Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 3
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Date: August 21, 2019 Day: Wednesday
Time: 2:00 PM - 2:05 PM Max. Marks: 5

Question. State whether the following function is true or false. Justify your answer. It is possible to find a function $f : \mathbb{R} \to \mathbb{R}$ such that

a. f is continuous on $[-7, 0]$
b. f is differentiable on $(-5, 0)$
c. $f'(x) \leq 2$
d. $f(-5) = -3$ and $f(0) = 8$

(2 marks for correct alternative; 3 marks for correct justification)

Solution. The given statement is **false**. (2 marks)

We shall prove this by contradiction. Suppose there does indeed exist an f satisfying all the above given conditions. Then f is continuous on $[-7, 0]$ (by (a)), and hence on $(-5, 0)$. Also, the function is differentiable on $(-5, 0)$ (by (b)). (1 mark)

This allows us to use Lagrange's Mean Value Theorem on the interval $[-5, 0]$, which tells us that there must exist a $c \in (-5, 0)$ such that (1 mark)

$$f'(c) = \frac{f(0) - f(-5)}{0 - (-5)} = \frac{8 - (-3)}{0 - (-5)} = \frac{11}{5} > 2$$

By (c), we have $f'(x) \leq 2$ ($\forall\ x \in \mathbb{R}$ should have been mentioned in the question), which means there cannot be such a c. (1 mark)

$\therefore$ Our assumption that such an f which satisfies all the given conditions exists, must be incorrect, implying that there is no such function. $\square$

General observations:

1. This time, all students had the right idea. The most common mistake was in not stating the interval that MVT is being applied on, just as in Short Quiz 2, where the interval in which IVT is used was not mentioned by a sizeable number of students.
2. A finer point here is that the problem statement says that it is possible to find such a function. Most students have just written that f is continuous and differentiable,

but the thing is, we have to check whether it is possible to find such an f. What should actually be written is that we first assume such an f exists, and then, by the conditions given, is continuous and differentiable and so on. This is just for being absolutely perfect though, and no marks have been dedcuted even for those who directly stated that f is continuous and differentiable.

3. Many students have directly used MVT, without talking about continuity and differentiability.
4. A few students have used the argument of drawing a line joining the two points given on an $x-y$ plane, with $y = f(x)$, and then claiming that this line has the maximum slope, i.e. they have argued in terms of geometry. But this course is about being rigorous in calculus, and such claims, if made, should be proven.
5. Some students have used the line argument without even making such a claim to support their answer, and have directly concluded that there is no such function by checking that the straight line doesn't satisfy all the given conditions. This only shows that the function given by the straight line is not a possible function here, but that does not mean there is NO such function, as is stated in the question.
