PHY 102 Introduction to Physics II

Spring Semester 2025

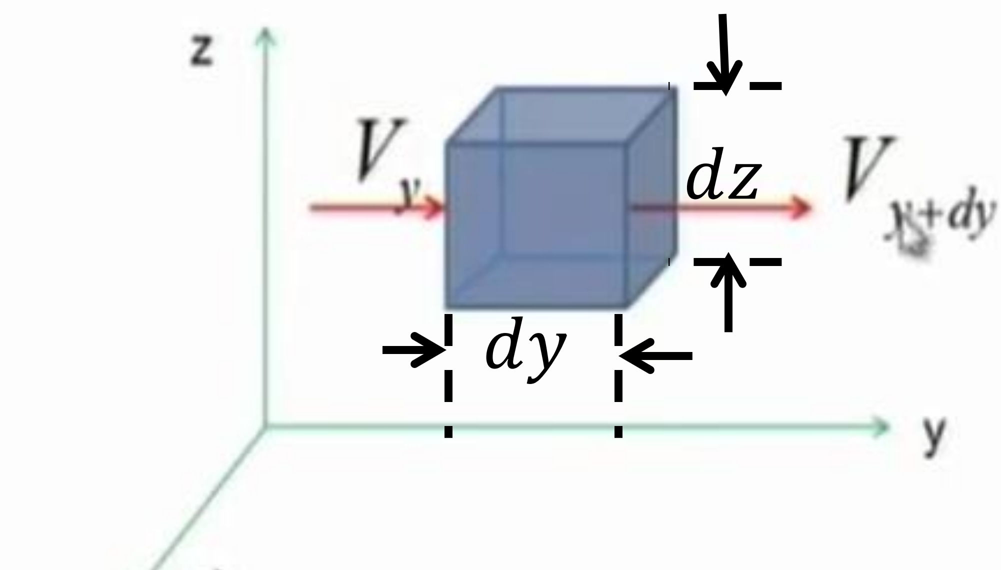
Lecture 3

- Significance of divergence and curl of a vector field
- Introduction to Curl

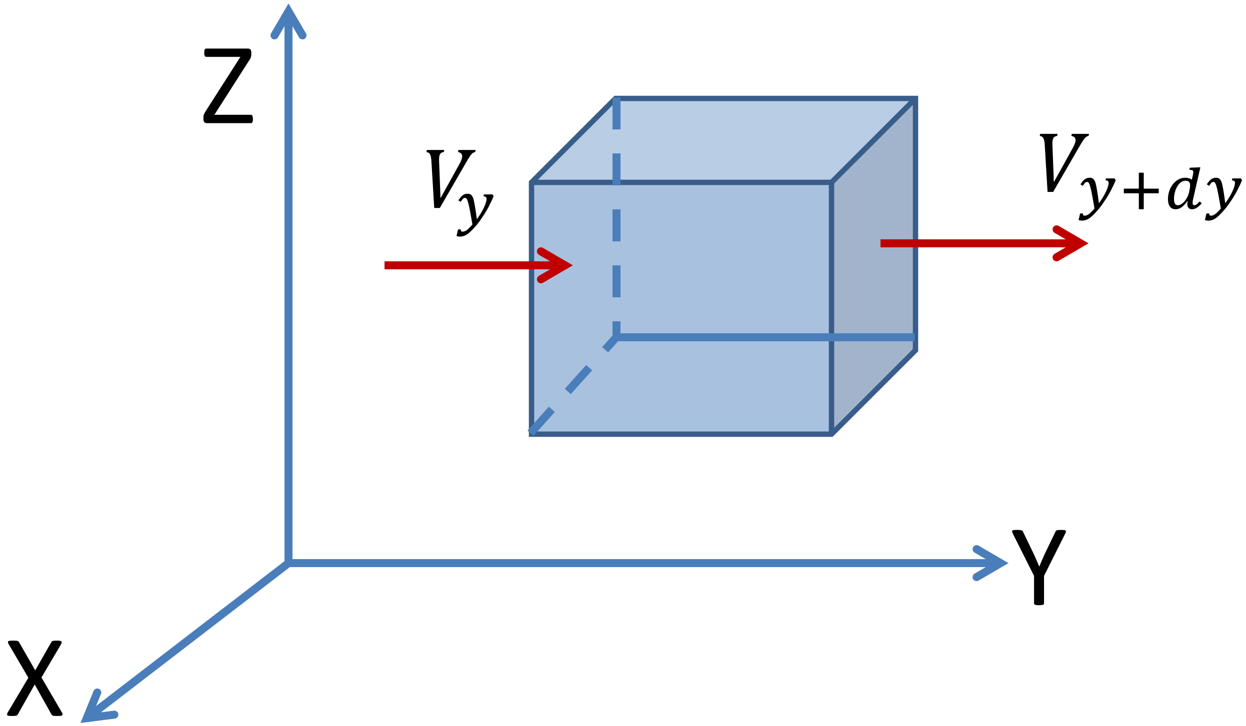
Significance of divergence of Vector field

Fluid passing through an elementary volume

Consider a fluid flowing with a velocity $\vec{\boldsymbol{v}}(x, y, z)$, at a point (x, y, z). The density of the fluid is given by $\rho(x, y, z)$(scalar). $\vec{V}$ is a momentum vector field combining $\vec{\boldsymbol{v}}(x, y, z)$ and $\rho(x, y, z)$.



$$\vec{V} = \vec{v}(x, y, z)\rho(x, y, z)$$

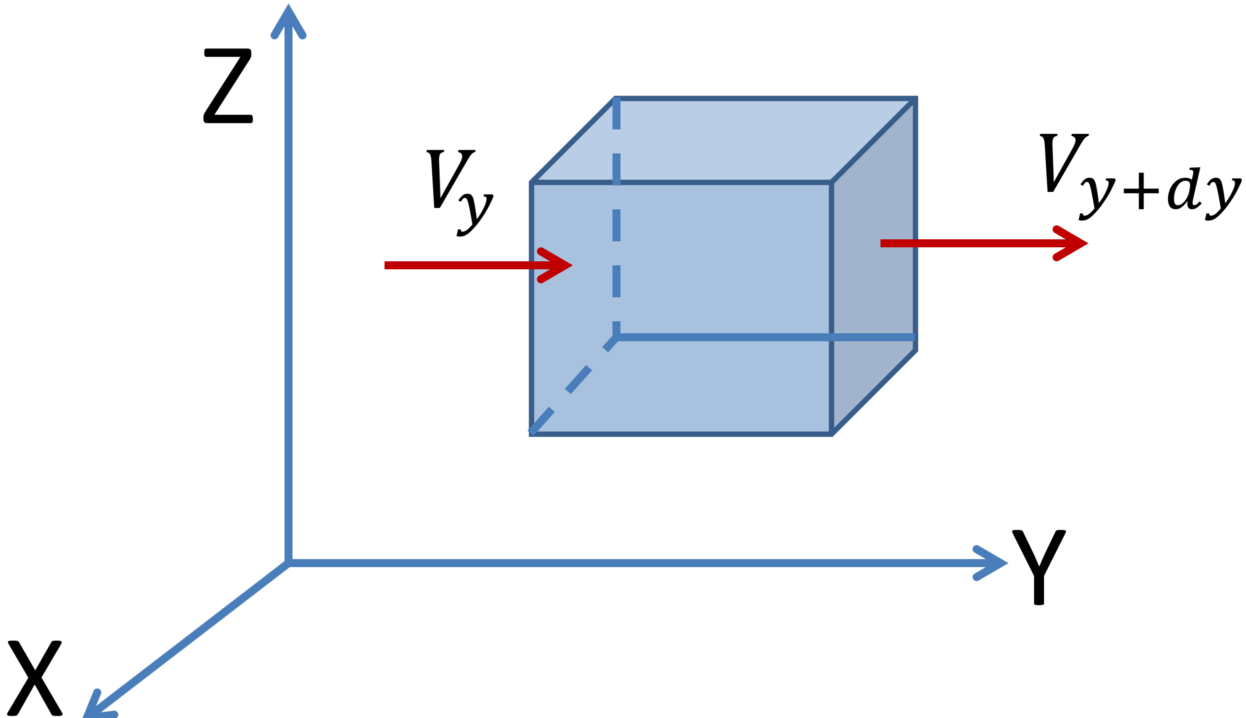


Mass of fluid passing through $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{j}}$ per unit time

$$\rho v_y dxdz = V_y dxdz$$

Mass of fluid passing through $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{j}}$ per unit time

$$\left(V_y + \frac{\partial V_y}{\partial y} dy\right) dxdz$$



Net increase in mass of fluid equals the mass flowing 'in' minus the mass flowing 'out'

$$= V_y\, dxdz - \left(V_y + \frac{\partial V_y}{\partial y} dy\right) dxdz$$

$$= -\frac{\partial V_y}{\partial y} dxdydz$$

From the six faces, the net increase in mass is:

$$= -\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dxdydz$$

$$= -\left(\vec{\nabla} \cdot \vec{V}\right) dxdydz$$

Rate of increase of mass

$$= \frac{\partial \rho}{\partial t} dxdydz$$

Equation of Continuity

$$\vec{\nabla} \cdot \vec{V} + \frac{\partial \rho}{\partial t} = 0$$

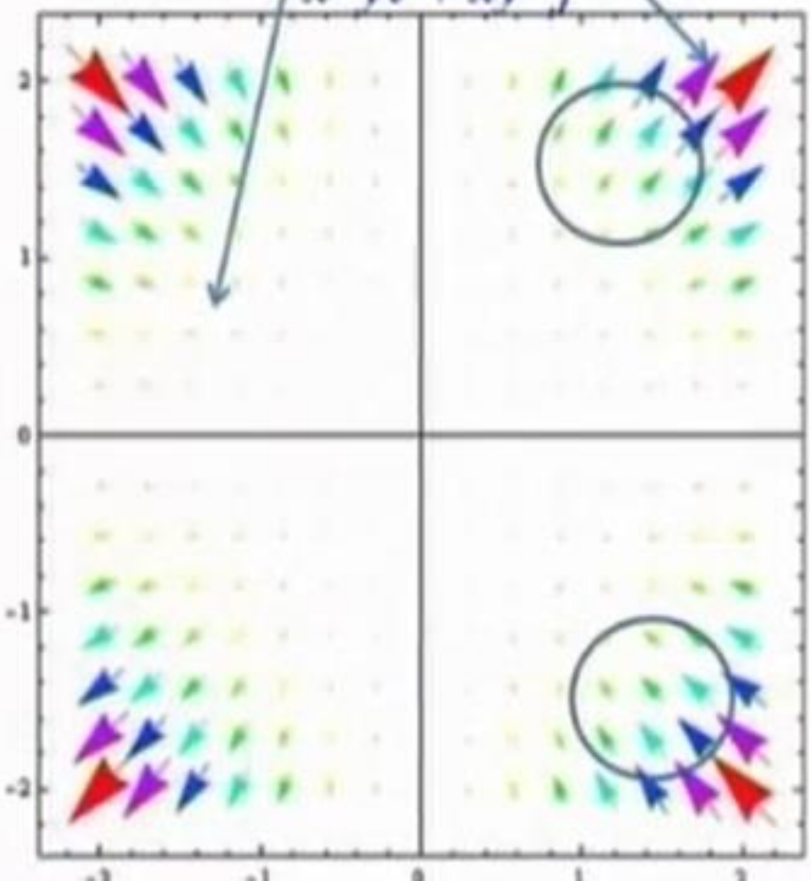
$$\frac{\partial \rho}{\partial t} > 0 \quad \text{divergence is negative (Inflow)}$$

$$\frac{\partial \rho}{\partial t} < 0 \quad \text{divergence is positive (Outflow)}$$

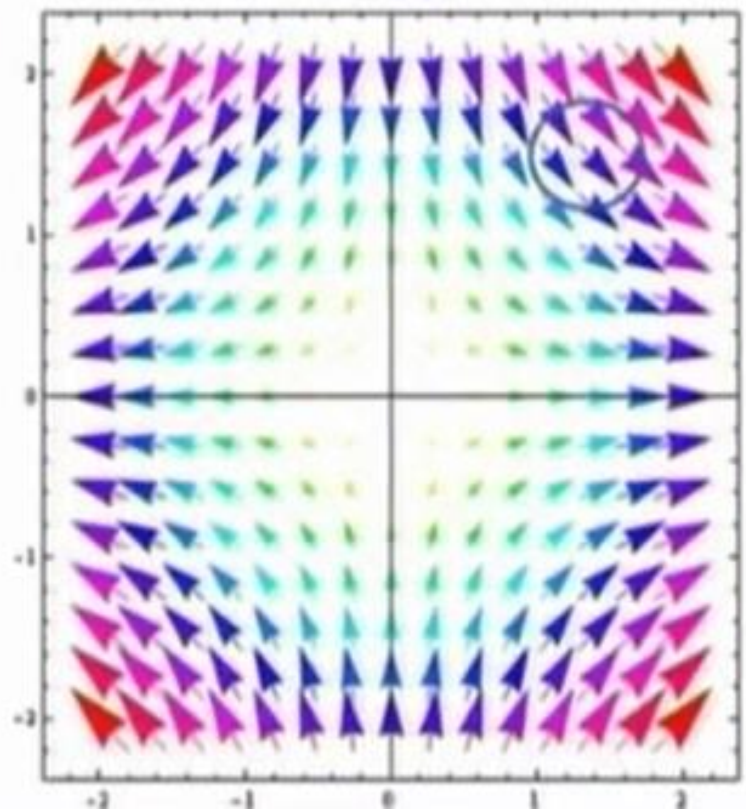
$$\frac{\partial \rho}{\partial t} = 0 \quad \text{divergence=0}$$

SOLENOIDAL

$x^2 y i + x y^2 j$



$x\hat{i} - y\hat{j}$

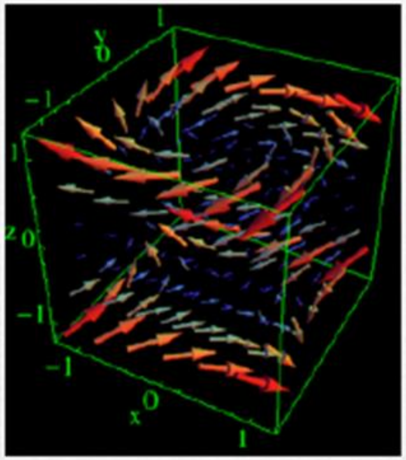


Curl (Rot)

The curl (or rotation) of a vector function $\mathbf{V}$ is obtained as

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$



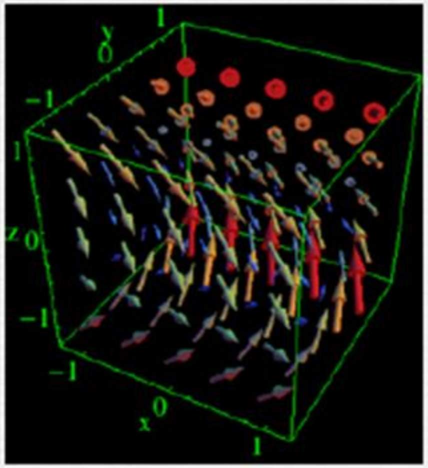
The vector function $\mathbf{V}$

For example, consider

$$\mathbf{V} \equiv \mathbf{V}(x, y, z) = \hat{i}\, yz + \hat{j}\, yz - \hat{k}\, xz^2$$

then

$$\nabla \times \mathbf{V} = \hat{i}(-y) + \hat{j}(y + z^2) + \hat{k}(-z)$$



Curl of $\mathbf{V}$

Geometrical Interpretation of Curl

The curl of a vector function $\mathbf{V}$ serves as the measure of how much the vector $\mathbf{V}$ "curls around" the concerned point.

The magnitude and direction of curl of a vector function $\mathbf{V}$ characterize the **infinitesimal** rotation of $\mathbf{V}$ at that point.The direction of the curl is the axis of rotation (locally), as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation (locally).

Imagine, again, standing at the edge of the pond. Float a tiny paddlewheel (or something similar); if it starts to rotate, then you placed it at a point of nonzero curl. (D. J. Griffiths)

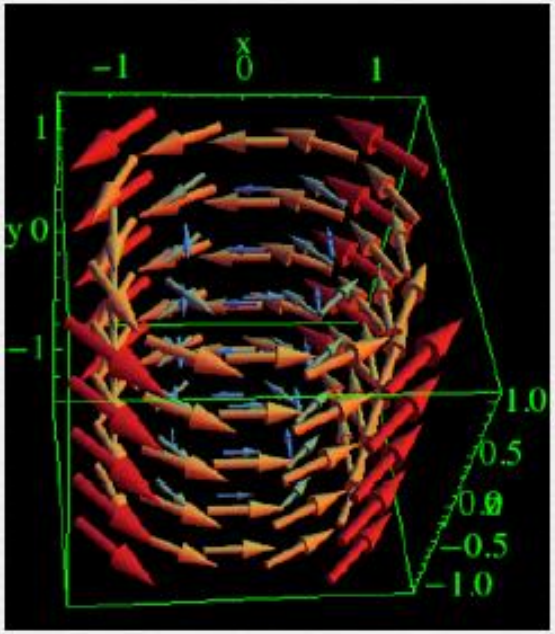
A rotating paddle wheel

Image Source: http://www.nature.com/nature/journal/v462/n7273/images/462584a-f1.2.jpg

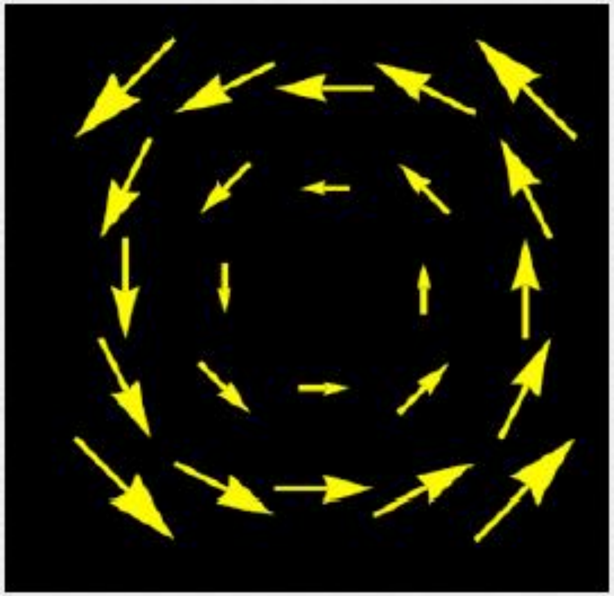
Curl

Consider $\mathbf{V} \equiv \mathbf{V}(x,y,z) = -\hat{i}\,y + \hat{j}\,x + \hat{k}$

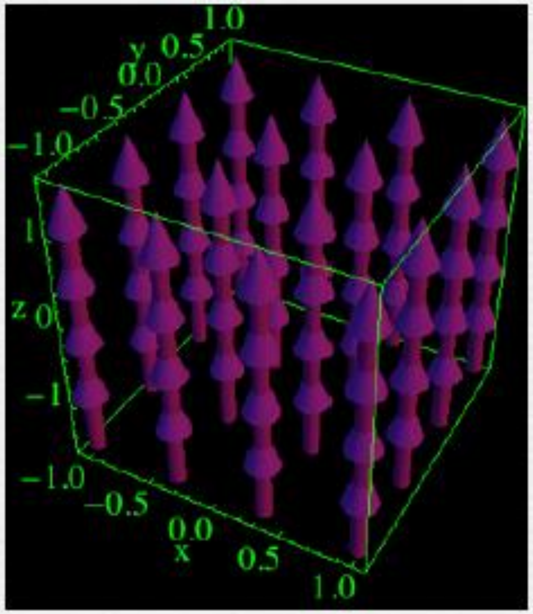
then $\boldsymbol{\nabla} \times \mathbf{V} = 2\hat{k}$



The vector field **V**

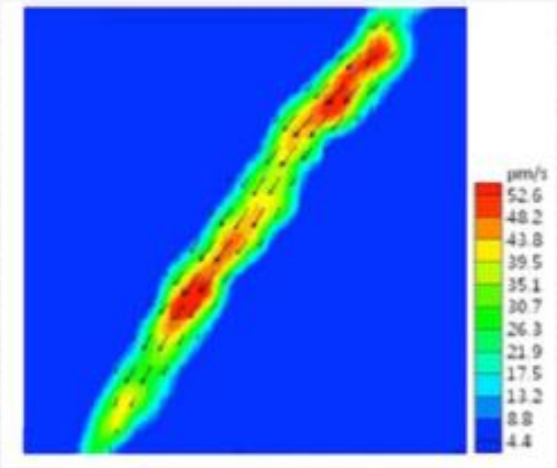


The vector field **V**
(2D View: XY Plane)



Curl of **V**

Curl: Real World Examples



Blood flow velocity vector field

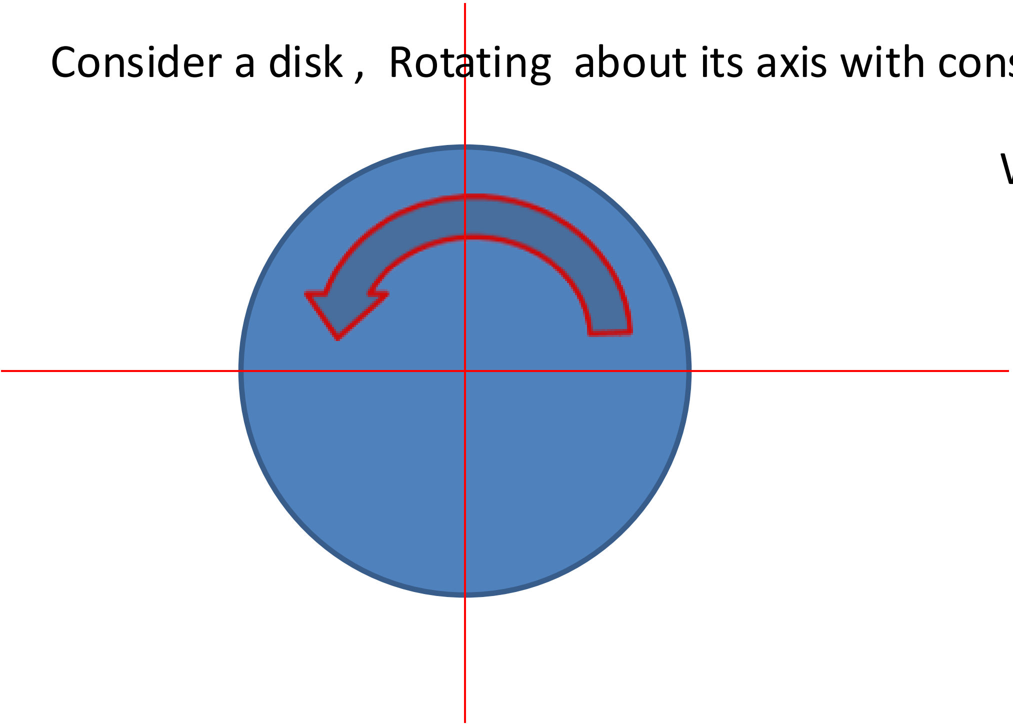
Whirlpool

Hurricane

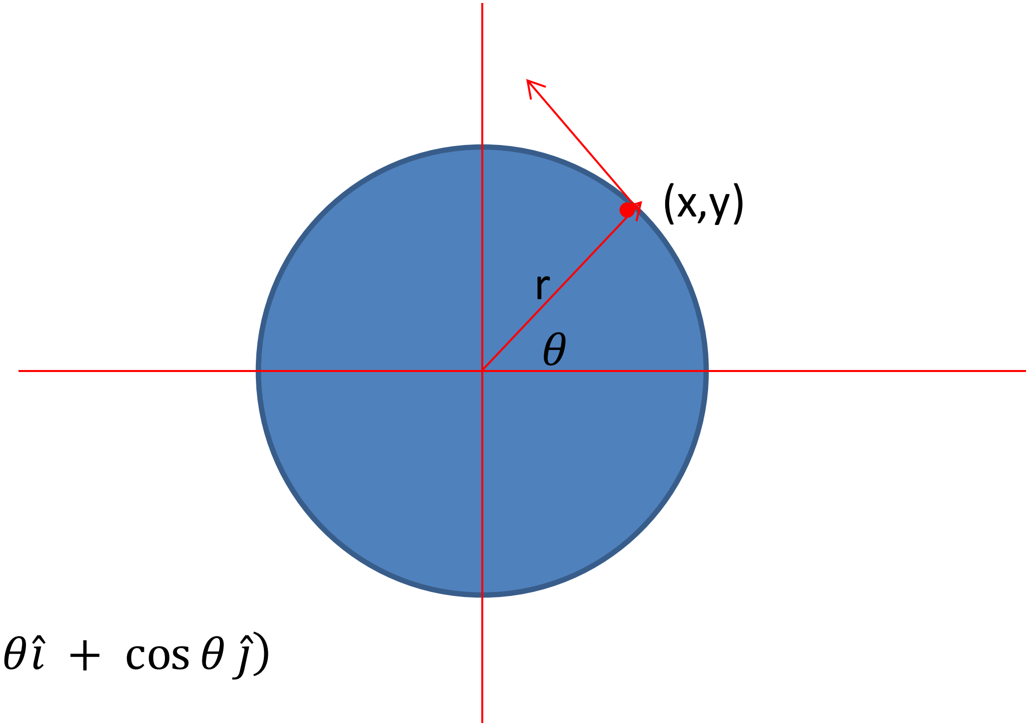
Remember that curl is a **local** quantity. The full pictures above **do not** depict curl. One has to examine the local behavior of the blood-velocity field, wind-velocity field or water-velocity field to figure out the curl at a desired point.

Image Sources:
http://dx.doi.org/10.1364/OE.19.004357
http://3.bp.blogspot.com/-xVteiueYwIc/UbWO66mkRnI/AAAAAAAAAps/90z3YhmR9So/s1600/storm+1.jpg
http://upload.wikimedia.org/wikipedia/commons/thumb/b/b7/Whirlpool.jpg/800px-Whirlpool.jpg

Consider a disk , Rotating about its axis with constant angular velocity

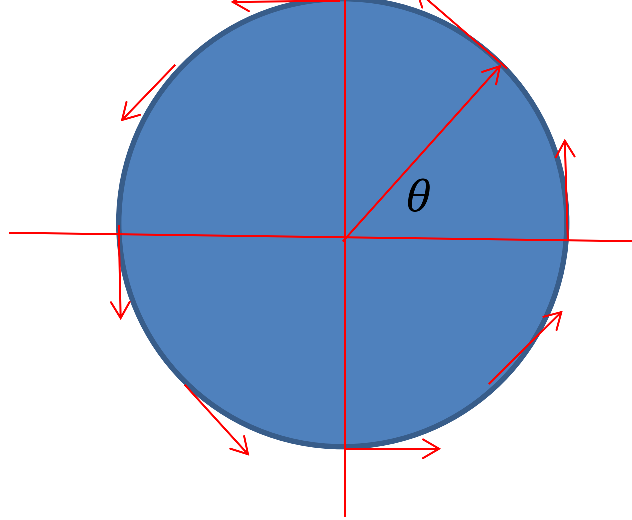


What is the velocity at the point (x,y) ?



The equation is $\boldsymbol{V} = r\,\omega\hat{\boldsymbol{\theta}} = r\,\omega(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$

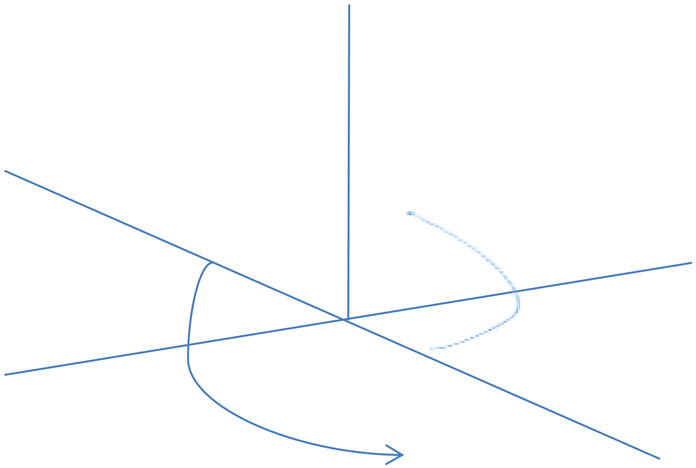
The equation is $V = r\,\omega\hat{\theta} = r\,\omega(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$



$$V = r\,\omega\hat{\theta} = r\,\omega\left(-\frac{y}{r}\hat{\imath} + \frac{x}{r}\hat{\jmath}\right) = \omega(-y\,\hat{\imath} + x\hat{\jmath})$$

$$\boldsymbol{\nabla} \times \boldsymbol{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & \mathbf{0} \end{vmatrix} = \left(\frac{\partial 0}{\partial y} - \frac{\partial \omega x}{\partial z}\right)\hat{x} - \left(\frac{\partial 0}{\partial x} - \frac{\partial(-\omega y)}{\partial z}\right)\hat{y} + \left(\frac{\partial(\omega x)}{\partial x} - \frac{\partial(-\omega y)}{\partial y}\right)\hat{z}$$

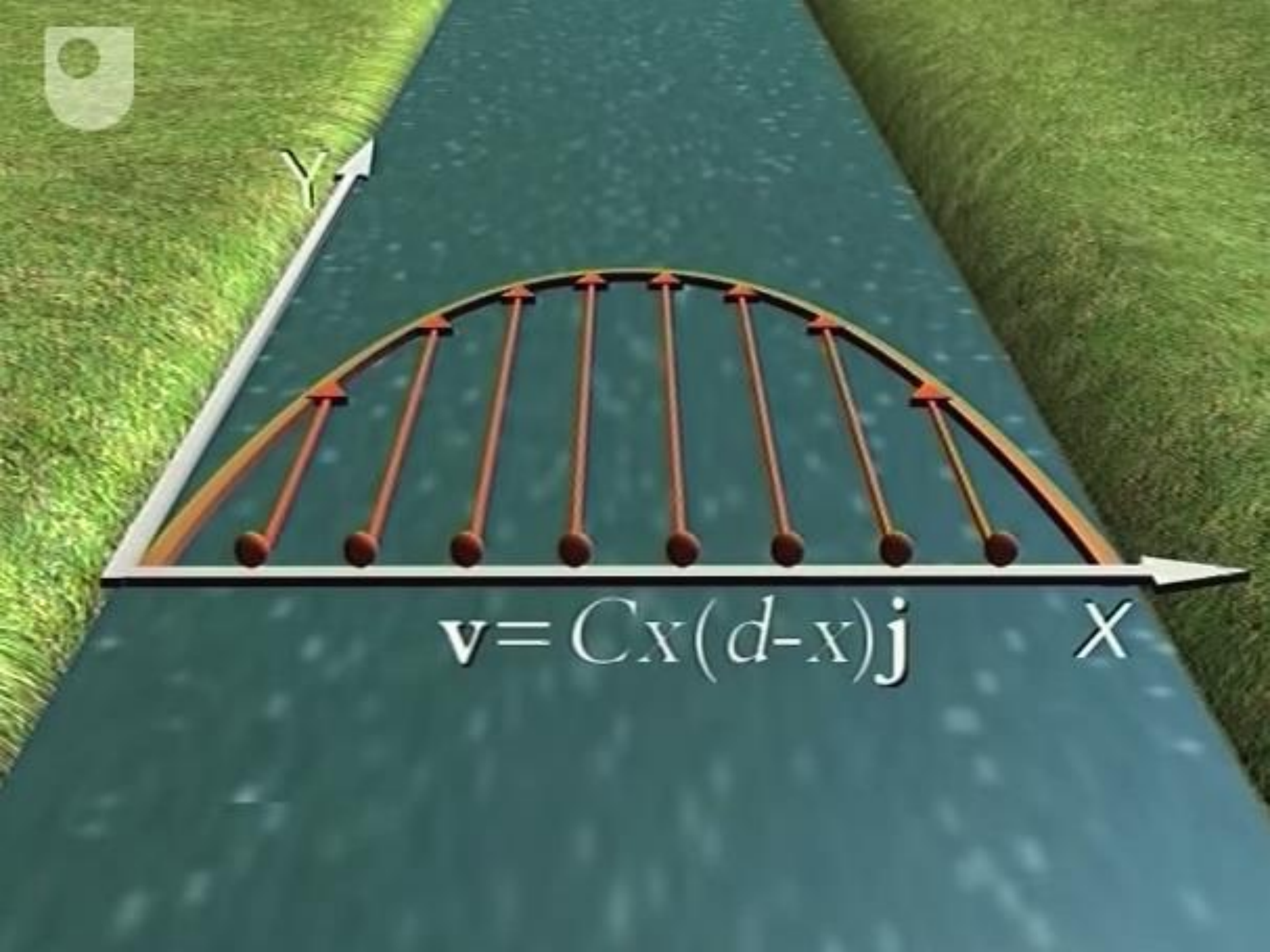
$$\boldsymbol{\nabla} \times \boldsymbol{V} = 2\omega\hat{z}$$

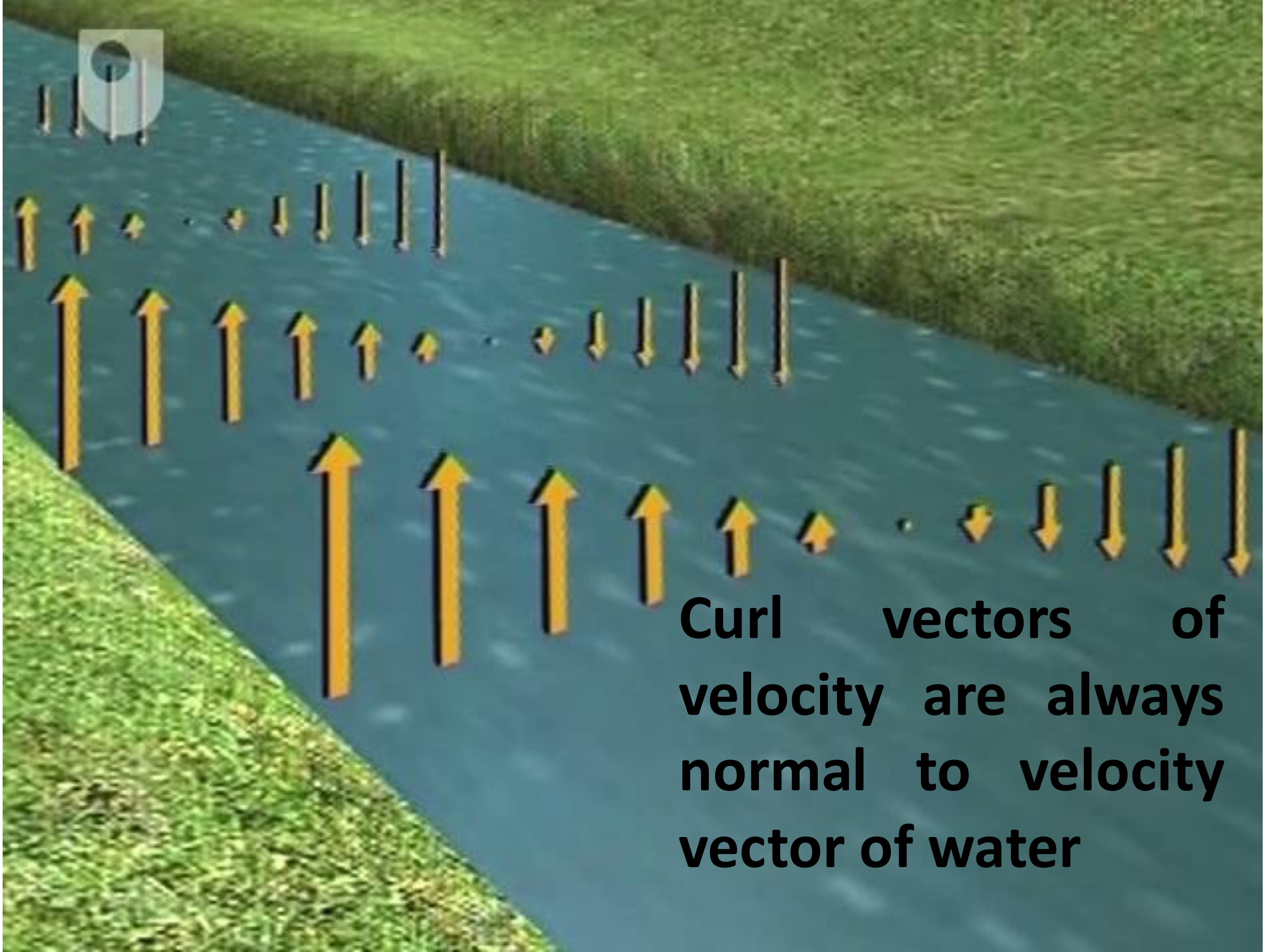


So $\nabla \times V$ at a point tells that the field is curling around axis given by the direction of $\nabla \times V$ with a strength given by its magnitude.

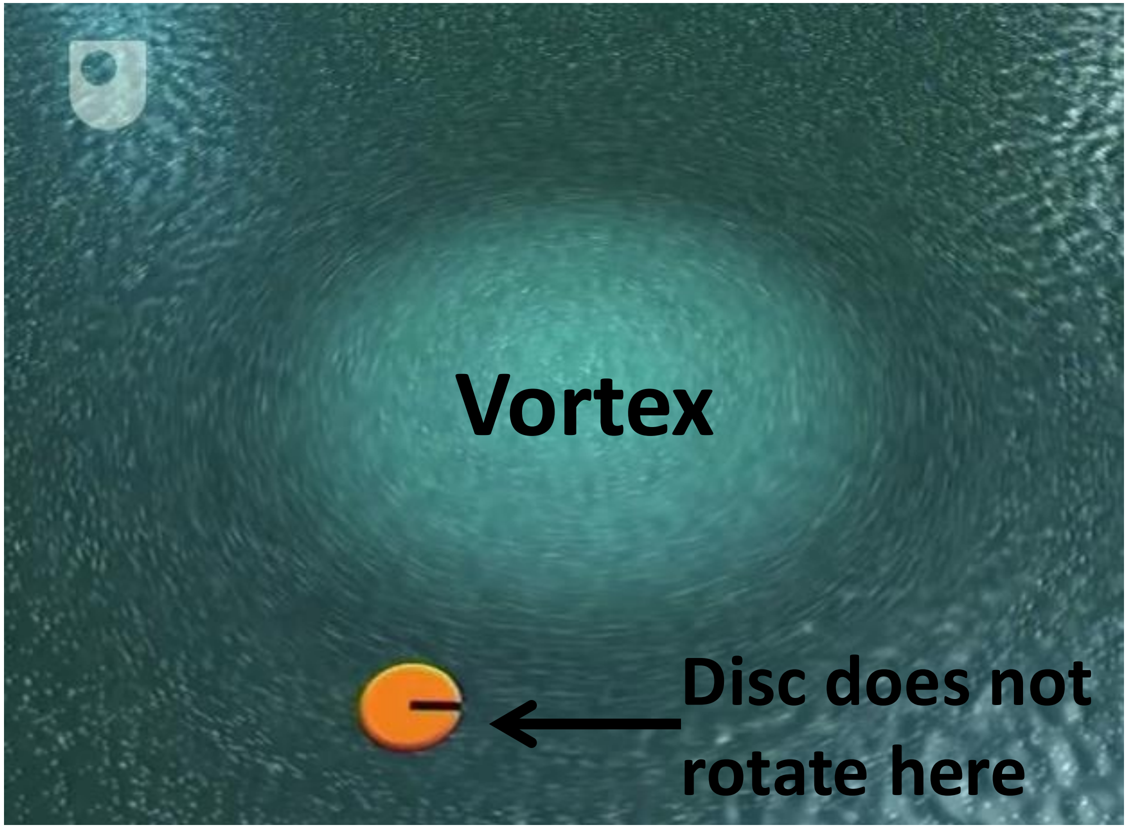
Imagine a floating disc on the river water. The disc would flow **downstream** with the force of the water and also it would ***rotate*** ?

What is the velocity field of the water on the surface that may cause such rotation?


Y
X
$\mathbf{v} = Cx(d-x)\mathbf{j}$



Curl vectors of velocity are always normal to velocity vector of water



Curl describes a *local rotation*, that is *rotation at each point* and NOT the bulk rotation of water that is seen at big river bends.

A disc placed near the center of the vortex rotates- there is local rotation caused by the velocity field, hence there is a curl associated with the vector field.

Not so outside the vortex although there is bulk rotation of the water.

Gradient, Divergence and Curl

- **Gradient of a scalar function f (∇f or grad f):**

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

- **Divergence of a vector function ($\nabla\cdot\mathrm{V}$ or div V):**

$$\nabla \cdot \mathrm{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

- **Curl of a vector function ($\nabla\times\mathrm{V}$ or curl V or rot V):**

$$\nabla \times \mathrm{V} = \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Keep in mind that when we say "scalar function f" and "vector function $\mathbf{V}$", it means

$f \equiv f(x,y,z)$ & $\mathbf{V} \equiv \mathbf{V}(x,y,z) = \hat{\mathbf{i}}V_x(x,y,z) + \hat{\mathbf{j}}\, V_y(x,y,z) + \mathbf{k}V_z(x,y,z)$

Some important terms

- If Divergence of a vector function is zero at all points, i.e., $\nabla\cdot\mathbf{V}=0$, we say that the vector function $\mathbf{V}$ is

  Solenoidal or Incompressible or Divergence-free or Divergence-less

- If Curl (or rotation) of a vector function is zero at all points, i.e., $\nabla\times\mathbf{V}=0$, we say that the vector function $\mathbf{V}$ is

  Irrotational or Rotation-free or Curl-free or Curl-less

Some important relations involving ∇ operator

Consider a constant-scalar k, two scalar functions f and g, and two vector functions $\mathbf{A}$ and $\mathbf{B}$, then

- $\nabla(k f) = k\,(\nabla f)$
- $\nabla\cdot(k\,\mathbf{A}) = k\,(\nabla\cdot\mathbf{A})$
- $\nabla\times(k\,\mathbf{A}) = k\,(\nabla\times\mathbf{A})$
- $\nabla(f+g) = \nabla f + \nabla g$
- $\nabla\cdot(\mathbf{A}+\mathbf{B}) = (\nabla\cdot\mathbf{A}) + (\nabla\cdot\mathbf{B})$
- $\nabla\times(\mathbf{A}+\mathbf{B}) = (\nabla\times\mathbf{A}) + (\nabla\times\mathbf{B})$

Some important relations involving ∇ operator

We can construct a scalar function

Using two scalar functions: $f\,g$

Using two vector functions: $\mathbf{A}\cdot\mathbf{B}$

The del operator can act on the above scalar functions $f\,g$ and $\mathbf{A}\cdot\mathbf{B}$ via the gradient operation: $\nabla(f\,g)$ and $\nabla(\mathbf{A}\cdot\mathbf{B})$. Then we have the following two relations:

- $\nabla\,(f\,g) = f\,(\nabla g) + g\,(\nabla f)$
- $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(Note that $A\cdot\nabla$ is actually an operator, $\mathbf{A}\cdot\nabla = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}$. $(A\cdot\nabla)B$ means that the operator $A\cdot\nabla$ acts on the vector function B. Similarly $B\cdot\nabla$ is also an operator.)

Some important relations involving ∇ operator

We can construct a vector function

Using a scalar function and a vector: $f\mathbf{A}$

Using two vector functions: $\mathbf{A}\times\mathbf{B}$

The del operator can act on the above vector functions $f\mathbf{A}$ and $\mathbf{A}\times\mathbf{B}$ via the divergence and curl operations: $\nabla\cdot(f\mathbf{A})$, $\nabla\cdot(\mathbf{A}\times\mathbf{B})$ and $\nabla\times(f\mathbf{A})$, $\nabla\times(\mathbf{A}\times\mathbf{B})$. Then we have the following **four** relations:

- $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$
- $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$
- $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$
- $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$