Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 5

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Date: September 4, 2019 Day: Wednesday

Time: 2:05 PM - 2:10 PM :p Max. Marks: 5

Question. State whether the following statement is true or false. Justify your answer. If the n^{th} Taylor polynomial of $f : (-\pi/2, \pi/2) \to \mathbb{R}$ defined by $f(x) = \tan x$ around 0 is given by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

then $3a_3 + 1 = 0$.
(2 marks for correct alternative; 3 marks for correct justification)

Solution. The given statement is **false**. (2 marks)

We are given $f(x) = \tan x$. Clearly, in the interval $(-\pi/2, \pi/2)$, derivatives of all orders exist. The point 0 belongs to this interval. Hence, by Taylor's formula, for all $x \neq 0$ in this interval, we can write, $f(x) = P_n(x) + R_n(x)$ where $P_n(x)$ is the Taylor polynomial of the n^{th} order, given by

$$P_n(x) := f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

and $R_n(x)$ is something that depends on x and n, (more precisely, it is given by $R_n(x) := \dfrac{f^{(n+1)}(c_x)}{(n+1)!}x^{n+1}$where c_x is some number between 0 and x).
Comparing this expression of the Taylor polynomial with that given, we have (1 mark)

$$a_k = \frac{f^{(k)}(0)}{k!}$$

Now we compute the derivatives of $f(x) = \tan x$, (1 mark)

$$f'(x) = \sec^2 x,$$

$$f''(x) = 2\sec^2 x \tan x,$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

From this, (1 mark)

$$f'''(0) = 2 \implies a_3 = \frac{f'''(0)}{3!} = \frac{2}{6} = \frac{1}{3} \implies 3a_3 + 1 = 2 \neq 0$$

$\therefore 3a_3 + 1 \neq 0$, and hence the statement is false. $\square$

General observation:

1. This was a very elementary problem and many students had the right idea, but the manner in which they expressed it was disappointing to say the least. A surprisingly large number wrote one of these two ridiculous statements: $\tan x = f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$, completely ignoring the remainder term, or the Taylor polynomial $P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$, an infinite sum. This is in spite of the slides and me stressing on the difference between the entire Taylor expansion and the Taylor polynomial repeatedly in class. I have deducted 0.5 marks for these blatantly wrong statements.
