NOTES ON L. C. EVANS: *PARTIAL DIFFERENTIAL EQUATIONS*

A. D. WENDLAND

Notes on chapters 5 and 6 of *Partial Differential Equations* by L. C. Evans [1].

CONTENTS

1. PRELIMINARIES

1.1. **Partial Differential Equations.** In this section we introduce the definition of a partial differential equation.

A *partial differential equation* is an equation that involves an unknown function of two or more variables and partial derivatives of this unknown function with respect to its independent variables.

For the remainder of this section, fix an open subset $\Omega \subset \mathbb{R}^n$.

Let us recall that if $u : \Omega \to \mathbb{R}$ is any function, we write

$$u(x) = u(x_1, \ldots, x_n), \quad x \in \Omega.$$

We may also consider vector–valued functions. If $\mathbf{u} : \Omega \to \mathbb{R}^m$, we write

$$\mathbf{u}(x) = (u^1(x), \ldots, u^m(x)), \quad x \in \Omega.$$

The function u^i is the $i-$th component of $\mathbf{u}$, $i = 1, \ldots, m$.

We will need some notation for the partial derivatives of a sufficiently smooth function $u : \Omega \to \mathbb{R}$. Let us first recall that

$$\mathcal{C}^k(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is } k - \text{times continuously differentiable}\}$$

and

$$\mathcal{C}^k(\overline{\Omega}) = \big\{u \in \mathcal{C}^k(\Omega) : D^\alpha u \text{ is uniformly continuous on bounded subsets of } \Omega,$$
$$\text{for all } |\alpha| \leq k\big\}.$$

See below for a review of multi–index notation. Recall also that we denote the partial derivative of u with respect to x_i by

$$u_{x_i}(x) = \frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h},$$

provided that this limit exists.

Next we recall multi–index notation.

Definition (Multi–index, Order of Multi–index)**.** *A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_i, $i = 1, \ldots, n$ is a nonnegative integer, is called a* **multi–index** *of* **order**

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

The purpose of multi–index notation is to write the partial derivatives of u more cleanly. Given a multi–index α, we define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x).$$

`ex:1.1-1` **Example 1.1.1.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is given by $u(x) := x_1^2 e^{3x_2}$ and $\alpha = (1, 2)$, then*

$$D^\alpha u(x) = \partial_{x_1} \partial_{x_2}^2 u(x) = 18 x_1 e^{3x_2}, \quad x \in \mathbb{R}^2.$$

Next, if $k \geq 0$ is an integer, we write

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\},$$

the set of all partial derivatives of order k. If we assign some ordering to the various partial derivatives, we can also regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$. We define

$$|D^k u| := \left(\sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{\frac{1}{2}}.$$

Before giving the definition of a partial differential equation, we first give some special cases of multi–index notation that will be useful to us later. If $k = 1$, we regard the elements of Du as being arranged in a vector, called the *gradient vector* of u :

$$Du := (u_{x_1}, \ldots, u_{x_n}).$$

Therefore Du gives a vector field in $\mathbb{R}^n$. If $k = 2$, we regard the elements of $D^2 u$ as being arranged in a matrix, called the *Hessian matrix* of u :

$$D^2 u := \begin{pmatrix} \partial_{x_1}^2 u & \cdots & \partial_{x_1 x_n} u \\ \vdots & \ddots & \vdots \\ \partial_{x_n x_1} u & \cdots & \partial_{x_n}^2 u \end{pmatrix}.$$

Therefore $D^2 u \in \mathbb{S}^n$, the space of real symmetric $n \times n$ matrices.

Definition (Partial Differential Equation)**.** *An expression of the form*

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0, \quad x \in \Omega \qquad (1.1.1) \quad \boxed{\text{\{eq:1.1-1}}}$$

*is called a $k-$**th order partial differential equation**, where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is the unknown.

Note that in (1.1.1) we do not actually assert that $u \in \mathcal{C}^k(\Omega)$, even though we are considering the k−th order partial derivatives of u. We will see why this makes sense in §2.2.

We say that we *solve* the PDE (1.1.1) if we find all functions u satisfying (1.1.1), possibly among only those functions satisfying certain auxiliary boundary conditions on some part Γ of the boundary $\partial\Omega$ of Ω. By "finding the solutions," we mean, ideally, obtaining simple, explicit solutions, or, failing that, deducing the existence and other properties of solutions.

We may also categorize PDE by their linearity/nonlinearity.

Definition (Linear PDE, Homogeneous). *The PDE (1.1.1) is called **linear** if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x) \tag{1.1.2}$$

*for given functions $a_\alpha(|\alpha| \leq k), f : \Omega \to \mathbb{R}$. Moreover, the linear PDE (1.1.2) is called **homogeneous** if $f \equiv 0$.*

Definition (Semilinear PDE). *The PDE (1.1.1) is called **semilinear** if it has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0. \tag{1.1.3}$$

Definition (Quasilinear PDE). *The PDE (1.1.1) is called **quasilinear** if it has the form*

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \ldots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0. \tag{1.1.4}$$

Definition (Fully Nonlinear PDE). *The PDE (1.1.1) is called **fully nonlinear** if it depends nonlinearly upon the highest order derivatives.*

Lastly, we may also consider *systems* of PDE, which we briefly present here. A system of partial differential equations is, informally speaking, a collection of several PDE for several unknown functions.

Definition (System of PDE). *An expression of the form*

$$\mathbf{F}(D^k\mathbf{u}(x), D^{k-1}\mathbf{u}(x), \ldots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0}, \quad x \in \Omega, \tag{1.1.5}$$

*is called a k−**th order system of partial differential equations**, where*

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$$

is given and

$$\mathbf{u} : \Omega \to \mathbb{R}^m, \quad \mathbf{u} = (u^1, \ldots, u^m)$$

is the unknown.

Here we are supposing that the system comprises the same number m of scalar equations as unknowns $(u^1, \ldots, u^m)$. This is the most common circumstance, although other systems may have fewer or more equations than unknowns. Systems are classified in the obvious way as being linear, semilinear, etc., as above. Note again that we do not assert that $\mathbf{u} \in \mathcal{C}^k(\Omega; \mathbb{R}^m)$.

1.2. **Convolution and Mollification.** We next want to introduce tools that will allow us to construct smooth approximations to certain functions. These will be important in the proofs of the Sobolev space approximation theorems in §2.3.

Definition (Ω_ϵ). *If $\Omega \subset \mathbb{R}^n$ is open and $\epsilon > 0$, we write*

$$\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}.$$

Thus we see that Ω_ϵ is the set of all points in Ω that are "away from the boundary."

Definition (Standard Mollifier). *We define the **standard mollifier** $\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ by*

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^n} \eta \, dx = 1$.

Definition (Mollifier). *For each $\epsilon > 0$, we define the **mollifier***

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

Note that for each $\epsilon > 0$, the functions η_ϵ belong to $\mathcal{C}^\infty(\mathbb{R}^n)$ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1, \quad \operatorname{supp}(\eta_\epsilon) \subset B(0, \epsilon).$$

Definition (Mollification). *If $f \in L^1_{\text{loc}}(\Omega)$, we define its mollification*

$$f_\epsilon := \eta_\epsilon * f \quad \text{in } \Omega_\epsilon.$$

Recalling the definition of the convolution of two functions, that is,

$$f_\epsilon(x) = \int_\Omega \eta_\epsilon(x - y) f(y) \, dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x - y) \, dy$$

for all $x \in \Omega_\epsilon$. Recall also that

$$\operatorname{supp}(f_\epsilon) = \operatorname{supp}(\eta_\epsilon + f) \subset \operatorname{supp}(\eta_\epsilon) + \operatorname{supp}(f).$$

Before presenting several properties of the mollification of a locally integrable function f, the intuitive idea is to "smooth out" or "average out" f. Note that we only require that f is locally integrable on Ω, and these functions may be very irregular. When we take the convolution of f with η_ϵ and take the limit as $\epsilon \to 0$, we are "smoothing out" sharp features of f while still remaining close – in a certain, specific sense, as we will see in the following theorem – to the original (nonsmoooth) function f.

`t1.2-1` **Theorem 1.2.1** (Properties of Mollifiers). *Let $\Omega \subset \mathbb{R}^n$ be open and let $f \in L^1_{\text{loc}}(\Omega)$. Then*

    (i) $f_\epsilon \in \mathcal{C}^\infty(\Omega_\epsilon)$;
    (ii) $f_\epsilon \to f \; \mathcal{L}^n-$a.e. as $\epsilon \to 0$;
    (iii) If $f \in \mathcal{C}(\Omega)$, then $f_\epsilon \to f$ uniformly on compact subsets of Ω;
    (iv) If $1 \leq p < +\infty$ and $f \in L^p_{\text{loc}}(\Omega)$, then $f_\epsilon \to f$ in $L^p_{\text{loc}}(\Omega)$.

Proof.

i. Fix $x \in \Omega_\epsilon$, $i \in \{1, \ldots, n\}$, and h so small that $x + he_i \in \Omega_\epsilon$. Since $\mathrm{supp}(\eta_\epsilon) \subset B(0, \epsilon)$, there exists an open set $U \subset\subset \Omega$ such that $\eta_\epsilon(x + he_i - y)$ and $\eta_\epsilon(x - y)$ both vanish for all $y \in \Omega \setminus U$. Thus

$$
\begin{aligned}
\frac{f_\epsilon(x + he_i) - f_\epsilon(x)}{h} &= \int_\Omega (\eta_\epsilon(x + he_i - y) - \eta_\epsilon(x - y)) f(y) \, dy \\
&= \frac{1}{\epsilon^n} \int_\Omega \frac{1}{h} \left[\eta\left(\frac{x + he_i - y}{h}\right) - \eta\left(\frac{x - y}{h}\right) \right] f(y) \, dy \\
&= \frac{1}{\epsilon^n} \int_U \frac{1}{h} \left[\eta\left(\frac{x + he_i - y}{h}\right) - \eta\left(\frac{x - y}{h}\right) \right] f(y) \, dy.
\end{aligned}
$$

Since

$$
\lim_{h \to 0} \frac{\eta\left(\frac{x + he_i - y}{\epsilon}\right) - \eta\left(\frac{x - y}{\epsilon}\right)}{h} = \frac{1}{\epsilon} \eta_{x_i}\left(\frac{x - y}{\epsilon}\right) = \epsilon^n \partial_{x_i} \eta_\epsilon(x - y).
$$

uniformly on U, Lebesgue's Dominated Convergence Theorem shows that the partial derivative $\partial_{x_i} f_\epsilon(x)$ and equals

$$
\partial_{x_i} f_\epsilon(x) \stackrel{L.D.C.}{=} \int_\Omega \partial_{x_i} \eta_\epsilon(x - y) f(y) \, dy
$$

for all $x \in \Omega_\epsilon$. Repeating this argument as necessary shows that $D^\alpha f_\epsilon(x)$ exists, and

$$
D^\alpha f_\epsilon(x) = \int_\Omega D^\alpha \eta_\epsilon(x - y) f(y) \, dy, \quad x \in \Omega
$$

for each multi-index α. This proves assertion (i).

ii. By Lebesgue's Differentiation Theorem,

$$
\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| \, dy = 0 \tag{1.2.1}
$$

`{eq:1.2-1`

for $\mathcal{L}^n$−a.e. $x \in \Omega$. Fix such a point $x \in \Omega$. Then, noting that $\eta\left(\frac{x-y}{\epsilon}\right) \leq Ce$ for all $y \in B(x, \epsilon)$,

$$
\begin{aligned}
|f_\epsilon(x) - f(x)| &= \left| \int_B (x, \epsilon) \eta_\epsilon(x - y) f(y) \, dy - f(x) \int_{B(x,\epsilon)} \eta_\epsilon(x - y) \, dy \right| \\
&= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x - y)[f(y) - f(x)] \, dy \right| \\
&\leq \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x - y}{\epsilon}\right) |f(y) - f(x)| \, dy \\
&\leq C \fint_{B(x,\epsilon)} |f(y) - f(x)| \, dy \to 0
\end{aligned}
$$

as $\epsilon \to 0$, by (1.2.1), as required. This proves assertion (ii).

iii. Assume now that $f \in \mathcal{C}(\Omega)$. Choose $U \subset\subset \Omega$, and then choose any $U \subset\subset V \subset\subset \Omega$ and note that f is uniformly continuous on V. Thus the limit in (1.2.1) holds uniformly for $x \in U$. Consequently the calculation in part (ii) implies that $f_\epsilon \to f$ uniformly on U.

iv. Next assume that $1 \leq p < +\infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Choose an open set $U \subset\subset \Omega$ and, as above, choose an open set V so that $U \subset\subset V \subset\subset \Omega$. We claim that for sufficiently small

$\epsilon > 0$,

$$\|f_\epsilon\|_{L^p(U)} \leq \|f\|_{L^p(W)}. \tag{1.2.2}$$ `{eq:1.2-2`

To see this, we note that if $1 \leq p < +\infty$ and $x \in V$, then

$$
\begin{aligned}
|f_\epsilon(x)| &= \left| \int_{B(x,\epsilon)} \eta_\epsilon(x-y) f(y) \, dy \right| \\
&\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{1-\frac{1}{p}} \eta_\epsilon(x-y)^{\frac{1}{p}} |f(y)| \, dy \\
&\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) \, dy \right)^{1-\frac{1}{p}} \cdot \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p \, dy \right)^{\frac{1}{p}},
\end{aligned}
$$

where we have used Hölder's inequality with conjugate exponents p and $\frac{p}{p-1}$ on the RHS. Since $\int_{B(x,\epsilon)} \eta_\epsilon(x-y) = 1$, this inequality implies that

$$
\begin{aligned}
\int_U |f_\epsilon(x)|^p \, dx &\leq \int_U \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y) |f(y)|^p \, dy \right) \, dx \\
&\leq \int_V |f(y)|^p \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y) \, dx \right) \, dy \\
&= \int_W |f(y)|^p \, dy
\end{aligned}
$$

for all $\epsilon > 0$ sufficiently small. This is (1.2.2).

v. Now fix $U \subset\subset V \subset\subset \Omega$, $\delta > 0$, and choose $g \in \mathcal{C}(V)$ such that

$$\|f - g\|_{L^p(V)} < \delta. \tag{1.2.3}$$ `{eq:1.2-3`

Then by Minkowski's inequality and (1.2.2), we have

$$
\begin{aligned}
\|f^\epsilon - f\|_{L^p(U)} &\leq \|f_\epsilon - g_\epsilon\|_{L^p(U)} + \|g_\epsilon - g\|_{L^p(U)} + \|g - f\|_{L^p(U)} \\
&\leq 2\|f - g\|_{L^p(V)} + \|g_\epsilon - g\|_{L^p(U)} \\
&\leq 2\delta + \|g_\epsilon - g\|_{L^p(U)}.
\end{aligned}
$$

Since $g_\epsilon \to g$ uniformly on U by assertion (iii) of the theorem, we have

$$\limsup_{\epsilon \to 0} \|f_\epsilon - f\|_{L^p(U)} \leq 2\delta,$$

as required, since U and V were arbitrary. The proof is complete. $\qquad\square$

Remark.

   (i) If $f \in L^p_{\text{loc}}(\Omega)$, inequality (1.2.2) asserts that mollification reduces the L^p_{loc}−norm of f.

   (ii) Notice that Theorem 1.2.1(iv) **does not** hold in the case $p = +\infty$. This is because $\mathcal{C}(\Omega)$ is not dense in $L^\infty(\Omega)$, so we may not necessarily be able to find $g \in \mathcal{C}(\Omega)$ such that (1.2.3) holds.

1.3. **Boundaries.** In this section we give the definition of a $\mathcal{C}^k-$ smooth boundary and discuss its properties. We let Ω be an open, bounded subset of $\mathbb{R}^n$, and $k \in \mathbb{N}$.

Definition ($\mathcal{C}^k$ Boundary). *We say that the boundary $\partial\Omega$ of Ω is $\mathcal{C}^k$ if for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a $\mathcal{C}^k$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$
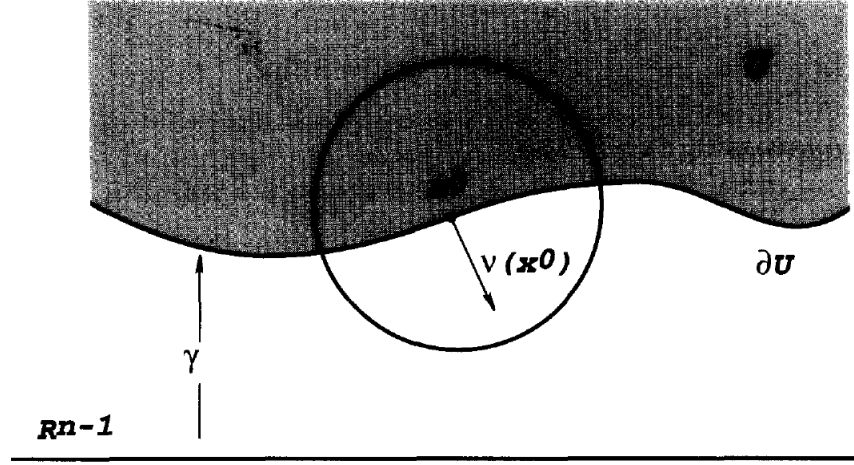
We sometimes also write $\partial\Omega \in \mathcal{C}^k$.

FIGURE 1.3.1. The Boundary of Ω.

Intuitively, the definition of a $\mathcal{C}^k$ boundary $\partial\Omega$ states that $\partial\Omega$ is given by the graph of a $\mathcal{C}^k(\mathbb{R}^{n-1})$ function γ and is one–sided, that is, no part of the domain Ω can lie on both sides of any part of the boundary $\partial\Omega$.

ex1.3-1 | **Example 1.3.1** (A Smooth Boundary). *Any open ball $B(x, r)$ in $\mathbb{R}^n$ has a smooth ($\mathcal{C}^\infty$) boundary for all $n \geq 1$.*
We show this for the open unit ball $\Omega := B(0, 1)$ in $\mathbb{R}^2$. To see this, fix any point $(x_0, y_0) \in \partial\Omega$, choose $r = 1 > 0$, and reorient the coordinate axes so that $(x_0, y_0) = (0, -1)$. Then

$$\Omega \cap B((x_0, y_0), r) = \Omega \cap B((0, -1), 1)$$

$$= \left\{ (x, y) \in \mathbb{R}^2 : -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2}, \ -\sqrt{1 - x^2} < y < +\sqrt{1 - x^2} - 1 \right\}$$

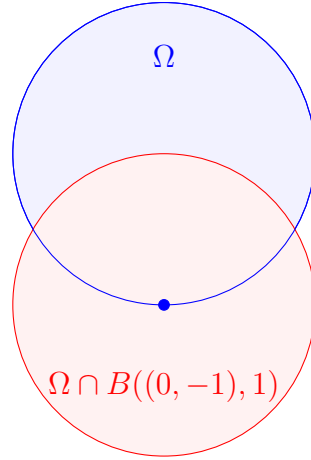*Define then the function $\gamma : \mathbb{R} \to \mathbb{R}$ by*

$$\gamma(x) := -\sqrt{1 - x^2}.$$

Clearly $\gamma \in \mathcal{C}^\infty(\mathbb{R})$. It remains to show that

$$\Omega_1 := \Omega \cap B((0, -1), 1) = \{(x, y) \in B((0, -1), 1) : y > \gamma(x)\} =: \Omega_2.$$

It is obvious that $\Omega_1 \subseteq \Omega_2$. For the reverse inclusion, let $(x, y) \in \Omega_2$. First note that if $|x| \geq \frac{\sqrt{3}}{2}$, then

$$y > \gamma(x) = -\sqrt{1 - x^2} \geq -\sqrt{1 - \frac{3}{4}} = -\frac{1}{2}.$$

FIGURE 1.3.2. Ω and $\Omega \cap B((0,-1),1)$.

But this is impossible since $(x,y) \in B((0,-1),1)$, and

$$\sqrt{x^2 + (y+1)^2} \geq \sqrt{\frac{3}{4} + \left(1 - \frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1.$$

So then $-\frac{-\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2}$. Now if $y \geq \sqrt{1 - x^2} - 1$, we obtain another contradiction, for
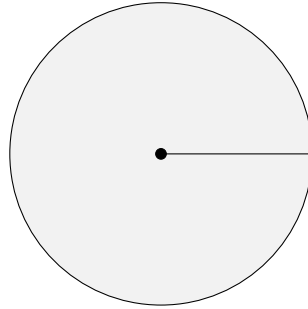
$$\sqrt{x^2 + (y+1)^2} \geq \sqrt{x^2 + (1 - x^2)} = 1.$$

Thus $\Omega_2 \subseteq \Omega_1$, so that $\Omega_1 = \Omega_2$, as required.

ex1.3-2 **Example 1.3.2** (A Nonsmooth Boundary)**.** *We now give an example of a boundary that is not $\mathcal{C}^1$. Define $\Omega \subset \mathbb{R}^2$ by*

$$\Omega := B(0,1) \setminus [0,1).$$

FIGURE 1.3.3. $B(0,1) \setminus [0,1)$.

To see that $\partial\Omega$ is not $\mathcal{C}^1$, first note that $x_0 := 0 \in \partial\Omega$. Let $r > 0$. Then

$$\Omega \cap B(0,r) = B(0, \min\{1,r\}) \setminus [0, \min\{1,r\}).$$

Suppose by contradiction that $\partial\Omega$ is $\mathcal{C}^1$. Put $r_0 := \frac{\min\{1,r\}}{2}$ and choose any sequence $\{x_m\}_{m=1}^{+\infty}$ such that $x_m > 0$, $x_m \to 0$, and $(x_m, r_0) \in \Omega \cap B(0,r)$ for all $m \in \mathbb{N}$. Since $\partial\Omega$ is $\mathcal{C}^1$, there exists $\gamma \in \mathcal{C}^1(\mathbb{R})$ such that

$$y > \gamma(x) \quad \text{for all } (x,y) \in B(0,r).$$

By continuity,

$$r_0 > \lim_{m \to +\infty} \gamma(x_m) = \gamma(0).$$

But note also that $(x_m, -r_0) \in \Omega \cap B(0, r)$ *for all* $m \in \mathbb{N}$. *Thus*

$$-r_0 > \lim_{m \to +\infty} \gamma(x_m) = \gamma(0),$$

a contradiction, because $r_0 > 0$.

The definition of an "analytic boundary" is the obvious one.

Definition (Analytic Boundary). *We say that the boundary* $\partial\Omega$ *of* Ω *is* **analytic** *if for each point* $x_0 \in \partial\Omega$ *there exist* $r > 0$ *and an analytic function* $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

We will often have occasion to consider the outer unit normal when applying the Gauss–Green theorem or the divergence theorem.

Definition (Outer Unit Normal). *If* $\partial\Omega$ *is* $\mathcal{C}^1$, *then along* $\partial\Omega$ *is defined the* **outward pointing unit normal vector field**

$$\nu = (\nu^1, \dots, \nu^n).$$

Definition (Outer Normal Derivative). *Let* $u \in \mathcal{C}^1(\Omega)$. *We call*

$$\frac{\partial u}{\partial \nu} := \nu \cdot Du$$

the **outward normal derivative** *of* u.

We will often consider BVPs with Dirichlet boundary conditions, that is, equations of the form

$$\begin{cases} F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = g, \end{cases}$$

where g is some given function. In this case the outer unit normal along $\partial\Omega$ is

$$\frac{Du}{|Du|} = \frac{1}{|Du|} \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right),$$

provided that $Du \not\equiv 0$.

Flattening the Boundary. We will frequently need to change coordinates near a point of $\partial\Omega$ so as to "flatten out" the boundary. To be specific, fix $x_0 \in \partial\Omega$, and choose $r > 0$ and $\gamma \in \mathcal{C}^k(\mathbb{R}^{n-1})$ as above. Define then

$$\begin{cases} y_i = x_i =: \Phi^i(x), & i = 1, \dots, n-1, \\ y_n = x_n - \gamma(x_1, \dots, x_{n-1}) =: \Phi^n(x), \end{cases}$$

and write

$$y = \Phi(x).$$

Similarly, we set

$$\begin{cases} x_i = y_i =: \Psi^i(y), & i = 1, \dots, n-1, \\ x_n = y_n + \gamma(y_1, \dots, y_{n-1}) =: \Psi^n(y), \end{cases}$$

and write

$$x = \Psi(y).$$
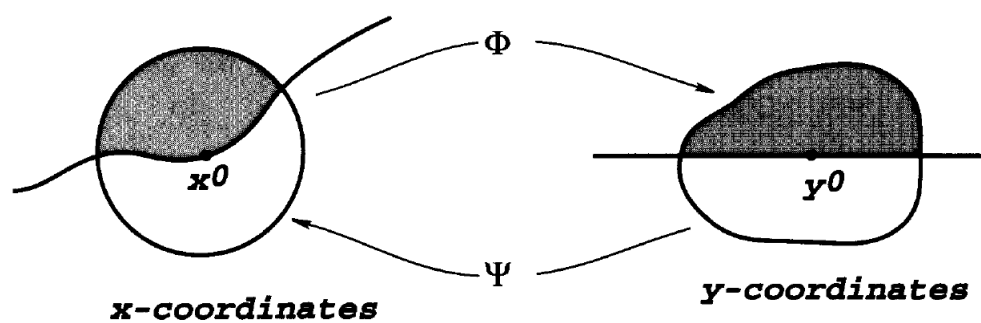
FIGURE 1.3.4. Flatting out the Boundary.

Then $\Phi = \Psi^{-1}$, and the mapping $x \mapsto \Phi(x) = y$ "straightens out $\partial\Omega$" near x_0. Observe also that

$$D\Phi = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ -\gamma_{x_1} & \cdots & -\gamma_{x_{n-1}} & 1 \end{pmatrix}$$

and

$$D\Psi = \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \gamma_{y_1} & \cdots & \gamma_{y_{n-1}} & 1 \end{pmatrix},$$

so that $\det D\Phi = \det D\Psi = 1$.

2. SOBOLEV SPACES

In this section we mostly develop the theory of *Sobolev spaces*, which we will see to be the proper settings in which to apply ideas of functional analysis when considering PDE.

Keeping in mind eventual applications to wide classes of PDEs (we want a theory that deals with linear/nonlinear elliptic, parabolic, and hyperbolic equations), we sketch out here our overall point of view. The intention is to take various specific PDEs and recast them abstractly as operators acting on appropriate normed linear spaces. We can symbolically write

$$A : X \to Y,$$

where the operator A describes the structure of the PDEs, including possibly boundary conditions, and X, Y are normed linear spaces of functions. The advantage of this formulation is that once our PDE problem has been interpreted in this form, we often can employ the general principles of functional analysis to study the properties (including solvability, existence, and uniqueness) of various equations involving the differential operator A. We will see that the most difficult work is not so much the invocation of functional analysis, but finding the "right" function spaces X and Y and the "right" differential operators A. Sobolev spaces are designed specifically to make this choices work out nicely.

As mentioned above, Sobolev spaces are useful for studying linear elliptic, parabolic, and hyperbolic PDE, as well as nonlinear elliptic and parabolic PDE.

2.1. **Hölder Spaces.** Before studying Sobolev spaces, we first consider the simpler Hölder spaces.

Throughout this section, we assume that $\Omega \subseteq \mathbb{R}^n$ is open and $0 < \gamma \le 1$.

We first recall the definition of Lipschitz continuity:

Definition (Lipschitz Continuity). *A function $u : \Omega \to \mathbb{R}$ is said to be **Lipschitz continuous** if there exists some constant $C > 0$ such that*

$$|u(x) - u(y)| \le C|x - y|, \quad x, y \in \Omega. \tag{2.1.1}$$

Note that if (2.1.1) holds, then clearly u is continuous, and more importantly, (2.1.1) provides a modulus of continuity. It turns out to be useful to consider also functions satisfying a variant of (2.1.1).

Definition (Hölder Continuity). *A function $u : \Omega \to \mathbb{R}^n$ is said to be **Hölder continuous with exponent** γ if there exists $C > 0$ such that*

$$|u(x) - u(y)| \le C|x - y|^\gamma, \quad x, y \in \Omega. \tag{2.1.2}$$

Let us recall here a simple but important definition from functional analysis.

Definition (Norm). *Let X be a linear space. A function $\| \cdot \| : X \to [0, +\infty)$ is called a **norm** on X if the following three conditions hold:*

   (i) $\|u\| = 0$ if and only if $u = 0$;
   (ii) $\|\gamma u\| = |\gamma| \|u\|$ for all $u \in X$ and $\gamma \in \mathbb{R}$ (or $\mathbb{C}$);
   (iii) $\|u + v\| \le \|u\| + \|v\|$ for all $u, v \in X$ (triangle inequality).

Definition ($\|\cdot\|_{\mathcal{C}(\overline{\Omega})}$)**.** *If* $u : \Omega \to \mathbb{R}$ *is bounded and continuous, we write*

$$\|u\|_{\mathcal{C}(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)|.$$

Definition ($\gamma-$th Hölder Seminorm $|\cdot|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}$)**.** *The* $\gamma-$*th Hölder seminorm of a function* $u :$ $\Omega \to \mathbb{R}$ *is*

$$|u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|} \right\}.$$

Note that $|\cdot|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}$ is indeed only a seminorm, and not a norm, as any nonzero constant function $u : \Omega \to \mathbb{R}$ satisfies $|u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} = 0$.

Definition ($\gamma-$th Hölder Norm $\|\cdot\|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}$)**.** *The* $\gamma-$*th Hölder norm of a function* $u : \Omega \to \mathbb{R}$ *is*

$$\|u\|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} := \|u\|_{\mathcal{C}(\overline{\Omega})} + |u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}.$$

Definition (Hölder Space $\mathcal{C}^{k,\gamma}(\overline{\Omega})$)**.** *The Hölder space*

$$\mathcal{C}^{k,\gamma}(\overline{\Omega})$$

consists of all functions $u \in \mathcal{C}^k(\overline{\Omega})$ *for which the norm*

$$\|u\|_{\mathcal{C}^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\mathcal{C}(\overline{\Omega})} + \sum_{|\alpha| = k} |D^\alpha u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} \tag{2.1.3}$$ `{eq:2.1-3`

is finite.

Note that the space $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ consists of all functions $u : \Omega \to \mathbb{R}$ that are $k-$times continuously differentiable and whose $k-$th partial derivatives are bounded and Hölder continuous with exponent γ. These functions are well–behaved, and furthermore we want to show that $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is a *Banach space*.

Definition (Banach Space)**.** *A* **Banach space** *is a normed linear space which is complete.*

`t2.1-1` **Theorem 2.1.1.** *The Hölder space* $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ *is a Banach space.*

Proof. It is clear that $\|\cdot\|_{\mathcal{C}^{k,\gamma}(\overline{\Omega})}$ is a norm on $\mathcal{C}^{k,\gamma}(\overline{\Omega})$.

It remains to show that $\mathcal{C}^{k,\gamma}(\overline{\Omega})$ is complete. Let $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}^{k,\gamma}(\overline{\Omega})$ be a Cauchy sequence. Recalling that $\mathcal{C}^k(\overline{\Omega})$ is a Banach space under the norm

$$\|u\|_{\mathcal{C}^k(\overline{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\mathcal{C}(\overline{\Omega})}$$

and $\mathcal{C}^{k,\gamma}(\overline{\Omega}) \subset \mathcal{C}^k(\overline{\Omega})$, define $u : \Omega \to \mathbb{R}$ by

$$u := \lim_{m \to +\infty} u_m \quad \text{in } \mathcal{C}^k(\overline{\Omega}). \tag{2.1.4}$$ `{eq:2.1-4`

We must first show that $u \in \mathcal{C}^{k,\gamma}(\overline{\Omega})$. Fix a multi–index α with $|\alpha| = k$. Note that by (2.1.4),

$$D^\alpha u_m \to D^\alpha u \quad \text{uniformly on } \Omega.$$

Thus, for any $x, y \in \Omega$, $x \neq y$ and $m \in \mathbb{N}$,

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma} \leq \frac{|D^\alpha u(x) - D^\alpha u_m(x)|}{|x - y|^\gamma} + \frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x - y|^\gamma} +$$

$$\frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma}$$

$$\leq \frac{|D^\alpha u(x) - D^\alpha u_m(x)|}{|x-y|^\gamma} + |D^\alpha u_m|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} + \frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma}.$$

By the uniform convergence, we may choose $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$,

$$\frac{|D^\alpha u(x) - D^\alpha u_{m_0}(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_{m_0}(y) - D^\alpha u(y)|}{|x-y|^\gamma} \leq 1.$$

Hence,

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha} \leq 1 + |D^\alpha u_{m_0}|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})}.$$

This is for all $x, y \in \Omega$, $x \neq y$ so that taking the supremum on the LHS over all $x, y \in \Omega$, $x \neq y$, and multi–indices $|\alpha| = k$ gives

$$|D^\alpha u|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} \leq 1 + |D^\alpha u_{m_0}|_{\mathcal{C}^{0,\gamma}(\overline{\Omega})} < +\infty,$$

as required.

Finally, we show that $\{D^\alpha u_m\}_{m=1}^{+\infty}$ converges to $D^\alpha u$ in the Hölder seminorm. Choose any $x, y \in \Omega$, $x \neq y$, and $m \in \mathbb{N}$. Then

$$\frac{|(D^\alpha u_m - D^\alpha u)(x) - (D^\alpha u_m - D^\alpha u)(y)|}{|x-y|^\gamma} = \frac{|D^\alpha u_m(x) - D^\alpha u(x) - D^\alpha u_m(y) + D^\alpha u(y)|}{|x-y|^\gamma}$$

$$\leq \frac{|D^\alpha u_m(x) - D^\alpha u(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma}$$

$$= \lim_{l \to +\infty} \left\{ \frac{|D^\alpha u_l(x) - D^\alpha u_m(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_l(y) - D^\alpha u_m(y)|}{|x-y|^\gamma} \right\},$$

and, since $\{D^\alpha u_m\}_{m=1}^{+\infty}$ is Cauchy in $\mathcal{C}^{0,\gamma}(\overline{\Omega})$, the RHS vanishes by taking the limit as $m \to +\infty$. Hence

$$D^\alpha u_m \to D^\alpha u \quad \text{in } \mathcal{C}^{k,\gamma}(\overline{\Omega}),$$

as required. The proof is complete. $\qquad\square$

2.2. **Sobolev Spaces.** Hölder spaces as introduced in §5.1 are unfortunately not often suitable settings for PDE theory, as we generally cannot make good enough analytic estimates to demonstrate that the solutions we construct actually belong to such spaces. What are needed are some other kinds of spaces, containing less smooth functions. In practice we must strike a balance, by designing spaces comprising functions which have some, but not too great, smoothness properties.

What we will end up defining is a space of functions $u : \Omega \to \mathbb{R}^n$ with k "weak derivatives." That is, the functions belonging to this space may not have derivatives in the classical sense, but have "derivatives" that behave nicely with respect to integration against a certain class of functions.

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2.2.1. *Weak Derivatives.* As mentioned, we start by weakening the notion of partial derivatives.

Definition ($\mathcal{C}_c^\infty(\Omega)$, Test Function). *We denote by*

$$\mathcal{C}_c^\infty(\Omega)$$

the space of all infinitely differentiable functions $\phi : \Omega \to \mathbb{R}$ with compact support in Ω. We call a function $\phi \in \mathcal{C}_c^\infty(\Omega)$ a **test function**.

Motivation for Definition of Weak Derivative. Assume that we are given a function $u \in \mathcal{C}^1(\Omega)$. Then if $\phi \in \mathcal{C}_c^\infty(\Omega)$, we see from the integration by parts formula that

$$\int_\Omega u\phi_{x_i}\, dx = -\int_\Omega u_{x_i}\phi\, dx + \int_{\partial\Omega} u v \nu^i\, dS = -\int_\Omega u_{x_i}\phi\, dx, \quad i = 1, \ldots, n, \qquad (2.2.1) \quad \boxed{\texttt{\{eq:2.2-1}}}$$

where the boundary term vanishes because ϕ has compact support in Ω. More generally now, if k is a positive integer, $u \in \mathcal{C}^k(\Omega)$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi–index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$, then

$$\int_\Omega u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi\, dx. \qquad (2.2.2) \quad \boxed{\texttt{\{eq:2.2-2}}}$$

Note that the equality in (2.2.2) holds because

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi,$$

and we may apply formula (2.2.1) $|\alpha|$ times.

We next assume that formula (2.2.2) holds for some function $u : \Omega \to \mathbb{R}$ and every test function ϕ. Note that in (2.2.2) we required that u be $k-$times continuously differentiable, and we consider now if this requirement may be weakened, that is, if (2.2.2) may still be true even if u is not $\mathcal{C}^k-$smooth. Note that the LHS of (2.2.2) makes sense if u is only locally integrable, as ϕ and all its derivatives have compact support in Ω. The problem is the RHS: if u is not $\mathcal{C}^k-$smooth, then the expression "$D^\alpha u$" has no obvious meaning. We resolve this issue by formulating the definition of a "weak derivative" of u, that is, a locally integrable function $v : \Omega \to \mathbb{R}$ for which formula (2.2.2) holds, with v in place of $D^\alpha u$.

Definition (Weak Derivative). *Suppose that $u, v \in L^1_{\text{loc}}(\Omega)$ and α is a multi–index. We say that v is the $\alpha^{th}-$weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided that

$$\int_\Omega u D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega v \phi\, dx \qquad (2.2.3) \quad \boxed{\texttt{\{eq:2.2-3}}}$$

holds for all test functions $\phi \in \mathcal{C}_c^\infty(\Omega)$.

That is, if we are given a function $u \in L^1_{\text{loc}}(\Omega)$ and if there happens to exist a function $v \in L^1_{\text{loc}}(\Omega)$ which satisfies (2.2.3) for all test functions ϕ, we then say that $D^\alpha u = v$ in the weak sense. If on the other hand there does not exist such a function v, then evidently u does not possess an $\alpha^{\text{th}}-$weak partial derivative.

Recall that if u has a classical derivative, it is clear that this derivative is unique. We show that the same is true for weak derivatives, at least up to sets of measure zero.

`l2.2-1` **Lemma 2.2.1** (Uniqueness of Weak Derivatives)**.** *Suppose that $u \in L^1_{\text{loc}}(\Omega)$ possesses a weak α^{th} partial derivative v. Then v is uniquely defined up to a set of measure zero.*

Proof. Suppose that $v, w \in L^1_{\text{loc}}(\Omega)$ satisfy

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v\phi \, dx = (-1)^{|\alpha|} \int_\Omega w\phi \, dx$$

for all $\phi \in \mathcal{C}^\infty_c(\Omega)$. Then evidently

$$\int_\Omega (v - w)\phi \, dx = 0 \tag{2.2.4}$$

`{eq:2.2-4`

for all $\phi \in \mathcal{C}^\infty_c(\Omega)$, so that $v = w$ a.e. The proof is complete. $\qquad\square$

`ex:2.2-1` **Example 2.2.1.** *Let $n = 1$, $\Omega = (0, 2)$, and*

$$u(x) := \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases}$$

Note that u is not differentiable at $x = 1$ in the classical sense. However, differentiating the piecewise components of u, we might expect that

$$u'(x) = \begin{cases} 1, & 0 < x \le 1, \\ 0, & 1 < x < 2 \end{cases}$$

to be the weak derivative of u.
To verify this, choose any $\phi \in \mathcal{C}^\infty_c(\Omega)$. By (2.2.3), we must show that

$$\int_0^2 u\phi' \, dx = -\int_0^2 u'\phi \, dx.$$

Integrating by parts and using the fact that ϕ vanishes at $x = 0$ and $x = 2$, we calculate

$$\int_0^2 u\phi' \, dx = \int_0^1 x\phi' \, dx + \int_1^2 \phi' \, dx$$

$$= \phi(1) - \phi(0) - \int_0^1 \phi \, dx + \phi(2) - \phi(1)$$

$$= -\int_0^1 \phi \, dx$$

$$= -\int_0^2 u'\phi \, dx,$$

as required.

`ex:2.2-2` **Example 2.2.2.** *Let $n = 1$, $\Omega = (0, 2)$, and*

$$u(x) := \begin{cases} x, & 0 < x \le 1, \\ 2, & 1 < x < 2. \end{cases}$$

Note again that u is not differentiable at $x = 1$. Based on the procedure in Example 2.2.1, we might expect that

$$v(x) := \begin{cases} 1, & 0 < x \le 1, \\ 0, & 1 < x < 2 \end{cases}$$

is the weak derivative of u. However, let us show that u does not have a weak derivative.

For motivation, let us first choose a test function $\phi \in C_c^\infty(\Omega)$ and observe that

$$\int_0^2 u\phi' \, dx = \int_0^1 x\phi' \, dx + \int_1^2 2\phi' \, dx$$

$$= \phi(1) - \phi(0) - \int_0^1 \phi \, dx + 2\phi(2) - 2\phi(1)$$

$$= -\int_0^1 v\phi \, dx - \phi(1)$$

$$= -\int_0^1 v\phi \, dx - \delta_1(\phi).$$

Intuitively, the problem is that the Dirac delta distribution is not a function – it is a distribution.

To verify that u possesses no weak derivative, suppose by contradiction that there exists $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_0^2 u\phi' \, dx = -\int_0^2 v\phi \, dx \qquad\qquad (2.2.5) \quad \boxed{\texttt{\{eq:2.2-5}}}$$

for all $\phi \in C_c^\infty(\Omega)$. Then

$$-\int_0^2 v\phi \, dx = \int_0^2 u\phi' \, dx$$

$$= \int_0^1 x\phi' \, dx + 2\int_1^2 \phi' \, dx$$

$$= \phi(1) - \phi(1) - \int_0^1 \phi \, dx + 2\phi(2) - \phi(1)$$

$$= -\int_0^1 \phi \, dx - \phi(1). \qquad\qquad (2.2.6) \quad \boxed{\texttt{\{eq:2.2-6}}}$$

Choose then a sequence $\{\phi_m\}_{m=1}^{+\infty} \subset C_c^\infty(\Omega)$ such that

$$0 \le \phi_m \le 1, \quad \phi_m(1) = 1, \quad \phi_m(x) \to 0 \text{ for all } x \ne 1.$$

But then replacing ϕ with ϕ_m in (2.2.6) and taking the limit as $\phi \to +\infty$, we find

$$1 = \lim_{m\to+\infty} \phi_m(1) = \lim_{m\to+\infty} \left[\int_0^2 v\phi_m \, dx - \int_0^1 \phi_m \, dx \right] \overset{L.D.C.}{=} 0,$$

a contradiction.

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2.2.2. *Definition of Sobolev Spaces.* Fix $1 \le p \le +\infty$ and let k be a nonnegative integer. With the definition of a weak derivative in mind, we now define certain function spaces comprised of functions which have weak derivatives of various orders lying in various L^p spaces.

Definition (Sobolev Space $W^{k,p}(\Omega)$). *The* **Sobolev space**

$$W^{k,p}(\Omega)$$

consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multi–index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Note that choosing $|\alpha| = 0$ implies that $u \in L^p(\Omega)$.

Remark.

   (i) If $p = 2$, we write

$$H^k(\Omega) := W^{k,2}(\Omega), \quad k \in \mathbb{N}_0.$$

   The letter H is used, since, as we will see later, $H^k(\Omega)$ is a Hilbert space. Note also that $H^0(\Omega) = L^2(\Omega)$.

   (ii) We identify functions in $W^{k,p}(\Omega)$ which coincide $\mathcal{L}^n$–a.e.

Definition ($\| \cdot \|_{W^{k,p}(\Omega)}$). *If $u \in W^{k,p}(\Omega)$, we define the $W^{k,p}(\Omega)$–norm of u by*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \displaystyle\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

Notice that the $W^{k,p}(\Omega)$–norm of a function $u \in W^{k,p}(\Omega)$ is the Minkowski norm of the $L^p(\Omega)$–norms of all weak derivatives of u if $1 \leq p < +\infty$, and the sum of all $L^\infty(\Omega)$–norms of all weak derivatives of u if $p = +\infty$.

Another choice of norm on $W^{k,p}(\Omega)$ is given by

$$\|u\|'_{W^{k,p}(\Omega)} := \begin{cases} \displaystyle\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}, & 1 \leq p < +\infty, \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

It may be shown that the norms $\| \cdot \|_{W^{k,p}(\Omega)}$ and $\| \cdot \|'_{W^{k,p}(\Omega)}$ are equivalent.

Definition (Convergence in $W^{k,p}(\Omega)$). *Let $\{u_m\}_{m=1}^{+\infty}, u \in W^{k,p}(\Omega)$. We say that u_m* **converges** *to u in $W^{k,p}(\Omega)$, and write*

$$u_m \to u \quad in \ W^{k,p}(\Omega),$$

provided that

$$\lim_{m \to +\infty} \|u_m - u\|_{W^{k,p}(\Omega)} \to 0.$$

Definition (Convergence in $W^{k,p}_{\text{loc}}(\Omega)$). *Let $\{u_m\}_{m=1}^{+\infty}, u \in W^{k,p}(\Omega)$. We say that u_m* **converges** *to u in $W^{k,p}_{\text{loc}}(\Omega)$, and write*

$$u_m \to u \quad in \ W^{k,p}_{\text{loc}}(\Omega),$$

provided that

$$u_m \to u \quad in \ W^{k,p}(U)$$

for all $U \subset\subset \Omega$.

Definition ($W^{k,p}_0(\Omega)$). *We define $W^{k,p}_0(\Omega)$ to be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$, that is,*

$$W^{k,p}_0(\Omega) := W^{k,p} \cap \overline{\mathcal{C}_c^\infty(\Omega)}.$$

Thus $u \in W_0^{k,p}(\Omega)$ if and only if there exists a sequence $\{u_m\}_{m=1}^{+\infty} \subset \mathcal{C}_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$. We will see later that we can think of $W_0^{k,p}(\Omega)$ functions as $W^{k,p}(\Omega)$ functions whose first $k-1$ partial derivatives "vanish at the boundary of Ω" (specifically, we will see that they have zero *trace*), that is, all functions $u \in W^{k,p}(\Omega)$ such that

$$\text{``}D^\alpha u = 0 \text{ on } \partial\Omega\text{''} \text{ for all } |\alpha| \le k-1.$$

Remark (Notation). *We will write*

$$H_0^k(\Omega) := W_0^{k,2}(\Omega).$$

In fact if $n = 1$ and Ω is an open interval in $\mathbb{R}$, then $u \in W^{1,p}(\Omega)$ if and only if u is equal $\mathcal{L}^1$–a.e. to an absolutely continuous function whose ordinary derivative (which exists $\mathcal{L}^1$–a.e.) belongs to $L^p(\Omega)$. This simple characterization is however only available for $n = 1$. In general a function can belong to a Sobolev space and yet be discontinuous and/or unbounded.

ex:2.2-3 **Example 2.2.3.** *Take $\Omega := B(0,1)$, the open unit ball in $\mathbb{R}^n$, and let*

$$u(x) := |x|^{-\alpha}, \quad x \in \Omega, \ x \ne 0.$$

We consider the values of $\alpha > 0$, n, and p for which u belongs to $W^{1,p}(\Omega)$. Note first that u is smooth away from $x = 0$, with

$$\frac{\partial u}{\partial x_i}(x) = -\frac{\alpha}{2|x|^{\alpha-2}} \cdot 2x_i = \frac{-\alpha x_i}{|x|^{\alpha+2}}, \quad x \ne 0,$$

and thus

$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}}.$$

Now fix $\phi \in \mathcal{C}_c^\infty(\Omega)$ and $\epsilon > 0$. Then integration by parts gives

$$\int_{\Omega \setminus B(0,\epsilon)} u\phi_{x_i} \, dx = -\int_{\Omega \setminus B(0,\epsilon)} u_{x_i}\phi \, dx + \int_{\partial B(0,\epsilon)} u\phi\nu^i \, dS,$$

where $\nu := (\nu^1, \dots, \nu^n)$ denotes the inward pointing unit normal on $\partial B(0,\epsilon)$. Now if $\alpha + 1 < n$, then $|Du(x)| \in L^1(\Omega)$. In this case

$$\left| \int_{\partial B(0,\epsilon)} u\phi\nu^i \, dS \right| \le \int_{\partial B(0,\epsilon)} |u\phi\nu^i| \, d\mathcal{H}^{n-1} \le \|\phi\|_{L^\infty(\Omega)} \epsilon^{-\alpha} \int_{\partial B(0,\epsilon)} d\mathcal{H}^{n-1}$$

$$\le C\epsilon^{n-1-\alpha} \quad \to 0 \text{ as } \epsilon \to 0$$

since $n - 1 - \alpha > 0$. Thus

$$\int_\Omega u\phi_{x_i} \, dx = -\int_\Omega u_{x_i}\phi \, dx$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, provided that $0 \le \alpha < n-1$. Furthermore $|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \in L^p(\Omega)$ if and only if $(\alpha+1)p < n$. Consequently $u \in W^{1,p}(\Omega)$ if and only if $\alpha < \frac{n-p}{p}$, since $\frac{n-p}{p} \le n-1$. In particular $u \notin W^{1,p}(\Omega)$ for any $p \ge n$.

ex:2.2-4 **Example 2.2.4.** *Let $\{r_k\}_{k=1}^{+\infty}$ be a countable, dense subset of $\Omega = B(0,1)$. Write*

$$u(x) := \sum_{k=1}^{+\infty} \frac{1}{2^k} |x - r_k|^{-\alpha}, \quad x \in \Omega.$$

Then $u \in W^{1,p}(\Omega)$ for $\alpha < \frac{n-p}{p}$. If $0 < \alpha < \frac{n-p}{p}$, we see that u belongs to $W^{1,p}(\Omega)$ and yet is unbounded on each open subset of Ω.

The last example shows that although a function u belonging to a Sobolev space possesses certain smoothness properties, it can still be rather badly behaved in other ways.

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2.2.3. *Elementary Properties.* Next we show certain properties of weak derivatives. Note here that these rules are obvious for smooth functions, but functions belonging to Sobolev spaces are not necessarily regular, and thus we have to rely only on the definition of weak derivatives.

`t2.2-1` **Theorem 2.2.1** (Properties of Weak Derivatives). *Assume that $u, v \in W^{k,p}(\Omega)$ and let α be a multi–index with $|\alpha| \leq k$. Then*

(i) *$D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multi–indices α, β with $|\alpha| + |\beta| \leq k$;*

(ii) *For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$;*

(iii) *If U is an open subset of Ω, then $u \in W^{k,p}(U)$;*

(iv) *If $\zeta \in \mathcal{C}_c^\infty(\Omega)$, then $\zeta u \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \quad \text{(Leibniz's formula)}, \tag{2.2.7}$$ `{eq:2.2-7`

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

Proof.

i. The first assertion in (i) is clear by choosing any multi–index β such that $|\alpha| + |\beta| = k$ and applying the definition of weak derivatives and $W^{k,p}(\Omega)$.

To prove the second assertion, fix $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then $D^\beta \phi \in \mathcal{C}_c^\infty(\Omega)$, and so

$$\int_\Omega D^\alpha u D^\beta \phi \, dx = (-1)^{|\alpha|} \int_\Omega u D^{\alpha+\beta} \phi \, dx$$

$$= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_\Omega (D^{\alpha+\beta} u)\phi \, dx$$

$$= (-1)^{|\beta|}(-1)^{2|\alpha+\beta|} \int_\Omega u D^{\alpha+\beta} \phi \, dx$$

$$= (-1)^{|\beta|} \int_\Omega u D^{\alpha+\beta} \phi \, dx.$$

Thus $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ in the weak sense. Similarly $D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ in the weak sense, which proves (i).

ii. The first assertion in (ii) simply states that $W^{k,p}(\Omega)$ is a real linear space, which is clear from linearity of the integral. The second assertion also follows by linearity of the integral as follows: for any $\phi \in \mathcal{C}_c^\infty(\Omega)$, we have

$$\int_\Omega D^\alpha(\lambda u + \mu v)\phi \, dx = (-1)^{|\alpha|} \int_\Omega (\lambda u + \mu v) D^\alpha \phi \, dx$$

$$= (-1)^{|\alpha|}\lambda \int_\Omega u D^\alpha \phi \, dx + (-1)^{|\alpha|}\mu \int_\Omega v D^\alpha \phi \, dx$$

$$= \lambda \int_\Omega (D^\alpha u)\phi \, dx + \mu \int_\Omega (D^\alpha v)\phi \, dx$$

$$= \int_\Omega (\lambda D^\alpha u + \mu D^\alpha v)\phi \, dx.$$

Since this is for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, (ii) follows.

iii. Assertion (iii) follows immediately by taking the restrictions of u and $D^\alpha u$ from Ω to U.

iv. To prove (2.2.7), we use induction on $|\alpha|$. First suppose that $|\alpha| = 1$. Choose any $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\int_\Omega \zeta u D^\alpha \phi \, dx = \int_\Omega u D^\alpha(\zeta\phi) - u(D^\alpha\zeta)\phi \, dx$$

$$= -\int_\Omega (D^\alpha u \zeta + u D^\alpha \zeta)\phi \, dx.$$

Thus $D^\alpha(\zeta u) = D^\alpha u \zeta + u D^\alpha \zeta$, as required. Note that on the LHS we have used the product rule on $\zeta\phi$ since we cannot necessarily apply the product rule on $u\zeta$.

Assume now for the induction hypothesis that $l < k$ and that formula (2.2.7) holds for all $|\alpha| \le l$ and all functions $\zeta \in \mathcal{C}_c^\infty(\Omega)$. Choose a multi–index α with $|\alpha| = l + 1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l, |\gamma| = 1$. Then for any $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega \zeta u D^\alpha \phi \, dx = \int_\Omega D^\beta(D^\gamma\phi) \, dx$$

$$= (-1)^{|\beta|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \zeta D^{\beta-\sigma} u D^\gamma \phi \, dx$$

(by the induction hypothesis)

$$= (-1)^{|\beta|+|\gamma|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\gamma(D^\sigma \zeta D^{\beta-\sigma} u)\phi \, dx$$

(by the induction hypothesis again)

$$= (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[D^\rho \zeta D^{\alpha-\rho} u + D^\sigma \zeta D^{\alpha-\sigma} u \right] \phi \, dx$$

(where $rho := \sigma + \gamma$)

$$= (-1)^{|\alpha|} \int_\Omega \left[\sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \zeta D^{\alpha-\sigma} u \right] \phi \, dx,$$

since

$$\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}.$$

The proof is complete. $\qquad\square$

Theorem 2.2.1 shows that many of the familiar rules of calculus apply to weak deriva-
tives. The following theorem states that the Sobolev spaces are in fact Banach spaces.

$\boxed{\texttt{t2.2-2}}$ **Theorem 2.2.2.** *For each $k \in \mathbb{N}_0$ and $1 \leq p \leq +\infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Proof.
i. We first of all check that $\| \cdot \|_{W^{k,p}(\Omega)}$ is a norm. Clearly

$$\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda| \|u\|_{W^{k,p}(\Omega)}$$

for all $\lambda \in \mathbb{R}$, and

$$\|u\|_{W^{k,p}(\Omega)} = 0 \text{ if and only if } u = 0 \; \mathcal{L}^n - \text{a.e.}$$

Next assume that $u, v \in W^{k,p}(\Omega)$. Then if $1 \leq p < +\infty$, Minkowski's inequality implies
that

$$
\begin{aligned}
\|u + v\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \leq k} \|D^\alpha u + D^\alpha v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \\
&\overset{M.I.}{\leq} \left(\sum_{|\alpha| \leq k} \left(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha v\|_{L^p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\
&\overset{M.I.}{\leq} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} + \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \\
&= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)}.
\end{aligned}
$$

The case $p = +\infty$ follows immediately from the fact that $\| \cdot \|_{L^\infty(\Omega)}$ is a norm. Thus
$\| \cdot \|_{W^{k,p}(\Omega)}$ is in fact a norm on $W^{k,p}(\Omega)$.

ii. It remains to show that $W^{k,p}(\Omega)$ is complete. Assume that $\{u_m\}_{m=1}^{+\infty} \subset W^{k,p}(\Omega)$ is a
Cauchy sequence. Then for each $|\alpha| \leq k$, $\{D^\alpha u_m\}_{m=1}^{+\infty}$ is a Cauchy sequence in $L^p(\Omega)$. Since
$L^p(\Omega)$ is complete, for each multi–index α with $|\alpha| \leq k$ there exist functions $u_\alpha \in L^p(\Omega)$
such that

$$D^\alpha u_m \to u_\alpha \quad \text{in } L^p(\Omega).$$

In particular, notice that

$$u_m \to u_{(0,\dots,0)} =: u \quad \text{in } L^p(\Omega).$$

iii. We now claim that

$$u \in W^{k,p}(\Omega), \quad D^\alpha u = u_\alpha, \quad |\alpha| \leq k. \tag{2.2.8} \quad \boxed{\texttt{\{eq:2.2-8}}}$$

To show this, fix $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then

$$
\begin{aligned}
\int_\Omega u D^\alpha \phi \, dx &\overset{L.D.C.}{=} \lim_{m \to +\infty} \int_\Omega u_m D^\alpha \phi \, dx \\
&= \lim_{m \to +\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m \phi \, dx
\end{aligned}
$$

$$\overset{L.\underline{D}.C.}{=} (-1)^{|\alpha|} \int_{\Omega} u_{\alpha} \phi \, dx.$$

Thus (2.2.8) is valid, which shows $u \in W^{k,p}(\Omega)$. But then $D^{\alpha} u_m \to D^{\alpha} u$ in $L^p(\Omega)$ for all multi–indices $|\alpha| \le k$, so that $u_m \to u$ in $W^{k,p}(\Omega)$, as required. The proof is complete. $\quad\square$

2.3. Approximation.

REFERENCES

1. L. C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, 2010.