MA 109: Assignment 2

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Question.
Do there exist functions with the following properties? (More than one option may be correct. Select all that are correct.) You must supplement your answers by writing out the justification for each answer.
Select one or more:

α. $f : [0, 1] \to \mathbb{R} : f$ convex and differentiable and such that f' is not differentiable at $\frac{1}{2}$.

β. $f : [0, 1] \to \mathbb{R} : f$ concave and discontinuous at $\frac{1}{2}$.

γ. $f : \mathbb{R} \to \mathbb{R} : f$ strictly convex and strictly decreasing.

δ. $f : [0, 1] \to \mathbb{R} : f$ convex and differentiable, $f'(\frac{1}{4}) = 2$, $f'(\frac{3}{4}) = -1$.

Solution.
(Note: The order of the options is not the same for all, which is why I have used $\alpha, \beta, \ldots$ instead of a, b, ...)

α. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} x^2 & 0 \le x < \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \le x \le 1 \end{cases}$$

We claim that this is a function which satisfies the required conditions, i.e. f is convex and differentiable, and f' is not differentiable at $\frac{1}{2}$.

Proof. We first show that f is differentiable.
For $0 < x < \frac{1}{2}$, there exists a neighbourhood around x such that $f(x) = x^2$, hence we have that $f'(x) = 2x$.
Similarly, for $\frac{1}{2} < x < 1$, there exists a neighbourhood around x such that $f(x) = x - \frac{1}{4}$, which gives us $f'(x) = 1$.
At $x = \frac{1}{2}$, the left hand derivative and right hand derivatives are to be computed separately (since f is defined differently on either side).

$$\text{LHD of } f \text{ at } \left(x = \frac{1}{2}\right) = \lim_{h \to 0^-} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0^-} \frac{\left(\frac{1}{2} + h\right)^2 - \left(\frac{1}{2} - \frac{1}{4}\right)}{h}$$
$$= \lim_{h \to 0^-} \frac{h^2 + h}{h} = 1$$

$$\text{RHD of } f \text{ at } \left(x = \frac{1}{2}\right) = \lim_{h \to 0^+} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0^+} \frac{\left(\frac{1}{2} + h - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{4}\right)}{h}$$
$$= \lim_{h \to 0^+} \frac{h}{h} = 1$$

We see that the left hand and right hand derivatives of f at $x = \frac{1}{2}$ are equal, hence f is differentiable at $x = \frac{1}{2}$, and thus on $(0, 1)$. We have

$$f'(x) = \begin{cases} 2x & 0 < x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x < 1 \end{cases}$$

Also note that f' is non-decreasing, which gives us that f is convex.
Finally, we show that f' is not differentiable at $x = \frac{1}{2}$.

$$\begin{aligned} \text{LHD of } f' \text{ at } \left(x = \frac{1}{2}\right) &= \lim_{h \to 0^-} \frac{f'\left(\frac{1}{2}+h\right) - f'\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0^-} \frac{2\left(\frac{1}{2}+h\right) - 1}{h} \\ &= \lim_{h \to 0^-} \frac{2h}{h} = 2 \end{aligned}$$

$$\text{RHD of } f' \text{ at } \left(x = \frac{1}{2}\right) = \lim_{h \to 0^+} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0$$

The left hand derivative of f' at $x = \frac{1}{2}$ is not equal to the right hand derivative of the same, hence the function f' is not differentiable at $\frac{1}{2}$, which completes the proof. □

β. *Claim:* Such a function does not exist.
Proof. Every concave function on an interval must be continuous at all interior points. Note that $\frac{1}{2}$ is an interior point of the interval $[0, 1]$, so any concave function on $[0, 1]$ must be continuous at $\frac{1}{2}$. Hence we have that there does not exist any function $f : [0, 1] \to \mathbb{R}$ such that f is concave and discontinuous at $\frac{1}{2}$. □

γ. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = e^{-x}$. We claim that this function has the properties desired.
Proof. Note that f is infinitely differentiable, and $f'(x) = -e^{-x} < 0 \ \forall \ x \in \mathbb{R}$, hence f is strictly decreasing. Also, $f''(x) = e^{-x} > 0 \ \forall \ x \in \mathbb{R}$, hence f is strictly convex. Thus f is an example of strictly convex and strictly decreasing function from $\mathbb{R}$ to $\mathbb{R}$ as required. □

δ. *Claim:* No such function exists.
Proof. Suppose such a function exists, and let $f : [0, 1] \to \mathbb{R}$ be it. Since f is given to be convex and differentiable, $f'(x)$ exists for all $x \in (0, 1)$ and f' is non-decreasing, which gives us that for any $x_1, x_2 \in (0, 1)$ such that $x_1 < x_2$, $f'(x_1) \leq f'(x_2)$.
In particular,

$$\frac{1}{4} < \frac{3}{4} \implies f'\left(\frac{1}{4}\right) \leq f'\left(\frac{3}{4}\right)$$

But we are given that $f'\left(\frac{1}{4}\right) = 2 > -1 = f'\left(\frac{3}{4}\right)$, which gives a contradiction.
Hence, there exists no function $f : [0, 1] \to \mathbb{R} : f$ convex and differentiable, $f'(\frac{1}{4}) = 2$, $f'(\frac{3}{4}) = -1$. □

General comments:

α. Many of you found this question difficult to solve, and even the ones who got very close made some small errors.

i. For starters, some of you misread the question entirely and found "convex and differentiable" functions that were "not differentiable at $\frac{1}{2}$" (the examples were, of course, incorrect, since this is impossible). Note that f' is to be not differentiable at $\frac{1}{2}$, not f.

ii. Most of you said that "since f is convex, $f''(x) > 0$ for all x" (which is not necessary) and concluded that f'' must exist everywhere. The definition of convexity says that for f to be convex on an interval I, for all $x_1, x_2 \in I$ and for all $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

There is no mention of differentiabilty. Now *if* a function *also happens to be twice differentiable*, we can say that $f''(x) \geq 0 \;\forall\; x \in I$. Hence it is possible to have functions that are convex but not differentiable, or differentiable but not twice differentiable.

iii. Many of you have also made a mistake in differentiating your example. If you want to obtain the derivative of a function by differentiating the expression, you must ensure that the function takes that expression for some neighbourhood around that point at least (think back to the definition of the derivative, the limit should exist). You cannot simply calculate the derivative of a piece-wise function at the points where the definition changes by differentiating the expression to one side. You must calculate the left hand and right hand derivatives. Consider this rather absurd example of a function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x & x \neq 1 \\ 1 & x = 1 \end{cases}$$

This function is just the identity function, $f(x) = x \;\forall\; x \in \mathbb{R}$, which is differentiable everywhere and has derivative $f'(x) = 1 \;\forall\; x \in \mathbb{R}$, but if you were to differentiate the individual expressions you would get

$$f'(x) = \begin{cases} 1 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

which is obviously incorrect.

iv. The mistake made by the most of the ones who came closest was in showing convexity. $f''(x) > 0$ everywhere only ensures convexity if f'' exists everywhere. Here, clearly $f''(\frac{1}{2})$ does not exist, so $f''(x)$ being positive for $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ does not mean anything. As a counter-example, consider the following function

$$f(x) = \begin{cases} \left(x - \frac{1}{4}\right)^2 & 0 \leq x < \frac{1}{2} \\ \left(x - \frac{3}{4}\right)^2 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Here $f''(x) > 0$ for all $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, but this function is not convex.

v. Of course, the example given in the solution above is not the only one possible - there are many others.

β. This makes use of a result straight from the slides (the result given in the slides is not exactly true, and the correction was discussed by the Professor in a following lecture) - all concave (and convex) functions on intervals are continuous at interior points of the interval.

i. Some of you have tried to force concave functions on $[0, 1]$ that are discontinuous at $\frac{1}{2}$ by drawing a graph - but of course, ended up drawing functions that are not concave without realising - hopefully now you see why the graphical method is not acceptable. We don't define properties based on how they look when plotted - we define properties based on mathematical relations, which might happen to look a certain way on a graph. We could have cases where the graph might look like it satisfies a certain property, but it might not because the mathematical definition is violated. Do not say "from looking at the graph it is clear that...", because no - it is not clear, it may even be false.

ii. Many of you did not mention the fact that $[0, 1]$ is an interval and $\frac{1}{2}$ is an interior point of $[0, 1]$, which is why we are able to use this result. (I have not cut marks here though since the result in the slides did not mention this, but note that in future evaluations marks will be deducted if you simply mention that "convex (or concave) functions are continuous".)

γ. This seemed to be the easiest part, many of you did get it right.

i. Some of you seemed to be confused between concavity and convexity.

ii. A few of you said that "since the function is convex, $f''(x) > 0 \; \forall \; x \in \mathbb{R}$, and hence f' must keep increasing, so will eventually become positive". The first error with this statement is in assuming that f'' exists. Secondly, not all increasing functions necessarily become positive - consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -e^{-x}$.

iii. A small number of you took an example of a convex function which is not strictly decreasing (such as $f(x) = x^2$), and used that to incorrectly conclude that no such functions are possible, which is completely wrong. Consider the statement - "Dogs exist". Would I be right to look at only cats, and hence conclude that dogs do not exist at all? (Hint: No.)

iv. One thing which some of you missed (possibly due to me giving unclear instructions on what to write when giving the example; I'm clearing it up now for future reference) is that when you give an example for something, you should show why it is what you claim it to be. In this case, you should show how your function is strictly convex and strictly decreasing.

δ. This was also something many of you were able to do correctly, or were on the right track at least.

i. Some of you tried to draw a graph and explain that the function must be concave, which is not necessarily true - you cannot conclude that by only knowing the value of f' at two points. For all we know the function could have multiple inflection points in between, or could even have discontinuous derivatives.

ii. A common mistake was in concluding that since f is convex and differentiable, $f''(x) > 0$ for all $x \in [0, 1]$ - note that f is not given to be necessarily twice differentiable, so f'' might not even exist.