Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 11

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Date: October 23, 2019 | Day: Wednesday

Time: 2:00 PM - 2:05 PM | Max. Marks: 5

Question. Using the substitution $u = y - x$ and $v = x + y$, express the integral $\int\int_D e^{\frac{y-x}{y+x}} d(x,y)$, where D is the triangluar region with vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the following form:

$$\int_a^b \int_{c(v)}^{d(v)} f(u,v)\ dudv$$

. (5 marks)

Solution. Consider the mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\phi(u,v) = (\frac{v-u}{2}, \frac{v+u}{2})$, for u and v as given in the question, which gives $\phi(u,v) = (x,y) = (\phi_1(u,v), \phi_2(u,v))$ This mapping is clearly one-one and continuous, and partial derivatives of u and v with respect to x and y all exist. (1 mark)

The Jacobian of this transformation is given by (1 mark)

$$J(\phi)(u,v) = \begin{vmatrix} \frac{\partial \phi_1}{\partial u}(u,v) & \frac{\partial \phi_1}{\partial v}(u,v) \\ \frac{\partial \phi_2}{\partial u}(u,v) & \frac{\partial \phi_2}{\partial v}(u,v) \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -1/2$$

Now consider the region given by $E := \{(u,v) \in \mathbb{R}^2 : 0 \leq v \leq 1, -v \leq u \leq v\}$. Check that $\phi(E) = D$, and also that the Jacobian $= -1/2$ is never zero in E. (1 mark)

Then, by the change of variables formula, (1 mark)

$$\int\int_D e^{\frac{y-x}{y+x}} d(x,y) = \int\int_E e^{\frac{u}{v}}\ |J(\phi)(u,v)|\ d(u,v)$$

By Fubini's theorem, this can be written as the iterated integral,

$$\int_0^1 \int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}}\ dudv$$

(1 mark)

which is precisely the form we want. $\square$

General observations:

1. WRONG FORMULA: Many students have written the incorrect formula for the Jacobian and gotten it to be -2 instead of -1/2. A way to remember the right formula is to think of it in the following way: When converting to an integral in $d(u, v)$, the Jacobian has u and v in the denominator, which can be "cancelled" with the u and the v in the "numerator" $((d(u, v))$to get some form of $d(x, y)$. Note that this is just a way to remember and is in no way a mathematically correct thing to do.
2. WRONG LIMITS: Many students have just found the maximum and minimum values of u and v independently of each other and written these as the limits. Even in the expression given in the question, the limits of u are given to be functions of u. Finding the limits can be done in many ways, the best is to find the image of the vertices and join them by straight lines (this is correct because the transformation is linear, it wouldn't work otherwise).
3. A surprisingly large number of students had no clue what to do. Obviously the first thing to do with this question (in terms of trying to solve it) is to find the Jacobian, especially since they have given the substitution directly.
4. Almost no one has mentioned Fubini's theorem, or the conditions required for it, so no marks have been deducted for it. In fact, there is a subtle point that no one has observed, which is that this function is not defined everywhere in D, and it's not even a removable discontinuity. So the integral given is an improper integral which you probably haven't covered. The right way to compute this integral would be to take the lower limit of v as some $c \in \mathbb{R}$, then take the limit as c approaches zero.
