Applications of Fourier and Hilbert transforms in Communication Systems

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Outline

Signals and Systems

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System

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Linear systems

Systems which posses the property of superposition, i.e.,if $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then for any $a, b \in \mathbb{C}$

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Time-Invariant systems

Systems for which a time-shift in the input causes the same time-shift in the output, i.e.,if $x(t) \to y(t)$, then for any valid t_0,

$$x(t - t_0) \to y(t - t_0)$$

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We represent this as a signal $\delta(t)$ satisfying

$$\int_a^b x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0) & t_0 \in (a, b) \\ 0 & \text{else} \end{cases}$$

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Relevant properties:

1. Replication : $x(t) * \delta(t - t_0) = x(t - t_0)$
2. Sampling : $\int_{\mathbb{R}} x(t)\delta(t - t_0)\, dt = x(t_0)$

Starting to come together

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This leads us to define the Fourier transform of $h(t)$ as

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$H(f)$ is called the *Transfer function* of the system $\mathcal{S}$.

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3. $\text{sinc}\left(Wt\right) \leftrightarrow \frac{1}{W}\text{rect}\left(\frac{f}{W}\right)$

Hilbert transform

Definition

The Hilbert transform of a signal $m(t)$ is the output signal obtained when $m(t)$ is passed through a system with the transfer function

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It is easy to check that $h_Q(t) = \frac{1}{\pi t} \leftrightarrow -i\,\mathrm{sgn}(f)$. We denote the Hilbert transform of $m(t)$ by

$$m_h(t) = (m * h_Q)(t)$$

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Each frequency component undergoes a -90° phase shift!

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1. Send multiple signals over the same channel, by dividing the frequency available into bands.
2. $v = \lambda f$, antenna size $\approx \frac{\lambda}{10}$.

Amplitude Modulation

The amplitude of the transmitted signal contains the message signal $m(t)$, i.e. the transmitted signal is

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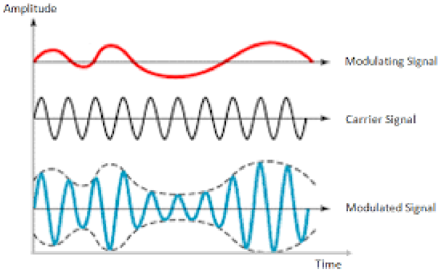

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Problem: High power and waste of bandwidth

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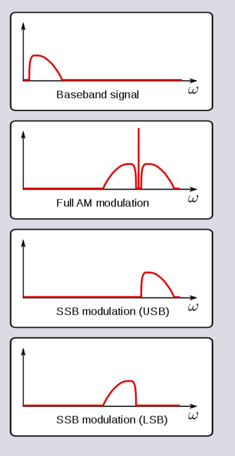
Back in the time domain, this is the signal $A_c e^{i2\pi f_c t}\left(1 + \mu m(t)\right)$.
Multiplying by $e^{-i2\pi f_c t}$ and removing the constant (by subtracting the average) will give us the original message $m(t)$ (up to a scaling).

Single Sideband Modulation

The aim is to get rid of the impulses and an unnecessary sideband.



Appplication 2: Single Sideband Modulation

Define

$$M_+(f) = \begin{cases} M(f) & f > 0 \\ 0 & f < 0 \end{cases}, \quad M_-(f) = \begin{cases} 0 & f > 0 \\ M(f) & f < 0 \end{cases}$$

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$$M_+(f) = \frac{1}{2}[M(f) + iM_h(f)], \quad M_-(f) = \frac{1}{2}[M(f) - iM_h(f)]$$

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$$S_{\mathrm{USB}}(f) = \frac{1}{4}[M(f - f_c) + M(f + f_c)] - \frac{1}{4i}[M_h(f - f_c) - M_h(f + f_c)]$$

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Thank you