MA 109: Assignment 1

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Question.
Take the last digit of your roll number. Call it w. Take the second last digit of your roll number. Call it z. Let $a = w + 10$ and $b = z + 10$. Evaluate the limit.

$$\lim_{n\to\infty} \frac{an+1}{bn+2}$$

Justify your answer using the $\epsilon - N$ definition of the limit.

Solution.
(Note: w and z are both single digit integers, so they belong to $\{0, 1, \cdots, 9\}$, hence a and b will be integers in $\{10, 11, \cdots, 19\}$. Here I will solve for the general case with the safe assumption that $a, b > 0$ and $b - 2a \neq 0$ (check that these conditions are satisfied for all possible combinations of a and b). This makes the calculation a little messy - when you use numbers it's not as bad.)
Claim. The limit is $\frac{a}{b}$.

Proof. Fix some $\epsilon > 0$, and let $N = \left\lfloor \frac{|b-2a|}{b^2\epsilon} \right\rfloor + 1$ be a natural number.
(where $\lfloor x \rfloor$ denotes the greater integer lesser than or equal to x).
Then, for all $n \in \mathbb{N}$ such that $n > N$,

$$\begin{aligned}
\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| &= \left|\frac{\cancel{anb} + b - \cancel{bna} - 2a}{b\,(bn+2)}\right| \\
&= \left|\frac{b-2a}{b\,(bn+2)}\right| \\
&< \left|\frac{b-2a}{b^2 n}\right| \qquad (\because |bn+2| > |bn| \text{ for all } n \in \mathbb{N} \text{ when } b > 0) \\
&< \left|\frac{b-2a}{b^2 N}\right| \qquad (\because n > N) \\
&= \left|\frac{b-2a}{b^2}\right| \left|\frac{1}{\left\lfloor \frac{|b-2a|}{b^2\epsilon} \right\rfloor + 1}\right| \qquad \left(\because N = \left\lfloor \frac{|b-2a|}{b^2\epsilon} \right\rfloor + 1\right) \\
&< \epsilon \qquad \left(\because \left\lfloor \frac{|b-2a|}{b^2\epsilon} \right\rfloor + 1 > \left|\frac{b-2a}{b^2\epsilon}\right|\right)
\end{aligned}$$

Thus we see that for any given $\epsilon > 0$, there is a corresponding $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon$$

and hence, by definition,

$$\lim_{n\to\infty} \frac{an+1}{bn+2} = \frac{a}{b}$$

□

General comments:

1. Almost all of you have first used some imprecise Class 12 argument to first arrive at the limit $\frac{a}{b}$, then used the $\epsilon - N$ definition to prove that it is indeed the limit. This is completely unnecessary - it does not matter how you came to the limit as long as you are able to prove it. You can make any sort of ridiculous, unintuitive claim, *as long as you support it with a correct proof*, which in some ways is part of the beauty of mathematics - nothing is taken for granted, nothing is "obvious".

2. Many of you seem unclear about the general flow of the proof, so here it is in short: to show that $\lim_{n\to\infty} a_n = L$, we have to show that for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$.
   To do this, first fix an $\epsilon > 0$, then find the corresponding $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$. Now that we have this, and ϵ was chosen arbitrarily, we have that this is true for every positive ϵ, and we are done.

3. In simplifying the expression, many of you have incorrectly done the following:

$$\left| \frac{an+1}{bn+2} - \frac{a}{b} \right| < \left| \frac{an+1}{bn} - \frac{a}{b} \right|$$

   In general, $|x| < |y|$ does not imply $|x - z| < |y - z|$, even for $x, y, z > 0$ - take the simple example of $x = 2, y = 3, z = 5$ to see this.
   Some of you have also used inequalities that are not true in general but happen to be true here because additional conditions were satisfied - in such cases, you should mention those lucky conditions that happened to fall your way to let you use that inequality. This shows that you have actually checked whether the statement you are making is true.

4. Many of you have just written symbols and inequalities without using words to justify what you're doing. When you introduce variables for the first time, describe them properly - where are they coming from, are you assuming them, are you fixing them to be something, are you imposing any constraints on them, are you splitting something into two separate cases, and so on. A proof should flow like a smooth story and never leave the reader confused.