Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 10
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Date: October 16, 2019 Day: Wednesday
Time: 2:00 PM - 2:05 PM Max. Marks: 5

Question. Compute the global maximum and minimum of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := xy$ subject to the constraint $g(x, y) = 3x^2 + 2y^2 - 1 = 0$.

Solution. The function f is continuous on $\mathbb{R}^2$ and hence, is continuous on the domain constrained by $g(x, y) = 3x^2 + 2y^2 - 1 = 0$. The set $D := \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$ is non-empty, closed and bounded. Hence, by the extreme value theorem, f attains a global maximum and a global minimum on the D. (1 mark)
Using the method of Lagrange multipliers, we have $\nabla f(x, y) = \lambda \nabla g(x, y)$, and $g(x, y) = 0$, at the point where we have an extremum. Also, $\nabla f(x, y) = (y, x)$, and $\nabla g(x, y) = (6x, 4y)$. This gives us the following equations: (1 mark)

$$y = 6\lambda x$$

$$x = 4\lambda y$$

$$3x^2 + 2y^2 = 1$$

Check that $x = 0 \iff y = 0$, but $(x, y) \notin D$. The first two equations give us $xy = 24\lambda^2 xy$, and hence $\lambda^2 = 1/24$. (1 mark)
Plugging these into the last equation gives us $x = \pm 1/\sqrt{6}$, and $y = \pm 1/2$. The values of f at these points are $\pm 1/\sqrt{24}$. (1 mark)
We should also check at the points where $g = 0$ and $\nabla g = \mathbf{0}$, which happens only at $(x, y) = (0, 0) \notin D$. (1 mark)
This allows us to conculde beyond doubt that the global maximum is $1/\sqrt{24}$ and global minimum is $-1/\sqrt{24}$. $\square$

General observations:

1. Most students have not stated that the domain we are constrained to is closed and bounded, which is why we can be sure that there does indeed exist a global maximum and a global minimum, and that the solution that we get by Lagrange's method gives us these global extrema.
2. A few students have solved the question using different methods, such as using the A.M-G.M inequality, converting it to one variable, trigonometric substitution, and so on. If done correctly, these have also been given full marks.
3. A common mistake made in converting it to one variable was in checking the extrema only at points where the derivative is zero, and not at all critical points and boundary points.

4. Some students did not even consider the case where $g(x, y) = 0$; though here there are no points in D at which g is zero, it is important to show that it has been considered, then discarded.
