Indian Institute of Technology Bombay

MA 105 Calculus

Solution to Short Quiz 7
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Date: September 25, 2019 Day: Wednesday
Time: 2:00 PM - 2:05 PM Max. Marks: 5

Question. Let $a, b \in \mathbb{R}$ with $a > b > 0$. Set up the surface area of the oblate ellipsoid formed by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around the y-axis as a Riemann integral, that is, express the surface area in the form

$$\int_c^d \varphi(y)dy$$

(5 marks)

Solution. The surface area is given by

$$S = \int_{-b}^{b} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

(2 marks)

Also, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$$

Since we are rotating the figure about the y-axis, we only care about the distance from the y-axis, so we have $x = \frac{a}{b}\sqrt{b^2 - y^2}$ and

$$\left|\frac{dx}{dy}\right| = \left|\frac{-ay}{b\sqrt{b^2 - y^2}}\right|$$

(2 marks)

Substituting this into the above expression for S gives us

$$S = \int_{-b}^{b} 2\pi \left(\frac{a}{b}\sqrt{b^2 - y^2}\right) \sqrt{1 + \left(\frac{ay}{b\sqrt{b^2 - y^2}}\right)^2} dy = \int_0^b 4\pi a \sqrt{1 + \left(\frac{a^2}{b^2} - 1\right)\frac{y^2}{b^2}} dy$$

(1 mark)

$\therefore$ We have expressed the surface area of the ellipsoid in the form $\int_c^d \varphi(y)dy$ as required, with

$$\varphi(y) = 4\pi a \sqrt{1 + \left(\frac{a^2}{b^2} - 1\right)\frac{y^2}{b^2}}, \ d = b, \ c = 0$$

$\square$

General observations:

1. The most common mistake made was in writing the expression for the surface area. So many students have carelessly interchanged x and y, sometimes even inconsistently, i.e. they have written the integrand as $2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$ or $2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ among other combinations, and some have even integrated with respect to x. Some students have also written the integrand as just $2\pi x$.
2. Some students have left the final integral in terms of variables other than y alone.
3. Many students have written the integral without mentioning the limits of integration or even the variable of integration.
4. A few students have written the integral as

$$S = \int_{-b}^{b} 2\pi x\sqrt{(dy)^2 + (dx)^2}$$

   which makes no sense at all considering the way we have defined integrals.
5. Many students have not found $\frac{dx}{dy}$, but instead, have found $\frac{dy}{dx}$ and blindly substituted its reciprocal into the formula. Interestingly, at $y = 0$ (and we are integrating from $y = -b$ to $y = b$), $\frac{dx}{dy}$ is 0, so $\frac{dy}{dx}$ is not defined. This is why we use $\frac{dx}{dy}$, which is non-zero throughout the range of integration except at the end points, which can be dealt with by considering the limit of the integral, which actually does exist.
6. Some students have tried substituting some θ somewhere into the equation, and have gotten unnecessaruly convoluted expressions consisting of x, y, r and θ, which only makes the problem harder to solve.
7. A few students have parameterized the ellipse as $(a\cos t, b\sin t)$, $t \in [0, 2\pi]$, which also yields the same answer if solved properly.
