PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 31

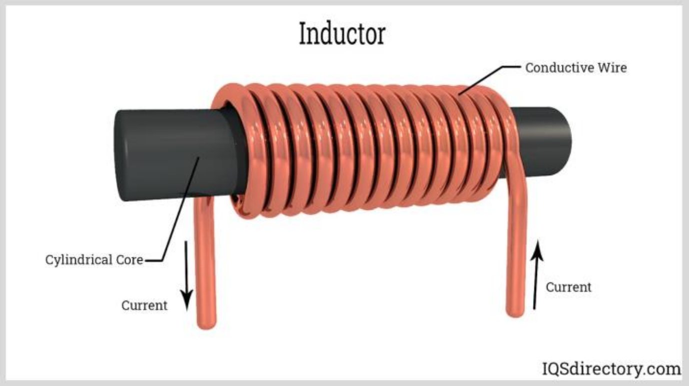
Inductor and Inductance
MAXWELL'S EQUATIONS

Inductor and Inductance

Inductor is an electrical device that is based on **Faraday's Flux Law**

If the current through the inductor changes, the magnetic flux through the wire will also change and will induce an EMF (from Faraday's flux law). The emf, in turn, induces current to flow through the inductor.

From Lenz's law, the induced current flows in a direction to oppose the initial change in the current.



Inductors are magnetic equivalent of the electrical capacitor, where they store magnetic energy.

Inductor and Inductance

Self- Inductance

The self-inductance 'L [H]' of an inductor is defined as the ratio of magnetic flux Ø linking the inductor to the current flowing through it:

$$L = \frac{\emptyset}{I}$$

Inductance is measured in henries (**H**); a henry is a volt-second per ampere.

The value of L depends on the geometry of the inductor.

***HW*:** *Determining the self-inductance of (a) N-turn solenoidal, (b) Parallel wires in the same circuit, c) Coaxial cable*

For inductors whose inductance is dependent on the current, the *differential* or *small signal inductance*

$$L_d = \frac{d\emptyset}{dI}$$

Inductor and Inductance

Back-emf

The induced emf in the inductor is called **back-emf.**
From Lenz's law, the **back-emf** acts in a direction that is opposite to the change.

From Faraday's Flux Law, the back emf is

$$\xi = -\frac{d\emptyset}{dt}$$

$$= -\frac{d\emptyset}{dI}\frac{dI}{dt}$$

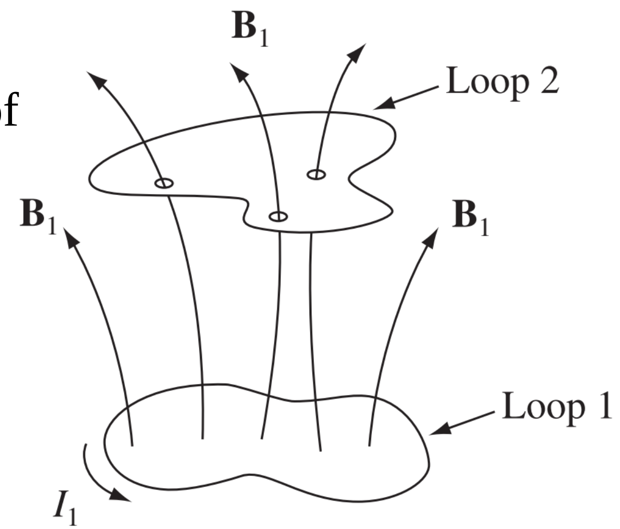
$$= -L\frac{dI}{dt}$$

"Self-inductance is the effect by which a changing current in a loop causes an emf in itself"

Inductor and Inductance

Mutual Inductance

Suppose you have two loops of wire, at rest. If we run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B_1}$. Some of the field lines pass through loop 2.



From Biot-Savart law

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\imath}}}{\imath^2}.$$

Let $\emptyset_2$ be the flux of $\mathbf{B_1}$ through 2.

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2$$

The change in $\mathbf{B_1}$ leads to a change in $\emptyset_2$, thus inducing an emf and current I_1

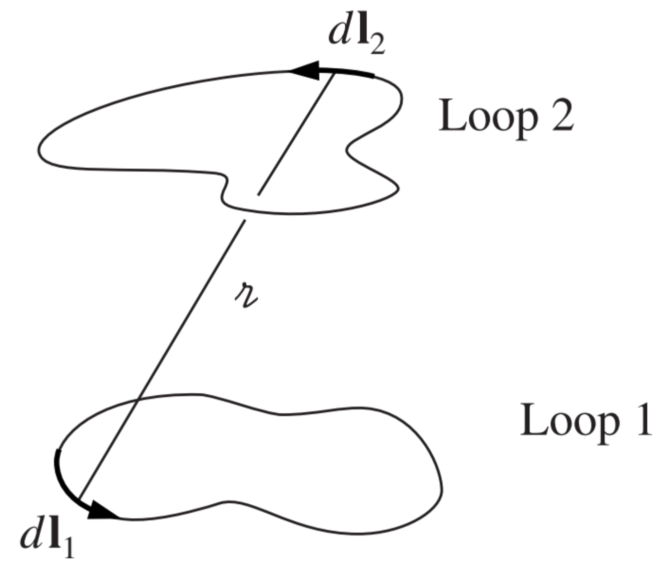
$$\xi = -\frac{d\emptyset_2}{dt} = -\frac{d\emptyset_2}{dI_1}\frac{dI_1}{dt} = M_{21}\frac{dI_1}{dt}$$

$\mathbf{M_{21}}$ is the constant of proportionality; it is known as the mutual inductance of the two loops

"Mutual Inductance is the effect by which a changing current in one loop causes an emf in another loop. This effect is symmetrical between the loops"

Mutual Inductance

There is a cute formula for the mutual inductance, which you can derive by expressing the flux in terms of the vector potential, and invoking Stokes' theorem:



$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2.$$

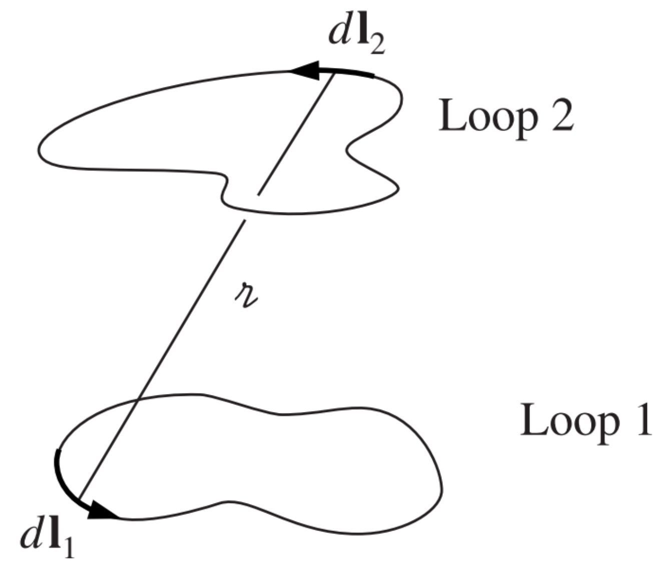
$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}}$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}.$$

This is the **Neumann formula**; it involves a double line integral—one integration around loop 1, the other around loop 2.

Inductor and Inductance

Mutual Inductance



Neumann formula

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}.$$

1. M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

2. The integral is unchanged if we switch the roles of loops 1 and 2; it follows that

$$M_{21} = M_{12}$$

Whatever the shapes and positions of the loops, the flux through 2 when we run a current *I* around 1 is identical to the flux through 1 when we send the same current I around 2.

Energy in Magnetic Fields

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\xi$

The amount of charge per unit time passing down the wire is I.

So the total work done per unit time is:

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}.$$

Total work done

$$\boxed{W = \frac{1}{2}LI^2.}$$

Energy in Magnetic Fields

Let Ø be the flux through the loop

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l},$$

where the line integral is around the perimeter of the loop. Thus

$$LI = \oint \mathbf{A} \cdot d\mathbf{l},$$

Therefore

$$W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, dl.$$

Energy in Magnetic Fields

In this form, the generalization to volume currents is obvious:

$$W = \frac{1}{2}\int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J})\, d\tau$$

From Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$,

$$W = \frac{1}{2\mu_0}\int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

From the Product rule

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B}).$$

Energy in Magnetic Fields

$$W = \frac{1}{2\mu_0}\left[\int B^2\, d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau\right]$$

$$= \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\, d\tau - \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}\right],$$

where $\mathcal{S}$ is the surface bounding the volume $\mathcal{V}$.

When integrating over all space, the volume integral dominates over the surface integral

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau.}$$

The energy is "**stored in the magnetic field**"

Comparison of energy stored in electrostatic field and magnetic field

$$W_{\mathrm{elec}} = \frac{1}{2}\int (V\rho)\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau,$$

$$W_{\mathrm{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\,d\tau = \frac{1}{2\mu_0}\int B^2\,d\tau.$$

Electrodynamics Before Maxwell

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \text{(Gauss's law),}$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0 \qquad \text{(no name),}$$

$$\text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law),}$$

$$\text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{(Ampère's law).}$$