MA 109: Main Quiz

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Question 1.
Using the $\epsilon - \delta$ definition of the limit of a function, determine whether the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} A \text{ if } x \leq 0 \\ x \text{ if } x > 0 \end{cases}$$

Justify your answer using the $\epsilon - N$ definition of the limit. (6)

Solution.

Claim. The function is NOT continuous at $x = 0$. (2)

Proof. Suppose f is continuous at 0. Then we have that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon \text{ whenever } 0 < |x - 0| = |x| < \delta$$

Take $\epsilon = 0.5$, and let δ_1 be the δ corresponding to this ϵ such that the above inequality holds. Then, for all $x \in (0, \delta_1)$, we must have

$$|f(x) - f(0)| = |x - A| < 0.5 \qquad (2)$$

(In all cases, A is a number in $\{1, \cdots, 10\}$, so $A - 0.9 > 0$, which is needed for the next step.)
Checking at $x = \min\{\frac{\delta_1}{2}, A - 0.9\} \in (0, \delta_1)$, we see that

$$|x - A| = A - x \geq 0.9 > 0.5$$

which is a contradiction to the previous inequality. (2)

Hence we have that f is $\boxed{\text{NOT continuous at } 0}$. □

Comments.

1. Very few of you have correctly taken a value of ϵ and shown the contradiction that arises assuming continuity.
2. Most of you calculated the right hand and left hand limits directly (without using the $\epsilon - \delta$ definition) and concluded discontinuity, in which case you get only 2 marks.
3. Some of you have calculated the right hand and left hand limits explicitly using the $\epsilon - \delta$ definition, which gives you full credit. The benefit of the above solution is that it is direct from the basic definition of a limit only, and nothing else.

Question 2.
True or False (if true give reasons, if false give an explicit counter-example): Let $f(x)$ be a twice differentiable function on $(0, 2B)$. If $f''(x) > 0$ on $(0, B)$ and $f''(B) = 0$, then B must be an inflection point for the curve $y = f(x)$. (4)

Solution.
The statement is False. (1)

Example. The following function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $(0, 2B)$, $f''(x) > 0$ on $(0, B)$ and $f''(B) = 0$, but B is not an inflection point for $y = f(x)$.

$$\boxed{f(x) = (x - B)^4} \qquad (1)$$

Proof. Being a polynomial function, f is infinitely differentiable on all $\mathbb{R}$, hence twice differentiable on $(0, 2B)$.

$$f'(x) = 4(x - B)^3$$
$$f''(x) = 12(x - B)^2$$

$f''(x) = 12(x - B)^2 > 0$ at $x \neq B$, hence $f''(x) > 0$ on $(0, B)$, and $f''(B) = (B - B)^4 = 0$. (1)

Now note that $f''(x) = 12(x - B)^4 > 0$ on $(B, 2B)$ as well. Hence we have that f is convex and not concave on $(0, B)$, and f is convex and not concave on $(B, 2B)$, implying that the concavity/convexity of f remains unchanged at B. (1)

Hence, B is NOT an inflection point. □

Comments.

1. A small number of you incorrectly used $f''(B) = 0$ to immediately conclude that B is an inflection point.
2. Many of you did manage to find a good example, but some of you did not justify why the function is a good example, so you get only 2 or 3 marks depending on how much you missed.
3. An extremely fine point that none of you picked up on (and probably wasn't expected either, so marks have not been cut here) is that a function being convex does not necessarily exclude it from being concave. Only noting that the function goes from convex to convex without checking what happens to concavity technically is not enough. You could have examples where the function goes from convex and not concave on one side to convex and concave on the other side - in this case the point is an inflection point! Some of you actually gave examples where this happened, so there is a need to be extra careful. Only one mark has been deducted in such cases even though the example is not correct.

Question 3.
Let $f(x) = x^3 + Ax + B$. Show that $f(x)$ has exactly one real root. (4)

Solution.

Proof. Since f is a polynomial function, it is continuous and differentiable everywhere. Observe the following:

$$f(0) = B > 0$$
$$f(-3) = -27 - 3A + B < 0$$

(Since A and B are both numbers in $\{1, \cdots, 10\}$, in all cases, A, B are positive and satisfy $B - 3A - 27 < 0$ (this is easy to check).)
Since $f(-3) < 0 < f(0)$ and f is continuous on the interval $(-3, 0)$, IVP gives us that there must exist a $c \in (-3, 0)$ such that $f(c) = 0$. Hence we have that f has at least one real root, c.(2)

Also, we have $f'(x) = 3x^2 + A > 0$, hence the function is strictly increasing, implying that for $x_1 < x_2 \in \mathbb{R}$, $f(x_1) < f(x_2)$. (1)
In particular, for $r \in \mathbb{R}$ such that $r \neq c$, either $r < c$, in which case $f(r) < f(c) = 0$ or $r > c$, in which case $f(r) > f(c) = 0$. Hence we have that $f(r) \neq 0$ for any $r \neq c$, giving us that the root is unique. (1)

Thus we see that $f(x)$ has exactly one real root. $\square$

Comments.

1. The most common mistake was in not using an interval while evoking IVP. The infinity at infinity and negative infinity at negative infinity claim does not let you use IVP since you do not have an interval where the function must take all values. No marks were cut though.
2. Many of you only checked for at least one root or at most - you had to show that exactly one root exists, so you had to do both. Marks have been deducted appropriately for this.
3. Most of you concluded uniqueness by saying that the function is "monotonically increasing", which is not enough. We need it to be *strictly* increasing, otherwise we only have that for $x_1 < x_2 \in \mathbb{R}$, $f(x_1) \leq f(x_2)$, which does not allow us to conclude uniqueness. No marks were deducted for this either.
4. Some of you used Rolle's theorem to properly prove the uniqueness of the root, which is perfectly correct (in fact, $f'(x) > 0$ implies strictly increasing comes from Rolle's theorem.)

Question 4.

Let (x_n) be a convergent sequence of non-negative real numbers. Let

$$x = \lim_{n\to\infty} b + x_n.$$

Prove (using the $\epsilon - N$ definition of limits) that $x \geq b$. (4)

Solution.

Proof. We need to show that $x \geq b$. The best way to do this is by contradiction. Suppose $\lim_{n\to\infty} = x < b$. By the $\epsilon - N$ definition of the limit, we have that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|b + x_n - x| < \epsilon \text{ for all } n > N$$

Choose $\epsilon = \frac{b-x}{2}$, and let N_1 be the corresponding N such that the above inequality holds. (2)
Since x_n is a non-negative sequence, we have that $x_n \geq 0$ for all $n \in \mathbb{N}$, and hence for all $n > N_1$,

$$0 < b - x \leq b + x_n - x < |b + x_n - x| < \epsilon = \frac{b-x}{2}$$
$$\implies b - x < \frac{b-x}{2}$$

which is false for $x < b$, giving us a contradiction. (2)

Thus our assumption that $x < b$ is incorrect, giving us $x \geq b$. □

Comments.

1. Many of you did not use the $\epsilon - N$ definition. Simply stating the definition and saying "hence by definition we have..." is not correct.

2. This was the lowest scoring question in the paper, most of you misread what was to be done. What was expected was essentially to show that the limit of a non-negative sequence is non-negative, but all of you took this for granted.

3. Those of you who properly used the $\epsilon - N$ definition to show that the limit can be split into sum of individual limits, then used the fact that the limit of a non-negative sequence is non-negative get 3 marks, since it is sometimes unclear what can be assumed in a question and what can not.

4. Those of you who tried to use the $\epsilon - N$ definition in a sensible way have been awarded 1 or 2 marks depending on how much progress was made.

Question 5.
Let

$$S_n = \frac{1}{n}\sum_{k=1}^{n} \cos\left(\frac{(a+1)k\pi}{(a+2)n}\right)$$

Evaluate $\lim_{n\to\infty} S_n$ by identifying it as a Riemann sum for a certain continuous function on a certain interval, and with respect to a certain (tagged/marked) partition. You must explicitly give the continuous function, the interval and the tagged/marked partition that you are using. You must then justify all further steps. (6)

Solution.

$$S_n = \frac{1}{n}\sum_{k=1}^{n} \cos\left(\frac{(a+1)k\pi}{(a+2)n}\right) = \sum_{k=1}^{n} \cos\left(\frac{(a+1)\pi}{(a+2)}\frac{k}{n}\right)\left(\frac{k}{n} - \frac{k-1}{n}\right)$$

It is clear that this is the Riemann sum $R(f, P_n, t) = \sum_{k=1}^{n} f(t_k)[x_k - x_{k-1}]$ for

$$\text{the function } f(x) = \cos\left(\frac{(a+1)\pi}{(a+2)}x\right) \text{ on } [0,1] \qquad (1)$$

$$\text{the partition } P_n \text{ of } [0,1] = \left\{x_0 = 0, x_1 = \frac{1}{n}, \cdots, x_{n-1} = \frac{n-1}{n}, x_n = 1\right\} \qquad (1)$$

$$\text{tags } t \text{ for } P_n,\ t_k = \frac{k}{n} \qquad (1)$$

Note that f is continuous on $[0,1]$ and hence is Riemann integrable. Thus we have

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{||P_n||\to 0} R(f, P_n, t) = \int_0^1 f(x)dx && (1)\\
&= \int_0^1 \cos\left(\frac{(a+1)\pi}{(a+2)}x\right) dx && (1)\\
&= \boxed{\frac{(a+2)}{(a+1)\pi}\sin\left(\frac{(a+1)\pi}{(a+2)}x\right)} && (1)
\end{aligned}$$

Comments.

1. Many of you did not explicitly write down the continuous function, the partition, and the tags for the Riemann sum though it was mentioned in the question. Some of you even wrote them incorrectly.

2. Some of you did not write that the function is integrable, but directly proceeded to integrate - no marks have been cut for this though.

3. Some of you made mistakes in the integration variable and limits of integration.

4. Minor calculation mistakes in the final integral are excused only if nothing else is correctly done (just so that you don't get 0/6).

Question 6.
Let $C = \frac{B}{B+1}$. Let $f(x) = \cos x$ on the interval $[0, \pi]$. Let $P_3(x)$ be the Taylor polynomial of $f(x)$ of degree 3 around the point $\frac{\pi}{2}$.
Write down the Taylor polynomial $P_3(x)$ and state if the following statement is true or false:

$$|f(x) - P_3(x)| < \frac{2C}{3}$$

for all $x \in [\pi/2, \pi]$. Justify your answer. You may use the fact that $\pi/2 < 1.6$. (6)

Solution. $f(x) = \cos x$ is infinitely differentiable on $[0, \pi]$. We are to find a Taylor polynomial about the point $\frac{\pi}{2}$, for which we require:

$$f^{(n)}\left(\frac{\pi}{2}\right) = \begin{cases} \cos\frac{\pi}{2} & n = 4k,\ k \text{ is some integer} \\ -\sin\frac{\pi}{2} & n = 4k+1,\ k \text{ is some integer} \\ -\cos\frac{\pi}{2} & n = 4k+2,\ k \text{ is some integer} \\ \sin\frac{\pi}{2} & n = 4k+3,\ k \text{ is some integer} \end{cases} = \begin{cases} 0 & n \text{ is even} \\ -1 & n = 4k+1,\ k \text{ is some integer} \\ 1 & n = 4k+3,\ k \text{ is some integer} \end{cases}$$

The Taylor polynomial $P_n(x)$ about $\frac{\pi}{2}$ is given by

$$P_n(x) = f\left(\frac{\pi}{2}\right) + f^{(1)}\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \cdots + \frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x - \frac{\pi}{2}\right)^n$$

Hence, we have that

$$P_3(x) = -\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3 \qquad (1)$$

The difference between the function and the Taylor polynomial, is given by the remainder term. Using Taylor's theorem, it can be written as

$$f(x) - P_n(x) = R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\left(x - \frac{\pi}{2}\right)^{n+1} \text{ for some } c \in \left(\frac{\pi}{2}, x\right)$$

Hence we have that the remainder for $P_3(x)$ is given by

$$R_3(x) = \frac{f^{(4)}(c)}{4!}\left(x - \frac{\pi}{2}\right)^4 = \frac{\cos c}{4!}\left(x - \frac{\pi}{2}\right)^4 \text{ for some } c \in \left(\frac{\pi}{2}, x\right) \qquad (2)$$

Now we can write

$$\begin{aligned} |f(x) - P_3(x)| = |R_3(x)| &= \left|\frac{\cos c}{4!}\left(x - \frac{\pi}{2}\right)^4\right| \\ &< \frac{(1.6)^4}{4!} \qquad \left(\because |\cos c| \leq 1 \text{ and } \left|x - \frac{\pi}{2}\right| \leq \frac{\pi}{2} < 1.6 \text{ for } x \in \left[\frac{\pi}{2}, \pi\right]\right) && (1) \\ &< 0.27 \text{ using calculator} && (1) \end{aligned}$$

(Since B is in $\{1, \cdots, 10\}$, the least value of $C = \frac{B}{B+1}$ occurs at $B = 1$, where $C = \frac{1}{2}$. Hence $C \geq \frac{1}{2} \implies \frac{2C}{3} \geq \frac{1}{3} > 0.33 > 0.27$, implying that our result is true for any value of B.)
Thus we have that

$$|f(x) - P_3(x)| = |R_3(x)| < \frac{2C}{3} \qquad (1)$$

and the statement is True.

Comments.

1. Many of you used incorrect expressions for the Taylor polynomial - some of you had only one term, some calculated the derivatives at 0 or some c. Some of you even had $\cos x$ and $\sin x$ terms in your Taylor polynomials, which is absurd because they are *polynomials*!.

2. Some of you did not use the formula for the remainder term, and tried to deal with the $\cos -$ polynomial expression. Of course, it is not hard to find the maximum of this over the range $[\pi/2, \pi]$ and show that it is lesser than $\frac{2C}{3}$, but none of you did this. In most cases the value was calculated at some points and that was used to conclude, which is incorrect.

3. Many of you were also careless in your simplification of the expression for $|R_3(x)|$, forgetting to put $|\cdot|$ in many places, and not justifying why $\left|x - \frac{\pi}{2}\right| < \frac{\pi}{2}$. No marks were deducted for these though, as long as the final bounding value was correct.
