

NOTES ON L. C. EVANS AND R. F. GARIEPY: MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

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Notes on chapters 2, 3, and 5 of *Measure Theory and Fine Properties of Functions* by L. C. Evans and R. F. Gariepy. All references are from [1] unless indicated otherwise.

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1. GENERAL MEASURE THEORY

1.1. Weak Convergence and Compactness for Radon Measures.

t1.9-1

Theorem 1.1.1. Let $\mu, \{\mu_k\}_{k=1}^{+\infty}$ be Radon measures on \mathbb{R}^n . The following three statements are equivalent:

- (i) $\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}^n} f d\mu$ for all $f \in C_c(\mathbb{R}^n)$;
- (ii) $\limsup_{k \rightarrow +\infty} \mu_k(K) \leq \mu(K)$ for each compact set $K \subseteq \mathbb{R}^n$ and $\mu(U) \leq \liminf_{k \rightarrow +\infty} \mu_k(U)$ for each open set $U \subseteq \mathbb{R}^n$;
- (iii) $\lim_{k \rightarrow +\infty} \mu_k(B) = \mu(B)$ for each bounded Borel set $B \subseteq \mathbb{R}^n$ with $\mu(\partial B) = 0$.

Remark. Recall that Radon measures on \mathbb{R}^n are characterized by inner and outer regularity. Let $B \subseteq \mathbb{R}^n$ be a Borel set, and let $K \subseteq B \subseteq U$ with K compact and U open. If $\{\mu_k\}_{k=1}^{+\infty}$ is converging to μ in any sense, we should expect $\mu_k(K) \leq \mu(K)$ for all $k \in \mathbb{N}$ and $\mu_k(U) \geq \mu(U)$ for all $k \in \mathbb{N}$. Conditions (ii) and (iii) tell us that this in fact holds up to a subsequence.

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Definition 1.1.1 (Weak Convergence of Radon Measures). Let $\mu, \{\mu_k\}_{k=1}^{+\infty}$ be Radon measures on \mathbb{R}^n . We say that $\{\mu_k\}_{k=1}^{+\infty}$ converges weakly to μ , and write

$$\mu_k \rightharpoonup \mu,$$

if

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\mu$$

for every $f \in \mathcal{C}_c(\mathbb{R}^n)$.

Proof. Assume first that (i) holds. Let $U \subseteq \mathbb{R}^n$ be open, and choose a compact set $K \subseteq U$. Next apply Urysohn's Lemma to choose a function $f \in \mathcal{C}_c(\mathbb{R}^n)$ such that

$$0 \leq f \leq 1, \quad \text{supp}(f) \subseteq U, \quad \text{and} \quad f \equiv 1 \text{ on } K.$$

Then

$$\begin{aligned} \mu(K) &= \int_K d\mu = \int_K f \, d\mu \leq \int_{\mathbb{R}^n} f \, d\mu = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k \leq \liminf_{k \rightarrow +\infty} \int_U d\mu_k \\ &= \liminf_{k \rightarrow +\infty} \mu_k(U). \end{aligned}$$

Thus

$$\begin{aligned} \mu(U) &= \sup\{\mu(K) : K \text{ compact}, K \subseteq U\} \\ &\leq \liminf_{k \rightarrow +\infty} \mu_k(U). \end{aligned}$$

This proves the second part of (ii). The first part is similar.

Next suppose that (ii) holds. Let $B \subseteq \mathbb{R}^n$ be a bounded Borel set, $\mu(\partial B) = 0$. Then by (ii),

$$\begin{aligned} \mu(B) &= \mu(B^\circ) \leq \liminf_{k \rightarrow +\infty} \mu_k(B^\circ) \\ &\leq \limsup_{k \rightarrow +\infty} \mu_k(\overline{B}) \\ &\leq \mu(\overline{B}) \\ &= \mu(B). \end{aligned}$$

Since $\mu_k(B^\circ) = \mu_k(B) = \mu_k(\overline{B})$ for all $k \in \mathbb{N}$ since $\mu(\partial B) = 0$, it follows

$$\liminf_{k \rightarrow +\infty} \mu_k(B) = \limsup_{k \rightarrow +\infty} \mu_k(B).$$

Thus $\lim_{k \rightarrow +\infty} \mu_k(B)$ exists, and

$$\lim_{k \rightarrow +\infty} \mu_k(B) = \mu(B),$$

as required.

Finally assume that (iii) holds. Fix $\epsilon > 0$ and $f \in \mathcal{C}_c^+(\mathbb{R}^n)$. Let $R > 0$ be such that $\text{supp}(f) \subseteq B(0, R)$ and $\mu(\partial B(0, R)) = 0$. Choose a partition

$$0 := t_0 < t_1 < \cdots < t_N = 2\|f\|_{L^\infty(\mathbb{R}^n)}$$

of $[0, 2\|f\|_{L^\infty(\mathbb{R}^n)}]$ such that $0 < t_i - t_{i-1} < \epsilon$, and $\mu(f^{-1}\{t_i\}) = 0$ for each $i = 1, \dots, N$. Put $B_i := f^{-1}((t_{i-1}, t_i])$, $i = 2, \dots, N$. Then $\mu(\partial B_i) = 0$ for each $i \geq 2$. Now

$$\begin{aligned} \sum_{i=2}^N t_{i-1} \mu_k(B_i) &= \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu_k \leq \sum_{i=2}^N \int_{B_i} f d\mu_k \\ &\leq \int_{\mathbb{R}^n} f d\mu_k \\ &\leq \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0, R)), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=2}^N t_{i-1} \mu(B_i) &= \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu \leq \sum_{i=2}^N \int_{B_i} f d\mu \\ &\leq \int_{\mathbb{R}^n} f d\mu \\ &\leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B(0, R)). \end{aligned}$$

Thus (iii) implies

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \left| \int_{\mathbb{R}^n} f d\mu_k - \int_{\mathbb{R}^n} f d\mu \right| &\leq \limsup_{k \rightarrow +\infty} \left| \left\{ \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0, R)) \right\} - \sum_{i=2}^N t_{i-1} \mu(B_i) \right| \\ &\leq \limsup_{k \rightarrow +\infty} \sum_{i=2}^N |t_i \mu_k(B_i) - t_{i-1} \mu(B_i)| + \limsup_{k \rightarrow +\infty} t_1 \mu_k(B(0, R)) \\ &= \sum_{i=2}^N |t_i - t_{i-1}| \mu(B_i) + t_1 \mu(B(0, R)) \\ &\leq 2\epsilon \mu(B(0, R)). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, taking the limit at $\epsilon \rightarrow 0$ shows that

$$\limsup_{k \rightarrow +\infty} \left| \int_{\mathbb{R}^n} f d\mu_k - \int_{\mathbb{R}^n} f d\mu \right| = 0,$$

and hence

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}^n} f d\mu.$$

The proof is complete. □

t1.9-2

Theorem 1.1.2 (Weak Compactness for Measures). *Let $\{\mu_k\}_{k=1}^{+\infty}$ be a sequence of Radon measures on \mathbb{R}^n satisfying*

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

for each compact set $K \subseteq \mathbb{R}^n$. Then there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{+\infty}$ and a Radon measure μ on \mathbb{R}^n such that

$$\mu_{k_j} \rightharpoonup \mu \quad \text{as } j \rightarrow +\infty.$$

Proof.

(i). Assume first that

$$\sup_{k \in \mathbb{N}} \mu_k(\mathbb{R}^n) < +\infty. \quad (1.1.1)$$

{eq:1.9-1}

(ii). Let $\{f_k\}_{k=1}^{+\infty}$ be a countable dense subset of $\mathcal{C}_c(\mathbb{R}^n)$. Note that (1.1.1) implies that the sequence $\{\int_{\mathbb{R}^n} f_1 d\mu_j\}_{j=1}^{+\infty}$ is bounded, for

$$\left| \int_{\mathbb{R}^n} f_1 d\mu_j \right| \leq \int_{\mathbb{R}^n} |f_1| d\mu_j \leq \max_{x \in \text{supp}(f)} |f(x)| \mu_j(\mathbb{R}^n) < +\infty.$$

Thus we may find a subsequence $\{\mu_j^1\}_{j=1}^{+\infty}$ and $a_1 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f_1 d\mu_j^1 \rightarrow a_1 \quad \text{as } j \rightarrow +\infty.$$

Continuing, we find subsequences $\{\mu_j^k\}_{j=1}^{+\infty}$ of $\{\mu_j^{k-1}\}_{j=1}^{+\infty}$ and numbers $a_k \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f_k d\mu_j^k \rightarrow a_k \quad \text{as } j \rightarrow +\infty$$

for each $k \in \mathbb{N}$. Set $\nu_j := \mu_j^j$. Then

$$\int_{\mathbb{R}^n} f_k d\nu_j \rightarrow a_k \quad \text{as } j \rightarrow +\infty$$

for all $k \in \mathbb{N}$, for if $j \geq k$, then $\nu_j = \mu_j^j \in \{\mu_j^k\}_{j=1}^{+\infty}$. Define $L(f_k) := a_k$, and note that L is linear and

$$|L(f_k)| \leq M \|f_k\|_{L^\infty(\mathbb{R}^n)}$$

by (1.1.1), where

$$M := \sup_{k \in \mathbb{N}} \mu_k(\mathbb{R}^n).$$

By the Hahn–Banach Theorem, L may be uniquely extended to a bounded linear functional \bar{L} defined on all of $\mathcal{C}_c(\mathbb{R}^n)$. Then, by the Riesz Representation Theorem, there exists a unique Radon measure μ on \mathbb{R}^n such that

$$\bar{L}(f) = \int_{\mathbb{R}^n} f d\mu$$

for all $f \in \mathcal{C}_c(\mathbb{R}^n)$.

(iii). Choose any $f \in \mathcal{C}_c(\mathbb{R}^n)$. Since $\{f_k\}_{k=1}^{+\infty}$ is dense in $\mathcal{C}_c(\mathbb{R}^n)$, there exists a subsequence $\{f_{k_i}\}_{i=1}^{+\infty}$ such that $f_i \rightarrow f$ uniformly. Fix $\epsilon > 0$ and then choose $i \in \mathbb{N}$ so large that

$$\|f_{k_i} - f\|_{L^\infty(\mathbb{R}^n)} < \frac{\epsilon}{4M}. \quad (1.1.2)$$

{eq:1.9-2}

Next choose $J \in \mathbb{N}$ so that for all $j > J$,

$$\left| \int_{\mathbb{R}^n} f_{k_i} d\nu_j - \int_{\mathbb{R}^n} f_{k_i} d\mu \right| < \frac{\epsilon}{2}.$$

Then for any $j > J$, we have by (I.1.2) ^{leq:1.9-2} and the Principle of Uniform Boundedness

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f \, d\nu_j - \int_{\mathbb{R}^n} f \, d\mu \right| &\leq \left| \int_{\mathbb{R}^n} f - f_{k_i} \, d\nu_j \right| + \left| \int_{\mathbb{R}^n} f_{k_i} \, d\nu_j - \int_{\mathbb{R}^n} f_{k_i} \, d\mu \right| + \\ &\quad \left| \int_{\mathbb{R}^n} f_{k_i} - f \, d\mu \right| \\ &\leq \frac{\epsilon}{2} + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \nu_j(\mathbb{R}^n) + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \mu(\mathbb{R}^n) \\ &< \epsilon, \end{aligned}$$

as required.

(iv). In the general case that (I.1.1) ^{leq:1.9-1} fails to hold, but

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

for each compact set $K \subseteq \mathbb{R}^n$, we apply the above argument to the measures

$$\mu_k^l := \mu_k \llcorner \overline{B(0, l)}, \quad k, l = 1, 2, \dots,$$

and use a diagonalization argument. The proof is complete. \square

For the remainder of this section, we assume that

- (i) $U \subseteq \mathbb{R}^n$ is open;
- (ii) $1 \leq p < +\infty$.

Definition 1.1.2 (Weak Convergence in $L^p(U)$). A sequence $\{f_k\}_{k=1}^{+\infty} \subset L^p(U)$ is said to converge weakly to $f \in L^p(U)$, written

$$f_k \rightharpoonup f \quad \text{in } L^p(U),$$

if

$$\lim_{k \rightarrow +\infty} \int_U f_k g \, d\mathcal{L}^n = \int_U f g \, d\mathcal{L}^n$$

for each $g \in L^q(U)$, where p and q are conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q \leq +\infty$.

t1.9-3

Theorem 1.1.3 (Weak Compactness in L^p). Suppose that $1 < p < +\infty$. Let $\{f_k\}_{k=1}^{+\infty} \subseteq L^p(U)$ satisfying

$$\sup_{k \in \mathbb{N}} \|f_k\|_{L^p(U)} < +\infty.$$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{+\infty}$ of $\{f_k\}_{k=1}^{+\infty}$ and a function $f \in L^p(U)$ such that

$$f_{k_j} \rightharpoonup f \quad \text{in } L^p(U) \quad \text{as } j \rightarrow +\infty.$$

Remark. This assertion is in general false for $p = 1$. The key property here is reflexivity. Recall that $L^p(U)$ is reflexive if and only if $1 < p < +\infty$.

Definition 1.1.3. We denote by

$$\nu := \mu \llcorner f$$

the signed measure with density f with respect to μ , that is, the signed measure

$$\nu(K) = \int_K f \, d\mu,$$

provided that this holds for all compact sets $K \subseteq \mathbb{R}^n$.

Proof.

(i). If $U \neq \mathbb{R}^n$, we extend each function f_k to \mathbb{R}^n by setting $f_k = 0$ on $\mathbb{R}^n \setminus U$. This done, we may assume that $U = \mathbb{R}^n$. We may also assume that

$$f_k \geq 0 \quad \mathcal{L}^n - \text{a.e.},$$

for otherwise we could apply the following analysis to f_k^+ and f_k^- .

(ii). Define the Radon measures

$$\mu_k := \mathcal{L}^n \llcorner f_k, \quad k \in \mathbb{N}.$$

Then for each compact set $K \subseteq \mathbb{R}^n$, by Hölder's inequality, we have

$$\mu_k(K) = \int_K f_k d\mathcal{L}^n \leq \|f_k\|_{L^p(K)} \cdot \mathcal{L}^n(K)^{\frac{p-1}{p}} < +\infty,$$

and thus

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty.$$

Therefore, we may apply Theorem (I.1.2) to obtain a Radon measure μ on \mathbb{R}^n and a subsequence

$$\mu_{k_j} \rightarrow \mu.$$

(iii). We now show that $\mu \ll \mathcal{L}^n$. Let $A \subseteq \mathbb{R}^n$ be bounded with $\mathcal{L}^n(A) = 0$. Fix $\epsilon > 0$ and choose an open bounded set $V \supseteq A$ such that $\mathcal{L}^n(V) < \epsilon$. Then by Theorem (I.1.1) and Hölder's inequality,

$$\begin{aligned} \mu(A) &\leq \mu(V) \leq \liminf_{j \rightarrow +\infty} \mu_{k_j}(V) = \liminf_{j \rightarrow +\infty} \int_V f_{k_j} d\mathcal{L}^n \\ &\leq \liminf_{j \rightarrow +\infty} \|f_{k_j}\|_{L^p(V)} \cdot \mathcal{L}^n(V)^{\frac{p-1}{p}} \\ &\leq C \epsilon^{\frac{p-1}{p}}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary and $\frac{p-1}{p} > 0$, $\mu(A) = 0$, as required. Therefore $\mu \ll \mathcal{L}^n$.

(iv). By the Radon–Nikodym Theorem, there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\mu(A) = \int_A f d\mathcal{L}^n$$

for every Borel set $A \subseteq \mathbb{R}^n$.

(v). We prove that $f \in L^p(\mathbb{R}^n)$. Let $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} f \phi d\mathcal{L}^n &= \int_{\mathbb{R}^n} \phi d\mu = \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi d\mu_{k_j} \\ &= \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi f_{k_j} d\mathcal{L}^n \\ &\leq \sup_{k \in \mathbb{N}} \|f_{k_j}\|_{L^p(\mathbb{R}^n)} \|\phi\|_{L^q(\mathbb{R}^n)} \\ &\leq C \|\phi\|_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Thus

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{\substack{\phi \in \mathcal{C}_c(\mathbb{R}^n) \\ \|\phi\|_{L^q(\mathbb{R}^n)}=1}} \left| \int_{\mathbb{R}^n} f \phi d\mathcal{L}^n \right| \leq C < +\infty,$$

and we see that $f \in L^p(\mathbb{R}^n)$.

(vi). Finally, we show that $f_{k_j} \rightharpoonup f$ in $L^p(\mathbb{R}^n)$. Fix $\epsilon > 0$. By the above,

$$\int_{\mathbb{R}^n} f_{k_j} \phi \, d\mathcal{L}^n \rightarrow \int_{\mathbb{R}^n} f \phi \, d\mathcal{L}^n$$

as $j \rightarrow +\infty$ for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Thus we may choose $J \in \mathbb{N}$ so large so that for all $j > J$,

$$\left| \int_{\mathbb{R}^n} f_{k_j} \phi - f \phi \, d\mathcal{L}^n \right| < \epsilon \quad (1.1.3) \quad \boxed{\text{eq:1.9-3}}$$

for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Given $g \in L^q(\mathbb{R}^n)$, choose by the density of $\mathcal{C}_c(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$ a function $\phi \in \mathcal{C}_c(\mathbb{R}^n)$ such that

$$\|g - \phi\|_{L^q(\mathbb{R}^n)} < \epsilon.$$

Then by ^{eq:1.9-3}(1.1.3), Hölder's inequality, and the Principle of Uniform Boundedness, we have for all $j > J$

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_{k_j} g \, d\mathcal{L}^n - \int_{\mathbb{R}^n} f g \, d\mathcal{L}^n \right| &\leq \int_{\mathbb{R}^n} |f_{k_j} g - f_{k_j} \phi| \, d\mathcal{L}^n + \left| \int_{\mathbb{R}^n} f_{k_j} \phi - f \phi \, d\mathcal{L}^n \right| + \\ &\quad \int_{\mathbb{R}^n} |f \phi - f g| \, d\mathcal{L}^n \\ &\leq \epsilon + \int_{\mathbb{R}^n} |f_{k_j}| |g - \phi| \, d\mathcal{L}^n + \int_{\mathbb{R}^n} |f| |\phi - g| \, d\mathcal{L}^n \\ &\leq \epsilon + \epsilon \|f_{k_j}\|_{L^p(\mathbb{R}^n)} + \epsilon \|f\|_{L^p(\mathbb{R}^n)} \\ &\leq (2C + 1)\epsilon. \end{aligned}$$

The proof is complete. □

2. HAUSDORFF MEASURE

2.1. Definitions and Elementary Properties; Hausdorff Dimension.

Definition 2.1.1 (\mathcal{H}_δ^s). Let $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$, $0 < \delta \leq +\infty$. We define

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \text{diam } C_j \leq \delta \right\},$$

where

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(1 + \frac{s}{2})}$$

denotes the volume of the unit ball in \mathbb{R}^s .

Note in the above definition that s need not be an integer.

Definition 2.1.2 (\mathcal{H}^s , s -Dimensional Hausdorff Measure). Let $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. We define the s -dimensional Hausdorff measure \mathcal{H}^s on \mathbb{R}^n by

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A).$$

Note that taking the limit as $\delta \rightarrow 0$ coincides with taking the supremum over $\delta > 0$, for, as $\delta \rightarrow 0$, we are taking the infimum over smaller and smaller sets. That is, if $\delta_1 < \delta_2$, then there exist coverings $\{C_j\}_{j=1}^{+\infty}$ of A such that $\text{diam } C_j \leq \delta_2$ but $\text{diam } C_j > \delta_1$.

Remark.

- (i) Requiring $\delta \rightarrow 0$ forces the coverings to “follow the local geometry” of the set A ;
- (ii) Recall that

$$\mathcal{L}^n(B(x, r)) = \alpha(n)r^n$$

for all balls $B(x, r) \subseteq \mathbb{R}^n$. In fact if $s = k$ is an integer, then \mathcal{H}^k coincides with the ordinary “ k -dimensional surface area” on nice sets. This is the reason that the normalizing constant $\alpha(s)$ is included in the definition of \mathcal{H}_δ^s .

t2.1-1

Theorem 2.1.1. \mathcal{H}^s is a Borel regular measure, $0 \leq s < +\infty$.

Remark.

- (i) Recall that this means that \mathcal{H}^s is Borel and for each $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

- (ii) \mathcal{H}^s is **not** a Radon measure if $0 \leq s < n$, since \mathbb{R}^n is not σ -finite with respect to \mathcal{H}^s .

Proof.

(i). \mathcal{H}_δ^s is a measure. Choose $\{A_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}^n$ and suppose that $A_k \subseteq \bigcup_{j=1}^{+\infty} C_j^k$, where $\text{diam } C_j^k \leq \delta$. Then $\{C_j^k\}_{j,k=1}^{+\infty}$ covers $\bigcup_{k=1}^{+\infty} A_k$. Thus

$$\mathcal{H}_\delta^s \left(\bigcup_{k=1}^{+\infty} A_k \right) \leq \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j^k)^s.$$

Taking infima over all such covers $\{C_j^k\}_{j,k=1}^{+\infty}$ of A_k , we find

$$\mathcal{H}_\delta^s \left(\bigcup_{k=1}^{+\infty} A_k \right) \leq \sum_{k=1}^{+\infty} \mathcal{H}_\delta^s(A_k),$$

as required.

(ii). \mathcal{H}^s is a measure. Choose $\{A_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}^n$. Since $\mathcal{H}^s(\cup_{k=1}^{+\infty} A_k) = \sup_{\delta > 0} \mathcal{H}_\delta^s(\cup_{k=1}^{+\infty} A_k)$, we have

$$\mathcal{H}_\delta^s\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}_\delta^s(A_k) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

Taking the limit as $\delta \rightarrow 0$ on the LHS shows that

$$\mathcal{H}^s\left(\bigcup_{k=1}^{+\infty} A_k\right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

(iii). \mathcal{H}^s is a Borel measure. Choose $A, B \subseteq \mathbb{R}^n$ with $\text{dist}(A, B) > 0$. Select $0 < \delta < \frac{1}{4} \text{dist}(A, B)$. Let $A \cup B \subseteq \cup_{k=1}^{+\infty} C_k$ with $\text{diam } C_k \leq \delta$.

Put

$$\mathcal{A} := \{C_j : C_j \cap A \neq \emptyset\}$$

and

$$\mathcal{B} := \{C_j : C_j \cap B \neq \emptyset\}.$$

Then $A \subseteq \cup_{C_j \in \mathcal{A}} C_j$ and $B \subseteq \cup_{C_j \in \mathcal{B}} C_j$, with $C_i \cap C_j = \emptyset$ if $C_i \in \mathcal{A}, C_j \in \mathcal{B}$. Thus

$$\begin{aligned} \sum -j = 1^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s &\geq \sum_{C_j \in \mathcal{A}} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s + \sum_{C_j \in \mathcal{B}} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \\ &\geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B). \end{aligned}$$

Taking the infimum over all such sets $\{C_j\}_{j=1}^{+\infty}$, $0 < \delta < \frac{1}{4} \text{dist}(A, B)$, we find

$$\mathcal{H}_\delta^s(A \cup B) \geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B).$$

Letting $\delta \rightarrow 0$, we obtain

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Consequently

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$$

for all $A, B \subseteq \mathbb{R}^n$ with $\text{dist}(A, B) > 0$. By Caratheodory's Criterion, \mathcal{H}^s is a Borel measure.

(iv). \mathcal{H}^s is Borel regular. First note that $\text{diam } \overline{C} = \text{diam } C$ for all $C \subseteq \mathbb{R}^n$. Thus

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \text{diam } C_j \leq \delta, C_j \text{ closed} \right\}.$$

Choose $A \subseteq \mathbb{R}^n$ such that $\mathcal{H}^s(A) < +\infty$. Then $\mathcal{H}_\delta^s(A) < +\infty$ for all $\delta > 0$. For each $k \geq 1$, choose closed sets $\{C_j^k\}_{j=1}^{+\infty}$ so that $\text{diam } C_j^k \leq \frac{1}{k}$, $A \subseteq \cup_{j=1}^{+\infty} C_j^k$, and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j^k)^s \leq \mathcal{H}_{1/k}^s(A) + \frac{1}{k}.$$

Put $A_k := \cup_{j=1}^{+\infty} C_j^k$ and $B := \cap_{k=1}^{+\infty} A_k$. Then B is Borel. Also $A \subseteq A_k$ for each $k \in \mathbb{N}$, so $A \subseteq B$. Moreover, since $B \subseteq A_k$ for each k ,

$$\mathcal{H}_{1/k}^s(B) \leq \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j^k)^s \leq \mathcal{H}_{1/k}^s(A) + \frac{1}{k}.$$

Letting $k \rightarrow +\infty$, we find

$$\mathcal{H}^s(B) \leq \mathcal{H}^s(A).$$

But since $A \subseteq B$, we have by monotonicity

$$\mathcal{H}^s(A) = \mathcal{H}^s(B).$$

The proof is complete. \square

t2.1-2

Theorem 2.1.2 (Elementary Properties of Hausdorff Measure).

- (i) \mathcal{H}^0 is counting measure;
- (ii) $\mathcal{H}^1 = \mathcal{L}^1$ on \mathbb{R} ;
- (iii) $\mathcal{H}^s \equiv 0$ on \mathbb{R}^n for all $s > n$;
- (iv) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for all $\lambda > 0$, $A \subseteq \mathbb{R}^n$;
- (v) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ for each affine isometry $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$.

Proof.

(iv). Fix $0 < \delta \leq +\infty$, and suppose that $A \subseteq \bigcup_{j=1}^{+\infty} C_j$, with $\text{diam } C_j \leq \delta$. Then $\lambda A \subseteq \bigcup_{j=1}^{+\infty} \lambda C_j$, and $\text{diam } \lambda C_j = \lambda \text{diam } C_j \leq \lambda \delta$. Thus

$$\begin{aligned} \lambda^s \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s &= \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\lambda \text{diam } C_j)^s \\ &\geq \mathcal{H}_{\lambda\delta}^s(\lambda A). \end{aligned}$$

Taking the infimum over all such covers $\{C_j\}_{j=1}^{+\infty}$ of A , we deduce

$$\lambda^s \mathcal{H}_\delta^s(A) \geq \mathcal{H}_{\lambda\delta}^s(\lambda A),$$

and taking the limit as $\delta \rightarrow 0$ shows

$$\lambda^s \mathcal{H}^s(A) \geq \mathcal{H}^s(\lambda A).$$

The reverse inequality may be shown similarly.

(v). This follows at once from (iv) along with the translation invariance of \mathcal{H}^s .

(i). First note that $\alpha(0) = 1$. Thus obviously $\mathcal{H}^0(\{a\}) = 1$ for all $a \in \mathbb{R}^n$, and (i) follows.

(ii). Choose $A \subseteq \mathbb{R}$ and $\delta > 0$. Then

$$\begin{aligned} \mathcal{L}^1(A) &= \inf \left\{ \sum_{j=1}^{+\infty} \text{diam } C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j \right\} \\ &\leq \inf \left\{ \sum_{j=1}^{+\infty} \text{diam } C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \text{diam } C_j \leq \delta \right\} \\ &= \mathcal{H}_\delta^1(A) \\ &\leq \mathcal{H}^1(A). \end{aligned}$$

On the other hand, set $I_k := [k\delta, (k+1)\delta]$, $k \in \mathbb{Z}$. Then $\text{diam}(C_j \cap I_k) \leq \delta$, and, since $\bigcup_{k=1}^{+\infty} C_j \cap I_k = C_j$,

$$\sum_{k=-\infty}^{+\infty} \text{diam}(C_j \cap I_k) \leq \text{diam } C_j.$$

Hence,

$$\begin{aligned}\mathcal{L}^1(A) &= \inf \left\{ \sum_{j=1}^{+\infty} \text{diam } C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j \right\} \\ &\geq \inf \left\{ \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} \text{diam}(C_j \cap I_k) : A \subseteq \bigcup_{j=1}^{+\infty} C_j \right\} \\ &= \mathcal{H}_\delta^1(A).\end{aligned}$$

Therefore $\mathcal{L}^1 = \mathcal{H}_\delta^1$ for all $\delta > 0$, so that taking the supremum over all $\delta > 0$, we have $\mathcal{L}^1 = \mathcal{H}^1$ on \mathbb{R} .

(iii). Fix an integer $m \geq 1$. The unit cube $Q(n)$ in \mathbb{R}^n may be decomposed into m^n cubes with side length $\frac{1}{m}$ and diameter $\frac{\sqrt{n}}{m}$. Thus

$$\mathcal{H}_{\sqrt{n}/m}^s(Q(n)) \leq \sum_{j=1}^{m^n} \alpha(s) \left(\frac{\sqrt{n}}{m} \right)^s = \alpha(s) n^{\frac{s}{2}} m^{n-s},$$

and the RHS tends to zero as $m \rightarrow +\infty$ if $s > n$. Hence $\mathcal{H}^s(Q(n)) = 0$, so $\mathcal{H}^s \equiv 0$. The proof is complete. \square

A convenient way to check that \mathcal{H}^s vanishes on a set $A \subseteq \mathbb{R}^n$ is the following lemma.

12-1-1

Lemma 2.1.1. *If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}_\delta^s(A) = 0$ for some $0 < \delta \leq +\infty$, then $\mathcal{H}^s(A) = 0$.*

Proof. The conclusion is obvious if $s = 0$, and so we may assume that $s > 0$.

Fix $\epsilon > 0$. There exist sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \bigcup_{j=1}^{+\infty} C_j$ and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \leq \epsilon.$$

In particular for each $j \in \mathbb{N}$,

$$\text{diam } C_j \leq 2 \left(\frac{\epsilon}{\alpha(s)} \right)^{\frac{1}{s}} =: \delta(\epsilon).$$

Hence $\mathcal{H}_{\delta(\epsilon)}^s < \epsilon$. But since $\delta(\epsilon) \rightarrow 0$ and $\epsilon \rightarrow 0$, we have

$$\mathcal{H}^s(A) = 0.$$

The proof is complete. \square

We next want to define the *Hausdorff dimension* of a subset of \mathbb{R}^n .

12.1-2

Lemma 2.1.2. *Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < +\infty$.*

(i) *If $\mathcal{H}^s(A) < +\infty$, then $\mathcal{H}^t(A) = 0$;*

(ii) *If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.*

Proof.

(i). Let $\mathcal{H}^s(A) < +\infty$ and $\delta > 0$. Then there exist sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \bigcup_{j=1}^{+\infty} C_j$, $\text{diam } C_j \leq \delta$, and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \leq \mathcal{H}_\delta^s(A) + 1 \leq \mathcal{H}^s(A) + 1.$$

Then

$$\begin{aligned}\mathcal{H}_\delta^t(A) &\leq \sum_{j=1}^{+\infty} \frac{\alpha(t)}{2^t} (\text{diam } C_j)^t \\ &= \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \cdot (\text{diam } C_j)^{t-s} \\ &\leq \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \delta^{t-s} (\mathcal{H}^s(A) + 1).\end{aligned}$$

Sending $\delta \rightarrow 0$, we conclude that $\mathcal{H}^t(A) = 0$. This proves (i).

(ii). Assertion (ii) follows at once from (i), by contrapositive. The proof is complete. \square

Definition 2.1.3 (Hausdorff Dimension). *We define the Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ by*

$$\mathcal{H}_{\dim}(A) := \inf\{0 \leq s < +\infty : \mathcal{H}^s(A) = 0.\}$$

Remark. *Observe for any set $A \subseteq \mathbb{R}^n$ that $\mathcal{H}_{\dim}(A) \leq n$. Let $s := \mathcal{H}_{\dim}(A)$. Then by the preceding lemma, $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = +\infty$ for all $t < s$. Moreover, $\mathcal{H}^s(A)$ may be any number between 0 and $+\infty$, inclusive. The point is that $s = \mathcal{H}_{\dim}$ is the only number such that $\mathcal{H}^s(A)$ can be a positive finite number for any $A \subseteq \mathbb{R}^n$.*

Also note that $\mathcal{H}_{\dim}(A)$ need not be an integer. Even if $\mathcal{H}_{\dim}(A) = k$ is an integer and $0 < \mathcal{H}^k(A) < +\infty$, A need not be a “ k -dimensional surface” in any sense, and may be extremely complicated geometrically. Examples include Cantor-like subsets A of \mathbb{R}^n and other fractals.

2.2. Isodiametric Inequality; $\mathcal{H}^n = \mathcal{L}^n$. We want to prove that $\mathcal{H}^n = \mathcal{L}^n$ on \mathbb{R}^n , where $n \in \mathbb{N}$. Recall that \mathcal{L}^n is defined as the n -fold product of one-dimensional Lebesgue measure \mathcal{L}^1 , so that

$$\mathcal{L}^1(A) := \inf \left\{ \sum_{i=1}^n \mathcal{L}^1(Q_i) : Q_i \text{ cubes}, A \subseteq \bigcup_{i=1}^n Q_i \right\}.$$

On the other hand, \mathcal{H}^n is computed in terms of arbitrary coverings of small diameter.

12.2-1

Lemma 2.2.1. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be \mathcal{L}^n -measurable. Then the region “under the graph” of f ,*

$$A := \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}, 0 \leq y \leq f(x)\}$$

is \mathcal{L}^{n+1} -measurable.

Proof. Define

$$B := \{x \in \mathbb{R}^n : f(x) = +\infty\}$$

and

$$C := \{x \in \mathbb{R}^n : 0 \leq f(x) < +\infty.\}$$

Also define

$$C_{j,k} := \left\{ x \in C : \frac{j}{k} \leq f(x) < \frac{j+1}{k} \right\}, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{N},$$

so that $C = \bigcup_{j=0}^{+\infty} C_{j,k}$. Finally, put

$$D_k := \bigcup_{j=0}^{+\infty} \left(C_{j,k} \times \left[0, \frac{j}{k} \right] \right) \cup (B \times [0, +\infty]),$$

$$E_k := \bigcup_{j=0}^{+\infty} \left(C_{j,k} \times \left[0, \frac{j+1}{k} \right] \right) \cup (B \times [0, +\infty]).$$

Clearly D_k and E_k are \mathcal{L}^{n+1} measurable, and we have for each $k \in \mathbb{N}$ $D_k \subseteq A \subseteq E_k$. Write $D := \bigcup_{k=1}^{+\infty} D_k$ and $E := \bigcap_{k=1}^{+\infty} E_k$. Then also $D \subseteq A \subseteq E$, with D and E both \mathcal{L}^{n+1} -measurable. Now for any \mathcal{L}^{n+1} -measurable set F with $\mathcal{L}^{n+1}(F) < +\infty$,

$$\mathcal{L}^{n+1}((E \setminus D) \cap F) \leq \mathcal{L}^{n+1}((E_k \setminus D_k) \cap F) \leq \frac{1}{k} \mathcal{L}^n(F),$$

and the RHS tends to zero as $k \rightarrow +\infty$. Thus $\mathcal{L}^{n+1}((E \setminus D) \cap F) = 0$, and, because F was arbitrary, $\mathcal{L}^{n+1}(E \setminus D) = 0$. Hence $\mathcal{L}^{n+1}(A \setminus D) = 0$, and consequently A is \mathcal{L}^{n+1} -measurable. \square

We now define the process of Steiner symmetrization, which takes a bounded Borel-measurable set $A \subseteq \mathbb{R}^n$ and transforms A into a set \tilde{A} having the same Lebesgue measure such that $\text{diam}(\tilde{A}) \leq \text{diam}(A)$.

Fix $a, b \in \mathbb{R}^n$, $\|a\| = 1$. We define

$$L_b^a := \{b + ta : t \in \mathbb{R}\}, \text{ the line through } b \text{ in the direction of } a,$$

and

$$P_a := \{x \in \mathbb{R}^n : x \cdot a = 0\}, \text{ the plane through the origin perpendicular to } a.$$

Definition 2.2.1 (Steiner Symmetrization). Choose $a \in \mathbb{R}^n$ with $\|a\| = 1$, and let $A \subseteq \mathbb{R}^n$. We define the Steiner symmetrization of A with respect to the hyperplane P_a to be the set

$$S_a(A) := \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \left\{ b + ta : \|t\| \leq \frac{1}{2} \mathcal{H}^1(A \cap L_b^a) \right\}.$$

Note that the Steiner symmetrization is the union of all line segments $b + ta$ of length less than $\mathcal{H}^1(A \cap L_b^a)$, where b is in the plane through the origin perpendicular to a and there exists $x \in A$ such that $b + ta = x$.

12.2-2

Lemma 2.2.2 (Properties of Steiner Symmetrization).

- (i) $\text{diam } S_a(A) \leq \text{diam } A$.
- (ii) If A is \mathcal{L}^n -measurable, then so is $S_a(A)$, and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

Proof.

(i). Statement (i) is trivial if $\text{diam } A = +\infty$, so we may assume that $\text{diam } A < +\infty$. We may also suppose that A is closed, for

$$\text{diam } A^\circ = \text{diam } A = \text{diam } \overline{A}.$$

Fix $\epsilon > 0$ and choose $x, y \in S_a(A)$ such that

$$\text{diam } S_a(A) \leq \|x - y\| + \epsilon.$$

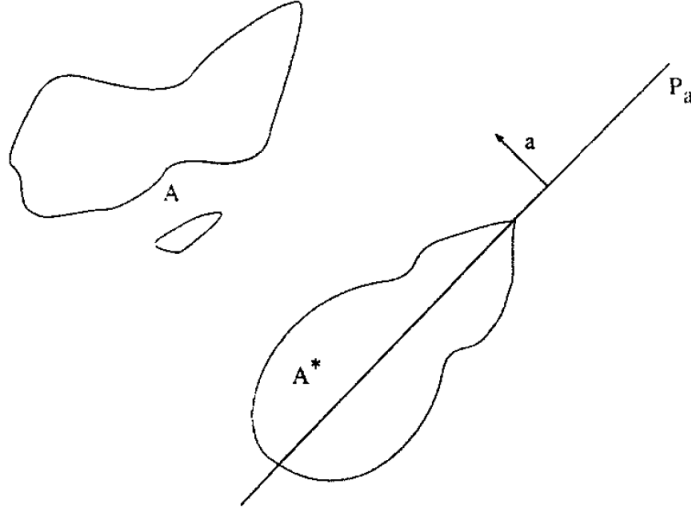


FIGURE 2.2.1. Steiner Symmetrization.

Write $b := x - (x \cdot a)a$ and $c := y - (y \cdot a)a$. Then $b, c \in P_a$. Put

$$\begin{aligned} r &:= \inf\{t : b + ta \in A\}, \\ s &:= \sup\{t : b + ta \in A\}, \\ u &:= \inf\{t : c + ta \in A\}, \\ v &:= \sup\{t : c + ta \in A\}. \end{aligned}$$

Without loss of generality, we may assume that $v - r \geq s - u$. Then

$$\begin{aligned} v - r &\geq \frac{1}{2}(v - r) + \frac{1}{2}(s - u) \\ &= \frac{1}{2}(s - r) + \frac{1}{2}(v - u) \\ &\geq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) + \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

Now, $|x \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)$, $|y \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_c^a)$, and consequently,

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a|.$$

Hence,

$$\begin{aligned} (\text{diam } S_a(A) - \epsilon)^2 &\leq \|x - y\|^2 \\ &= \|x\|^2 - 2x \cdot y + \|y\|^2 \\ &= \|b\|^2 + 2(x \cdot a)(b \cdot a) + |x \cdot a|^2 - 2(b + (x \cdot a)a) \cdot (c + (y \cdot a)a) + \|c\|^2 + \\ &\quad 2(y \cdot a)(b \cdot a) + |y \cdot a|^2 \\ &= (\|b\|^2 - 2b \cdot c + \|c\|^2) + (|x \cdot a|^2 - 2(x \cdot a)(y \cdot a) + |y \cdot a|^2) + \\ &\quad 2(x \cdot a)(b \cdot a) - 2(b \cdot a)(y \cdot a) - 2(c \cdot a)(x \cdot a) + 2(y \cdot a)(b \cdot a) \\ &= \|b - c\|^2 + \|x \cdot a - y \cdot a\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|b - c\|^2 + (v - r)^2 \\
&= \|b\|^2 - 2b \cdot c + \|c\|^2 + v^2 - 2rv + r^2 \\
&= (\|b\|^2 + 2b \cdot ra + \|ra\|^2) - 2(b \cdot c - b \cdot va - c \cdot ra - rv\|a\|^2) + \\
&\quad (\|c\|^2 + 2c \cdot va + \|va\|^2) \\
&= \|(b + ra) - (c + va)\|^2 \\
&\leq (\text{diam } A)^2,
\end{aligned}$$

since $b, c \perp a$ and A is closed, so that $b + ra, c + va \in A$. Thus $\text{diam } S_a(A) - \epsilon \leq \text{diam } A$, and since $\epsilon > 0$ was arbitrary, this proves (i).

(ii). Since \mathcal{L}^n is rotation invariant, we may assume that $a = e_n$. Then $P_a = P_{e_n} = \mathbb{R}^{n-1}$. Since $\mathcal{L}^1 = \mathcal{H}^1$ on \mathbb{R} , Tonelli's Theorem implies that the map $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ defined by $f(b) = \mathcal{H}^1(A \cap L_b^a)$ is \mathcal{L}^{n-1} -measurable and $\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b)$, for

$$\int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(A \cap L_b^a) d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).$$

Therefore

$$S_a(A) = \left\{ (b, y) : 0 \leq |y| \leq \frac{f(b)}{2} \right\} \setminus \{(b, 0) : L_b^a \cap A = \emptyset\}$$

is \mathcal{L}^n -measurable by Lemma (2.2.1), and

$$\begin{aligned}
\mathcal{L}^n(S_a(A)) &= \int_{\mathbb{R}} \mathbb{1}_{S_a(A)} d\mathcal{L}^n = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathbb{1}_{S_a(A)} d\mathcal{L}^1 d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\mathbb{1}_{S_a(A)})_{(e_1, \dots, e_{n-1})}(y) d\mathcal{L}^1(y) d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \int_{-f(b)/2}^{f(b)/2} d\mathcal{L}^1 d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).
\end{aligned}$$

The proof is complete. □

Remark. In proving $\mathcal{H}^n = \mathcal{L}^n$ below, notice that we use only statement (ii) above in the special case that a is a standard coordinate vector. Since \mathcal{H}^n is obviously rotation invariant, we in fact prove that \mathcal{L}^n is rotation invariant also.

t2.2-1

Theorem 2.2.1 (Isodiametric Inequality). For all sets $A \subseteq \mathbb{R}^n$,

$$\mathcal{L}^n(A) \leq \frac{\alpha(n)}{2^n} (\text{diam } A)^n.$$

Remark.

- (i) Geometrically, the isodiametric inequality says that of all sets of fixed diameter in \mathbb{R}^n , the n -sphere has greatest volume.
- (ii) This inequality is particularly interesting because it is not necessarily the case that A is contained in a ball of diameter $\text{diam } A$, for in \mathbb{R}^2 consider the case of an equilateral triangle

with side length 1. The smallest closed ball B which inscribes the triangle has radius $1/\sqrt{3}$, so

$$\text{diam } B = \frac{2}{\sqrt{3}} > 1.$$

Proof. If $\text{diam } A = +\infty$, the inequality is trivial. Therefore we may assume that $\text{diam } A < +\infty$.

Let $\{e_1, \dots, e_n\}$ be the standard basis for \mathbb{R}^n . Define $A_1 := S_{e_1}(A)$, $A_2 := S_{e_2}(A_1), \dots$, $A_n := S_{e_n}(A_{n-1})$. Write $A^* := A_n$.

(i). We first show that A^* is symmetric with respect to the origin. We use induction. Clearly A_1 is symmetric with respect to P_{e_1} . Let k be an integer such that $1 \leq k < n$ and suppose that A_k is symmetric with respect to P_{e_1}, \dots, P_{e_k} . Clearly $A_{k+1} = S_{e_{k+1}}(A_k)$ is symmetric with respect to $P_{e_{k+1}}$. Fix $1 \leq j < k$ and let $S_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the reflection through P_{e_j} . Let $b \in P_{e_{k+1}}$. Since A_k is symmetric with respect to P_{e_1}, \dots, P_{e_k} by the induction hypothesis and $1 \leq j \leq k$, we have $S_j(A_k) = A_k$, and so

$$\mathcal{H}^1(A_k \cap L_b^{e_{k+1}}) = \mathcal{H}^1(A_k \cap L_{S_j b}^{e_{k+1}}).$$

Consequently

$$\{t \in \mathbb{R} : b + te_{k+1} \in A_{k+1}\} = \{t \in \mathbb{R} : S_j b + te_{k+1} \in A_{k+1}\}.$$

Thus $S_j(A_{k+1}) = A_{k+1}$, that is, A_{k+1} is symmetric with respect to P_{e_j} . Since j was arbitrary, $A^* = A_n$ is symmetric with respect to P_{e_1}, \dots, P_{e_n} , and so with respect to the origin.

(ii). We show that

$$\mathcal{L}^n(A^*) \leq \frac{\alpha(n)}{2^n} (\text{diam } A^*)^n.$$

Choose $x \in A^*$. Then $-x \in A^*$ by (i), and so $\text{diam } A^* \geq 2|x|$. Thus $A^* \subseteq B(0, \frac{1}{2} \text{diam } A^*)$, and it follows by monotonicity of the Lebesgue measure

$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B\left(0, \frac{1}{2} \text{diam } A^*\right)\right) = \frac{\alpha(n)}{2^n} (\text{diam } A^*)^n.$$

(iii). We now prove the isodiametric inequality. Note that \overline{A} is \mathcal{L}^n -measurable, and thus the above Lemma (2.2.2) implies that

$$\mathcal{L}^n((\overline{A})^*) = \mathcal{L}^n(\overline{A}),$$

as well as

$$\text{diam}(\overline{A})^* \leq \text{diam } \overline{A}.$$

Hence, monotonicity of the Lebesgue measure together with (ii) give

$$\begin{aligned} \mathcal{L}^n(A) &\leq \mathcal{L}^n(\overline{A}) = \mathcal{L}^n((\overline{A})^*) \\ &\leq \frac{\alpha(n)}{2^n} (\text{diam}(\overline{A})^*)^n \\ &\leq \frac{\alpha(n)}{2^n} (\text{diam } \overline{A})^n \\ &= \frac{\alpha(n)}{2^n} (\text{diam } A)^n. \end{aligned}$$

The proof is complete. □

t2.2-2

Theorem 2.2.2. On \mathbb{R}^n , $\mathcal{L}^n = \mathcal{H}^n$.

Proof. (i). We first show that $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$ for all $A \subseteq \mathbb{R}^n$. Fix $\delta > 0$. Choose sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \bigcup_{j=1}^{+\infty} C_j$ and $\text{diam } C_j \leq \delta$. Then by monotonicity and the Isodiametric Inequality (cf. (2.2.1)),

$$\mathcal{L}^n(A) \leq \sum_{j=1}^{+\infty} \mathcal{L}^n(C_j) \leq \sum_{j=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } C_j)^n.$$

Taking the infimum of the RHS over all cover countable covers of A with diameter less than δ , we obtain $\mathcal{L}^n(A) \leq H_\delta^n(A)$. Taking the limit as $\delta \rightarrow 0$, we have

$$\mathcal{L}^n(A) \leq \mathcal{H}_\delta^n(A) \leq \mathcal{H}^n(A),$$

as required.

(ii). From the definition of \mathcal{L}^n as the n -fold product of $\mathcal{L}^1 \times \cdots \times \mathcal{L}^1$, we see that for all $A \subseteq \mathbb{R}^n$ and $\delta > 0$,

$$\mathcal{L}^n(A) = \inf \left\{ \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) : Q_i \text{ cubes, } A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \text{diam } Q_i \leq \delta \right\}.$$

We may consider only cubes parallel to the coordinate axes in \mathcal{L}^n .

(iii). We now show that \mathcal{H}^n is absolutely continuous with respect to \mathcal{L}^n . Set $C_n := \frac{\alpha(n)}{2^n}$. Then for each cube $Q \subseteq \mathbb{R}^n$,

$$\frac{\alpha(n)}{2^n} (\text{diam } Q)^n = C_n \mathcal{L}^n(Q).$$

Thus for any $A \subseteq \mathbb{R}^n$,

$$\begin{aligned} \mathcal{H}_\delta^n(A) &= \inf \left\{ \sum_{i=1}^n \frac{\alpha(n)}{2^n} (\text{diam } U_i)^n : A \subseteq \bigcup_{i=1}^{+\infty} U_i, \text{diam } U_i \leq \delta \right\} \\ &\leq \inf \left\{ \sum_{i=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } Q_i)^n : Q_i \text{ cubes, } A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \text{diam } Q_i \leq \delta \right\} \\ &= C_n \mathcal{L}^n(A). \end{aligned}$$

Taking the supremum over all $\delta > 0$, we've:

$$\mathcal{H}^n(A) \leq C_n \mathcal{L}^n(A).$$

Thus $\mathcal{H}^n(A) = 0$ whenever $\mathcal{L}^n(A) = 0$. This proves (iii).

(iv). We now show that $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$ for all $A \subseteq \mathbb{R}^n$. To this end, fix $\delta > 0$ and $\epsilon > 0$. We may choose cubes $\{Q_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ such that $A \subseteq \bigcup_{i=1}^{+\infty} Q_i$, $\text{diam } Q_i \leq \delta$, and

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon.$$

Now for each $i \in \mathbb{N}$ there exist disjoint closed balls $\{B_k^i\}_{k=1}^{+\infty} \subseteq Q_i^\circ$ such that

$$\text{diam } B_k^i \leq \delta$$

and

$$\mathcal{L}^n \left(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i \right) = \mathcal{L}^n \left(Q_i^\circ \setminus \bigcup_{k=1}^{+\infty} B_k^i \right) = 0.$$

Since $\mathcal{H}^n, \mathcal{H}_\delta^n$ are absolutely continuous with respect to \mathcal{L}^n by (iii), $\mathcal{H}^n(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i) = \mathcal{H}_\delta^n(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i) = 0$. Therefore $\mathcal{H}^n(Q_i) = \mathcal{H}^n(\bigcup_{k=1}^{+\infty} B_k^i)$ and $\mathcal{H}_\delta^n(Q_i) = \mathcal{H}_\delta^n(\bigcup_{k=1}^{+\infty} B_k^i)$, and we have

$$\begin{aligned} \mathcal{H}_\delta^n(A) &\leq \sum_{i=1}^{+\infty} \mathcal{H}_\delta^n(Q_i) = \sum_{i=1}^{+\infty} \mathcal{H}_\delta^n \left(\bigcup_{k=1}^{+\infty} B_k^i \right) \leq \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{H}_\delta^n(B_k^i) \leq \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{H}^n(B_k^i) \\ &= \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } B_k^i)^n = \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{L}^n(B_k^i) = \sum_{i=1}^{+\infty} \mathcal{L}^n \left(\bigcup_{k=1}^{+\infty} B_k^i \right) \\ &= \sum_{i=1}^{+\infty} \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$. The proof is complete. \square

2.3. Densities. We first recall the Lebesgue Density Theorem:

Theorem (Lebesgue Density Theorem). *Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. For any $r > 0$ and $x \in \mathbb{R}^n$, define the approximate Lebesgue density of E in the r -neighborhood of x by*

$$d_r(x) := \frac{\mathcal{L}^n(B(x, r) \cap E)}{\alpha(n)r^n}.$$

Further define the Lebesgue density of E at x by

$$d(x) := \lim_{r \rightarrow 0} d_r(x).$$

Then

$$d(x) = \lim_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x, r) \cap E)}{\alpha(n)r^n} = \begin{cases} 1, & \text{for } \mathcal{L}^n\text{-a.e. } x \in E, \\ 0, & \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Since $\mathcal{H}^n = \mathcal{L}^n$ for $n \in \mathbb{N}$, the above result clearly holds for \mathcal{H}^n as well. We want to develop some analogous results for lower-dimensional Hausdorff measures. Thus we assume throughout this section that $0 < s < n$.

We first establish a theorem that tells us the lower-dimensional Hausdorff density of a set at a.e. point outside the set is zero.

t2.3-1 **Theorem 2.3.1.** *Assume that $E \subseteq \mathbb{R}^n$ with E \mathcal{H}^s -measurable and $\mathcal{H}^s(E) < +\infty$. Then*

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} = 0$$

for \mathcal{H}^s -a.e. $x \in \mathbb{R}^n \setminus E$.

Proof. Fix $t > 0$ and define

$$A_t := \left\{ x \in \mathbb{R}^n \setminus E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

It suffices to show that $\mathcal{H}^s(A_t) = 0$.

Note that $\mathcal{H}^s \llcorner E$ is a Radon measure, and so, if we fix $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that

$$\mathcal{H}^s(E \setminus K) \leq \epsilon.$$

Set $U := \mathbb{R}^n \setminus K$. Then U is open and $A_t \subseteq U$ because $K \subseteq E$. Fix $\delta > 0$ and consider

$$\mathcal{F} := \left\{ B(x, r) : B(x, r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

By the Vitali Covering Lemma, there exists a countable family of balls $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ such that

$$A_t \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

Thus by monotonicity

$$\begin{aligned} \mathcal{H}_{10\delta}^s(A_t) &\leq \mathcal{H}_{10\delta}^s\left(\bigcup_{i=1}^{+\infty} B(x_i, 5r_i)\right) \leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (10r_i)^s \leq \sum_{i=1}^{+\infty} 5^s \alpha(s) r_i^s \\ &\leq \frac{5^s}{t} \sum_{i=1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \leq \frac{5^s}{t} \mathcal{H}^s(U \cap E) = \frac{5^s}{t} \mathcal{H}^s(E \setminus K) \\ &\leq \frac{5^s}{t} \epsilon. \end{aligned}$$

Letting $\delta \rightarrow 0$, we obtain $\mathcal{H}^s(A_t) \leq \frac{5^s}{t} \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\mathcal{H}^s(A_t) = 0$ for each $t > 0$. The proof is complete. \square

Now we prove that the lower-dimensional Hausdorff density of a set at a.e. point in the set is nonzero. Note that this contrasts with the Lebesgue Density Theorem: the density may not be 1. However, it is bounded below if we replace the limit with limit superior.

t2.3-2

Theorem 2.3.2. Assume that $E \subseteq \mathbb{R}^n$ with $E\mathcal{H}^s$ -measurable and $\mathcal{H}^s(E) < +\infty$. Then

$$\frac{1}{2^s} \leq \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} \leq 1$$

for \mathcal{H}^s -a.e. $x \in E$.

Remark. It is possible to have

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} < 1$$

and

$$\liminf_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} = 0$$

for \mathcal{H}^s -a.e. $x \in E$, even if $0 < \mathcal{H}^s(E) < +\infty$.

Proof. (i) We first show the upper inequality. Fix $\epsilon > 0$, $t > 1$, and define

$$B_t := \left\{ x \in E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

Since $\mathcal{H}^s \llcorner E$ is Radon, there exists an open set U containing B_t such that

$$\mathcal{H}^s(U \cap E) \leq \mathcal{H}^s(B_t) + \epsilon.$$

Define

$$\mathcal{F} := \left\{ B(x, r) : B(x, r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

By a corollary of the Vitali Covering Lemma, there exists a countable family of disjoint balls $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ such that

$$B_t \subseteq \left(\bigcup_{i=1}^m B(x_i, r_i) \right) \cup \left(\bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right).$$

Thus

$$\begin{aligned} \mathcal{H}_{10\delta}^s(B_t) &\leq \mathcal{H}_{10\delta}^s \left(\bigcup_{i=1}^m B(x_i, r_i) \right) + \mathcal{H}_{10\delta}^s \left(\bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right) \\ &\leq \sum_{i=1}^m \frac{\alpha(s)}{2^s} (2r_i)^s + \sum_{i=m+1}^{+\infty} \frac{\alpha(s)}{2^s} (10r_i)^s \\ &\leq \sum_{i=1}^m \alpha(s)r^s + \sum_{i=m+1}^{+\infty} 5^s \alpha(s)r^s \\ &\leq \frac{1}{t} \sum_{i=1}^m \mathcal{H}^s(B(x_i, r_i) \cap E) + \frac{5^s}{t} \sum_{i=m+1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \\ &\leq \frac{1}{t} \mathcal{H}^s(U \cap E) + \frac{5^s}{t} \mathcal{H}^s \left(\bigcup_{i=m+1}^{+\infty} B(x_i, r_i) \cap E \right). \end{aligned}$$

Note that this holds for each $m = 1, 2, \dots$. Thus taking the limit as $m \rightarrow \infty$ gives

$$\mathcal{H}_{10\delta}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(U \cap E) \leq \frac{1}{t} (\mathcal{H}^s(B_t) + \epsilon).$$

Letting $\delta \rightarrow 0$, we obtain

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} (\mathcal{H}^s(B_t) + \epsilon),$$

and then taking the limit as $\epsilon \rightarrow 0$ gives

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(B_t).$$

Since $\mathcal{H}^s(B_t) \leq \mathcal{H}^s(E) < +\infty$, this implies that $\mathcal{H}^s(B_t) = 0$ for each $t > 1$, as required.

(ii) We now show that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}$$

for \mathcal{H}^s -a.e. $x \in E$.

For any $\delta > 0$ and $0 < \tau < 1$, denote by $E(\delta, \tau)$ the set of all points $x \in E$ such that

$$\mathcal{H}_\delta^s(C \cap E) \leq \frac{\alpha(s)}{2^s} \tau (\text{diam } C)^s,$$

whenever $C \subseteq \mathbb{R}^n$, $x \in C$, and $\text{diam } C \leq \delta$. Then if $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ with $\text{diam } C_i \leq \delta$, $E(\delta, \tau) \subseteq \cup_{i=1}^{+\infty} C_i$, and $C_i \cap E(\delta, \tau) \neq \emptyset$, we have

$$\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \sum_{i=1}^{+\infty} \mathcal{H}_\delta^s(C_i \cap E(\delta, \tau)) \leq \tau \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_i)^s.$$

Taking the infimum over all such covers $\{C_i\}_{i=1}^{+\infty}$ of $E(\delta, \tau)$, we see that

$$\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \tau \mathcal{H}_\delta^s(E(\delta, \tau)),$$

and so $\mathcal{H}_\delta^s(E(\delta, \tau)) = 0$, since $0 < \tau < 1$ and $\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \mathcal{H}_\delta^s(E) \leq \mathcal{H}^s(E) < +\infty$. In particular,

$$\mathcal{H}^s(E(1 - \delta, \delta)) = 0 \tag{2.3.1}$$

for any $0 < \delta < 1$. Now if $x \in E$ and

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1}{2^s},$$

there exists $\delta > 0$ such that

$$\frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1 - \delta}{2^s} \tag{2.3.2}$$

for all $0 < r \leq \delta$. Thus if $x \in C$ and $\text{diam } C \leq \delta$,

$$\begin{aligned} \mathcal{H}_\delta^s(C \cap E) &= \mathcal{H}_\infty^s(C \cap E) \\ &\leq \mathcal{H}_\infty^s(B(x, \text{diam } C) \cap E) \\ &\leq (1 - \delta) \frac{\alpha(s)}{2^s} (\text{diam } C)^s, \end{aligned}$$

by (2.3.2). Consequently $x \in E(\delta, 1 - \delta)$, and it follows

$$\left\{ x \in E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1}{2^s} \right\} \subseteq \left\{ \bigcup_{k=2}^{+\infty} E\left(\frac{1}{k}, 1 - \frac{1}{k}\right) \right\}.$$

But since the RHS has \mathcal{H}^s -measure zero by (2.3.1), this proves (ii).

(iii) Since $\mathcal{H}^s(B(x, r) \cap E) \geq \mathcal{H}_\infty^s(B(x, r) \cap E)$ for any $x \in E$ and $r > 0$, (ii) immediately gives the required lower estimate

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}.$$

The proof is complete. □

2.4. Hausdorff Measure and Elementary Properties of Functions. We establish some properties relating the behavior of certain functions and Hausdorff measure.

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2.4.1. Hausdorff Measure and Lipschitz Mappings.

Definition 2.4.1 (Lipschitz). A function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called Lipschitz if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Definition 2.4.2 (Lipschitz Constant). We define the Lipschitz constant of a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$\text{Lip}(f) := \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Note that for any Lipschitz function f ,

$$|f(x) - f(y)| \leq \text{Lip}(f)|x - y|.$$

c2.4-1

Theorem 2.4.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. Then

$$\mathcal{H}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}^s(A).$$

Proof. Fix $\delta > 0$ and choose sets $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ such that $\text{diam } C_i \leq \delta$, $A \subseteq \bigcup_{i=1}^{+\infty} C_i$. Then

$$\text{diam } f(C_i) \leq \text{Lip}(f) \text{diam } C_i \leq \delta \text{Lip}(f),$$

and $f(A) \subseteq f(\bigcup_{i=1}^{+\infty} C_i) = \bigcup_{i=1}^{+\infty} f(C_i)$. Thus

$$\begin{aligned} \mathcal{H}_{\delta \text{Lip}(f)}^s(f(A)) &\leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } f(C_i))^s \\ &\leq (\text{Lip}(f))^s \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_i)^s. \end{aligned}$$

Taking the infimum over all such sets $\{C_i\}_{i=1}^{+\infty}$ which cover A , we find on the RHS

$$\mathcal{H}_{\delta \text{Lip}(f)}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}_\delta^s(A).$$

Taking the limit as $\delta \rightarrow 0$, we obtain

$$\mathcal{H}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}^s(A),$$

as required. The proof is complete. □

c2.4-1

Corollary 2.4.1. Suppose that $n > k$. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be the usual projection, $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. Then

$$\mathcal{H}^s(P(A)) \leq \mathcal{H}^s(A).$$

Proof. Since P is the standard projection map from \mathbb{R}^n to \mathbb{R}^k , $\text{Lip}(P) = 1$. Applying the above theorem (cf. (2.4.1)) gives the required estimate. □

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2.4.2. Graphs of Lipschitz Functions.

Definition 2.4.3 (Graph). For $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, we define the graph $\Gamma(f; A)$ of f over A by

$$\Gamma(f; A) := \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$

t2.4-2

Theorem 2.4.2. Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\mathcal{L}^n(A) > 0$.

- (i) Then $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$;
- (ii) If f is Lipschitz, then $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$.

Remark. We thus see that the graph of a Lipschitz function f has the expected Hausdorff dimension (think of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$). We will see from the Area Formula that $\mathcal{H}^s(\Gamma(f; A))$ can be computed according to the usual rules of calculus.

Proof.

- (i). Let $P : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be the usual projection. Then by (2.4.1),

$$\mathcal{H}^n(\Gamma(f; A)) \geq \mathcal{H}^n(A) > 0.$$

Thus $\mathcal{H}^n(\Gamma(f; A)) > 0$, so that $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$.

- (ii). Let Q denote any cube in \mathbb{R}^n of side length 1. Subdivide Q into k^n subcubes $\{Q_1, \dots, Q_{k^n}\}$ of side length $\frac{1}{k}$. Note that $\text{diam } Q_i = \frac{\sqrt{n}}{k}$ for each $i = 1, \dots, k^n$. Define

$$a_j^i := \min_{x \in Q_j} f^i(x), \quad b_j^i := \max_{x \in Q_j} f^i(x),$$

where $i = 1, \dots, m$ and $j = 1, \dots, k^n$. Since f is Lipschitz,

$$|b_j^i - a_j^i| \leq \text{Lip}(f) \text{diam } Q_j = \text{Lip}(f) \frac{\sqrt{n}}{k}.$$

For each $j = 1, \dots, k^n$, put

$$C_j := Q_j \times \prod_{i=1}^m (a_j^i, b_j^i).$$

Then

$$\Gamma(f; Q_j \cap A) = \{(x, f(x)) : x \in Q_j \cap A\} \subseteq C_j,$$

and $\text{diam } C_j \leq \frac{C}{k}$ for some constant $C > 0$. Since

$$\Gamma(f; A \cap Q) = \Gamma(f; A \cap \bigcup_{j=1}^{k^n} Q_j) = \bigcup_{j=1}^{k^n} \Gamma(f; A \cap Q_j) \subseteq \bigcup_{j=1}^{k^n} C_j,$$

we have by monotonicity

$$\begin{aligned} \mathcal{H}_{C/k}^n(G(f; A \cap Q)) &\leq \sum_{j=1}^{k^n} \frac{\alpha(n)}{2^n} (\text{diam } C_j)^n \\ &\leq \frac{k^n \alpha(n)}{2^n} \left(\frac{C}{k}\right)^n = \frac{C^n \alpha(n)}{2^n}. \end{aligned}$$

Then upon letting $k \rightarrow +\infty$, we find $\mathcal{H}^n(\Gamma(f; A \cap Q)) < +\infty$, and so $\mathcal{H}_{\dim}(\Gamma(f; A \cap Q)) \leq n$. Recall that this estimate is valid for each cube $Q \subseteq \mathbb{R}^n$ of side length 1. Consequently $\mathcal{H}_{\dim}(\Gamma(f; A)) \leq n$. Applying (i), it follows $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$. The proof is complete. \square

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2.4.3. *The Set Where an Integrable Function is Large.* If a function f is locally integrable, we can estimate the Hausdorff measure of the set where f is locally large.

2.4-3

Theorem 2.4.3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let $0 \leq s < n$, and define

$$\Lambda_s := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| d\mathcal{L}^n(y) > 0 \right\}$$

Then

$$\mathcal{H}^s(\Lambda_s) = 0.$$

Proof. We may as well assume that $f \in L^1(\mathbb{R}^n)$. By the Lebesgue Differentiation Theorem,

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f(y)| d\mathcal{L}^n(y) = |f(x)|$$

for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$, and thus

$$\lim_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| d\mathcal{L}^n(y) = \lim_{r \rightarrow 0} \alpha(n) r^{n-s} \int_{B(x,r)} |f(y)| d\mathcal{L}^n(y) = \lim_{r \rightarrow 0} \alpha(n) r^{n-s} |f(x)| = 0$$

for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$, since $0 \leq s < n$. Hence

$$\mathcal{L}^n(\Lambda_s) = 0.$$

Fix $\epsilon > 0$, $\delta > 0$, $\sigma > 0$. Since f is \mathcal{L}^n -integrable, there exists $\eta > 0$ such that $\mathcal{L}^n(\Omega) \leq \eta$ implies

$$\int_{\Omega} |f(x)| d\mathcal{L}^n(x) < \sigma.$$

Define

$$\Lambda_s^\epsilon := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| d\mathcal{L}^n(y) > \epsilon \right\}.$$

By the above analysis,

$$\mathcal{L}^n(\Lambda_s^\epsilon) = 0.$$

Thus there exists an open set $\Omega \subseteq \mathbb{R}^n$ such that $\Lambda_s^\epsilon \subseteq \Omega$ and $\mathcal{L}^n(\Omega) < \eta$. Put

$$\mathcal{F} := \left\{ B(x, r) : x \in \Lambda_s^\epsilon, 0 < r < \delta, B(x, r) \subseteq \Omega, \int_{B(x,r)} |f(y)| d\mathcal{L}^n(y) > \epsilon r^s \right\}.$$

By the Vitali Covering Lemma, there exists a countable family $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ of disjoint balls in \mathcal{F} such that

$$\Lambda_s^\epsilon \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

We thus compute

$$\begin{aligned} \mathcal{H}_{10\delta}^s(\Lambda_s^\epsilon) &\leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } B(x_i, 5r_i))^s \leq \sum_{i=1}^{+\infty} \alpha(s) (5r_i)^s \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \sum_{i=1}^{+\infty} \int_{B(x_i, r_i)} |f(y)| d\mathcal{L}^n(y) \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \int_{\Omega} |f(y)| d\mathcal{L}^n(y) \end{aligned}$$

$$\leq \frac{\alpha(s)5^s}{\epsilon}\sigma.$$

Taking the limit as $\delta \rightarrow 0$, we have

$$\mathcal{H}^s(\Lambda_s^\epsilon) \leq \frac{\alpha(s)5^s}{\epsilon}\sigma,$$

and then upon sending $\sigma \rightarrow 0$ we obtain

$$\mathcal{H}^s(\Lambda_s^\epsilon) = 0.$$

Since $\epsilon > 0$ was arbitrary, it follows

$$\mathcal{H}^s(\Lambda_s) = 0.$$

The proof is complete. □

3. AREA AND COAREA FORMULAS

3.1. Lipschitz Functions, Rademacher's Theorem.

Definition 3.1.1 (Lipschitz). Let $A \subseteq \mathbb{R}^n$. A function $f : A \rightarrow \mathbb{R}^m$ is called Lipschitz provided that

$$|f(x) - f(y)| \leq C|x - y| \quad (3.1.1)$$

for some constant $C > 0$ and all $x, y \in A$. The smallest constant C such that (3.1.1) holds for all $x, y \in A$ is denoted

$$\text{Lip}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y \right\}.$$

Definition 3.1.2 (Locally Lipschitz). A function $f : A \rightarrow \mathbb{R}^m$ is called locally Lipschitz if for each compact set $K \subseteq A$, there exists a constant $C_K > 0$ such that

$$|f(x) - f(y)| \leq C_K|x - y|$$

for all $x, y \in K$.

t3.1-1

Theorem 3.1.1 (Extension of Lipschitz Functions). Assume that $A \subseteq \mathbb{R}^n$, and let $f : A \rightarrow \mathbb{R}^m$ be Lipschitz. There exists a Lipschitz function $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

- (i) $\bar{f} = f$ on A ;
- (ii) $\text{Lip}(\bar{f}) \leq \sqrt{m} \text{Lip}(f)$.

Proof.

(i). First assume that $f : A \rightarrow \mathbb{R}$. Define

$$\bar{f}(x) := \inf_{a \in A} \{f(a) + \text{Lip}(f)|x - a|\}.$$

If $b \in A$, then we have $\bar{f}(b) = f(b)$. This follows because if $b \in A$, then

$$\bar{f}(b) \leq f(b) + \text{Lip}(f)|b - b| = f(b).$$

On the other hand, for all $a \in A$, we've:

$$f(a) + \text{Lip}(f)|b - a| \geq f(a) + \frac{f(b) - f(a)}{|b - a|}|b - a| = f(b).$$

Taking the infimum over all $a \in A$ on the LHS thus gives $\bar{f}(b) \geq f(b)$. Now if $x, y \in \mathbb{R}^n$, then

$$\begin{aligned} \bar{f}(x) &\leq \inf_{a \in A} \{f(a) + \text{Lip}(f)(|x - y| + |y - a|)\} \\ &= \inf_{a \in A} \{f(a) + \text{Lip}(f)|y - a|\} + \text{Lip}(f)|x - y| \\ &= \bar{f}(y) + \text{Lip}(f)|x - y|. \end{aligned}$$

Similarly

$$\bar{f}(y) \leq \bar{f}(x) + \text{Lip}(f)|x - y|.$$

Therefore

$$\frac{|\bar{f}(x) - \bar{f}(y)|}{|x - y|} \leq \text{Lip}(f)$$

for all $x, y \in A$. This proves the result for functions $f : A \rightarrow \mathbb{R}$.

{eq:3.1-1}

(ii). In the general case $f : A \rightarrow \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, define $\bar{f} := (\bar{f}^1, \dots, \bar{f}^m)$, where \bar{f}^i , $i = 1, \dots, m$, are defined as in (i). Then

$$|\bar{f}(x) - \bar{f}(y)|^2 = \sum_{i=1}^m \left| \bar{f}^i(x) - \bar{f}^i(y) \right|^2 \leq m(\text{Lip}(f))^2 |x - y|^2.$$

Taking square roots,

$$|\bar{f}(x) - \bar{f}(y)| \leq \sqrt{m} \text{Lip}(f) |x - y|,$$

as required. The proof is complete. \square

Remark. In fact there exists an extension \bar{f} of f with $\text{Lip}(\bar{f}) = \text{Lip}(f)$. This is Kirszbraun's Theorem.

We now prove Rademacher's Theorem, which states that a locally Lipschitz function is differentiable \mathcal{L}^n -a.e. Note that the inequality

$$|f(x) - f(y)| \leq \text{Lip}(f) |x - y|$$

says nothing about the possibility of locally approximating f by a linear map.

Definition 3.1.3 (Differentiable). The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that

$$\lim_{y \rightarrow x} \frac{|f(y) - f(x) - L(x - y)|}{|x - y|} = 0,$$

or, equivalently,

$$f(y) = f(x) + L(x - y) + o(|y - x|), \quad y \rightarrow x.$$

Remark.

- (i) Note that this is actually the definition of the Fréchet derivative.
- (ii) If such a linear mapping L exists, it is unique, and we write

$$Df(x)$$

for L . We call $Df(x)$ the derivative of f at x .

t3.1-2

Theorem 3.1.2 (Rademacher's Theorem). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable \mathcal{L}^n -a.e.

Proof.

(i). We may assume that $m = 1$, for otherwise, repeat the below argument m times. Since differentiability is a local property, we may as well also suppose that f is Lipschitz.

(ii). Fix any $v \in \mathbb{R}^n$ with $|v| = 1$, and for any $x \in \mathbb{R}^n$, define the Gateaux derivative

$$D_v f(x) := \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

at x , provided that this limit exists.

(iii). We show that $D_v f(x)$ exists for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. Since f is continuous,

$$\overline{D}_v f(x) = \limsup_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

$$= \lim_{k \rightarrow +\infty} \sup_{\substack{0 < |t| < \frac{1}{k} \\ t \in \mathbb{Q}}} \frac{f(x + tv) - f(x)}{t}$$

is Borel measurable, as is

$$\underline{D}_v f(x) := \liminf_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Thus

$$\begin{aligned} A_v &:= \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist}\} \\ &= \{x \in \mathbb{R}^n : \underline{D}_v f(x) < \overline{D}_v f(x)\}, \end{aligned}$$

being the complement of the set of all points of which the pointwise limit of measurable functions exists, is Borel measurable.

Now, for each $x, v \in \mathbb{R}^n$ with $|v| = 1$, define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(t) := f(x + tv).$$

Note that for any $t \in \mathbb{R}$,

$$\begin{aligned} |\phi(t) - \phi(s)| &= |f(x + tv) - f(x + sv)| \leq \text{Lip}(f)|(x + tv) - (x + sv)| \\ &= \text{Lip}(f)|t - s|, \end{aligned}$$

so that ϕ is Lipschitz. Therefore ϕ is absolutely continuous, and thus differentiable \mathcal{L}^1 -a.e. Thus for any line L parallel to v , the set of all points on L such that f is not differentiable has Lebesgue measure zero. That is,

$$\mathcal{H}^1(A_v \cap L) = 0$$

for each line L parallel to v . Thus the Fubini–Tonelli Theorem implies

$$\mathcal{L}^n(A_v) = 0,$$

as required.

(iv). Noting that

$$\frac{\partial}{\partial x_j} f(x) = D_{e_j} f(x) = \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t}$$

for each $j = 1, \dots, n$, we have by (iii) that

$$\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \dots, \frac{\partial}{\partial x_n} f(x) \right)$$

exists for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$.

(v). Next we show that $D_v f(x) = v \cdot \nabla f(x)$ for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. Let $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left[\frac{f(x + tv) - f(x)}{t} \right] \zeta(x) dx &= \frac{1}{t} \left[\int_{\mathbb{R}^n} f(x + tv) \zeta(x) dx - \int_{\mathbb{R}^n} f(x) \zeta(x) dx \right] \\ &= \frac{1}{t} \left[\int_{\mathbb{R}^n} f(x) \zeta(x - tv) dx - \int_{\mathbb{R}^n} f(x) \zeta(x) dx \right] \\ &= - \int_{\mathbb{R}^n} f(x) \left[\frac{\zeta(x) - \zeta(x - tv)}{t} \right] dx. \end{aligned}$$

This is the integration by parts formula for difference quotients. Let $t = \frac{1}{k}$ for $k = 1, 2, \dots$, in the above equality and note that

$$\frac{|f(x + \frac{1}{k}v) - f(x)|}{\frac{1}{k}} \leq \text{Lip}(f).$$

Thus, by Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} D_v f(x) \zeta(x) \, dx &\stackrel{LDC}{=} - \int_{\mathbb{R}^n} f(x) D_v \zeta(x) \, dx \\ &= - \sum_{j=1}^n v_j \int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_j} \zeta(x) \, dx \\ &= \sum_{j=1}^n v_j \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} f(x) \zeta(x) \, dx \\ &= \int_{\mathbb{R}^n} (v \cdot \nabla f(x)) \zeta(x) \, dx, \end{aligned}$$

where we have used integration by parts and the partial derivatives on f are understood in the a.e. sense. Since the above equality holds for every $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have $D_v f = v \cdot \nabla f$ \mathcal{L}^n -a.e.

(vi). Choose $\{v_k\}_{k=1}^{+\infty}$ to be a countable, dense subset of $\partial B(0, 1)$. Set

$$A_k := \{x \in \mathbb{R}^n : D_{v_k} f(x), \nabla f(x) \text{ exist and } D_{v_k} f(x) = v_k \cdot \nabla f(x)\}$$

for each $k \in \mathbb{N}$. Note that by (iii)-(v), $\mathcal{L}^n(\mathbb{R}^n \setminus A_k) = 0$ for each $k \in \mathbb{N}$. Define

$$A := \bigcap_{k=1}^{+\infty} A_k$$

and observe that

$$\mathcal{L}^n(\mathbb{R}^n \setminus A) = \mathcal{L}^n(\mathbb{R}^n \setminus \bigcap_{k=1}^{+\infty} A_k) = \mathcal{L}^n(\bigcup_{k=1}^{+\infty} (\mathbb{R}^n \setminus A_k)) = 0.$$

(vii). We now show that f is differentiable at each point $x \in A$. Fix any $x \in A$. Choose $v \in \partial B(0, 1)$, $t \in \mathbb{R}$, $t \neq 0$, and write

$$Q(x, v, t) := \frac{f(x + tv) - f(x)}{t} - v \cdot \nabla f(x).$$

Then if $w \in \partial B(0, 1)$, we have

$$\begin{aligned} |Q(x, v, t) - Q(x, w, t)| &= \left| \frac{f(x + tv) - f(x + tw)}{t} - (v - w) \cdot \nabla f(x) \right| \\ &\leq \left| \frac{f(x + tv) - f(x + tw)}{t} \right| + |(v - w) \cdot \nabla f(x)| \\ &\leq \text{Lip}(f)|v - w| + |\nabla f(x)||v - w| \\ &\leq (1 + \sqrt{n}) \text{Lip}(f)|v - w|. \end{aligned} \tag{3.1.2}$$

Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ so large that if $v \in \partial B(0, 1)$, then

$$|v - v_k| \leq \frac{\epsilon}{2(1 + \sqrt{n}) \text{Lip}(f)}$$

for some $k = 1, \dots, N$. Note that since $x \in A$,

$$\begin{aligned}\lim_{t \rightarrow 0} Q(x, v_k, t) &= \lim_{t \rightarrow 0} \left\{ \frac{f(x + tv_k) - f(x)}{t} - v_k \cdot \nabla f(x) \right\} \\ &= D_{v_k} f(x) - v_k \cdot \nabla f(x) \\ &= 0\end{aligned}$$

for each $k = 1, \dots, N$. Thus there exists $\delta > 0$ so that for all $0 < |t| < \delta$,

$$|Q(x, v_k, t)| < \frac{\epsilon}{2} \quad (3.1.3)$$

{eq:3.1-3}

holds for each $k = 1, \dots, N$. Consequently for each $v \in \partial B(0, 1)$ there exists $k \in \{1, \dots, N\}$ such that

$$\begin{aligned}|Q(x, v, t)| &\leq |Q(x, v, t) - Q(x, v_k, t)| + |Q(x, v_k, t)| \\ &< (1 + \sqrt{n}) \text{Lip}(f) |v - v_k| + \frac{\epsilon}{2} \\ &< \epsilon,\end{aligned}$$

by (3.1.2) and (3.1.3), provided that $0 < |t| < \delta$. Note that this is the same $\delta > 0$ for all $v \in \partial B(0, 1)$.

Now choose any $x, y \in \mathbb{R}^n$, $y \neq x$. Write

$$v := \frac{y - x}{|y - x|},$$

so that $y = x + tv$, where $t := |x - y|$. Then

$$\begin{aligned}|f(y) - f(x) - \nabla f(x) \cdot (y - x)| &= |f(x + tv) - f(x) - \nabla f(x) \cdot tv| \\ &= |Q(x, t, v)| |t| \\ &< \epsilon |t|,\end{aligned}$$

so that

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) = o(t) = o(|x - y|), \quad y \rightarrow x.$$

Hence, f is differentiable at x , with

$$Df(x) = \nabla f(x).$$

The proof is complete. □

c3.1-1

Corollary 3.1.1.

(i) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be locally Lipschitz, and

$$\mathcal{Z} := \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Then $Df(x) = 0$ for \mathcal{L}^n -a.e. $x \in \mathcal{Z}$.

(ii) Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, and

$$Y := \{x \in \mathbb{R}^n : g(f(x)) = x\}.$$

Then

$$Dg(f(x))Df(x) = I$$

for \mathcal{L}^n -a.e. $x \in Y$.

Proof.

- (i). We may assume that $m = 1$ in (i), otherwise, repeat the following argument m times.
- (ii). Choose $x \in \mathcal{Z}$ so that $Df(x)$ exists, and

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(\mathcal{Z} \cap B(x, r))}{\mathcal{L}^n(B(x, r))} = 1. \quad (3.1.4) \quad \boxed{\text{eq:3.1-4}}$$

Note that this holds for \mathcal{L}^n -a.e. $x \in \mathcal{Z}$. Since $x \in \mathcal{Z}$, it follows

$$f(y) = Df(x) \cdot (y - x) + o(|y - x|). \quad (3.1.5) \quad \boxed{\text{eq:3.1-5}}$$

By contradiction, suppose that $Df(x) = \alpha \neq 0$, and set

$$S := \left\{ v \in \partial B(0, 1) : \alpha \cdot v \geq \frac{1}{2}|\alpha| \right\}.$$

Note that S is nonempty, for otherwise $Df(x) = 0$. Now for each $v \in S$ and $t > 0$, set $y := x + tv$ in (3.1.5) to obtain

$$\begin{aligned} f(x + tv) &= \alpha \cdot tv + o(|tv|) \\ &\geq \frac{|\alpha|}{2}t + o(t). \end{aligned}$$

Hence, there exists $\delta > 0$ such that for all $0 < t < \delta$ and all $v \in S$,

$$f(x + tv) > 0.$$

But this contradicts (3.1.4), since for all $0 < r < \delta$, $B(x, r) \cap \mathcal{Z} = \{x\}$. This proves (i).

- (iii). We now show (ii). Define

$$\text{dom } Df := \{x \in \mathbb{R}^n : Df(x) \text{ exists}\}$$

and

$$\text{dom } Dg := \{x \in \mathbb{R}^n : Dg(x) \text{ exists}\}.$$

Put

$$X := Y \cap \text{dom } Df \cap f^{-1}(\text{dom } Dg).$$

Then

$$\begin{aligned} Y \setminus X &= Y \cap (Y^C \cup (\text{dom } Df)^C \cup (f^{-1}(\text{dom } Dg))^C) \\ &= (Y \setminus \text{dom } Df) \cup (Y \setminus f^{-1}(\text{dom } Dg)) \\ &\subseteq (\mathbb{R}^n \setminus \text{dom } Df) \cup g(\mathbb{R}^n \setminus \text{dom } Dg). \end{aligned} \quad (3.1.6) \quad \boxed{\text{eq:3.1-6}}$$

This follows since if $x \in Y \setminus f^{-1}(\text{dom } Dg)$, then $f(x) \in f(Y) \subseteq \mathbb{R}^n$, and $f(x) \notin \text{dom } Dg$, so that

$$f(x) \in \mathbb{R}^n \setminus \text{dom } Dg.$$

Thus

$$x = g(f(x)) \in g(\mathbb{R}^n \setminus \text{dom } Dg.)$$

By Rademacher's Theorem (cf. (3.1.2)),

$$\mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Df) = 0$$

and

$$\mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Dg) = 0.$$

Moreover, since g is Lipschitz (cf. (2.4.1)), we have

$$\mathcal{L}^n(g(\mathbb{R}^n \setminus \text{dom } Dg)) \leq (\text{Lip}(g))^n \mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Dg) = 0.$$

Thus, by (3.1.6),

$$\mathcal{L}^n(Y \setminus X) = 0.$$

Now if $x \in X$, $Dg(f(x))$ and $Df(x)$ exist, and so the chain rule implies

$$Dg(f(x))Df(x) = D(g \circ f)(x)$$

exists. Finally, since $(g \circ f)(x) - x = g(f(x)) - x = 0$ on Y , assertion (i) gives

$$Dg(f(x))Df(x) = D(g \circ f)(x) = I$$

\mathcal{L}^n -a.e. on Y . The proof is complete. \square

3.2. Linear Maps and Jacobians. We first review some basic linear algebra. Our goal in this section is to define the Jacobian of a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

3.2.1. Linear Maps.

Definition 3.2.1 (Orthogonal Linear Map). *A linear map $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is orthogonal if*

$$Ox \cdot Oy = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

Definition 3.2.2 (Symmetric Linear Map). *A linear map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric if*

$$x \cdot Sy = Sx \cdot y$$

for all $x, y \in \mathbb{R}^n$.

Definition 3.2.3 (Diagonal Linear Map). *A linear map $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is diagonal if there exist $d_1, \dots, d_n \in \mathbb{R}$ such that*

$$Dx = (d_1x_1, \dots, d_nx_n)$$

for all $x \in \mathbb{R}^n$.

Definition 3.2.4 (Adjoint). *Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. The adjoint of A is the linear map $A^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$ defined by*

$$x \cdot A^*y = Ax \cdot y$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Recall that the existence of adjoints in Euclidean space with the Euclidean metric is guaranteed, and, since \mathbb{R}^n is a Hilbert space under the Euclidean metric, the adjoint operator has the above form by the Riesz Representation Theorem.

t3.2-1

Theorem 3.2.1.

- (i) $A^{**} = A$;
- (ii) $(A \circ B)^* = B^* \circ A^*$;
- (iii) If $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal, then $O^* = O^{-1}$;
- (iv) If $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric, then $S^* = S$;

(v) If $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric, there exists an orthogonal map $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a diagonal map $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$S = O \circ D \circ O^{-1};$$

(vi) If $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is orthogonal, then $n \leq m$ and

$$\begin{aligned} O^* \circ O &= I \quad \text{on } \mathbb{R}^n, \\ O \circ O^* &= I \quad \text{on } O(\mathbb{R}^n). \end{aligned}$$

Proof.

(i). Since the dot product is symmetric, we have for all $x, y \in \mathbb{R}^n$ that

$$\begin{aligned} x \cdot (A^{**}y) &= x \cdot (A^*)^*y = A^*x \cdot y = y \cdot A^*x = Ay \cdot x \\ &= x \cdot Ay. \end{aligned}$$

Since this is for all $x \in \mathbb{R}^n$, assertion (i) follows.

(ii). For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \cdot (A \circ B)^*y &= (A \circ B)x \cdot y = A(Bx) \cdot y = Bx \cdot A^*y \\ &= x \cdot B^*(A^*y). \end{aligned}$$

This is for all $x \in \mathbb{R}^n$, so this proves (ii).

(iii). Let $x, y \in \mathbb{R}^n$. Then

$$x \cdot y = Ox \cdot Oy = x \cdot O^*(Oy),$$

and

$$x \cdot y = O(O^{-1}x) \cdot y = O^{-1}x \cdot O^*y = x \cdot O(O^*y).$$

This shows $O^* = O^{-1}$.

(iv). If $x, y \in \mathbb{R}^n$, then

$$x \cdot Sy = Sx \cdot y = x \cdot S^*y,$$

and since this is for all $x \in \mathbb{R}^n$, assertion (iv) follows. \square

t3.2-2

Theorem 3.2.2 (Polar Decomposition). Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear mapping.

(i) If $n \leq m$, there exists a symmetric map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and an orthogonal map $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$L = O \circ S.$$

(ii) If $n \geq m$, there exists a symmetric map $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that

$$L = S \circ O^*.$$

Proof.

(i). First suppose $n \leq m$. Consider the mapping $C := L^* \circ L : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Now for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} Cx \cdot y &= (L^* \circ L)x \cdot y = L^*(Lx) \cdot y = Lx \cdot Ly = x \cdot L^*(Ly) = x \cdot (L^* \circ L)y \\ &= x \cdot Cy, \end{aligned}$$

and also

$$Cx \cdot x = (L^* \circ L)x \cdot x = L^*(Lx) \cdot x = Lx \cdot Lx \geq 0.$$

Thus C is symmetric and positive semidefinite. Hence there exist $\mu_1, \dots, \mu_n \geq 0$ and an orthonormal basis $\{x_k\}_{k=1}^n$ of \mathbb{R}^n such that

$$Cx_k = \mu_k x_k,$$

$k = 1, \dots, n$. Write $\mu_k := \lambda_k^2$, $\lambda_k \geq 0$, $k = 1, \dots, n$.

(ii). We show that there exists an orthonormal set $\{z_k\}_{k=1}^n$ in \mathbb{R}^m such that

$$Lx_k = \lambda_k z_k,$$

$k = 1, \dots, n$. To see this, if $\lambda_k \neq 0$, define

$$z_k := \frac{1}{\lambda_k} Lx_k.$$

Then if $\lambda_k, \lambda_l \neq 0$,

$$\begin{aligned} z_k \cdot z_l &= \frac{1}{\lambda_k} Lx_k \cdot \frac{1}{\lambda_l} Lx_l = \frac{1}{\lambda_k \lambda_l} Lx_k \cdot Lx_l = \frac{1}{\lambda_k \lambda_l} x_k \cdot L^*(Lx_l) = \frac{1}{\lambda_k \lambda_l} x_k \cdot Cx_l \\ &= \frac{\lambda_l^2}{\lambda_k \lambda_l} x_k \cdot x_l \\ &= \frac{\lambda_l}{\lambda_k} \delta_{kl}, \end{aligned}$$

by (i) and the fact that $\{x_k\}_{k=1}^n$ is an orthonormal set. Thus the set $\{z_k : \lambda_k \neq 0\}$ is orthonormal. If $\lambda_k = 0$, define z_k to be any unit vector such that the set $\{z_k\}_{k=1}^n$ is orthonormal, applying the Gram–Schmidt process if necessary.

(iii). Define $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$Sx_k := \lambda_k x_k,$$

$k = 1, \dots, n$ and $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$Ox_k := z_k,$$

$k = 1, \dots, n$. Then

$$(O \circ S)x_k = O(Sx_k) = O(\lambda_k x_k) = \lambda_k Ox_k = \lambda_k z_k = Lx_k,$$

and, since $\{x_k\}_{k=1}^n$ is a basis for \mathbb{R}^n ,

$$L = O \circ S.$$

Notice that the mapping S is clearly symmetric. Moreover, O is orthogonal because

$$Ox_k \cdot Ox_l = z_k \cdot z_l = \delta_{kl} = x_k \cdot x_l.$$

This proves assertion (i) of the theorem.

(iv). To prove assertion (ii), we apply assertion (i) to L^* and apply (3.2.1) to obtain

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

The proof is complete. □

We now define the Jacobian of a linear map.

Definition 3.2.5 (Jacobian). *Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map.*

(i) *If $n \leq m$, write $L = O \circ S$ (cf. (3.2.2)), and we define the Jacobian of L to be*

$$[L] := |\det S|;$$

- (ii) If $n \geq m$, write $L = S \circ O^*$ (cf. ^{t3.2-2}(3.2.2)), and we define the Jacobian of L to be
- $$\llbracket L \rrbracket := |\det S|.$$

Remark.

- (i) It will follow from Theorem ^{t3.2-3}(3.2.3) below that the definition of $\llbracket L \rrbracket$ is independent of the particular choices of O and S .
(ii) Note that if, say, $n \leq m$, then $L = O \circ S$ implies

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

This is the same O and S , and it clearly follows

$$\llbracket L \rrbracket = \llbracket L^* \rrbracket.$$

t3.2-3

Theorem 3.2.3.

- (i) If $n \leq m$,
- $$\llbracket L \rrbracket^2 = \det(L^* \circ L);$$
- (ii) If $n \geq m$,
- $$\llbracket L \rrbracket^2 = \det(L \circ L^*).$$

Proof.

- (i). Assume that $n \leq m$, and apply Theorem ^{t3.2-2}(3.2.2) to write

$$L = O \circ S$$

and

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

Then

$$L^* \circ L = (S \circ O^*) \circ (O \circ S) = S \circ (O^* \circ O) \circ S = S \circ S = S^2$$

(cf. ^{t3.2-1}(3.2.1)). Hence,

$$\det(L^* \circ L) = \det(S^2) = (\det S)^2 = \llbracket L \rrbracket^2,$$

as required.

- (ii). The proof of (ii) is similar. The proof is complete. \square

Theorem ^{t3.2-3}(3.2.3) provides us with a nice way to compute the Jacobian $\llbracket L \rrbracket$ of a linear map. We augment this with the Binet–Cauchy formula below.

Definition 3.2.6 ($\Lambda(m, n)$). If $n \leq m$, we define

$$\Lambda(m, n) := \{\lambda : \{1, \dots, n\} \rightarrow \{1, \dots, m\} : \lambda \text{ strictly increasing}\}.$$

Note that this is the set of all functions λ that take $\{1, \dots, n\}$ to $\{1, \dots, m\}$ such that $\lambda(k) > \lambda(l)$ if $k > l$, $k, l \in \{1, \dots, n\}$.

Definition 3.2.7 (P_λ). If $n \leq m$, for each $\lambda \in \Lambda(m, n)$, we define $P_\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$ by

$$P_\lambda(x_1, \dots, x_m) := (x_{\lambda(1)}, \dots, x_{\lambda(n)}).$$

We may think of P_λ as a mapping that “deletes” points from (x_1, \dots, x_m) .

Remark. For each $\lambda \in \Lambda(m, n)$, there exists an n –dimensional subspace

$$S_\lambda := \text{span}\{e_{\lambda(1)}, \dots, e_{\lambda(n)}\} \subseteq \mathbb{R}^m$$

such that P_λ is the projection of \mathbb{R}^m onto S_λ .

t3.2-4

Theorem 3.2.4 (Binet–Cauchy Formula). *Let $n \leq m$ and let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then*

$$\llbracket L \rrbracket^2 = \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda \circ L))^2.$$

Remark.

- (i) *To calculate $\llbracket L \rrbracket$, we compute the sums of the squares of the determinants of each $n \times n$ submatrix of the $m \times n$ matrix representing L , with respect to the standard bases of \mathbb{R}^n and \mathbb{R}^m ;*
- (ii) *This is a kind of higher dimensional version of the Pythagorean Theorem.*

Proof.

(i). Identifying linear maps with their matrices with respect to the standard bases of \mathbb{R}^n and \mathbb{R}^m , we write

$$L : +((l_{ij}))_{m \times n}, \quad A := L^* \circ L = ((a_{ij}))_{n \times n};$$

so that

$$a_{ij} = \sum_{k=1}^m l_{ki} l_{kj}, \quad i, j = 1, \dots, n.$$

(ii). Note that

$$\llbracket L \rrbracket^2 = \det A = \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)},$$

where Σ denotes the set of all permutations of $\{1, \dots, n\}$. Thus

$$\begin{aligned} \llbracket L \rrbracket^2 &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n \sum_{k=1}^m l_{ki} l_{k\sigma(i)} \\ &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\phi \in \Phi} \prod_{i=1}^n l_{\phi(i)i} l_{\phi(i)\sigma(i)}, \end{aligned}$$

where Φ denotes the set of all one-to-one mappings of $\{1, \dots, n\}$ into $\{1, \dots, m\}$.

(iii). Now for each $\phi \in \Phi$, we can uniquely write $\phi := \lambda \circ \theta$, where $\theta \in \Sigma$ and $\lambda \in \Lambda(m, n)$. Consequently we have

$$\begin{aligned} \llbracket L \rrbracket^2 &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \prod_{i=1}^n l_{\lambda \circ \theta(i), i} l_{\lambda \circ \theta(i), \sigma(i)} \\ &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \prod_{i=1}^n l_{\lambda(i), \theta^{-1}(i)} l_{\lambda(i), \sigma \circ \theta^{-1}(i)} \\ &= \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n l_{\lambda(i), \theta(i)} l_{\lambda(i), \sigma \circ \theta(i)}. \end{aligned}$$

Set $\rho := \sigma \circ \theta$. Then

$$\llbracket L \rrbracket^2 = \sum_{\lambda \in \Lambda(m,n)} \sum_{\rho \in \Sigma} \sum_{\theta \in \Sigma} \operatorname{sgn}(\theta) \operatorname{sgn}(\rho) \prod_{i=1}^n l_{\lambda(i), \theta(i)} l_{\lambda(i), \rho(i)}$$

$$\begin{aligned}
&= \sum_{\lambda \in \Lambda(m,n)} \left(\sum_{\theta \in \Sigma} \operatorname{sgn}(\theta) \prod_{i=1}^n l_{\lambda(i), \theta(i)} \right)^2 \\
&= \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda) \circ L)^2,
\end{aligned}$$

as required. The proof is complete. \square

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3.2.2. Jacobians. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz mapping. By Rademacher's Theorem (cf. (3.1.2)), f is differentiable \mathcal{L}^n -a.e., and therefore $Df(x)$ exists and may be regarded as a linear mapping from \mathbb{R}^n into \mathbb{R}^m for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. We recall the definition of a gradient matrix.

Definition 3.2.8 (Gradient Matrix). *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $f = (f^1, \dots, f^m)$, we define the gradient matrix*

$$Df(x) := \begin{bmatrix} \frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f^m(x) & \cdots & \frac{\partial}{\partial x_n} f^m(x) \end{bmatrix}.$$

Definition 3.2.9 (Jacobian). *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, the Jacobian of f is*

$$Jf(x) := \llbracket Df(x) \rrbracket, \quad \mathcal{L}^n - a.e.$$

Note that in view of Theorem (3.2.3), we have

$$(Jf(x))^2 = \det(Df(x)^* \circ Df(x)) = \det(Df(x) \circ Df(x)^*).$$

3.3. The Area Formula. Throughout this section we assume that

$$n \leq m.$$

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3.3.1. Preliminaries.

Lemma 3.3.1. *Suppose that $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, $n \leq m$. Then*

$$\mathcal{H}^n(L(A)) = \llbracket L \rrbracket \mathcal{L}^n(A)$$

for all $A \subseteq \mathbb{R}^n$.

Proof.

(i). Write $L := O \circ S$, where $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an orthogonal map and $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a symmetric map (cf (3.2.2)). Recall that $\llbracket L \rrbracket = |\det S|$.

(ii). If $\llbracket L \rrbracket = 0$, then $\dim S(\mathbb{R}^n) \leq n - 1$, and so $\dim L(\mathbb{R}^n) \leq n - 1$. Consequently $\mathcal{H}^n(L(A)) = 0$, and the inequality is trivial.

(iii). If $\llbracket L \rrbracket > 0$, then

$$\begin{aligned} \frac{\mathcal{H}^n(L(B(x, r)))}{\mathcal{L}^n(B(x, r))} &= \frac{\mathcal{L}^n(O^* \circ L(B(x, r)))}{\mathcal{L}^n(B(x, r))} \\ &= \frac{\mathcal{L}^n(O^* \circ O \circ S(B(x, r)))}{\mathcal{L}^n(B(x, r))} \\ &= \frac{\mathcal{L}^n(S(B(x, r)))}{\mathcal{L}^n(B(x, r))} \\ &= \frac{\mathcal{L}^n(S(B(0, 1)))}{\alpha(n)} \\ &= |\det S| = \llbracket L \rrbracket. \end{aligned}$$

(iv). Define $\nu(A) := \mathcal{H}^n(L(A))$ for all $A \subseteq \mathbb{R}^n$. Then ν is a Radon measure, $\nu \ll \mathcal{L}^n$, and

$$D_{\mathcal{L}^n} \nu(x) = \lim_{r \rightarrow 0} \frac{\nu(B(x, r))}{\mathcal{L}^n(B(x, r))} = \llbracket L \rrbracket$$

by (iii). Thus for all Borel sets $B \subseteq \mathbb{R}^n$,

$$\mathcal{H}^n(L(B)) = \llbracket L \rrbracket \mathcal{L}^n(B).$$

Since ν and \mathcal{L}^n are Radon measures, the same identity holds for all sets $A \subseteq \mathbb{R}^n$. The proof is complete. \square

For the remainder of the section we assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz.

13.3-2

Lemma 3.3.2. *Let $A \subseteq \mathbb{R}^n$ be \mathcal{L}^n -measurable. Then*

- (i) $f(A)$ is \mathcal{H}^n -measurable;
- (ii) The mapping $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is \mathcal{H}^n -measurable on \mathbb{R}^m ;
- (iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n \leq (\text{Lip}(f))^n \mathcal{L}^n(A)$.

Proof.

(i). We may assume without loss of generality that A is bounded.

(ii). There exist compact sets $K_i \subseteq A$ such that

$$\mathcal{L}^n(K_i) \geq \mathcal{L}^n(A) - \frac{1}{i}, \quad i = 1, \dots, n.$$

Since $\mathcal{L}^n(A) < +\infty$ by the assumption and A is \mathcal{L}^n -measurable, $\mathcal{L}^n(A \setminus K_i) \leq \frac{1}{i}$. Since f is continuous, $f(K_i)$ is compact and thus \mathcal{H}^n -measurable. Hence, $f(\bigcup_{i=1}^{+\infty} K_i) = \bigcup_{i=1}^{+\infty} f(K_i)$ is \mathcal{H}^n -measurable. Moreover

$$\begin{aligned} \mathcal{H}^n \left(f(A) \setminus f \left(\bigcup_{i=1}^{+\infty} K_i \right) \right) &\leq \mathcal{H}^n \left(f \left(A \setminus \bigcup_{i=1}^{+\infty} K_i \right) \right) \\ &\leq (\text{Lip}(f))^n \mathcal{L}^n \left(A \setminus \bigcup_{i=1}^{+\infty} K_i \right) = 0. \end{aligned}$$

Thus $f(A)$ is \mathcal{H}^n -measurable. This proves (i).

(iii). Put

$$\mathcal{B}_k := \left\{ Q : Q = (a_1, b_1] \times \cdots \times (a_n, b_n], a_i := \frac{c_i}{k}, b_i := \frac{c_i + 1}{k}, c_i \in \mathbb{Z}, i = 1, \dots, n \right\},$$

and notice that

$$\mathbb{R}^n = \bigcup_{Q \in \mathcal{B}_k} Q.$$

Define

$$g_k := \sum_{Q \in \mathcal{B}_k} \mathbb{1}_{f(A \cap Q)},$$

and note that g_k is \mathcal{H}^n -measurable by assertion (i). Also $g_k(y)$ gives the number of cubes $Q \in \mathcal{B}_k$ such that $f^{-1}(y) \cap (A \cap Q) \neq \emptyset$. Thus

$$g_k(y) \rightarrow \mathcal{H}^0(A \cap f^{-1}(y)) \quad \text{as } k \rightarrow +\infty$$

for each $y \in \mathbb{R}^m$, and so $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is \mathcal{H}^n -measurable.

(iv). Note that g_k as defined in (iii) satisfies

$$0 \leq g_1 \leq g_2 \leq \cdots.$$

Thus by the Monotone Convergence Theorem,

$$\begin{aligned} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y) &= \int_{\mathbb{R}^m} \lim_{k \rightarrow +\infty} g_k(y) d\mathcal{H}^n(y) \\ &\stackrel{MCT}{=} \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^m} g_k(y) d\mathcal{H}^n(y) \\ &= \lim_{k \rightarrow +\infty} \sum_{Q \in \mathcal{B}_k} \mathcal{H}^n(f(A \cap Q)) \\ &\leq \limsup_{k \rightarrow +\infty} \sum_{Q \in \mathcal{B}_k} (\text{Lip}(f))^n(A \cap Q) \\ &= (\text{Lip}(f))^n \mathcal{L}^n(A), \end{aligned}$$

as required. The proof is complete. □

13.3-3

Lemma 3.3.3. *Let $t > 1$ and define*

$$B := \{x \in \mathbb{R}^n : Df(x) \text{ exists, } Jf(x) > 0\}.$$

Then there is a countable collection $\{E_k\}_{k=1}^{+\infty}$ of Borel subsets of \mathbb{R}^n such that

- (i) $B = \bigcup_{k=1}^{+\infty} E_k$;
- (ii) $f|_{E_k}$ is one-to-one, $k = 1, 2, \dots$;
- (iii) For each $k = 1, 2, \dots$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\begin{aligned} \text{Lip}((f|_{E_k}) \circ T_k^{-1}) &\leq t, \quad \text{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t, \\ t^{-n} |\det T_k| &\leq Jf|_{E_k} \leq t^n |\det T_k|. \end{aligned}$$

Proof.

(i). Fix $\epsilon > 0$ such that

$$\frac{1}{t} + \epsilon < 1 < t - \epsilon.$$

Let C be a countable dense subset of B and let S be a countable dense subset of the symmetric automorphisms of \mathbb{R}^n .

(ii). Then for each $c \in C$ and $T \in S$, and $i = 1, 2, \dots$, define $E(c, T, i)$ to be the set of all $b \in B \cap B(c, \frac{1}{i})$ satisfying

$$\left(\frac{1}{t} + \epsilon\right) |Tv| \leq |Df(b)v| \leq (t - \epsilon) |Tv| \quad (3.3.1) \quad \{\text{eq3.3-1}\}$$

for all $v \in \mathbb{R}^n$ and

$$|f(a) - f(b) - Df(b) \cdot (a - b)| \leq \epsilon |T(a - b)| \quad (3.3.2) \quad \{\text{eq3.3-2}\}$$

for all $a \in B(b, \frac{2}{3t})$. Note that $E(c, T, i)$ is a Borel set since Df is Borel measurable. Note that from (3.3.1) and (3.3.2) follows the estimate

$$\frac{1}{t} |T(a - b)| \leq |f(a) - f(b)| \leq t |T(a - b)| \quad (3.3.3) \quad \{\text{eq3.3-3}\}$$

holding for all $b \in E(c, T, i)$ and $a \in B(b, \frac{2}{3t})$.

(iii). We next show that if $b \in E(c, T, i)$, then

$$\left(\frac{1}{t} + \epsilon\right)^n |\det T| \leq Jf(b) \leq (t - \epsilon)^n |\det T|.$$

To see this, first note that Df is a linear map. Thus there exists an orthogonal map $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a symmetric map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (cf. (3.2.2)) such that $Df = O \circ S$. Then

$$Jf(b) = \llbracket Df(b) \rrbracket = |\det S|.$$

By (3.3.1),

$$\left(\frac{1}{t} + \epsilon\right) |Tv| \leq |(O \circ S)v| = |Sv| \leq (t - \epsilon) |Tv|$$

for all $v \in \mathbb{R}^n$, and so

$$\left(\frac{1}{t} + \epsilon\right) |v| \leq |(S \circ T^{-1})v| \leq (t - \epsilon) |v|$$

for all $v \in \mathbb{R}^n$. Thus

$$(S \circ T^{-1})(B(0, 1)) \subset B(0, t - \epsilon),$$

so that

$$|\det(S \circ T^{-1})| \alpha(n) \leq \mathcal{L}^n(B(0, t - \epsilon)) = \alpha(n)(t - \epsilon)^n,$$

and hence

$$|\det S| \leq (t - \epsilon)^n |\det T|.$$

The proof of the reverse inequality follows from the fact that

$$|(S \circ T^{-1})v| \geq \left(\frac{1}{t} + \epsilon\right) |v|,$$

and thus

$$B\left(0, \frac{1}{t} + \epsilon\right) \subset (S \circ T^{-1})(B(0, 1)).$$

(iv). Relabel the countable collection $\{E(c, T, i) : c \in C, T \in S, i \in \mathbb{N}\}$ as $\{E_k\}_{k=1}^{+\infty}$. Choose any $b \in B$, write $Df = O \circ S$, and choose $T \in S$ such that

$$\text{Lip}(T \circ S^{-1}) \leq \left(\frac{1}{t} + \epsilon\right)^{-1}, \quad \text{Lip}(S \circ T^{-1}) \leq t - \epsilon.$$

Now choose $i \in \mathbb{N}$ and $c \in C$ such that $|b - c| < \frac{1}{i}$ and

$$|f(a) - f(b) - Df(b) \cdot (a - b)| \leq \frac{\epsilon}{\text{Lip}(T^{-1})} |a - b| \leq \epsilon |T(a - b)|$$

for all $a \in B(b, \frac{2}{i})$. Then by (iii), $b \in E(c, T, i)$. Since this holds for all $b \in B$, this proves assertion (i).

(v). Next choose any set $E_k = E(c, T, i)$. Let $T_k := T$. By (3.3.3),

$$\frac{1}{t} |T_k(a - b)| \leq |f(a) - f(b)| \leq t |T_k(a - b)|$$

for all $b \in E_k$, $a \in B(b, \frac{2}{i})$. Since $E_k \subset B(c, \frac{1}{i}) \subset B(b, \frac{2}{i})$, we have

$$\frac{1}{t} |T_k(a - b)| \leq |f(a) - f(b)| \leq t |T_k(a - b)| \quad (3.3.4) \quad \{\text{eq3.3-4}\}$$

holding for all $a, b \in E_k$. Thus $f|_{E_k}$ is one-to-one.

(vi). Finally notice that (3.3.4) implies

$$\text{Lip}((f|_{E_k}) \circ T_k^{-1}) \leq t, \quad \text{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t.$$

Thus (iii) provides the estimate

$$t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|,$$

which proves assertion (iii). The proof is complete. \square

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3.3.2. Proof of the Area Formula.

t3.3-1

Theorem 3.3.1 (The Area Formula). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each \mathcal{L}^n -measurable subset $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf(x) d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y).$$

Proof.

(i). In view of Rademacher's Theorem (cf. (3.1.2)), we may assume that $Df(x)$ and $Jf(x)$ exist for all $x \in A$. We may also assume that $\mathcal{L}^n(A) < +\infty$, for otherwise both sides of the equality are $+\infty$.

(ii). Suppose now that $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$. Fix $t > 1$ and choose Borel sets $\{E_k\}_{k=1}^{+\infty}$ as in Lemma (3.3.3). That is,

- (1) $B = \cup_{k=1}^{+\infty} E_k$,
- (2) $f|_{E_k}$ is one-to-one, $k = 1, 2, \dots$,
- (3) For each $k = 1, 2, \dots$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\text{Lip}((f|_{E_k}) \circ T_k^{-1}) \leq t, \quad \text{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t,$$

and

$$t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|.$$

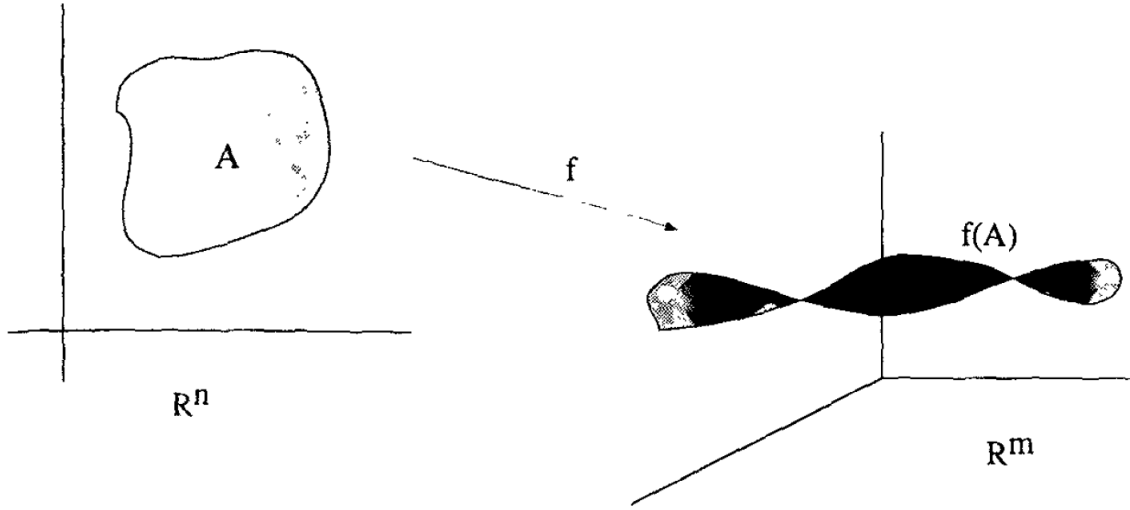


FIGURE 3.3.1. The Area Formula.

Upon passing to the collection $F_k := E_k \setminus (\cup_{i=1}^{k-1} E_i)$ if necessary, we may also suppose that the set $\{E_k\}_{k=1}^{+\infty}$ are disjoint. Define \mathcal{B}_k as in the proof of Lemma (3.3.2), that is,

$$\mathcal{B}_k := \{Q : Q = (a_1, b_1] \times \cdots \times (a_n, b_n], a_i := \frac{c_i}{k}, b_i := \frac{c_i + 1}{k}, c_i \in \mathbb{Z}, i = 1, \dots, n\}.$$

Set

$$F_j^i := E_j \cap Q_i \cap A, \quad Q_i \in \mathcal{B}_k, \quad j = 1, \dots, n.$$

Then the sets F_j^i are disjoint because $\{E_k\}_{k=1}^{+\infty}$ is disjoint, and $A = \cup_{i,j=1}^{+\infty} F_j^i$.

(iii). We claim that

$$\lim_{k \rightarrow +\infty} \sum_{i,j=1}^{+\infty} \mathcal{H}^n(f(F_j^i)) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y).$$

To see this, put

$$g_k := \sum_{i,j=1}^{+\infty} \mathbb{1}_{f(F_j^i)}.$$

Note that $g_k(y)$ is equal to the number of sets $\{F_j^i\}$ such that $F_j^i \cap f^{-1}(y) \neq \emptyset$. Then $g_k(y) \rightarrow \mathcal{H}^0(A \cap f^{-1}(y))$ as $k \rightarrow +\infty$. Notice that this is also an increasing sequence. Thus by the Monotone Convergence Theorem,

$$\begin{aligned} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y) &= \int_{\mathbb{R}^m} \lim_{k \rightarrow +\infty} g_k(y) d\mathcal{H}^n(y) \\ &\stackrel{MCT}{=} \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^m} g_k(y) d\mathcal{H}^n(y) \\ &= \lim_{k \rightarrow +\infty} \sum_{i,j=1}^{+\infty} \mathcal{H}^n(f(F_j^i)), \end{aligned}$$

where the last inequality follows from the fact that $\{F_j^i\}$ is disjoint.

(iv). Next note that

$$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(f|_{E_j}(F_j^i)) = \mathcal{H}^n(f|_{E_j} \circ T_j^{-1} \circ T_j(F_j^i)) \leq t^n \mathcal{L}^n(T_j(F_j^i))$$

and

$$\mathcal{L}^n(T_j(F_j^i)) = \mathcal{H}^n(T_j \circ (f|_{E_j})^{-1} \circ f|_{E_j}(F_j^i)) \leq t^n \mathcal{H}^n(f(F_j^i))$$

by Lemma (3.3.3) (cf. (2.4.1)). Thus

$$\begin{aligned} t^{-2n} \mathcal{H}^n(f(F_j^i)) &\leq t^{-n} \mathcal{L}^n(T_j(F_j^i)) \\ &= t^{-n} |\det T_j| \mathcal{L}^n(F_j^i) \\ &\leq \int_{F_j^i} Jf(x) d\mathcal{L}^n(x) \\ &\leq t^n |\det T_j| \mathcal{L}^n(F_j^i) \\ &= t^n \mathcal{L}^n(T_j(F_j^i)) \\ &\leq t^{2n} \mathcal{H}^n(f(F_j^i)) \end{aligned}$$

(cf. Lemmas (3.3.1) and (3.3.3)). Now summing on i and j , and recalling that $A = \cup_{i,j=1}^{+\infty} F_j^i$, we have

$$t^{-2n} \sum_{i,j=1}^{+\infty} \mathcal{H}^n(f(F_j^i)) \leq \int_A Jf(x) d\mathcal{L}^n(x) \leq t^{2n} \sum_{i,j=1}^{+\infty} \mathcal{H}^n(f(F_j^i)).$$

Letting $k \rightarrow +\infty$, we have by (iii) that

$$t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y) \leq \int_A Jf(x) d\mathcal{L}^n(x) \leq t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y).$$

Finally, taking the limit as $t \rightarrow 1^+$ shows that

$$\int_A Jf(x) d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n(y),$$

which completes the proof for the case $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$.

(v). Now consider the case $A \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$. Fix $\epsilon > 0$. We factor $f := p \circ g$, where

$$g : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n, \quad g(x) := (f(x), \epsilon x), \quad x \in \mathbb{R}^n,$$

and

$$p : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad p(y, z) := y, \quad y \in \mathbb{R}^m, z \in \mathbb{R}^n.$$

(vi). We now claim that there exists a constant $C > 0$ such that

$$0 < Jg(x) \leq C\epsilon$$

for all $x \in A$. To prove this claim, write $g = (f^1, \dots, f^m, \epsilon x_1, \dots, \epsilon x_m)$. Then

$$Dg(x) = \begin{bmatrix} Df(x) \\ \epsilon I \end{bmatrix}.$$

Since $Jg(x)^2$ equals the sum of squares of the $(n \times n)$ subdeterminants of $Dg(x)$ according to the Binet–Cauchy Formula (cf. (3.2.4)), we see that

$$Jg(x)^2 \geq \epsilon^{2n} > 0.$$

Moreover, since $|Df| \leq \text{Lip}(f) < +\infty$, we may use the Binet–Cauchy formula to also compute

$$Jg(x)^2 = Jf(x)^2 + \{\text{sum of squares of terms each involving at least one } \epsilon\} \leq C\epsilon^2$$

for each $x \in A$.

(vii). Since $p : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a projection, $\text{Lip}(p) \leq 1$, and we can compute using the first case $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$

$$\begin{aligned} \mathcal{H}^n(f(A)) &\leq \mathcal{H}^n(g(A)) \\ &\leq \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}(y, z)) \, d\mathcal{H}^n(y, z) \\ &= \int_A Jg(x) \, d\mathcal{L}^n(x) \\ &\leq C\epsilon \mathcal{L}^n(A). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we conclude that $\mathcal{H}^n(f(A)) = 0$, and thus

$$\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y) = 0,$$

since $\text{supp } \mathcal{H}^0(A \cap f^{-1}(y)) \subset f(A)$. But then since $Jf(x) = 0$ on A by the assumption, it follows

$$\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y) = 0 = \int_A Jf(x) \, d\mathcal{L}^n(x),$$

as required.

(viii). In the general case, write $A := A_1 \cup A_2$, with $A_1 \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$, $A_2 \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$, and apply the above arguments. The proof is complete. \square

3.3.3. Change of Variables Formula.

t3.3-2

Theorem 3.3.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each \mathcal{L}^n –integrable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}^n} g(x) Jf(x) \, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] d\mathcal{H}^n(y).$$

Proof.

(i). Consider first the case $g \geq 0$. Recall that the sequence $\{s_n\}_{n=1}^{+\infty}$ of simple functions defined by

$$s_j(x) := \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) + j \mathbb{1}_{g^{-1}[j, +\infty)}(x)$$

satisfies $s_j \rightarrow g$ as $j \rightarrow +\infty$ and

$$0 \leq s_1 \leq s_2 \leq \cdots.$$

Thus the Monotone Convergence Theorem implies that

$$\int_{\mathbb{R}^n} g(x) Jf(x) \, d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \lim_{j \rightarrow +\infty} s_j(x) Jf(x) \, d\mathcal{L}^n(x)$$

$$\begin{aligned}
& \stackrel{MCT}{=} \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} s_j(x) Jf(x) \, d\mathcal{L}^n(x) \\
& \stackrel{B.L.}{=} \lim_{j \rightarrow +\infty} \sum_{k=1}^{j2^j} \frac{k}{2^j} \int_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})} Jf(x) \, d\mathcal{L}^n(x) \\
& = \lim_{j \rightarrow +\infty} \sum_{k=1}^{j2^j} \frac{k}{2^j} \int_{\mathbb{R}^m} \mathcal{H}^0 \left(g^{-1} \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right) \cap f^{-1}(y) \right) \, d\mathcal{H}^n(y) \\
& \stackrel{B.L.}{=} \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^m} \sum_{k=1}^{j2^j} \frac{k}{2^j} \sum_{x \in f^{-1}(y)} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) \, d\mathcal{H}^n(y) \\
& \stackrel{MCT}{=} \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} \lim_{j \rightarrow +\infty} \sum_{k=1}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) \, d\mathcal{H}^n(y) \\
& = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] \, d\mathcal{H}^n(y).
\end{aligned}$$

(ii). Now in the case that g is any \mathcal{L}^n -integrable function, write $g = g^+ - g^-$ and apply the above case (i). The proof is complete. \square

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3.3.4. Applications.

Example 3.3.1 (Length of a Curve ($n = 1, m \geq 1$)). Assume that $f : \mathbb{R} \rightarrow \mathbb{R}^m$ is Lipschitz and one-to-one. Write

$$f = (f^1, \dots, f^m), \quad Df = (\dot{f}^1, \dots, \dot{f}^m),$$

so that

$$Jf = |Df| = |\dot{f}|.$$

For any $-\infty < a < b < +\infty$, define the curve

$$C := f([a, b]) \subset \mathbb{R}^m.$$

Then by the Area Formula

$$\begin{aligned}
\int_a^b |\dot{f}(t)| \, dt &= \int_{[a, b]} Jf(x) \, d\mathcal{L}^1(x) \\
&= \int_{\mathbb{R}^m} \mathcal{H}^0([a, b] \cap f^{-1}(y)) \, d\mathcal{L}^1(y) \\
&= \mathcal{H}^1(C).
\end{aligned}$$

Example 3.3.2 (Surface Area of a Graph ($n \geq 1, m = n + 1$)). Assume that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz and define $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ by

$$f(x) := (x, g(x)).$$

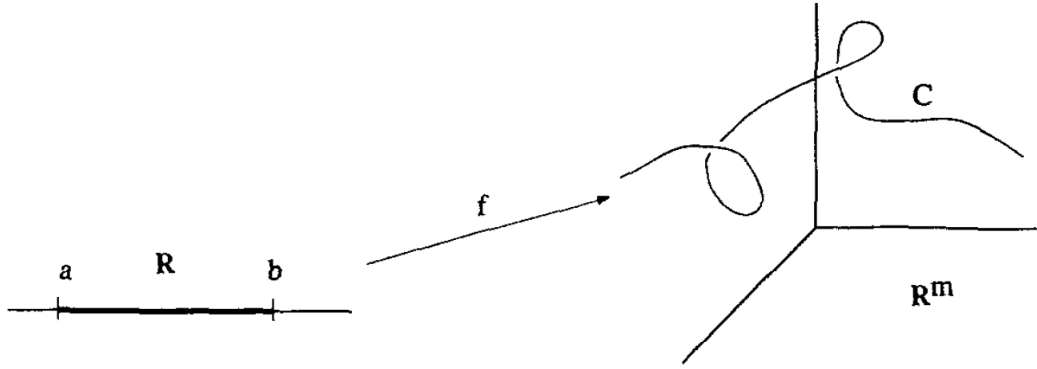


FIGURE 3.3.2. Length of a Curve.

Note that $f = \Gamma(g)$. Then

$$Df(x) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial}{\partial x_1} g(x) & \cdots & \frac{\partial}{\partial x_n} g(x) \end{bmatrix}.$$

By the Binet–Cauchy formula,

$$(Jf)^2 = \text{sum of squares of } n \times n \text{ subdeterminants} = 1 + |Dg|^2,$$

so that $Jf = (1 + |Dg|^2)^{1/2}$. Now for each open set $\Omega \subset \mathbb{R}^n$, recall the graph of g over Ω :

$$\Gamma(g, \Omega) = \{(x, f(x)) : x \in \Omega\} \subset \mathbb{R}^{n+1}.$$

Then by the Area Formula

$$\begin{aligned} \int_{\Omega} (1 + |Dg(x)|^2)^{1/2} d\mathcal{L}^n(x) &= \int_{\Omega} Jf(x) d\mathcal{L}^n(x) \\ &= \int_{\mathbb{R}^{n+1}} \mathcal{H}^0(\Omega \cap f^{-1}(y)) d\mathcal{H}^n(y) \\ &= \mathcal{H}^n(\Gamma(g, \Omega)). \end{aligned}$$

Example 3.3.3 (Surface Area of a Parametric Hypersurface ($n \geq 1$, $m = n + 1$)). Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ is one-to-one and Lipschitz. Write

$$f = (f^1, \dots, f^{n+1})$$

and

$$Df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f^{n+1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{n+1}(x) \end{bmatrix}.$$

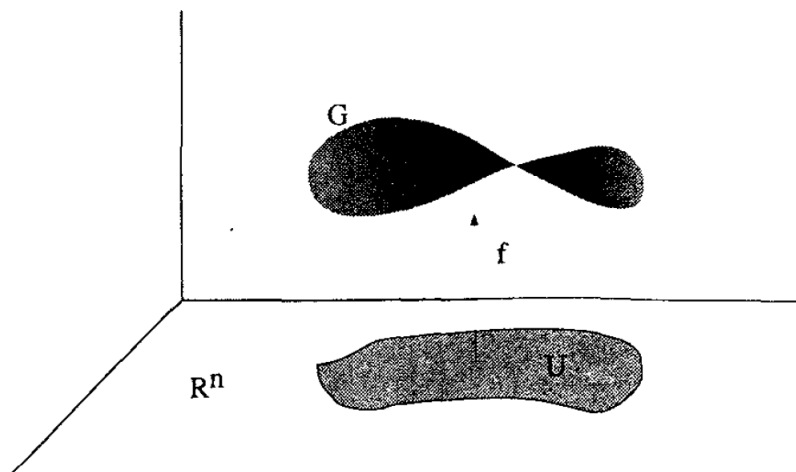


FIGURE 3.3.3. Surface Area of a Graph.

Then by the Binet–Cauchy formula,

$$\begin{aligned} (Jf)^2 &= \text{sum of squares of } n \times n \text{ subdeterminants} \\ &= \sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \dots, f^{k-1}, f^{k+1}, \dots, f^{n+1})}{\partial x_1, \dots, x_n} \right]^2, \end{aligned}$$

where

$$\frac{\partial(f^1, \dots, f^{k-1}, f^{k+1}, \dots, f^{n+1})}{\partial x_1, \dots, x_n}$$

denotes the Jacobian of the function with gradient matrix

$$\begin{bmatrix} \frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} f^{k-1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{k-1}(x) \\ \frac{\partial}{\partial x_1} f^{k+1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{k+1}(x) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1} f^{n+1}(x) & \cdots & \frac{\partial}{\partial x_n} f^{n+1}(x) \end{bmatrix}.$$

For each open set $\Omega \subset \mathbb{R}^n$, write

$$S := f(\Omega) \subset \mathbb{R}^{n+1}.$$

Then by the Area Formula

$$\begin{aligned}
 \int_{\Omega} \left(\sum_{k=1}^{n+1} \left[\frac{\partial(f^1, \dots, f^{k-1}, f^{k+1}, \dots, f^{n+1})}{\partial x_1, \dots, x_n} \right]^2 \right)^{\frac{1}{2}} d\mathcal{L}^n(x) &= \int_{\Omega} Jf(x) d\mathcal{L}^n(x) \\
 &= \int_{\mathbb{R}^{n+1}} \mathcal{H}^0(\Omega \cap f^{-1}(y)) d\mathcal{H}^n(y) \\
 &= \mathcal{H}^n(S).
 \end{aligned}$$

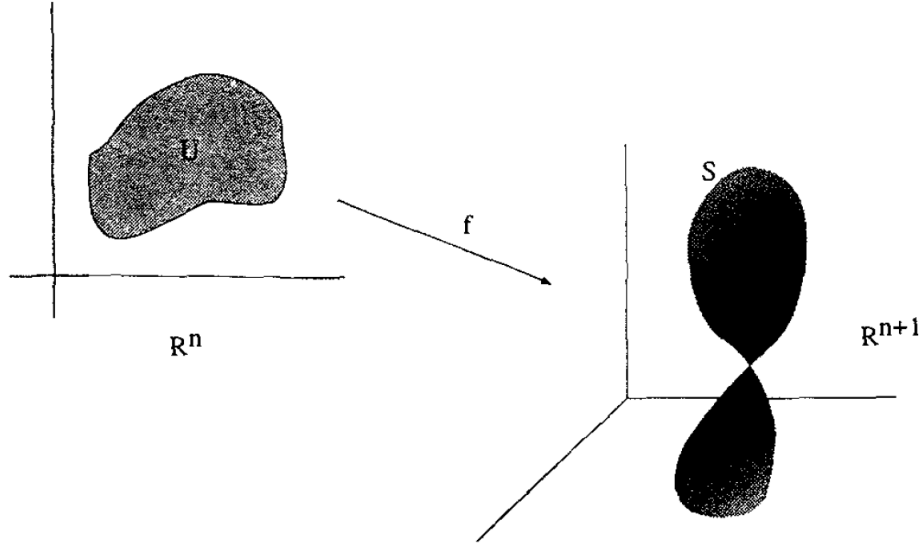


FIGURE 3.3.4. Surface Area of a Parametric Hypersurface.

Example 3.3.4 (Submanifolds). Let $M \subset \mathbb{R}^m$ be a Lipschitz n -dimensional embedded submanifold. Suppose that $\Omega \subset \mathbb{R}^n$ and let $f : \Omega \rightarrow M$ be coordinates for M . Let $A \subset f(\Omega)$. Let $A \subset f(\Omega) \subset M$, A Borel, and let $B := f^{-1}(A) \subset \Omega$. Define the metric $g : M \rightarrow \mathbb{R}$ on M by

$$g_{ij} = g \left(\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right) := \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}, \quad i, j = 1, \dots, n,$$

and

$$g := \det((g_{ij})_{n \times n}).$$

Then

$$Df \circ (Df)^* = (g_{ij})_{n \times n},$$

and so

$$Jf = (\det(Df \circ (Df)^*))^{\frac{1}{2}} = g^{\frac{1}{2}}.$$

Thus by the Area Formula,

$$\begin{aligned}
 \int_B g^{\frac{1}{2}} d\mathcal{L}^n(x) &= \int_B Jf(x) d\mathcal{L}^n(x) \\
 &= \int_{\mathbb{R}^m} \mathcal{H}^0(B \cap f^{-1}(y)) d\mathcal{H}^n(y)
 \end{aligned}$$

$$= \mathcal{H}^n(A).$$

Here $\mathcal{H}^n(A)$ represents the “volume” of A in M .

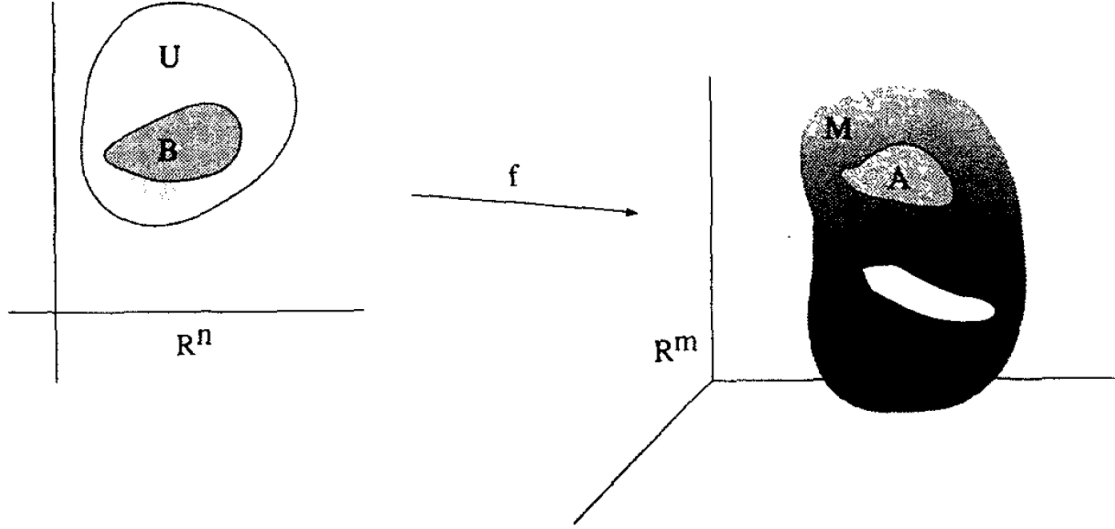


FIGURE 3.3.5. Volume of a Submanifold.

3.4. The Coarea Formula. Throughout this section we assume that

$$n \geq m.$$

3.4.1. Preliminaries.

13.4-1

Lemma 3.4.1. Suppose that $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, $n \geq m$, and $A \subseteq \mathbb{R}^n$ is \mathcal{L}^n -measurable. Then

- (i) The mapping $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(y))$ is \mathcal{L}^m -measurable;
- (ii) $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(y)) d\mathcal{L}^m(y) = \|L\| \mathcal{L}^n(A).$

Proof.

(i). First suppose that $\dim L(\mathbb{R}^n) < m$. In this case $A \cap L^{-1}(y) = \emptyset$ and consequently $\mathcal{H}^{n-m}(A \cap L^{-1}(y)) = 0$ for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$. Also if we write $L = S \circ O^*$ as in the Polar Decomposition Theorem (cf. (3.2.2)) we have $L(\mathbb{R}^n) = S(\mathbb{R}^m)$. Thus $\dim S(\mathbb{R}^m) < m$, and hence $\|L\| = |\det S| = 0$.

(ii). Now suppose that $L = P$, where P is an orthogonal projection of \mathbb{R}^n onto \mathbb{R}^m . In this case, for each $y \in \mathbb{R}^m$, $P^{-1}(y)$ is an $(n - m)$ -dimensional affine subspace of \mathbb{R}^n , a translation of $P^{-1}(0)$. By Fubini's Theorem,

$$y \mapsto \mathcal{H}^{n-m}(A \cap P^{-1}(y)) \text{ is } \mathcal{L}^m\text{-measurable}$$

and

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap P^{-1}(y)) d\mathcal{L}^m(y) = \mathcal{L}^n(A), \quad (3.4.1)$$

{eq:3.4-1}

as required.

(iii). Now consider the general case that $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\dim L(\mathbb{R}^n) = m$. Again applying the Polar Decomposition Theorem (cf. (3.2.2)) we can write

$$L := S \circ O^*,$$

where $S: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is symmetric and $O: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is orthogonal. Recall that, since S evidently is not singular,

$$\llbracket L \rrbracket = |\det S| > 0.$$

(iv). We claim that $O^* = P \circ Q$, where P is the orthogonal projection of \mathbb{R}^n onto \mathbb{R}^m and $Q: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal. To see this, let Q be any orthogonal map of \mathbb{R}^n onto \mathbb{R}^n such that

$$Q^*(x_1, \dots, x_m, 0, \dots, 0) = O(x_1, \dots, x_m)$$

for all $x \in \mathbb{R}^m$. Note that

$$P^*(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0) \in \mathbb{R}^n$$

for all $x \in \mathbb{R}^m$. Thus

$$(Q^* \circ P^*)(x_1, \dots, x_m) = Q^*(x_1, \dots, x_m, 0, \dots, 0) = O(x_1, \dots, x_m),$$

so that $O = Q^* \circ P^*$, and hence $O^* = P \circ Q$.

(v). Now $L^{-1}(0)$ is an $(n - m)$ -dimensional subspace of \mathbb{R}^n and $L^{-1}(y)$ is a translation of $L^{-1}(0)$ for each $y \in \mathbb{R}^m$. Thus by Fubini's Theorem,

$$y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(y)) \text{ is } \mathcal{L}^m\text{-measurable}$$

and by (3.4.1) we may calculate

$$\begin{aligned} \mathcal{L}^n(A) &= \mathcal{L}^n(Q(A)) \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(Q(A) \cap P^{-1}(y)) d\mathcal{L}^m(y) \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap Q^{-1} \circ P^{-1}(y)) d\mathcal{L}^m(y). \end{aligned}$$

Now set $z := Sy$ to calculate using the change of variables formula (cf. (3.3.2))

$$|\det S| \mathcal{L}^n(A) = \int_A JS(x) d\mathcal{L}^n(x) = |\det S| \mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap Q^{-1} \circ P^{-1} \circ S^{-1}(z)) d\mathcal{H}^m(z).$$

but $L = S \circ O^* = S \circ P \circ Q$, and so, since $\llbracket L \rrbracket = |\det S|$,

$$\llbracket L \rrbracket \mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(z)) d\mathcal{L}^m(z),$$

as required. The proof is complete. \square

13.4-2

Lemma 3.4.2. Assume that $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz. Let $A \subseteq \mathbb{R}^n$ be \mathcal{L}^n -measurable, $n \geq m$. Then

- (i) $f(A)$ is \mathcal{L}^m -measurable;
- (ii) $A \cap f^{-1}(y)$ is \mathcal{H}^{n-m} -measurable for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$;
- (iii) The mapping $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$ is \mathcal{L}^m -measurable;

$$(iv) \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y) \leq \frac{(\alpha(n-m)\alpha(m))}{\alpha(n)} (\text{Lip } f)^m \mathcal{L}^n(A).$$

Proof.

(i). Assertion (i) is proved exactly in the same way as the corresponding statement of Lemma (3.3.2) (cf. §3.3).

(ii). Next, for each $j = 1, 2, \dots$, there exist closed balls $\{B_i^j\}_{i=1}^{+\infty}$ such that

$$A \subset \bigcup_{i=1}^{+\infty} B_i^j, \quad \text{diam } B_i^j \leq \frac{1}{j},$$

and

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(B_i^j) \leq \mathcal{L}^n(A) + \frac{1}{j}.$$

Define now $g_i^j : \mathbb{R}^m \rightarrow \mathbb{R}$ by

$$g_i^j(x) := \alpha(n-m) \left(\frac{\text{diam } B_i^j}{2} \right)^{n-m} \mathbb{1}_{f(B_i^j)}(x).$$

By assertion (i) of Lemma (3.3.2), g_i^j is \mathcal{L}^m -measurable. Note also that for all $y \in \mathbb{R}^m$,

$$\mathcal{H}_{1/j}^{n-m}(A \cap f^{-1}(y)) \leq \sum_{i=1}^{+\infty} g_i^j(y).$$

Indeed, recall that

$$\mathcal{H}_{1/j}^{n-m}(A \cap f^{-1}(y)) = \inf \left\{ \sum_{i=1}^{+\infty} \frac{\alpha(n-m)}{2^{n-m}} (\text{diam } C_i)^{n-m} : A \cap f^{-1}(y) \subseteq \bigcup_{i=1}^{+\infty} C_i, \text{diam } C_i \leq \frac{1}{j} \right\}.$$

On the other hand,

$$g_i^j(y) = \begin{cases} \frac{\alpha(n-m)}{2^{n-m}} (\text{diam } B_i^j)^{n-m}, & y \in f^{-1}(B_i^j), \\ 0, & \text{otherwise.} \end{cases}$$

Now since $\text{diam } B_i^j \leq \frac{1}{j}$ and $A \subset \bigcup_{j=1}^{+\infty} B_i^j$, $\sum_{j=1}^{+\infty} g_i^j(y)$ is contained in the set of series the infimum is taken over. Thus using Fatou's Lemma and the Isodiametric Inequality (cf. Theorem (2.2.1)), we calculate

$$\begin{aligned} & \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^n(y) \\ &= \int_{\mathbb{R}^m} \lim_{j \rightarrow +\infty} \mathcal{H}_{1/j}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y) \\ &\leq \int_{\mathbb{R}^m} \liminf_{j \rightarrow +\infty} \sum_{i=1}^{+\infty} g_i^j(y) d\mathcal{L}^m(y) \\ &\stackrel{F.L.}{\leq} \liminf_{j \rightarrow +\infty} \sum_{i=1}^{+\infty} \int_{\mathbb{R}^m} g_i^j(y) d\mathcal{L}^m(y) \end{aligned}$$

$$\begin{aligned}
&= \liminf_{j \rightarrow +\infty} \sum_{i=1}^{+\infty} \alpha(n-m) \left(\frac{\text{diam } B_i^j}{2} \right)^{n-m} \mathcal{L}^m(f(B_i^j)) \\
&\leq \liminf_{j \rightarrow +\infty} \sum_{i=1}^{+\infty} \alpha(n-m) \left(\frac{\text{diam } B_i^j}{2} \right)^{n-m} \alpha(m) \left(\frac{\text{diam } f(B_i^j)}{2} \right)^m \\
&= \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} \liminf_{j \rightarrow +\infty} \sum_{i=1}^{+\infty} \left(\frac{\text{diam } f(B_i^j)}{\text{diam } B_i^j} \right)^m \alpha(n) \left(\frac{\text{diam } B_i^j}{2} \right)^n \\
&\leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\text{Lip } f)^m \liminf_{j \rightarrow +\infty} \sum_{i=1}^{+\infty} \mathcal{L}^n(B_i^j) \\
&\leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\text{Lip } f)^m \mathcal{L}^n(A).
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y) \leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\text{Lip } f)^m \mathcal{L}^n(A). \quad (3.4.2)$$

{eq:3.4-2}

This will prove assertion (iv) once we establish (ii) and (iii).

(iii). *Case #1: A is compact.*

Fix $t \geq 0$, and for each positive integer i , let U_i be the set of all points $y \in \mathbb{R}^m$ for which there exist finitely many open sets S_1, \dots, S_l such that

$$\begin{cases} A \cap f^{-1}(y) \subset \bigcup_{j=1}^l S_j, \\ \text{diam } S_j \leq \frac{1}{i}, \quad j = 1, 2, \dots, l, \\ \sum_{j=1}^l \alpha(n-m) \left(\frac{\text{diam } S_j}{2} \right)^{n-m} \leq t + \frac{1}{i}. \end{cases}$$

(iv). We claim that U_i is open. To see this, assume that $y \in U_i$. Then $A \cap f^{-1}(y) \subset \bigcup_{j=1}^l S_j$, as above. Then since f is continuous and A is compact,

$$A \cap f^{-1}(z) \subset \bigcup_{j=1}^l S_j$$

for all z sufficiently close to y .

(v). We next claim that

$$\{y \in \mathbb{R}^m : \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t\} = \bigcap_{i=1}^{+\infty} U_i,$$

and hence the LHS is a Borel set.

Let $\mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t$. Then for each $\delta > 0$,

$$\mathcal{H}_\delta^{n-m}(A \cap f^{-1}(y)) \leq t.$$

Given i , choose $\delta \in (0, \frac{1}{i})$. Then there exist sets $\{S_j\}_{j=1}^{+\infty}$ such that

$$\begin{cases} A \cap f^{-1}(y) \subset \bigcup_{j=1}^{+\infty} S_j, \\ \text{diam } S_j \leq \delta < \frac{1}{i}, \\ \sum_{j=1}^{+\infty} \alpha(n-m) \left(\frac{\text{diam } S_j}{2} \right)^{n-m} < t + \frac{1}{i}. \end{cases}$$

We may assume that the sets S_j , $j = 1, 2, \dots$, are open. Since $A \cap f^{-1}(y)$ is compact, a finite subcollection $\{S_1, \dots, S_l\}$ covers $A \cap f^{-1}(y)$, and hence $y \in U_i$. We may apply the same argument for each $i = 1, 2, \dots$, and thus

$$\{y \in \mathbb{R}^m : \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t\} \subset \bigcap_{i=1}^{+\infty} U_i.$$

Now let $y \in \bigcap_{i=1}^{+\infty} U_i$. Then for each i ,

$$\begin{aligned} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) &\leq \mathcal{H}_{1/i}^{n-m} \left(\bigcup_{j=1}^l S_j \right) \\ &\leq t + \frac{1}{i}, \end{aligned}$$

and so

$$\mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t.$$

Therefore

$$\{y \in \mathbb{R}^m : \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \leq t\} = \bigcap_{i=1}^{+\infty} U_i,$$

as required.

(vi). In view of (v), for compact sets A , the mapping

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is Borel measurable, and thus \mathcal{H}^{n-m} -measurable.

(vii). *Case #2: A is open.*

If A is open, there exist compact sets $K_1 \subset K_2 \subset \dots \subset A$ such that

$$A = \bigcup_{i=1}^{+\infty} K_i.$$

This is an increasing sequence, and so for each $y \in \mathbb{R}^m$,

$$\mathcal{H}^{n-m}(A \cap f^{-1}(y)) = \lim_{i \rightarrow +\infty} \mathcal{H}^{n-m}(K_i \cap f^{-1}(y)).$$

Thus the mapping

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is Borel measurable, as needed.

(viii). *Case #3: $\mathcal{L}^n(A) < +\infty$.*

In this case there exist open sets $V_1 \supset V_2 \supset \dots \supset A$ such that

$$\lim_{i \rightarrow +\infty} \mathcal{L}^n(V_i \setminus A) = 0, \quad \mathcal{L}^n(V_1) < +\infty.$$

Now

$$\begin{aligned}\mathcal{H}^{n-m}(V_i \cap f^{-1}(y)) &= \mathcal{H}^{n-m}((A \cup (V_i \setminus A)) \cap f^{-1}(y)) \\ &\leq \mathcal{H}^{n-m}(A \cap f^{-1}(y)) + \mathcal{H}^{n-m}((V_i \setminus A) \cap f^{-1}(y)),\end{aligned}$$

and thus by ^{eq:3.4-2}(3.4.2),

$$\begin{aligned}\limsup_{i \rightarrow +\infty} \int_{\mathbb{R}^m} |\mathcal{H}^{n-m}(V_i \cap f^{-1}(y)) - \mathcal{H}^{n-m}(A \cap f^{-1}(y))| d\mathcal{L}^m(y) \\ \leq \limsup_{i \rightarrow +\infty} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}((V_i \setminus A) \cap f^{-1}(y)) d\mathcal{L}^m(y) \\ \leq \limsup_{i \rightarrow +\infty} \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\text{Lip } f)^m \mathcal{L}^n(V_i \setminus A) \\ = 0.\end{aligned}$$

Consequently

$$\mathcal{H}^{n-m}(V_i \cap f^{-1}(y)) \rightarrow \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$, and so according to (vii), the mapping

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is \mathcal{L}^m -measurable, being the pointwise a.e. limit of the mappings

$$y \mapsto \mathcal{H}^{n-m}(V_i \cap f^{-1}(y)).$$

In addition, we see that $\mathcal{H}^{n-m}((V_i \setminus A) \cap f^{-1}(y)) \rightarrow 0$ for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$, and so $A \cap f^{-1}(y)$ is \mathcal{H}^{n-m} measurable for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$.

(ix). *Case #4: $\mathcal{L}^n(A) = +\infty$.*

In this case we may write A as a union of an increasing sequence of bounded \mathcal{L}^n -measurable sets and apply (viii) to prove that

$$A \cap f^{-1}(y) \text{ is } \mathcal{H}^{n-m} \text{ - measurable for } \mathcal{L}^m \text{ - a.e. } y \in \mathbb{R}^m,$$

and

$$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(y))$$

is \mathcal{L}^m -measurable.

(x). Parts (iii) through (ix) prove assertions (ii) and (iii) of the theorem. In view of ^{eq:3.4-2}(3.4.2), this proves assertion (iv) as well. The proof is complete. \square

Remark. A proof similar to that of assertion (iv) of Lemma ^{13.4-2}(3.4.2) shows that

$$\int_{\mathbb{R}^m} \mathcal{H}^k(A \cap f^{-1}(y)) d\mathcal{H}^l(y) \leq \frac{\alpha(k)\alpha(l)}{\alpha(k+l)} (\text{Lip } f)^l \mathcal{H}^{k+l}(A)$$

for each $A \subseteq \mathbb{R}^m$.

13.4-3

Lemma 3.4.3. Let $t > 1$, assume that $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz, and set

$$B := \{x \in \mathbb{R}^n : Dg(x) \text{ exists, } Jg(x) > 0\}.$$

Then there exists a countable collection $\{D_k\}_{k=1}^{+\infty}$ of Borel subsets of \mathbb{R}^n such that

- (i) $\mathcal{L}^n(B \setminus \cup_{k=1}^{+\infty} D_k) = 0$;
- (ii) $g|_{D_k}$ is one-to-one for $k = 1, 2, \dots$;

(iii) For each $k = 1, 2, \dots$, there exists a symmetric automorphism $S_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\begin{aligned} \text{Lip}(S_k^{-1} \circ (g|_{D_k})) &\leq t, & \text{Lip}((g|_{D_k})^{-1} \circ S_k) &\leq t, \\ t^{-n} |\det S_k| &\leq Jg|_{D_k} \leq t^n |\det S_k|. \end{aligned}$$

Proof.

(i). We may apply Lemma (3.3.3) (cf. §3.3) to find Borel sets $\{E_k\}_{k=1}^{+\infty}$ and symmetric automorphisms $T_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

- (i) $B = \cup_{k=1}^{+\infty} E_k$,
- (ii) $g|_{E_k}$ is one-to-one,
- (iii)

$$\begin{cases} \text{Lip}((g|_{E_k}) \circ T_k^{-1}) \leq t, & \text{Lip}(T_k \circ (g|_{E_k})^{-1}) \leq t, \\ t^{-n} |\det T_k| \leq Jg|_{E_k} \leq t^n |\det T_k|, & k = 1, 2, \dots \end{cases}$$

By (iii), $(g|_{E_k})^{-1}$ is Lipschitz and thus by Theorem (3.1.1) (cf. §3.1.1, extension of Lipschitz functions) there exists a Lipschitz mapping $g_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $g_k = (h|_{E_k})^{-1}$ on $g(E_k)$.

(ii). We claim that $Jg_k > 0$ \mathcal{L}^n -a.e. on $g(E_k)$. To see this, first note that since $g_k \circ g(x) = x$ for $x \in E_k$, Corollary (3.1.1) (cf. §3.1.2) implies

$$Dg_k(g(x)) \circ Dg(x) = I, \quad \mathcal{L}^n - \text{a.e. on } E_k,$$

and so

$$Jg_k(g(x))Jg(x) = 1 \quad \mathcal{L}^n - \text{a.e. on } E_k.$$

In view of (iii), this implies $Jg_k(g(x)) > 0$ for \mathcal{L}^n -a.e. $x \in E_k$, and (ii) follows because g is Lipschitz.

(iii). Now applying Lemma (3.3.3) (cf. §3.3) to g_k , there exist Borel sets $\{F_j^k\}_{j=1}^{+\infty}$ and symmetric automorphisms $\{R_j^k\}_{j=1}^{+\infty}$ such that

- (i) $\mathcal{L}^n(g(E_k) \setminus \cup_{j=1}^{+\infty} F_j^k) = 0$,
- (ii) $g_k|_{F_j^k}$ is one-to-one,
- (iii)

$$\begin{cases} \text{Lip}((g_k|_{F_j^k}) \circ (R_j^k)^{-1}) \leq t, & \text{Lip}(R_j^k \circ (g_k|_{F_j^k})^{-1}) \leq t, \\ t^{-n} |\det R_j^k| \leq Jg_k|_{F_j^k} \leq t^n |\det R_j^k|, & k = 1, 2, \dots \end{cases}$$

Put

$$D_j^k := E_k \cap g^{-1}(F_j^k), \quad S_j^k := (R_j^k)^{-1}, \quad k = 1, 2, \dots$$

(iv). We next claim that $\mathcal{L}^n(B \setminus \cup_{k,j=1}^{+\infty} D_j^k) = 0$. Note that

$$\begin{aligned} g_k \left(g(E_k) \setminus \bigcup_{j=1}^{+\infty} F_j^k \right) &= g^{-1} \left(g(E_k) \setminus \bigcup_{j=1}^{+\infty} F_j^k \right) \\ &= E_k \setminus \bigcup_{j=1}^{+\infty} D_j^k. \end{aligned}$$

Thus, by (i) and the fact that the image of a set of Lebesgue measure zero has Lebesgue measure zero,

$$\mathcal{L}^n \left(E_k \setminus \bigcup_{j=1}^{+\infty} D_j^k \right) = 0, \quad k = 1, 2, \dots$$

By (i) in part (i), this proves (iv).

(v). Clearly (ii) in part (i) implies that $g|_{D_j^k}$ is one-to-one, for $D_j^k \subseteq E_k$, $k = 1, 2, \dots$

(vi). We lastly claim that for $k, j = 1, 2, \dots$, we have

$$\begin{aligned} \text{Lip}((S_j^k)^{-1} \circ (g|_{D_j^k})) &\leq t, \quad \text{Lip}((g|_{D_j^k})^{-1} \circ S_j^k) \leq t, \\ t^{-n} |\det S_j^k| &\leq Jg|_{D_j^k} \leq t^n |\det S_j^k|. \end{aligned}$$

Observe that

$$\begin{aligned} \text{Lip}((S_j^k)^{-1} \circ (g|_{D_j^k})) &= \text{Lip}(R_j^k \circ (g|_{D_j^k})) \\ &\leq \text{Lip}(R_j^k \circ (g_k|_{F_j^k})^{-1}) \\ &\leq t, \end{aligned}$$

because $D_j^k \subseteq g^{-1}(F_j^k)$. Similarly

$$\begin{aligned} \text{Lip}((g|_{D_j^k})^{-1} \circ S_j^k) &= \text{Lip}((g|_{D_j^k})^{-1} \circ (R_j^k)^{-1}) \\ &\leq \text{Lip}((g_k|_{F_j^k}) \circ (R_j^k)^{-1}) \\ &\leq t. \end{aligned}$$

Moreover, as noted above,

$$Jg_k(g(x))Jg(x) = 1 \quad \mathcal{L}^n - \text{a.e. on } D_j^k.$$

Thus (iii) in part (iii) of the proof implies

$$t^{-n} |\det S_j^k| = t^{-n} |\det R_j^k|^{-1} \leq Jg|_{D_j^k} \leq t^n |\det R_j^k|^{-1} = t^n |\det S_j^k|,$$

as required. The proof is complete. \square

3.4.2. Proof of the Coarea Formula.

t3.4-1

Theorem 3.4.1 (Coarea Formula). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then for each \mathcal{L}^n -measurable set $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf(x) d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y).$$

Remark.

- (i) The Coarea Formula allows us to integrate $Jf(x)$ over A by integrating the $(n-m)$ -dimensional Hausdorff measure of the fibers of f .
- (ii) Observe that the Coarea Formula is a kind of “curvilinear” generalization of Fubini’s Theorem.
- (iii) Applying the Coarea Formula to $A := \{x \in \mathbb{R}^n : Jf(x) = 0\}$, we find

$$\mathcal{H}^{n-m}(\{x \in \mathbb{R}^n : Jf(x) = 0\} \cap f^{-1}(y)) = 0 \tag{3.4.3}$$

for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$. This is a weak variant of the Morse–Sard Theorem, which asserts

$$\{x \in \mathbb{R}^n : Jf(x) = 0\} \cap f^{-1}(y) = \emptyset$$

{eq:3.4-3}

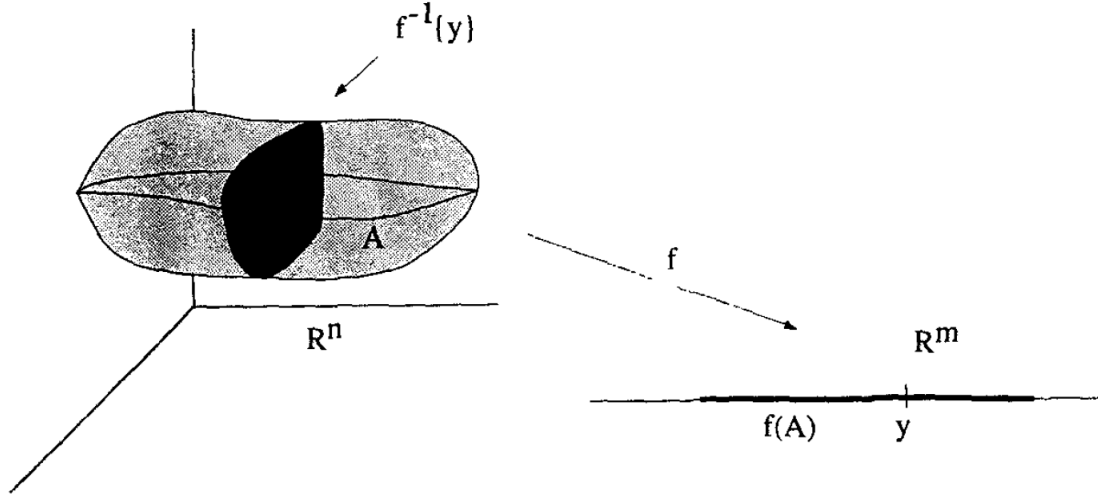


FIGURE 3.4.1. The Coarea Formula.

for \mathcal{L}^m -a.e. $y \in \mathbb{R}^m$, provided that $f \in C^k(\mathbb{R}^n; \mathbb{R}^m)$, for

$$k = 1 + n - m.$$

On the other hand, ^{leq: 3.4-3}(3.4.3) required only that f is Lipschitz.

Proof.

(i). By Rademacher's Theorem (cf. Theorem ^{§3.1-2}(3.1.2)) and Lemma ^{§3.4-2}(3.4.2), we may assume that $Df(x)$, and thus $Jf(x)$, exist for all $x \in A$ and that $\mathcal{L}^n(A) < +\infty$.

(ii). *Case #1:* $A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$.

For each $\lambda \in \Lambda(n, n-m)$, write

$$f := q \circ h_\lambda,$$

where

$$h_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}, \quad h_\lambda(x) := (f(x), P_\lambda(x)), \quad x \in \mathbb{R}^n,$$

and

$$q : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m, \quad q(y, z) := y, \quad y \in \mathbb{R}^m, z \in \mathbb{R}^{n-m},$$

and P_λ is the projection

$$P_\lambda(x_1, \dots, x_n) := (x_{\lambda(1)}, \dots, x_{\lambda(n-m)})$$

(cf. §3.2.1). Set

$$\begin{aligned} A_\lambda &:= \{x \in A : \det Dh_\lambda \neq 0\} \\ &= \{x \in A : P_\lambda|_{[Df(x)]^{-1}(0)} \text{ is injective}\}. \end{aligned}$$

Now

$$A = \bigcup_{\lambda \in \Lambda(n, n-m)} A_\lambda,$$

and therefore we may as well for simplicity assume that $A = A_\lambda$ for some $\lambda \in \Lambda(n, n-m)$.

(iii). Fix $t > 1$. Applying Lemma ^{§3.4-3}(3.4.3) to $h := h_\lambda$, we obtain disjoint Borel sets $\{D_k\}_{k=1}^{+\infty}$ and symmetric automorphisms $\{S_k\}_{k=1}^{+\infty}$ such that

(i) $\mathcal{L}^n(A \setminus \bigcup_{k=1}^{+\infty} D_k) = 0$;

- (ii) $h|_{D_k}$ is one-to-one for $k = 1, 2, \dots$;
- (iii) For each $k = 1, 2, \dots$,

$$\begin{aligned} \text{Lip}(S_k^{-1} \circ (h|_{D_k})) &\leq t, & \text{Lip}((h|_{D_k})^{-1} \circ S_k) &\leq t, \\ t^{-n} |\det S_k| &\leq Jh_{D_k} \leq t^n |\det S_k|. \end{aligned}$$

Set $G_k := A \cap D_k$.

(iv). We claim that

$$t^{-n} \llbracket q \circ S_k \rrbracket \leq Jf|_{G_k} \leq t^n \llbracket q \circ S_k \rrbracket.$$

To see this, first note that since $f = q \circ h$, we have \mathcal{L}^n -a.e. that

$$\begin{aligned} Df &= Dq(h) \cdot Dh = q \circ Dh \\ &= q \circ S_k \circ S_k^{-1} \circ Dh \\ &= q \circ S_k \circ D(S_k^{-1} \circ h) \\ &= q \circ S_k \circ C, \end{aligned}$$

where $C := D(S_k^{-1} \circ h)$.
Thus by Lemma (3.4.3),

$$t^{-1} \leq \text{Lip}(S_k^{-1} \circ h) = \text{Lip}(C) \leq t \quad \text{on } G_k. \quad (3.4.4)$$

Now write

$$\begin{aligned} Df &:= S \circ O^*, \\ q \circ S_k &:= T \circ P^* \end{aligned}$$

for symmetric maps $S, T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and orthogonal maps $O, P : \mathbb{R}^m \rightarrow \mathbb{R}^n$ (cf. Theorem (3.2.2)).

We have then

$$S \circ O^* = T \circ P^* \circ C. \quad (3.4.5)$$

Consequently

$$S = T \circ P^* \circ C \circ O.$$

Since $G_k \subset A \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$, $\det S \neq 0$ and thus $\det T \neq 0$.

Thus if $v \in \mathbb{R}^m$, we have by (3.4.4)

$$\begin{aligned} |T^{-1} \circ Sv| &= |T^{-1} \circ T \circ P^* \circ C \circ Ov| \\ &= |P^* \circ C \circ Ov| \\ &\leq |C \circ Ov| \\ &\leq t|Ov| \\ &= t|v|. \end{aligned}$$

Therefore

$$(T^{-1} \circ S)(B(0, 1)) \subset B(0, t),$$

and so

$$Jf = |\det S| \leq t^n |\det T| = t^n \llbracket q \circ S_k \rrbracket.$$

Similarly, if $v \in \mathbb{R}^m$, we have by (3.4.5) and (3.4.4)

$$\begin{aligned} |S^{-1} \circ Tv| &= |O^* \circ C^{-1} \circ P \circ T^{-1} \circ Tv| \\ &= |O^* \circ C^{-1} \circ Pv| \end{aligned}$$

$$\begin{aligned}
&\leq |C^{-1} \circ Pv| \\
&\leq t|Pv| \\
&= t|v|.
\end{aligned}$$

Thus

$$(S^{-1} \circ T)(B(0, 1)) \subset B(0, t),$$

so evidently

$$\llbracket q \circ S_k \rrbracket = |\det T| \leq t^n |\det S| = t^n Jf.$$

This establishes the claim.

(v). We now calculate by Lemmas (3.4.1) and (3.4.3) and Theorem (2.4.1)

$$\begin{aligned}
&t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(G_k \cap f^{-1}(y)) \, d\mathcal{L}^m(y) \\
&= t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(h^{-1}(h(G_k) \cap q^{-1}(y))) \, d\mathcal{L}^m(y) \\
&\leq t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(S_k^{-1}(h(G_k) \cap q^{-1}(y))) \, d\mathcal{L}^m(y) \\
&= t^{-2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(S_k^{-1} \circ h(G_k) \cap (q \circ S_k)^{-1}(y)) \, d\mathcal{L}^m(y) \\
&= t^{-2n} \llbracket q \circ S_k \rrbracket \mathcal{L}^n(S_k^{-1} \circ h(G_k)) \\
&\leq t^{-n} \llbracket q \circ S_k \rrbracket \mathcal{L}^n(G_k) \\
&= \int_{G_k} t^{-n} \llbracket q \circ S_k \rrbracket \, d\mathcal{L}^n(x) \\
&\leq \int_{G_k} Jf(x) \, d\mathcal{L}^n(x) \\
&\leq \int_{G_k} t^n \llbracket q \circ S_k \rrbracket \, d\mathcal{L}^n(x) \\
&= t^n \llbracket q \circ S_k \rrbracket \mathcal{L}^n(G_k) \\
&\leq t^{2n} \llbracket q \circ S_k \rrbracket \mathcal{L}^n(S_k^{-1} \circ h(G_k)) \\
&= t^{2n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(S_k^{-1} \circ h(G_k) \cap (q \circ S_k)^{-1}(y)) \, d\mathcal{L}^m(y) \\
&\leq t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(h^{-1}(h(G_k) \cap q^{-1}(y))) \, d\mathcal{L}^m(y) \\
&= t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(G_k \cap f^{-1}(y)) \, d\mathcal{L}^m(y).
\end{aligned}$$

Since

$$\mathcal{L}^n \left(A \setminus \bigcup_{k=1}^{+\infty} G_k \right) = 0,$$

we may sum on k to obtain

$$t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \, d\mathcal{L}^m(y) \leq \int_A Jf(x) \, d\mathcal{L}^n(x)$$

$$\leq t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y).$$

Letting $t \rightarrow 1^+$, we conclude that

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y) = \int_A Jf(x) d\mathcal{L}^n(x),$$

which completes the proof for this case.

(vi). *Case #2:* $A \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$.

In this case fix $\epsilon > 0$ and define

$$\begin{aligned} g : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^m, & g(x, y) &:= f(x) + \epsilon y, \\ p : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^m, & p(x, y) &:= y, \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m. \end{aligned}$$

We claim that there exists a constant $C > 0$ such that

$$0 < Jg(x) \leq C\epsilon$$

for all $x \in A$. Notice that

$$Dg(x) = (Df(x), \epsilon I).$$

By the Binet–Cauchy Formula (cf. (3.2.4)), $Jg(x)^2$ equals the sum of squares of all $(m \times m)$ subdeterminants of $Dg(x)$, so

$$Jg(x)^2 \geq \epsilon^{2m} > 0.$$

Moreover, since $|Df| \leq \text{Lip}(f) < +\infty$, the Binet–Cauchy formula also gives

$$Jg(x) = Jf(x)^2 + \{\text{sum of squares of terms involving at least one } \epsilon\} \leq C\epsilon^2$$

for each $x \in A$. Thus

$$\epsilon^m \leq Jg = \|Dg\| \leq C\epsilon.$$

(vii). Observe that

$$\begin{aligned} &\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) d\mathcal{L}^m(y) \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y - \epsilon w)) d\mathcal{L}^m(y) \quad \text{for all } w \in \mathbb{R}^m \\ &= \frac{1}{\alpha(m)} \int_{B(0,1)} \int_{\mathbb{R}^m} \mathcal{H}^{n-1}(A \cap f^{-1}(y - \epsilon w)) d\mathcal{L}^m(y) d\mathcal{L}^m(w). \end{aligned}$$

(viii). Fix $y, w \in \mathbb{R}^m$, and set $B := A \times B(0, 1) \subset \mathbb{R}^{n+m}$. We claim that

$$B \cap g^{-1}(y) \cap p^{-1}(w) = \begin{cases} \emptyset, & w \notin B(0, 1), \\ (A \cap f^{-1}(y - \epsilon w)) \times \{w\}, & w \in B(0, 1). \end{cases}$$

To see this, note that we have $(x, z) \in B \cap g^{-1}(y) \cap p^{-1}(w)$ if and only if

$$x \in A, \quad z \in B(0, 1), \quad f(x) + \epsilon z = y, \quad z = w.$$

Moreover, this holds if and only if

$$x \in A, \quad z = w \in B(0, 1), \quad f(x) = y - \epsilon w.$$

Finally, the above holds if and only if

$$w \in B(0, 1), \quad (x, z) \in (A \cap f^{-1}(y - \epsilon w)) \times \{w\}.$$

This proves (viii).

(ix). We use (viii) to continue the calculation from (vii), and obtain by Lemma (3.4.2)^[13.4-2] and Case #1

$$\begin{aligned}
& \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \, d\mathcal{L}^m(y) \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(B \cap g^{-1}(y) \cap p^{-1}(w)) \, d\mathcal{L}^m(w) \, d\mathcal{L}^m(y) \\
&\leq \frac{1}{\alpha(m)} \frac{\alpha(m)\alpha(n-m)}{\alpha(n)} (\text{Lip } p)^m \int_{\mathbb{R}^m} \mathcal{H}^n(B \cap g^{-1}(y)) \, d\mathcal{L}^m(y) \\
&= \frac{\alpha(n-m)}{\alpha(n)} \int_{\mathbb{R}^m} \mathcal{H}^n(B \cap g^{-1}(y)) \, d\mathcal{L}^m(y) \\
&= \frac{\alpha(n-m)}{\alpha(n)} \int_B Jg(x, z) \, d\mathcal{L}^n(x) \, d\mathcal{L}^m(z) \\
&\leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} \mathcal{L}^n(A) \sup_B Jg(x, z) \\
&\leq C \mathcal{L}^n(A) \epsilon.
\end{aligned}$$

Letting $\epsilon \rightarrow 0$, we obtain

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y)) \, d\mathcal{L}^m(y) = 0 = \int_A Jf(x) \, d\mathcal{L}^n(x),$$

as required.

(x). In the general case we write $A := A_1 \cup A_2$, where $A_1 \subset \{x \in \mathbb{R}^n : Jf(x) > 0\}$ and $A_2 \subset \{x \in \mathbb{R}^n : Jf(x) = 0\}$, and apply Cases #1 and #2 above. The proof is complete. \square

3.4.3. Change of Variables Formula.

t3.4-2

Theorem 3.4.2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then for each \mathcal{L}^n -integrable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$g|_{f^{-1}(y)} \text{ is } \mathcal{H}^{n-m} \text{--integrable for } \mathcal{L}^m \text{--a.e. } y \in \mathbb{R}^m,$$

and

$$\int_{\mathbb{R}^n} g(x) Jf(x) \, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\int_{f^{-1}(y)} g \, d\mathcal{H}^{n-m} \right] \, d\mathcal{L}^m(y).$$

Proof.

(i). Case #1: $g \geq 0$.

Define the sequence $\{s_j\}_{j=1}^{+\infty}$ by

$$s_j(x) := \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) + j \mathbb{1}_{g^{-1}[j, +\infty)}(x).$$

Recall that $s_j \rightarrow g$ as $j \rightarrow +\infty$ and

$$0 \leq s_1 \leq s_2 \leq \cdots$$

Hence, by the Monotone Convergence Theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x) Jf(x) d\mathcal{L}^n(x) &= \int_{\mathbb{R}^n} \lim_{j \rightarrow +\infty} s_j(x) Jf(x) d\mathcal{L}^n(x) \\
&\stackrel{MCT}{=} \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} s_n(x) Jf(x) d\mathcal{L}^n(x) \\
&= \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \left(\sum_{k=0}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) + j \mathbb{1}_{g^{-1}[j, +\infty)}(x) \right) Jf(x) d\mathcal{L}^n(x) \\
&\stackrel{B.L.}{=} \lim_{j \rightarrow +\infty} \sum_{k=0}^{j2^j} \frac{k}{2^j} \int_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})} Jf(x) d\mathcal{L}^n(x) \\
&= \lim_{j \rightarrow +\infty} \sum_{k=0}^{j2^j} \frac{k}{2^j} \int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(g^{-1} \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right) \cap f^{-1}(y) \right) d\mathcal{L}^m(y) \\
&\stackrel{B.L.}{=} \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^m} \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathcal{H}^{n-m} \left(g^{-1} \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right) \cap f^{-1}(y) \right) d\mathcal{L}^m(y) \\
&\stackrel{MCT}{=} \int_{\mathbb{R}^m} \lim_{j \rightarrow +\infty} \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathcal{H}^{n-m} \left(g^{-1} \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right) \cap f^{-1}(y) \right) d\mathcal{L}^m(y) \\
&= \int_{\mathbb{R}^m} \left[\int_{f^{-1}(y)} \lim_{j \rightarrow +\infty} \sum_{k=0}^{j2^j} \frac{k}{2^j} \mathbb{1}_{g^{-1}[\frac{k}{2^j}, \frac{k+1}{2^j})}(x) d\mathcal{H}^{n-m}(x) \right] d\mathcal{L}^m(y) \\
&= \int_{\mathbb{R}^m} \left[\int_{f^{-1}(y)} g(x) d\mathcal{H}^{n-m}(x) \right] d\mathcal{L}^m(y),
\end{aligned}$$

as required.

(ii). *Case #2: g is any \mathcal{L}^n -integrable function.* In this case, write $g := g^+ - g^-$ and apply Case #1. The proof is complete. \square

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3.4.4. Applications.

p3.4-1

Proposition 3.4.1 (Polar Coordinates). *Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be \mathcal{L}^n -integrable. Then*

$$\int_{\mathbb{R}^n} g(x) d\mathcal{L}^n(x) = \int_0^{+\infty} \left[\int_{\partial B(0,r)} g(x) d\mathcal{H}^{n-1}(x) \right] dr.$$

In particular, we see that

$$\frac{d}{dr} \left[\int_{B(0,r)} g(x) d\mathcal{L}^n(x) \right] = \int_{\partial B(0,r)} g(x) d\mathcal{H}^{n-1}(x)$$

for \mathcal{L}^1 -a.e. $r > 0$.

Proof. Define $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(x) := |x|$. Then for all $x \neq 0$, we have

$$Df(x) = \frac{x}{|x|}, \quad Jf(x) = 1.$$

Thus the Change of Variables Formula (cf. ^{p3.4-2}(3.4.2)) gives

$$\begin{aligned} \int_{\mathbb{R}^n} g(x) d\mathcal{L}^n(x) &= \int_{\mathbb{R}} \left[\int_{f^{-1}(r)} g(x) d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(r) \\ &= \int_0^{+\infty} \left[\int_{\partial B(0,r)} g(x) d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(r), \end{aligned}$$

as required.

For the second assertion, observe first that

$$\int_{B(0,r)} g(x) d\mathcal{L}^n(x) = \int_0^r \left[\int_{\partial B(0,s)} g(x) d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s).$$

Hence, by the Fundamental Theorem of Calculus for Lebesgue Integrals,

$$\frac{d}{dr} \left(\int_{B(0,r)} g(x) d\mathcal{L}^n(x) \right) = \int_{\partial B(0,r)} g(x) d\mathcal{H}^{n-1}(x).$$

The proof is complete. □

p3.4-2

Proposition 3.4.2 (Level Sets). Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitz. Then

$$\int_{\mathbb{R}^n} |Df(x)| d\mathcal{L}^n(x) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\{f = t\}) d\mathcal{L}^1(t).$$

Proof. Noting that $Jf(x) = |Df(x)|$, we have directly by the Coarea Formula

$$\begin{aligned} \int_{\mathbb{R}^n} |Df(x)| d\mathcal{L}^n(x) &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(f^{-1}(t)) d\mathcal{L}^1(t) \\ &= \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\{f = t\}) d\mathcal{L}^1(t). \end{aligned}$$

The proof is complete. □

Remark. Compare Proposition ^{p3.4-2}(3.4.2) with the Coarea Formula for BV functions which will be proved in §5.5.

p3.4-3

Proposition 3.4.3 (Level Sets). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be Lipschitz, with

$$\operatorname{ess\,inf}_{x \in \mathbb{R}^n} |Df(x)| > 0.$$

Suppose also that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is \mathcal{L}^n -integrable. Then

$$\int_{\{f > t\}} g(x) d\mathcal{L}^n(x) = \int_t^{+\infty} \left[\int_{\{f=s\}} \frac{g(x)}{|Df(x)|} d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s).$$

In particular, we see that

$$\frac{d}{dt} \left[\int_{\{f > t\}} g(x) d\mathcal{L}^n(x) \right] = - \int_{\{f=t\}} \frac{g(x)}{|Df(x)|} d\mathcal{H}^{n-1}(x).$$

Proof. Again recall that $Jf(x) = |Df(x)|$. Write $E_t := \{x \in \mathbb{R}^n : f(x) > t\}$. By the Change of Variables Formula (cf. (3.4.2)), we have

$$\begin{aligned}
 \int_{\{f>t\}} g(x) d\mathcal{L}^n(x) &= \int_{\mathbb{R}^n} \frac{g(x)}{|Df(x)|} \mathbb{1}_{E_t}(x) Jf(x) d\mathcal{L}^n(x) \\
 &= \int_{\mathbb{R}} \left[\int_{f^{-1}(s)} \frac{g(x)}{|Df(x)|} \mathbb{1}_{E_t}(x) d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s) \\
 &= \int_{-\infty}^{+\infty} \left[\int_{\{f=s\}} \frac{g(x)}{|Df(x)|} \mathbb{1}_{E_t}(x) d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s) \\
 &= \int_t^{+\infty} \left[\int_{\{f=s\}} \frac{g(x)}{|Df(x)|} d\mathcal{H}^{n-1}(x) \right] d\mathcal{L}^1(s),
 \end{aligned}$$

as required.

Applying the Fundamental Theorem for Lebesgue Integrals gives

$$\frac{d}{dt} \left[\int_{\{f>t\}} g(x) d\mathcal{L}^n(x) \right] = - \int_{\{f>t\}} \frac{g(x)}{|Df(x)|} d\mathcal{H}^{n-1}(x).$$

The proof is complete. □

4. BV FUNCTIONS AND SETS OF FINITE PERIMETER

Throughout this chapter, Ω will denote an open subset of \mathbb{R}^n .

4.1. Definitions; Structure Theorem.

Definition 4.1.1 (Bounded Variation). *A function $f \in L^1(\Omega)$ is said to have **bounded variation** in Ω if*

$$\sup \left\{ \int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n(x) : \phi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^n), |\phi| \leq 1 \right\} < +\infty.$$

We will write

$$BV(\Omega)$$

to denote the space of functions of bounded variation on Ω .

Definition 4.1.2 (Finite Perimeter). *An \mathcal{L}^n -measurable subset $E \subset \mathbb{R}^n$ is said to have **finite perimeter** in Ω if*

$$\mathbb{1}_E \in BV(\Omega).$$

We also introduce the local versions of the above concepts.

Definition 4.1.3 (Locally Bounded Variation). *A function $f \in \mathcal{L}_{\text{loc}}^1(\Omega)$ is said to have **locally bounded variation** in Ω if for each open set $U \subset\subset \Omega$,*

$$\sup \left\{ \int_U f \operatorname{div} \phi \, d\mathcal{L}^n(x) : \phi \in \mathcal{C}_c^\infty(U; \mathbb{R}^n), |\phi| \leq 1 \right\} < +\infty.$$

We will write

$$BV_{\text{loc}}(\Omega)$$

to denote the space of functions of locally bounded variation on Ω .

Definition 4.1.4 (Locally Finite Perimeter). *An \mathcal{L}^n -measurable subset $E \subset \mathbb{R}^n$ is said to have **locally finite perimeter** in Ω if*

$$\mathbb{1}_E \in BV_{\text{loc}}(\Omega).$$

We now present the BV Structure Theorem, which asserts that the weak first partial derivatives of a function $f \in BV(\Omega)$ are Radon measures.

5.1-1

Theorem 4.1.1 (Structure Theorem for BV_{loc} Functions). *Let $f \in BV_{\text{loc}}(\Omega)$. Then there exists a Radon measure μ on Ω and a μ -measurable function $\sigma : \Omega \rightarrow \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ for μ -a.e. $x \in \Omega$;

(ii) $\int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n(x) = - \int_{\Omega} \phi \cdot \sigma \, d\mu$

for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$.

Proof. Define the linear functional

$$L : \mathcal{C}_c^1(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}$$

by

$$L(\phi) := - \int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n(x)$$

for $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. Since $f \in BV_{\text{loc}}(\Omega)$, we have for each open set $U \subset\subset \Omega$

$$\sup \{ L(\phi) : \phi \in \mathcal{C}_c^1(U; \mathbb{R}^n), |\phi| \leq 1 \} := C(U) < +\infty.$$

Thus

$$|L(\phi)| \leq C(U) \|\phi\|_{L^\infty(U)} \quad (4.1.1)$$

{eq:5.1-1}

for $\phi \in \mathcal{C}_c^1(U; \mathbb{R}^n)$.

Fix any compact set $K \subset \Omega$, and then choose an open set U such that $K \subset U \subset\subset \Omega$. For each $\phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n)$ with $\text{supp } \phi \subset K$, choose $\phi_k \in \mathcal{C}_c^1(U; \mathbb{R}^n)$, $k = 1, 2, \dots$, so that $\phi_k \rightarrow \phi$ uniformly on U . Define

$$\bar{L}(\phi) := \lim_{k \rightarrow +\infty} L(\phi_k).$$

By (4.1.1), \bar{L} is bounded, and thus the above limit exists and is independent of the choice of sequence $\{\phi_k\}_{k=1}^{+\infty}$ converging to ϕ . Since $\mathcal{C}_c^1(U; \mathbb{R}^n)$ is dense in

$$\{\phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n) : \text{supp } \phi \subset K\},$$

we have by the BLT Theorem that L uniquely extends to a bounded linear functional

$$\bar{L} : \mathcal{C}_c(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}$$

and

$$\sup \{\bar{L}(\phi) : \phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n) : |\phi| \leq 1, \text{supp } \phi \subset K\} < +\infty$$

for each compact set $K \subset \Omega$. By the Riesz Representation Theorem, there exists a Radon measure μ on Ω and a μ -measurable function $\sigma : \Omega \rightarrow \mathbb{R}^n$ such that

- (i) $|\sigma(x)| = 1$ for μ -a.e. $x \in \Omega$;
- (ii) $\bar{L}(\phi) = \int_{\Omega} \phi \cdot \sigma \, d\mu$

for all $\phi \in \mathcal{C}_c(\Omega; \mathbb{R}^n)$. Since \bar{L} is an extension of L from $\mathcal{C}_c^1(\Omega; \mathbb{R}^n)$ to $\mathcal{C}_c(\Omega; \mathbb{R}^n)$, it follows that $\bar{L}(\phi) = L(\phi)$ whenever $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. Hence,

$$\int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n(x) = -L(\phi) = - \int_{\Omega} \phi \cdot \sigma \, d\mu$$

for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. The proof is complete. □

Remark (Notation).

(i) If $f \in BV_{\text{loc}}(\Omega)$, we will write

$$\|Df\|$$

for the measure μ , and

$$[Df] := \|Df\| \llcorner \sigma,$$

where $[Df] = \|Df\| \llcorner \sigma$ denotes that $[Df]$ is the measure with density σ with respect to $\|Df\|$, that is,

$$[Df](K) = \int_K \sigma \, d\|Df\|$$

for all compact sets $K \subset \Omega$. Thus assertion (ii) in the Structure Theorem (4.1.1) reads

$$\int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n(x) = - \int_{\Omega} \phi \cdot \sigma \, d\|Df\| = - \int_{\Omega} \phi \cdot d[Df]$$

for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$. Note in particular that $\|Df\|, [Df]$ are vector-valued measures.

(ii) Similarly if $f = \mathbb{1}_E$, and E is a set of locally finite perimeter in Ω , we write

$$\|\partial E\|$$

for the measure μ , and

$$\nu_E := -\sigma.$$

Consequently the Structure Theorem gives

$$\int_E \operatorname{div} \phi \, d\mathcal{L}^n(x) = \int_{\Omega} \phi \cdot \nu_E \, d\|\partial E\|$$

for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$.

Remark (More Notation). If $f \in BV_{\text{loc}}(\Omega)$, we write

$$\mu^i := \|Df\| \llcorner \sigma^i, \quad i = 1, \dots, n$$

for $\sigma = (\sigma^1, \dots, \sigma^n)$. By Lebesgue's Decomposition Theorem, we may further set

$$\mu^i = \mu_{ac}^i + \mu_s^i,$$

where

$$\mu_{ac}^i \ll \mathcal{L}^n, \quad \mu_s^i \perp \mathcal{L}^n.$$

Then by the Radon–Nikodym Theorem,

$$\mu_{ac}^i = \mathcal{L}^n \llcorner f^i$$

for some function $f^i \in L^1_{\text{loc}}(\Omega)$, $i = 1, \dots, n$. Write

$$\begin{cases} \frac{\partial f}{\partial x_i} := f^i, & i = 1, \dots, n, \\ Df := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right), \\ [Df]_{ac} := (\mu_{ac}^1, \dots, \mu_{ac}^n) = \mathcal{L}^n \llcorner Df, \\ [Df]_s := (\mu_s^1, \dots, \mu_s^n). \end{cases}$$

Thus

$$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s,$$

so that $Df \in L^1_{\text{loc}}(\Omega; \mathbb{R}^n)$ is the density of the absolutely continuous part of $[Df]$.

Remark.

- (i) $\|Df\|$ is the **variation measure** of f , $\|\partial E\|$ is the **perimeter measure** of E , and $\|\partial E\|(\Omega)$ is the **perimeter** of E in Ω .
- (ii) If $f \in BV_{\text{loc}}(\Omega) \cap L^1(\Omega)$, then $f \in BV(\Omega)$ if and only if $\|Df\|(\Omega) < +\infty$. To see this, first let $f \in BV(\Omega)$. Then by the Structure Theorem

$$\begin{aligned} \|Df\|(\Omega) &= \int_{\Omega} \sigma \, d\|Df\| \\ &= \sup \left\{ \int_{\Omega} \phi \cdot \sigma \, d\|Df\| : \phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n), |\phi| \leq 1 \right\} \\ &= \sup \left\{ - \int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n : \phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n), |\phi| \leq 1 \right\} < +\infty \end{aligned}$$

Now if $\|Df\|(\Omega) < +\infty$, we have again by the Structure Theorem

$$\begin{aligned} - \int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n &= \int_{\Omega} \phi \cdot \sigma \, d\|Df\| \\ &\leq \int_{\Omega} d\|Df\| < +\infty, \end{aligned}$$

for all $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$ with $|\phi| \leq 1$, so that $f \in BV(\Omega)$.

In this case we define the BV norm of f by

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \|Df\|(\Omega).$$

(iii). From the proof of the Riesz Representation Theorem, we see that

$$\begin{aligned} \|Df\|(U) &= \sup \left\{ \int_U f \operatorname{div} \phi \, d\mathcal{L}^n : \phi \in \mathcal{C}_c^1(U; \mathbb{R}^n), |\phi| \leq 1 \right\}, \\ \|\partial E\|(U) &= \sup \left\{ \int_U \operatorname{div} \phi \, d\mathcal{L}^n : \phi \in \mathcal{C}_c^1(U; \mathbb{R}^n), |\phi| \leq 1 \right\} \end{aligned}$$

for each $U \subset\subset \Omega$. Here we have used the fact that the dual of the space of vector-valued Radon measures on Ω is $\mathcal{C}_c(\Omega; \mathbb{R}^n)$ along the Hahn–Banach Theorem and Structure Theorem.

Example 4.1.1. Let $f \in W_{\text{loc}}^{1,1}(\Omega)$. Then for each $U \subset\subset \Omega$ and $\phi \in \mathcal{C}_c^1(U; \mathbb{R}^n)$, with $|\phi| \leq 1$, we have

$$\int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n = - \int_{\Omega} Df \cdot \phi \, d\mathcal{L}^n \leq \int_U |Df| \, d\mathcal{L}^n < +\infty.$$

Furthermore, if we put

$$\|Df\| = \mathcal{L}^n \llcorner |Df|,$$

and

$$\sigma(x) = \begin{cases} \frac{Df(x)}{|Df(x)|}, & Df(x) \neq 0, \\ 0, & Df(x) = 0 \end{cases} \quad \mathcal{L}^n - a.e.,$$

we see that $\|Df\|$ is a Radon measure and $|\sigma(x)| = 1$ \mathcal{L}^n -a.e. Moreover

$$\begin{aligned} - \int_{\Omega} \phi \cdot \sigma \, d\|Df\| &= - \int_U \phi \cdot \frac{Df}{|Df|} |Df| \, d\mathcal{L}^n \\ &= \int_{\Omega} f \operatorname{div} \phi \, d\mathcal{L}^n \\ &= \int_U f \operatorname{div} \phi \, d\mathcal{L}^n. \end{aligned}$$

Hence

$$W_{\text{loc}}^{1,1}(\Omega) \subset BV_{\text{loc}}(\Omega),$$

and similarly

$$W^{1,1}(\Omega) \subset BV(\Omega).$$

In particular,

$$W_{\text{loc}}^{1,p}(\Omega) \subset BV_{\text{loc}}(\Omega)$$

for $1 \leq p \leq +\infty$, for $f \in W_{\text{loc}}^{1,p}(\Omega)$ implies $W_{\text{loc}}^{1,1}(\Omega)$. That is, each Sobolev function has locally bounded variation.

Example 4.1.2. Let $\Omega = \mathbb{R}^2$, and let $B = B(0, 1)$ be the open unit ball in \mathbb{R}^2 . Then the BV Structure Theorem gives for each $\phi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^n)$, $|\phi| \leq 1$,

$$\int_B \operatorname{div} \phi \, d\mathcal{L}^n = \int_\Omega \mathbb{1}_B \operatorname{div} \phi \, d\mathcal{L}^n = - \int_\Omega \phi \cdot \nu_B \, d\|\partial B\|.$$

On the other hand, by the Divergence Theorem, we obtain

$$\int_B \operatorname{div} \phi \, d\mathcal{L}^n = \int_{\partial B} \phi \cdot \nu \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial B) < +\infty,$$

where ν denotes the outward-pointing unit normal vector on ∂B . Hence B has finite perimeter in \mathbb{R}^2 . Moreover we see that if we put

$$\nu_B := \nu,$$

then evidently

$$\|\partial B\| = \mathcal{H}^{n-1} \llcorner \mathbb{1}_{\partial B} \nu.$$

Example 4.1.3. Let E be a smooth, open subset of \mathbb{R}^n and assume that $\mathcal{H}^{n-1}(\partial E \cap K) < +\infty$ for each $K \subset \Omega$. Then for each $U \subset\subset \Omega$ and each $\phi \in \mathcal{C}_c^1(U; \mathbb{R}^n)$ with $|\phi| \leq 1$, we have by the Divergence Theorem

$$\int_E \operatorname{div} \phi \, d\mathcal{L}^n(x) = \int_{\partial E} \phi \cdot \nu \, d\mathcal{H}^{n-1}$$

where ν denotes the outward-pointing unit normal along ∂E .

Thus

$$\int_E \operatorname{div} \phi \, d\mathcal{L}^n = \int_{\partial E \cap U} \phi \cdot \nu \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial E \cap U) < +\infty.$$

That is, E has locally finite perimeter in Ω . Furthermore

$$\|\partial E\|(\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$$

and

$$\nu_E = \nu \quad \mathcal{H}^{n-1} - \text{a.e. on } \partial E \cap \Omega.$$

Thus $\|\partial E\|(\Omega)$ measures the “size” of ∂E in Ω . Since $\mathbb{1}_E \notin W_{\text{loc}}^{1,1}(\Omega)$, we see that

$$W_{\text{loc}}^{1,1}(\Omega) \subsetneq BV_{\text{loc}}(\Omega)$$

and similarly

$$W^{1,1}(\Omega) \subsetneq BV(\Omega).$$

That is, not every function of (locally) bounded variation is a Sobolev function.

Remark. If $f \in BV_{\text{loc}}(\Omega)$, we can write as above

$$[Df] = [Df]_{ac} + [Df]_s = \mathcal{L}^n \llcorner Df + [Df]_s.$$

Consequently, $f \in BV_{\text{loc}}(\Omega)$ belongs to $W_{\text{loc}}^{1,p}(\Omega)$ if and only if

$$f \in L_{\text{loc}}^p(\Omega), \quad [Df]_s = 0, \quad Df \in L_{\text{loc}}^p(\Omega).$$

We see by the above remark that the theory of BV functions is more subtle than the theory of Sobolev functions, since we have to keep track of the singular part $[Df]_s$ of the vector-valued measure Df .

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