MA 109: Assignment/Quiz 3

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Question 1.
Do there exist differentiable functions with the following properties? (More than one option may be correct. Select all that are correct.)
Select one or more:

a. $f : [0,1] \to \mathbb{R}$, $f(0) = -2, f(1) = 3, f'(x) \geq 10$ for all $x \in (0,1)$.

b. $f : [0,1] \to \mathbb{R}$, $|f(x)| \leq 1$ for all $x \in (0,1)$ and $\exists\ x \in (0,1) : f'(x) \geq 2$.

c. $f : [0,1] \to \mathbb{R}$, $f(0) = f(1)$, and $f(x) \neq f\left(x + \frac{1}{3}\right)$ for all $x \in \left(0, \frac{2}{3}\right)$.

d. $f : [0,1] \to \mathbb{R}$, $f(0) = f(1)$, and $f(x) \neq f\left(x + \frac{3}{4}\right)$ for all $x \in \left(0, \frac{1}{4}\right)$.

Solution.

a. *Claim.* Such a function does not exist.

Proof. Suppose there does indeed exist a function $f : [0,1] \to \mathbb{R}$, $f(0) = -2, f(1) = 3, f'(x) \geq 10$ for all $x \in (0,1)$. Since f is differentiable on $(0,1)$, applying the Mean Value Theorem to f between 0 and 1, we have that there exists a $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{3 - (-2)}{1 - 0} = 5 < 10$$

which immediately fails the requirement that $f'(x) \geq 10$ for all $x \in (0,1)$, leaving us with no functions satisiying the given conditions. $\square$

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b. Consider the function $f : [0,1] \to \mathbb{R}$ defined as $f(x) = 2x - 1$. We claim that this function has the properties desired.

Proof. Firstly, f is a linear polynomial, hence is differentiable. For any $x \in [0,1]$, we have

$$0 \leq x \leq 1 \iff 0 \leq 2x \leq 2 \iff -1 \leq 2x - 1 = f(x) \leq 1 \iff |f(x)| \leq 1$$

and $f'(x) = 2 \geq 2\ \forall\ x \in [0,1]$, hence any $x \in [0,1]$ can be taken as an example point where $f'(x) \geq 2$.
Thus f is an example of a differentiable function defined on $[0,1]$ satisfying $|f(x)| \leq 1$ for all $x \in (0,1)$ and $\exists\ x \in (0,1) : f'(x) \geq 2$. $\square$

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c. *Claim.* No such function exists.

Proof. Suppose such a function exists, and let $f : [0, 1] \to \mathbb{R}$ be it. Consider the function $g : \left[0, \frac{2}{3}\right] \to \mathbb{R}$, given by $g(x) := f(x) - f\left(x + \frac{1}{3}\right)$. In particular, we have

$$\begin{aligned} g(0) &= f(0) - f\left(\tfrac{1}{3}\right) \\ g\left(\tfrac{1}{3}\right) &= f\left(\tfrac{1}{3}\right) - f\left(\tfrac{2}{3}\right) \\ g\left(\tfrac{2}{3}\right) &= f\left(\tfrac{2}{3}\right) - f(1) \end{aligned}$$

Adding the three, we get $g(0) + g\left(\frac{1}{3}\right) + g\left(\frac{2}{3}\right) = f(0) - f(1) = 0$.
Since f is continuous on $[0, 1]$, we have that g is continuous on $\left[0, \frac{2}{3}\right]$, and from this we conclude that there must be a $c \in \left(0, \frac{2}{3}\right)$ such that $g(c) = 0$ as follows.
Since $g(0) + g\left(\frac{1}{3}\right) + g\left(\frac{2}{3}\right) = 0$, either

i. $g(0) = g\left(\frac{1}{3}\right) = g\left(\frac{2}{3}\right) = 0$, or

ii. we have $x_1, x_2 \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$ such that $g(x_1) < 0 < g(x_2)$. Applying IVP on the interval between x_1 and x_2 (which exists because $x_1 \neq x_2$), we have that there is some x_0 lying strictly between x_1 and x_2 such that $g(x_0) = 0$.

In either case, $g(c) = 0$ for some $c \in \left(0, \frac{2}{3}\right)$, which gives us that $f(c) - f\left(c + \frac{1}{3}\right) = 0$, and hence $f(c) = f\left(c + \frac{1}{3}\right)$ for some $c \in \left(0, \frac{2}{3}\right)$, contradicting the requirement that $f(x) \neq f\left(x + \frac{1}{3}\right)$ for all $x \in \left(0, \frac{2}{3}\right)$.
Thus we have that no function could possibly satisfy the given conditions. □

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d. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sin(2\pi x)$. We now show that this is a satisfactory example as needed.

Proof. It is clear that f is a differentiable function, being a composition of differentiable functions.
Also note that $f(0) = \sin 0 = 0 = \sin 2\pi = f(1)$.
Finally, note that for $x \in \left(0, \frac{1}{4}\right)$, $\sin(2\pi x) > 0$ and $\sin\left[2\pi\left(x + \frac{3}{4}\right)\right] < 0$, implying that $\sin(2\pi x) \neq \sin\left[2\pi\left(x + \frac{3}{4}\right)\right]$ for all $x \in \left(0, \frac{1}{4}\right)$.
Thus f is a differentiable function defined on $[0, 1]$ satisfying $f(0) = f(1)$, and $f(x) \neq f\left(x + \frac{3}{4}\right)$ for all $x \in \left(0, \frac{1}{4}\right)$. □

General comments:

a. This was a question that almost all of you got right. A small number of you were unclear with the statement of MVT and incorrectly said things along the lines of "$f'(x) = 5$ for all x".

b. Again, most of you did manage to find a suitable example. Some of you did not justify each part.

c. Only a small number of you managed to come up with the right solution, many of you tried to use a combination of Rolle's theorem and IVP but fell short. Some of you even misread the question - what we require is that for *any* $x \in \left(0, \frac{2}{3}\right)$, $f(x) \neq f\left(x + \frac{1}{3}\right)$, not that there is some $x \in \left(0, \frac{2}{3}\right)$ where $f(x) \neq f\left(x + \frac{1}{3}\right)$.

d. Many of you said that this is the same as part c, so the answer must stay the same. Some of you even used the exact same argument as in part c to conclude that no function exists - this should also make it clear that your argument for c is wrong, since in this case we have a simple example. Again there was the same error as in c, where some of you misread the question to mean that there is some $x \in \left(0, \frac{1}{4}\right)$ such that $f(x) \neq f\left(x + \frac{3}{4}\right)$.