Mathematics in Electrical Engineering: Transform Analysis

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Overview

1. Getting started
   - Signals
   - Signals and Systems
   - Impulse and Step
   - Convolutions
   - Eigenfunctions of LTI systems
2. Transforms
   - Laplace
   - Fourier and more
   - z
3. Applications

Transforms are used in Electrical Engineering primarily to analyze and understand signals and responses of systems. Before we can look at some of the transforms used, we shall first look at the mathematical representation of signals and systems. We will then limit our analysis to a particularly interesting class of signals, which satisfy the properties of linearity and time-invariance. Finally, we will look at the commonly used transforms and some of their applications.

Signals

Signals are represented mathematically as functions of independent variables, usually time. This independent variable can come from a continuous data set or from a set of discrete values. For example, the temperature of a geographic region is a function of the coordinates of the place, which continuously vary, and hence, can be modelled as a continuous-time signal. On the other hand, the number of students who obtained a particular score in an exam will be a function of the score, which is usually a discrete data set. This can be modelled as a discrete-time signal.
Both these classes of signals can be studied in the same manner since they are analogous, the only real difference will be in their mathematical treatments.

Types of Signals

Continuous-time signals

The signals for which the independent variable belongs to a continuum of values are together called **continuous-time signals**. They are represented as $x(\cdot)$, where the independent variable, generally denoted by t, takes the position of $\cdot$. For example, $x(t) = \sin t, \;\; t \in \mathbb{R}$.

Discrete-time signals

The signals for which the independent variable belongs to a discrete set of values are together called **discrete-time signals**. They are represented as $x[\cdot]$, where the independent variable, generally denoted by n, takes the position of $\cdot$. For example, $x[n] = 2n, \;\; n \in \mathbb{Z}$.

Systems

A **system** in general, is any process that results in the transformation of signals (may be continuous-time or discrete-time). Every system has an input signal, $x(t)$ (or $x[n]$), and an output signal, $y(t)$ (or $y[n]$), which is some function of the input signal. This is represented by $x(t) \to y(t)$ (or $x[n] \to y[n]$). We will look at a particular class of systems, called **Linear Time-Invariant systems**, or **LTI** systems, which satisfy the properties of linearity and time-invariance.

Linear, Time-Invariant Systems

Linear systems

The systems which posses the property of superposition are called **Linear systems.**, i.e.,if $x_1(t) \to y_1(t)$ (or $x_1[n] \to y_1[n]$) and $x_2(t) \to y_2(t)$ (or $x_2[n] \to y_2[n]$), then for any $a, b \in \mathbb{C}$

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$
$$(or \;\; ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n])$$

Time-Invariant systems

The systems for which a time-shift in the input causes the same time-shift in the output are called **Time-Invariant systems.**, i.e.,if $x(t) \to y(t)$ (or $x[n] \to y[n]$), then for any t_0 (or n_0),

$$x(t - t_0) \to y(t - t_0) \; (or \; x[n - n_0] \to y[n - n_0])$$

Some "functions"?

We first look at the impulse and step in discrete-time.

Definition

We define the **Unit Impulse** function as

$$\delta[n] = \begin{cases} 1 & n0 \\ 0 & n \neq 0 \end{cases}$$

And the **Unit Step** function as

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

Observe that

$$u[n] = \sum_{j=0}^{\infty} \delta[n-j]$$

Now we extend this into continuous-time.

Definition

We define the **Unit Step** function as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

And the **Unit Impulse** "function" such that

$$u(t) = \int_{-\infty}^{t} \delta(u) du \implies \delta(t) = \frac{du(t)}{dt}, \forall\ t \in \mathbb{R}$$

What δ(t) refers to is a function which is zero at all values except $t = 0$, such that $\int_{-\infty}^{t} \delta(u) du = 0$ for all $t < 0$, and $\int_{-\infty}^{t} \delta(u) du = 1$ for all $t \geq 0$. $\delta(0)$ can be thought to take the value ∞, but strictly speaking, δ is NOT a function with any meaning outside the integral.

A * is born...

Observe that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

If we have an LTI (linear, time-invariant) system, which has the response $h[n]$ to a unit impulse function, then we can write the response of this system to the input $x[n]$ as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

which is called the **Convolution sum**, represented by $x[n] * h[n]$. This can be extended to continuous-time also.

...and another

In continuous-time, we have

$$x(t) = \int_{-\infty}^{\infty} x(u)\delta(t-u)du$$

Again, if we have an LTI (linear, time-invariant) system, which has the response $h(t)$ to a unit impulse function, then we can write the response of this system to the input $x(t)$ as

$$y(t) = \int_{-\infty}^{\infty} x(u)h(t-u)du$$

which is called the **Convolution integral**, represented by $x(t) * h(t)$, which can to be easily shown to be commutative as well.

It's him again!

If we have an LTI system and provide input signals of the form $x(t) = e^{st}$ for continuous-time and $x[n] = z^n$ in discrete-time, (both complex exponentials), we have,

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)du \implies y(t) = e^{st}H(s)$$

where $H(s) = \int_{-\infty}^{\infty} h(u)e^{-su}du$; and

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \implies y[n] = z^n H(z)$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$.
Hence we see that complex exponentials are **Eigenfunctions** for LTI systems.

The Laplace Transform

Definition

We define the (bilateral) **Laplace transform** of a continuous-time signal $x(t)$ as, for $s \in \mathbb{C}$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

This is represented as $\mathcal{L}\{x(t)\} = X(s)$.

Now look at $y(t) = H(s)e^{st}$ for some LTI system. Taking the Laplace transform of both sides of this equation, we have,

$$Y(s) = H(s)X(s)$$

where $H(s)$ is the Laplace transform of the response of the system to a unit impulse, also called the *system function*.

The Fourier Transform

Definition

We define the **Fourier transform** of a continuous-time signal $x(t)$ as a special case of the Laplace transform, with $s = j\omega, \omega \in \mathbb{R}$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

This is represented as $\mathcal{F}\{x(t)\} = X(j\omega)$.

Observing that applying the Fourier transform twice to $x(t)$ gives us $X(-t)$ leads us to think of the Fourier transform as an operator which rotates a signal $x(t)$ on the time-frequency plane, clockwise by an angle of $\pi/2$. Now maybe we can find a generalization of this transform which rotates a signal by *any* angle α, not just $\pi/2$. It turns out, there is such a transformation, called the **Fractional Fourier Transform**.

The Fractional Fourier Transform

Definition

We define the **Fractional Fourier transform** as

$$X_\alpha(\omega) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,\omega) dt$$

where

$$K_\alpha(t,\omega) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{j\frac{t^2+\omega^2}{2}\cot\alpha - j\omega t\csc\alpha} & \alpha \neq n\pi \\ \delta(t-\omega) & \alpha = 2n\pi \\ \delta(t+\omega) & \alpha = (2n+1)\pi \end{cases}$$

It is easy to see that $\alpha = \pi/2$ returns the Fractional Fourier transform to the classical Fourier transform.

The z-Transform

Definition

We define the z-**transform** of a discrete-time signal $x[n]$ as, for $z \in \mathbb{C}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

This is represented as $\mathcal{Z}\{x(t)\} = X(z)$.

Now look at $y[n] = H(z)z^n$ for some LTI system. Taking the z-transform of both sides of this equation, we have,

$$Y(z) = H(z)X(z)$$

where $H(z)$ is the z-transform of the response of the system to a unit impulse, also called the *system function*.

Applications Of Laplace transform

Example (Solving Differential Equations)

Suppose we have an LTI system described by the following differential equation, and we want to find the response of this system to the input $x(t) = u(t)$.

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

Taking the Laplace transform of this equation, we have, with ROC $\mathcal{R}\{s\} > 0$

$$Y(s) = \frac{X(s)}{s^2 + 3s + 2} = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

Now, taking the inverse Laplace transform of the above equation, we have

$$y(t) = \left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right] u(t)$$

Applications Of Laplace transform (contd.)

Example (Checking the stability and causality of the system)

Suppose we have an LTI system described by some differential equation, and taking the Laplace transform of this equation gives us

$$Y(s) = \frac{X(s)\cdot(s-1)}{(s+1)(s-2)} \implies H(s) = \frac{s-1}{(s+1)(s-2)}$$

This system function has three regions of convergence, and depending on the region of convergence we chose, we get different solutions. The system will be stable only if the ROC includes the imaginary axis, and will be causal only if the region being considered is on the right. Of course, we could make these inferences by looking at the solution, but this is a neat way to understand the general nature of the solution.

Application Of Fourier transform

Example (Frequency-selective filtering)

Every periodic signal can be written as a sum of complex exponentials, each of which has a particular frequency.
Now suppose we have an input signal comprised of several frequencies, and we want to filter this signal to frequencies around a particular ω_0 only. The best way to do this is to convert it to the frequency domain by applying the Fourier transform. Now we have an input in the frequency domain, which can be multiplied with a suitable window function centered at ω_0, to get a signal with frequencies only from our desired range. Converting this back into the time-domain will give us our required output, with only those components of $x(t)$ included, which satisfy our restriction on the frequency.

Thank you