# Static Program Analysis

## Part 5 – widening and narrowing

http://cs.au.dk/~amoeller/spa/

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# Interval analysis

- Compute upper and lower bounds for integers
- Possible applications:
  - array bounds checking
  - integer representation
  - …
- Lattice of intervals:

$$\mathit{Interval} = \mathit{lift}(\{\ [\mathrm{l}\,,\mathrm{h}]\ \mid \mathrm{l},\mathrm{h} \in N \wedge \mathrm{l} \leq \mathrm{h}\ \})$$

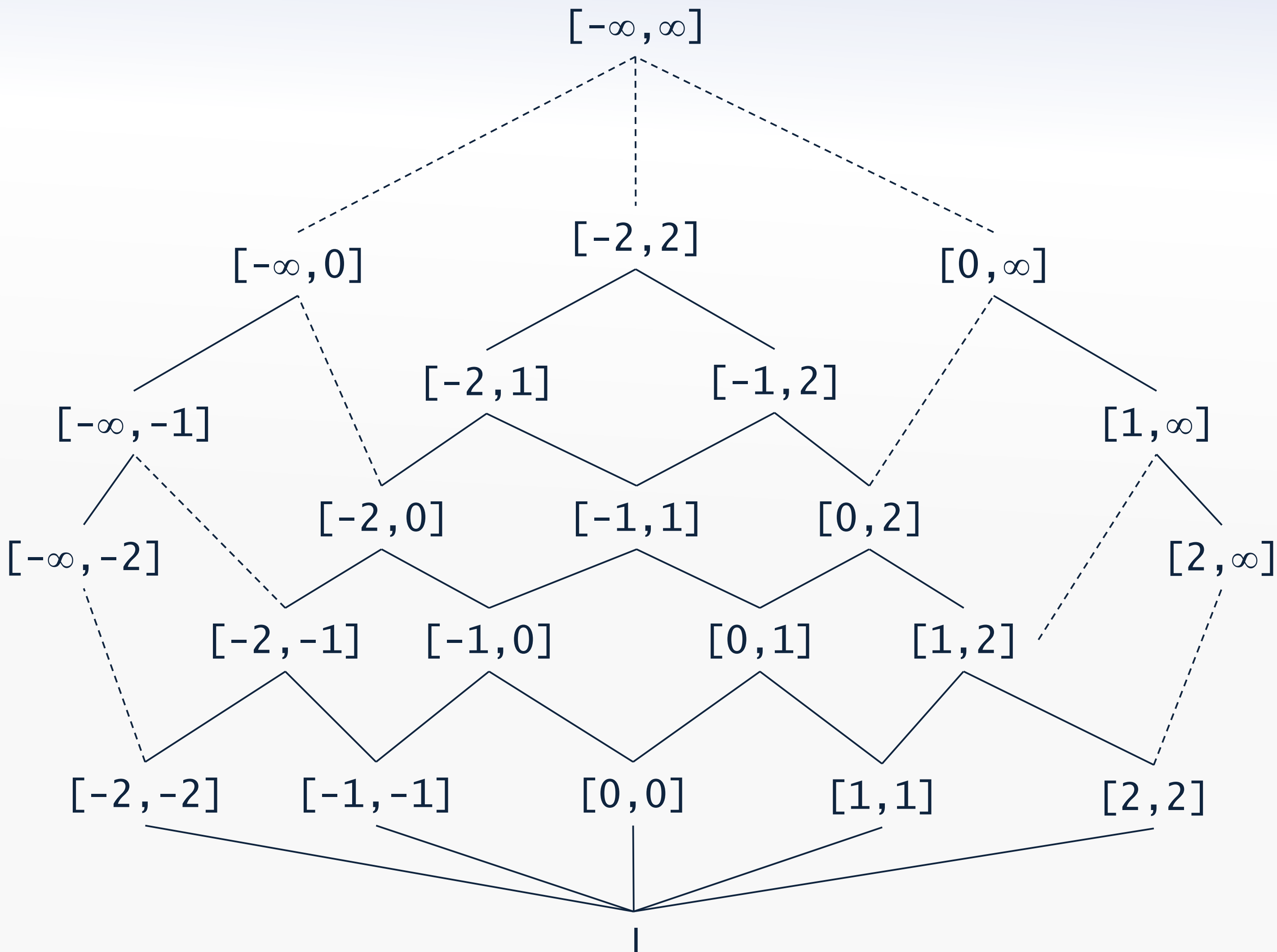
where

$$N = \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$$

and intervals are ordered by inclusion:

$$[\mathrm{l}_1\,,\mathrm{h}_1] \sqsubseteq [\mathrm{l}_2\,,\mathrm{h}_2] \text{ iff } \mathrm{l}_2 \leq \mathrm{l}_1 \wedge \mathrm{h}_1 \leq \mathrm{h}_2$$

# The interval lattice




bottom element here interpreted as “not an integer”

# Interval analysis lattice

- The total lattice for a program point is

  $$L = \mathit{Vars} \rightarrow \mathit{Interval}$$

  that provides bounds for each (integer) variable

- If using the worklist solver that initializes the worklist with only the *entry* node, use the lattice *lift*(L)
  - bottom value of *lift*(L) represents "unreachable program point"
  - bottom value of L represents "maybe reachable, but all variables are non-integers"

- This lattice has *infinite height*, since the chain

  $$[0,0] \sqsubseteq [0,1] \sqsubseteq [0,2] \sqsubseteq [0,3] \sqsubseteq [0,4] \ldots$$

  occurs in *Interval*

# Interval constraints

- For assignments:

$$[\![ x = E ]\!] = \mathit{JOIN}(\mathrm{v})[x \rightarrow \mathit{eval}(\mathit{JOIN}(\mathrm{v}), E)]$$

- For all other nodes:

$$[\![\mathrm{v}]\!] = \mathit{JOIN}(\mathrm{v})$$

where $\mathit{JOIN}(\mathrm{v}) = \bigsqcup_{\mathrm{w} \in \mathit{pred}(\mathrm{v})} [\![\mathrm{w}]\!]$

# Evaluating intervals

- The *eval* function is an *abstract evaluation*:
  - $eval(\sigma, x) = \sigma(x)$
  - $eval(\sigma, intconst) = [intconst, intconst]$
  - $eval(\sigma, E_1 \text{ op } E_2) = \overline{\text{op}}(eval(\sigma,E_1),eval(\sigma,E_2))$
- Abstract arithmetic operators:
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) =$

not trivial to implement!

$$[\min_{x \in [l_1, h_1],\, y \in [l_2, h_2]} x \text{ op } y,\ \max_{x \in [l_1, h_1],\, y \in [l_2, h_2]} x \text{ op } y]$$

- Abstract comparison operators (could be improved):
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) = [0, 1]$

# Fixed-point problems

- The lattice has infinite height, so the fixed-point algorithm does not work ☹

- In $L^n$, the sequence of approximants

  $$f^i(\bot, \bot, \ldots, \bot)$$

  is not guaranteed to converge
- (Exercise: give an example of a program where this happens)

- Restricting to 32 bit integers is not a practical solution
- *Widening* gives a useful solution…

# Widening

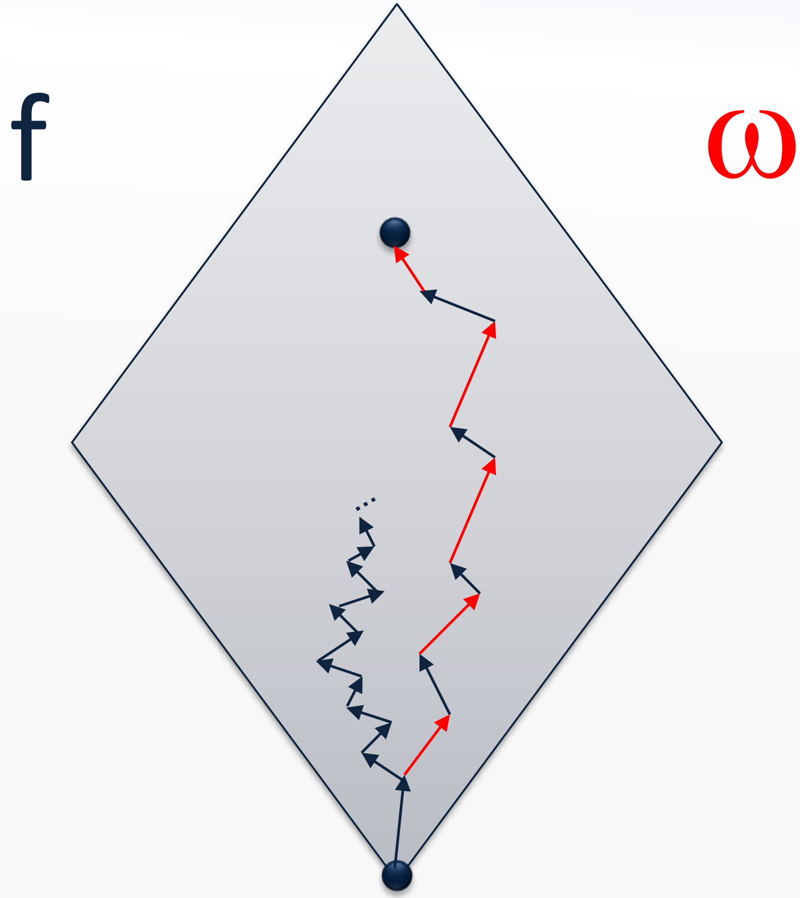
- Introduce a *widening* function $\omega: L^n \rightarrow L^n$ so that

  $$(\omega \circ f)^i(\perp, \perp, \ldots, \perp)$$

  converges on a fixed-point that is a safe approximation of each $f^i(\perp, \perp, \ldots, \perp)$

- i.e. the function $\omega$ coarsens the information

# Turbo charging the iterations



# Widening for intervals

- The function $\omega$ is defined pointwise on $L^n$
- Parameterized with a fixed finite subset $B \subset N$
  - must contain $-\infty$ and $\infty$ (to retain the $\top$ element)
  - typically seeded with all integer constants occurring in the given program
- Idea: Find the nearest enclosing allowed interval
- On single elements from *Interval* :

$$\omega([a, b]) = [\ max\{i \in B \mid i \leq a\},\ min\{i \in B \mid b \leq i\}\ ]$$

$$\omega(\perp) = \perp$$

# Divergence in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [8,8], y \to [0,1]]$
$[x \to [8,8], y \to [0,2]]$
$[x \to [8,8], y \to [0,3]]$
…

# Widening in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
   x = 7;
   x = x+1;
   y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

# Correctness of widening

- Widening works when:
  - $\omega$ is an *extensive* and *monotone* function, and
  - $\omega(L)$ is a *finite-height* lattice
- Safety: $\forall i: f^i(\bot, \bot, \ldots, \bot) \sqsubseteq (\omega \circ f)^i(\bot, \bot, \ldots, \bot)$
  since f is monotone and $\omega$ is extensive
- $\omega \circ f$ is a monotone function $\omega(L) \to \omega(L)$
  so the fixed-point exists
- Almost "correct by definition"!
- When used in the worklist algorithm, it suffices to apply widening on back-edges in the CFG

# Narrowing

- Widening generally shoots over the target
- *Narrowing* may improve the result by applying f
- Define:

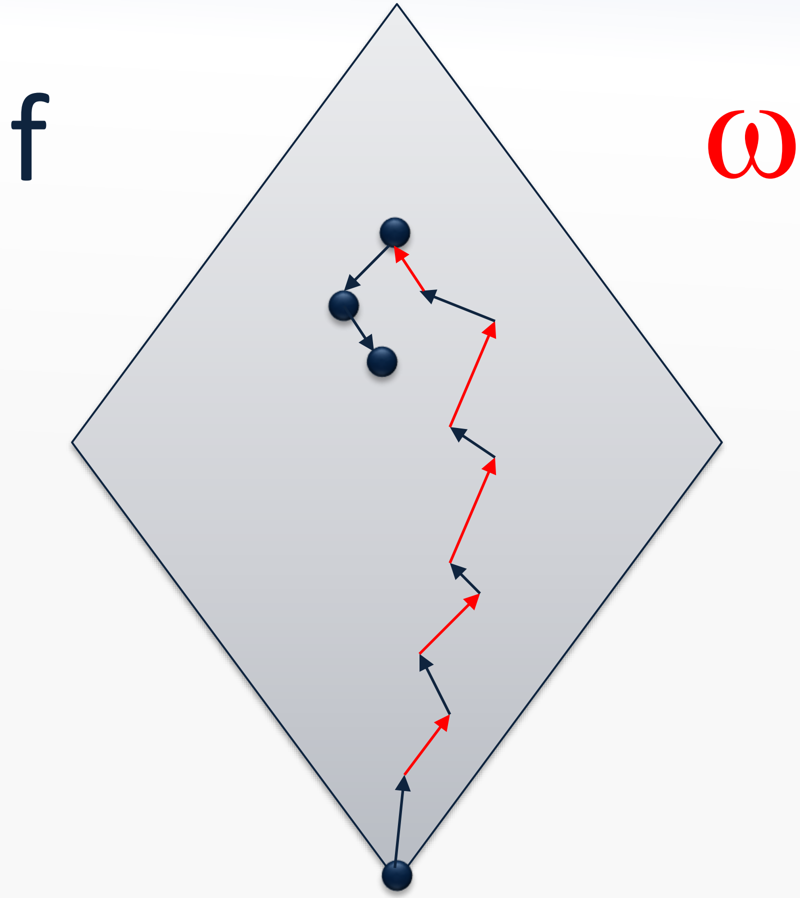
  $$fix = \bigsqcup f^i(\bot, \bot, \ldots, \bot) \qquad fix\omega = \bigsqcup (\omega \circ f)^i(\bot, \bot, \ldots, \bot)$$

  then $fix \sqsubseteq fix\omega$
- But we also have that

  $$fix \sqsubseteq f(fix\omega) \sqsubseteq fix\omega$$

  so applying f again may improve the result and remain sound!
- This can be iterated arbitrarily many times
  – may diverge, but safe to stop anytime

# Backing up



# Narrowing in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$
…
$[x \to [8,8], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

# Correctness of (repeated) narrowing

- $f(\mathit{fix}\omega) \sqsubseteq \omega(f(\mathit{fix}\omega)) = (\omega \circ f)(\mathit{fix}\omega) = \mathit{fix}\omega$
  since $\omega$ is extensive
  - by induction we also have, for all i:

    $f^{i+1}(\mathit{fix}\omega) \sqsubseteq f^{i}(\mathit{fix}\omega) \sqsubseteq \mathit{fix}\omega$
  - i.e. $f^{i+1}(\mathit{fix}\omega)$ is at least as precise as $f^{i}(\mathit{fix}\omega)$
- $\mathit{fix} \sqsubseteq \mathit{fix}\omega$ hence $f(\mathit{fix}) = \mathit{fix} \sqsubseteq f(\mathit{fix}\omega)$
  by monotonicity of f
  - by induction we also have, for all i:

    $\mathit{fix} \sqsubseteq f^{i}(\mathit{fix}\omega)$
  - i.e. $f^{i}(\mathit{fix}\omega)$ is a sound approximation of *fix*

# More powerful widening

- Defining the widening function based on constants occurring in the given program may not work

```
f(x) { // "McCarthy's 91 function"
  var r;
  if (x > 100) {
    r = x - 10;
  } else {
    r = f(f(x + 11));
  }
  return r;
}
```

https://en.wikipedia.org/wiki/McCarthy_91_function

- Note: this example requires interprocedural analysis…

# More powerful widening

- A *widening* is a function $\nabla$: $L \times L \rightarrow L$ that is extensive in both arguments and satisfies the following property:
  for all increasing chains $z_0 \sqsubseteq z_1 \sqsubseteq \ldots$,
  the sequence $y_0 = z_0, \ldots, y_{i+1} = y_i \nabla z_{i+1}, \ldots$ converges
  (i.e. stabilizes after a finite number of steps)

- Now replace the basic fixed point solver by computing $x_0 = \bot, \ldots, x_{i+1} = x_i \nabla F(x_i), \ldots$ until convergence

# More powerful widening for interval analysis

Extrapolates unstable bounds to B:

$$\bot \; \nabla \; y = y$$

$$x \; \nabla \; \bot = x$$

$$[a_1, b_1] \; \nabla \; [a_2, b_2] =$$

$$[\text{if } a_1 \le a_2 \text{ then } a_1 \text{ else } \mathit{max}\{i \in B \mid i \le a_2\},$$

$$\text{if } b_2 \le b_1 \text{ then } b_1 \text{ else } \mathit{min}\{i \in B \mid b_2 \le i\}]$$

The $\nabla$ operator on L is then defined pointwise down to individual intervals

For the small example program, we now get the same result as with simple widening plus narrowing (but now without using narrowing)