PHY101: Introduction to Physics I

Monsoon Semester 2024

Lecture 19

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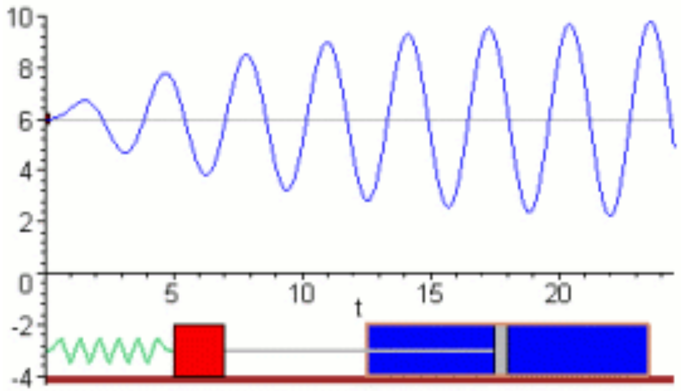
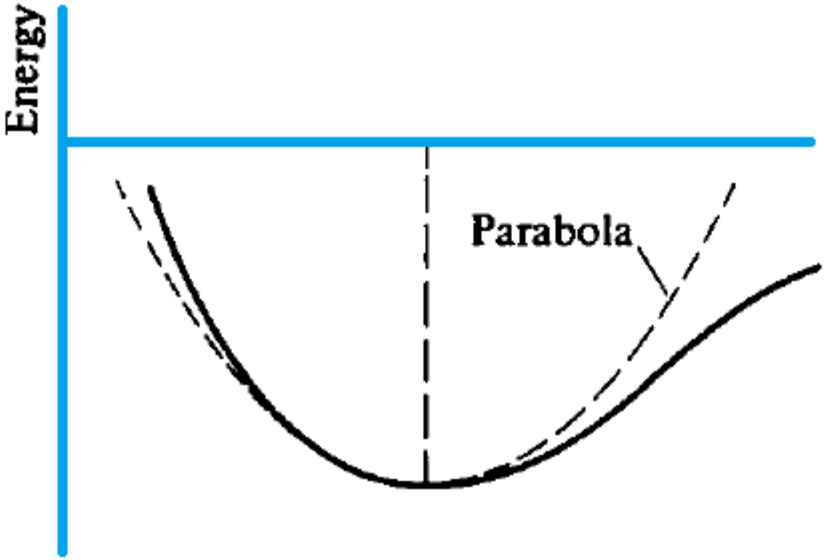
Previous Lecture

Driven harmonic motion



This Lecture

Energy involved in harmonic motion
Bound system in a potential
Small oscillation



Energy associated with a particle in one-dimension motion

Consider a one-dimensional system with a given energy E (conserved) and potential energy $U(x)$.

Energy conservation gives the kinetic energy as

$$K(x) = E - U(x)$$

$$\Rightarrow v = \sqrt{\frac{2}{m}(E - U(x))}$$

As KE cannot be negative (v cannot be imaginary) the motion must be restricted to a region where $\boldsymbol{E} \geq \boldsymbol{U}(\boldsymbol{x})$.

Energy associated with a particle in S. H. M.

Consider the potential energy associated with Simple Harmonic Motion (S.H.M.)

$$U(x) = \frac{1}{2}kx^2; \;\; k > 0$$

If E is the total energy, then

$$K(x) = E - U(x) = E - \frac{1}{2}kx^2$$

As $U(x) \geq 0$, for a positive $K(x)$ we must have

$E > 0$ for this system.

($E = 0$ is not of our interest)

Bound system in a potential (the general expression)

Consider a bound system with potential energy $U(x)$.

The equilibrium (stable) point is obtained for

$$\left.\frac{dU(x)}{dx}\right|_{x=x_0} \equiv U'(x_0) = 0$$

Assuming that $U(x)$ is a *well behaved function,* it can be Taylor-expanded around x_0, i.e.,

$$\boldsymbol{U(x) = U(x_0) + (x - x_0)U'(x_0) + \frac{1}{2!}(x - x_0)^2 U''(x_0) + \cdots}$$

Bound system in a potential (in presence of small perturbation)

Since $U'(x_0) = 0$, we get

$$\boldsymbol{U(x) = U(x_0) + \frac{1}{2!}(x - x_0)^2 U''(x_0) + \frac{1}{3!}(x - x_0)^3 U'''(x_0) + \cdots}$$

Consider small displacement around the equilibrium point x_0, i.e., $x - x_0 \sim 0$. Then we may neglect $(x - x_0)^3$ and higher order terms ,

$$\boldsymbol{U(x) = U(x_0) + \frac{1}{2!}(x - x_0)^2 U''(x_0)}$$

$U(x_0)$ is just a constant, $U(x)$ can be shifted to absorb it.

Potential energy associated with S. H. M.

Thus we have

$$\boldsymbol{U(x) = \frac{1}{2} U''(x_0)(x - x_0)^2}$$

Comparing this with the potential energy associated with a simple harmonic oscillator with natural length position $\boldsymbol{x_0}$ and spring constant $\boldsymbol{k}$,

$$\boldsymbol{U(x) = \frac{1}{2} k(x - x_0)^2,}$$

we conclude that $\boldsymbol{k = U''(x_0)} = \left.\frac{d^2U(x)}{dx^2}\right]_{x=x_0}$.

Dependence of important parameters on $U''(x)$ for S. H. M.

Thus, for small oscillations around x_0:

Spring constant, $\boldsymbol{k = U''(x_0) = \left.\frac{d^2U(x)}{dx^2}\right]_{x=x_0}}$.

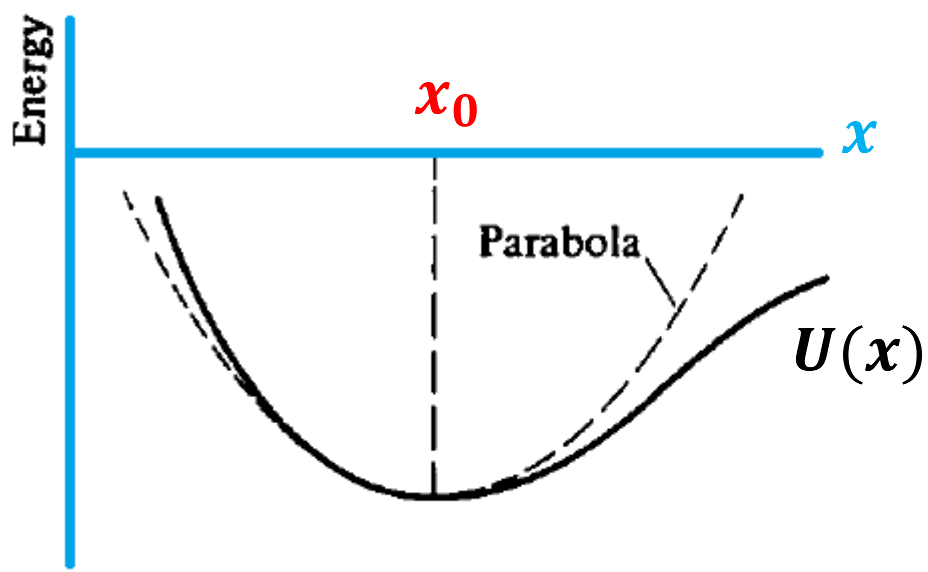
Angular frequency, $\boldsymbol{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}}$

Linear frequency, $\boldsymbol{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{U''(x_0)}{m}}}$

Time period, $\boldsymbol{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{U''(x_0)}}}$

Bound system in a potential (general property)

Nearly all bound systems oscillate like a simple harmonic oscillator when slightly perturbed from its equilibrium position: The potential energy curve looks like a simple harmonic potential (parabolic).



Questions

A particle of mass m is constrained to move along the positive x-axis under the influence of a single force whose potential energy function is

$$U(x) = -a\,x^2 \exp(-x^2/b^2),$$

where a, b are positive constants. (a) Find the equilibrium point(s). (b) For each stable equilibrium, calculate the frequency of small oscillations.

Solution: **(a)** At the equilibrium point(s), $F = -\frac{dU}{dx} = 0$

$$\Rightarrow -\frac{d}{dx}\left(-ax^2 e^{-\frac{x^2}{b^2}}\right) = 0$$

$$\Rightarrow \frac{2ae^{-\frac{x^2}{b^2}}}{b^2} x(b^2 - x^2) = 0$$

$$\Rightarrow x = 0, x = -b, x = b.$$

These are the equilibrium points.

Stability analysis:

We have $$\frac{d^2U}{dx^2} = -\frac{2ae^{-\frac{x^2}{b^2}}}{b^4}(b^4 - 5b^2x^2 + 2x^4).$$

Therefore, $\left[\frac{d^2U}{dx^2}\right]_{x=0} = -2a < 0$, since $a > 0$.

$\Rightarrow$ x= 0 is a point of unstable equilibrium.

$\left[\frac{d^2U}{dx^2}\right]_{x=\pm b} = \frac{4a}{e} > 0$, since $a > 0$.

$\Rightarrow$Both $x=-b$ and $x=b$ are a points of stable equilibrium.

(b) Time period of small oscillations around the equilibrium point (x_0) is given by

$$T = 2\pi \sqrt{\frac{m}{\left[\frac{d^2U}{dx^2}\right]_{x=x_0}}}$$

Here $x_0 = \pm b$ and $\left[\frac{d^2U}{dx^2}\right]_{x=\pm b} = \frac{4a}{e}$, so $T = 2\pi\sqrt{\frac{e\,m}{4a}}$.

Next Lecture

Centre of Mass