

NOTES ON L. C. EVANS AND R. F. GARIEPY: MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

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Notes on chapters 2, 3, and 5 of *Measure Theory and Fine Properties of Functions* by L. C. Evans and R. F. Gariepy. All references are from [1] ^{eg: measure} unless indicated otherwise.

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1. GENERAL MEASURE THEORY

1.1. Weak Convergence and Compactness for Radon Measures. We want to define what it means for a sequence $\{\mu_k\}_{k=1}^{+\infty}$ of Radon measures to converge weakly.

t1.9-1 **Theorem 1.1.1.** *Let $\mu, \{\mu_k\}_{k=1}^{+\infty}$ be Radon measures on \mathbb{R}^n . The following three statements are equivalent:*

- (i) $\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\mu$ for all $f \in \mathcal{C}_c(\mathbb{R}^n)$;
- (ii) $\limsup_{k \rightarrow +\infty} \mu_k(K) \leq \mu(K)$ for each compact set $K \subseteq \mathbb{R}^n$ and $\mu(U) \leq \liminf_{k \rightarrow +\infty} \mu_k(U)$ for each open set $U \subseteq \mathbb{R}^n$;
- (iii) $\lim_{k \rightarrow +\infty} \mu_k(B) = \mu(B)$ for each bounded Borel set $B \subseteq \mathbb{R}^n$ with $\mu(\partial B) = 0$.

Remark. Recall that Radon measures on \mathbb{R}^n are characterized by inner and outer regularity. Let $B \subseteq \mathbb{R}^n$ be a Borel set, and let $K \subseteq B \subseteq U$ with K compact and U open. If $\{\mu_k\}_{k=1}^{+\infty}$ is converging to μ in any sense, we should expect $\mu_k(K) \leq \mu(K)$ for all $k \in \mathbb{N}$ and $\mu_k(U) \geq \mu(U)$ for all $k \in \mathbb{N}$. Conditions (ii) and (iii) tell us that this in fact holds up to a subsequence.

Definition 1.1.1 (Weak Convergence of Radon Measures). *Let $\mu, \{\mu_k\}_{k=1}^{+\infty}$ be Radon measures on \mathbb{R}^n . We say that $\{\mu_k\}_{k=1}^{+\infty}$ converges weakly to μ , and write*

$$\mu_k \rightharpoonup \mu,$$

if

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\mu$$

for every $f \in \mathcal{C}_c(\mathbb{R}^n)$.

Proof. Assume first that (i) holds. Let $U \subseteq \mathbb{R}^n$ be open, and choose a compact set $K \subseteq U$. Next apply Urysohn's Lemma to choose a function $f \in \mathcal{C}_c(\mathbb{R}^n)$ such that

$$0 \leq f \leq 1, \quad \text{supp}(f) \subseteq U, \quad \text{and} \quad f \equiv 1 \text{ on } K.$$

Then

$$\begin{aligned} \mu(K) &= \int_K d\mu = \int_{\mathbb{R}^n} f \, d\mu \leq \int_{\mathbb{R}^n} f \, d\mu = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k \leq \liminf_{k \rightarrow +\infty} \int_U d\mu_k \\ &= \liminf_{k \rightarrow +\infty} \mu_k(U). \end{aligned}$$

Thus

$$\begin{aligned} \mu(U) &= \sup\{\mu(K) : K \text{ compact, } K \subseteq U\} \\ &\leq \liminf_{k \rightarrow +\infty} \mu_k(U). \end{aligned}$$

This proves the second part of (ii). The first part is similar.

Next suppose that (ii) holds. Let $B \subseteq \mathbb{R}^n$ be a bounded Borel set, $\mu(\partial B) = 0$. Then by (ii),

$$\begin{aligned} \mu(B) &= \mu(B^\circ) \leq \liminf_{k \rightarrow +\infty} \mu_k(B^\circ) \\ &\leq \limsup_{k \rightarrow +\infty} \mu_k(\overline{B}) \\ &\leq \mu(\overline{B}) \\ &= \mu(B). \end{aligned}$$

Since $\mu_k(B^\circ) = \mu_k(B) = \mu_k(\overline{B})$ for all $k \in \mathbb{N}$ since $\mu(\partial B) = 0$, it follows

$$\liminf_{k \rightarrow +\infty} \mu_k(B) = \limsup_{k \rightarrow +\infty} \mu_k(B).$$

Thus $\lim_{k \rightarrow +\infty} \mu_k(B)$ exists, and

$$\lim_{k \rightarrow +\infty} \mu_k(B) = \mu(B),$$

as required.

Finally assume that (iii) holds. Fix $\epsilon > 0$ and $f \in \mathcal{C}_c^+(\mathbb{R}^n)$. Let $R > 0$ be such that $\text{supp}(f) \subseteq B(0, R)$ and $\mu(\partial B(0, R)) = 0$. Choose a partition

$$0 := t_0 < t_1 < \cdots < t_N = 2\|f\|_{L^\infty(\mathbb{R}^n)}$$

of $[0, 2\|f\|_{L^\infty(\mathbb{R}^n)}]$ such that $0 < t_i - t_{i-1} < \epsilon$, and $\mu(f^{-1}\{t_i\}) = 0$ for each $i = 1, \dots, N$. Put $B_i := f^{-1}((t_{i-1}, t_i])$, $i = 2, \dots, N$. Then $\mu(\partial B_i) = 0$ for each $i \geq 2$. Now

$$\begin{aligned} \sum_{i=2}^N t_{i-1} \mu_k(B_i) &= \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu_k \leq \sum_{i=2}^N \int_{B_i} f d\mu_k \\ &\leq \int_{\mathbb{R}^n} f d\mu_k \\ &\leq \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0, R)), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=2}^N t_{i-1} \mu(B_i) &= \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu \leq \sum_{i=2}^N \int_{B_i} f d\mu \\ &\leq \int_{\mathbb{R}^n} f d\mu \\ &\leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B(0, R)). \end{aligned}$$

Thus (iii) implies

$$\begin{aligned} &\limsup_{k \rightarrow +\infty} \left| \int_{\mathbb{R}^n} f d\mu_k - \int_{\mathbb{R}^n} f d\mu \right| \\ &\leq \limsup_{k \rightarrow +\infty} \left| \left\{ \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0, R)) \right\} - \sum_{i=2}^N t_{i-1} \mu(B_i) \right| \\ &\leq \limsup_{k \rightarrow +\infty} \sum_{i=2}^N |t_i \mu_k(B_i) - t_{i-1} \mu(B_i)| + \limsup_{k \rightarrow +\infty} t_1 \mu_k(B(0, R)) \\ &= \sum_{i=2}^N |t_i - t_{i-1}| \mu(B_i) + t_1 \mu(B(0, R)) \\ &\leq 2\epsilon \mu(B(0, R)). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, taking the limit at $\epsilon \rightarrow 0$ shows that

$$\limsup_{k \rightarrow +\infty} \left| \int_{\mathbb{R}^n} f d\mu_k - \int_{\mathbb{R}^n} f d\mu \right| = 0,$$

and hence

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}^n} f d\mu.$$

The proof is complete. □

t1.9-2

Theorem 1.1.2 (Weak Compactness for Measures). *Let $\{\mu_k\}_{k=1}^{+\infty}$ be a sequence of Radon measures on \mathbb{R}^n satisfying*

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

for each compact set $K \subseteq \mathbb{R}^n$. Then there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{+\infty}$ and a Radon measure μ on \mathbb{R}^n such that

$$\mu_{k_j} \rightharpoonup \mu \quad \text{as } j \rightarrow +\infty.$$

Proof.

(i). Assume first that

$$\sup_{k \in \mathbb{N}} \mu_k(\mathbb{R}^n) < +\infty. \quad (1.1.1)$$

{eq:1.9-1}

(ii). Let $\{f_k\}_{k=1}^{+\infty}$ be a countable dense subset of $\mathcal{C}_c(\mathbb{R}^n)$. Note that (1.1.1) implies that the sequence $\{\int_{\mathbb{R}^n} f_1 d\mu_j\}_{j=1}^{+\infty}$ is bounded, for

$$\left| \int_{\mathbb{R}^n} f_1 d\mu_j \right| \leq \int_{\mathbb{R}^n} |f_1| d\mu_j \leq \max_{x \in \text{supp}(f)} |f(x)| \mu_j(\mathbb{R}^n) < +\infty.$$

Thus we may find a subsequence $\{\mu_j^1\}_{j=1}^{+\infty}$ and $a_1 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f_1 d\mu_j^1 \rightarrow a_1 \quad \text{as } j \rightarrow +\infty.$$

Continuing, we find subsequences $\{\mu_j^k\}_{j=1}^{+\infty}$ of $\{\mu_j^{k-1}\}_{j=1}^{+\infty}$ and numbers $a_k \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} f_k d\mu_j^k \rightarrow a_k \quad \text{as } j \rightarrow +\infty$$

for each $k \in \mathbb{N}$. Set $\nu_j := \mu_j^j$. Then

$$\int_{\mathbb{R}^n} f_k d\nu_j \rightarrow a_k \quad \text{as } j \rightarrow +\infty$$

for all $k \in \mathbb{N}$, for if $j \geq k$, then $\nu_j = \mu_j^j \in \{\mu_j^k\}_{j=1}^{+\infty}$. Define $L(f_k) := a_k$, and note that L is linear and

$$|L(f_k)| \leq M \|f_k\|_{L^\infty(\mathbb{R}^n)}$$

by (1.1.1), where

$$M := \sup_{k \in \mathbb{N}} \mu_k(\mathbb{R}^n).$$

By the Hahn–Banach Theorem, L may be uniquely extended to a bounded linear functional \bar{L} defined on all of $\mathcal{C}_c(\mathbb{R}^n)$. Then, by the Riesz Representation Theorem, there exists a unique Radon measure μ on \mathbb{R}^n such that

$$\bar{L}(f) = \int_{\mathbb{R}^n} f d\mu$$

for all $f \in \mathcal{C}_c(\mathbb{R}^n)$.

(iii). Choose any $f \in \mathcal{C}_c(\mathbb{R}^n)$. Since $\{f_k\}_{k=1}^{+\infty}$ is dense in $\mathcal{C}_c(\mathbb{R}^n)$, there exists a subsequence $\{f_{k_i}\}_{i=1}^{+\infty}$ such that $f_i \rightarrow f$ uniformly. Fix $\epsilon > 0$ and then choose $i \in \mathbb{N}$ so large that

$$\|f_{k_i} - f\|_{L^\infty(\mathbb{R}^n)} < \frac{\epsilon}{4M}. \quad (1.1.2)$$

{eq:1.9-2}

Next choose $J \in \mathbb{N}$ so that for all $j > J$,

$$\left| \int_{\mathbb{R}^n} f_{k_i} d\nu_j - \int_{\mathbb{R}^n} f_{k_i} d\mu \right| < \frac{\epsilon}{2}.$$

Then for any $j > J$, we have by ^(eq:1.9-2)(1.1.2) and the Principle of Uniform Boundedness

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f \, d\nu_j - \int_{\mathbb{R}^n} f \, d\mu \right| &\leq \left| \int_{\mathbb{R}^n} f - f_{k_i} \, d\nu_j \right| + \left| \int_{\mathbb{R}^n} f_{k_i} \, d\nu_j - \int_{\mathbb{R}^n} f_{k_i} \, d\mu \right| + \\ &\quad \left| \int_{\mathbb{R}^n} f_{k_i} - f \, d\mu \right| \\ &\leq \frac{\epsilon}{2} + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \nu_j(\mathbb{R}^n) + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \mu(\mathbb{R}^n) \\ &< \epsilon, \end{aligned}$$

as required.

(iv). In the general case that ^(eq:1.9-1)(1.1.1) fails to hold, but

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

for each compact set $K \subseteq \mathbb{R}^n$, we apply the above argument to the measures

$$\mu_k^l := \mu_k \llcorner \overline{B(0, l)}, \quad k, l = 1, 2, \dots,$$

and use a diagonalization argument. The proof is complete. \square

For the remainder of this section, we assume that

- (i) $U \subseteq \mathbb{R}^n$ is open;
- (ii) $1 \leq p < +\infty$.

Definition 1.1.2 (Weak Convergence in $L^p(U)$). *A sequence $\{f_k\}_{k=1}^{+\infty} \subset L^p(U)$ is said to converge weakly to $f \in L^p(U)$, written*

$$f_k \rightharpoonup f \quad \text{in } L^p(U),$$

if

$$\lim_{k \rightarrow +\infty} \int_U f_k g \, d\mathcal{L}^n = \int_U f g \, d\mathcal{L}^n$$

for each $g \in L^q(U)$, where p and q are conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q \leq +\infty$.

t1.9-3

Theorem 1.1.3 (Weak Compactness in L^p). *Suppose that $1 < p < +\infty$. Let $\{f_k\}_{k=1}^{+\infty} \subseteq L^p(U)$ satisfying*

$$\sup_{k \in \mathbb{N}} \|f_k\|_{L^p(U)} < +\infty.$$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{+\infty}$ of $\{f_k\}_{k=1}^{+\infty}$ and a function $f \in L^p(U)$ such that

$$f_{k_j} \rightharpoonup f \quad \text{in } L^p(U) \quad \text{as } j \rightarrow +\infty.$$

Remark. *This assertion is in general false for $p = 1$. The key property here is reflexivity. Recall that $L^p(U)$ is reflexive if and only if $1 < p < +\infty$.*

Definition 1.1.3. *We denote by*

$$\nu := \mu \llcorner f$$

the signed measure with density f with respect to μ , that is, the signed measure

$$\nu(K) = \int_K f \, d\mu,$$

provided that this holds for all compact sets $K \subseteq \mathbb{R}^n$.

Proof.

(i). If $U \neq \mathbb{R}^n$, we extend each function f_k to \mathbb{R}^n by setting $f_k = 0$ on $\mathbb{R}^n \setminus U$. This done, we may assume that $U = \mathbb{R}^n$. We may also assume that

$$f_k \geq 0 \quad \mathcal{L}^n - \text{a.e.},$$

for otherwise we could apply the following analysis to f_k^+ and f_k^- .

(ii). Define the Radon measures

$$\mu_k := \mathcal{L}^n \llcorner f_k, \quad k \in \mathbb{N}.$$

Then for each compact set $K \subseteq \mathbb{R}^n$, by Hölder's inequality, we have

$$\mu_k(K) = \int_K f_k d\mathcal{L}^n \leq \|f_k\|_{L^p(K)} \cdot \mathcal{L}^n(K)^{\frac{p-1}{p}} < +\infty,$$

and thus

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty.$$

Therefore, we may apply Theorem (I.1.2) to obtain a Radon measure μ on \mathbb{R}^n and a subsequence

$$\mu_{k_j} \rightharpoonup \mu.$$

(iii). We now show that $\mu \ll \mathcal{L}^n$. Let $A \subseteq \mathbb{R}^n$ be bounded with $\mathcal{L}^n(A) = 0$. Fix $\epsilon_1 > 0$ and choose an open bounded set $V \supseteq A$ such that $\mathcal{L}^n(V) < \epsilon$. Then by Theorem (I.1.1) and Hölder's inequality,

$$\begin{aligned} \mu(A) &\leq \mu(V) \leq \liminf_{j \rightarrow +\infty} \mu_{k_j}(V) = \liminf_{j \rightarrow +\infty} \int_V f_{k_j} d\mathcal{L}^n \\ &\leq \liminf_{j \rightarrow +\infty} \|f_{k_j}\|_{L^p(V)} \cdot \mathcal{L}^n(V)^{\frac{p-1}{p}} \\ &\leq C\epsilon^{\frac{p-1}{p}}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary and $\frac{p-1}{p} > 0$, $\mu(A) = 0$, as required. Therefore $\mu \ll \mathcal{L}^n$.

(iv). By the Radon–Nikodym Theorem, there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\mu(A) = \int_A f d\mathcal{L}^n$$

for every Borel set $A \subseteq \mathbb{R}^n$.

(v). We prove that $f \in L^p(\mathbb{R}^n)$. Let $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} f\phi d\mathcal{L}^n &= \int_{\mathbb{R}^n} \phi d\mu = \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi d\mu_{k_j} \\ &= \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi f_{k_j} d\mathcal{L}^n \\ &\leq \sup_{k \in \mathbb{N}} \|f_{k_j}\|_{L^p(\mathbb{R}^n)} \|\phi\|_{L^q(\mathbb{R}^n)} \\ &\leq C \|\phi\|_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Thus

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{\substack{\phi \in \mathcal{C}_c(\mathbb{R}^n) \\ \|\phi\|_{L^q(\mathbb{R}^n)}=1}} \left| \int_{\mathbb{R}^n} f\phi d\mathcal{L}^n \right| \leq C < +\infty,$$

and we see that $f \in L^p(\mathbb{R}^n)$.

(vi). Finally, we show that $f_{k_j} \rightharpoonup f$ in $L^p(\mathbb{R}^n)$. Fix $\epsilon > 0$. By the above,

$$\int_{\mathbb{R}^n} f_{k_j} \phi \, d\mathcal{L}^n \rightarrow \int_{\mathbb{R}^n} f \phi \, d\mathcal{L}^n$$

as $j \rightarrow +\infty$ for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Thus we may choose $J \in \mathbb{N}$ so large so that for all $j > J$,

$$\left| \int_{\mathbb{R}^n} f_{k_j} \phi - f \phi \, d\mathcal{L}^n \right| < \epsilon \quad (1.1.3)$$

{eq:1.9-3}

for all $\phi \in \mathcal{C}_c(\mathbb{R}^n)$. Given $g \in L^q(\mathbb{R}^n)$, choose by the density of $\mathcal{C}_c(\mathbb{R}^n)$ in $L^q(\mathbb{R}^n)$ a function $\phi \in \mathcal{C}_c(\mathbb{R}^n)$ such that

$$\|g - \phi\|_{L^q(\mathbb{R}^n)} < \epsilon.$$

Then by (1.1.3), Hölder's inequality, and the Principle of Uniform Boundedness, we have for all $j > J$

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_{k_j} g \, d\mathcal{L}^n - \int_{\mathbb{R}^n} f g \, d\mathcal{L}^n \right| &\leq \int_{\mathbb{R}^n} |f_{k_j} g - f_{k_j} \phi| \, d\mathcal{L}^n + \left| \int_{\mathbb{R}^n} f_{k_j} \phi - f \phi \, d\mathcal{L}^n \right| + \\ &\quad \int_{\mathbb{R}^n} |f \phi - f g| \, d\mathcal{L}^n \\ &\leq \epsilon + \int_{\mathbb{R}^n} |f_{k_j}| |g - \phi| \, d\mathcal{L}^n + \int_{\mathbb{R}^n} |f| |\phi - g| \, d\mathcal{L}^n \\ &\leq \epsilon + \epsilon \|f_{k_j}\|_{L^p(\mathbb{R}^n)} + \epsilon \|f\|_{L^p(\mathbb{R}^n)} \\ &\leq (2C + 1)\epsilon. \end{aligned}$$

The proof is complete. □

2. HAUSDORFF MEASURE

2.1. Definitions and Elementary Properties; Hausdorff Dimension.

Definition 2.1.1 (\mathcal{H}_δ^s). Let $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$, $0 < \delta \leq +\infty$. We define

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^2 : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \text{diam } C_j \leq \delta \right\},$$

where

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(1 + \frac{s}{2})}$$

denotes the volume of the unit ball in \mathbb{R}^s .

Note in the above definition that s need not be an integer.

Definition 2.1.2 (\mathcal{H}^s , s -Dimensional Hausdorff Measure). Let $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. We define the s -dimensional Hausdorff measure \mathcal{H}^s on \mathbb{R}^n by

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A).$$

Note that taking the limit as $\delta \rightarrow 0$ coincides with taking the supremum over $\delta > 0$, for, as $\delta \rightarrow 0$, we are taking the infimum over smaller and smaller sets. That is, if $\delta_1 < \delta_2$, then there exist coverings $\{C_j\}_{j=1}^{+\infty}$ of A such that $\text{diam } C_j \leq \delta_2$ but $\text{diam } C_j > \delta_1$.

Remark.

- (i) Requiring $\delta \rightarrow 0$ forces the coverings to “follow the local geometry” of the set A ;
- (ii) Recall that

$$\mathcal{L}^n(B(x, r)) = \alpha(n)r^n$$

for all balls $B(x, r) \subseteq \mathbb{R}^n$. In fact if $s = k$ is an integer, then \mathcal{H}^k coincides with the ordinary “ k -dimensional surface area” on nice sets. This is the reason that the normalizing constant $\alpha(s)$ is included in the definition of \mathcal{H}_δ^s .

t2.1-1 Theorem 2.1.1. \mathcal{H}^s is a Borel regular measure, $0 \leq s < +\infty$.

Remark.

- (i) Recall that this means that \mathcal{H}^s is Borel and for each $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.
- (ii) \mathcal{H}^s is **not** a Radon measure if $0 \leq s < n$, since \mathbb{R}^n is not σ -finite with respect to \mathcal{H}^s .

Proof.

(i). \mathcal{H}_δ^s is a measure. Choose $\{A_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}^n$ and suppose that $A_k \subseteq \bigcup_{j=1}^{+\infty} C_j^k$, where $\text{diam } C_j^k \leq \delta$. Then $\{C_j^k\}_{j,k=1}^{+\infty}$ covers $\bigcup_{k=1}^{+\infty} A_k$. Thus

$$\mathcal{H}_\delta^s \left(\bigcup_{k=1}^{+\infty} A_k \right) \leq \sum_{k=1}^{+\infty} \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j^k)^s.$$

Taking infima over all such covers $\{C_j^k\}_{k=1}^{+\infty}$ of A_k , we find

$$\mathcal{H}_\delta^s \left(\bigcup_{k=1}^{+\infty} A_k \right) \leq \sum_{k=1}^{+\infty} \mathcal{H}_\delta^s(A_k),$$

as required.

(ii). \mathcal{H}^s is a measure. Choose $\{A_k\}_{k=1}^{+\infty} \subseteq \mathbb{R}^n$. Since $\mathcal{H}^s(\bigcup_{k=1}^{+\infty} A_k) = \sup_{\delta > 0} \mathcal{H}_\delta^s(\bigcup_{k=1}^{+\infty} A_k)$, we have

$$\mathcal{H}_\delta^s \left(\bigcup_{k=1}^{+\infty} A_k \right) \leq \sum_{k=1}^{+\infty} \mathcal{H}_\delta^s(A_k) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

Taking the limit as $\delta \rightarrow 0$ on the LHS shows that

$$\mathcal{H}^s \left(\bigcup_{k=1}^{+\infty} A_k \right) \leq \sum_{k=1}^{+\infty} \mathcal{H}^s(A_k).$$

(iii). \mathcal{H}^s is a Borel measure. Choose $A, B \subseteq \mathbb{R}^n$ with $\text{dist}(A, B) > 0$. Select $0 < \delta < \frac{1}{4} \text{dist}(A, B)$. Let $A \cup B \subseteq \bigcup_{k=1}^{+\infty} C_k$ with $\text{diam } C_k \leq \delta$.

Put

$$\mathcal{A} := \{C_j : C_j \cap A \neq \emptyset\}$$

and

$$\mathcal{B} := \{C_j : C_j \cap B \neq \emptyset\}.$$

Then $A \subseteq \bigcup_{C_j \in \mathcal{A}} C_j$ and $B \subseteq \bigcup_{C_j \in \mathcal{B}} C_j$, with $C_i \cap C_j = \emptyset$ if $C_i \in \mathcal{A}$, $C_j \in \mathcal{B}$. Thus

$$\begin{aligned} \sum -j &= 1^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \geq \sum_{C_j \in \mathcal{A}} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s + \sum_{C_j \in \mathcal{B}} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \\ &\geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B). \end{aligned}$$

Taking the infimum over all such sets $\{C_j\}_{j=1}^{+\infty}$, $0 < \delta < \frac{1}{4} \text{dist}(A, B)$, we find

$$\mathcal{H}_\delta^s(A \cup B) \geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B).$$

Letting $\delta \rightarrow 0$, we obtain

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Consequently

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$$

for all $A, B \subseteq \mathbb{R}^n$ with $\text{dist}(A, B) > 0$. By Caratheodory's Criterion, \mathcal{H}^s is a Borel measure.

(iv). \mathcal{H}^s is Borel regular. First note that $\text{diam } \overline{C} = \text{diam } C$ for all $C \subseteq \mathbb{R}^n$. Thus

$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \text{diam } C_j \leq \delta, C_j \text{ closed} \right\}.$$

Choose $A \subseteq \mathbb{R}^n$ such that $\mathcal{H}^s(A) < +\infty$. Then $\mathcal{H}_\delta^s(A) < +\infty$ for all $\delta > 0$. For each $k \geq 1$, choose closed sets $\{C_j^k\}_{j=1}^{+\infty}$ so that $\text{diam } C_j^k \leq \frac{1}{k}$, $A \subseteq \bigcup_{j=1}^{+\infty} C_j^k$, and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j^k)^s \leq \mathcal{H}_{1/k}^s(A) + \frac{1}{k}.$$

Put $A_k := \cup_{j=1}^{+\infty} C_j^k$ and $B := \cap_{k=1}^{+\infty} A_k$. Then B is Borel. Also $A \subseteq A_k$ for each $k \in \mathbb{N}$, so $A \subseteq B$. Moreover, since $B \subseteq A_k$ for each k ,

$$\mathcal{H}_{1/k}^s(B) \leq \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j^k)^s \leq \mathcal{H}_{1/k}^s(A) + \frac{1}{k}.$$

Letting $k \rightarrow +\infty$, we find

$$\mathcal{H}^s(B) \leq \mathcal{H}^s(A).$$

But since $A \subseteq B$, we have by monotonicity

$$\mathcal{H}^s(A) = \mathcal{H}^s(B).$$

The proof is complete. □

τ2.1-2 **Theorem 2.1.2** (Elementary Properties of Hausdorff Measure).

- (i) \mathcal{H}^0 is counting measure;
- (ii) $\mathcal{H}^1 = \mathcal{L}^1$ on \mathbb{R} ;
- (iii) $\mathcal{H}^s \equiv 0$ on \mathbb{R}^n for all $s > n$;
- (iv) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for all $\lambda > 0$, $A \subseteq \mathbb{R}^n$;
- (v) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ for each affine isometry $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$.

Proof.

(iv). Fix $0 < \delta \leq +\infty$, and suppose that $A \subseteq \cup_{j=1}^{+\infty} C_j$, with $\text{diam } C_j \leq \delta$. Then $\lambda A \subseteq \cup_{j=1}^{+\infty} \lambda C_j$, and $\text{diam } \lambda C_j = \lambda \text{diam } C_j \leq \lambda \delta$. Thus

$$\begin{aligned} \lambda^s \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s &= \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\lambda \text{diam } C_j)^s \\ &\geq \mathcal{H}_{\lambda \delta}^s(\lambda A). \end{aligned}$$

Taking the infimum over all such covers $\{C_j\}_{j=1}^{+\infty}$ of A , we deduce

$$\lambda^s \mathcal{H}_\delta^s(A) \geq \mathcal{H}_{\lambda \delta}^s(\lambda A),$$

and taking the limit as $\delta \rightarrow 0$ shows

$$\lambda^s \mathcal{H}^s(A) \geq \mathcal{H}^s(\lambda A).$$

The reverse inequality may be shown similarly.

- (v). This follows at once from (iv) along with the translation invariance of \mathcal{H}^s .
- (i). First note that $\alpha(0) = 1$. Thus obviously $\mathcal{H}^0(\{a\}) = 1$ for all $a \in \mathbb{R}^n$, and (i) follows.
- (ii). Choose $A \subseteq \mathbb{R}$ and $\delta > 0$. Then

$$\begin{aligned} \mathcal{L}^1(A) &= \inf \left\{ \sum_{j=1}^{+\infty} \text{diam } C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j \right\} \\ &\leq \inf \left\{ \sum_{j=1}^{+\infty} \text{diam } C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j, \text{diam } C_j \leq \delta \right\} \\ &= \mathcal{H}_\delta^1(A) \\ &\leq \mathcal{H}^1(A). \end{aligned}$$

On the other hand, set $I_k := [k\delta, (k+1)\delta]$, $k \in \mathbb{Z}$. Then $\text{diam}(C_j \cap I_k) \leq \delta$, and, since $\bigcup_{k=1}^{+\infty} C_j \cap I_k = C_j$,

$$\sum_{k=-\infty}^{+\infty} \text{diam}(C_j \cap I_k) \leq \text{diam } C_j.$$

Hence,

$$\begin{aligned} \mathcal{L}^1(A) &= \inf \left\{ \sum_{j=1}^{+\infty} \text{diam } C_j : A \subseteq \bigcup_{j=1}^{+\infty} C_j \right\} \\ &\geq \inf \left\{ \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} \text{diam}(C_j \cap I_k) : A \subseteq \bigcup_{j=1}^{+\infty} C_j \right\} \\ &= \mathcal{H}_\delta^1(A). \end{aligned}$$

Therefore $\mathcal{L}^1 = \mathcal{H}_\delta^1$ for all $\delta > 0$, so that taking the supremum over all $\delta > 0$, we have $\mathcal{L}^1 = \mathcal{H}^1$ on \mathbb{R} .

(iii). Fix an integer $m \geq 1$. The unit cube $Q(n)$ in \mathbb{R}^n may be decomposed into m^n cubes with side length $\frac{1}{m}$ and diameter $\frac{\sqrt{n}}{m}$. Thus

$$\mathcal{H}_{\sqrt{n}/m}^s(Q(n)) \leq \sum_{j=1}^{m^n} \alpha(s) \left(\frac{\sqrt{n}}{m} \right)^s = \alpha(s) n^{\frac{s}{2}} m^{n-s},$$

and the RHS tends to zero as $m \rightarrow +\infty$ if $s > n$. Hence $\mathcal{H}^s(Q(n)) = 0$, so $\mathcal{H}^s \equiv 0$. The proof is complete. \square

A convenient way to check that \mathcal{H}^s vanishes on a set $A \subseteq \mathbb{R}^n$ is the following lemma.

12-1-1 Lemma 2.1.1. *If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}_\delta^s(A) = 0$ for some $0 < \delta \leq +\infty$, then $\mathcal{H}^s(A) = 0$.*

Proof. The conclusion is obvious if $s = 0$, and so we may assume that $s > 0$.

Fix $\epsilon > 0$. There exist sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \bigcup_{j=1}^{+\infty} C_j$ and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \leq \epsilon.$$

In particular for each $j \in \mathbb{N}$,

$$\text{diam } C_j \leq 2 \left(\frac{\epsilon}{\alpha(s)} \right)^{\frac{1}{s}} =: \delta(\epsilon).$$

Hence $\mathcal{H}_{\delta(\epsilon)}^s < \epsilon$. But since $\delta(\epsilon) \rightarrow 0$ and $\epsilon \rightarrow 0$, we have

$$\mathcal{H}^s(A) = 0.$$

The proof is complete. \square

We next want to define the *Hausdorff dimension* of a subset of \mathbb{R}^n .

12.1-2 Lemma 2.1.2. *Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < +\infty$.*

- (i) *If $\mathcal{H}^s(A) < +\infty$, then $\mathcal{H}^t(A) = 0$;*
- (ii) *If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.*

Proof.

(i). Let $\mathcal{H}^s(A) < +\infty$ and $\delta > 0$. Then there exist sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \cup_{j=1}^{+\infty} C_j$, $\text{diam } C_j \leq \delta$, and

$$\sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \leq \mathcal{H}_\delta^s(A) + 1 \leq \mathcal{H}^s(A) + 1.$$

Then

$$\begin{aligned} \mathcal{H}_\delta^t(A) &\leq \sum_{j=1}^{+\infty} \frac{\alpha(t)}{2^t} (\text{diam } C_j)^t \\ &= \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sum_{j=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_j)^s \cdot (\text{diam } C_j)^{t-s} \\ &\leq \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \delta^{t-s} (\mathcal{H}^s(A) + 1). \end{aligned}$$

Sending $\delta \rightarrow 0$, we conclude that $\mathcal{H}^t(A) = 0$. This proves (i).

(ii). Assertion (ii) follows at once from (i), by contrapositive. The proof is complete. \square

Definition 2.1.3 (Hausdorff Dimension). *We define the Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ by*

$$\mathcal{H}_{\dim}(A) := \inf\{0 \leq s < +\infty : \mathcal{H}^s(A) = 0.\}$$

Remark. *Observe for any set $A \subseteq \mathbb{R}^n$ that $\mathcal{H}_{\dim}(A) \leq n$. Let $s := \mathcal{H}_{\dim}(A)$. Then by the preceding lemma, $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = +\infty$ for all $t < s$. Moreover, $\mathcal{H}^s(A)$ may be any number between 0 and $+\infty$, inclusive. Furthermore, $\mathcal{H}_{\dim}(A)$ need not be an integer. Even if $\mathcal{H}_{\dim}(A) = k$ is an integer and $0 < \mathcal{H}^k(A) < +\infty$, A need not be a “ k -dimensional surface” in any sense, and may be extremely complicated geometrically. Examples include Cantor-like subsets A of \mathbb{R}^n and other fractals.*

2.2. Isodiametric Inequality; $\mathcal{H}^n = \mathcal{L}^n$. We want to prove that $\mathcal{H}^n = \mathcal{L}^n$ on \mathbb{R}^n , where $n \in \mathbb{N}$. Recall that \mathcal{L}^n is defined as the n -fold product of one-dimensional Lebesgue measure \mathcal{L}^1 , so that

$$\mathcal{L}^1(A) := \inf \left\{ \sum_{i=1}^n \mathcal{L}^1(Q_i) : Q_i \text{ cubes}, A \subseteq \bigcup_{i=1}^n Q_i \right\}.$$

On the other hand, \mathcal{H}^n is computed in terms of arbitrary coverings of small diameter.

12.2-1

Lemma 2.2.1. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be \mathcal{L}^n -measurable. Then the region “under the graph” of f ,*

$$A := \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}, 0 \leq y \leq f(x)\}$$

is \mathcal{L}^{n+1} -measurable.

Proof. Define

$$B := \{x \in \mathbb{R}^n : f(x) = +\infty\}$$

and

$$C := \{x \in \mathbb{R}^n : 0 \leq f(x) < +\infty.\}$$

Also define

$$C_{j,k} := \left\{ x \in C : \frac{j}{k} \leq f(x) < \frac{j+1}{k} \right\}, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{N},$$

so that $C = \bigcup_{j=0}^{+\infty} C_{j,k}$. Finally, put

$$D_k := \bigcup_{j=0}^{+\infty} \left(C_{j,k} \times \left[0, \frac{j}{k} \right] \right) \cup (B \times [0, +\infty]),$$

$$E_k := \bigcup_{j=0}^{+\infty} \left(C_{j,k} \times \left[0, \frac{j+1}{k} \right] \right) \cup (B \times [0, +\infty]).$$

Clearly D_k and E_k are \mathcal{L}^{n+1} measurable, and we have for each $k \in \mathbb{N}$ $D_k \subseteq A \subseteq E_k$. Write $D := \bigcup_{k=1}^{+\infty} D_k$ and $E := \bigcap_{k=1}^{+\infty} E_k$. Then also $D \subseteq A \subseteq E$, with D and E both \mathcal{L}^{n+1} –measurable. Now for any \mathcal{L}^{n+1} –measurable set F with $\mathcal{L}^{n+1}(F) < +\infty$,

$$\mathcal{L}^{n+1}((E \setminus D) \cap F) \leq \mathcal{L}^{n+1}((E_k \setminus D_k) \cap F) \leq \frac{1}{k} \mathcal{L}^n(F),$$

and the RHS tends to zero as $k \rightarrow +\infty$. Thus $\mathcal{L}^{n+1}((E \setminus D) \cap F) = 0$, and, because F was arbitrary, $\mathcal{L}^{n+1}(E \setminus D) = 0$. Hence $\mathcal{L}^{n+1}(A \setminus D) = 0$, and consequently A is \mathcal{L}^{n+1} –measurable. \square

We now define the process of Steiner symmetrization, which takes a bounded Borel–measurable set $A \subseteq \mathbb{R}^n$ and transforms A into a set \tilde{A} having the same Lebesgue measure such that $\text{diam}(\tilde{A}) \leq \text{diam}(A)$.

Fix $a, b \in \mathbb{R}^n$, $\|a\| = 1$. We define

$$L_b^a := \{b + ta : t \in \mathbb{R}\}, \text{ the line through } b \text{ in the direction of } a,$$

and

$$P_a := \{x \in \mathbb{R}^n : x \cdot a = 0\}, \text{ the plane through the origin perpendicular to } a.$$

Definition 2.2.1 (Steiner Symmetrization). *Choose $a \in \mathbb{R}^n$ with $\|a\| = 1$, and let $A \subseteq \mathbb{R}^n$. We define the Steiner symmetrization of A with respect to the hyperplane P_a to be the set*

$$S_a(A) := \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \left\{ b + ta : \|t\| \leq \frac{1}{2} \mathcal{H}^1(A \cap L_b^a) \right\}.$$

Note that the Steiner symmetrization is the union of all line segments $b + ta$ of length less than $\mathcal{H}^1(A \cap L_b^a)$, where b is in the plane through the origin perpendicular to a and there exists $x \in A$ such that $b + ta = x$.

12.2-2 Lemma 2.2.2 (Properties of Steiner Symmetrization).

- (i) $\text{diam } S_a(A) \leq \text{diam } A$.
- (ii) If A is \mathcal{L}^n –measurable, then so is $S_a(A)$, and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

Proof.

(i). Statement (i) is trivial if $\text{diam } A = +\infty$, so we may assume that $\text{diam } A < +\infty$. We may also suppose that A is closed, for

$$\text{diam } A^\circ = \text{diam } A = \text{diam } \overline{A}.$$

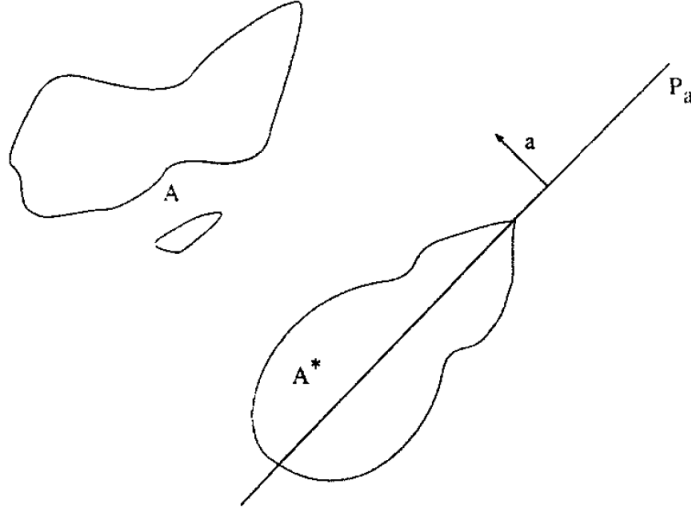


FIGURE 1. Steiner Symmetrization.

Fix $\epsilon > 0$ and choose $x, y \in S_a(A)$ such that

$$\text{diam } S_a(A) \leq \|x - y\| + \epsilon.$$

Write $b := x - (x \cdot a)a$ and $c := y - (y \cdot a)a$. Then $b, c \in P_a$. Put

$$\begin{aligned} r &:= \inf\{t : b + ta \in A\}, \\ s &:= \sup\{t : b + ta \in A\}, \\ u &:= \inf\{t : c + ta \in A\}, \\ v &:= \sup\{t : c + ta \in A\}. \end{aligned}$$

Without loss of generality, we may assume that $v - r \geq s - u$. Then

$$\begin{aligned} v - r &\geq \frac{1}{2}(v - r) + \frac{1}{2}(s - u) \\ &= \frac{1}{2}(s - r) + \frac{1}{2}(v - u) \\ &\geq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) + \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

Now, $|x \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)$, $|y \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_c^a)$, and consequently,

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a|.$$

Hence,

$$\begin{aligned} (\text{diam } S_a(A) - \epsilon)^2 &\leq \|x - y\|^2 \\ &= \|x\|^2 - 2x \cdot y + \|y\|^2 \\ &= \|b\|^2 + 2(x \cdot a)(b \cdot a) + |x \cdot a|^2 - 2(b + (x \cdot a)a) \cdot (c + (y \cdot a)a) + \|c\|^2 + \\ &\quad 2(y \cdot a)(b \cdot a) + |y \cdot a|^2 \\ &= (\|b\|^2 - 2b \cdot c + \|c\|^2) + (|x \cdot a|^2 - 2(x \cdot a)(y \cdot a) + |y \cdot a|^2) + \end{aligned}$$

$$\begin{aligned}
& 2(x \cdot a)(b \cdot a) - 2(b \cdot a)(y \cdot a) - 2(c \cdot a)(x \cdot a) + 2(y \cdot a)(b \cdot a) \\
&= \|b - c\|^2 + \|x \cdot a - y \cdot a\|^2 \\
&\leq \|b - c\|^2 + (v - r)^2 \\
&= \|b\|^2 - 2b \cdot c + \|c\|^2 + v^2 - 2rv + r^2 \\
&= (\|b\|^2 + 2b \cdot ra + \|ra\|^2) - 2(b \cdot c - b \cdot va - c \cdot ra - rv\|a\|^2) + \\
&\quad (\|c\|^2 + 2c \cdot va + \|va\|^2) \\
&= \|(b + ra) - (c + va)\|^2 \\
&\leq (\text{diam } A)^2,
\end{aligned}$$

since $b, c \perp a$ and A is closed, so that $b + ra, c + va \in A$. Thus $\text{diam } S_a(A) - \epsilon \leq \text{diam } A$, and since $\epsilon > 0$ was arbitrary, this proves (i).

(ii). Since \mathcal{L}^n is rotation invariant, we may assume that $a = e_n$. Then $P_a = P_{e_n} = \mathbb{R}^{n-1}$. Since $\mathcal{L}^1 = \mathcal{H}^1$ on \mathbb{R} , Tonelli's Theorem implies that the map $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ defined by $f(b) = \mathcal{H}^1(A \cap L_b^a)$ is \mathcal{L}^{n-1} -measurable and $\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b)$, for

$$\int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(A \cap L_b^a) d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).$$

Therefore

$$S_a(A) = \left\{ (b, y) : 0 \leq |y| \leq \frac{f(b)}{2} \right\} \setminus \{(b, 0) : L_b^a \cap A = \emptyset\}$$

is \mathcal{L}^n -measurable by Lemma (2.2.1), and

$$\begin{aligned}
\mathcal{L}^n(S_a(A)) &= \int_{\mathbb{R}} \mathbb{1}_{S_a(A)} d\mathcal{L}^n = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathbb{1}_{S_a(A)} d\mathcal{L}^1 d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\mathbb{1}_{S_a(A)})_{(e_1, \dots, e_{n-1})}(y) d\mathcal{L}^1(y) d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \int_{-f(b)/2}^{f(b)/2} d\mathcal{L}^1 d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).
\end{aligned}$$

The proof is complete. □

Remark. In proving $\mathcal{H}^n = \mathcal{L}^n$ below, notice that we use only statement (ii) above in the special case that a is a standard coordinate vector. Since \mathcal{H}^n is obviously rotation invariant, we in fact prove that \mathcal{L}^n is rotation invariant also.

t2.2-1

Theorem 2.2.1 (Isodiametric Inequality). *For all sets $A \subseteq \mathbb{R}^n$,*

$$\mathcal{L}^n(A) \leq \frac{\alpha(n)}{2^n} (\text{diam } A)^n.$$

Remark.

(i) Geometrically, the isodiametric inequality says that of all sets of fixed diameter in \mathbb{R}^n , the n -sphere has greatest volume.

(ii) *This inequality is particularly interesting because it is not necessarily the case that A is contained in a ball of diameter $\text{diam } A$, for in \mathbb{R}^2 consider the case of an equilateral triangle with side length 1. The smallest closed ball B which inscribes the triangle has radius $1/\sqrt{3}$, so*

$$\text{diam } B = \frac{2}{\sqrt{3}} > 1.$$

Proof. If $\text{diam } A = +\infty$, the inequality is trivial. Therefore we may assume that $\text{diam } A < +\infty$.

Let $\{e_1, \dots, e_n\}$ be the standard basis for \mathbb{R}^n . Define $A_1 := S_{e_1}(A)$, $A_2 := S_{e_2}(A_1), \dots$, $A_n := S_{e_n}(A_{n-1})$. Write $A^* := A_n$.

(i). We first show that A^* is symmetric with respect to the origin. We use induction. Clearly A_1 is symmetric with respect to P_{e_1} . Let k be an integer such that $1 \leq k < n$ and suppose that A_k is symmetric with respect to P_{e_1}, \dots, P_{e_k} . Clearly $A_{k+1} = S_{e_{k+1}}(A_k)$ is symmetric with respect to $P_{e_{k+1}}$. Fix $1 \leq j < k$ and let $S_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the reflection through P_{e_j} . Let $b \in P_{e_{k+1}}$. Since A_k is symmetric with respect to P_{e_1}, \dots, P_{e_k} by the induction hypothesis and $1 \leq j \leq k$, we have $S_j(A_k) = A_k$, and so

$$\mathcal{H}^1(A_k \cap L_b^{e_{k+1}}) = \mathcal{H}^1(A_k \cap L_{S_j b}^{e_{k+1}}).$$

Consequently

$$\{t \in \mathbb{R} : b + te_{k+1} \in A_{k+1}\} = \{t \in \mathbb{R} : S_j b + te_{k+1} \in A_{k+1}\}.$$

Thus $S_j(A_{k+1}) = A_{k+1}$, that is, A_{k+1} is symmetric with respect to P_{e_j} . Since j was arbitrary, $A^* = A_n$ is symmetric with respect to P_{e_1}, \dots, P_{e_n} , and so with respect to the origin.

(ii). We show that

$$\mathcal{L}^n(A^*) \leq \frac{\alpha(n)}{2^n} (\text{diam } A^*)^n.$$

Choose $x \in A^*$. Then $-x \in A^*$ by (i), and so $\text{diam } A^* \geq 2|x|$. Thus $A^* \subseteq B(0, \frac{1}{2} \text{diam } A^*)$, and it follows by monotonicity of the Lebesgue measure

$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B\left(0, \frac{1}{2} \text{diam } A^*\right)\right) = \frac{\alpha(n)}{2^n} (\text{diam } A^*)^2.$$

(iii). We now prove the isodiametric inequality. Note that \bar{A} is \mathcal{L}^n -measurable, and thus the above Lemma (2.2.2) implies that

$$\mathcal{L}^n((\bar{A})^*) = \mathcal{L}^n(\bar{A}),$$

as well as

$$\text{diam}(\bar{A})^* \leq \text{diam } \bar{A}.$$

Hence, monotonicity of the Lebesgue measure together with (ii) give

$$\begin{aligned} \mathcal{L}^n(A) &\leq \mathcal{L}^n(\bar{A}) = \mathcal{L}^n((\bar{A})^*) \\ &\leq \frac{\alpha(n)}{2^n} (\text{diam}(\bar{A})^*)^n \\ &\leq \frac{\alpha(n)}{2^n} (\text{diam } \bar{A})^n \\ &= \frac{\alpha(n)}{2^n} (\text{diam } A)^n. \end{aligned}$$

The proof is complete. \square

t2.2-2 **Theorem 2.2.2.** *On \mathbb{R}^n , $\mathcal{L}^n = \mathcal{H}^n$.*

Proof. (i). We first show that $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$ for all $A \subseteq \mathbb{R}^n$. Fix $\delta > 0$. Choose sets $\{C_j\}_{j=1}^{+\infty}$ such that $A \subseteq \bigcup_{j=1}^{+\infty} C_j$ and $\text{diam } C_j \leq \delta$. Then by monotonicity and the Isodiametric Inequality (cf. (2.2.1)),

$$\mathcal{L}^n(A) \leq \sum_{j=1}^{+\infty} \mathcal{L}^n(C_j) \leq \sum_{j=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } C_j)^n.$$

Taking the infimum of the RHS over all cover countable covers of A with diameter less than δ , we obtain $\mathcal{L}^n(A) \leq H_\delta^n(A)$. Taking the limit as $\delta \rightarrow 0$, we have

$$\mathcal{L}^n(A) \leq \mathcal{H}_\delta^n(A) \leq \mathcal{H}^n(A),$$

as required.

(ii). From the definition of \mathcal{L}^n as the n -fold product of $\mathcal{L}^1 \times \cdots \times \mathcal{L}^1$, we see that for all $A \subseteq \mathbb{R}^n$ and $\delta > 0$,

$$\mathcal{L}^n(A) = \inf \left\{ \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) : Q_i \text{ cubes, } A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \text{diam } Q_i \leq \delta \right\}.$$

We may consider only cubes parallel to the coordinate axes in \mathcal{L}^n .

(iii). We now show that \mathcal{H}^n is absolutely continuous with respect to \mathcal{L}^n . Set $C_n := \frac{\alpha(n)}{2^n}$. Then for each cube $Q \subseteq \mathbb{R}^n$,

$$\frac{\alpha(n)}{2^n} (\text{diam } Q)^n = C_n \mathcal{L}^n(Q).$$

Thus for any $A \subseteq \mathbb{R}^n$,

$$\begin{aligned} \mathcal{H}_\delta^n(A) &= \inf \left\{ \sum_{i=1}^n \frac{\alpha(n)}{2^n} (\text{diam } U_i)^n : A \subseteq \bigcup_{i=1}^{+\infty} U_i, \text{diam } U_i \leq \delta \right\} \\ &\leq \inf \left\{ \sum_{i=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } Q_i)^n : Q_i \text{ cubes, } A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \text{diam } Q_i \leq \delta \right\} \\ &= C_n \mathcal{L}^n(A). \end{aligned}$$

Taking the supremum over all $\delta > 0$, we've:

$$\mathcal{H}^n(A) \leq C_n \mathcal{L}^n(A).$$

Thus $\mathcal{H}^n(A) = 0$ whenever $\mathcal{L}^n(A) = 0$. This proves (iii).

(iv). We now show that $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$ for all $A \subseteq \mathbb{R}^n$. To this end, fix $\delta > 0$ and $\epsilon > 0$. We may choose cubes $\{Q_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ such that $A \subseteq \bigcup_{i=1}^{+\infty} Q_i$, $\text{diam } Q_i \leq \delta$, and

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon.$$

Now for each $i \in \mathbb{N}$ there exist disjoint closed balls $\{B_k^i\}_{k=1}^{+\infty} \subseteq Q_i^\circ$ such that

$$\text{diam } B_k^i \leq \delta$$

and

$$\mathcal{L}^n \left(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i \right) = \mathcal{L}^n \left(Q_i^\circ \setminus \bigcup_{k=1}^{+\infty} B_k^i \right) = 0.$$

Since $\mathcal{H}^n, \mathcal{H}_\delta^n$ are absolutely continuous with respect to \mathcal{L}^n by (iii), $\mathcal{H}^n(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i) = \mathcal{H}_\delta^n(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i) = 0$. Therefore $\mathcal{H}^n(Q_i) = \mathcal{H}^n(\bigcup_{k=1}^{+\infty} B_k^i)$ and $\mathcal{H}_\delta^n(Q_i) = \mathcal{H}_\delta^n(\bigcup_{k=1}^{+\infty} B_k^i)$, and we have

$$\begin{aligned} \mathcal{H}_\delta^n(A) &\leq \sum_{i=1}^{+\infty} \mathcal{H}_\delta^n(Q_i) = \sum_{i=1}^{+\infty} \mathcal{H}_\delta^n \left(\bigcup_{k=1}^{+\infty} B_k^i \right) \leq \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{H}_\delta^n(B_k^i) \leq \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{H}^n(B_k^i) \\ &= \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } B_k^i)^n = \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{L}^n(B_k^i) = \sum_{i=1}^{+\infty} \mathcal{L}^n \left(\bigcup_{k=1}^{+\infty} B_k^i \right) \\ &= \sum_{i=1}^{+\infty} \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$. The proof is complete. \square

2.3. Densities. We first recall the Lebesgue Density Theorem:

Theorem (Lebesgue Density Theorem). *Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. For any $r > 0$ and $x \in \mathbb{R}^n$, define the approximate Lebesgue density of E in the r -neighborhood of x by*

$$d_r(x) := \frac{\mathcal{L}^n(B(x, r) \cap E)}{\alpha(n)r^n}.$$

Further define the Lebesgue density of E at x by

$$d(x) := \lim_{r \rightarrow 0} d_r(x).$$

Then

$$d(x) = \lim_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x, r) \cap E)}{\alpha(n)r^n} = \begin{cases} 1, & \text{for } \mathcal{L}^n - \text{a.e. } x \in E, \\ 0, & \text{for } \mathcal{L}^n - \text{a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Since $\mathcal{H}^n = \mathcal{L}^n$ for $n \in \mathbb{N}$, the above result clearly holds for \mathcal{H}^n as well. We want to develop some analogous results for lower-dimensional Hausdorff measures. Thus we assume throughout this section that $0 < s < n$.

We first establish a theorem that tells us the lower-dimensional Hausdorff density of a set at a.e. point outside the set is zero.

t2.3-1 Theorem 2.3.1. *Assume that $E \subseteq \mathbb{R}^n$ with E \mathcal{H}^s -measurable and $\mathcal{H}^s(E) < +\infty$. Then*

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} = 0$$

for \mathcal{H}^s -a.e. $x \in \mathbb{R}^n \setminus E$.

Proof. Fix $t > 0$ and define

$$A_t := \left\{ x \in \mathbb{R}^n \setminus E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

It suffices to show that $\mathcal{H}^s(A_t) = 0$.

Note that $\mathcal{H}^s \llcorner E$ is a Radon measure, and so, if we fix $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that

$$\mathcal{H}^s(E \setminus K) \leq \epsilon.$$

Set $U := \mathbb{R}^n \setminus K$. Then U is open and $A_t \subseteq U$ because $K \subseteq E$. Fix $\delta > 0$ and consider

$$\mathcal{F} := \left\{ B(x, r) : B(x, r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

By the Vitali Covering Lemma, there exists a countable family of balls $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ such that

$$A_t \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

Thus by monotonicity

$$\begin{aligned} \mathcal{H}_{10\delta}^s(A_t) &\leq \mathcal{H}_{10\delta}^s\left(\bigcup_{i=1}^{+\infty} B(x_i, 5r_i)\right) \leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (10r_i)^s \leq \sum_{i=1}^{+\infty} 5^s \alpha(s) r_i^s \\ &\leq \frac{5^s}{t} \sum_{i=1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \leq \frac{5^s}{t} \mathcal{H}^s(U \cap E) = \frac{5^s}{t} \mathcal{H}^s(E \setminus K) \\ &\leq \frac{5^s}{t} \epsilon. \end{aligned}$$

Letting $\delta \rightarrow 0$, we obtain $\mathcal{H}^s(A_t) \leq \frac{5^s}{t} \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\mathcal{H}^s(A_t) = 0$ for each $t > 0$. The proof is complete. \square

Now we prove that the lower-dimensional Hausdorff density of a set at a.e. point in the set is nonzero. Note that this contrasts with the Lebesgue Density Theorem: the density may not be 1. However, it is bounded below if we replace the limit with limit superior.

t2.3-2 **Theorem 2.3.2.** *Assume that $E \subseteq \mathbb{R}^n$ with $E\mathcal{H}^s$ -measurable and $\mathcal{H}^s(E) < +\infty$. Then*

$$\frac{1}{2^s} \leq \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} \leq 1$$

for \mathcal{H}^s -a.e. $x \in E$.

Remark. *It is possible to have*

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} < 1$$

and

$$\liminf_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} = 0$$

for \mathcal{H}^s -a.e. $x \in E$, even if $0 < \mathcal{H}^s(E) < +\infty$.

Proof. (i) We first show the upper inequality. Fix $\epsilon > 0$, $t > 1$, and define

$$B_t := \left\{ x \in E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

Since $\mathcal{H}^s \llcorner E$ is Radon, there exists an open set U containing B_t such that

$$\mathcal{H}^s(U \cap E) \leq \mathcal{H}^s(B_t) + \epsilon.$$

Define

$$\mathcal{F} := \left\{ B(x, r) : B(x, r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

By a corollary of the Vitali Covering Lemma, there exists a countable family of disjoint balls $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ such that

$$B_t \subseteq \left(\bigcup_{i=1}^m B(x_i, r_i) \right) \cup \left(\bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right).$$

Thus

$$\begin{aligned} \mathcal{H}_{10\delta}^s(B_t) &\leq \mathcal{H}_{10\delta}^s \left(\bigcup_{i=1}^m B(x_i, r_i) \right) + \mathcal{H}_{10\delta}^s \left(\bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right) \\ &\leq \sum_{i=1}^m \frac{\alpha(s)}{2^s} (2r_i)^s + \sum_{i=m+1}^{+\infty} \frac{\alpha(s)}{2^s} (10r_i)^s \\ &\leq \sum_{i=1}^m \alpha(s)r_i^s + \sum_{i=m+1}^{+\infty} 5^s \alpha(s)r_i^s \\ &\leq \frac{1}{t} \sum_{i=1}^m \mathcal{H}^s(B(x_i, r_i) \cap E) + \frac{5^s}{t} \sum_{i=m+1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \\ &\leq \frac{1}{t} \mathcal{H}^s(U \cap E) + \frac{5^s}{t} \mathcal{H}^s \left(\bigcup_{i=m+1}^{+\infty} B(x_i, r_i) \cap E \right). \end{aligned}$$

Note that this holds for each $m = 1, 2, \dots$. Thus taking the limit as $m \rightarrow \infty$ gives

$$\mathcal{H}_{10\delta}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(U \cap E) \leq \frac{1}{t} (\mathcal{H}^s(B_t) + \epsilon).$$

Letting $\delta \rightarrow 0$, we obtain

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} (\mathcal{H}^s(B_t) + \epsilon),$$

and then taking the limit as $\epsilon \rightarrow 0$ gives

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(B_t).$$

Since $\mathcal{H}^s(B_t) \leq \mathcal{H}^s(E) < +\infty$, this implies that $\mathcal{H}^s(B_t) = 0$ for each $t > 1$, as required.

(ii) We now show that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}$$

for \mathcal{H}^s -a.e. $x \in E$.

For any $\delta > 0$ and $0 < \tau < 1$, denote by $E(\delta, \tau)$ the set of all points $x \in E$ such that

$$\mathcal{H}_\delta^s(C \cap E) \leq \frac{\alpha(s)}{2^s} \tau (\text{diam } C)^s,$$

whenever $C \subseteq \mathbb{R}^n$, $x \in C$, and $\text{diam } C \leq \delta$. Then if $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ with $\text{diam } C_i \leq \delta$, $E(\delta, \tau) \subseteq \cup_{i=1}^{+\infty} C_i$, and $C_i \cap E(\delta, \tau) \neq \emptyset$, we have

$$\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \sum_{i=1}^{+\infty} \mathcal{H}_\delta^s(C_i \cap E(\delta, \tau)) \leq \tau \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_i)^s.$$

Taking the infimum over all such covers $\{C_i\}_{i=1}^{+\infty}$ of $E(\delta, \tau)$, we see that

$$\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \tau \mathcal{H}_\delta^s(E(\delta, \tau)),$$

and so $\mathcal{H}_\delta^s(E(\delta, \tau)) = 0$, since $0 < \tau < 1$ and $\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \mathcal{H}_\delta^s(E) \leq \mathcal{H}^s(E) < +\infty$. In particular,

$$\mathcal{H}^s(E(1 - \delta, \delta)) = 0 \tag{2.3.1}$$

for any $0 < \delta < 1$. Now if $x \in E$ and

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1}{2^s},$$

there exists $\delta > 0$ such that

$$\frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1 - \delta}{2^s} \tag{2.3.2}$$

for all $0 < r \leq \delta$. Thus if $x \in C$ and $\text{diam } C \leq \delta$,

$$\begin{aligned} \mathcal{H}_\delta^s(C \cap E) &= \mathcal{H}_\infty^s(C \cap E) \\ &\leq \mathcal{H}_\infty^s(B(x, \text{diam } C) \cap E) \\ &\leq (1 - \delta) \frac{\alpha(s)}{2^s} (\text{diam } C)^s, \end{aligned}$$

by (2.3.2). Consequently $x \in E(\delta, 1 - \delta)$, and it follows

$$\left\{ x \in E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1}{2^s} \right\} \subseteq \left\{ \bigcup_{k=2}^{+\infty} E\left(\frac{1}{k}, 1 - \frac{1}{k}\right) \right\}.$$

But since the RHS has \mathcal{H}^s -measure zero by (2.3.1), this proves (ii).

(iii) Since $\mathcal{H}^s(B(x, r) \cap E) \geq \mathcal{H}_\infty^s(B(x, r) \cap E)$ for any $x \in E$ and $r > 0$, (ii) immediately gives the required lower estimate

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}.$$

The proof is complete. □

2.4. Hausdorff Measure and Elementary Properties of Functions. We establish some properties relating the behavior of certain functions and Hausdorff measure.

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2.4.1. Hausdorff Measure and Lipschitz Mappings.

Definition 2.4.1 (Lipschitz). A function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called Lipschitz if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Definition 2.4.2 (Lipschitz Constant). We define the Lipschitz constant of a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$\text{Lip}(f) := \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Note that for any Lipschitz function f ,

$$|f(x) - f(y)| \leq \text{Lip}(f)|x - y|.$$

t2.4-1 Theorem 2.4.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. Then

$$\mathcal{H}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}^s(A).$$

Proof. Fix $\delta > 0$ and choose sets $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$ such that $\text{diam } C_i \leq \delta$, $A \subseteq \cup_{i=1}^{+\infty} C_i$. Then

$$\text{diam } f(C_i) \leq \text{Lip}(f) \text{diam } C_i \leq \delta \text{Lip}(f),$$

and $f(A) \subseteq f(\cup_{i=1}^{+\infty} C_i) = \cup_{i=1}^{+\infty} f(C_i)$. Thus

$$\begin{aligned} \mathcal{H}_{\delta \text{Lip}(f)}^s(f(A)) &\leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } f(C_i))^s \\ &\leq (\text{Lip}(f))^s \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_i)^s. \end{aligned}$$

Taking the infimum over all such sets $\{C_i\}_{i=1}^{+\infty}$ which cover A , we find on the RHS

$$\mathcal{H}_{\delta \text{Lip}(f)}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}_{\delta}^s(A).$$

Taking the limit as $\delta \rightarrow 0$, we obtain

$$\mathcal{H}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}^s(A),$$

as required. The proof is complete. □

c2.4-1 Corollary 2.4.1. Suppose that $n > k$. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be the usual projection, $A \subseteq \mathbb{R}^n$, $0 \leq s < +\infty$. Then

$$\mathcal{H}^s(P(A)) \leq \mathcal{H}^s(A).$$

Proof. Since P is the standard projection map from \mathbb{R}^n to \mathbb{R}^k , $\text{Lip}(P) = 1$. Applying the above theorem (cf. (2.4.1)) gives the required estimate. □

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2.4.2. Graphs of Lipschitz Functions.

Definition 2.4.3 (Graph). For $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, we define the graph $\Gamma(f; A)$ of f over A by

$$\Gamma(f; A) := \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$

t2.4-2

Theorem 2.4.2. Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\mathcal{L}^n(A) > 0$.

- (i) Then $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$;
- (ii) If f is Lipschitz, then $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$.

Remark. We thus see that the graph of a Lipschitz function f has the expected Hausdorff dimension (think of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$). We will see from the Area Formula that $\mathcal{H}^s(\Gamma(f; A))$ can be computed according to the usual rules of calculus.

Proof.

- (i). Let $P : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be the usual projection. Then by (2.4.1),

$$\mathcal{H}^n(\Gamma(f; A)) \geq \mathcal{H}^n(A) > 0.$$

Thus $\mathcal{H}^n(\Gamma(f; A)) > 0$, so that $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$.

- (ii). Let Q denote any cube in \mathbb{R}^n of side length 1. Subdivide Q into k^n subcubes $\{Q_1, \dots, Q_{k^n}\}$ of side length $\frac{1}{k}$. Note that $\text{diam } Q_i = \frac{\sqrt{n}}{k}$ for each $i = 1, \dots, k^n$. Define

$$a_j^i := \min_{x \in Q_j} f^i(x), \quad b_j^i := \max_{x \in Q_j} f^i(x),$$

where $i = 1, \dots, m$ and $j = 1, \dots, k^n$. Since f is Lipschitz,

$$|b_j^i - a_j^i| \leq \text{Lip}(f) \text{diam } Q_j = \text{Lip}(f) \frac{\sqrt{n}}{k}.$$

For each $j = 1, \dots, k^n$, put

$$C_j := Q_j \times \prod_{i=1}^m (a_j^i, b_j^i).$$

Then

$$\Gamma(f; Q_j \cap A) = \{(x, f(x)) : x \in Q_j \cap A\} \subseteq C_j,$$

and $\text{diam } C_j \leq \frac{C}{k}$ for some constant $C > 0$. Since

$$\Gamma(f; A \cap Q) = \Gamma(f; A \cap \bigcup_{j=1}^{k^n} Q_j) = \bigcup_{j=1}^{k^n} \Gamma(f; A \cap Q_j) \subseteq \bigcup_{j=1}^{k^n} C_j,$$

we have by monotonicity

$$\begin{aligned} \mathcal{H}_{C/k}^n(G(f; A \cap Q)) &\leq \sum_{j=1}^{k^n} \frac{\alpha(n)}{2^n} (\text{diam } C_j)^n \\ &\leq \frac{k^n \alpha(n)}{2^n} \left(\frac{C}{k}\right)^n = \frac{C^n \alpha(n)}{2^n}. \end{aligned}$$

Then upon letting $k \rightarrow +\infty$, we find $\mathcal{H}^n(\Gamma(f; A \cap Q)) < +\infty$, and so $\mathcal{H}_{\dim}(\Gamma(f; A \cap Q)) \leq n$. Recall that this estimate is valid for each cube $Q \subseteq \mathbb{R}^n$ of side length 1. Consequently $\mathcal{H}_{\dim}(\Gamma(f; A)) \leq n$. Applying (i), it follows $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$. The proof is complete. \square

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2.4.3. *The Set Where an Integrable Function is Large.* If a function f is locally integrable, we can estimate the Hausdorff measure of the set where f is locally large.

t2.4-3 **Theorem 2.4.3.** *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let $0 \leq s < n$, and define*

$$\Lambda_s := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) > 0. \right\}$$

Then

$$\mathcal{H}^s(\Lambda_s) = 0.$$

Proof. We may as well assume that $f \in L^1(\mathbb{R}^n)$. By the Lebesgue Differentiation Theorem,

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) = |f(x)|$$

for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$, and thus

$$\lim_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) = \lim_{r \rightarrow 0} \alpha(n) r^{n-s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) = \lim_{r \rightarrow 0} \alpha(n) r^{n-s} |f(x)| = 0$$

for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$, since $0 \leq s < n$. Hence

$$\mathcal{L}^n(\Lambda_s) = 0.$$

Fix $\epsilon > 0$, $\delta > 0$, $\sigma > 0$. Since f is \mathcal{L}^n -integrable, there exists $\eta > 0$ such that $\mathcal{L}^n(\Omega) \leq \eta$ implies

$$\int_{\Omega} |f(x)| \, d\mathcal{L}^n(x) < \sigma.$$

Define

$$\Lambda_s^\epsilon := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) > \epsilon \right\}.$$

By the above analysis,

$$\mathcal{L}^n(\Lambda_s^\epsilon) = 0.$$

Thus there exists an open set $\Omega \subseteq \mathbb{R}^n$ such that $\Lambda_s^\epsilon \subseteq \Omega$ and $\mathcal{L}^n(\Omega) < \eta$. Put

$$\mathcal{F} := \left\{ B(x, r) : x \in \Lambda_s^\epsilon, 0 < r < \delta, B(x, r) \subseteq \Omega, \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) > \epsilon r^s \right\}.$$

By the Vitali Covering Lemma, there exists a countable family $\{B(x_i, r_i)\}_{i=1}^{+\infty}$ of disjoint balls in \mathcal{F} such that

$$\Lambda_s^\epsilon \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

We thus compute

$$\begin{aligned} \mathcal{H}_{10\delta}^s(\Lambda_s^\epsilon) &\leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } B(x_i, 5r_i))^s \leq \sum_{i=1}^{+\infty} \alpha(s) (5r_i)^s \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \sum_{i=1}^{+\infty} \int_{B(x_i, r_i)} |f(y)| \, d\mathcal{L}^n(y) \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \int_{\Omega} |f(y)| \, d\mathcal{L}^n(y) \end{aligned}$$

$$\leq \frac{\alpha(s)5^s}{\epsilon}\sigma.$$

Taking the limit as $\delta \rightarrow 0$, we have

$$\mathcal{H}^s(\Lambda_s^\epsilon) \leq \frac{\alpha(s)5^s}{\epsilon}\sigma,$$

and then upon sending $\sigma \rightarrow 0$ we obtain

$$\mathcal{H}^s(\Lambda_s^\epsilon) = 0.$$

Since $\epsilon > 0$ was arbitrary, it follows

$$\mathcal{H}^s(\Lambda_s) = 0.$$

The proof is complete. □

3. AREA AND COAREA FORMULAS

3.1. Lipschitz Functions, Rademacher's Theorem.

Definition 3.1.1 (Lipschitz). Let $A \subseteq \mathbb{R}^n$. A function $f : A \rightarrow \mathbb{R}^m$ is called *Lipschitz* provided that

$$|f(x) - f(y)| \leq C|x - y| \quad (3.1.1)$$

for some constant $C > 0$ and all $x, y \in A$. The smallest constant C such that (3.1.1) holds for all $x, y \in A$ is denoted

$$\text{Lip}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y \right\}.$$

Definition 3.1.2 (Locally Lipschitz). A function $f : A \rightarrow \mathbb{R}^m$ is called *locally Lipschitz* if for each compact set $K \subseteq A$, there exists a constant $C_K > 0$ such that

$$|f(x) - f(y)| \leq C_K|x - y|$$

for all $x, y \in K$.

t3.1-1

Theorem 3.1.1 (Extension of Lipschitz Functions). Assume that $A \subseteq \mathbb{R}^n$, and let $f : A \rightarrow \mathbb{R}^m$ be Lipschitz. There exists a Lipschitz function $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

- (i) $\bar{f} = f$ on A ;
- (ii) $\text{Lip}(\bar{f}) \leq \sqrt{m} \text{Lip}(f)$.

Proof.

(i). First assume that $f : A \rightarrow \mathbb{R}$. Define

$$\bar{f}(x) := \inf_{a \in A} \{f(a) + \text{Lip}(f)|x - a|\}.$$

If $b \in A$, then we have $\bar{f}(b) = f(b)$. This follows because if $b \in A$, then

$$\bar{f}(b) \leq f(b) + \text{Lip}(f)|b - b| = f(b).$$

On the other hand, for all $a \in A$, we've:

$$f(a) + \text{Lip}(f)|b - a| \geq f(a) + \frac{f(b) - f(a)}{|b - a|}|b - a| = f(b).$$

Taking the infimum over all $a \in A$ on the LHS thus gives $\bar{f}(b) \geq f(b)$. Now if $x, y \in \mathbb{R}^n$, then

$$\begin{aligned} \bar{f}(x) &\leq \inf_{a \in A} \{f(a) + \text{Lip}(f)(|x - y| + |y - a|)\} \\ &= \inf_{a \in A} \{f(a) + \text{Lip}(f)|y - a|\} + \text{Lip}(f)|x - y| \\ &= \bar{f}(y) + \text{Lip}(f)|x - y|. \end{aligned}$$

Similarly

$$\bar{f}(y) \leq \bar{f}(x) + \text{Lip}(f)|x - y|.$$

Therefore

$$\frac{|\bar{f}(x) - \bar{f}(y)|}{|x - y|} \leq \text{Lip}(f)$$

for all $x, y \in A$. This proves the result for functions $f : A \rightarrow \mathbb{R}$.

(ii). In the general case $f : A \rightarrow \mathbb{R}^m$, $f = (f^1, \dots, f^m)$, define $\bar{f} := (\bar{f}^1, \dots, \bar{f}^m)$, where \bar{f}^i , $i = 1, \dots, m$, are defined as in (i). Then

$$|\bar{f}(x) - \bar{f}(y)|^2 = \sum_{i=1}^m \left| \bar{f}^i(x) - \bar{f}^i(y) \right|^2 \leq m(\text{Lip}(f))^2 |x - y|^2.$$

Taking square roots,

$$|\bar{f}(x) - \bar{f}(y)| \leq \sqrt{m} \text{Lip}(f) |x - y|,$$

as required. The proof is complete. \square

Remark. In fact there exists an extension \bar{f} of f with $\text{Lip}(\bar{f}) = \text{Lip}(f)$. This is Kirszbraun's Theorem.

We now prove Rademacher's Theorem, which states that a locally Lipschitz function is differentiable \mathcal{L}^n -a.e. Note that the inequality

$$|f(x) - f(y)| \leq \text{Lip}(f) |x - y|$$

says nothing about the possibility of locally approximating f by a linear map.

Definition 3.1.3 (Differentiable). *The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping*

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that

$$\lim_{y \rightarrow x} \frac{|f(y) - f(x) - L(x - y)|}{|x - y|} = 0,$$

or, equivalently,

$$f(y) = f(x) + L(x - y) + o(|y - x|), \quad y \rightarrow x.$$

Remark.

- (i) Note that this is actually the definition of the Fréchet derivative.
- (ii) If such a linear mapping L exists, it is unique, and we write

$$Df(x)$$

for L . We call $Df(x)$ the derivative of f at x .

t3.1-2

Theorem 3.1.2 (Rademacher's Theorem). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable \mathcal{L}^n -a.e.*

Proof.

(i). We may assume that $m = 1$, for otherwise, repeat the below argument m times. Since differentiability is a local property, we may as well also suppose that f is Lipschitz.

(ii). Fix any $v \in \mathbb{R}^n$ with $|v| = 1$, and for any $x \in \mathbb{R}^n$, define the Gateaux derivative

$$D_v f(x) := \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

at x , provided that this limit exists.

(iii). We show that $D_v f(x)$ exists for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. Since f is continuous,

$$\overline{D}_v f(x) = \limsup_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

$$= \lim_{k \rightarrow +\infty} \sup_{\substack{0 < |t| < \frac{1}{k} \\ t \in \mathbb{Q}}} \frac{f(x + tv) - f(x)}{t}$$

is Borel measurable, as is

$$\underline{D}_v f(x) := \liminf_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Thus

$$\begin{aligned} A_v &:= \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist}\} \\ &= \{x \in \mathbb{R}^n : \underline{D}_v f(x) < \overline{D}_v f(x)\}, \end{aligned}$$

being the complement of the set of all points of which the pointwise limit of measurable functions exists, is Borel measurable.

Now, for each $x, v \in \mathbb{R}^n$ with $|v| = 1$, define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(t) := f(x + tv).$$

Note that for any $t \in \mathbb{R}$,

$$\begin{aligned} |\phi(t) - \phi(s)| &= |f(x + tv) - f(x + sv)| \leq \text{Lip}(f)|(x + tv) - (x + sv)| \\ &= \text{Lip}(f)|t - s|, \end{aligned}$$

so that ϕ is Lipschitz. Therefore ϕ is absolutely continuous, and thus differentiable \mathcal{L}^1 -a.e. Thus for any line L parallel to v , the set of all points on L such that f is not differentiable has Lebesgue measure zero. That is,

$$\mathcal{H}^1(A_v \cap L) = 0$$

for each line L parallel to v . Thus the Fubini–Tonelli Theorem implies

$$\mathcal{L}^n(A_v) = 0,$$

as required.

(iv). Noting that

$$\frac{\partial}{\partial x_j} f(x) = D_{e_j} f(x) = \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t}$$

for each $j = 1, \dots, n$, we have by (iii) that

$$\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \dots, \frac{\partial}{\partial x_n} f(x) \right)$$

exists for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$.

(v). Next we show that $D_v f(x) = v \cdot \nabla f(x)$ for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. Let $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left[\frac{f(x + tv) - f(x)}{t} \right] \zeta(x) \, dx &= \frac{1}{t} \left[\int_{\mathbb{R}^n} f(x + tv) \zeta(x) \, dx - \int_{\mathbb{R}^n} f(x) \zeta(x) \, dx \right] \\ &= \frac{1}{t} \left[\int_{\mathbb{R}^n} f(x) \zeta(x - tv) \, dx - \int_{\mathbb{R}^n} f(x) \zeta(x) \, dx \right] \\ &= - \int_{\mathbb{R}^n} f(x) \left[\frac{\zeta(x) - \zeta(x - tv)}{t} \right] \, dx. \end{aligned}$$

This is the integration by parts formula for difference quotients. Let $t = \frac{1}{k}$ for $k = 1, 2, \dots$, in the above equality and note that

$$\frac{|f(x + \frac{1}{k}v) - f(x)|}{\frac{1}{k}} \leq \text{Lip}(f).$$

Thus, by Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} D_v f(x) \zeta(x) \, dx &\stackrel{LDC}{=} - \int_{\mathbb{R}^n} f(x) D_v \zeta(x) \, dx \\ &= - \sum_{j=1}^n v_j \int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_j} \zeta(x) \, dx \\ &= \sum_{j=1}^n v_j \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} f(x) \zeta(x) \, dx \\ &= \int_{\mathbb{R}^n} (v \cdot \nabla f(x)) \zeta(x) \, dx, \end{aligned}$$

where we have used integration by parts and the partial derivatives on f are understood in the a.e. sense. Since the above equality holds for every $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have $D_v f = v \cdot \nabla f$ \mathcal{L}^n -a.e.

(vi). Choose $\{v_k\}_{k=1}^{+\infty}$ to be a countable, dense subset of $\partial B(0, 1)$. Set

$$A_k := \{x \in \mathbb{R}^n : D_{v_k} f(x), \nabla f(x) \text{ exist and } D_{v_k} f(x) = v_k \cdot \nabla f(x)\}$$

for each $k \in \mathbb{N}$. Note that by (iii)-(v), $\mathcal{L}^n(\mathbb{R}^n \setminus A_k) = 0$ for each $k \in \mathbb{N}$. Define

$$A := \bigcap_{k=1}^{+\infty} A_k$$

and observe that

$$\mathcal{L}^n(\mathbb{R}^n \setminus A) = \mathcal{L}^n(\mathbb{R}^n \setminus \bigcap_{k=1}^{+\infty} A_k) = \mathcal{L}^n(\bigcup_{k=1}^{+\infty} (\mathbb{R}^n \setminus A_k)) = 0.$$

(vii). We now show that f is differentiable at each point $x \in A$. Fix any $x \in A$. Choose $v \in \partial B(0, 1)$, $t \in \mathbb{R}$, $t \neq 0$, and write

$$Q(x, v, t) := \frac{f(x + tv) - f(x)}{t} - v \cdot \nabla f(x).$$

Then if $w \in \partial B(0, 1)$, we have

$$\begin{aligned} |Q(x, v, t) - Q(x, w, t)| &= \left| \frac{f(x + tv) - f(x + tw)}{t} - (v - w) \cdot \nabla f(x) \right| \\ &\leq \left| \frac{f(x + tv) - f(x + tw)}{t} \right| + |(v - w) \cdot \nabla f(x)| \\ &\leq \text{Lip}(f)|v - w| + |\nabla f(x)||v - w| \\ &\leq (1 + \sqrt{n}) \text{Lip}(f)|v - w|. \end{aligned} \tag{3.1.2}$$

Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ so large that if $v \in \partial B(0, 1)$, then

$$|v - v_k| \leq \frac{\epsilon}{2(1 + \sqrt{n}) \text{Lip}(f)}$$

for some $k = 1, \dots, N$. Note that since $x \in A$,

$$\begin{aligned}\lim_{t \rightarrow 0} Q(x, v_k, t) &= \lim_{t \rightarrow 0} \left\{ \frac{f(x + tv_k) - f(x)}{t} - v_k \cdot \nabla f(x) \right\} \\ &= D_{v_k} f(x) - v_k \cdot \nabla f(x) \\ &= 0\end{aligned}$$

for each $k = 1, \dots, N$. Thus there exists $\delta > 0$ so that for all $0 < |t| < \delta$,

$$|Q(x, v_k, t)| < \frac{\epsilon}{2} \tag{3.1.3}$$

{eq:3.1-3}

holds for each $k = 1, \dots, N$. Consequently for each $v \in \partial B(0, 1)$ there exists $k \in \{1, \dots, N\}$ such that

$$\begin{aligned}|Q(x, v, t)| &\leq |Q(x, v, t) - Q(x, v_k, t)| + |Q(x, v_k, t)| \\ &< (1 + \sqrt{n}) \text{Lip}(f) |v - v_k| + \frac{\epsilon}{2} \\ &< \epsilon,\end{aligned}$$

by [\(3.1.2\)](#) and [\(3.1.3\)](#), provided that $0 < |t| < \delta$. Note that this is the same $\delta > 0$ for all $v \in \partial B(0, 1)$.

Now choose any $x, y \in \mathbb{R}^n$, $y \neq x$. Write

$$v := \frac{y - x}{|y - x|},$$

so that $y = x + tv$, where $t := |x - y|$. Then

$$\begin{aligned}|f(y) - f(x) - \nabla f(x) \cdot (y - x)| &= |f(x + tv) - f(x) - \nabla f(x) \cdot tv| \\ &= |Q(x, t, v)| |t| \\ &< \epsilon |t|,\end{aligned}$$

so that

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) = o(t) = o(|x - y|), \quad y \rightarrow x.$$

Hence, f is differentiable at x , with

$$Df(x) = \nabla f(x).$$

The proof is complete. □

c3.1-1

Corollary 3.1.1.

(i) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be locally Lipschitz, and

$$\mathcal{Z} := \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Then $Df(x) = 0$ for \mathcal{L}^n -a.e. $x \in \mathcal{Z}$.

(ii) Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, and

$$Y := \{x \in \mathbb{R}^n : g(f(x)) = x\}.$$

Then

$$Dg(f(x))Df(x) = I$$

for \mathcal{L}^n -a.e. $x \in Y$.

Proof.

- (i). We may assume that $m = 1$ in (i), otherwise, repeat the following argument m times.
- (ii). Choose $x \in \mathcal{Z}$ so that $Df(x)$ exists, and

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(\mathcal{Z} \cap B(x, r))}{\mathcal{L}^n(B(x, r))} = 1. \quad (3.1.4) \quad \boxed{\text{eq:3.1-4}}$$

Note that this holds for \mathcal{L}^n -a.e. $x \in \mathcal{Z}$. Since $x \in \mathcal{Z}$, it follows

$$f(y) = Df(x) \cdot (y - x) + o(|y - x|). \quad (3.1.5) \quad \boxed{\text{eq:3.1-5}}$$

By contradiction, suppose that $Df(x) = \alpha \neq 0$, and set

$$S := \left\{ v \in \partial B(0, 1) : \alpha \cdot v \geq \frac{1}{2}|\alpha| \right\}.$$

Note that S is nonempty, for otherwise $Df(x) = 0$. Now for each $v \in S$ and $t > 0$, set $y := x + tv$ in (3.1.5) to obtain

$$\begin{aligned} f(x + tv) &= \alpha \cdot tv + o(|tv|) \\ &\geq \frac{|\alpha|}{2}t + o(t). \end{aligned}$$

Hence, there exists $\delta > 0$ such that for all $0 < t < \delta$ and all $v \in S$,

$$f(x + tv) > 0.$$

But this contradicts (3.1.4), since for all $0 < r < \delta$, $B(x, r) \cap \mathcal{Z} = \{x\}$. This proves (i).

- (iii). We now show (ii). Define

$$\text{dom } Df := \{x \in \mathbb{R}^n : Df(x) \text{ exists}\}$$

and

$$\text{dom } Dg := \{x \in \mathbb{R}^n : Dg(x) \text{ exists}\}.$$

Put

$$X := Y \cap \text{dom } Df \cap f^{-1}(\text{dom } Dg).$$

Then

$$\begin{aligned} Y \setminus X &= Y \cap (Y^C \cup (\text{dom } Df)^C \cup (f^{-1}(\text{dom } Dg))^C) \\ &= (Y \setminus \text{dom } Df) \cup (Y \setminus f^{-1}(\text{dom } Dg)) \\ &\subseteq (\mathbb{R}^n \setminus \text{dom } Df) \cup g(\mathbb{R}^n \setminus \text{dom } Dg). \end{aligned} \quad (3.1.6) \quad \boxed{\text{eq:3.1-6}}$$

This follows since if $x \in Y \setminus f^{-1}(\text{dom } Dg)$, then $f(x) \in f(Y) \subseteq \mathbb{R}^n$, and $f(x) \notin \text{dom } Dg$, so that

$$f(x) \in \mathbb{R}^n \setminus \text{dom } Dg.$$

Thus

$$x = g(f(x)) \in g(\mathbb{R}^n \setminus \text{dom } Dg.)$$

By Rademacher's Theorem (cf. (3.1.2)),

$$\mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Df) = 0$$

and

$$\mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Dg) = 0.$$

Moreover, since g is Lipschitz (cf. (t2.4-1)), we have

$$\mathcal{L}^n(g(\mathbb{R}^n \setminus \text{dom } Dg)) \leq (\text{Lip}(g))^n \mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Dg) = 0.$$

Thus, by (eq:3.1-6),

$$\mathcal{L}^n(Y \setminus X) = 0.$$

Now if $x \in X$, $Dg(f(x))$ and $Df(x)$ exist, and so the chain rule implies

$$Dg(f(x))Df(x) = D(g \circ f)(x)$$

exists. Finally, since $(g \circ f)(x) - x = g(f(x)) - x = 0$ on Y , assertion (i) gives

$$Dg(f(x))Df(x) = D(g \circ f)(x) = I$$

\mathcal{L}^n -a.e. on Y . The proof is complete. \square

3.2. Linear Maps and Jacobians. We first review some basic linear algebra. Our goal in this section is to define the Jacobian of a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

3.2.1. Linear Maps.

Definition 3.2.1 (Orthogonal Linear Map). *A linear map $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is orthogonal if*

$$Ox \cdot Oy = x \cdot y$$

for all $x, y \in \mathbb{R}^n$.

Definition 3.2.2 (Symmetric Linear Map). *A linear map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric if*

$$x \cdot Sy = Sx \cdot y$$

for all $x, y \in \mathbb{R}^n$.

Definition 3.2.3 (Diagonal Linear Map). *A linear map $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is diagonal if there exist $d_1, \dots, d_n \in \mathbb{R}$ such that*

$$Dx = (d_1x_1, \dots, d_nx_n)$$

for all $x \in \mathbb{R}^n$.

Definition 3.2.4 (Adjoint). *Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. The adjoint of A is the linear map $A^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$ defined by*

$$x \cdot A^*y = Ax \cdot y$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Recall that the existence of adjoints in Euclidean space with the Euclidean metric is guaranteed, and, since \mathbb{R}^n is a Hilbert space under the Euclidean metric, the adjoint operator has the above form by the Riesz Representation Theorem.

t3.2-1

Theorem 3.2.1.

- (i) $A^{**} = A$;
- (ii) $(A \circ B)^* = B^* \circ A^*$;
- (iii) If $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal, then $O^* = O^{-1}$;
- (iv) If $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric, then $S^* = S$;

(v) If $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is symmetric, there exists an orthogonal map $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a diagonal map $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$S = O \circ D \circ O^{-1};$$

(vi) If $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is orthogonal, then $n \leq m$ and

$$\begin{aligned} O^* \circ O &= I \quad \text{on } \mathbb{R}^n, \\ O \circ O^* &= I \quad \text{on } O(\mathbb{R}^n). \end{aligned}$$

Proof.

(i). Since the dot product is symmetric, we have for all $x, y \in \mathbb{R}^n$ that

$$\begin{aligned} x \cdot (A^{**}y) &= x \cdot (A^*)^*y = A^*x \cdot y = y \cdot A^*x = Ay \cdot x \\ &= x \cdot Ay. \end{aligned}$$

Since this is for all $x \in \mathbb{R}^n$, assertion (i) follows.

(ii). For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \cdot (A \circ B)^*y &= (A \circ B)x \cdot y = A(Bx) \cdot y = Bx \cdot A^*y \\ &= x \cdot B^*(A^*y). \end{aligned}$$

This is for all $x \in \mathbb{R}^n$, so this proves (ii).

(iii). Let $x, y \in \mathbb{R}^n$. Then

$$x \cdot y = Ox \cdot Oy = x \cdot O^*(Oy),$$

and

$$x \cdot y = O(O^{-1}x) \cdot y = O^{-1}x \cdot O^*y = x \cdot O(O^*y).$$

This shows $O^* = O^{-1}$.

(iv). If $x, y \in \mathbb{R}^n$, then

$$x \cdot Sy = Sx \cdot y = x \cdot S^*y,$$

and since this is for all $x \in \mathbb{R}^n$, assertion (iv) follows. \square

t3.2-2

Theorem 3.2.2 (Polar Decomposition). *Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear mapping.*

(i) *If $n \leq m$, there exists a symmetric map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and an orthogonal map $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that*

$$L = O \circ S.$$

(ii) *If $n \geq m$, there exists a symmetric map $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and an orthogonal map $O : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that*

$$L = S \circ O^*.$$

Proof.

(i). First suppose $n \leq m$. Consider the mapping $C := L^* \circ L : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Now for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} Cx \cdot y &= (L^* \circ L)x \cdot y = L^*(Lx) \cdot y = Lx \cdot Ly = x \cdot L^*(Ly) = x \cdot (L^* \circ L)y \\ &= x \cdot Cy, \end{aligned}$$

and also

$$Cx \cdot x = (L^* \circ L)x \cdot x = L^*(Lx) \cdot x = Lx \cdot Lx \geq 0.$$

Thus C is symmetric and positive semidefinite. Hence there exist $\mu_1, \dots, \mu_n \geq 0$ and an orthonormal basis $\{x_k\}_{k=1}^n$ of \mathbb{R}^n such that

$$Cx_k = \mu_k x_k,$$

$k = 1, \dots, n$. Write $\mu_k := \lambda_k^2$, $\lambda_k \geq 0$, $k = 1, \dots, n$.

(ii). We show that there exists an orthonormal set $\{z_k\}_{k=1}^n$ in \mathbb{R}^m such that

$$Lx_k = \lambda_k z_k,$$

$k = 1, \dots, n$. To see this, if $\lambda_k \neq 0$, define

$$z_k := \frac{1}{\lambda_k} Lx_k.$$

Then if $\lambda_k, \lambda_l \neq 0$,

$$\begin{aligned} z_k \cdot z_l &= \frac{1}{\lambda_k} Lx_k \cdot \frac{1}{\lambda_l} Lx_l = \frac{1}{\lambda_k \lambda_l} Lx_k \cdot Lx_l = \frac{1}{\lambda_k \lambda_l} x_k \cdot L^*(Lx_l) = \frac{1}{\lambda_k \lambda_l} x_k \cdot Cx_l \\ &= \frac{\lambda_l^2}{\lambda_k \lambda_l} x_k \cdot x_l \\ &= \frac{\lambda_l}{\lambda_k} \delta_{kl}, \end{aligned}$$

by (i) and the fact that $\{x_k\}_{k=1}^n$ is an orthonormal set. Thus the set $\{z_k : \lambda_k \neq 0\}$ is orthonormal. If $\lambda_k = 0$, define z_k to be any unit vector such that the set $\{z_k\}_{k=1}^n$ is orthonormal, applying the Gram–Schmidt process if necessary.

(iii). Define $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$Sx_k := \lambda_k x_k,$$

$k = 1, \dots, n$ and $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$Ox_k := z_k,$$

$k = 1, \dots, n$. Then

$$(O \circ S)x_k = O(Sx_k) = O(\lambda_k x_k) = \lambda_k Ox_k = \lambda_k z_k = Lx_k,$$

and, since $\{x_k\}_{k=1}^n$ is a basis for \mathbb{R}^n ,

$$L = O \circ S.$$

Notice that the mapping S is clearly symmetric. Moreover, O is orthogonal because

$$Ox_k \cdot Ox_l = z_k \cdot z_l = \delta_{kl} = x_k \cdot x_l.$$

This proves assertion (i) of the theorem.

(iv). To prove assertion (ii), we apply assertion (i) to L^* and apply $\text{\texttt{3.2.1}}$ to obtain

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

The proof is complete. □

We now define the Jacobian of a linear map.

Definition 3.2.5 (Jacobian). *Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map.*

(i) *If $n \leq m$, write $L = O \circ S$ (cf. $\text{\texttt{3.2.2}}$), and we define the Jacobian of L to be*

$$\llbracket L \rrbracket := |\det S|;$$

(ii) If $n \geq m$, write $L = S \circ O^*$ (cf. t3.2-2), and we define the Jacobian of L to be

$$\llbracket L \rrbracket := |\det S|.$$

Remark.

- (i) It will follow from Theorem t3.2-3 below that the definition of $\llbracket L \rrbracket$ is independent of the particular choices of O and S .
- (ii) Note that if, say, $n \leq m$, then $L = O \circ S$ implies

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

This is the same O and S , and it clearly follows

$$\llbracket L \rrbracket = \llbracket L^* \rrbracket.$$

t3.2-3 Theorem 3.2.3.

- (i) If $n \leq m$,
- $$\llbracket L \rrbracket^2 = \det(L^* \circ L);$$
- (ii) If $n \geq m$,
- $$\llbracket L \rrbracket^2 = \det(L \circ L^*).$$

Proof.

- (i). Assume that $n \leq m$, and apply Theorem t3.2-2 to write

$$L = O \circ S$$

and

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

Then

$$L^* \circ L = (S \circ O^*) \circ (O \circ S) = S \circ (O^* \circ O) \circ S = S \circ S = S^2$$

(cf. t3.2-1). Hence,

$$\det(L^* \circ L) = \det(S^2) = (\det S)^2 = \llbracket L \rrbracket^2,$$

as required.

- (ii). The proof of (ii) is similar. The proof is complete. \square

Theorem t3.2-3 provides us with a nice way to compute the Jacobian $\llbracket L \rrbracket$ of a linear map. We augment this with the Binet–Cauchy formula below.

Definition 3.2.6 ($\Lambda(m, n)$). If $n \leq m$, we define

$$\Lambda(m, n) := \{\lambda : \{1, \dots, n\} \rightarrow \{1, \dots, m\} : \lambda \text{ strictly increasing}\}.$$

Note that this is the set of all functions λ that take $\{1, \dots, n\}$ to $\{1, \dots, m\}$ such that $\lambda(k) > \lambda(l)$ if $k > l$, $k, l \in \{1, \dots, n\}$.

Definition 3.2.7 (P_λ). If $n \leq m$, for each $\lambda \in \Lambda(m, n)$, we define $P_\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$ by

$$P_\lambda(x_1, \dots, x_m) := (x_{\lambda(1)}, \dots, x_{\lambda(n)}).$$

We may think of P_λ as a mapping that “deletes” points from (x_1, \dots, x_m) .

Remark. For each $\lambda \in \Lambda(m, n)$, there exists an n -dimensional subspace

$$S_\lambda := \text{span}\{e_{\lambda(1)}, \dots, e_{\lambda(n)}\} \subseteq \mathbb{R}^m$$

such that P_λ is the projection of \mathbb{R}^m onto S_λ .

t3.2-4

Theorem 3.2.4 (Binet–Cauchy Formula). *Let $n \leq m$ and let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then*

$$\llbracket L \rrbracket^2 = \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda \circ L))^2.$$

Remark.

- (i) *To calculate $\llbracket L \rrbracket$, we compute the sums of the squares of the determinants of each $n \times n$ submatrix of the $m \times n$ matrix representing L , with respect to the standard bases of \mathbb{R}^n and \mathbb{R}^m ;*
- (ii) *This is a kind of higher dimensional version of the Pythagorean Theorem.*

Proof.

(i). Identifying linear maps with their matrices with respect to the standard bases of \mathbb{R}^n and \mathbb{R}^m , we write

$$L : ((l_{ij}))_{m \times n}, \quad A := L^* \circ L = ((a_{ij}))_{n \times n};$$

so that

$$a_{ij} = \sum_{k=1}^m l_{ki} l_{kj}, \quad i, j = 1, \dots, n.$$

(ii). Note that

$$\llbracket L \rrbracket^2 = \det A = \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)},$$

where Σ denotes the set of all permutations of $\{1, \dots, n\}$. Thus

$$\begin{aligned} \llbracket L \rrbracket^2 &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n \sum_{k=1}^m l_{ki} l_{k\sigma(i)} \\ &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\phi \in \Phi} \prod_{i=1}^n l_{\phi(i)i} l_{\phi(i)\sigma(i)}, \end{aligned}$$

where Φ denotes the set of all one-to-one mappings of $\{1, \dots, n\}$ into $\{1, \dots, m\}$.

(iii). Now for each $\phi \in \Phi$, we can uniquely write $\phi := \lambda \circ \theta$, where $\theta \in \Sigma$ and $\lambda \in \Lambda(m, n)$. Consequently we have

$$\begin{aligned} \llbracket L \rrbracket^2 &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \prod_{i=1}^n l_{\lambda \circ \theta(i), i} l_{\lambda \circ \theta(i), \sigma(i)} \\ &= \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \prod_{i=1}^n l_{\lambda(i), \theta^{-1}(i)} l_{\lambda(i), \sigma \circ \theta^{-1}(i)} \\ &= \sum_{\lambda \in \Lambda(m,n)} \sum_{\theta \in \Sigma} \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n l_{\lambda(i), \theta(i)} l_{\lambda(i), \sigma \circ \theta(i)}. \end{aligned}$$

Set $\rho := \sigma \circ \theta$. Then

$$\llbracket L \rrbracket^2 = \sum_{\lambda \in \Lambda(m,n)} \sum_{\rho \in \Sigma} \sum_{\theta \in \Sigma} \operatorname{sgn}(\theta) \operatorname{sgn}(\rho) \prod_{i=1}^n l_{\lambda(i), \theta(i)} l_{\lambda(i), \rho(i)}$$

$$\begin{aligned}
&= \sum_{\lambda \in \Lambda(m,n)} \left(\sum_{\theta \in \Sigma} \operatorname{sgn}(\theta) \prod_{i=1}^n l_{\lambda(i), \theta(i)} \right)^2 \\
&= \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda) \circ L)^2,
\end{aligned}$$

as required. The proof is complete. \square

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3.2.2. Jacobians. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz mapping. By Rademacher's Theorem (cf. (3.1.2)), f is differentiable \mathcal{L}^n -a.e., and therefore $Df(x)$ exists and may be regarded as a linear mapping from \mathbb{R}^n into \mathbb{R}^m for \mathcal{L}^n -a.e. $x \in \mathbb{R}^n$. We recall the definition of a gradient matrix.

Definition 3.2.8 (Gradient Matrix). *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $f = (f^1, \dots, f^m)$, we define the gradient matrix*

$$Df(x) := \begin{bmatrix} \frac{\partial}{\partial x_1} f^1(x) & \cdots & \frac{\partial}{\partial x_n} f^1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f^m(x) & \cdots & \frac{\partial}{\partial x_n} f^m(x) \end{bmatrix}.$$

Definition 3.2.9 (Jacobian). *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, the Jacobian of f is*

$$Jf(x) := \llbracket Df(x) \rrbracket, \quad \mathcal{L}^n - a.e.$$

Note that in view of Theorem (3.2.3), we have

$$(Jf(x))^2 = \det(Df(x)^* \circ Df(x)) = \det(Df(x) \circ Df(x)^*).$$

3.3. The Area Formula. Throughout this section we assume that

$$n \leq m.$$

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3.3.1. Preliminaries.

13.3-1 Lemma 3.3.1. *Suppose that $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, $n \leq m$. Then*

$$\mathcal{H}^n(L(A)) = \llbracket L \rrbracket \mathcal{L}^n(A)$$

for all $A \subseteq \mathbb{R}^n$.

Proof.

(i). Write $L := O \circ S$, where $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an orthogonal map and $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a symmetric map (cf (3.2.2)). Recall that $\llbracket L \rrbracket = |\det S|$.

(ii). If $\llbracket L \rrbracket = 0$, then $\dim S(\mathbb{R}^n) \leq n - 1$, and so $\dim L(\mathbb{R}^n) \leq n - 1$. Consequently $\mathcal{H}^n(L(A)) = 0$, and the inequality is trivial.

(iii). If $\llbracket L \rrbracket > 0$, then

$$\begin{aligned} \frac{\mathcal{H}^n(L(B(x, r)))}{\mathcal{L}^n(B(x, r))} &= \frac{\mathcal{L}^n(O^* \circ L(B(x, r)))}{\mathcal{L}^n(B(x, r))} \\ &= \frac{\mathcal{L}^n(O^* \circ O \circ S(B(x, r)))}{\mathcal{L}^n(B(x, r))} \\ &= \frac{\mathcal{L}^n(S(B(x, r)))}{\mathcal{L}^n(B(x, r))} \\ &= \frac{\mathcal{L}^n(S(B(0, 1)))}{\alpha(n)} \\ &= |\det S| = \llbracket L \rrbracket. \end{aligned}$$

(iv). Define $\nu(A) := \mathcal{H}^n(L(A))$ for all $A \subseteq \mathbb{R}^n$. Then ν is a Radon measure, $\nu \ll \mathcal{L}^n$, and

$$D_{\mathcal{L}^n} \nu(x) = \lim_{r \rightarrow 0} \frac{\nu(B(x, r))}{\mathcal{L}^n(B(x, r))} = \llbracket L \rrbracket$$

by (iii). Thus for all Borel sets $B \subseteq \mathbb{R}^n$,

$$\mathcal{H}^n(L(B)) = \llbracket L \rrbracket \mathcal{L}^n(B).$$

Since ν and \mathcal{L}^n are Radon measures, the same identity holds for all sets $A \subseteq \mathbb{R}^n$. The proof is complete. \square

For the remainder of the section we assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz.

13.3-2 Lemma 3.3.2. *Let $A \subseteq \mathbb{R}^n$ be \mathcal{L}^n -measurable. Then*

- (i) $f(A)$ is \mathcal{H}^n -measurable;
- (ii) The mapping $u \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is \mathcal{H}^n -measurable on \mathbb{R}^m ;
- (iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) d\mathcal{H}^n \leq (\text{Lip}(f))^n \mathcal{L}^n(A)$.

Proof.

(i). We may assume without loss of generality that A is bounded.

(ii). There exist compact sets $K_i \subseteq A$ such that

$$\mathcal{L}^n(K_i) \geq \mathcal{L}^n(A) - \frac{1}{i}, \quad i = 1, \dots, n.$$

Since $\mathcal{L}^n(A) < +\infty$ by the assumption and A is \mathcal{L}^n -measurable, $\mathcal{L}^n(A \setminus K_i) \leq \frac{1}{i}$. Since f is continuous, $f(K_i)$ is compact and thus \mathcal{H}^n -measurable. Hence, $f(\cup_{i=1}^{+\infty} K_i) = \cup_{i=1}^{+\infty} f(K_i)$ is \mathcal{H}^n -measurable. Moreover

$$\begin{aligned} \mathcal{H}^n \left(f(A) \setminus f \left(\bigcup_{i=1}^{+\infty} K_i \right) \right) &\leq \mathcal{H}^n \left(f \left(A \setminus \bigcup_{i=1}^{+\infty} K_i \right) \right) \\ &\leq (\text{Lip}(f))^n \mathcal{L}^n \left(A \setminus \bigcup_{i=1}^{+\infty} K_i \right) = 0. \end{aligned}$$

Thus $f(A)$ is \mathcal{H}^n -measurable. This proves (i).

(iii). Put

$$\mathcal{B}_k := \left\{ Q : Q = (a_1, b_1] \times \cdots \times (a_n, b_n], a_i := \frac{c_i}{k}, b_i := \frac{c_i + 1}{k}, c_i \in \mathbb{Z}, i = 1, \dots, n \right\},$$

and notice that

$$\mathbb{R}^n = \bigcup_{Q \in \mathcal{B}_k} Q.$$

Define

$$g_k := \sum_{Q \in \mathcal{B}_k} \mathbb{1}_{f(A \cap Q)},$$

and note that g_k is \mathcal{H}^n -measurable by assertion (i). Also $g_k(y)$ gives the number of cubes $Q \in \mathcal{B}_k$ such that $f^{-1}(y) \cap (A \cap Q) \neq \emptyset$. Thus

$$g_k(y) \rightarrow \mathcal{H}^0(A \cap f^{-1}(y)) \quad \text{as } k \rightarrow +\infty$$

for each $y \in \mathbb{R}^m$, and so $y \mapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is \mathcal{H}^n -measurable.

(iv). Note that g_k as defined in (iii) satisfies

$$0 \leq g_1 \leq g_2 \leq \cdots.$$

Thus by the Monotone Convergence Theorem,

$$\begin{aligned} \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y) &= \int_{\mathbb{R}^m} \lim_{k \rightarrow +\infty} g_k(y) \, d\mathcal{H}^n(y) \\ &\stackrel{MCT}{=} \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^m} g_k(y) \, d\mathcal{H}^n(y) \\ &= \lim_{k \rightarrow +\infty} \sum_{Q \in \mathcal{B}_k} \mathcal{H}^n(f(A \cap Q)) \\ &\leq \limsup_{k \rightarrow +\infty} \sum_{Q \in \mathcal{B}_k} (\text{Lip}(f))^n(A \cap Q) \\ &= (\text{Lip}(f))^n \mathcal{L}^n(A), \end{aligned}$$

as required. The proof is complete. \square

13.3-3 **Lemma 3.3.3.** *Let $t > 1$ and define*

$$B := \{x \in \mathbb{R}^n : Df(x) \text{ exists, } Jf(x) > 0\}.$$

Then there is a countable collection $\{E_k\}_{k=1}^{+\infty}$ of Borel subsets of \mathbb{R}^n such that

- (i) $B = \bigcup_{k=1}^{+\infty} E_k$;
- (ii) $f|_{E_k}$ is one-to-one, $k = 1, 2, \dots$;
- (iii) For each $k = 1, 2, \dots$, there exists a symmetric automorphism $T_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\begin{aligned} \text{Lip}((f|_{E_k}) \circ T_k^{-1}) &\leq t, \quad \text{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t, \\ t^{-n} |\det T_k| &\leq Jf|_{E_k} \leq t^n |\det T_k|. \end{aligned}$$

Proof.

(i). Fix $\epsilon > 0$ such that

$$\frac{1}{t} + \epsilon < 1 < t - \epsilon.$$

Let C be a countable dense subset of B and let S be a countable dense subset of the symmetric automorphisms of \mathbb{R}^n .

(ii). Then for each $c \in C$ and $T \in S$, and $i = 1, 2, \dots$, define $E(c, T, i)$ to be the set of all $b \in B \cap B(c, \frac{1}{i})$ satisfying

$$\left(\frac{1}{t} + \epsilon\right) |Tv| \leq |Df(b)v| \leq (t - \epsilon) |Tv| \quad (3.3.1) \quad \{\text{eq3.3-1}\}$$

for all $v \in \mathbb{R}^n$ and

$$|f(a) - f(b) - Df(b) \cdot (a - b)| \leq \epsilon |T(a - b)| \quad (3.3.2) \quad \{\text{eq3.3-2}\}$$

for all $a \in B(b, \frac{2}{i})$. Note that $E(c, T, i)$ is a Borel set since Df is Borel measurable. Note that from (3.3.1) and (3.3.2) follows the estimate

$$\frac{1}{t} |T(a - b)| \leq |f(a) - f(b)| \leq t |T(a - b)| \quad (3.3.3) \quad \{\text{eq3.3-3}\}$$

holding for all $b \in E(c, T, i)$ and $a \in B(b, \frac{2}{i})$.

(iii). We next show that if $b \in E(c, T, i)$, then

$$\left(\frac{1}{t} + \epsilon\right)^n |\det T| \leq Jf(b) \leq (t - \epsilon)^n |\det T|.$$

To see this, first note that Df is a linear map. Thus there exists an orthogonal map $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a symmetric map $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (cf. (3.2.2)) such that $Df = O \circ S$. Then

$$Jf(b) = \llbracket Df(b) \rrbracket = |\det S|.$$

By (3.3.1),

$$\left(\frac{1}{t} + \epsilon\right) |Tv| \leq |(O \circ S)v| = |Sv| \leq (t - \epsilon) |Tv|$$

for all $v \in \mathbb{R}^n$, and so

$$\left(\frac{1}{t} + \epsilon\right) |v| \leq |(S \circ T^{-1})v| \leq (t - \epsilon) |v|$$

for all $v \in \mathbb{R}^n$. Thus

$$(S \circ T^{-1})(B(0, 1)) \subset B(0, t - \epsilon),$$

so that

$$|\det(S \circ T^{-1})| \alpha(n) \leq \mathcal{L}^n(B(0, t - \epsilon)) = \alpha(n)(t - \epsilon)^n,$$

and hence

$$|\det S| \leq (t - \epsilon)^n |\det T|.$$

The proof of the reverse inequality follows from the fact that

$$|(S \circ T^{-1})v| \geq \left(\frac{1}{t} + \epsilon\right) |v|,$$

and thus

$$B\left(0, \frac{1}{t} + \epsilon\right) \subset (S \circ T^{-1})(B(0, 1)).$$

(iv). Relabel the countable collection $\{E(c, T, i) : c \in C, T \in S, i \in \mathbb{N}\}$ as $\{E_k\}_{k=1}^{+\infty}$. Choose any $b \in B$, write $Df = O \circ S$, and choose $T \in S$ such that

$$\text{Lip}(T \circ S^{-1}) \leq \left(\frac{1}{t} + \epsilon\right)^{-1}, \quad \text{Lip}(S \circ T^{-1}) \leq t - \epsilon.$$

Now choose $i \in \mathbb{N}$ and $c \in C$ such that $|b - c| < \frac{1}{i}$ and

$$|f(a) - f(b) - Df(b) \cdot (a - b)| \leq \frac{\epsilon}{\text{Lip}(T^{-1})} |a - b| \leq \epsilon |T(a - b)|$$

for all $a \in B(b, \frac{2}{i})$. Then by (iii), $b \in E(c, T, i)$. Since this holds for all $b \in B$, this proves assertion (i).

(v). Next choose any set $E_k = E(c, T, i)$. Let $T_k := T$. By $\frac{\text{eq3.3-3}}{(3.3.3)}$,

$$\frac{1}{t} |T_k(a - b)| \leq |f(a) - f(b)| \leq t |T_k(a - b)|$$

for all $b \in E_k$, $a \in B(b, \frac{2}{i})$. Since $E_k \subset B(c, \frac{1}{i}) \subset B(b, \frac{2}{i})$, we have

$$\frac{1}{t} |T_k(a - b)| \leq |f(a) - f(b)| \leq t |T_k(a - b)| \tag{3.3.4} \quad \boxed{\text{eq3.3-4}}$$

holding for all $a, b \in E_k$. Thus $f|_{E_k}$ is one-to-one.

(vi). Finally notice that $\frac{\text{eq3.3-4}}{(3.3.4)}$ implies

$$\text{Lip}((f|_{E_k}) \circ T_k^{-1}) \leq t, \quad \text{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t.$$

Thus (iii) provides the estimate

$$t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|,$$

which proves assertion (iii). The proof is complete. \square

3.3.2. Proof of the Area Formula.

t3.3-1

Theorem 3.3.1 (The Area Formula). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each \mathcal{L}^n -measurable subset $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf(x) \, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y).$$

Proof. \square

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