NOTES ON THE CALCULUS OF VARIATIONS

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Notes on chapter 8 of *Partial Differential Equations* by L. C. Evans [1].

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1. Introduction

1.1. **Basic Ideas.** We suppose that we wish to solve some particular partial differential equation, which we write in the abstract form

$$A[u] = 0. \tag{1.1}$$

{eq:1-1}

In this above equation, $A[\cdot]$ denotes a given, possibly nonlinear partial differential operator and u is the unknown. Recall that there is of course no general theory for solving all such PDE.

The *calculus of variations* identifies an important class of such nonlinear problems that may be solved using relatively simple techniques from nonlinear functional analysis. We call this class of problems the *variational problems*, that is, PDE of the form (1.1), where the nonlinear differential operator $A[\cdot]$ is the "derivative" of some appropriate *energy functional* $I[\cdot]$. Symbolically, we write

$$A[\cdot] = I'[\cdot]. \tag{1.2}$$

{eq:1-2}

Then problem (1.1) becomes

$$I'[u] = 0. \tag{1.3}$$

{eq:1-3}

The idea of the formulation in (1.3) is that we can now recognize solutions of the (possibly nonlinear) PDE (1.1) as being critical points of $I[\cdot]$. In certain circumstances, these critical points may be relatively easy to find: if, for instance, the functional $I[\cdot]$ has a minimum at u, then presumably (1.3) holds and thus u is a solution of the original PDE (1.1). *The idea is that on the one hand, it is usually extremely difficult to solve* (1.1) *directly. On the other, it may be much easier to discover minimizers (or other critical points) of the functional* $I[\cdot]$.

Additionally, many of the laws of physics and other scientific disciplines arise directly as variational principles.

Date: March 13, 2023.

1.2. **First Variation, Euler-Lagrange Equation.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set with $\mathcal{C}^\infty$ boundary $\partial\Omega$.

Definition 1.1 (Lagrangian)**.** *The* **Lagrangian** *is a* $\mathcal{C}^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *function,*

$$L : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

We will write

$$L = L(x, z, p) = L(x_1, \dots, x_n, z, p_1, \dots, p_n)$$

for $x \in \Omega$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^n$. Here "z" is the variable for which we substitute $w(x)$ below, and "p" is the variable for which we substitute $Dw(x)$. We also set

$$\begin{cases} D_x L := \left(\frac{\partial}{\partial x_1} L, \dots, \frac{\partial}{\partial x_n} L\right) = (L_{x_1}, \dots, L_{x_n}), \\ D_z L := \frac{\partial}{\partial z} L = L_z, \\ D_p L := \left(\frac{\partial}{\partial p_1} L, \dots, \frac{\partial}{\partial p_n} L\right) = (L_{p_1}, \dots, L_{p_n}). \end{cases}$$

We now assume that the energy functional in (1.2) has the explicit form

$$I[w] := \int_\Omega L(x, w(x), Dw(x))\ dx, \tag{1.4}$$ {eq:1-4}

for all smooth functions $w : \overline{\Omega} \to \mathbb{R}$ satisfying some given boundary condition, say,

$$w|_{\partial\Omega} = g. \tag{1.5}$$ {eq:1-5}

We now additionally assume that some particular smooth function $u : \overline{\Omega} \to \mathbb{R}$, satisfying the boundary condition $u|_{\partial\Omega} = g$, is a minimizer of $I[\cdot]$ among all functions w satisfying the boundary condition (1.5). We will show that this function u is then automatically a solution of a certain nonlinear PDE, called the *Euler–Lagrange Equation*.

To prove this, choose any smooth function $\phi \in \mathcal{C}_c^\infty(\Omega)$ and consider the real–valued function

$$i(\tau) := I[u + \tau\phi], \quad \tau \in \mathbb{R}. \tag{1.6}$$ {eq:1-6}

We call the term $\tau\phi$ the *variation* of the function u.

Since u is a minimizer of $I[\cdot]$ and $u + \tau\phi = u = g$ on $\partial\Omega$ (because ϕ has compact support on Ω), we observe that $i(\cdot)$ has a minimum at $\tau = 0$. Therefore

$$i'(0) = 0. \tag{1.7}$$ {eq:1-7}

Computing this first derivative explicitly by writing out

$$i(\tau) = \int_\Omega L(x, u + \tau\phi, Du + \tau D\phi)\ dx, \tag{1.8}$$ {eq:1-8}

we have by the chain rule and differentiation under the integral sign

$$\begin{aligned} i'(\tau) &= \frac{d}{d\tau}\left[\int_\Omega L(x, u + \tau\phi, Du + \tau D\phi)\ dx\right] \\ &= \int_\Omega \frac{\partial}{\partial\tau}\left[L(x, u + \tau\phi, Du + \tau D\phi)\right]\ dx \\ &= \int_\Omega L_z(x, u + \tau\phi, Du + \tau D\phi)\phi + \sum_{i=1}^n L_{p_i}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\ dx. \end{aligned}$$

Definition 1.2 (First Variation)**.** *The derivative $i'(\tau)$ of $i(\tau)$ as defined in (1.8), that is,*

$$i'(\tau) = \frac{\partial}{\partial \tau} I[u + \tau\phi],$$

is called the **first variation** *of the functional $I[\cdot]$.*

We note here that the first variation of $I[\cdot]$ is recognizable as the Gateaux derivative of $I[\cdot]$.

Letting $\tau = 0$, we see from (1.7) and the assumption that u is a minimizer of $I[\cdot]$ that

$$0 = i'(0) = \int_\Omega L_z(x,u,Du)\phi + \sum_{i=1}^n L_{p_i}(x,u,Du)\phi_{x_i}\ dx.$$

Since ϕ has compact support in Ω, integration by parts on the second term gives

$$\begin{aligned} 0 = i'(0) &= \int_\Omega L_z(x,u,Du)\phi - \sum_{i=1}^n (L_{p_i}(x,u,Du))_{x_i}\phi\ dx + \int_{\partial\Omega} \sum_{i=1}^n L_{p_i}(x,u,Du)\phi\nu^i\ dS \\ &= \int_\Omega \left[L_z(x,u,Du) - \sum_{i=1}^n (L_{p_i}(x,u,Du))_{x_i} \right] \phi\ dx, \end{aligned}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ as usual denotes the outward pointing unit normal vector field along $\partial\Omega$. Since this equality holds for all text functions $\phi \in \mathcal{C}_c^\infty(\Omega)$, we conclude that u solves the (possibly) nonlinear PDE

$$L_z(x,u,Du) - \sum_{i=1}^n (L_{p_i}(x,u,Du))_{x_i} = 0 \quad \text{on } \Omega. \tag{1.9}$$

{eq:1-9}

Definition 1.3 (Euler–Lagrange Equation)**.** *For the energy functional $I[\cdot]$ as defined in (1.4), the equation*

$$L_z(x,u,Du) - \sum_{i=1}^n (L_{p_i}(x,u,Du))_{x_i} = 0 \quad \text{on } \Omega$$

is called the **Euler–Lagrange equation** *associated with $I[\cdot]$.*

We observe that the Euler–Lagrange equation (1.9) is a quasilinear, second–order PDE in divergence form.

In summary, any smooth minimizer u of $I[\cdot]$ is a solution of the Euler–Lagrange equation. Conversely, we can try to find a solution of the Euler–Lagrange PDE (1.9) by finding minimizers of the energy functional $I[\cdot]$ as defined in (1.4).

Example 1.1 (Dirichlet's Principle)**.** *Put*

$$L(x,z,p) := \frac{1}{2}|p|^2.$$

Then $L_{p_i}(x,z,p) = p_i$, $i = 1, \ldots, n$ and $L_z(x,z,p) = 0$. Thus the Euler–Lagrange equation associated with the functional

$$I[w] := \frac{1}{2}\int_\Omega |Dw|^2\ dx = \int_\Omega L(x,w,Dw)\ dx$$

is

$$\Delta u = 0 \quad \text{on } \Omega.$$

This is Dirichlet's principle.

Example 1.2. *Sometimes we wish to convert a given PDE into a variational problem, that is, to recover a Lagrangian from a given PDE. Motivated by the previous example, consider the Laplacian*

$$\Delta u = 0 \quad \text{on } \Omega.$$

Thus we want to define a function L such that

$$0 = \Delta w = L_z(x, w, Dw) - \sum_{i=1}^{n} (L_{p_i}(x, w, Dw))_{x_i}.$$

We "guess" that

$$\sum_{i=1}^{n} (L_{p_i}(x, w, Dw))_{x_i} = \Delta w.$$

Taking $(L_{p_i}(x, w, Dw))_{x_i} := \partial^2_{x_i} w$ and integrating with respect to x_i, we have

$$L_{p_i}(x, w, Dw) = \partial_{x_i} w + C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Thus we have

$$\sum_{i=1}^{n} L_{p_i}(x, w, Dw) = \operatorname{div} w.$$

Now taking $L_{p_i}(x, w, Dw) := \partial_{x_i} w$ and integrating with respect to $\partial_{x_i} w$, it follows

$$L(x, w, Dw) = \frac{1}{2}(\partial_{x_i} w)^2 + C(\partial_{x_1} w, \ldots, \partial_{x_{i-1}} w, \partial_{x_{i+1}} w, \ldots, \partial_{x_n} w).$$

Hence,

$$L(x, w, Dw) = \frac{1}{2}|Dw|^2,$$

which is the Lagrangian from the previous example.

Example 1.3 (Generalized Dirichlet's Principle)**.** *Take*

$$L(x, z, p) := \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(x) p_i p_j - z f(x),$$

where $a^{ij} = a^{ji}$, $i, j = 1, \ldots, n$. Then

$$L_{p_i}(x, z, p) = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} a^{ij} p_j = \sum_{j=1}^{n} a^{ij} p_j,$$

$j = 1, \ldots, n$, so that

$$(L_z(x, z, p))_{x_i} = \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{n} a^{ij} p_j \right] = \sum_{j=1}^{n} (a^{ij} p_j)_{x_i},$$

and

$$L_z(x, z, p) = -f(x).$$

Thus the Euler–Lagrange equation associated with the functional

$$I[w] := \int_\Omega \frac{1}{2} \sum_{i,j=1}^{n} a^{ij} w_{x_i} w_{x_j} - w f \; dx$$

is the divergence structure linear equation

$$-\sum_{i,j}^{n}(a^{ij}u_{x_j})_{x_i} = f \quad \text{on } \Omega.$$

We will see later that the uniform ellipticity condition on the a^{ij}, $i,j = 1,\ldots,n$ is a natural further assumption required to prove the existence of a minimizer. Consequently from the nonlinear viewpoint of the calculus of variations, the divergence structure form of a linear second–order elliptic PDE is completely natural.

Example 1.4 (Nonlinear Poisson Equation)**.** *Assume that we are given a smooth function* $f:\mathbb{R}\to\mathbb{R}$, *and define its antiderivative* $F(z) := \int_0^z f(\xi)\, d\xi$. *Take*

$$L(x,z,p) := \frac{1}{2}|p|^2 - F(z).$$

Then $L_{p_i}(x,z,p) = p_i$, $i = 1,\ldots,n$, *so that* $(L_{p_i}(x,z,p))_{x_i} = \partial_{x_i}p_i$, $i = 1,\ldots,n$, *and* $L_z(x,z,p) = -f(z)$. *Thus the Euler–Lagrange equation associated with the functional*

$$I[w] := \int_\Omega \frac{1}{2}|Dw|^2 - F(w)\, dx$$

is the nonlinear Poisson equation

$$-\Delta u = f(u) \quad \text{on } \Omega.$$

Example 1.5 (Minimal Surfaces)**.** *Put*

$$L(x,z,p) := (1+|p|^2)^{1/2},$$

so that

$$I[w] := \int_\Omega (1+|Dw|^2)^{1/2}\, dx$$

is the area of the graph of the function $w:\Omega\to\mathbb{R}$. *We compute*

$$L_{p_i}(x,z,p) = \frac{1}{2}(1+|p|^2)^{1/2}\cdot 2p_i = \frac{p_i}{(1+|p|^2)^{1/2}},$$

$i = 1,\ldots,n$, *and* $L_z(x,z,p) = 0$, *so that the associated Euler–Lagrange equation is*

$$\sum_{i=1}^{n}\left(\frac{u_{x_i}}{(1+|Du|^2)^{1/2}}\right)_{x_i} = 0 \quad \text{on } \Omega. \tag{1.10}$$ {eq:1-10}

This PDE is called the minimal surface equation. The expression

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right)$$

on the LHS of (1.10) *is* n *times the mean curvature of* u. *Thus a minimal surface has zero mean curvature.*

1.3. **Second Variation.** We continue the calculations from §1.2 by computing the *second variation* of the functional $I[\cdot]$ at the function u. This we find by observing that since u gives a minimum for $I[\cdot]$, we must have

$$i''(0) \geq 0,$$

where i is defined as above by (1.6). Recall from that the first variation of $I[\cdot]$ is given by

$$i'(\tau) = \int_\Omega L_z(x, u + \tau\phi, Du + \tau\phi)\phi + \sum_{i=1}^n L_{p_i}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\ dx.$$

Calculating the second derivative explicitly, once again by applying the chain rule and differentiation under the integral sign, we find

$$\begin{aligned} i''(\tau) &= \frac{d}{d\tau}\left[\int_\Omega L_z(x, u + \tau\phi, Du + \tau\phi)\phi + \sum_{i=1}^n L_{p_i}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\ dx\right] \\ &= \int_\Omega \frac{\partial}{\partial\tau}\left[L_z(x, u + \tau\phi, Du + \tau D\phi)\phi\right] + \sum_{i=1}^n \frac{\partial}{\partial\tau}\left[L_{p_i}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\right]\ dx \\ &= \int_\Omega L_{zz}(x, u + \tau\phi, Du + \tau D\phi)\phi^2 + \sum_{i=1}^n L_{zp_i}(x, u + \tau\phi, Du + \tau D\phi)\phi\phi_{x_i} + \\ &\quad \sum_{i=1}^n L_{p_i z}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\phi + \sum_{i=1}^n\sum_{j=1}^n L_{p_i p_j}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\phi_{x_j}\ dx \\ &= \int_\Omega L_{zz}(x, u + \tau\phi, Du + \tau D\phi)\phi^2 + 2\sum_{i=1}^n L_{p_i z}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\phi + \\ &\quad \sum_{i,j}^n L_{p_i p_j}(x, u + \tau\phi, Du + \tau D\phi)\phi_{x_i}\phi_{x_j}\ dx. \end{aligned}$$

Definition 1.4 (Second Variation)**.** *The second derivative $i''(\tau)$ of $i(\tau)$ as defined in (1.8), that is,*

$$i''(\tau) = \frac{\partial^2}{\partial\tau^2} I[u + \tau\phi],$$

is called the **second variation** *of the functional $I[\cdot]$.*

Again now letting $\tau = 0$, we obtain the inequality

$$0 \leq i''(0) = \int_\Omega L_{zz}(x, u, Du)\phi^2 + 2\sum_{i=1}^n L_{p_i z}(x, u, Du)\phi_{x_i}\phi + \sum_{i,j=1}^n L_{p_i p_j}(x, u, Du)\phi_{x_i}\phi_{x_j}\ dx. \tag{1.11}$$

{eq:1-11}

This holds for all test functions $\phi \in \mathcal{C}_c^\infty(\Omega)$.

We can extract useful information from the inequality (1.11) as follows. First, note that after a standard approximation argument that the estimate (1.11) is valid for any Lipschitz continuous function ϕ vanishing on $\partial\Omega$. This is because for an open, bounded set $\Omega \subseteq \mathbb{R}^n$ with $\partial\Omega \in \mathcal{C}^1$, $\mathrm{Lip}(\Omega, \mathbb{R}) = W^{1,\infty}(\Omega)$. We then fix $\xi \in \mathbb{R}^n$ and define

$$v(x) := \epsilon\rho\left(\frac{\xi \cdot x}{\epsilon}\right)\zeta(x), \quad x \in \Omega, \tag{1.12}$$

{eq:1-12}

where $\zeta \in \mathcal{C}_c^\infty(\Omega)$ and $\rho : \mathbb{R} \to \mathbb{R}$ is the "periodic triangular function" defined by

$$\rho(x) := \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1 - x, & \frac{1}{2} \le x \le 1, \end{cases} \qquad \rho(x+1) = \rho(x), \quad x \in \mathbb{R}.$$

Note that

$$|\rho'| = 1 \quad \mathcal{L}^1 - \text{a.e.} \tag{1.13}$$

{eq:1-13}

Observe further that by the chain rule,

$$\begin{aligned} v_{x_i}(x) &= \rho'\left(\frac{\xi \cdot x}{\epsilon}\right) \xi_i \zeta(x) + \epsilon \rho \left(\frac{\xi \cdot x}{\epsilon}\right) \zeta_{x_i}(x) \\ &= \rho'\left(\frac{\xi \cdot x}{\epsilon}\right) \xi_i \zeta(x) + \mathcal{O}(\epsilon), \quad \epsilon \to 0, \end{aligned}$$

since $|\rho(x)| \le 1$ for all $x \in \mathbb{R}$ and $\max_{x \in \Omega} |\zeta_{x_i}(x)| < +\infty$ because $\zeta \in \mathcal{C}_c^\infty(\Omega)$. Similarly, note also that

$$v(x) = \mathcal{O}(\epsilon), \quad \epsilon \to 0.$$

Thus, substituting v (cf. (1.12)) into the estimate (1.11) yields

$$0 \le \int_\Omega \sum_{i,j=1}^n L_{p_i p_j}(x, u, Du)(\rho')^2 \xi_i \xi_j \zeta^2 + \mathcal{O}(\epsilon)\ dx.$$

Recalling (1.13) and taking the limit as $\epsilon \to 0$, we obtain the inequality

$$0 \le \int_\Omega \sum_{i,j=1}^n L_{p_i p_j}(x, u, Du) \xi_i \xi_j \zeta^2\ dx.$$

Since this estimate holds for all $\zeta \in \mathcal{C}_c^\infty(\Omega)$, it follows

$$\sum_{i,j=1}^n L_{p_i p_j}(x, u, Du) \xi_i \xi_j \ge 0, \quad \xi \in \mathbb{R}^n, \quad x \in \Omega. \tag{1.14}$$

{eq:1-14}

We will see later that this necessary condition suggests the assumption that the Lagrangian L is convex in its third argument, which is required for the existence theory.

1.4. **Systems.**

1.4.1. *Euler–Langrange Equations.* The previous considerations generalize easily to the case of systems. Recall that $\mathbb{R}^{m \times n}$ denotes the space of real $m \times n$ matrices. Assume that the smooth Lagrangian

$$L : \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$$

is given.

We will write

$$L = L(x, z, P) = L(x_1, \dots, x_n, z^1, \dots, z^m, p_1^1, \dots, p_n^m)$$

for $x \in \Omega$, $z \in \mathbb{R}^m$, and $P \in \mathbb{R}^{m \times n}$, where

$$P := \begin{pmatrix} p_1^1 & \cdots & p_n^1 \\ & \ddots & \\ p_1^m & \cdots & p_n^m \end{pmatrix}.$$

We are employing superscripts to denote rows, as this notational convention simplifies the following formulas.

We associate with L the functional

$$I[\boldsymbol{w}] := \int_\Omega L(x, \boldsymbol{w}(x), D\boldsymbol{w}(x))\ dx, \tag{1.15}$$

{eq:1-15}

defined for smooth functions $\boldsymbol{w} : \overline{\Omega} \to \mathbb{R}^m$, $\boldsymbol{w} = (w^1, \dots, w^m)$, satisfying some given boundary condition

$$\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{g},$$

where $\boldsymbol{g} : \partial\Omega \to \mathbb{R}^m$. Note here that

$$D\boldsymbol{w}(x) = \begin{pmatrix} w^1_{x_1}(x) & \dots & w^1_{x_n}(x) \\ & \ddots & \\ w^m_{x_1}(x) & \dots & w^m_{x_n}(x) \end{pmatrix}$$

is the gradient matrix of $\boldsymbol{w}$ at x.

We proceed as we did in the previous sections and show that any smooth minimizer $\boldsymbol{u} = (u^1, \dots, u^m)$ of $I[\cdot]$, given now by (1.15), taking among all functions satisfying the boundary condition $\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{g}$, must solve a certain *system* of nonlinear PDEs. We therefore fix a function $\boldsymbol{\phi} = (\phi^1, \dots, \phi^m) \in \mathcal{C}^\infty_c(\Omega, \mathbb{R}^m)$ and write

$$i(\tau) := I[\boldsymbol{u} + \tau\boldsymbol{\phi}].$$

As before,

$$i'(0) = 0.$$

Calculating the first variation explicitly, we find

$$\begin{aligned} i'(\tau) &= \frac{d}{d\tau}\left[\int_\Omega L(x, \boldsymbol{w} + \tau\boldsymbol{\phi}, D\boldsymbol{w} + \tau D\boldsymbol{\phi})\ dx\right] = \int_\Omega \frac{\partial}{\partial\tau}\left[L(x, \boldsymbol{w} + \tau\boldsymbol{\phi}, D\boldsymbol{w} + \tau D\boldsymbol{\phi})\right]\ dx \\ &= \int_\Omega \sum_{k=1}^m L_{z^k}(x, \boldsymbol{w} + \tau\boldsymbol{\phi}, D\boldsymbol{w} + \tau D\boldsymbol{\phi})\phi^k + \sum_{i=1}^n \sum_{k=1}^m L_{p^k_i}(x, \boldsymbol{w} + \tau\boldsymbol{\phi}, D\boldsymbol{w} + \tau D\boldsymbol{\phi})\phi^k_{x_i}\ dx. \end{aligned}$$

Setting $\tau = 0$, we derive the equality

$$0 = i'(0) = \int_\Omega \sum_{k=1}^m L_{z^k}(x, \boldsymbol{u}, D\boldsymbol{u})\phi^k + \sum_{i=1}^n \sum_{k=1}^m L_{p^k_i}(x, \boldsymbol{u}, D\boldsymbol{u})\phi^k_{x_i}\ dx.$$

Integrating by parts, we have

$$0 = i'(0) = \int_\Omega \sum_{k=1}^m L_{z^k}(x, \boldsymbol{u}, D\boldsymbol{u})\phi^k - \sum_{i=1}^n \sum_{k=1}^m (L_{p^k_i}(x, \boldsymbol{u}, D\boldsymbol{u}))_{x_i}\phi^k\ dx$$

holding for all test functions $\boldsymbol{\phi} \in \mathcal{C}^\infty_c(\Omega, \mathbb{R}^m)$. Thus we conclude that $\boldsymbol{u}$ solves the coupled, quasilinear system of PDE

$$L_{z^k}(x, \boldsymbol{u}, D\boldsymbol{u}) - \sum_{i=1}^n \Big(L_{p^k_i}(x, \boldsymbol{u}, D\boldsymbol{u})\Big)_{x_i} = 0 \quad \text{on } \Omega, \quad k = 1, \dots, m. \tag{1.16}$$

{eq:1-16}

Definition 1.5 (Euler–Lagrange Equations)**.** *For the energy functional $I[\cdot]$ as defined in* eq:1-15 (1.15)*, the equations*

$$L_{z^k}(x,\boldsymbol{u},D\boldsymbol{u}) - \sum_{i=1}^{n}\Big(L_{p_i^k}(x,\boldsymbol{u},D\boldsymbol{u})\Big)_{x_i} = 0 \quad \text{on } \Omega, \quad k=1,\dots,m,$$

are called the **Euler–Lagrange equations** *associated with $I[\cdot]$.*

. . . .

1.4.2. *Null Lagrangians.* It turns out to be interesting to study certain systems of nonlinear PDEs for which *every* smooth function is a solution.

Definition 1.6 (Null Lagrangian)**.** *The function $L : \overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}$ is called a* **null Lagrangian** *if the system of Euler–Lagrange equations*

$$L_{z^k}(x,\boldsymbol{u},D\boldsymbol{u}) - \sum_{i=1}^{n}\Big(L_{p_i^k}(x,\boldsymbol{u},D\boldsymbol{u})\Big)_{x_i} = 0 \quad \text{on } \Omega, \quad k=1,\dots,m, \tag{1.17}$$ {eq:1-17}

is solved by all smooth functions $u : \Omega\to\mathbb{R}^m$.

The importance of null Lagrangians is that the corresponding energy functional

$$I[\boldsymbol{w}] = \int_\Omega L(x,\boldsymbol{w},D\boldsymbol{w})\,dx$$

depends only on the boundary conditions.

t1.1 **Theorem 1.1** (Null Lagrangians and Boundary Conditions)**.** *Let $L : \overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}$ be a null Lagrangian. Assume that $\boldsymbol{u},\widetilde{\boldsymbol{u}}\in\mathcal{C}^2(\overline{\Omega},\mathbb{R}^m)$ are two functions such that*

$$\boldsymbol{u}\equiv\widetilde{\boldsymbol{u}} \quad \text{on } \partial\Omega. \tag{1.18}$$ {eq:1-18}

Then

$$I[\boldsymbol{u}] = I[\widetilde{\boldsymbol{u}}]. \tag{1.19}$$ {eq:1-19}

Proof. Define

$$i(\tau) := I[\tau\boldsymbol{u}+(1-\tau)\widetilde{\boldsymbol{u}}], \quad \tau\in[0,1].$$

Note that $\tau\boldsymbol{u}+(1-\tau)\widetilde{\boldsymbol{u}}\in\mathcal{C}^2(\overline{\Omega},\mathbb{R}^m)$, and thus satisfies the system of Euler–Lagrange equations

$$L_{z^k}(x,\tau\boldsymbol{u}+(1-\tau\widetilde{\boldsymbol{u}}),\tau D\boldsymbol{u}+(1-\tau)D\widetilde{\boldsymbol{u}}) - \sum_{i=1}^{n}\Big(L_{p_i^k}(x,\tau\boldsymbol{u}+(1-\tau\widetilde{\boldsymbol{u}}),\tau D\boldsymbol{u}+(1-\tau)D\widetilde{\boldsymbol{u}})\Big)_{x_i} = 0$$

on Ω, $k=1,\dots,m$. Therefore,

$$\begin{aligned} i'(\tau) = \int_\Omega \sum_{k=1}^{m} & L_{z^k}(x,\tau\boldsymbol{u}+(1-\tau\widetilde{\boldsymbol{u}}),\tau D\boldsymbol{u}+(1-\tau)D\widetilde{\boldsymbol{u}})(u^k-\widetilde{u}^k) + \\ & \sum_{i=1}^{n}\sum_{k=1}^{m} L_{p_i^k}(x,\tau\boldsymbol{u}+(1-\tau\widetilde{\boldsymbol{u}}),\tau D\boldsymbol{u}+(1-\tau)D\widetilde{\boldsymbol{u}})(u^k_{x_i}-\widetilde{u}^k_{x_i})\,dx. \end{aligned}$$

Integrating by parts on the second term, we obtain

$$
\begin{aligned}
i'(\tau) = &\int_\Omega \sum_{k=1}^{m} L_{z^k}(x, \tau\boldsymbol{u} + (1-\tau\widetilde{\boldsymbol{u}}), \tau D\boldsymbol{u} + (1-\tau)D\widetilde{\boldsymbol{u}})(u^k - \widetilde{u}^k) \; - \\
&\quad \sum_{i=1}^{n}\sum_{k=1}^{m} \Big(L_{p_i^k}(x, \tau\boldsymbol{u} + (1-\tau\widetilde{\boldsymbol{u}}), \tau D\boldsymbol{u} + (1-\tau)D\widetilde{\boldsymbol{u}}) \Big)_{x_i} (u^k - \widetilde{u}^k)\; dx \; + \\
&\quad \int_{\partial\Omega} \sum_{i=1}^{n}\sum_{k=1}^{m} L_{p_i^k}(x, \tau\boldsymbol{u} + (1-\tau\widetilde{\boldsymbol{u}}), \tau D\boldsymbol{u} + (1-\tau)D\widetilde{\boldsymbol{u}})(u^k - \widetilde{u}^k)\nu^i \; dS \\
= &\sum_{k=1}^{m} \int_\Omega \Big[L_{z^k}(x, \tau\boldsymbol{u} + (1-\tau\widetilde{\boldsymbol{u}}), \tau D\boldsymbol{u} + (1-\tau)D\widetilde{\boldsymbol{u}}) \; - \\
&\quad \sum_{k=1}^{m} \Big(L_{p_i^k}(x, \tau\boldsymbol{u} + (1-\tau\widetilde{\boldsymbol{u}}), \tau D\boldsymbol{u} + (1-\tau)D\widetilde{\boldsymbol{u}}) \Big) \Big] (u^k - \widetilde{u}^k)\; dx \; + \\
&\quad \int_{\partial\Omega} \sum_{i=1}^{n}\sum_{k=1}^{m} L_{p_i^k}(x, \tau\boldsymbol{u} + (1-\tau\widetilde{\boldsymbol{u}}), \tau D\boldsymbol{u} + (1-\tau)D\widetilde{\boldsymbol{u}})(u^k - \widetilde{u}^k)\nu^i \; dS \\
= &\, 0,
\end{aligned}
$$

where the last equality holds by the assumption (eq:1-18 1.18) and the fact that $\tau\boldsymbol{u} + (1-\,)\widetilde{\boldsymbol{u}}$ solves the system of Euler–Lagrange equations. Thus identity (eq:1-19 1.19) follows by observing

$$I[\boldsymbol{u}] = i(1) = i(0) = I[\widetilde{\boldsymbol{u}}].$$

The proof is complete. □

In the scalar case $m = 1$, the only null Lagrangians are the cases where L is linear in the variable p. For the case of systems, that is, when $m > 1$, however, there are certain nontrivial examples.

We explain a bit of notation for the following result. If A is an $n \times n$ matrix, we denote by

$$\operatorname{cof} A$$

the *cofactor* matrix, whose $(k,i)^{\text{th}}$ entry is $(\operatorname{cof} A)_i^k = (-1)^{i+k} d(A)_i^k$, where $d(A)_i^k$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the k^{th} row and i^{th} column of A.

l1.1 **Lemma 1.1** (Divergence–Free Rows). *Let* $\boldsymbol{u}\mathbb{R}^n \to \mathbb{R}^n$ *be a smooth function. Then*

$$\sum_{i=1}^{n} (\operatorname{cof} D\boldsymbol{u})^n_{i,x_i} = 0, \quad k = 1, \ldots, n. \tag{1.20}$$ {eq:1-20}

Proof.

(i). From linear algebra we recall the identity

$$(\det P) I = P^T (\operatorname{cof} P), \quad P \in \mathbb{R}^{n\times n}, \tag{1.21}$$ {eq:1-21}

that is,

$$(\det P)\delta_{ij} = \sum_{i=1}^{n} p_i^k (\operatorname{cof} P)_j^k, \quad i,j = 1, \ldots, n. \tag{1.22}$$ {eq:1-22}

Thus in particular

$$\partial_{p^k_m} \det P = (\operatorname{cof} P)^k_m, \quad k, m = 1, \dots, n. \tag{1.23}$$ {eq:1-23}

(ii). Now set $P = D\boldsymbol{u}$ in (1.22), differentiate with respect to x_j, and sum $j = 1, \dots, n$, to find

$$\sum_{j,k,m=1}^{n} \delta_{ij} (\operatorname{cof} D\boldsymbol{u})^k_m u^k_{x_m x_j} = \sum_{k,j=1}^{n} u^k_{x_i x_j} (\operatorname{cof} D\boldsymbol{u})^k_j + u^k_{x_i} (\operatorname{cof} D\boldsymbol{u})^k_{j,x_j},$$

for $i = 1, \dots, n$. This identity simplifies to

$$\sum_{i=1}^{n} u^n_{x_i} \left(\sum_{j=1}^{n} (\operatorname{cof} D\boldsymbol{u})^k_{j,x_j} \right) = 0, \quad i = 1, \dots, n. \tag{1.24}$$ {eq:1-24}

(iii). Now if $\det D\boldsymbol{u}(x_0) \neq 0$, we deduce from (1.24) that

$$\sum_{j=1}^{n} (\operatorname{cof} D\boldsymbol{u})^k_{j,x_j} = 0, \quad k = 1, \dots, n$$

at x_0. If instead $\det D\boldsymbol{u}(x_0) = 0$, we choose a number $\epsilon > 0$ so small that $\det(D\boldsymbol{u}(x_0) + \epsilon I) \neq 0$, apply steps (i) and (ii) to $\widetilde{\boldsymbol{u}} := \boldsymbol{u} + \epsilon x$, and send $\epsilon \to 0$. The proof is complete. $\square$

t1.2 **Theorem 1.2** (Determinants as Null Lagrangians)**.** *The determinant function*

$$L(P) = \det P, \quad P \in \mathbb{R}^{n \times n},$$

is a null Lagrangian.

Proof. We must show that for any smooth function $\boldsymbol{u} : \Omega \to \mathbb{R}^n$,

$$\sum_{i=1}^{n} \Big(L_{p^k_i}(D\boldsymbol{u}) \Big)_{x_i} = 0, \quad k = 1, \dots, n.$$

By (1.23), we have $L_{p^k_i} = (\operatorname{cof} P)^k_i$, $i, k = 1, \dots, n$. But then it follows by Lemma (1.1) that

$$\sum_{i=1}^{n} \Big(L_{p^k_i}(D\boldsymbol{u}) \Big)_{x_i} = \sum_{i=1}^{n} (\operatorname{cof} D\boldsymbol{u})^k_{i,x_i} = 0, \quad k = 1, \dots, n,$$

as required. The proof is complete. $\square$

. . . .

1.4.3. *Application.* A nice application is an analytic proof of Brouwer's Fixed Point Theorem.

t1.3 **Theorem 1.3** (Brouwer's Fixed Point Theorem)**.** *Assume that*

$$\boldsymbol{u} : \overline{B(0,1)} \to \overline{B(0,1)}$$

is continuous, where $B(0,1)$ denotes the open unit ball in $\mathbb{R}^n$. Then $\boldsymbol{u}$ has a fixed point, that is, there exists a point $x \in \overline{B(0,1)}$ such that

$$\boldsymbol{u}(x) = x.$$

Proof.

(i). Write $B := \overline{B(0,1)}$. We first of all show that there does not exist a smooth function

$$\boldsymbol{w} : B \to \partial B \tag{1.25}$$

such that

$$\boldsymbol{w}(x) = x \tag{1.26}$$

for all $x \in \partial B$.

(ii). Suppose by contradiction that such a function $\boldsymbol{w}$ exists. Denote by $\widetilde{\boldsymbol{w}}$ the identity function on B, so that $\widetilde{\boldsymbol{w}}(x) = x$ for all $x \in B$. By (1.26), $\boldsymbol{w} = \widetilde{\boldsymbol{w}}$ on ∂B. Since the determinant is a null Lagrangian (cf. (1.2)), we have by Theorem (1.1)

$$\int_B \det D\boldsymbol{w}\, dx = \int_B \det D\widetilde{\boldsymbol{w}}\, dx = \mathcal{L}^n(B) \neq 0. \tag{1.27}$$

On the other hand, (1.25) implies that $|\boldsymbol{w}(x)| = 1$ for all $x \in B$, so that $|\boldsymbol{w}|^2 \equiv 1$. Differentiating, we find

$$(D\boldsymbol{w})^T \boldsymbol{w} = \mathbf{0}. \tag{1.28}$$

Since $|\boldsymbol{w}| = 1$, (1.28) implies that 0 is an eigenvalue of $(D\boldsymbol{w})^T$ for each $x \in B$. Therefore $\det D\boldsymbol{w} \equiv 0$ in B. This contradicts (1.27), and therefore no smooth function $\boldsymbol{w}$ satisfying (1.25) and (1.26) can exist.

(iii). Next we show that there does not exist any continuous function satisfying (1.25) and (1.26). Suppose again that such a function $\boldsymbol{w}$ does exist. We may then continuously extend $\boldsymbol{w}$ by setting $\boldsymbol{w}(x) := x$ if $x \in \mathbb{R}^n \setminus B$. Observe that $\boldsymbol{w}(x) \neq 0$ for all $x \in \mathbb{R}^n$. Fix $\epsilon > 0$ so small that $\boldsymbol{w}_1 := \eta_\epsilon * \boldsymbol{w}$ satisfies $\boldsymbol{w}_1(x) \neq 0$ for all $x \in \mathbb{R}^n$. Note also that since η_ϵ is radial, we have $\boldsymbol{w}_1(x) = x$ if $x \in \mathbb{R}^n \setminus \overline{B(0,2)}$, for $\epsilon > 0$ sufficiently small. Then

$$\boldsymbol{w}_2 := \frac{2\boldsymbol{w}_1}{|\boldsymbol{w}_1|}$$

would be a smooth mapping satisfying (1.25) and (1.26) with the ball $\overline{B(0,2)}$ replacing $\overline{B(0,1)}$, a contradiction to steps (i) and (ii).

(iv). Finally, suppose that $\boldsymbol{u} : B \to B$ is continuous but has no fixed point. Define the mapping $\boldsymbol{w} : B \to \partial B$ by setting $\boldsymbol{w}(x)$ to be the point on ∂B hit by the ray emanating from $\boldsymbol{u}(x)$ and passing through x. Note that this mapping is well–defined since $\boldsymbol{u}(x) \neq x$ for all $x \in B$. Moreover, $\boldsymbol{w}$ is continuous and satisfies (1.25) and (1.26).

But this is a contradiction to step (iii). The proof is complete. □

2. Existence of Minimizers

References

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