Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 2

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Date: August 14, 2019 Day: Wednesday
Time: 2:00 PM - 2:05 PM Max. Marks: 5

Question. State whether the following statement is true or false. Justify your answer. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = x^3 + sin(x) + \frac{x}{1+x^2} \quad for\ x \in \mathbb{R}.$$

Then there exists a real number c such that $f(c) = 2019$.
(2 marks for correct alternative; 3 marks for correct justification)

Solution. The given statement is **true**. (2 marks)

x^3, $sin(x)$, x, and $1 + x^2$ are all continuous functions on $\mathbb{R}$. Also, $1 + x^2 > 0,\ \forall\ x \in \mathbb{R}$ imlpying that $\dfrac{x}{1+x^2}$ is also continuous at all $x \in \mathbb{R}$.
$f(x)$ is thus the sum of continuous functions, and hence, is itself continuous. (1 mark)

Observe that $f(0) = 0$, and $f(2019\pi) = (2019\pi)^3 + sin(2019\pi) + \dfrac{2019\pi}{1+(2019\pi)^2}$.
It is clear that $sin(2019\pi) = 0$ and $\dfrac{2019\pi}{1+(2019\pi)^2} > 0$,

$\implies f(2019\pi) > (2019\pi)^3 > 2019\pi > 2019$ $[\because x > 1 \implies x^3 > 1\ \forall\ x \in \mathbb{R}\ \&\ \pi, 2019 > 1]$
Hence we have $f(0) < 2019 < f(2019\pi)$. (1 mark)

According to the Intermediate Value Theorem, since f is a continuous function on $\mathbb{R}$, f satisfies the Intermediate Value Property on $\mathbb{R}$.
Hence if we take two values $0,\ 2019\pi \in \mathbb{R}$, and $2019 \in [f(0), f(2019\pi)]$ there must be a $c \in [0, 2019\pi]$, such that $f(c) = 2019$. (1 mark)

$\therefore$ we have $c \in [0, 2019\pi] \subset \mathbb{R} \implies c \in \mathbb{R}$, such that $f(c) = 2019$, which is what we are required to prove. $\square$

General observations:

1. Barring a few, almost all students seem to have had the right idea. The most common mistake was in not stating that $1 + x^2 > 0$ in $\mathbb{R}$, which is necessary to say that $\frac{x}{1+x^2}$ is continuous throughout the domain. Many students seem to have overlooked this fact, and have just stated that $f(x)$ is a sum of continuous functions and is hence continuous.

2. Some students have simply stated that f is continuous without any justification.
3. It is necessary to show that there are some $c_1,\ c_2 \in \mathbb{R}$ such that $f(c_1) < 2019 < f(c_2)$. Some students have not done this, and have used only the fact that f is increasing and continuous to argue that f returns all values in $\mathbb{R}$ in its co-domain. This is neither necessary nor sufficient, as we can easily construct continuous non-increasing functions with co-domain $= \mathbb{R}$ (for example, any cubic equation with 3 real roots), and we can just as easily construct continuous increasing functions with co-domain a strict subset of $\mathbb{R}$ (for example, $f(x) = 1 - e^{-x}$ does not take any value greater than 1).
4. Some students have just said that f is increasing (without talking about continuity) and so, must take all values in $\mathbb{R}$, which is immediately shown to be false by the previous point, and also, lack of continuity means that it is extremely obvious that there can be increasing functions which skip over any particular point (2019, in this case). For example, consider:

$$f(x) = \begin{cases} x & x \neq 2019 \\ 2020 & x = 2019 \end{cases}$$

   It is clear that this function never takes the value 2019 for any value of x.
5. An exceedingly small number of students have found that the statement is false, by saying that since 2019 is an integer, the induvidual terms adding up to get $f(c) = 2019$ must also be rational, which is an incorrect assertion, as can be seen by adding two irrational numbers, say $\sqrt{2}$ and $2019 - \sqrt{2}$, which gives us 2019, which is rational (also an integer). It is not necessary that induvidual terms in a sum adding up to a rational must be rational.
