Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 4
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Date: August 28, 2019 Day: Wednesday
Time: 2:00 PM - 2:05 PM Max. Marks: 5

Question. Recall that a cubic polynomial is a function $q : \mathbb{R} \to \mathbb{R}$ given by $q(x) = ax^3 + bx^2 + cx + d$ for a, b, c, $d \in \mathbb{R}$ and $a \neq 0$.
State whether the following statement is true or false. Justify your answer.
Every cubic polynomial has an inflection point.
(2 marks for correct alternative; 3 marks for correct justification)

Solution. The given statement is **true**. (2 marks)

First note that any polynomial is infinitely differentiable (or at least, a cubic polynomial is thrice differentiable), implying that for the given function q which is a polynomial, q', q'', q''' and so on all exist. (1 mark)

Hence we can compute these derivatives, (1 mark)

$$q'(x) = 3ax^2 + 2bx + c,$$

$$q''(x) = 6ax + 2b,$$
$$q'''(x) = 6a$$

Since it is given that $a \neq 0$ (which is also necessary for $q(x)$ to be a cubic polynomial), we have $\frac{-b}{3a} \in \mathbb{R}$, and from the above expressions of the derivatives it is clear that $q''\left(\frac{-b}{3a}\right) = 0$, and $q'''\left(\frac{-b}{3a}\right) = 6a \neq 0$ since $a \neq 0$, which is sufficient to say that $x = \frac{-b}{3a}$ is an inflection point of $q(x)$. (1 mark)

$\therefore$ Every cubic polynomial does indeed have an inflection point. $\square$

General observations:

1. Again, most of the students had the right idea. The most common mistake was in not stating that by virtue of being a polynomial, $q(x)$ is infinitely differentiable. Only then can we compute the derivative by differentiating. Many students have either ignored this entirely, or have only said that q is differentiable, without talking about higher order derivatives.

2. Many students have shown that the signs of $q''(x)$ in a neighbourhood of the point $x = \dfrac{-b}{3a}$ are opposite on either side, which is also a perfectly correct solution, and is in fact a more general check. However, in this case, since $q'''(x) \neq 0$ at the candidate for the inflection point, it is immediate that it is indeed an inflection point. If $q'''(x)$ had come out to be zero, then checking the sign of $q''(x)$ would have been the best way.
3. Many students have only shown that there is an $x_0 \in \mathbb{R}$ such that $q''(x_0) = 0$, without talking about the third derivative or the signs of the second derivative. Remember that $q''(x_0) = 0$ is only a necessary condition for x_0 to be differentiable (if the function is at least twice differentiable), and not enough to conclude that x_0 is indeed a point of inflection.
4. Now some general points for which I have not deducted marks, but are more for the aesthetics. Some students have said that since $q(x)$ is a polynomial, it is differentiable once and then computed the derivative, which is again a polynomial and so, is differentiable, and so on, which is absolutely correct, but it looks neater to say that polynomials are infinitely differentiable and hence all derivatives exist, the proof of which, of course, is that the derivative is again a polynomial. Note that both of these are correct and is only a matter of personal preference. Also, many students have named the candidate for the inflection point c, when c is already the coefficient of x in $q(x)$, as given in the question or as used previously in their own solution. This is actually a mistake, but a very minor one which will go unpunished.
