

# NOTES ON L. C. EVANS AND R. F. GARIEPY: MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

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Notes on chapters 2, 3, and 5 of *Measure Theory and Fine Properties of Functions* by L. C. Evans and R. F. Gariepy. All references are from [1] <sup>eg: measure</sup> unless indicated otherwise.

## CONTENTS

1. General Measure Theory	1
1.1. Weak Convergence and Compactness for Radon Measures	1
2. Hausdorff Measure	8
2.1. Definitions and Elementary Properties; Hausdorff Dimension	8
2.2. Isodiametric Inequality; $\mathcal{H}^n = \mathcal{L}^n$	8
2.3. Densities	14
2.4. Hausdorff Measure and Elementary Properties of Functions	17
3. Area and Coarea Formulas	21
3.1. Lipschitz Functions, Rademacher's Theorem	21
3.2. Linear Maps and Jacobians	27
References	32

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## 1. GENERAL MEASURE THEORY

**1.1. Weak Convergence and Compactness for Radon Measures.** We want to define what it means for a sequence  $\{\mu_k\}_{k=1}^{+\infty}$  of Radon measures to converge weakly.

t1.9-1

**Theorem 1.1.1.** *Let  $\mu, \{\mu_k\}_{k=1}^{+\infty}$  be Radon measures on  $\mathbb{R}^n$ . The following three statements are equivalent:*

- (i)  $\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\mu$  for all  $f \in \mathcal{C}_c(\mathbb{R}^n)$ ;
- (ii)  $\limsup_{k \rightarrow +\infty} \mu_k(K) \leq \mu(K)$  for each compact set  $K \subseteq \mathbb{R}^n$  and  $\mu(U) \leq \liminf_{k \rightarrow +\infty} \mu_k(U)$  for each open set  $U \subseteq \mathbb{R}^n$ ;
- (iii)  $\lim_{k \rightarrow +\infty} \mu_k(B) = \mu(B)$  for each bounded Borel set  $B \subseteq \mathbb{R}^n$  with  $\mu(\partial B) = 0$ .

**Remark.** Recall that Radon measures on  $\mathbb{R}^n$  are characterized by inner and outer regularity. Let  $B \subseteq \mathbb{R}^n$  be a Borel set, and let  $K \subseteq B \subseteq U$  with  $K$  compact and  $U$  open. If  $\{\mu_k\}_{k=1}^{+\infty}$  is converging to  $\mu$  in any sense, we should expect  $\mu_k(K) \leq \mu(K)$  for all  $k \in \mathbb{N}$  and  $\mu_k(U) \geq \mu(U)$  for all  $k \in \mathbb{N}$ . Conditions (ii) and (iii) tell us that this in fact holds up to a subsequence.

**Definition 1.1.1** (Weak Convergence of Radon Measures). *Let  $\mu, \{\mu_k\}_{k=1}^{+\infty}$  be Radon measures on  $\mathbb{R}^n$ . We say that  $\{\mu_k\}_{k=1}^{+\infty}$  converges weakly to  $\mu$ , and write*

$$\mu_k \rightharpoonup \mu,$$

if

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\mu$$

for every  $f \in \mathcal{C}_c(\mathbb{R}^n)$ .

*Proof.* Assume first that (i) holds. Let  $U \subseteq \mathbb{R}^n$  be open, and choose a compact set  $K \subseteq U$ . Next apply Urysohn's Lemma to choose a function  $f \in \mathcal{C}_c(\mathbb{R}^n)$  such that

$$0 \leq f \leq 1, \quad \text{supp}(f) \subseteq U, \quad \text{and} \quad f \equiv 1 \text{ on } K.$$

Then

$$\begin{aligned} \mu(K) &= \int_K d\mu = \int_{\mathbb{R}^n} f \, d\mu \leq \int_{\mathbb{R}^n} f \, d\mu = \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f \, d\mu_k \leq \liminf_{k \rightarrow +\infty} \int_U d\mu_k \\ &= \liminf_{k \rightarrow \infty} \mu_k(U). \end{aligned}$$

Thus

$$\begin{aligned} \mu(U) &= \sup\{\mu(K) : K \text{ compact}, K \subseteq U\} \\ &\leq \liminf_{k \rightarrow +\infty} \mu_k(U). \end{aligned}$$

This proves the second part of (ii). The first part is similar.

Next suppose that (ii) holds. Let  $B \subseteq \mathbb{R}^n$  be a bounded Borel set,  $\mu(\partial B) = 0$ . Then by (ii),

$$\begin{aligned} \mu(B) &= \mu(B^\circ) \leq \liminf_{k \rightarrow +\infty} \mu_k(B^\circ) \\ &\leq \limsup_{k \rightarrow +\infty} \mu_k(\overline{B}) \\ &\leq \mu(\overline{B}) \\ &= \mu(B). \end{aligned}$$

Since  $\mu_k(B^\circ) = \mu_k(B) = \mu_k(\overline{B})$  for all  $k \in \mathbb{N}$  since  $\mu(\partial B) = 0$ , it follows

$$\liminf_{k \rightarrow +\infty} \mu_k(B) = \limsup_{k \rightarrow +\infty} \mu_k(B).$$

Thus  $\lim_{k \rightarrow +\infty} \mu_k(B)$  exists, and

$$\lim_{k \rightarrow +\infty} \mu_k(B) = \mu(B),$$

as required.

Finally assume that (iii) holds. Fix  $\epsilon > 0$  and  $f \in \mathcal{C}_c^+(\mathbb{R}^n)$ . Let  $R > 0$  be such that  $\text{supp}(f) \subseteq B(0, R)$  and  $\mu(\partial B(0, R)) = 0$ . Choose a partition

$$0 := t_0 < t_1 < \cdots < t_N = 2\|f\|_{L^\infty(\mathbb{R}^n)}$$

of  $[0, 2\|f\|_{L^\infty(\mathbb{R}^n)}]$  such that  $0 < t_i - t_{i-1} < \epsilon$ , and  $\mu(f^{-1}\{t_i\}) = 0$  for each  $i = 1, \dots, N$ . Put  $B_i := f^{-1}((t_{i-1}, t_i])$ ,  $i = 2, \dots, N$ . Then  $\mu(\partial B_i) = 0$  for each  $i \geq 2$ . Now

$$\begin{aligned} \sum_{i=2}^N t_{i-1} \mu_k(B_i) &= \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu_k \leq \sum_{i=2}^N \int_{B_i} f \, d\mu_k \\ &\leq \int_{\mathbb{R}^n} f \, d\mu_k \end{aligned}$$

$$\leq \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0, R)),$$

and

$$\begin{aligned} \sum_{i=2}^N t_{i-1} \mu(B_i) &= \sum_{i=2}^N t_{i-1} \int_{B_i} d\mu \leq \sum_{i=2}^N \int_{B_i} f d\mu \\ &\leq \int_{\mathbb{R}^n} f d\mu \\ &\leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B(0, R)). \end{aligned}$$

Thus (iii) implies

$$\begin{aligned} &\limsup_{k \rightarrow +\infty} \left| \int_{\mathbb{R}^n} f d\mu_k - \int_{\mathbb{R}^n} f d\mu \right| \\ &\leq \limsup_{k \rightarrow +\infty} \left| \left\{ \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(0, R)) \right\} - \sum_{i=2}^N t_{i-1} \mu(B_i) \right| \\ &\leq \limsup_{k \rightarrow +\infty} \sum_{i=2}^N |t_i \mu_k(B_i) - t_{i-1} \mu(B_i)| + \limsup_{k \rightarrow +\infty} t_1 \mu_k(B(0, R)) \\ &= \sum_{i=2}^N |t_i - t_{i-1}| \mu(B_i) + t_1 \mu(B(0, R)) \\ &\leq 2\epsilon \mu(B(0, R)). \end{aligned}$$

Since  $\epsilon > 0$  was arbitrary, taking the limit at  $\epsilon \rightarrow 0$  shows that

$$\limsup_{k \rightarrow +\infty} \left| \int_{\mathbb{R}^n} f d\mu_k - \int_{\mathbb{R}^n} f d\mu \right| = 0,$$

and hence

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}^n} f d\mu.$$

The proof is complete. □

**t1.9-2**

**Theorem 1.1.2** (Weak Compactness for Measures). *Let  $\{\mu_k\}_{k=1}^{+\infty}$  be a sequence of Radon measures on  $\mathbb{R}^n$  satisfying*

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

*for each compact set  $K \subseteq \mathbb{R}^n$ . Then there exists a subsequence  $\{\mu_{k_j}\}_{j=1}^{+\infty}$  and a Radon measure  $\mu$  on  $\mathbb{R}^n$  such that*

$$\mu_{k_j} \rightharpoonup \mu \quad \text{as } j \rightarrow +\infty.$$

*Proof.*

(i). Assume first that

$$\sup_{k \in \mathbb{N}} \mu_k(\mathbb{R}^n) < +\infty.$$

(1.1.1)

**{eq:1.9-1}**

(ii). Let  $\{f_k\}_{k=1}^{+\infty}$  be a countable dense subset of  $\mathcal{C}_c(\mathbb{R}^n)$ . Note that (eq:1.9-1) implies that the sequence  $\{\int_{\mathbb{R}^n} f_1 d\mu_j\}_{j=1}^{+\infty}$  is bounded, for

$$\left| \int_{\mathbb{R}^n} f_1 d\mu_j \right| \leq \int_{\mathbb{R}^n} |f_1| d\mu_j \leq \max_{x \in \text{supp}(f)} |f(x)| \mu_j(\mathbb{R}^n) < +\infty.$$

Thus we may find a subsequence  $\{\mu_j^1\}_{j=1}^{+\infty}$  and  $a_1 \in \mathbb{R}$  such that

$$\int_{\mathbb{R}^n} f_1 d\mu_j^1 \rightarrow a_1 \quad \text{as } j \rightarrow +\infty.$$

Continuing, we find subsequences  $\{\mu_j^k\}_{j=1}^{+\infty}$  of  $\{\mu_j^{k-1}\}_{j=1}^{+\infty}$  and numbers  $a_k \in \mathbb{R}$  such that

$$\int_{\mathbb{R}^n} f_k d\mu_j^k \rightarrow a_k \quad \text{as } j \rightarrow +\infty$$

for each  $k \in \mathbb{N}$ . Set  $\nu_j := \mu_j^j$ . Then

$$\int_{\mathbb{R}^n} f_k d\nu_j \rightarrow a_k \quad \text{as } j \rightarrow +\infty$$

for all  $k \in \mathbb{N}$ , for if  $j \geq k$ , then  $\nu_j = \mu_j^j \in \{\mu_j^k\}_{j=1}^{+\infty}$ . Define  $L(f_k) := a_k$ , and note that  $L$  is linear and

$$|L(f_k)| \leq M \|f_k\|_{L^\infty(\mathbb{R}^n)}$$

by (eq:1.9-1), where

$$M := \sup_{k \in \mathbb{N}} \mu_k(\mathbb{R}^n).$$

By the Hahn–Banach Theorem,  $L$  may be uniquely extended to a bounded linear functional  $\bar{L}$  defined on all of  $\mathcal{C}_c(\mathbb{R}^n)$ . Then, by the Riesz Representation Theorem, there exists a unique Radon measure  $\mu$  on  $\mathbb{R}^n$  such that

$$\bar{L}(f) = \int_{\mathbb{R}^n} f d\mu$$

for all  $f \in \mathcal{C}_c(\mathbb{R}^n)$ .

(iii). Choose any  $f \in \mathcal{C}_c(\mathbb{R}^n)$ . Since  $\{f_k\}_{k=1}^{+\infty}$  is dense in  $\mathcal{C}_c(\mathbb{R}^n)$ , there exists a subsequence  $\{f_{k_i}\}_{i=1}^{+\infty}$  such that  $f_i \rightarrow f$  uniformly. Fix  $\epsilon > 0$  and then choose  $i \in \mathbb{N}$  so large that

$$\|f_{k_i} - f\|_{L^\infty(\mathbb{R}^n)} < \frac{\epsilon}{4M}. \quad (1.1.2)$$

{eq:1.9-2}

Next choose  $J \in \mathbb{N}$  so that for all  $j > J$ ,

$$\left| \int_{\mathbb{R}^n} f_{k_i} d\nu_j - \int_{\mathbb{R}^n} f_{k_i} d\mu \right| < \frac{\epsilon}{2}.$$

Then for any  $j > J$ , we have by (eq:1.9-2) and the Principle of Uniform Boundedness

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f d\nu_j - \int_{\mathbb{R}^n} f d\mu \right| &\leq \left| \int_{\mathbb{R}^n} f - f_{k_i} d\nu_j \right| + \left| \int_{\mathbb{R}^n} f_{k_i} d\nu_j - \int_{\mathbb{R}^n} f_{k_i} d\mu \right| + \\ &\quad \left| \int_{\mathbb{R}^n} f_{k_i} - f d\mu \right| \\ &\leq \frac{\epsilon}{2} + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \nu_j(\mathbb{R}^n) + \|f - f_{k_i}\|_{L^\infty(\mathbb{R}^n)} \mu(\mathbb{R}^n) \\ &< \epsilon, \end{aligned}$$

as required.

(iv). In the general case that (I.I.I) fails to hold, but

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty$$

for each compact set  $K \subseteq \mathbb{R}^n$ , we apply the above argument to the measures

$$\mu_k^l := \mu_k \llcorner \overline{B(0, l)}, \quad k, l = 1, 2, \dots,$$

and use a diagonalization argument. The proof is complete.  $\square$

For the remainder of this section, we assume that

- (i)  $U \subseteq \mathbb{R}^n$  is open;
- (ii)  $1 \leq p < +\infty$ .

**Definition 1.1.2** (Weak Convergence in  $L^p(U)$ ). A sequence  $\{f_k\}_{k=1}^{+\infty} \subset L^p(U)$  is said to converge weakly to  $f \in L^p(U)$ , written

$$f_k \rightharpoonup f \quad \text{in } L^p(U),$$

if

$$\lim_{k \rightarrow +\infty} \int_U f_k g \, d\mathcal{L}^n = \int_U f g \, d\mathcal{L}^n$$

for each  $g \in L^q(U)$ , where  $p$  and  $q$  are conjugate exponents,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 < q \leq +\infty$ .

**t1.9-3**

**Theorem 1.1.3** (Weak Compactness in  $L^p$ ). Suppose that  $1 < p < +\infty$ . Let  $\{f_k\}_{k=1}^{+\infty} \subseteq L^p(U)$  satisfying

$$\sup_{k \in \mathbb{N}} \|f_k\|_{L^p(U)} < +\infty.$$

Then there exists a subsequence  $\{f_{k_j}\}_{j=1}^{+\infty}$  of  $\{f_k\}_{k=1}^{+\infty}$  and a function  $f \in L^p(U)$  such that

$$f_{k_j} \rightharpoonup f \quad \text{in } L^p(U) \quad \text{as } j \rightarrow +\infty.$$

**Remark.** This assertion is in general false for  $p = 1$ . The key property here is reflexivity. Recall that  $L^p(U)$  is reflexive if and only if  $1 < p < +\infty$ .

**Definition 1.1.3.** We denote by

$$\nu := \mu \llcorner f$$

the signed measure with density  $f$  with respect to  $\mu$ , that is, the signed measure

$$\nu(K) = \int_K f \, d\mu,$$

provided that this holds for all compact sets  $K \subseteq \mathbb{R}^n$ .

*Proof.*

(i). If  $U \neq \mathbb{R}^n$ , we extend each function  $f_k$  to  $\mathbb{R}^n$  by setting  $f_k = 0$  on  $\mathbb{R}^n \setminus U$ . This done, we may assume that  $U = \mathbb{R}^n$ . We may also assume that

$$f_k \geq 0 \quad \mathcal{L}^n - \text{a.e.},$$

for otherwise we could apply the following analysis to  $f_k^+$  and  $f_k^-$ .

(ii). Define the Radon measures

$$\mu_k := \mathcal{L}^n \llcorner f_k, \quad k \in \mathbb{N}.$$

Then for each compact set  $K \subseteq \mathbb{R}^n$ , by Hölder's inequality, we have

$$\mu_k(K) = \int_K f_k d\mathcal{L}^n \leq \|f_k\|_{L^p(K)} \cdot \mathcal{L}^n(K)^{\frac{p-1}{p}} < +\infty,$$

and thus

$$\sup_{k \in \mathbb{N}} \mu_k(K) < +\infty.$$

Therefore, we may apply Theorem (I.I.2) to obtain a Radon measure  $\mu$  on  $\mathbb{R}^n$  and a subsequence

$$\mu_{k_j} \rightharpoonup \mu.$$

(iii). We now show that  $\mu \ll \mathcal{L}^n$ . Let  $A \subseteq \mathbb{R}^n$  be bounded with  $\mathcal{L}^n(A) = 0$ . Fix  $\epsilon > 0$  and choose an open bounded set  $V \supseteq A$  such that  $\mathcal{L}^n(V) < \epsilon$ . Then by Theorem (I.I.1) and Hölder's inequality,

$$\begin{aligned} \mu(A) &\leq \mu(V) \leq \liminf_{j \rightarrow +\infty} \mu_{k_j}(V) = \liminf_{j \rightarrow +\infty} \int_V f_{k_j} d\mathcal{L}^n \\ &\leq \liminf_{j \rightarrow +\infty} \|f_{k_j}\|_{L^p(V)} \cdot \mathcal{L}^n(V)^{\frac{p-1}{p}} \\ &\leq C\epsilon^{\frac{p-1}{p}}. \end{aligned}$$

Since  $\epsilon > 0$  was arbitrary and  $\frac{p-1}{p} > 0$ ,  $\mu(A) = 0$ , as required. Therefore  $\mu \ll \mathcal{L}^n$ .

(iv). By the Radon–Nikodym Theorem, there exists  $f \in L^1_{loc}(\mathbb{R}^n)$  such that

$$\mu(A) = \int_A f d\mathcal{L}^n$$

for every Borel set  $A \subseteq \mathbb{R}^n$ .

(v). We prove that  $f \in L^p(\mathbb{R}^n)$ . Let  $\phi \in \mathcal{C}_c(\mathbb{R}^n)$ . Then

$$\begin{aligned} \int_{\mathbb{R}^n} f\phi d\mathcal{L}^n &= \int_{\mathbb{R}^n} \phi d\mu = \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi d\mu_{k_j} \\ &= \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n} \phi f_{k_j} d\mathcal{L}^n \\ &\leq \sup_{k \in \mathbb{N}} \|f_{k_j}\|_{L^p(\mathbb{R}^n)} \|\phi\|_{L^q(\mathbb{R}^n)} \\ &\leq C \|\phi\|_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Thus

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{\substack{\phi \in \mathcal{C}_c(\mathbb{R}^n) \\ \|\phi\|_{L^q(\mathbb{R}^n)}=1}} \left| \int_{\mathbb{R}^n} f\phi d\mathcal{L}^n \right| \leq C < +\infty,$$

and we see that  $f \in L^p(\mathbb{R}^n)$ .

(vi). Finally, we show that  $f_{k_j} \rightharpoonup f$  in  $L^p(\mathbb{R}^n)$ . Fix  $\epsilon > 0$ . By the above,

$$\int_{\mathbb{R}^n} f_{k_j} \phi d\mathcal{L}^n \rightarrow \int_{\mathbb{R}^n} f \phi d\mathcal{L}^n$$

as  $j \rightarrow +\infty$  for all  $\phi \in \mathcal{C}_c(\mathbb{R}^n)$ . Thus we may choose  $J \in \mathbb{N}$  so large so that for all  $j > J$ ,

$$\left| \int_{\mathbb{R}^n} f_{k_j} \phi - f \phi d\mathcal{L}^n \right| < \epsilon \quad (1.1.3)$$

{eq:1.9-3}

for all  $\phi \in \mathcal{C}_c(\mathbb{R}^n)$ . Given  $g \in L^q(\mathbb{R}^n)$ , choose by the density of  $\mathcal{C}_c(\mathbb{R}^n)$  in  $L^q(\mathbb{R}^n)$  a function  $\phi \in \mathcal{C}_c(\mathbb{R}^n)$  such that

$$\|g - \phi\|_{L^q(\mathbb{R}^n)} < \epsilon.$$

Then by <sup>(eq:1.9-3)</sup>(1.1.3), Hölder's inequality, and the Principle of Uniform Boundedness, we have for all  $j > J$

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_{k_j} g \, d\mathcal{L}^n - \int_{\mathbb{R}^n} f g \, d\mathcal{L}^n \right| &\leq \int_{\mathbb{R}^n} |f_{k_j} g - f_{k_j} \phi| \, d\mathcal{L}^n + \left| \int_{\mathbb{R}^n} f_{k_j} \phi - f \phi \, d\mathcal{L}^n \right| + \\ &\quad \int_{\mathbb{R}^n} |f \phi - f g| \, d\mathcal{L}^n \\ &\leq \epsilon + \int_{\mathbb{R}^n} |f_{k_j}| |g - \phi| \, d\mathcal{L}^n + \int_{\mathbb{R}^n} |f| |\phi - g| \, d\mathcal{L}^n \\ &\leq \epsilon + \epsilon \|f_{k_j}\|_{L^p(\mathbb{R}^n)} + \epsilon \|f\|_{L^p(\mathbb{R}^n)} \\ &\leq (2C + 1)\epsilon. \end{aligned}$$

The proof is complete. □

## 2. HAUSDORFF MEASURE

## 2.1. Definitions and Elementary Properties; Hausdorff Dimension.

**Definition 2.1.1** ( $\mathcal{H}_\delta^s$ ).

**2.2. Isodiametric Inequality;  $\mathcal{H}^n = \mathcal{L}^n$ .** We want to prove that  $\mathcal{H}^n = \mathcal{L}^n$  on  $\mathbb{R}^n$ , where  $n \in \mathbb{N}$ . Recall that  $\mathcal{L}^n$  is defined as the  $n$ -fold product of one-dimensional Lebesgue measure  $\mathcal{L}^1$ , so that

$$\mathcal{L}^1(A) := \inf \left\{ \sum_{i=1}^n \mathcal{L}^1(Q_i) : Q_i \text{ cubes}, A \subseteq \bigcup_{i=1}^n Q_i \right\}.$$

On the other hand,  $\mathcal{H}^n$  is computed in terms of arbitrary coverings of small diameter.

12.2-1

**Lemma 2.2.1.** *Let  $f : \mathbb{R}^n \rightarrow [0, +\infty]$  be  $\mathcal{L}^n$ -measurable. Then the region “under the graph” of  $f$ ,*

$$A := \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}, 0 \leq y \leq f(x)\}$$

*is  $\mathcal{L}^{n+1}$ -measurable.*

*Proof.* Define

$$B := \{x \in \mathbb{R}^n : f(x) = +\infty\}$$

and

$$C := \{x \in \mathbb{R}^n : 0 \leq f(x) < +\infty\}.$$

Also define

$$C_{j,k} := \left\{ x \in C : \frac{j}{k} \leq f(x) < \frac{j+1}{k} \right\}, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{N},$$

so that  $C = \bigcup_{j=0}^{+\infty} C_{j,k}$ . Finally, put

$$D_k := \bigcup_{j=0}^{+\infty} \left( C_{j,k} \times \left[ 0, \frac{j}{k} \right] \right) \cup (B \times [0, +\infty]),$$

$$E_k := \bigcup_{j=0}^{+\infty} \left( C_{j,k} \times \left[ 0, \frac{j+1}{k} \right] \right) \cup (B \times [0, +\infty]).$$

Clearly  $D_k$  and  $E_k$  are  $\mathcal{L}^{n+1}$  measurable, and we have for each  $k \in \mathbb{N}$   $D_k \subseteq A \subseteq E_k$ . Write  $D := \bigcup_{k=1}^{+\infty} D_k$  and  $E := \bigcap_{k=1}^{+\infty} E_k$ . Then also  $D \subseteq A \subseteq E$ , with  $D$  and  $E$  both  $\mathcal{L}^{n+1}$ -measurable. Now for any  $\mathcal{L}^{n+1}$ -measurable set  $F$  with  $\mathcal{L}^{n+1}(F) < +\infty$ ,

$$\mathcal{L}^{n+1}((E \setminus D) \cap F) \leq \mathcal{L}^{n+1}((E_k \setminus D_k) \cap F) \leq \frac{1}{k} \mathcal{L}^n(F),$$

and the RHS tends to zero as  $k \rightarrow +\infty$ . Thus  $\mathcal{L}^{n+1}((E \setminus D) \cap F) = 0$ , and, because  $F$  was arbitrary,  $\mathcal{L}^{n+1}(E \setminus D) = 0$ . Hence  $\mathcal{L}^{n+1}(A \setminus D) = 0$ , and consequently  $A$  is  $\mathcal{L}^{n+1}$ -measurable.  $\square$



We now define the process of Steiner symmetrization, which takes a bounded Borel-measurable set  $A \subseteq \mathbb{R}^n$  and transforms  $A$  into a set  $\tilde{A}$  having the same Lebesgue measure such that  $\text{diam}(\tilde{A}) \leq \text{diam}(A)$ .

Fix  $a, b \in \mathbb{R}^n$ ,  $\|a\| = 1$ . We define

$$L_b^a := \{b + ta : t \in \mathbb{R}\}, \text{ the line through } b \text{ in the direction of } a,$$

and

$$P_a := \{x \in \mathbb{R}^n : x \cdot a = 0\}, \text{ the plane through the origin perpendicular to } a.$$

**Definition 2.2.1** (Steiner Symmetrization). *Choose  $a \in \mathbb{R}^n$  with  $\|a\| = 1$ , and let  $A \subseteq \mathbb{R}^n$ . We define the Steiner symmetrization of  $A$  with respect to the hyperplane  $P_a$  to be the set*

$$S_a(A) := \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \left\{ b + ta : \|t\| \leq \frac{1}{2} \mathcal{H}^1(A \cap L_b^a) \right\}.$$

Note that the Steiner symmetrization is the union of all line segments  $b + ta$  of length less than  $\mathcal{H}^1(A \cap L_b^a)$ , where  $b$  is in the plane through the origin perpendicular to  $a$  and there exists  $x \in A$  such that  $b + ta = x$ .

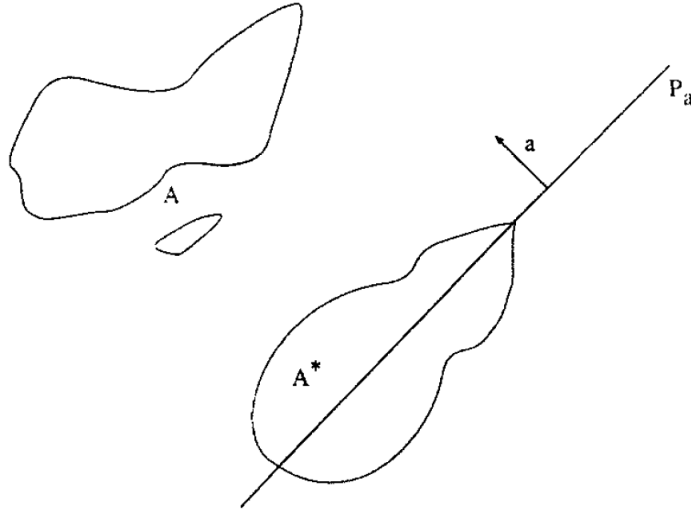


FIGURE 1. Steiner Symmetrization.

**12.2-2 Lemma 2.2.2** (Properties of Steiner Symmetrization).

- (i)  $\text{diam } S_a(A) \leq \text{diam } A$ .
- (ii) If  $A$  is  $\mathcal{L}^n$ -measurable, then so is  $S_a(A)$ , and  $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$ .

*Proof.*

(i). Statement (i) is trivial if  $\text{diam } A = +\infty$ , so we may assume that  $\text{diam } A < +\infty$ . We may also suppose that  $A$  is closed, for

$$\text{diam } A^\circ = \text{diam } A = \text{diam } \overline{A}.$$

Fix  $\epsilon > 0$  and choose  $x, y \in S_a(A)$  such that

$$\text{diam } S_a(A) \leq \|x - y\| + \epsilon.$$

Write  $b := x - (x \cdot a)a$  and  $c := y - (y \cdot a)a$ . Then  $b, c \in P_a$ . Put

$$\begin{aligned} r &:= \inf\{t : b + ta \in A\}, \\ s &:= \sup\{t : b + ta \in A\}, \\ u &:= \inf\{t : c + ta \in A\}, \\ v &:= \sup\{t : c + ta \in A\}. \end{aligned}$$

Without loss of generality, we may assume that  $v - r \geq s - u$ . Then

$$\begin{aligned} v - r &\geq \frac{1}{2}(v - r) + \frac{1}{2}(s - u) \\ &= \frac{1}{2}(s - r) + \frac{1}{2}(v - u) \\ &\geq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) + \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

Now,  $|x \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)$ ,  $|y \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_c^a)$ , and consequently,

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a|.$$

Hence,

$$\begin{aligned} (\text{diam } S_a(A) - \epsilon)^2 &\leq \|x - y\|^2 \\ &= \|x\|^2 - 2x \cdot y + \|y\|^2 \\ &= \|b\|^2 + 2(x \cdot a)(b \cdot a) + |x \cdot a|^2 - 2(b + (x \cdot a)a) \cdot (c + (y \cdot a)a) + \|c\|^2 + \\ &\quad 2(y \cdot a)(b \cdot a) + |y \cdot a|^2 \\ &= (\|b\|^2 - 2b \cdot c + \|c\|^2) + (|x \cdot a|^2 - 2(x \cdot a)(y \cdot a) + |y \cdot a|^2) + \\ &\quad 2(x \cdot a)(b \cdot a) - 2(b \cdot a)(y \cdot a) - 2(c \cdot a)(x \cdot a) + 2(y \cdot a)(b \cdot a) \\ &= \|b - c\|^2 + \|x \cdot a - y \cdot a\|^2 \\ &\leq \|b - c\|^2 + (v - r)^2 \\ &= \|b\|^2 - 2b \cdot c + \|c\|^2 + v^2 - 2rv + r^2 \\ &= (\|b\|^2 + 2b \cdot ra + \|ra\|^2) - 2(b \cdot c - b \cdot va - c \cdot ra - rv\|a\|^2) + \\ &\quad (\|c\|^2 + 2c \cdot va + \|va\|^2) \\ &= \|(b + ra) - (c + va)\|^2 \\ &\leq (\text{diam } A)^2, \end{aligned}$$

since  $b, c \perp a$  and  $A$  is closed, so that  $b + ra, c + va \in A$ . Thus  $\text{diam } S_a(A) - \epsilon \leq \text{diam } A$ , and since  $\epsilon > 0$  was arbitrary, this proves (i).

(ii). Since  $\mathcal{L}^n$  is rotation invariant, we may assume that  $a = e_n$ . Then  $P_a = P_{e_n} = \mathbb{R}^{n-1}$ . Since  $\mathcal{L}^1 = \mathcal{H}^1$  on  $\mathbb{R}$ , Tonelli's Theorem implies that the map  $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  defined by  $f(b) = \mathcal{H}^1(A \cap L_b^a)$  is  $\mathcal{L}^{n-1}$ -measurable and  $\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b)$ , for

$$\int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(A \cap L_b^a) d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A).$$

Therefore

$$S_a(A) = \left\{ (b, y) : 0 \leq |y| \leq \frac{f(b)}{2} \right\} \setminus \{(b, 0) : L_b^a \cap A = \emptyset\}$$

is  $\mathcal{L}^n$ -measurable by Lemma (12.2-1) (2.2.1), and

$$\begin{aligned} \mathcal{L}^n(S_a(A)) &= \int_{\mathbb{R}} \mathbb{1}_{S_a(A)} d\mathcal{L}^n = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \mathbb{1}_{S_a(A)} d\mathcal{L}^1 d\mathcal{L}^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (\mathbb{1}_{S_a(A)})_{(e_1, \dots, e_{n-1})}(y) d\mathcal{L}^1(y) d\mathcal{L}^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \int_{-f(b)/2}^{f(b)/2} d\mathcal{L}^1 d\mathcal{L}^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} f(b) d\mathcal{L}^{n-1}(b) = \mathcal{L}^n(A). \end{aligned}$$

The proof is complete.  $\square$

**Remark.** In proving  $\mathcal{H}^n = \mathcal{L}^n$  below, notice that we use only statement (ii) above in the special case that  $a$  is a standard coordinate vector. Since  $\mathcal{H}^n$  is obviously rotation invariant, we in fact prove that  $\mathcal{L}^n$  is rotation invariant also.

t2.2-1

**Theorem 2.2.1** (Isodiametric Inequality). For all sets  $A \subseteq \mathbb{R}^n$ ,

$$\mathcal{L}^n(A) \leq \frac{\alpha(n)}{2^n} (\text{diam } A)^n.$$

**Remark.**

- (i) Geometrically, the isodiametric inequality says that of all sets of fixed diameter in  $\mathbb{R}^n$ , the  $n$ -sphere has greatest volume.
- (ii) This inequality is particularly interesting because it is not necessarily the case that  $A$  is contained in a ball of diameter  $\text{diam } A$ , for in  $\mathbb{R}^2$  consider the case of an equilateral triangle with side length 1. The smallest closed ball  $B$  which inscribes the triangle has radius  $1/\sqrt{3}$ , so

$$\text{diam } B = \frac{2}{\sqrt{3}} > 1.$$

*Proof.* If  $\text{diam } A = +\infty$ , the inequality is trivial. Therefore we may assume that  $\text{diam } A < +\infty$ .

Let  $\{e_1, \dots, e_n\}$  be the standard basis for  $\mathbb{R}^n$ . Define  $A_1 := S_{e_1}(A)$ ,  $A_2 := S_{e_2}(A_1), \dots$ ,  $A_n := S_{e_n}(A_{n-1})$ . Write  $A^* := A_n$ .

(i). We first show that  $A^*$  is symmetric with respect to the origin. We use induction. Clearly  $A_1$  is symmetric with respect to  $P_{e_1}$ . Let  $k$  be an integer such that  $1 \leq k < n$  and suppose that  $A_k$  is symmetric with respect to  $P_{e_1}, \dots, P_{e_k}$ . Clearly  $A_{k+1} = S_{e_{k+1}}(A_k)$  is symmetric with respect to  $P_{e_{k+1}}$ . Fix  $1 \leq j < k$  and let  $S_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the reflection through  $P_{e_j}$ . Let  $b \in P_{e_{k+1}}$ . Since  $A_k$  is symmetric with respect to  $P_{e_1}, \dots, P_{e_k}$  by the induction hypothesis and  $1 \leq j \leq k$ , we have  $S_j(A_k) = A_k$ , and so

$$\mathcal{H}^1(A_k \cap L_b^{e_{k+1}}) = \mathcal{H}^1(A_k \cap L_{S_j b}^{e_{k+1}}).$$

Consequently

$$\{t \in \mathbb{R} : b + te_{k+1} \in A_{k+1}\} = \{t \in \mathbb{R} : S_j b + te_{k+1} \in A_{k+1}\}.$$

Thus  $S_j(A_{k+1}) = A_{k+1}$ , that is,  $A_{k+1}$  is symmetric with respect to  $P_{e_j}$ . Since  $j$  was arbitrary,  $A^* = A_n$  is symmetric with respect to  $P_{e_1}, \dots, P_{e_n}$ , and so with respect to the origin.

(ii). We show that

$$\mathcal{L}^n(A^*) \leq \frac{\alpha(n)}{2^n} (\text{diam } A^*)^n.$$

Choose  $x \in A^*$ . Then  $-x \in A^*$  by (i), and so  $\text{diam } A^* \geq 2|x|$ . Thus  $A^* \subseteq B(0, \frac{1}{2} \text{diam } A^*)$ , and it follows by monotonicity of the Lebesgue measure

$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B\left(0, \frac{1}{2} \text{diam } A^*\right)\right) = \frac{\alpha(n)}{2^n} (\text{diam } A^*)^2.$$

(iii). We now prove the isodiametric inequality. Note that  $\bar{A}$  is  $\mathcal{L}^n$ -measurable, and thus the above Lemma (2.2.2) implies that

$$\mathcal{L}^n((\bar{A})^*) = \mathcal{L}^n(\bar{A}),$$

as well as

$$\text{diam}(\bar{A})^* \leq \text{diam } \bar{A}.$$

Hence, monotonicity of the Lebesgue measure together with (ii) give

$$\begin{aligned} \mathcal{L}^n(A) &\leq \mathcal{L}^n(\bar{A}) = \mathcal{L}^n((\bar{A})^*) \\ &\leq \frac{\alpha(n)}{2^n} (\text{diam}(\bar{A})^*)^n \\ &\leq \frac{\alpha(n)}{2^n} (\text{diam } \bar{A})^n \\ &= \frac{\alpha(n)}{2^n} (\text{diam } A)^n. \end{aligned}$$

The proof is complete. □

**t2.2-2** **Theorem 2.2.2.** On  $\mathbb{R}^n$ ,  $\mathcal{L}^n = \mathcal{H}^n$ .

*Proof.* (i). We first show that  $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$  for all  $A \subseteq \mathbb{R}^n$ . Fix  $\delta > 0$ . Choose sets  $\{C_j\}_{j=1}^{+\infty}$  such that  $A \subseteq \bigcup_{j=1}^{+\infty} C_j$  and  $\text{diam } C_j \leq \delta$ . Then by monotonicity and the Isodiametric Inequality (cf. (2.2.1)),

$$\mathcal{L}^n(A) \leq \sum_{j=1}^{+\infty} \mathcal{L}^n(C_j) \leq \sum_{j=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } C_j)^n.$$

Taking the infimum of the RHS over all cover countable covers of  $A$  with diameter less than  $\delta$ , we obtain  $\mathcal{L}^n(A) \leq H_\delta^n(A)$ . Taking the limit as  $\delta \rightarrow 0$ , we have

$$\mathcal{L}^n(A) \leq \mathcal{H}_\delta^n(A) \leq \mathcal{H}^n(A),$$

as required.

(ii). From the definition of  $\mathcal{L}^n$  as the  $n$ -fold product of  $\mathcal{L}^1 \times \dots \times \mathcal{L}^1$ , we see that for all  $A \subseteq \mathbb{R}^n$  and  $\delta > 0$ ,

$$\mathcal{L}^n(A) = \inf \left\{ \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) : Q_i \text{ cubes, } A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \text{diam } Q_i \leq \delta \right\}.$$

We may consider only cubes parallel to the coordinate axes in  $\mathcal{L}^n$ .

(iii). We now show that  $\mathcal{H}^n$  is absolutely continuous with respect to  $\mathcal{L}^n$ . Set  $C_n := \frac{\alpha(n)}{2^n}$ . Then for each cube  $Q \subseteq \mathbb{R}^n$ ,

$$\frac{\alpha(n)}{2^n}(\text{diam } Q)^n = C_n \mathcal{L}^n(Q).$$

Thus for any  $A \subseteq \mathbb{R}^n$ ,

$$\begin{aligned} \mathcal{H}_\delta^n(A) &= \inf \left\{ \sum_{i=1}^n \frac{\alpha(n)}{2^n} (\text{diam } U_i)^n : A \subseteq \bigcup_{i=1}^{+\infty} U_i, \text{diam } U_i \leq \delta \right\} \\ &\leq \inf \left\{ \sum_{i=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } Q_i)^n : Q_i \text{ cubes}, A \subseteq \bigcup_{i=1}^{+\infty} Q_i, \text{diam } Q_i \leq \delta \right\} \\ &= C_n \mathcal{L}^n(A). \end{aligned}$$

Taking the supremum over all  $\delta > 0$ , we've:

$$\mathcal{H}^n(A) \leq C_n \mathcal{L}^n(A).$$

Thus  $\mathcal{H}^n(A) = 0$  whenever  $\mathcal{L}^n(A) = 0$ . This proves (iii).

(iv). We now show that  $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$  for all  $A \subseteq \mathbb{R}^n$ . To this end, fix  $\delta > 0$  and  $\epsilon > 0$ . We may choose cubes  $\{Q_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$  such that  $A \subseteq \bigcup_{i=1}^{+\infty} Q_i$ ,  $\text{diam } Q_i \leq \delta$ , and

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon.$$

Now for each  $i \in \mathbb{N}$  there exist disjoint closed balls  $\{B_k^i\}_{k=1}^{+\infty} \subseteq Q_i^\circ$  such that

$$\text{diam } B_k^i \leq \delta$$

and

$$\mathcal{L}^n \left( Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i \right) = \mathcal{L}^n \left( Q_i^\circ \setminus \bigcup_{k=1}^{+\infty} B_k^i \right) = 0.$$

Since  $\mathcal{H}^n, \mathcal{H}_\delta^n$  are absolutely continuous with respect to  $\mathcal{L}^n$  by (iii),  $\mathcal{H}^n(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i) = \mathcal{H}_\delta^n(Q_i \setminus \bigcup_{k=1}^{+\infty} B_k^i) = 0$ . Therefore  $\mathcal{H}^n(Q_i) = \mathcal{H}^n(\bigcup_{k=1}^{+\infty} B_k^i)$  and  $\mathcal{H}_\delta^n(Q_i) = \mathcal{H}_\delta^n(\bigcup_{k=1}^{+\infty} B_k^i)$ , and we have

$$\begin{aligned} \mathcal{H}_\delta^n(A) &\leq \sum_{i=1}^{+\infty} \mathcal{H}_\delta^n(Q_i) = \sum_{i=1}^{+\infty} \mathcal{H}_\delta^n \left( \bigcup_{k=1}^{+\infty} B_k^i \right) \leq \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{H}_\delta^n(B_k^i) \leq \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{H}^n(B_k^i) \\ &= \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{\alpha(n)}{2^n} (\text{diam } B_k^i)^n = \sum_{i=1}^{+\infty} \sum_{k=1}^{+\infty} \mathcal{L}^n(B_k^i) = \sum_{i=1}^{+\infty} \mathcal{L}^n \left( \bigcup_{k=1}^{+\infty} B_k^i \right) \\ &= \sum_{i=1}^{+\infty} \mathcal{L}^n(Q_i) < \mathcal{L}^n(A) + \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  was arbitrary, it follows  $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$ . The proof is complete.  $\square$

**2.3. Densities.** We first recall the Lebesgue Density Theorem:

**Theorem** (Lebesgue Density Theorem). *Let  $E \subseteq \mathbb{R}^n$  be a Lebesgue measurable set. For any  $r > 0$  and  $x \in \mathbb{R}^n$ , define the approximate Lebesgue density of  $E$  in the  $r$ -neighborhood of  $x$  by*

$$d_r(x) := \frac{\mathcal{L}^n(B(x, r) \cap E)}{\alpha(n)r^n}.$$

*Further define the Lebesgue density of  $E$  at  $x$  by*

$$d(x) := \lim_{r \rightarrow 0} d_r(x).$$

*Then*

$$d(x) = \lim_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x, r) \cap E)}{\alpha(n)r^n} = \begin{cases} 1, & \text{for } \mathcal{L}^n - \text{a.e. } x \in E, \\ 0, & \text{for } \mathcal{L}^n - \text{a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Since  $\mathcal{H}^n = \mathcal{L}^n$  for  $n \in \mathbb{N}$ , the above result clearly holds for  $\mathcal{H}^n$  as well. We want to develop some analogous results for lower-dimensional Hausdorff measures. Thus we assume throughout this section that  $0 < s < n$ .

We first establish a theorem that tells us the lower-dimensional Hausdorff density of a set at a.e. point outside the set is zero.

**t2.3-1 Theorem 2.3.1.** *Assume that  $E \subseteq \mathbb{R}^n$  with  $E$   $\mathcal{H}^s$ -measurable and  $\mathcal{H}^s(E) < +\infty$ . Then*

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} = 0$$

*for  $\mathcal{H}^s$ -a.e.  $x \in \mathbb{R}^n \setminus E$ .*

*Proof.* Fix  $t > 0$  and define

$$A_t := \left\{ x \in \mathbb{R}^n \setminus E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

It suffices to show that  $\mathcal{H}^s(A_t) = 0$ .

Note that  $\mathcal{H}^s \llcorner E$  is a Radon measure, and so, if we fix  $\epsilon > 0$ , there exists a compact set  $K \subseteq E$  such that

$$\mathcal{H}^s(E \setminus K) \leq \epsilon.$$

Set  $U := \mathbb{R}^n \setminus K$ . Then  $U$  is open and  $A_t \subseteq U$  because  $K \subseteq E$ . Fix  $\delta > 0$  and consider

$$\mathcal{F} := \left\{ B(x, r) : B(x, r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

By the Vitali Covering Lemma, there exists a countable family of balls  $\{B(x_i, r_i)\}_{i=1}^{+\infty}$  such that

$$A_t \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

Thus by monotonicity

$$\mathcal{H}_{10\delta}^s(A_t) \leq \mathcal{H}_{10\delta}^s\left(\bigcup_{i=1}^{+\infty} B(x_i, 5r_i)\right) \leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (10r_i)^s \leq \sum_{i=1}^{+\infty} 5^s \alpha(s) r_i^s$$

$$\begin{aligned}
&\leq \frac{5^s}{t} \sum_{i=1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \leq \frac{5^s}{t} \mathcal{H}^s(U \cap E) = \frac{5^s}{t} \mathcal{H}^s(E \setminus K) \\
&\leq \frac{5^s}{t} \epsilon.
\end{aligned}$$

Letting  $\delta \rightarrow 0$ , we obtain  $\mathcal{H}^s(A_t) \leq \frac{5^s}{t} \epsilon$ . Since  $\epsilon > 0$  was arbitrary, we have  $\mathcal{H}^s(A_t) = 0$  for each  $t > 0$ . The proof is complete.  $\square$

Now we prove that the lower-dimensional Hausdorff density of a set at a.e. point in the set is nonzero. Note that this contrasts with the Lebesgue Density Theorem: the density may not be 1. However, it is bounded below if we replace the limit with limit superior.

**t2.3-2** **Theorem 2.3.2.** *Assume that  $E \subseteq \mathbb{R}^n$  with  $E\mathcal{H}^s$ -measurable and  $\mathcal{H}^s(E) < +\infty$ . Then*

$$\frac{1}{2^s} \leq \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} \leq 1$$

for  $\mathcal{H}^s$ -a.e.  $x \in E$ .

**Remark.** *It is possible to have*

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} < 1$$

and

$$\liminf_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} = 0$$

for  $\mathcal{H}^s$ -a.e.  $x \in E$ , even if  $0 < \mathcal{H}^s(E) < +\infty$ .

*Proof.* (i) We first show the upper inequality. Fix  $\epsilon > 0$ ,  $t > 1$ , and define

$$B_t := \left\{ x \in E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

Since  $\mathcal{H}^s \llcorner E$  is Radon, there exists an open set  $U$  containing  $B_t$  such that

$$\mathcal{H}^s(U \cap E) \leq \mathcal{H}^s(B_t) + \epsilon.$$

Define

$$\mathcal{F} := \left\{ B(x, r) : B(x, r) \subseteq U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} > t \right\}.$$

By a corollary of the Vitali Covering Lemma, there exists a countable family of disjoint balls  $\{B(x_i, r_i)\}_{i=1}^{+\infty}$  such that

$$B_t \subseteq \left( \bigcup_{i=1}^m B(x_i, r_i) \right) \cup \left( \bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right).$$

Thus

$$\begin{aligned}
\mathcal{H}_{10\delta}^s(B_t) &\leq \mathcal{H}_{10\delta}^s \left( \bigcup_{i=1}^m B(x_i, r_i) \right) + \mathcal{H}_{10\delta}^s \left( \bigcup_{i=m+1}^{+\infty} B(x_i, 5r_i) \right) \\
&\leq \sum_{i=1}^m \frac{\alpha(s)}{2^s} (2r_i)^s + \sum_{i=m+1}^{+\infty} \frac{\alpha(s)}{2^s} (10r_i)^s
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^m \alpha(s) r^s + \sum_{i=m+1}^{+\infty} 5^s \alpha(s) r^s \\
&\leq \frac{1}{t} \sum_{i=1}^m \mathcal{H}^s(B(x_i, r_i) \cap E) + \frac{5^s}{t} \sum_{i=m+1}^{+\infty} \mathcal{H}^s(B(x_i, r_i) \cap E) \\
&\leq \frac{1}{t} \mathcal{H}^s(U \cap E) + \frac{5^s}{t} \mathcal{H}^s \left( \bigcup_{i=m+1}^{+\infty} B(x_i, r_i) \cap E \right).
\end{aligned}$$

Note that this holds for each  $m = 1, 2, \dots$ . Thus taking the limit as  $m \rightarrow \infty$  gives

$$\mathcal{H}_{10\delta}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(U \cap E) \leq \frac{1}{t} (\mathcal{H}^s(B_t) + \epsilon).$$

Letting  $\delta \rightarrow 0$ , we obtain

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} (\mathcal{H}^s(B_t) + \epsilon),$$

and then taking the limit as  $\epsilon \rightarrow 0$  gives

$$\mathcal{H}^s(B_t) \leq \frac{1}{t} \mathcal{H}^s(B_t).$$

Since  $\mathcal{H}^s(B_t) \leq \mathcal{H}^s(E) < +\infty$ , this implies that  $\mathcal{H}^s(B_t) = 0$  for each  $t > 1$ , as required.

(ii) We now show that

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s) r^s} \geq \frac{1}{2^s}$$

for  $\mathcal{H}^s$ -a.e.  $x \in E$ .

For any  $\delta > 0$  and  $0 < \tau < 1$ , denote by  $E(\delta, \tau)$  the set of all points  $x \in E$  such that

$$\mathcal{H}_\delta^s(C \cap E) \leq \frac{\alpha(s)}{2^s} \tau (\text{diam } C)^s,$$

whenever  $C \subseteq \mathbb{R}^n$ ,  $x \in C$ , and  $\text{diam } C \leq \delta$ . Then if  $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$  with  $\text{diam } C_i \leq \delta$ ,  $E(\delta, \tau) \subseteq \bigcup_{i=1}^{+\infty} C_i$ , and  $C_i \cap E(\delta, \tau) \neq \emptyset$ , we have

$$\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \sum_{i=1}^{+\infty} \mathcal{H}_\delta^s(C_i \cap E(\delta, \tau)) \leq \tau \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_i)^s.$$

Taking the infimum over all such covers  $\{C_i\}_{i=1}^{+\infty}$  of  $E(\delta, \tau)$ , we see that

$$\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \tau \mathcal{H}_\delta^s(E(\delta, \tau)),$$

and so  $\mathcal{H}_\delta^s(E(\delta, \tau)) = 0$ , since  $0 < \tau < 1$  and  $\mathcal{H}_\delta^s(E(\delta, \tau)) \leq \mathcal{H}_\delta^s(E) \leq \mathcal{H}^s(E) < +\infty$ . In particular,

$$\mathcal{H}^s(E(1 - \delta, \delta)) = 0 \tag{2.3.1}$$

for any  $0 < \delta < 1$ . Now if  $x \in E$  and

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s) r^s} < \frac{1}{2^s},$$

there exists  $\delta > 0$  such that

$$\frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s) r^s} < \frac{1 - \delta}{2^s} \tag{2.3.2}$$



for all  $0 < r \leq \delta$ . Thus if  $x \in C$  and  $\text{diam } C \leq \delta$ ,

$$\begin{aligned}\mathcal{H}_\delta^s(C \cap E) &= \mathcal{H}_\infty^s(C \cap E) \\ &\leq \mathcal{H}_\infty^s(B(x, \text{diam } C) \cap E) \\ &\leq (1 - \delta) \frac{\alpha(s)}{2^s} (\text{diam } C)^s,\end{aligned}$$

by (eq:2.3-2). Consequently  $x \in E(\delta, 1 - \delta)$ , and it follows

$$\left\{ x \in E : \limsup_{r \rightarrow 0} \frac{\mathcal{H}_\infty^s(B(x, r) \cap E)}{\alpha(s)r^s} < \frac{1}{2^s} \right\} \subseteq \left\{ \bigcup_{k=2}^{+\infty} E\left(\frac{1}{k}, 1 - \frac{1}{k}\right) \right\}.$$

But since the RHS has  $\mathcal{H}^s$ -measure zero by (eq:2.3-1), this proves (ii).

(iii) Since  $\mathcal{H}^s(B(x, r) \cap E) \geq \mathcal{H}_\infty^s(B(x, r) \cap E)$  for any  $x \in E$  and  $r > 0$ , (ii) immediately gives the required lower estimate

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(B(x, r) \cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}.$$

The proof is complete. □

**2.4. Hausdorff Measure and Elementary Properties of Functions.** We establish some properties relating the behavior of certain functions and Hausdorff measure.

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#### 2.4.1. Hausdorff Measure and Lipschitz Mappings.

**Definition 2.4.1** (Lipschitz). A function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called *Lipschitz* if there exists a constant  $C > 0$  such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all  $x, y \in \mathbb{R}^n$ .

**Definition 2.4.2** (Lipschitz Constant). We define the *Lipschitz constant* of a Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by

$$\text{Lip}(f) := \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Note that for any Lipschitz function  $f$ ,

$$|f(x) - f(y)| \leq \text{Lip}(f)|x - y|.$$

**t2.4-1 Theorem 2.4.1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be Lipschitz,  $A \subseteq \mathbb{R}^n$ ,  $0 \leq s < +\infty$ . Then

$$\mathcal{H}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}^s(A).$$

*Proof.* Fix  $\delta > 0$  and choose sets  $\{C_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}^n$  such that  $\text{diam } C_i \leq \delta$ ,  $A \subseteq \cup_{i=1}^{+\infty} C_i$ . Then

$$\text{diam } f(C_i) \leq \text{Lip}(f) \text{diam } C_i \leq \delta \text{Lip}(f),$$

and  $f(A) \subseteq f(\cup_{i=1}^{+\infty} C_i) = \cup_{i=1}^{+\infty} f(C_i)$ . Thus

$$\begin{aligned} \mathcal{H}_{\delta \text{Lip}(f)}^s(f(A)) &\leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } f(C_i))^s \\ &\leq (\text{Lip}(f))^s \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } C_i)^s. \end{aligned}$$

Taking the infimum over all such sets  $\{C_i\}_{i=1}^{+\infty}$  which cover  $A$ , we find on the RHS

$$\mathcal{H}_{\delta \text{Lip}(f)}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}_{\delta}^s(A).$$

Taking the limit as  $\delta \rightarrow 0$ , we obtain

$$\mathcal{H}^s(f(A)) \leq (\text{Lip}(f))^s \mathcal{H}^s(A),$$

as required. The proof is complete.  $\square$

**c2.4-1**

**Corollary 2.4.1.** Suppose that  $n > k$ . Let  $P : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be the usual projection,  $A \subseteq \mathbb{R}^n$ ,  $0 \leq s < +\infty$ . Then

$$\mathcal{H}^s(P(A)) \leq \mathcal{H}^s(A).$$

*Proof.* Since  $P$  is the standard projection map from  $\mathbb{R}^n$  to  $\mathbb{R}^k$ ,  $\text{Lip}(P) = 1$ . Applying the above theorem (cf. (2.4.1)) gives the required estimate.  $\square$

### 2.4.2. Graphs of Lipschitz Functions.

**Definition 2.4.3** (Graph). For  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $A \subseteq \mathbb{R}^n$ , we define the graph  $\Gamma(f; A)$  of  $f$  over  $A$  by

$$\Gamma(f; A) := \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$

**t2.4-2**

**Theorem 2.4.2.** Assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $\mathcal{L}^n(A) > 0$ .

- (i) Then  $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$ ;
- (ii) If  $f$  is Lipschitz, then  $\mathcal{H}_{\dim}(\Gamma(f; A)) = n$ .

**Remark.** We thus see that the graph of a Lipschitz function  $f$  has the expected Hausdorff dimension (think of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ). We will see from the Area Formula that  $\mathcal{H}^s(\Gamma(f; A))$  can be computed according to the usual rules of calculus.

*Proof.*

- (i). Let  $P : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  be the usual projection. Then by (c2.4-1) (2.4.1),

$$\mathcal{H}^n(\Gamma(f; A)) \geq \mathcal{H}^n(A) > 0.$$

Thus  $\mathcal{H}^n(\Gamma(f; A)) > 0$ , so that  $\mathcal{H}_{\dim}(\Gamma(f; A)) \geq n$ .

- (ii). Let  $Q$  denote any cube in  $\mathbb{R}^n$  of side length 1. Subdivide  $Q$  into  $k^n$  subcubes  $\{Q_1, \dots, Q_{k^n}\}$  of side length  $\frac{1}{k}$ . Note that  $\text{diam } Q_i = \frac{\sqrt{n}}{k}$  for each  $i = 1, \dots, k^n$ . Define

$$a_j^i := \min_{x \in Q_j} f^i(x), \quad b_j^i := \max_{x \in Q_j} f^i(x),$$

where  $i = 1, \dots, m$  and  $j = 1, \dots, k^n$ . Since  $f$  is Lipschitz,

$$|b_j^i - a_j^i| \leq \text{Lip}(f) \text{diam } Q_j = \text{Lip}(f) \frac{\sqrt{n}}{k}.$$

For each  $j = 1, \dots, k^n$ , put

$$C_j := Q_j \times \prod_{i=1}^m (a_j^i, b_j^i).$$

Then

$$\Gamma(f; Q_j \cap A) = \{(x, f(x)) : x \in Q_j \cap A\} \subseteq C_j,$$

and  $\text{diam } C_j \leq \frac{C}{k}$  for some constant  $C > 0$ . Since

$$\Gamma(f; A \cap Q) = \Gamma(f; A \cap \bigcup_{j=1}^{k^n} Q_j) = \bigcup_{j=1}^{k^n} \Gamma(f; A \cap Q_j) \subseteq \bigcup_{j=1}^{j_n} C_j,$$

we have by monotonicity

$$\begin{aligned} \mathcal{H}_{C/k}^n(G(f; A \cap Q)) &\leq \sum_{j=1}^{k^n} \frac{\alpha(n)}{2^n} (\text{diam } C_j)^n \\ &\leq \frac{k^n \alpha(n)}{2^n} \left(\frac{C}{k}\right)^n = \frac{C^n \alpha(n)}{2^n}. \end{aligned}$$

Then upon letting  $k \rightarrow +\infty$ , we find  $\mathcal{H}^n(\Gamma(f; A \cap Q)) < +\infty$ , and so  $\mathcal{H}_{\text{dim}}(\Gamma(f; A \cap Q)) \leq n$ . Recall that this estimate is valid for each cube  $Q \subseteq \mathbb{R}^n$  of side length 1. Consequently  $\mathcal{H}_{\text{dim}}(\Gamma(f; A)) \leq n$ . Applying (i), it follows  $\mathcal{H}_{\text{dim}}(\Gamma(f; A)) = n$ . The proof is complete.  $\square$

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2.4.3. *The Set Where an Integrable Function is Large.* If a function  $f$  is locally integrable, we can estimate the Hausdorff measure of the set where  $f$  is locally large.

**t2.4-3** **Theorem 2.4.3.** *Let  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ , let  $0 \leq s < n$ , and define*

$$\Lambda_s := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) > 0 \right\}$$

Then

$$\mathcal{H}^s(\Lambda_s) = 0.$$

*Proof.* We may as well assume that  $f \in L^1(\mathbb{R}^n)$ . By the Lebesgue Differentiation Theorem,

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) = |f(x)|$$

for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$ , and thus

$$\lim_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) = \lim_{r \rightarrow 0} \alpha(n) r^{n-s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) = \lim_{r \rightarrow 0} \alpha(n) r^{n-s} |f(x)| = 0$$

for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$ , since  $0 \leq s < n$ . Hence

$$\mathcal{L}^n(\Lambda_s) = 0.$$

Fix  $\epsilon > 0$ ,  $\delta > 0$ ,  $\sigma > 0$ . Since  $f$  is  $\mathcal{L}^n$ -integrable, there exists  $\eta > 0$  such that  $\mathcal{L}^n(\Omega) \leq \eta$  implies

$$\int_{\Omega} |f(x)| \, d\mathcal{L}^n(x) < \sigma.$$

Define

$$\Lambda_s^\epsilon := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) > \epsilon \right\}.$$

By the above analysis,

$$\mathcal{L}^n(\Lambda_s^\epsilon) = 0.$$

Thus there exists an open set  $\Omega \subseteq \mathbb{R}^n$  such that  $\Lambda_s^\epsilon \subseteq \Omega$  and  $\mathcal{L}^n(\Omega) < \eta$ . Put

$$\mathcal{F} := \left\{ B(x, r) : x \in \Lambda_s^\epsilon, 0 < r < \delta, B(x, r) \subseteq \Omega, \int_{B(x,r)} |f(y)| \, d\mathcal{L}^n(y) > \epsilon r^s \right\}.$$

By the Vitali Covering Lemma, there exists a countable family  $\{B(x_i, r_i)\}_{i=1}^{+\infty}$  of disjoint balls in  $\mathcal{F}$  such that

$$\Lambda_s^\epsilon \subseteq \bigcup_{i=1}^{+\infty} B(x_i, 5r_i).$$

We thus compute

$$\begin{aligned} \mathcal{H}_{10\delta}^s(\Lambda_s^\epsilon) &\leq \sum_{i=1}^{+\infty} \frac{\alpha(s)}{2^s} (\text{diam } B(x_i, 5r_i))^s \leq \sum_{i=1}^{+\infty} \alpha(s) (5r_i)^s \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \sum_{i=1}^{+\infty} \int_{B(x_i, r_i)} |f(y)| \, d\mathcal{L}^n(y) \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \int_{\Omega} |f(y)| \, d\mathcal{L}^n(y) \\ &\leq \frac{\alpha(s) 5^s}{\epsilon} \sigma. \end{aligned}$$

Taking the limit as  $\delta \rightarrow 0$ , we have

$$\mathcal{H}^s(\Lambda_s^\epsilon) \leq \frac{\alpha(s) 5^s}{\epsilon} \sigma,$$

and then upon sending  $\sigma \rightarrow 0$  we obtain

$$\mathcal{H}^s(\Lambda_s^\epsilon) = 0.$$

Since  $\epsilon > 0$  was arbitrary, it follows

$$\mathcal{H}^s(\Lambda_s) = 0.$$

The proof is complete. □

## 3. AREA AND COAREA FORMULAS

## 3.1. Lipschitz Functions, Rademacher's Theorem.

**Definition 3.1.1** (Lipschitz). Let  $A \subseteq \mathbb{R}^n$ . A function  $f : A \rightarrow \mathbb{R}^m$  is called Lipschitz provided that

$$|f(x) - f(y)| \leq C|x - y| \quad (3.1.1)$$

for some constant  $C > 0$  and all  $x, y \in A$ . The smallest constant  $C$  such that (3.1.1) holds for all  $x, y \in A$  is denoted

$$\text{Lip}(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y \right\}.$$

**Definition 3.1.2** (Locally Lipschitz). A function  $f : A \rightarrow \mathbb{R}^m$  is called locally Lipschitz if for each compact set  $K \subseteq A$ , there exists a constant  $C_K > 0$  such that

$$|f(x) - f(y)| \leq C_K|x - y|$$

for all  $x, y \in K$ .

t3.1-1

**Theorem 3.1.1** (Extension of Lipschitz Functions). Assume that  $A \subseteq \mathbb{R}^n$ , and let  $f : A \rightarrow \mathbb{R}^m$  be Lipschitz. There exists a Lipschitz function  $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

- (i)  $\bar{f} = f$  on  $A$ ;
- (ii)  $\text{Lip}(\bar{f}) \leq \sqrt{m} \text{Lip}(f)$ .

*Proof.*

(i). First assume that  $f : A \rightarrow \mathbb{R}$ . Define

$$\bar{f}(x) := \inf_{a \in A} \{f(a) + \text{Lip}(f)|x - a|\}.$$

If  $b \in A$ , then we have  $\bar{f}(b) = f(b)$ . This follows because if  $b \in A$ , then

$$\bar{f}(b) \leq f(b) + \text{Lip}(f)|b - b| = f(b).$$

On the other hand, for all  $a \in A$ , we've:

$$f(a) + \text{Lip}(f)|b - a| \geq f(a) + \frac{f(b) - f(a)}{|b - a|}|b - a| = f(b).$$

Taking the infimum over all  $a \in A$  on the LHS thus gives  $\bar{f}(b) \geq f(b)$ . Now if  $x, y \in \mathbb{R}^n$ , then

$$\begin{aligned} \bar{f}(x) &\leq \inf_{a \in A} \{f(a) + \text{Lip}(f)(|x - y| + |y - a|)\} \\ &= \inf_{a \in A} \{f(a) + \text{Lip}(f)|y - a|\} + \text{Lip}(f)|x - y| \\ &= \bar{f}(y) + \text{Lip}(f)|x - y|. \end{aligned}$$

Similarly

$$\bar{f}(y) \leq \bar{f}(x) + \text{Lip}(f)|x - y|.$$

Therefore

$$\frac{|\bar{f}(x) - \bar{f}(y)|}{|x - y|} \leq \text{Lip}(f)$$

for all  $x, y \in A$ . This proves the result for functions  $f : A \rightarrow \mathbb{R}$ .

(ii). In the general case  $f : A \rightarrow \mathbb{R}^m$ ,  $f = (f^1, \dots, f^m)$ , define  $\bar{f} := (\bar{f}^1, \dots, \bar{f}^m)$ , where  $\bar{f}^i$ ,  $i = 1, \dots, m$ , are defined as in (i). Then

$$|\bar{f}(x) - \bar{f}(y)|^2 = \sum_{i=1}^m \left| \bar{f}^i(x) - \bar{f}^i(y) \right|^2 \leq m(\text{Lip}(f))^2 |x - y|^2.$$

Taking square roots,

$$|\bar{f}(x) - \bar{f}(y)| \leq \sqrt{m} \text{Lip}(f) |x - y|,$$

as required. The proof is complete.  $\square$

**Remark.** In fact there exists an extension  $\bar{f}$  of  $f$  with  $\text{Lip}(\bar{f}) = \text{Lip}(f)$ . This is Kirszbraun's Theorem.

We now prove Rademacher's Theorem, which states that a locally Lipschitz function is differentiable  $\mathcal{L}^n$ -a.e. Note that the inequality

$$|f(x) - f(y)| \leq \text{Lip}(f) |x - y|$$

says nothing about the possibility of locally approximating  $f$  by a linear map.

**Definition 3.1.3** (Differentiable). *The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to be differentiable at  $x \in \mathbb{R}^n$  if there exists a linear mapping*

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

*such that*

$$\lim_{y \rightarrow x} \frac{|f(y) - f(x) - L(x - y)|}{|x - y|} = 0,$$

*or, equivalently,*

$$f(y) = f(x) + L(x - y) + o(|y - x|), \quad y \rightarrow x.$$

**Remark.**

- (i) Note that this is actually the definition of the Fréchet derivative.
- (ii) If such a linear mapping  $L$  exists, it is unique, and we write

$$Df(x)$$

for  $L$ . We call  $Df(x)$  the derivative of  $f$  at  $x$ .

**t3.1-2**

**Theorem 3.1.2** (Rademacher's Theorem). *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a locally Lipschitz function. Then  $f$  is differentiable  $\mathcal{L}^n$ -a.e.*

*Proof.*

(i). We may assume that  $m = 1$ , for otherwise, repeat the below argument  $m$  times. Since differentiability is a local property, we may as well also suppose that  $f$  is Lipschitz.

(ii). Fix any  $v \in \mathbb{R}^n$  with  $|v| = 1$ , and for any  $x \in \mathbb{R}^n$ , define the Gateaux derivative

$$D_v f(x) := \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

at  $x$ , provided that this limit exists.

(iii). We show that  $D_v f(x)$  exists for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$ . Since  $f$  is continuous,

$$\overline{D}_v f(x) = \limsup_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

$$= \lim_{k \rightarrow +\infty} \sup_{\substack{0 < |t| < \frac{1}{k} \\ t \in \mathbb{Q}}} \frac{f(x + tv) - f(x)}{t}$$

is Borel measurable, as is

$$\underline{D}_v f(x) := \liminf_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Thus

$$\begin{aligned} A_v &:= \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist}\} \\ &= \{x \in \mathbb{R}^n : \underline{D}_v f(x) < \overline{D}_v f(x)\}, \end{aligned}$$

being the complement of the set of all points of which the pointwise limit of measurable functions exists, is Borel measurable.

Now, for each  $x, v \in \mathbb{R}^n$  with  $|v| = 1$ , define  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi(t) := f(x + tv).$$

Note that for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} |\phi(t) - \phi(s)| &= |f(x + tv) - f(x + sv)| \leq \text{Lip}(f)|(x + tv) - (x + sv)| \\ &= \text{Lip}(f)|t - s|, \end{aligned}$$

so that  $\phi$  is Lipschitz. Therefore  $\phi$  is absolutely continuous, and thus differentiable  $\mathcal{L}^1$ -a.e. Thus for any line  $L$  parallel to  $v$ , the set of all points on  $L$  such that  $f$  is not differentiable has Lebesgue measure zero. That is,

$$\mathcal{H}^1(A_v \cap L) = 0$$

for each line  $L$  parallel to  $v$ . Thus the Fubini–Tonelli Theorem implies

$$\mathcal{L}^n(A_v) = 0,$$

as required.

(iv). Noting that

$$\frac{\partial}{\partial x_j} f(x) = D_{e_j} f(x) = \lim_{t \rightarrow 0} \frac{f(x + te_j) - f(x)}{t}$$

for each  $j = 1, \dots, n$ , we have by (iii) that

$$\nabla f(x) = \left( \frac{\partial}{\partial x_1} f(x), \dots, \frac{\partial}{\partial x_n} f(x) \right)$$

exists for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$ .

(v). Next we show that  $D_v f(x) = v \cdot \nabla f(x)$  for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n$ . Let  $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ . Then

$$\begin{aligned} \int_{\mathbb{R}^n} \left[ \frac{f(x + tv) - f(x)}{t} \right] \zeta(x) \, dx &= \frac{1}{t} \left[ \int_{\mathbb{R}^n} f(x + tv) \zeta(x) \, dx - \int_{\mathbb{R}^n} f(x) \zeta(x) \, dx \right] \\ &= \frac{1}{t} \left[ \int_{\mathbb{R}^n} f(x) \zeta(x - tv) \, dx - \int_{\mathbb{R}^n} f(x) \zeta(x) \, dx \right] \\ &= - \int_{\mathbb{R}^n} f(x) \left[ \frac{\zeta(x) - \zeta(x - tv)}{t} \right] \, dx. \end{aligned}$$

This is the integration by parts formula for difference quotients. Let  $t = \frac{1}{k}$  for  $k = 1, 2, \dots$ , in the above equality and note that

$$\frac{|f(x + \frac{1}{k}v) - f(x)|}{\frac{1}{k}} \leq \text{Lip}(f).$$

Thus, by Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} D_v f(x) \zeta(x) \, dx &\stackrel{LDC}{=} - \int_{\mathbb{R}^n} f(x) D_v \zeta(x) \, dx \\ &= - \sum_{j=1}^n v_j \int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_j} \zeta(x) \, dx \\ &= \sum_{j=1}^n v_j \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} f(x) \zeta(x) \, dx \\ &= \int_{\mathbb{R}^n} (v \cdot \nabla f(x)) \zeta(x) \, dx, \end{aligned}$$

where we have used integration by parts and the partial derivatives on  $f$  are understood in the a.e. sense. Since the above equality holds for every  $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ , we have  $D_v f = v \cdot \nabla f$   $\mathcal{L}^n$ -a.e.

(vi). Choose  $\{v_k\}_{k=1}^{+\infty}$  to be a countable, dense subset of  $\partial B(0, 1)$ . Set

$$A_k := \{x \in \mathbb{R}^n : D_{v_k} f(x), \nabla f(x) \text{ exist and } D_{v_k} f(x) = v_k \cdot \nabla f(x)\}$$

for each  $k \in \mathbb{N}$ . Note that by (iii)-(v),  $\mathcal{L}^n(\mathbb{R}^n \setminus A_k) = 0$  for each  $k \in \mathbb{N}$ . Define

$$A := \bigcap_{k=1}^{+\infty} A_k$$

and observe that

$$\mathcal{L}^n(\mathbb{R}^n \setminus A) = \mathcal{L}^n(\mathbb{R}^n \setminus \bigcap_{k=1}^{+\infty} A_k) = \mathcal{L}^n(\bigcup_{k=1}^{+\infty} (\mathbb{R}^n \setminus A_k)) = 0.$$

(vii). We now show that  $f$  is differentiable at each point  $x \in A$ . Fix any  $x \in A$ . Choose  $v \in \partial B(0, 1)$ ,  $t \in \mathbb{R}$ ,  $t \neq 0$ , and write

$$Q(x, v, t) := \frac{f(x + tv) - f(x)}{t} - v \cdot \nabla f(x).$$

Then if  $w \in \partial B(0, 1)$ , we have

$$\begin{aligned} |Q(x, v, t) - Q(x, w, t)| &= \left| \frac{f(x + tv) - f(x + tw)}{t} - (v - w) \cdot \nabla f(x) \right| \\ &\leq \left| \frac{f(x + tv) - f(x + tw)}{t} \right| + |(v - w) \cdot \nabla f(x)| \\ &\leq \text{Lip}(f)|v - w| + |\nabla f(x)||v - w| \\ &\leq (1 + \sqrt{n}) \text{Lip}(f)|v - w|. \end{aligned} \tag{3.1.2}$$

Fix  $\epsilon > 0$  and choose  $N \in \mathbb{N}$  so large that if  $v \in \partial B(0, 1)$ , then

$$|v - v_k| \leq \frac{\epsilon}{2(1 + \sqrt{n}) \text{Lip}(f)}$$



for some  $k = 1, \dots, N$ . Note that since  $x \in A$ ,

$$\begin{aligned}\lim_{t \rightarrow 0} Q(x, v_k, t) &= \lim_{t \rightarrow 0} \left\{ \frac{f(x + tv_k) - f(x)}{t} - v_k \cdot \nabla f(x) \right\} \\ &= D_{v_k} f(x) - v_k \cdot \nabla f(x) \\ &= 0\end{aligned}$$

for each  $k = 1, \dots, N$ . Thus there exists  $\delta > 0$  so that for all  $0 < |t| < \delta$ ,

$$|Q(x, v_k, t)| < \frac{\epsilon}{2} \tag{3.1.3}$$

{eq:3.1-3}

holds for each  $k = 1, \dots, N$ . Consequently for each  $v \in \partial B(0, 1)$  there exists  $k \in \{1, \dots, N\}$  such that

$$\begin{aligned}|Q(x, v, t)| &\leq |Q(x, v, t) - Q(x, v_k, t)| + |Q(x, v_k, t)| \\ &< (1 + \sqrt{n}) \text{Lip}(f) |v - v_k| + \frac{\epsilon}{2} \\ &< \epsilon,\end{aligned}$$

by [\(3.1.2\)](#) and [\(3.1.3\)](#), provided that  $0 < |t| < \delta$ . Note that this is the same  $\delta > 0$  for all  $v \in \partial B(0, 1)$ .

Now choose any  $x, y \in \mathbb{R}^n$ ,  $y \neq x$ . Write

$$v := \frac{y - x}{|y - x|},$$

so that  $y = x + tv$ , where  $t := |x - y|$ . Then

$$\begin{aligned}|f(y) - f(x) - \nabla f(x) \cdot (y - x)| &= |f(x + tv) - f(x) - \nabla f(x) \cdot tv| \\ &= |Q(x, t, v)| |t| \\ &< \epsilon |t|,\end{aligned}$$

so that

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) = o(t) = o(|x - y|), \quad y \rightarrow x.$$

Hence,  $f$  is differentiable at  $x$ , with

$$Df(x) = \nabla f(x).$$

The proof is complete. □

c3.1-1

### Corollary 3.1.1.

(i) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be locally Lipschitz, and

$$\mathcal{Z} := \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Then  $Df(x) = 0$  for  $\mathcal{L}^n$ -a.e.  $x \in \mathcal{Z}$ .

(ii) Let  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be locally Lipschitz, and

$$Y := \{x \in \mathbb{R}^n : g(f(x)) = x\}.$$

Then

$$Dg(f(x))Df(x) = I$$

for  $\mathcal{L}^n$ -a.e.  $x \in Y$ .

*Proof.*

- (i). We may assume that  $m = 1$  in (i), otherwise, repeat the following argument  $m$  times.
- (ii). Choose  $x \in \mathcal{Z}$  so that  $Df(x)$  exists, and

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(\mathcal{Z} \cap B(x, r))}{\mathcal{L}^n(B(x, r))} = 1. \quad (3.1.4) \quad \boxed{\text{eq:3.1-4}}$$

Note that this holds for  $\mathcal{L}^n$ -a.e.  $x \in \mathcal{Z}$ . Since  $x \in \mathcal{Z}$ , it follows

$$f(y) = Df(x) \cdot (y - x) + o(|y - x|). \quad (3.1.5) \quad \boxed{\text{eq:3.1-5}}$$

By contradiction, suppose that  $Df(x) = \alpha \neq 0$ , and set

$$S := \left\{ v \in \partial B(0, 1) : \alpha \cdot v \geq \frac{1}{2}|\alpha| \right\}.$$

Note that  $S$  is nonempty, for otherwise  $Df(x) = 0$ . Now for each  $v \in S$  and  $t > 0$ , set  $y := x + tv$  in (3.1.5) to obtain

$$\begin{aligned} f(x + tv) &= \alpha \cdot tv + o(|tv|) \\ &\geq \frac{|\alpha|}{2}t + o(t). \end{aligned}$$

Hence, there exists  $\delta > 0$  such that for all  $0 < t < \delta$  and all  $v \in S$ ,

$$f(x + tv) > 0.$$

But this contradicts (3.1.4), since for all  $0 < r < \delta$ ,  $B(x, r) \cap \mathcal{Z} = \{x\}$ . This proves (i).

- (iii). We now show (ii). Define

$$\text{dom } Df := \{x \in \mathbb{R}^n : Df(x) \text{ exists}\}$$

and

$$\text{dom } Dg := \{x \in \mathbb{R}^n : Dg(x) \text{ exists}\}.$$

Put

$$X := Y \cap \text{dom } Df \cap f^{-1}(\text{dom } Dg).$$

Then

$$\begin{aligned} Y \setminus X &= Y \cap (Y^C \cup (\text{dom } Df)^C \cup (f^{-1}(\text{dom } Dg))^C) \\ &= (Y \setminus \text{dom } Df) \cup (Y \setminus f^{-1}(\text{dom } Dg)) \\ &\subseteq (\mathbb{R}^n \setminus \text{dom } Df) \cup g(\mathbb{R}^n \setminus \text{dom } Dg). \end{aligned} \quad (3.1.6) \quad \boxed{\text{eq:3.1-6}}$$

This follows since if  $x \in Y \setminus f^{-1}(\text{dom } Dg)$ , then  $f(x) \in f(Y) \subseteq \mathbb{R}^n$ , and  $f(x) \notin \text{dom } Dg$ , so that

$$f(x) \in \mathbb{R}^n \setminus \text{dom } Dg.$$

Thus

$$x = g(f(x)) \in g(\mathbb{R}^n \setminus \text{dom } Dg.)$$

By Rademacher's Theorem (cf. (3.1.2)),

$$\mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Df) = 0$$

and

$$\mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Dg) = 0.$$

Moreover, since  $g$  is Lipschitz (cf. (t2.4-1)), we have

$$\mathcal{L}^n(g(\mathbb{R}^n \setminus \text{dom } Dg)) \leq (\text{Lip}(g))^n \mathcal{L}^n(\mathbb{R}^n \setminus \text{dom } Dg) = 0.$$

Thus, by (eq:3.1-6),

$$\mathcal{L}^n(Y \setminus X) = 0.$$

Now if  $x \in X$ ,  $Dg(f(x))$  and  $Df(x)$  exist, and so the chain rule implies

$$Dg(f(x))Df(x) = D(g \circ f)(x)$$

exists. Finally, since  $(g \circ f)(x) - x = g(f(x)) - x = 0$  on  $Y$ , assertion (i) gives

$$Dg(f(x))Df(x) = D(g \circ f)(x) = I$$

$\mathcal{L}^n$ -a.e. on  $Y$ . The proof is complete.  $\square$

**3.2. Linear Maps and Jacobians.** We first review some basic linear algebra. Our goal in this section is to define the Jacobian of a map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

### 3.2.1. Linear Maps.

**Definition 3.2.1** (Orthogonal Linear Map). *A linear map  $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is orthogonal if*

$$Ox \cdot Oy = x \cdot y$$

*for all  $x, y \in \mathbb{R}^n$ .*

**Definition 3.2.2** (Symmetric Linear Map). *A linear map  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is symmetric if*

$$x \cdot Sy = Sx \cdot y$$

*for all  $x, y \in \mathbb{R}^n$ .*

**Definition 3.2.3** (Diagonal Linear Map). *A linear map  $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is diagonal if there exist  $d_1, \dots, d_n \in \mathbb{R}$  such that*

$$Dx = (d_1x_1, \dots, d_nx_n)$$

*for all  $x \in \mathbb{R}^n$ .*

**Definition 3.2.4** (Adjoint). *Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. The adjoint of  $A$  is the linear map  $A^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined by*

$$x \cdot A^*y = Ax \cdot y$$

*for all  $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ .*

Recall that the existence of adjoints in Euclidean space with the Euclidean metric is guaranteed, and, since  $\mathbb{R}^n$  is a Hilbert space under the Euclidean metric, the adjoint operator has the above form by the Riesz Representation Theorem.

t3.2-1

**Theorem 3.2.1.**

- (i)  $A^{**} = A$ ;
- (ii)  $(A \circ B)^* = B^* \circ A^*$ ;
- (iii) If  $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is orthogonal, then  $O^* = O^{-1}$ ;
- (iv) If  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is symmetric, then  $S^* = S$ ;

(v) If  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is symmetric, there exists an orthogonal map  $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a diagonal map  $D : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$S = O \circ D \circ O^{-1};$$

(vi) If  $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is orthogonal, then  $n \leq m$  and

$$\begin{aligned} O^* \circ O &= I \quad \text{on } \mathbb{R}^n, \\ O \circ O^* &= I \quad \text{on } O(\mathbb{R}^n). \end{aligned}$$

*Proof.*

(i). Since the dot product is symmetric, we have for all  $x, y \in \mathbb{R}^n$  that

$$\begin{aligned} x \cdot (A^{**}y) &= x \cdot (A^*)^*y = A^*x \cdot y = y \cdot A^*x = Ay \cdot x \\ &= x \cdot Ay. \end{aligned}$$

Since this is for all  $x \in \mathbb{R}^n$ , assertion (i) follows.

(ii). For any  $x, y \in \mathbb{R}^n$ ,

$$\begin{aligned} x \cdot (A \circ B)^*y &= (A \circ B)x \cdot y = A(Bx) \cdot y = Bx \cdot A^*y \\ &= x \cdot B^*(A^*y). \end{aligned}$$

This is for all  $x \in \mathbb{R}^n$ , so this proves (ii).

(iii). Let  $x, y \in \mathbb{R}^n$ . Then

$$x \cdot y = Ox \cdot Oy = x \cdot O^*(Oy),$$

and

$$x \cdot y = O(O^{-1}x) \cdot y = O^{-1}x \cdot O^*y = x \cdot O(O^*y).$$

This shows  $O^* = O^{-1}$ .

(iv). If  $x, y \in \mathbb{R}^n$ , then

$$x \cdot Sy = Sx \cdot y = x \cdot S^*y,$$

and since this is for all  $x \in \mathbb{R}^n$ , assertion (iv) follows.  $\square$

**t3.2-2**

**Theorem 3.2.2** (Polar Decomposition). *Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear mapping.*

(i) *If  $n \leq m$ , there exists a symmetric map  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and an orthogonal map  $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that*

$$L = O \circ S.$$

(ii) *If  $n \geq m$ , there exists a symmetric map  $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$  and an orthogonal map  $O : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that*

$$L = S \circ O^*.$$

*Proof.*

(i). First suppose  $n \leq m$ . Consider the mapping  $C := L^* \circ L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Now for any  $x, y \in \mathbb{R}^n$ ,

$$\begin{aligned} Cx \cdot y &= (L^* \circ L)x \cdot y = L^*(Lx) \cdot y = Lx \cdot Ly = x \cdot L^*(Ly) = x \cdot (L^* \circ L)y \\ &= x \cdot Cy, \end{aligned}$$

and also

$$Cx \cdot x = (L^* \circ L)x \cdot x = L^*(Lx) \cdot x = Lx \cdot Lx \geq 0.$$

Thus  $C$  is symmetric and positive semidefinite. Hence there exist  $\mu_1, \dots, \mu_n \geq 0$  and an orthonormal basis  $\{x_k\}_{k=1}^n$  of  $\mathbb{R}^n$  such that

$$Cx_k = \mu_k x_k,$$

$k = 1, \dots, n$ . Write  $\mu_k := \lambda_k^2$ ,  $\lambda_k \geq 0$ ,  $k = 1, \dots, n$ .

(ii). We show that there exists an orthonormal set  $\{z_k\}_{k=1}^n$  in  $\mathbb{R}^m$  such that

$$Lx_k = \lambda_k z_k,$$

$k = 1, \dots, n$ . To see this, if  $\lambda_k \neq 0$ , define

$$z_k := \frac{1}{\lambda_k} Lx_k.$$

Then if  $\lambda_k, \lambda_l \neq 0$ ,

$$\begin{aligned} z_k \cdot z_l &= \frac{1}{\lambda_k} Lx_k \cdot \frac{1}{\lambda_l} Lx_l = \frac{1}{\lambda_k \lambda_l} Lx_k \cdot Lx_l = \frac{1}{\lambda_k \lambda_l} x_k \cdot L^*(Lx_l) = \frac{1}{\lambda_k \lambda_l} x_k \cdot Cx_l \\ &= \frac{\lambda_l^2}{\lambda_k \lambda_l} x_k \cdot x_l \\ &= \frac{\lambda_l}{\lambda_k} \delta_{kl}, \end{aligned}$$

by (i) and the fact that  $\{x_k\}_{k=1}^n$  is an orthonormal set. Thus the set  $\{z_k : \lambda_k \neq 0\}$  is orthonormal. If  $\lambda_k = 0$ , define  $z_k$  to be any unit vector such that the set  $\{z_k\}_{k=1}^n$  is orthonormal, applying the Gram–Schmidt process if necessary.

(iii). Define  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$Sx_k := \lambda_k x_k,$$

$k = 1, \dots, n$  and  $O : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by

$$Ox_k := z_k,$$

$k = 1, \dots, n$ . Then

$$(O \circ S)x_k = O(Sx_k) = O(\lambda_k x_k) = \lambda_k Ox_k = \lambda_k z_k = Lx_k,$$

and, since  $\{x_k\}_{k=1}^n$  is a basis for  $\mathbb{R}^n$ ,

$$L = O \circ S.$$

Notice that the mapping  $S$  is clearly symmetric. Moreover,  $O$  is orthogonal because

$$Ox_k \cdot Ox_l = z_k \cdot z_l = \delta_{kl} = x_k \cdot x_l.$$

This proves assertion (i) of the theorem.

(iv). To prove assertion (ii), we apply assertion (i) to  $L^*$  and apply  $\text{\texttt{3.2.1}}$  to obtain

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

The proof is complete. □

We now define the Jacobian of a linear map.

**Definition 3.2.5** (Jacobian). *Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map.*

(i) *If  $n \leq m$ , write  $L = O \circ S$  (cf.  $\text{\texttt{3.2.2}}$ ), and we define the Jacobian of  $L$  to be*

$$\llbracket L \rrbracket := |\det S|;$$

(ii) If  $n \geq m$ , write  $L = S \circ O^*$  (cf.  $\text{t3.2-2}$ ), and we define the Jacobian of  $L$  to be

$$\llbracket L \rrbracket := |\det S|.$$

**Remark.**

- (i) It will follow from Theorem  $\text{t3.2-3}$  below that the definition of  $\llbracket L \rrbracket$  is independent of the particular choices of  $O$  and  $S$ .
- (ii) Note that if, say,  $n \leq m$ , then  $L = O \circ S$  implies

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

This is the same  $O$  and  $S$ , and it clearly follows

$$\llbracket L \rrbracket = \llbracket L^* \rrbracket.$$

**t3.2-3 Theorem 3.2.3.**

- (i) If  $n \leq m$ ,
- $$\llbracket L \rrbracket^2 = \det(L^* \circ L);$$
- (ii) If  $n \geq m$ ,
- $$\llbracket L \rrbracket^2 = \det(L \circ L^*).$$

*Proof.*

- (i). Assume that  $n \leq m$ , and apply Theorem  $\text{t3.2-2}$  to write

$$L = O \circ S$$

and

$$L^* = (O \circ S)^* = S^* \circ O^* = S \circ O^*.$$

Then

$$L^* \circ L = (S \circ O^*) \circ (O \circ S) = S \circ (O^* \circ O) \circ S = S \circ S = S^2$$

(cf.  $\text{t3.2-1}$ ). Hence,

$$\det(L^* \circ L) = \det(S^2) = (\det S)^2 = \llbracket L \rrbracket^2,$$

as required.

- (ii). The proof of (ii) is similar. The proof is complete.  $\square$

Theorem  $\text{t3.2-3}$  provides us with a nice way to compute the Jacobian  $\llbracket L \rrbracket$  of a linear map. We augment this with the Binet–Cauchy formula below.

**Definition 3.2.6** ( $\Lambda(m, n)$ ). If  $n \leq m$ , we define

$$\Lambda(m, n) := \{\lambda : \{1, \dots, n\} \rightarrow \{1, \dots, m\} : \lambda \text{ strictly increasing}\}.$$

Note that this is the set of all functions  $\lambda$  that take  $\{1, \dots, n\}$  to  $\{1, \dots, m\}$  such that  $\lambda(k) > \lambda(l)$  if  $k > l$ ,  $k, l \in \{1, \dots, n\}$ .

**Definition 3.2.7** ( $P_\lambda$ ). If  $n \leq m$ , for each  $\lambda \in \Lambda(m, n)$ , we define  $P_\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$  by

$$P_\lambda(x_1, \dots, x_m) := (x_{\lambda(1)}, \dots, x_{\lambda(n)}).$$

We may think of  $P_\lambda$  as a mapping that “deletes” points from  $(x_1, \dots, x_m)$ .

**Remark.** For each  $\lambda \in \Lambda(m, n)$ , there exists an  $n$ –dimensional subspace

$$S_\lambda := \text{span}\{e_{\lambda(1)}, \dots, e_{\lambda(n)}\} \subseteq \mathbb{R}^m$$

such that  $P_\lambda$  is the projection of  $\mathbb{R}^m$  onto  $S_\lambda$ .

t3.2-4

**Theorem 3.2.4** (Binet–Cauchy Formula). *Let  $n \leq m$  and let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then*

$$\llbracket L \rrbracket^2 = \sum_{\lambda \in \Lambda(m,n)} (\det(P_\lambda \circ L))^2.$$

## REFERENCES

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