| $\delta_{p_b}$ | 0.025 | Nordhaus (2016), Decline rate of backstop price |
| $\theta_2$ | 2.6 | Nordhaus (2016), Exponent of control cost function |
| Carbon Intensity: $\sigma(t) = \sigma(t-1)e^{(-g_\sigma * (1-\delta_\sigma)^{5(t-1)} * 5)}$ | | |
| $g_\sigma$ | 0.0152 | Nordhaus (2016), Emission intensity base rate |
| $\delta_\sigma$ | 0.001 | Nordhaus (2016), Decline rate of emission intensity |

| Carbon Cycle Transition Matrix: $\Phi^{BEAM}$ | | |
|---|---|---|
| $k_a$ | 0.2 | Glotter (2014) |
| $k_d$ | 0.05 | Glotter (2014) |
| $k_l$ | 0.001 | Glotter (2014) |
| $k_h$ | 1230 | Glotter (2014) |
| $k_1$ | $8 \times 10^{-7}$ | Glotter (2014), mol/kg |
| $k_2$ | $4.63 \times 10^{-10}$ | Glotter (2014), mol/kg |
| $AM$ | $1.77 \times 10^{20}$ | Glotter (2014), moles |
| $OM$ | $7.8 \times 10^{22}$ | Glotter (2014), moles |
| $Alk$ | 767 | Glotter (2014), GtC |
| Temperature transition matrix: $\Phi^{TEMP}$ | | |
| $\mu_{AT}$ | 0.2 | Glotter (2014) |
| $\mu_{LO}$ | 0.05 | Glotter (2014) |
| $\gamma_{LO}$ | 0.3 | Glotter (2014) |
| $\kappa$ | 1.3 | Glotter (2014), General climate sensitivity |
| $\overline{M_{AT}}$ | 588 | Glotter (2014), Mass of carbon in the atmosphere in 1750 |

### 5.3 Results

This section presents the results of the optimal policy exercise across different scenarios. I interpret the Paris agreement scenario as $T_{AT}(t) \leq 2°C, \forall t$. I will also look at other temperature targets of $2.5°C$ and $3.5°C$. Business as usual is going to refer to the case when $\mu(t) = 0, \forall t$. Finally the "free optimal" refers to the scenario when the planner faces no temperature constraints and freely chooses optimal policies. Initially, I use the form of the damage function given by

$$R(T_{AT}) = a_2 T_{AT}^{a_3}$$

The above damage function is the one used in Nordhaus (2016). The path of the rise in atmospheric temperature over time in the different scenarios is depicted in Figure 4. The

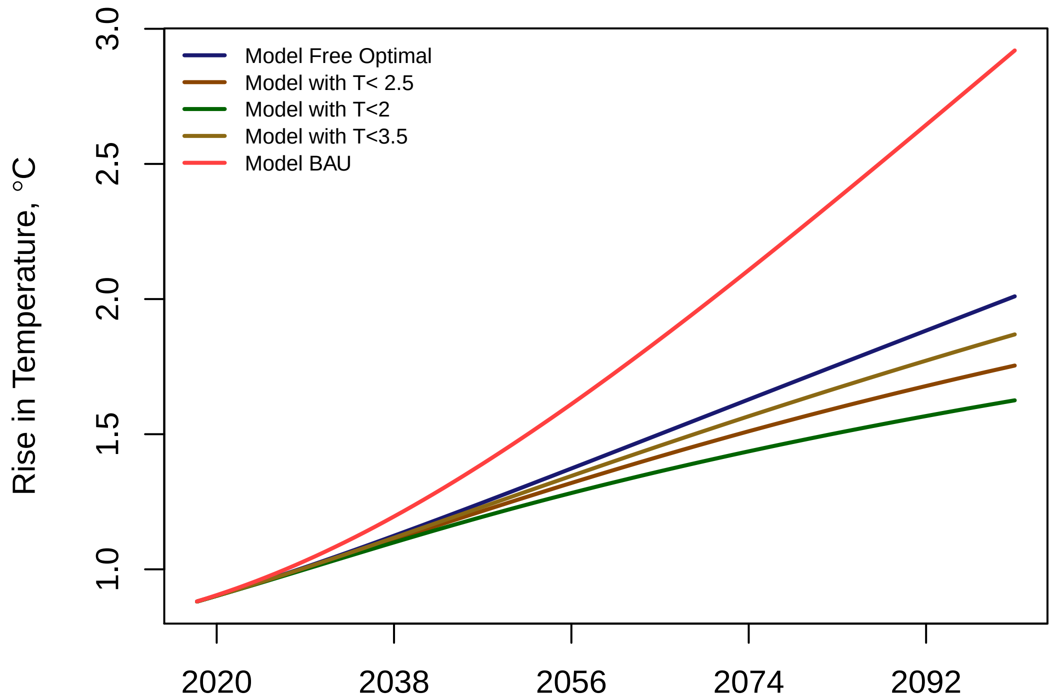


Figure 4: Rise in atmospheric temperature in different scenarios

most important thing to note in this picture is that if we implement carbon taxes that are consistent with the “free optimal" scenario, where the optimal tax is calculated without the inclusion of explicit temperature constraints in the planning problem, the atmospheric temperature would eventually rise $4°C$ above pre-industrial levels. Thus if we are to take temperature targets seriously, they must be included as separate constraints in the planning problem. The other point to note is not doing anything and going on a business as usual trajectory would mean a large increase in temperature, almost by $3°C$ by the end of the century. Given the inclusion of these temperature constraints, the relative change in the optimal tax across different scenarios is intuitive. A lower temperature target is associated with a higher optimal tax.

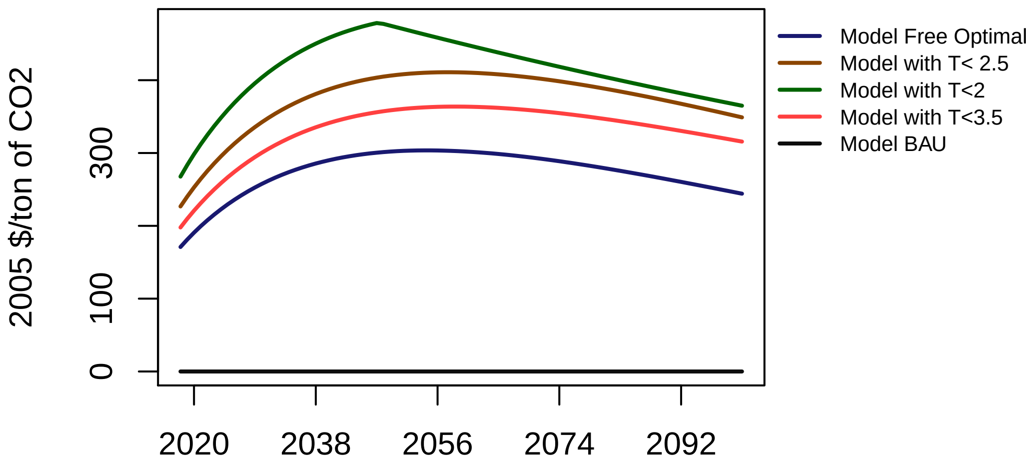


Figure 5: Optimal Carbon Tax in different scenarios

While the levels of the optimal taxes in these different scenarios reflect the role of temperature targets, the time profile shows the impact of the nonlinear dynamics. The optimal taxes I obtain rise much faster than those estimated in standard models like Nordhaus (2016) during the initial period. However, later on, these taxes stabilize at levels much lower than those prescribed by Nordhaus. This is because there are greater benefits to immediate action that are underestimated in models with linear dynamics. In particular, by taking steps to reduce emissions drastically by 2050, we can stabilize atmospheric carbon concentrations at lower levels compared to if we were to implement the same reductions by 2100. This aspect of timing is crucial in the design of environmental policy and the entire time path of these taxes is important, not just its value at a particular year.

When we look at the fraction of output lost as damages, we can get some idea about

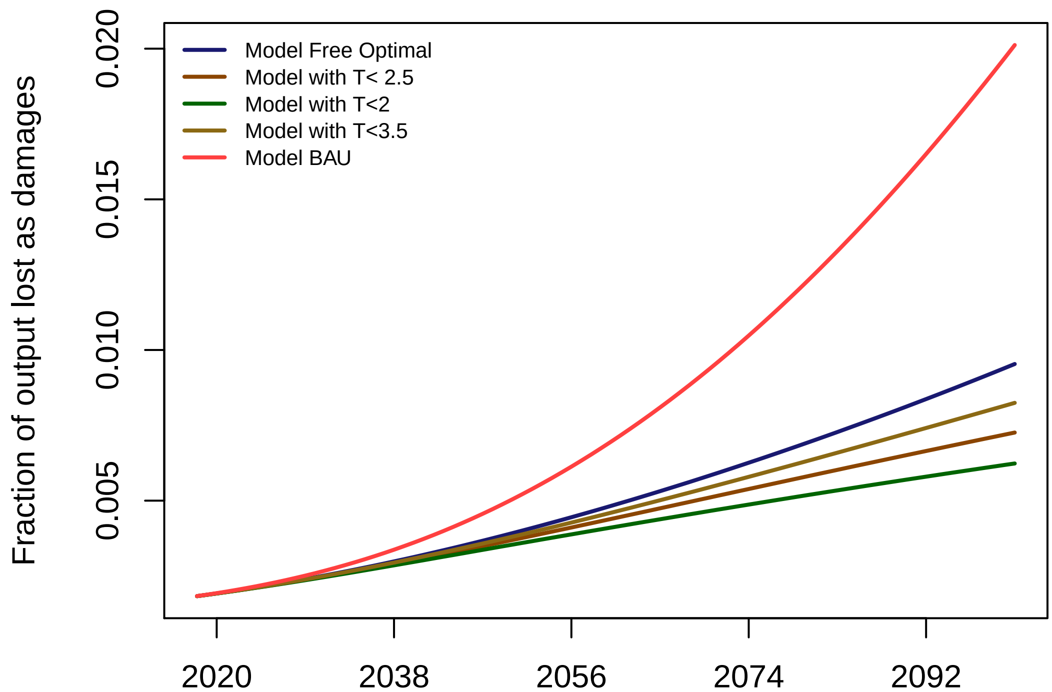

Figure 6: Fraction of output lost as damages in different scenarios

the impact of these different policies. Based on our discussion in Section 4.2, the choice of the damages function is crucial to understand the welfare impacts of these policies. If we take the Nordhaus (2016) damages function, we are looking at about a 2% loss in annual output by 2100 if we go along business as usual. By following policies to limit global temperatures within a $2°C$ rise over preindustrial levels, damages are only 0.62% of global output by 2100. Thus following Nordhaus' damages function, the fraction of output lost as damages would be 3.2 times higher if we didn't do anything compared to the $2°C$ scenario.

As discussed in Section 4.2, these damages are calibrated for low rises in temperature and are not appropriate for large increases in temperature. Thus, although it might estimate the damages with precision in scenarios where the rise in temperature is modest, its estimate in business as usual scenarios are underestimated.

To isolate the effects that nonlinear ocean dynamics on the estimation of the optimal carbon tax, I compare the results with an alternative carbon cycle as in Nordhaus (2016). This implies using the following equation for the law of motion for carbon:

$$\mathbf{M}(t) = \Phi^{DICE}\mathbf{M}(t-1) + (\mathcal{E}(k,\mu,t), 0, 0)^T$$

where

$$\Phi^{DICE} = \begin{bmatrix} 0.912 & 0.0383 & 0 \\ 0.088 & 0.9592 & 0.00033 \\ 0 & 0.0025 & 0.99966 \end{bmatrix}$$

From the results in Table 2, it is clear that rapid mitigation policies have costs in the short term. The high taxes reduce output relative to the output in a business as usual scenario in the short run. But with rapid mitigation policies that quickly bring down emission levels, the benefits through output increases are visible in the long run. By 2200, we can get significantly higher output compared to business as usual. If the additional temperature constraints are removed, the output is higher because of the effect of the increase in output due to lower taxes more than offsets the decrease in output due to higher damages from the rise in temperatures. Output in the business as usual scenario is much lower since the damages from higher atmospheric temperature rise are far greater than the loss in output from imposing carbon taxes.

We can see an interesting result when we compare the optimal carbon taxes of the model with the estimates from Nordhaus (2016). In the Nordhaus optimal scenario, there are no explicit temperature targets. The optimal taxes are much lower and there is a larger increase in atmospheric temperature leading to higher damages. While in the short run this leads to an increase in output relative to my model, in the long run output is lower. The other reason dragging down output in Nordhaus' model is the high taxes in the long run. In Nordhaus' simulations, taxes go up much more gradually but go up much higher in the long run. This ties to the initial point about the timing of environmental policy. Nordhaus proposes a more gradual policy since, from a cost-benefit analysis, rapid mitigation policy is too costly compared to the benefits. With a more accurate carbon cycle representation, the benefits turn out to be much higher leading to higher efforts at mitigation early on.

Table 2: Main Results

| | Carbon Tax* | | $\Delta$ Output %† | |
|---|---|---|---|---|
| Scenario | 2020 | 2050 | 2100 | 2200 |
| Model with $T_{AT} \leq 2°C$ | 298.3 | 472.5 | -2.1 | 17 |
| Model with $T_{AT} \leq 2.5°C$ | 252.8 | 407.3 | -1.2 | 19.2 |
| Model Free optimal | 190 | 302.5 | 1.2 | 22 |
| Nordhaus (2016), optimal | 36.7 | 103.6 | 1.5 | 8 |
| Nordhaus (2016), $T_{AT} \leq 2.5°C$ | 229.1 | 1006.2 | -5.2 | 1.5 |
| Stern (2007) | 266.5 | 629.2 | .. | .. |

* 2005 \$, † Annual change relative to BAU scenario

The fact that rapid mitigation benefits are underestimated in Nordhaus' model is more starkly visible when you compare the carbon taxes across the 2.5°C scenario. Since Nordhaus underestimates the rate of ocean absorption of $CO_2$, the taxes he estimates are higher than the taxes I estimate. Although they start lower in 2020, it increases much faster and by 250, his estimates are more than double of mine. Such high taxes have a significant effect on output leading to a much lower increase relative to business as usual compared to the other scenarios considered. There is no comparison for the 2°C scenario using Nordhaus' model since given inertia of the climate system, it is not feasible to remain within this target. This means that the land emissions and exogenous forcing considered in the model as exogenously given would be enough to give rise to a higher increase in temperature.

To compare these results to the other widely debated topic of discounting, I present a comparison with estimates by Stern (2007). A key difference between Nordhaus and Stern is that they use very different subjective discount rates. Nordhaus used a rate of 1.5% per year, mostly based on market measures. Stern, who added a "moral" concern for future generations, used a significantly lower rate of 0.1% per year. Since Stern values damages to the future much higher, he suggests a higher profile of carbon taxes. If I adjust the subjective discount rate down to the level advocated by Stern, we obtain an optimal tax that is about twice the size of his.

# 6 Conclusion

In this paper, I formulate a dynamic general equilibrium model of the world, considering it as a uniform region that is inhabited by a representative household. There is a global externality from carbon dioxide emission which is a by-product of output production. The first contribution of this paper is to show how estimates of the optimal tax on carbon change when we introduce nonlinear dynamics. This is done by incorporating a state of the art climate model into a standard neoclassical growth model. Using this model, I calculate the optimal carbon taxes with various temperature targets including the Paris agreement target of 2°C rise in atmospheric temperature. While Nordhaus (2016) claims reaching the Paris agreement target is not feasible, I show that the benefits of early mitigation efforts are higher with the incorporation of nonlinear dynamics and, therefore, sufficient to achieve the Paris targets.

I also show that a carbon tax alone does not decentralize the solution to the planner's problem. The market allocation has two distortions: there is over-investment in capital accumulation and no investment in mitigation efforts. To correct these two margins, the planner must use two taxes. I propose a decentralization with the use of a capital tax and an emission tax. While a similar result holds for many other models in the literature, including DICE models by Nordhaus, the issue of capital taxation has been largely ignored.

Given the computational challenge of solving this class of models, this paper shows the application cutting edge numerical techniques. By converting the optimal control problem into a nonlinear programming problem, we can create a structure that gives rise to a block sparse Jacobian and a block diagonal Hessian. This structure can be exploited to efficiently find the solution using open source solvers that can be implemented through the CasADi framework. This would, in particular, allow significant improvement in modeling many of the complexities, as this helps eliminate a major computational bottleneck. With greater ease of dealing with large state spaces with nonlinear policy functions, many new dimensions can be explored including the inclusion of uncertainty, multi-regional modeling and a richer description of the climate system.

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# A Appendix

## A.1 Planning Problem

In this section I will layout a simplified version of the planning problem. Instead of having 5 state variables to describe the climate dynamics, I will use only two, $T_{AT}(t), M_{AT}(t)$, which will denote the atmospheric temperature and the mass of carbon in the atmosphere at time $t$ respectively. Here the dynamics of carbon flow will be captured by a time varying nonlinear function $a(M_{AT}(t))$. This is a simpler way of capturing the nonlinear dynamics of carbon depreciation in a simpler notation. The planner wants to maximize discounted sum of utilities subject to the resource constraint and the law of motion for the stock of carbon dioxide in the atmosphere. The presence of temperature constraints in the original problem can be thought of as an upper bound on the amount of carbon in the atmosphere over the planning horizon. The general planning problem can be written as follows:

$$\max_{\{c(t),k(t+1),\mu(t)\}_{t=0}^{\infty}} \quad \sum_{0}^{\infty} \beta^t U(c(t), \ell(t))$$

$$\begin{aligned}
\text{s.t.} \quad & K(t+1) \leq F(t)\Big\{1 - R\Big(T_{AT}\Big(M_{AT}(t)\Big)\Big)\Big\} - c(t) - \theta_1(t)\mu(t)^{\theta_2}F(t) + (1-\delta)K(t) \\
& M_{AT}(t) = \sigma(t)F(t)\{1-\mu(t)\} + \{1 - a(t-1)\}M_{AT}(t-1) \\
& T_{AT}(t) = \mathcal{Z}\Big(M_{AT}(t)\Big) \\
& 0 \leq \mu(t) \leq 1 \quad \forall t \\
& K(0), M_{AT}(-1), T_{AT}(-1) \text{ given.}
\end{aligned}$$

The planner realizes that the choices it makes has an effect on future values of $M_{AT}(t)$. We can substitute $M_{AT}(t)$ from the second constraint into the first constraint. Let $\beta^t\lambda(t)$ denote the lagrange multiplier on the resource constraint. After recursively substituting the constraints, the Lagrangian takes the following form:

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(t) + \beta^0 \lambda(0) \Big[ & F(0) \Big\{ 1 - R\Big( T_{AT}\Big( \underbrace{\sigma(0)F(0) - \mu(0)\sigma(0)F(0) + (1-a(-1))M_{AT}(-1)}_{M_{AT}(0)} \Big)\Big) \Big\} \\
& - c(0) - \theta_1(0)\mu(0)^{\theta_2}F(0) + (1-\delta)K(0) - K(1) \Big] \\
+ \beta^1 \lambda(1) \Big[ & F(1) \Big\{ 1 - R\Big( T_{AT}\Big( \underbrace{\sigma(1)F(1) - \mu(1)\sigma(1)F(1) + (1-a(0))M_{AT}(0)}_{M_{AT}(1)} \Big)\Big) \Big\} \\
& - c(1) - \theta_1(1)\mu(1)^{\theta_2}F(1) + (1-\delta)K(1) - K(2) \Big] \\
+ \beta^2 \lambda(2) \Big[ & F(2) \Big\{ 1 - R\Big( T_{AT}\Big( \underbrace{\sigma(2)F(2) - \mu(2)\sigma(2)F(2) + (1-a(1))M_{AT}(1)}_{M_{AT}(2)} \Big)\Big) \Big\} \\
& - c(2) - \theta_1(2)\mu(2)^{\theta_2}F(2) + (1-\delta)K(2) - K(3) \Big] \\
+ \beta^3 \lambda(3) \Big[ & F(3) \Big\{ 1 - R\Big( T_{AT}\Big( \underbrace{\sigma(3)F(3) - \mu(3)\sigma(3)F(3) + (1-a(2))M_{AT}(2)}_{M_{AT}(3)} \Big)\Big) \Big\} \\
& - c(3) - \theta_1(3)\mu(3)^{\theta_2}F(3) + (1-\delta)K(3) - K(4) \Big] + ....
\end{aligned}
$$

Taking first order conditions, we get

$$
\begin{aligned}
[c(t)]: \quad & U_c(t) = \lambda(t) \\
[k(t+1)]: \quad & \lambda(t) = \beta\lambda(t+1) \Big[ F_K(t+1) \Big\{ 1 - R\Big( T_{AT}(t+1) \Big) \Big\} - \theta_1(t+1)\mu(t+1)^{\theta_2} F_K(t+1) + 1 - \delta \Big] \\
& \qquad - \sum_{i=1}^{\infty} \beta^i \lambda(t+i) F(t+i) R'\Big( T_{AT}(t+i) \Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \frac{\partial M_{AT}(t+i)}{\partial K(t+1)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \frac{U_c(t)}{\beta U_c(t+1)} = & F_K(t+1) \Big[ 1 - R(T_{AT}(t+1)) - \theta_1(t+1)\mu(t+1)^{\theta_2} \Big] + 1 - \delta \\
& - \underbrace{\sum_{i=1}^{\infty} \frac{\beta^{i-1} U_c(t+i)}{U_c(t+1)} F(t+i) R'\Big( T_{AT}(t+i) \Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \Omega_k(t,i)}_{\text{PV of future damages from Capital}}
\end{aligned} \tag{7}
$$

$$[\mu(t)]:\ \beta^t\lambda(t)\theta_1(t)\theta_2\mu(t)^{\theta_2-1}F(t)=\sum_{i=0}^{\infty}\beta^{t+i}\lambda(t+i)F(t+i)R'\Big(T_{AT}(t+i)\Big)\frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)}\frac{\partial M_{AT}(t+i)}{\partial\mu(t)}$$

$$\implies \underbrace{\theta_1(t)\theta_2\mu(t)^{\theta_2-1}F(t)}_{\text{MC of }\mu}=\underbrace{\sum_{i=0}^{\infty}\frac{\beta^i U_c(t+i)}{U_c(t)}F(t+i)R'\Big(T_{AT}(t+i)\Big)\frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)}\Omega_\mu(t,i)}_{\text{PV of marginal benefit of }\mu} \tag{8}$$

where

$$\Omega_k(t,i)=\frac{\partial M_{AT}(t+i)}{\partial K(t+1)}=\begin{cases}\Big(1-\mu(t+1)\Big)\sigma(t+1)F_K(t+1) & i=1\\ \prod_{j=1}^{t+i-1}\Big(1-a(j)\Big)\Big(1-\mu(t+1)\Big)\sigma(t+1)F_k(t+1) & i>1\end{cases}$$

and

$$\Omega_\mu(t,i)=\frac{\partial M_{AT}(t+i)}{\partial\mu(t)}=\begin{cases}\sigma(t)F(t) & i=0\\ \prod_{j=0}^{t+i-1}\Big(1-a(j)\Big)\sigma(t)F(t) & i>0\end{cases}$$

The optimal allocations can be solved by simultaneously solving the above three equations, resource constraint and the law of motion for $N$ along with the initial conditions. Let the optimal allocation that solves the planning problem be given by $\{c^*(t),\mu^*(t),K^*(t+1),N^*(t+1)\}_t$

## A.2 Decentralized Problem

Now, we want to write down a set of taxes that implements the solution to the planner's problem above. We show that one possible implementation involves the use of a capital tax that households need to pay, denoted by $\{\tau_K(t)\}_t$ and a subsidy on $\mu$ that firms get denoted by $\{\tau_\mu(t)\}_t$.

**Consumer's Problem**

$$\max_{\{c(t),K(t+1)\}_{t=0}^{\infty}} \quad \sum_{0}^{\infty} \beta^t U(c(t))$$

$$\text{subject to} \quad \sum_{0}^{\infty} p(t)\,(c(t)+K(t+1)) \leq \sum_{0}^{\infty} p(t)\,(w(t)N(t)+(1-\delta-\tau_K(t)+r(t))K(t)+T(t))$$

$$K(0) \text{ given}$$

Let $\chi$ denote the lagrange multiplier on the intertemporal budget constraint. The first order conditions of this problem are given by

$$\begin{aligned}
[c(t)]: &\quad \beta^t U_c(t) = \chi p(t) \\
[K(t+1)]: &\quad \chi p(t) = \chi p(t+1)\{1-\delta-\tau_K(t+1)+r(t+1)\} \\
\Longrightarrow &\quad \frac{U_c(t)}{\beta U_c(t+1)} = \left[1-\delta-\tau_K(t+1)+r(t+1)\right]
\end{aligned}$$

**Firm's Problem**

There is an arbitrary number of firms that employ labor and rent capital from households. They have access to the same neoclassical technology and produce a homogeneous good that they sell competitively to the households in the economy. The firms do not realize how their choice of $\mu(t)$ affects $M_{AT}(t+1)$. They take $M_{AT}(t)$ as given. Thus, firms solve the following static problem:

$$\max_{K,\mu} \quad \sum_{0}^{\infty} p(t)\left[F(t)\{1-R(M_{AT}(t))\}-r(t)k(t)-w(t)M_{AT}(t)+\tau_\mu(t)\mu(t)-\theta_1(t)\mu(t)^{\theta_2}F(t)\right]$$

The first order conditions of this problem for an interior solution are given by

$$\begin{aligned}
[K(t)]: &\quad F_K(t)\{1-R(M_{AT}(t)\}-r(t)-\theta_1(t)\mu(t)^{\theta_2}F_K(t)=0 \\
[\mu(t)]: &\quad \tau_\mu(t)=\theta_1(t)\theta_2\mu(t)^{\theta_2-1}F(t)
\end{aligned}$$

Combining the first order conditions from the consumer problem with the first order conditions from the firm problem gives us the following Euler equation:

$$\frac{U_c(t)}{\beta U_c(t+1)} = F_K(t+1)\Big[1 - R(M_{AT}(t+1)) - \theta_1(t+1)\mu(t+1)^{\theta_2}\Big] + 1 - \delta - \tau_K(t+1)$$

Given a sequence of taxes $\{\tau_K(t), \tau_\mu(t)\}_t$, let the competitive equilibrium allocation be given by $\{c^e(t), \mu^e(t), K^e(t+1), N^e(t+1)\}_t$. Comparing the first order conditions in the decentralized problem with the first order conditions in the planning problem, we can see that if we choose

$$\tau_K(t+1) = \sum_{i=1}^{\infty} \frac{\beta^{i-1} U_c(t+i)}{U_c(t+1)} F(t+i) R'\Big(T_{AT}(t+i)\Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \Omega_k(t,i)$$

$$\tau_\mu(t) = \sum_{i=0}^{\infty} \frac{\beta^{i} U_c(t+i)}{U_c(t)} F(t+i) R'\Big(T_{AT}(t+i)\Big) \frac{\partial T_{AT}(t+i)}{\partial M_{AT}(t+i)} \Omega_\mu(t,i)$$

Then, $c^e(t) = c^*(t), \mu^e(t) = \mu^*(t), K^e(t+1) = K^*(t+1), N^e(t+1) = N^*(t+1) \quad \forall t$

**Carbon Tax**

In the previous section, we showed what would be one implementation of the wedges in the planning problem. In the decentralization that was outlined, there was one tax on capital, $\tau_K(t)$ and a subsidy on the emission control rate, $\tau_\mu(t)$. In this section, I will show that the subsidy on the emission control rate, $\mu$, is equivalent to a tax on emission. This is what we call a carbon tax in the paper.

The total amount of dollars spent on the subsidy is given by $\tau_\mu(t)\mu(t)$. This subsidy helped reduce $\mu(t)\sigma(t)F(t)$ $GtCO_2$ emissions. Then the money spent on a unit reduction in emission of $CO_2$ is given by $\frac{\tau_\mu(t)}{\sigma(t)F(t)}$. From the planning problem, we know $\tau_\mu(t) = \theta_1(t)\theta_2\mu(t)^{\theta_2-1}F(t)$. Thus the money spent on a unit reduction of emission in time $t$ or the carbon tax is

$$\frac{\theta_1(t)\theta_2\mu(t)^{\theta_2-1}}{\sigma(t)} \ \$/GtCO_2$$

# B Temperature Model

In a one box model, we have

$$C_{eff}\frac{dT}{dt} = F - \lambda T$$

where $C_{eff}$ is the effective heat capacity of the system, $F$ is the radiative forcing and $\lambda$ is the feedback parameter. Basically the difference in incoming and outgoing heat energy will cause a rise in temperature of the system.

For a two box model, the above structure can be generalized as shown in Calel and Stainforth (2017), we have

$$\begin{aligned} C_{AT}\frac{dT_{AT}}{dt} &= F - \lambda T_{AT} - \beta(T_{AT} - T_{LO}) \\ C_{LO}\frac{dT_{LO}}{dt} &= \beta(T_{AT} - T_{LO}) \end{aligned}$$

If we discretize the above system of equation using the forward Euler method, we get

$$\begin{aligned} T_{AT}(t) &= T_{AT}(t-1) + \frac{\Delta t}{C_{AT}}\Big[F(t-1) - \lambda T_{AT}(t-1) - \beta(T_{AT}(t-1) - T_{LO}(t-1))\Big] \\ T_{LO}(t) &= T_{LO}(t-1) + \frac{\beta\Delta t}{C_{LO}}\Big[T_{AT}(t-1) - T_{LO}(t-1)\Big] \end{aligned}$$

After some change of notation, we arrive at

$$\begin{aligned} T_{AT}(t) &= T_{AT}(t-1) + \mu_{AT}\Big[\mathcal{F}(t-1) - \kappa T_{AT}(t-1) - \gamma\left(T_{AT}(t-1) - T_{LO}(t-1)\right)\Big] \\ T_{LO}(t) &= T_{LO}(t-1) + \mu_{LO}\gamma\Big[T_{AT}(t-1) - T_{LO}(t-1)\Big] \end{aligned}$$