Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 10

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1. A steady current I flows down a long cylindrical conductor of radius a. The current density at a distance r from the axis of the conductor is proportional to r. Calculate the magnetic field both inside and outside as a function of r.

Solution. Let us consider an Amperian loop in the form of a circle of radius $r (r < a)$ with its centre on the axis of the cylinder. Symmetry of the problem indicates that $\vec{B}$ is tangential to the Amperian loop everywhere and also of constant magnitude all over it. So by applying Ampere's law we get

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

$$\text{or} \quad B \cdot 2\pi r = \mu_0 \int_0^r 2\pi r dr \cdot J(r) = 2\pi\mu_0 K \int_0^r r^2 dr = 2\pi\mu_0 K \frac{r^3}{3},$$

where we assume that $J(r) = Kr$, K being a proportionality constant.

$$\therefore \quad B = \mu_0 K \frac{r^3}{3} \text{ for } r \leq a.$$

For any external point $r > a$, $I_{\text{enc}} = I$ and then

$$B \cdot 2\pi r = \mu_0 I \quad \text{or} \quad B = \frac{\mu_0 I}{2\pi r} \text{ for } r \geq a.$$

Total current

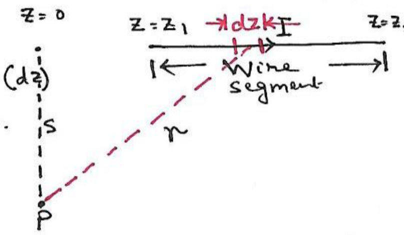
$$I = \int_0^a 2\pi r dr \cdot J(r) = 2\pi K \int_0^a r^2 dr = 2\pi K \frac{a^3}{3} \quad \text{or} \quad K = \frac{3I}{2\pi a^3}.$$

Thus,

$$B = \frac{\mu_0 I r^2}{2\pi a^3} \text{ for } r \leq a.$$

2. Find the magnetic vector potential of a finite segment of straight wire carrying a current I. Using the expression of the vector potential find the expression of magnetic field.

[Hint: Put the wire on the z-axis from z_1 to z_2 and use the relation $A = \frac{\mu_0}{4\pi}\int \frac{Idl}{r}$]



1. Let the wire be on z-axis and its length spans from $z = z_1$ to $z = z_2$. Consider an element $(d\vec{z})$ of current-carrying wire at a distance z from $z = 0$. We would like to calculate vector potential at P due to current element dz

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\,d\vec{l}'}{r}$$. Here I = constant, can be taken out of integral $d\vec{l} = dz(\hat{z})$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{z=z_1}^{z_2} \frac{dz}{\sqrt{s^2+z^2}}(\hat{z}) = \frac{\mu_0 I}{4\pi} \ln\left|z + \sqrt{s^2+z^2}\right|_{z_1}^{z_2}(\hat{z})$$

$$= \frac{\mu_0 I}{4\pi} \ln\left(\frac{z_2 + \sqrt{s^2+z_2^2}}{z_1 + \sqrt{s^2+z_1^2}}\right)\hat{z}$$

. We will calculate $\vec{B}$ due to $\vec{A}$ and check whether it matches with $\vec{B}$ due to finite straight wire as derived in the class. Use the relation $\vec{B} = \vec{\nabla}\times\vec{A}$ and use expression of curl in cylindrical co-ordinates (s, φ, z)

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\varphi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\varphi) - \frac{\partial A_s}{\partial\varphi}\right]\hat{z}$$

But in this problem, we only have $A_z(s)$

$$\vec{\nabla}\times\vec{A} = \vec{B} = -\frac{\partial A_z}{\partial s}\hat{\varphi}$$

$$\vec{B} = -\frac{\mu_0 I}{4\pi}\left[\frac{z_1+\sqrt{s^2+z_1^2}}{z_2+\sqrt{s^2+z_2^2}} \times \left\{\frac{(z_1+\sqrt{s^2+z_1^2}) \times \frac{2s}{2\sqrt{s^2+z_2^2}} - (z_2+\sqrt{s^2+z_2^2})\frac{s}{\sqrt{s^2+z_1^2}}}{(z_1+\sqrt{s^2+z_1^2})^2}\right\}\right] \times \hat{\varphi}$$

$$= -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{s^2+z_2^2}}\left(\frac{s}{\sqrt{s^2+z_2^2}}\right) - \frac{s}{\sqrt{s^2+z_1^2}}\left(\frac{1}{z_1+\sqrt{s^2+z_1^2}}\right)\right]\hat{\varphi}$$

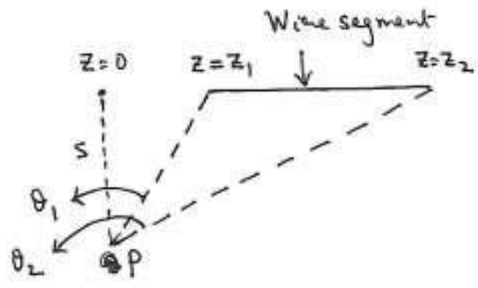
$\downarrow$ Multiply and divide by $(z_2 - \sqrt{s^2+z_2^2})$

$\downarrow$ Multiply and divide by $(z_1 - \sqrt{s^2+z_1^2})$

$$\vec{B} = -\frac{\mu_0 I}{4\pi}\,\hat{\varphi}\left[\frac{s}{\sqrt{s^2+z_2^2}}\,\frac{\left(z_2-\sqrt{s^2+z_2^2}\right)}{\left[\cancel{z_2^2}-(s^2+\cancel{z_2^2})\right]} - \frac{s}{\sqrt{s^2+z_1^2}}\,\frac{\left(z_1-\sqrt{s^2+z_1^2}\right)}{\left[\cancel{z_1^2}-(s^2+\cancel{z_1^2})\right]}\right]$$

$$= \frac{\mu_0 I s}{4\pi s^2}\left[\frac{1}{\sqrt{s^2+z_2^2}}\left(z_2-\sqrt{s^2+z_2^2}\right) - \frac{1}{\sqrt{s^2+z_1^2}}\left(z_1-\sqrt{s^2+z_1^2}\right)\right]\hat{\varphi}$$

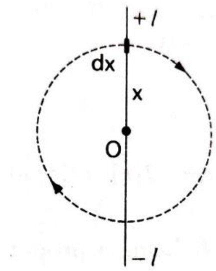
$$\vec{B} = \frac{\mu_0 I}{4\pi s}\left[\underbrace{\frac{z_2}{\sqrt{s^2+z_2^2}}}_{\sin\theta_2} - \underbrace{\frac{z_1}{\sqrt{s^2+z_1^2}}}_{\sin\theta_1}\right]\hat{\varphi}$$



$$\boxed{\vec{B} = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\varphi}}$$

which tallies with the expression of $\vec{B}$ for a finite length current-carrying wire obtained using Biot-Savart law, as done in class.

3. A straight wire of length $2l$ carries a charge λ per unit length. It rotates uniformly with an angular velocity ω about an axis passing through its mid-point and perpendicular to its length. Show that the equivalent magnetic dipole moment is of magnitude $(1/3)\lambda\omega l^3$.



Solution. Consider an element dx at a distance x from the centre O (Fig. 8.P-10). Charge on it is λdx and it is rotating $\omega/2\pi$ times per sec in a circle of radius x. So the current produced is

$$I = \lambda dx \times \frac{\omega}{2\pi}$$

and the associated magnetic moment is

$$I \times \pi x^2 = \tfrac{1}{2}\lambda\omega x^2 dx.$$

The total dipole moment is, therefore,

$$\int_{-l}^{+l} \tfrac{1}{2}\lambda\omega x^2 dx = \tfrac{1}{3}\lambda\omega l^3.$$

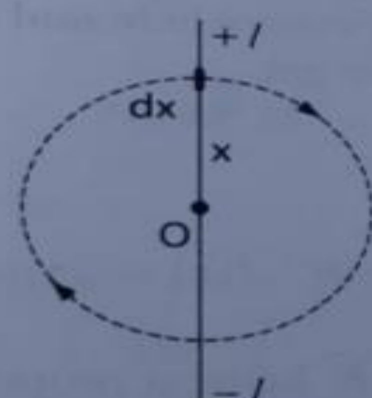
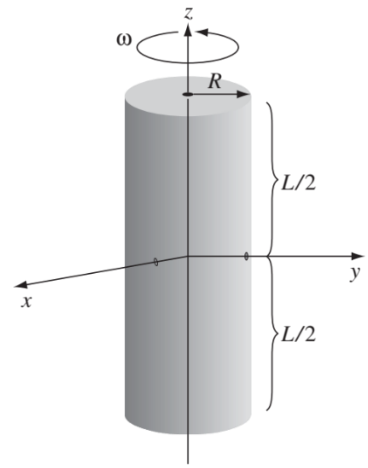


Fig 8.P-10

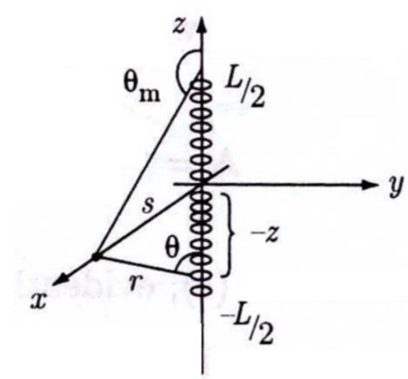
3. A thin glass rod of radius R and length L carries a uniform surface charge σ. It is set spinning about its axis, at an angular velocity ω. Find the magnetic field at a distance s ≫ R from the axis, in the xy plane as shown in the figure.
[Hint: treat it as a stack of magnetic dipoles.]



For a dipole at the origin and a field point in the xz plane ($\phi = 0$), we have

$$\mathrm{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{\mu_0}{4\pi}\frac{m}{r^3}[2\cos\theta(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}) + \sin\theta(\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{z}})]$$
$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}[3\sin\theta\cos\theta\,\hat{\mathbf{x}} + (2\cos^2\theta - \sin^2\theta)\,\hat{\mathbf{z}}].$$

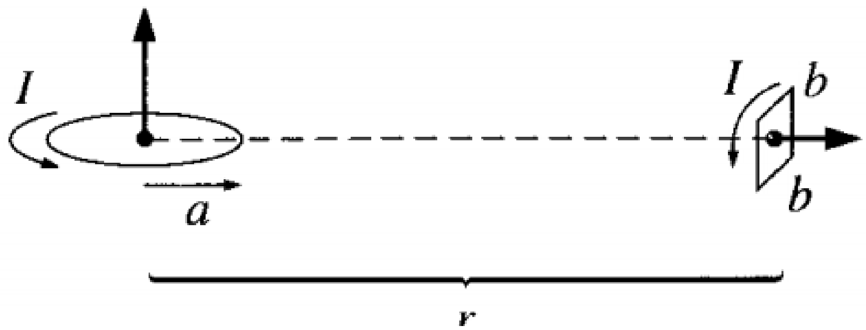
Here we have a *stack* of such dipoles, running from $z = -L/2$ to $z = +L/2$. Put the field point at s on the x axis. The $\hat{\mathbf{x}}$ components cancel (because of symmetrically placed dipoles above and below $z = 0$), leaving $\mathbf{B} = \frac{\mu_0}{4\pi} 2\mathcal{M}\,\hat{\mathbf{z}} \int_0^{L/2} \frac{(3\cos^2\theta - 1)}{r^3}\,dz$, where $\mathcal{M}$ is the dipole moment per unit length: $m = I\pi R^2 = (\sigma v h)\pi R^2 = \sigma\omega R\pi R^2 h \Rightarrow \mathcal{M} = \frac{m}{h} = \pi\sigma\omega R^3$. Now $\sin\theta = \frac{s}{r}$, so $\frac{1}{r^3} = \frac{\sin^3\theta}{s^3}$; $z = -s\cot\theta \Rightarrow dz = \frac{s}{\sin^2\theta}\,d\theta$. Therefore



$$\mathbf{B} = \frac{\mu_0}{2\pi}(\pi\sigma\omega R^3)\,\hat{\mathbf{z}} \int_{\pi/2}^{\theta_m} (3\cos^2\theta - 1)\frac{\sin^3\theta}{s^3}\frac{s}{\sin^2\theta}\,d\theta = \frac{\mu_0\sigma\omega R^3}{2s^2}\,\hat{\mathbf{z}} \int_{\pi/2}^{\theta_m} (3\cos^2\theta - 1)\sin\theta\,d\theta$$
$$= \frac{\mu_0\sigma\omega R^3}{2s^2}\,\hat{\mathbf{z}}\left(-\cos^3\theta + \cos\theta\right)\Big|_{\pi/2}^{\theta_m} = \frac{\mu_0\sigma\omega R^3}{2s^2}\cos\theta_m\left(1 - \cos^2\theta_m\right)\hat{\mathbf{z}} = \frac{\mu_0\sigma\omega R^3}{2s^2}\cos\theta_m\sin^2\theta_m\,\hat{\mathbf{z}}.$$

But $\sin\theta_m = \frac{s}{\sqrt{s^2 + (L/2)^2}}$, and $\cos\theta_m = \frac{-(L/2)}{\sqrt{s^2 + (L/2)^2}}$, so $\boxed{\mathbf{B} = -\frac{\mu_0\sigma\omega R^3 L}{4[s^2 + (L/2)^2]^{3/2}}\,\hat{\mathbf{z}}.}$

5. Calculate the torque exerted on the square loop shown in the figure, due to the circular loop (assume r is much larger than a or b). If the square loop is free to rotate, what will the equilibrium orientation be?



$$\mathbf{N} = \mathbf{m}_2 \times \mathbf{B}_1;\ \mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1\right];\ \hat{\mathbf{r}} = \hat{\mathbf{y}}; \mathbf{m}_1 = m_1\hat{\mathbf{z}};\ \mathbf{m}_2 = m_2\hat{\mathbf{y}}.\quad \mathbf{B}_1 = -\frac{\mu_0}{4\pi}\frac{m_1}{r^3}\hat{\mathbf{z}}.$$

$\mathbf{N} = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}(\hat{\mathbf{y}} \times \hat{\mathbf{z}}) = -\frac{\mu_0}{4\pi}\frac{m_1 m_2}{r^3}\hat{\mathbf{x}}$. Here $m_1 = \pi a^2 I$, $m_2 = b^2 I$. So $\boxed{\mathbf{N} = -\frac{\mu_0}{4}\frac{(abI)^2}{r^3}\hat{\mathbf{x}}.}$ Final orientation: $\boxed{\text{downward}}$ $(-\hat{\mathbf{z}})$.