On entropy-constrained Gaussian channel capacity via the moment problem

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joint work with Shlomo Shamai and Emre Telatar

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June 26, 2025

Outline

1. Entropy-constrained Gaussian channel
2. Moment problems
3. Low SNR capacity

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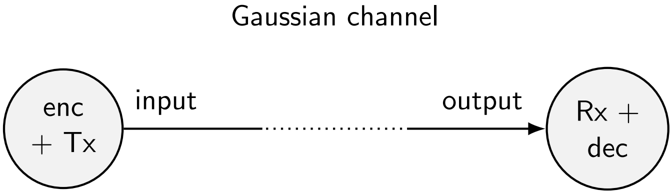
3. Low SNR capacity

Gaussian channel



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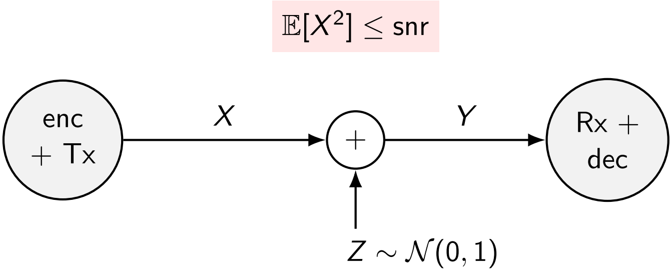
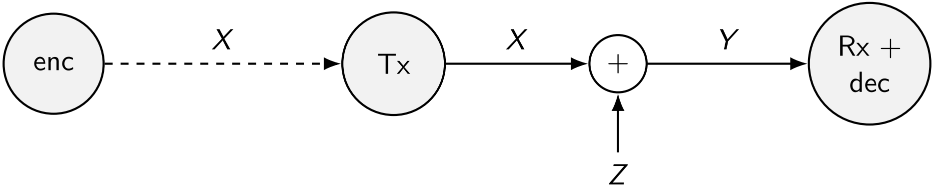


$$C(\mathsf{snr}) = \max_{X:\mathbb{E}[X^2]\leq \mathsf{snr}} I(X; X+Z) = \frac{1}{2}\log(1+\mathsf{snr})$$

Gaussian channel

finite-capacity noiseless link

$\mathbb{E}[X^2] \leq \mathsf{snr}$



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Entropy-constrained Gaussian channel

finite-capacity noiseless link

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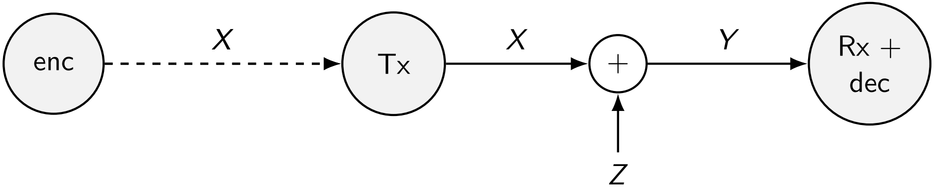

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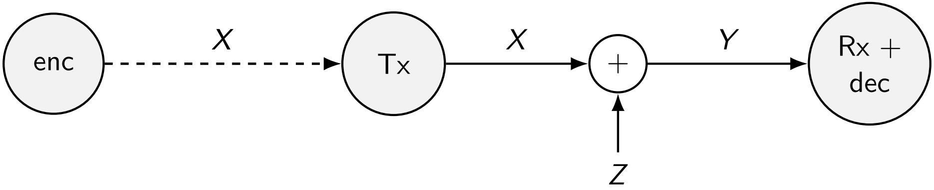

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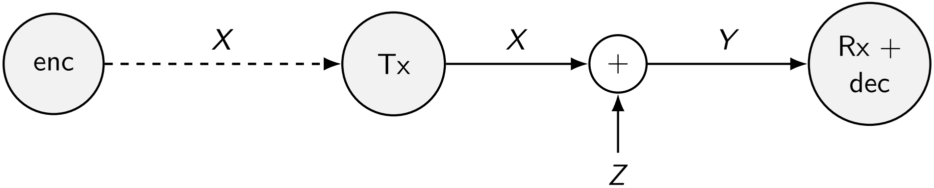

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- optimal distribution at h: discrete X with $H(X) \le h$ that is closest to $\mathcal{N}(0, 1 + \mathsf{snr})$ after "Gaussian smoothing"

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- MMSE of estimating X from $Y = \sqrt{\mathsf{snr}}X + Z$:

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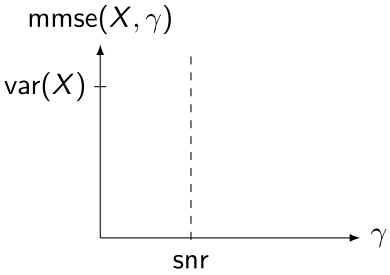

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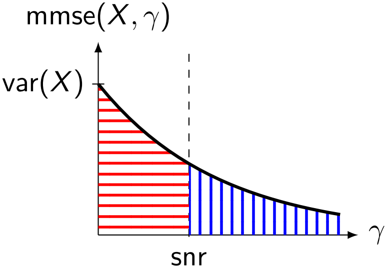

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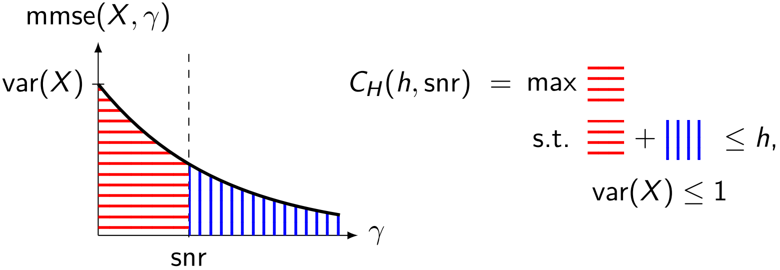

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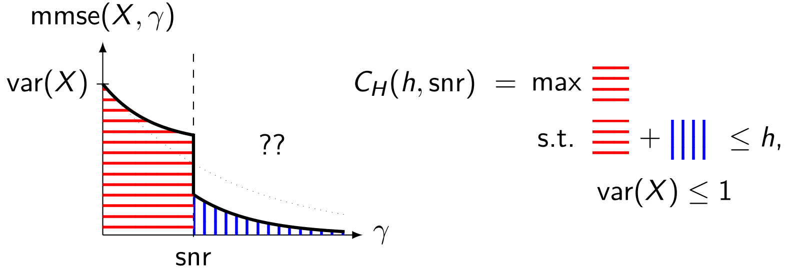

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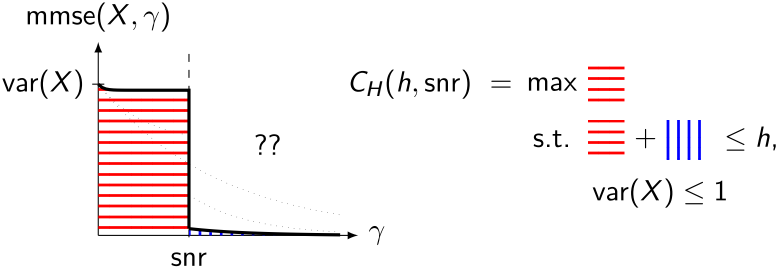

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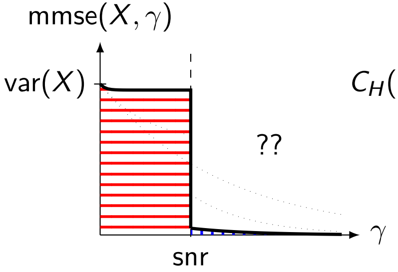

$$\begin{aligned} C_H(h, \mathsf{snr}) \ &= \ \max \\ &\quad \text{s.t.} \quad + \quad \le h, \\ &\quad \mathsf{var}(X) \le 1 \end{aligned}$$

- optimal distribution at snr: indistinguishable at SNR $<$ snr, distinguishable at SNR $>$ snr

Low SNR asymptotics via Taylor expansion

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  ($\mathcal{O}$ instead of Θ to allow for X with $\mathbb{E}[X^{2(k_h+2)}] = \infty$)

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  A: iff $(k$ odd$)$ there exists $\tilde{s}_{k+1}$ such that $\mathrm{H}_{\frac{k+1}{2}}(s_1, \ldots, s_k, \tilde{s}_{k+1}) \succeq 0$

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  finite support: at most $\lfloor k/2 \rfloor + 1$ atoms (if it exists)

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(i) *if X has $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3, 4$, then $H(X) \geq h_2(\eta_W)$.*

(ii) *for any $h > 0$, there is X with $H(X) \leq h$ and $\mathbb{E}[X^n] = \mathbb{E}[W^n]$ for $n = 1, 2, 3$.*

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$\eta_G = \frac{1}{3}$, *so for $h < h_2(\frac{1}{3})$, as* $\mathrm{snr} \to 0$, $C(\mathrm{snr}) - C_H(h, \mathrm{snr}) = \mathcal{O}(\mathrm{snr}^4)$.

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Thank you!