Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz ~~4~~ 8
by Adway Girish, D1-T3

Date: ~~August 28,2019~~ September 29, 2019 | Day: ~~Wednesday~~ Sunday
Time: ~~2:00 PM - 2:05 PM~~ 4:00 PM-4:05 PM | Max. Marks: 5

Question. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ ~~defines a~~ defined as follows: $f(x, y) = (x^4 + y^3)/(x^4 + y^2)$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = (0, 0)$.

State whether the following statements are true or false and justify your answer.
(a) The function f is continuous at the origin $(0, 0)$.
(b) Both the partial derivatives of f exist.

(2 marks for correct alternative; 3 marks for correct justification)

Solution. The given statements are both **false**. (2 marks, 1 for each)

(a) Consider the sequences of numbers $x_n := 1/n$ and $y_n := 0$ so that $(x_n, y_n) \to (0, 0)$. Consider

$$f(x_n, y_n) = f(1/n, 0) = \frac{\frac{1}{n^4} + 0}{\frac{1}{n^4} + 0} = 1, \ \forall\, n \in \mathbb{N}$$

Thus we have, for some $(x_n, y_n) \to (0, 0)$, $f(x_n, y_n) \to 1 \neq 0 = f(0, 0)$. (1 mark)
$\therefore$ the function f is NOT continuous at the origin $(0, 0)$. $\square$

(b) The partial derivative of f wrt x at $(0, 0)$ is given by (1 mark)

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \frac{\frac{h^4+0}{h^4+0} - 0}{h} = \lim_{h \to 0} \frac{1}{h}$$

which does not exist. (1 mark)
$\therefore$ it is NOT true that both partial derivatives exist. $\square$

General observations:

1. Many students have checked only along particular sequences in part (a), gotten the limit along those sequences to be $= 0 = f(0, 0)$ and incorrectly concluded that f is continuous. They have taken sequences y_n and x_n such that $y_n = \alpha x_n$ for some $\alpha \in \mathbb{R}$. While it may seem like these are general sequences, they are being restricted to only those special sequences which are of the type $(a_n, \alpha a_n)$, when we should be considering sequences (a_n, b_n) where a_n and b_n are defined completely independent of each other to prove that f is continuous. Getting the same limit for even all of these sequences does not prove that f is continuous, as there are many other sequences which DO NOT give the same limit.

2. Some students have correctly gotten the limit not equal to 0 along some sequence (x_n, y_n), and then have directly stated that it is continuous without even mentioning that $f(0,0) = 0$. It is necessary to mention this, or at least find another sequence for which $f(x_n, y_n)$ converges to a different value.
3. A surprisingly large number of students have incorrectly computed the expression for f_x as $\lim_{h\to 0} \frac{h^4}{h^4}$, and conculded that f_x does exist. How so many students made the exact same mistake is beyond me.
4. The statement "both partial derivatives exist" is true only if BOTH f_x and f_y exist at ALL points in $\mathbb{R}^2$. A lot of students have first correctly found that f_x does not exist, and then for some reason proceeded to compute f_y, which is completely unnecessary since it has already been shown that the statement is false.
5. Many students are using expressions very loosely, particularly with reference to writing limits of functions vs. values of functions, here are some examples:
   (a) $x = 1/n, y = 0$ (since these are sequences, write them as $x_n = 1/n, y = 0$).
   (b) $f(x,y) = \mu$ where μ is the limit of the function obtained on substiting the sequences into $f(x,y)$. (this should instead read $f(x_n, y_n) \to \mu$)
   (c) $f_x(0,0) = \frac{f(h,0)-f(0,0)}{h}$ (what is h?)
   (d) $f_x(x,y) = \frac{f(h,0)-f(0,0)}{h}$ (the partial derivative at (x,y) is independent of x and y? And what is h?)
