Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 1

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Date: August 7, 2019 Day: Wednesday
Time: 2:00 PM - 2:05 PM Max. Marks: 5

Question. State whether the following statement is true or false. Justify your answer.
If S is a non-empty subset of $\mathbb{R}$ such that S is bounded above and if $c := \sup S$, then there exists a sequence (x_n) of elements of S such that (x_n) is convergent and $x_n \to c$.
(2 marks for correct alternative; 3 marks for correct justification)

Solution. The given statement is **true**.

For $n \in \mathbb{N}$, $c - \dfrac{1}{n} < c$. Since c is the supremum (least upper bound), $c - \dfrac{1}{n}$ is not an upper bound.
Consider the sequence (x_n) with $c - \dfrac{1}{n} < x_n \leq c$, such that $x_n \in S$ for all $n \in \mathbb{N}$, which is possible since S is non-empty, and $c - \dfrac{1}{n}$ is not an upper bound.
Then, it follows immediately from Sandwich Theorem that as n goes to ∞, as both $c - \dfrac{1}{n}$ and c go to c, x_n will also converge to c.
Hence, there does indeed exist a sequence (x_n) of elements of S such that (x_n) is convergent and converges to c. $\square$

General observations:

1. Some students have shown that for some particular S there is a sequence which does not converge to the supremum of the set, and have incorrectly concluded that the statement is false. The statement given to us reads, "...there exists **a** sequence...", which means that all we have to show is the existence of some sequence in S which converges to its supremum. Note that it does not say that **all** sequences in S will converge to its supremum.
2. Some students have taken the sequence to be $x_n = c - \frac{1}{n}$. However, we cannot guarantee that $c - \frac{1}{n}$, while lesser than c, will necessarily lie in S. This is why in the solution, we take the sequence (x_n), with each term forced to be in S.
3. Some students have stated that an increasing sequence in a set with a supremum will converge to its supremum, but this is an incorrect statment. For example, let $S = [0, 2)$, and let the increasing sequence be $1 - \frac{1}{n}$. This sequence clearly converges to 1, while the supremum of S in $\mathbb{R}$ is 2.
