Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 5

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1. Suppose the electric field in some region is found to be $\mathbf{E} = \mathrm{k}r^3\hat{\mathbf{r}}$ in spherical coordinates (k is some constant).

a) Find the charge density ρ.

b) Find the total charge contained in a sphere of radius R, centered at the origin.

c)

(a) $\rho = \epsilon_0 \nabla\cdot\mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \epsilon_0 \frac{1}{r^2} k(5r^4) = \boxed{5\epsilon_0 k r^2.}$

(b) *By Gauss's law:* $Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \epsilon_0(kR^3)(4\pi R^2) = \boxed{4\pi\epsilon_0 k R^5.}$

By direct integration: $Q_{\text{enc}} = \int \rho\, d\tau = \int_0^R (5\epsilon_0 k r^2)(4\pi r^2 dr) = 20\pi\epsilon_0 k \int_0^R r^4 dr = 4\pi\epsilon_0 k R^5.$✓

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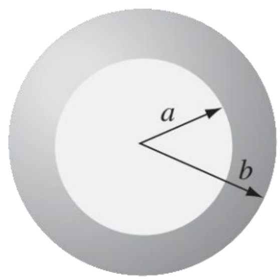
2. A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2}\ ;\ (a \le r \le b)$$



a) Find the electric field in the three regions:

(i) r < a, (ii) a < r < b, (iii) r > b.

b) Plot **|E|** as a function of ***r***, for the case ***b = 2a***

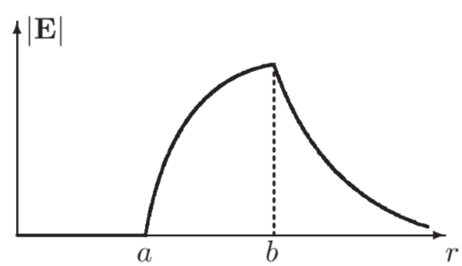
(i) $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$

(ii) $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int \frac{k}{\bar{r}^2}\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\phi$

$= \frac{4\pi k}{\epsilon_0}\int_a^r d\bar{r} = \frac{4\pi k}{\epsilon_0}(r-a)\ \therefore\ \boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.}$

(iii) $E(4\pi r^2) = \frac{4\pi k}{\epsilon_0}\int_a^b d\bar{r} = \frac{4\pi k}{\epsilon_0}(b-a)$, so

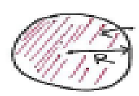
$\boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.}$



3. The volume charge density of a solid sphere of radius R varies as $\rho = \rho_0\left(\frac{r}{R}\right)$, where ρ_0 is a constant (of appropriate unit) and r is the radial distance measured from the center of the sphere. Find out the electric field at a distance $'s'$ from the center of the sphere using the ***Gauss's law***. Consider both $0 \le s \le R$ and $s \ge R$ cases.

1. The solid sphere has a volume charge density $\rho = \rho_0 \frac{r}{R}$ where r is the radial distance in spherical polar co-ordinates

Inside sphere: $0 \leq s \leq R$



Consider a Gaussian sphere inside the sphere of radius R. Apply Gauss's law over the Gaussian surface:

$$\oiint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} \quad \text{—— (1)}$$

To find Q_{enc} within the sphere of radius 's', consider volume element $d\tau$ at $\vec{r}'$ from O

$$E_{in}\, 4\pi s^2 = \frac{1}{\epsilon_0}\int \rho(r')\, d\tau'$$ [primed variables refers to point inside 's']

$$= \frac{1}{\epsilon_0}\int_{r'=0}^{s} \rho_0 \frac{r'}{R}\, r'^2\, dr' \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$ [$d\tau' = r'^2 dr' \sin\theta\, d\theta\, d\phi$]

$$= \frac{1}{\epsilon_0}\int_0^{s}\left(\frac{\rho_0 r'}{R}\right) 4\pi r'^2\, dr' = \frac{4\pi\rho_0}{\epsilon_0 R}\int_0^{s} r'^3\, dr' = \frac{4\pi\rho_0}{\epsilon_0 R}\frac{s^4}{4}$$

$$\therefore \boxed{\vec{E}_{in} = \frac{\rho_0 s^2}{\epsilon_0 R}\hat{s}} \quad \text{—— (2)}$$ [here $\hat{s}$ is also the radial direction]

Outside sphere $s > R$

Consider Gaussian sphere of radius 's' such that $s > R$. Apply Gauss's law over this surface:

$$\oiint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}.$$

$Q_{enc} = \int \rho(r')\, d\tau' = \int_{r'=0}^{R} \rho(r')\, 4\pi r'^2\, dr'$ [4π factor from integration of θ and ϕ in $d\tau'$]

Similar to above, $d\tau'$ volume element can be considered to be at $\vec{r}'$ from O inside the sphere - except limits of r' will be from 0 to R because volume charge density (source) is only confined with the sphere. (1)

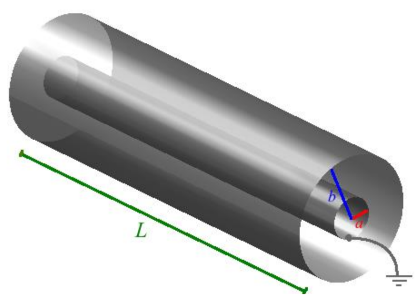
$$\therefore E_{out}\, 4\pi s^2 = \frac{1}{\epsilon_0}\int_{r'=0}^{R} \rho(\vec{r}')\, 4\pi r'^2\, dr' = \frac{\rho_0}{\epsilon_0}\frac{4\pi}{R}\int_0^{R} r'^3\, dr'$$

$$= \frac{4\pi\rho_0}{\epsilon_0 R}\frac{R^4}{4}$$

$$\therefore \boxed{\vec{E}_{out} = \frac{\rho_0 R^3}{\epsilon_0 s^2}\hat{r}} \quad \text{—— (3)}$$

Results (2) and (3) are easy to arrive at compared to the integration method adopted to the same problem in Tutorial 6. This explains the power of Gauss's law when symmetry is involved in the problem.

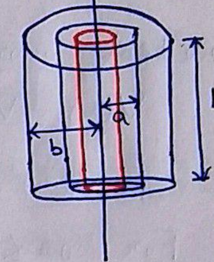
4. (a) Consider two conducting coaxial cylindrical shells of radii a and b, (a < b), as shown in figure below. The length of both cylinders is L which much larger than (b-a), the separation between the cylinders, so that edge effects can be neglected. The inner cylinder is grounded (electric potential = 0), while the outer cylinder is supplied a charge -Q which gets distributed uniformly on the surface (Again, we are neglecting edge effects). As a consequence a charge +Q (drawn from the ground) gets induced uniformly on the inner cylinder. Calculate the electric field as well as the electric potential in the regions (i) $0 \leq r < a$, (ii) $a < r < b$, and (iii) $r \geq b$. Here r is the radial (perpendicular) distance measured from the common axis of the cylinders.
[For the regions (i) and (iii) assume, in addition, that L >>a, b. Moreover, for region (iii) the observation point should be close to the outer cylinder | it should not be at a distance comparable to- or larger than the system dimension, i.e., $r - b << L$.]



(b) Verify the discontinuity of electric field and continuity of the electric potential at the surfaces of the two cylindrical shells in the above problem.

(c) What is the capacitance of the coaxial cylinder system? Does the answer depend on Q?

(a) Region (i) $0 \leq r < a$.



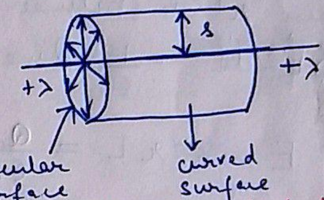
The inner cylinder has a charge $+Q$ while the outer cylinder has charge $-Q$.

Consider a Gaussian cylinder of radius $r < a$ and length L. Apply Gauss's law

$$\oint\!\!\oint \vec{E} \cdot \vec{da} = \frac{1}{\epsilon_0} Q_{enc} \Rightarrow E \cdot 2\pi r L = 0$$

Thus $E = 0$

Note that in a cylindrical symmetry, the field acts radially outward. For e.g. consider a thick wire of line charge λ. Direction of field are as shown in arrows. It is clear that there is contribution of electric flux through the curved surface only and not through the circular surfaces. This has been discussed in class.



(do not contribute to flux as lines of $\vec{E}$ are perpendicular to $\vec{da}$)

(contributes to flux)

To calculate potential in cylindrical symmetry, we have to be careful choosing the reference point. Normally from the definition of potential $V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dl} = V(\vec{r})$, we choose 'infinity' as the reference point because $V(r \to \infty) = 0$. But in cylindrical cases, because of logarithmic dependence of potential V on 's' (like in the above case of a straight wire of line charge density λ), choosing 'infinity' as the reference point is no good idea.

Let's choose $V(a)$ (which is zero) to be the reference point.

So for $r < a$, $V(r) - V(a) = -\int_{a}^{r} \vec{E} \cdot \vec{dl}$. Note that $\vec{dl}$ is

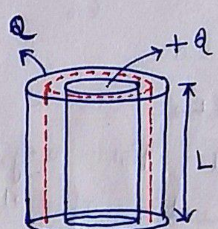
referred to as line element in cylindrical co-ordinates
$\vec{dl} = ds(\hat{s}) + s\,d\varphi(\hat{\varphi}) + dz(\hat{z})$. For $\vec{E}$ depending on radial co-ordinate ($\hat{s}$) only, only first term in $\vec{dl}$ appears.

$$V(r) - V(a) = -\int_a^r \vec{E}\cdot\vec{dl} = 0 \quad [\because E = 0 \text{ inside inner cylinder}]$$

$$\therefore V(r) = V(a) = 0 \quad [\because \text{inner cylinder is grounded}]$$

For $0 \le r < a$: $\boxed{\begin{matrix} V = 0 \\ E = 0 \end{matrix}}$

(ii) Region (ii) $a < r < b$



Consider a Gaussian cylinder of radius 'r' in between the two cylinders

$$\oiint_{\text{curved}} \vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0} Q_{enc}$$

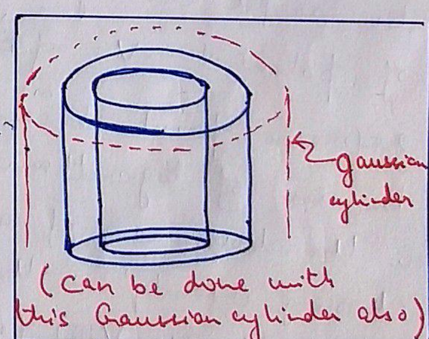
$$E \times 2\pi r L = \frac{Q}{\epsilon_0} \Rightarrow \boxed{E = \frac{Q}{2\pi\epsilon_0 r L}}$$

To calculate potential in region $a < r < b$

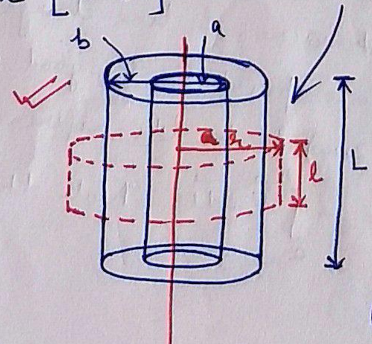
$$V(r) - V(a) = -\int_a^r \vec{E}\cdot\vec{dl} = -\int_a^r \frac{Q}{2\pi\epsilon_0 r L}\,dr = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{r}\right)$$

$$\therefore \boxed{V(r) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{r}\right)} \quad a < r < b$$

(iii) Region (iii) $r \geq b$



Consider this time a Gaussian cylinder of length l $[l < L]$ outside the coaxial cylinders



Apply Gauss's law over this surface

$$\underset{\text{curved}}{\oiint} \vec{E}\cdot\vec{da} = \frac{1}{\epsilon_0} Q_{enc}$$

$$E\cdot 2\pi r l = \frac{1}{\epsilon_0}\left[\frac{Q}{2\pi a L}\times 2\pi a l + -\frac{Q}{2\pi b L}\times 2\pi b l\right]$$

↑ surface charge density on inner conducting cylinderical shell

↑ surface charge density due to outer conducting cylinderical shell

$$\therefore E\cdot 2\pi r l = \frac{1}{\epsilon_0}\left(Q\frac{l}{L} - Q\frac{l}{L}\right) = 0$$

$$\therefore \boxed{E = 0}$$

[Note that you could have used the same procedure, i.e considering Gaussian cylinder of length 'l' ($l < L$) to calculate the field $\vec{E}$ in region (ii) above – it would give you the same result]

To calculate potential in the region $r \geq b$, we can do it in two ways

i) continuity of electrical potential across a charged interface – Since we know in region (ii) $a < r < b$, $V(r) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{r}\right)$

So on the outer surface (charged) $-Q$, the potential is;

$V(r = b) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)$. So in principle outside $r = b$, the potential would be the same i.e $\boxed{V(r) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)}$, $r \geq b$

iii) $V(r) - V(a) = -\int_a^r \vec{E}\cdot\vec{dl}$; $r \geq b$

$$= -\int_a^b \vec{E}\cdot\vec{dl} - \underbrace{\int_b^r \vec{E}\cdot\vec{dl}}_{0} = -\int_a^b \vec{E}\cdot\vec{dl} = -\int_a^b \frac{Q}{2\pi\epsilon_0 r L}dr = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)$$

$$\therefore V(r) - V(a) = \boxed{V(r) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)}$$

(b) At the surface of inner ~~[illegible]~~ cylinder (radius 'a')

the discontinuity in $\vec{E}$ will be given by

$$\lim_{\epsilon \to 0} \left[\vec{E}(r = a+\epsilon) - \vec{E}(r = a-\epsilon) \right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{Q}{2\pi\epsilon_0 L(a+\epsilon)} \hat{r} - 0 \right] \qquad [\because \vec{E} = 0 \text{ for } r < a]$$

$$= \frac{Q}{2\pi\epsilon_0 L a} \hat{r} = \hat{r} \left(\frac{Q/2\pi a L}{\epsilon_0} \right) = \frac{\sigma}{\epsilon_0} \hat{r}$$

$[\sigma =$ surface charge density on inner cylinder $= \frac{Q}{2\pi a L}]$

which is consistent with the general result $(\sigma/\epsilon_0)\, \hat{n}$

For electric potential on the surface of inner cylinder $(r = a)$

$$\lim_{\epsilon \to 0} \left[V(r = a+\epsilon) - V(r = a-\epsilon) \right] = \lim_{\epsilon \to 0} \left[\frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{a+\epsilon}\right) - 0 \right]$$

$$= \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{a}\right) \qquad \left[\because V = 0 \text{ inside } r = a\right]$$

$$= 0$$

Thus V is continuous as expected

Now we have to prove for outer cylinder (radius, $r = b$):

Discontinuity in electric field is given by

$$\lim_{\epsilon \to 0} \left[E(r = b+\epsilon) - E(r = b-\epsilon) \right]$$

$$= \lim_{\epsilon \to 0} \left[0 - \frac{Q}{2\pi\epsilon_0 L(b-\epsilon)} \right] = -\frac{Q}{2\pi\epsilon_0 L b} = \frac{-Q/2\pi L b}{\epsilon_0} = \frac{\sigma_b}{\epsilon_0}$$

$\left[\because \sigma_b = \frac{-Q}{2\pi b L} =\right.$ surface charge density of outer cylinder with charge $-Q]$

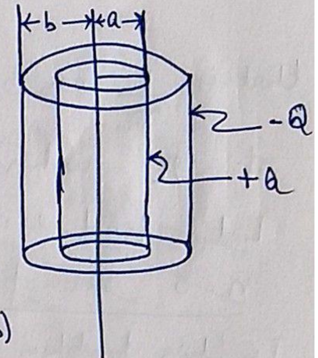
For potential

$$\lim_{\epsilon \to 0} \left[V(r = b+\epsilon) - V(r = b-\epsilon) \right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b}\right) - \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b-\epsilon}\right) \right]$$

$$= \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b}\right) - \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{a}{b}\right) = 0$$

(c) To find the capacitance of the co-axial cylinder system (inner cylinder $+Q$, outer cylinder $-Q$)



Let's calculate the voltage between two cylinders.

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot \vec{dl}$$

We have already computed the electric field in the region between two cylinders $(a < r < b)$ to be $\vec{E} = \dfrac{Q}{2\pi \epsilon_0 r L} \hat{r}$

$$V = -\int_{r=b}^{a} \frac{Q}{2\pi \epsilon_0 r L} \, dr = \frac{Q}{2\pi \epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

$$= \frac{Q}{C}$$

$$\therefore \boxed{C = \frac{2\pi \epsilon_0 L}{\ln(b/a)}}$$

Capacitance is independent of total charge Q - it depends upon the geometry of the system (i.e shapes/sizes of conductors)