PHY 102 Introduction to Physics II

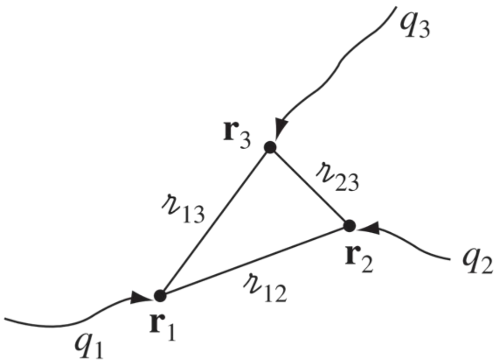
Spring Semester 2025

Lecture 13

Work and Energy in Electrostatics

The Energy of a Point Charge Distribution

How much work would it take to assemble an entire collection of point charges?



The first charge, q_1, takes *no* work → $W_1 = 0$

The second charge, q_2 → $W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{r_{12}}\right)$

The third charge, q_3 → $W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \longrightarrow W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$

The *total* work necessary to assemble 4 charges,

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_1q_4}{r_{14}} + \frac{q_2q_3}{r_{23}} + \frac{q_2q_4}{r_{24}} + \frac{q_3q_4}{r_{34}}\right)$$

$$\xrightarrow[\text{In general,}]{} W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{r_{ij}} \xrightarrow{\text{Divided by 2}} W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}}$$

$$\longrightarrow W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) \longrightarrow \boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)}$$

Energy of a continuous charge distribution

For a continuous charge distribution we need to replace the summation in the expression for W on the previous page by integral, thus

$$W = \frac{1}{2}\sum_{j=1}^{n} q_j V(\mathbf{r}_j) \rightarrow \frac{1}{2}\int_{\mathcal{V}} dq\, V(\mathbf{r}).$$

If $\rho(\mathbf{r})$ be the volume charge density, then $dq=\rho(\mathbf{r})\, d\tau$ and therefore

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau.$$

The corresponding expressions for a surface charge distribution σ, and a line charge distribution λ would be

$$W = \frac{1}{2}\int_{S} \sigma(\mathbf{r})\, V(\mathbf{r})\, dS, \text{ and } \quad W = \frac{1}{2}\int_{C} \lambda(\mathbf{r})\, V(\mathbf{r})\, dl.$$

Energy of a continuous charge distribution

We will now express the result for the energy of a continuous charge distribution in terms of electric field, instead of electric potential. For that we will need certain result which is essentially the idea of **integration by parts** (or **partial integration**) carried over to scalar and vector fields.

We already know that for a scalar function f and a vector function $\mathbf{A}$ we have,

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}f)\,.$$

Therefore considering the volume integral,

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(f\mathbf{A})\,d\tau = \int_{\mathcal{V}} f(\boldsymbol{\nabla}\cdot\mathbf{A})\,d\tau + \int_{\mathcal{V}} \mathbf{A}\cdot(\boldsymbol{\nabla}f)\,d\tau.$$

While, according to the Gauss's divergence theorem we have

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(f\mathbf{A})\,d\tau = \int_{\mathcal{S}} f\mathbf{A}\cdot d\mathbf{S}\,.$$

Energy of a continuous charge distribution

Comparing the last two equations we obtain,

$$\int_{\mathcal{V}} f(\nabla\cdot\mathbf{A})\, d\tau + \int_{\mathcal{V}} \mathbf{A}\cdot(\nabla f)\, d\tau = \int_{\mathcal{S}} f\mathbf{A}\cdot d\mathbf{S}.$$

$$\Rightarrow \int_{\mathcal{V}} f(\nabla\cdot\mathbf{A})\, d\tau = -\int_{\mathcal{V}} \mathbf{A}\cdot(\nabla f)\, d\tau + \int_{\mathcal{S}} f\mathbf{A}\cdot d\mathbf{S}.$$

Coming back to our original problem, using Gauss's law $\nabla\cdot\mathbf{E}=\rho/\epsilon_0$ to eliminate ρ we obtain

$$W = \frac{1}{2}\int \rho V\, d\tau = \frac{\epsilon_0}{2}\int (\nabla\cdot\mathbf{E})\, V\, d\tau.$$

Comparing with the second equation above and using $f = V$ and $\mathbf{A}=\mathbf{E}$, we obtain

$$W = \frac{\epsilon_0}{2}\int_{\mathcal{V}} (\nabla\cdot\mathbf{E})\, V\, d\tau = -\frac{\epsilon_0}{2}\int_{\mathcal{V}} \mathbf{E}\cdot(\nabla V)\, d\tau + \frac{\epsilon_0}{2}\int_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{S}.$$

Energy of a continuous charge distribution

But $-\nabla V = \mathbf{E}$ therefore when used in the first integral on the RHS (the volume integral), we get

$$W = \frac{\epsilon_0}{2}\int_{\mathcal{V}} E^2\, d\tau + \frac{\epsilon_0}{2}\int_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{S}\,,$$

since $\mathbf{E}\cdot\mathbf{E}=(E)(E)\cos 0 = E^2$ where $E=|\,\mathbf{E}\,|$. Now if we examine the original expression for W involving the charge density and potential, viz.

$$W = \frac{1}{2}\int_{\mathcal{V}} \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau,$$

we conclude that we may increase the domain of integration from $\mathcal{V}$ to a larger volume without changing the value of W since $\rho=0$ outside $\mathcal{V}$. In fact we may consider $\mathcal{V}\rightarrow\infty$.

Energy of a continuous charge distribution

Therefore we can consider $\mathcal{V}\rightarrow\infty$ (and correspondingly the surface enclosing the volume $\mathcal{V}$, $\mathcal{S}\rightarrow\infty$) in

$$W = \frac{\epsilon_0}{2}\int_{\mathcal{V}} E^2\, d\tau + \frac{\epsilon_0}{2}\int_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{S}\,,$$

as well, without changing the value of W (LHS). However, the two terms on the RHS may adjust their values to keep the sum constant. In fact the volume integral will go on increasing as we increase $\mathcal{V}$, since E^2 is strictly positive ($\mathbf{E}=0$ case is uninteresting). Correspondingly the surface integral must decrease to keep the sum intact. In fact, in 3 dimensions for finite charge systems, on the surface V behaves as $1/r$, E behaves as $1/r^2$ while the surface area has dependence r^2 *. Thus in the limit $\mathcal{V}\rightarrow\infty$ the surface integral vanishes as $1/r$; the contribution to W comes from the volume integral only. (*Note that for the volume integral the contribution comes from the volume and not the surface, therefore we can't do a similar analysis in this case.)

Energy of a continuous charge distribution

Therefore in the limit $\mathcal{V}\to\infty$ (all space) we obtain,

$$W = \frac{\epsilon_0}{2} \int_{(\infty)} E^2 \, d\tau \, .$$

The notation (∞) represents that the volume integral is over the entire space.

We may refer to $(\epsilon_0/2)E^2$ as the energy stored per unit volume. In this way we see the energy being stored in the field. Similarly from

$$W = \frac{1}{2} \int_{\mathcal{V}} \rho(\mathbf{r}) V(\mathbf{r}) \, d\tau.$$

we can say that $(1/2)\rho V$ is the energy stored per unit volume. In this interpretation the energy can be thought of being stored in the charge.

In the radiation theory it is useful (in general theory of relativity it is essential) to consider the energy being stored in the field. In the electrostatics both are okay.

A Paradox?

If we examine the expression

$$W = \frac{1}{2}\sum_{j=1}^{n} q_j V(\mathbf{r}_j).$$

we conclude that it can be positive or negative (or zero). For instance if we consider $q_1 = +q$ and $q_2 = -q$ placed respectively at $\mathbf{r}_1$ and $\mathbf{r}_2$ then we obtain (show this),

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$

This is clearly negative. On the other hand the expression

$$W = \frac{\epsilon_0}{2}\int_{(\infty)} E^2\, d\tau\,.$$

is always non-negative (positive or zero). **Where did we go wrong?**

Resolution of the Paradox & Energy of a point charge

The resolution to this seemingly paradoxical observation is that the expression

$$W = \frac{1}{2}\sum_{j=1}^{n} q_j V(\mathbf{r}_j)$$

does not take into account the work required to construct the point charges in the first place: it **avoids the "self-energy"** contributions. In fact, it can be seen from the expression

$$W = \frac{\epsilon_0}{2}\int_{(\infty)} E^2\, d\tau\,,$$

which does take into account the **self-energy also**, that the energy of a point charge is infinite.

$$W = \frac{\epsilon_0}{2}\int_{(\infty)} E^2\, d\tau = \frac{\epsilon_0}{2}\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi) = \frac{q^2}{8\pi\epsilon_0}\int_0^{\infty}\frac{dr}{r^2} = \infty.$$

Resolution of the Paradox

The "inconsistency" crept in when we switched from the summation to integral:

$$W = \frac{1}{2}\sum_{j=1}^{n} q_j V(\mathbf{r}_j) \to \frac{1}{2}\int_{\mathcal{V}} dq\, V(\mathbf{r}).$$

In the summation formula, the contribution at $\mathbf{r}_j$ due to q_j is avoided, while in the integral formula $V(\mathbf{r})$ is the full electric potential at $\mathbf{r}$, i.e., takes into account contribution from dq present at $\mathbf{r}$ also. Note that as long as we are dealing with a continuous charge distribution there is no distinction (and hence no problem), since the amount of charge *exactly at the point* $\mathbf{r}$ is vanishingly small, and its contribution to the potential is zero.

Energy of a charge system

Therefore,

- Whenever we are interested in calculating the energy associated with a system of discrete charges, we should use

$$W = \frac{1}{2}\sum_{j=1}^{n} q_j V(\mathbf{r}_j), \quad \text{with} \quad V(\mathbf{r}_j) = \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_k}{|\mathbf{r}_j - \mathbf{r}_k|},$$

  since we want to avoid the *"purposeless"* infinite contribution associated with the formation of individual point charges.

- Whenever we are interested in calculating the energy associated with a continuous charge distribution we should use

$$W = \frac{1}{2}\int_{(\infty)} \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau \quad \text{or} \quad W = \frac{\epsilon_0}{2}\int_{(\infty)} E^2\, d\tau\,.$$

Energy of a point charge

If we use the charge density expression for discrete point charges in

$$W = \frac{1}{2}\int_{(\infty)} \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau$$

using Dirac delta functions, then we will obtain infinity as the answer since then the contributions from the positions where the charges are situated are taken into account.

For example, for a point charge present at origin $\rho(\mathbf{r}) = q\delta^3(\mathbf{r})$ (*not a continuous charge distribution!*) we have

$$W = \frac{1}{2}\int_{(\infty)} \rho V\, d\tau = \frac{1}{2}\int_{(\infty)} (q\delta^3(\mathbf{r}))\left(\frac{q}{4\pi\epsilon_0 r}\right) d\tau = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{\delta^3(\mathbf{r})\, d\tau}{r} = \infty,$$

in complete agreement with the result obtained using the expression involving E^2

Superposition principle

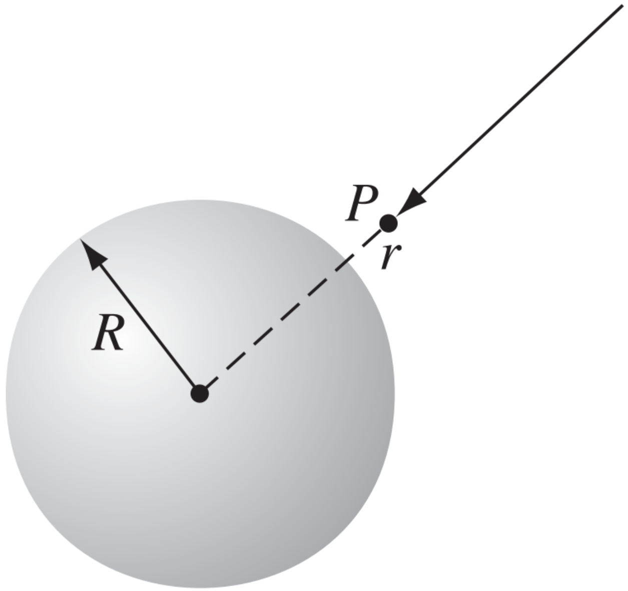
The electrostatic potential energy involves E^2 (quadratic in E). As a consequence it does not obey superposition principle. The energy of a compound system is not the sum of energies of its constituents considered separately. For instance, if we consider two constituents then

$$\begin{aligned}
W_{\text{total}} &= \frac{\epsilon_0}{2}\int_{(\infty)} E^2\, d\tau = \frac{\epsilon_0}{2}\int_{(\infty)} (\mathbf{E}_1 + \mathbf{E}_2)^2\, d\tau \\
&= \frac{\epsilon_0}{2}\int_{(\infty)} (\mathbf{E}_1 + \mathbf{E}_2)\cdot(\mathbf{E}_1 + \mathbf{E}_2)\, d\tau \\
&= \frac{\epsilon_0}{2}\int_{(\infty)} (E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2)\, d\tau \\
&= \frac{\epsilon_0}{2}\int_{(\infty)} E_1^2\, d\tau + \frac{\epsilon_0}{2}\int_{(\infty)} E_2^2\, d\tau + \epsilon_0\int_{(\infty)} (\mathbf{E}_1\cdot\mathbf{E}_2)\, d\tau \\
&= W_1 + W_2 + \epsilon_0\int_{(\infty)} (\mathbf{E}_1\cdot\mathbf{E}_2)\, d\tau \neq W_1 + W_2.
\end{aligned}$$

Optional Slides

Electric Potential of a Spherical Shell

P1. Find the potential inside and outside a spherical shell of radius **R** that carries a uniform surface charge. Set the reference point at infinity.



Solution

From Gauss's law, the field outside is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}},$$

where q is the total charge on the sphere. The field inside is zero.

Electric Potential of a Spherical Shell

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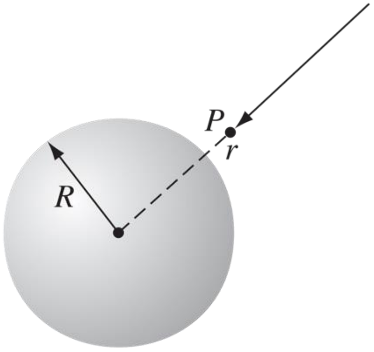


For points outside the sphere (r > R),

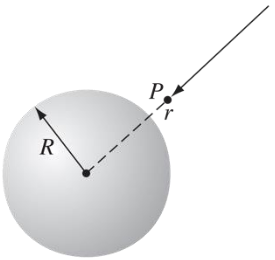
$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{q}{r'^2}\, dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

To find the potential inside the sphere ($r < R$), we must break the integral into two pieces, using in each region the field that prevails there:

$$V(r) = \frac{-1}{4\pi\epsilon_0} \int_{\infty}^{R} \frac{q}{r'^2}\, dr' - \int_{R}^{r} (0)\, dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r'}\bigg|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0} \frac{q}{R}.$$

Electric Potential of a Spherical Shell

P1. Find the potential inside and outside a spherical shell of radius **R** that carries a uniform surface charge. Set the reference point at infinity.



Comments

- The potential is not zero inside the shell even though the field is.
- V is constant inside the sphere, so the work done is **zero** when moving a charge inside the sphere.
- You cannot figure out the potential inside the sphere based on the electric field there alone. The potential inside the sphere is sensitive to what's going on outside the sphere as well.
- If we placed a second uniformly charged shell out at radius R' > R, the potential inside R would change, even though the field would still be zero.

Electric Potential

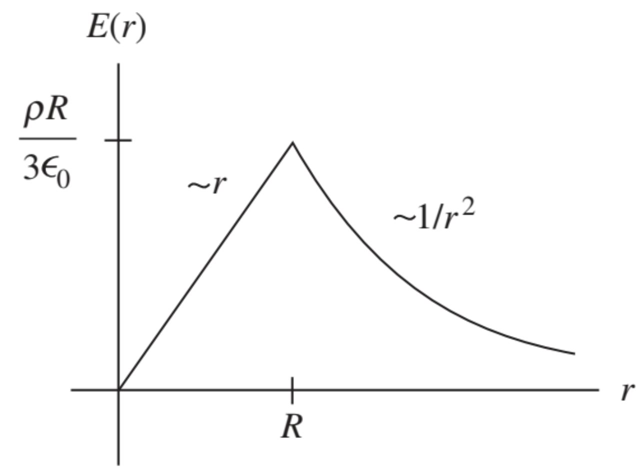
Electric Field and Potential of a uniform sphere of charge

P2. A spherical charge distribution has a density **ρ** that is constant from **r = 0** out to **r = R** and is zero beyond. What is the electric field for all values of **r**, both less than and greater than **R**?

$$E(r) = \frac{(4\pi R^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho R^3}{3\epsilon_0 r^2} \qquad (r \geq R)$$

$$Q = (4\pi R^3/3)\rho$$

$$E(r) = \frac{(4\pi r^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0} \qquad (r \leq R).$$



$E(r)$ is continuous at $r = R$, where it takes on the value $\rho R/3\epsilon_0$

The field goes to zero at the center, so it is continuous there also.

Electric Potential

Electric Field and Potential of a uniform sphere of charge

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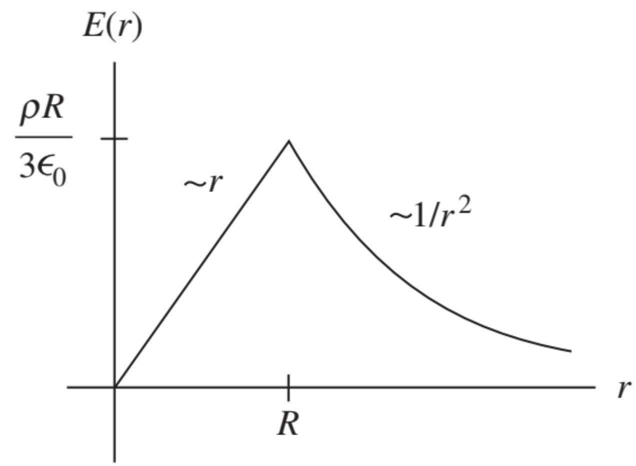
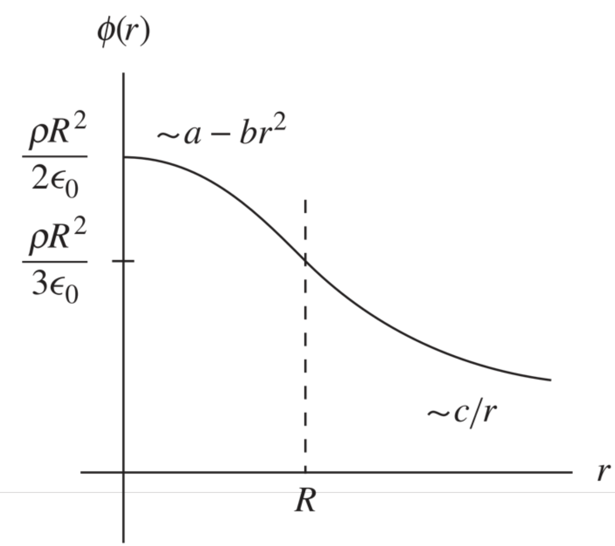


The potential outside the sphere is

$$\phi_{\text{out}}(r) = -\int_{\infty}^{r} E(r')\, dr' = -\int_{\infty}^{r} \frac{\rho R^3}{3\epsilon_0 r'^2}\, dr' = \frac{\rho R^3}{3\epsilon_0 r}.$$

$$\phi_{\text{out}}(r) = Q/4\pi\epsilon_0 r \qquad \Big| \qquad Q = (4\pi R^3/3)\rho$$



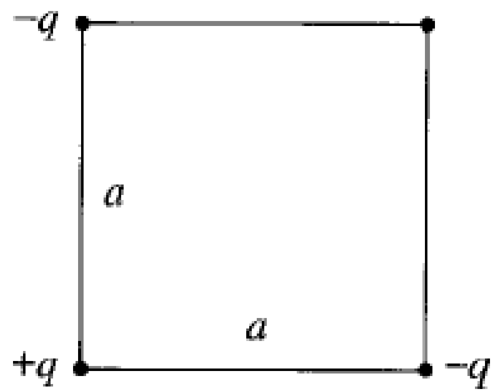
To find the potential inside the sphere, we must break the integral into two pieces:

$$\phi_{\text{in}}(r) = -\int_{\infty}^{R} E(r')\, dr' - \int_{R}^{r} E(r')\, dr' = -\int_{\infty}^{R} \frac{\rho R^3}{3\epsilon_0 r'^2}\, dr' - \int_{R}^{r} \frac{\rho r'}{3\epsilon_0}\, dr'$$

$$= \frac{\rho R^3}{3\epsilon_0 R} - \frac{\rho}{6\epsilon_0}(r^2 - R^2) = \frac{\rho R^2}{2\epsilon_0} - \frac{\rho r^2}{6\epsilon_0}.$$

(a) Three charges are situated at the corners of a square (side a), How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corner?

(b) How much work does it take to assemble the whole configuration of four charges?



(a) $V = \frac{1}{4\pi\epsilon_0}\sum \frac{q_i}{r_{ij}} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q}{a}+\frac{q}{\sqrt{2}a}+\frac{-q}{a}\right\} = \frac{q}{4\pi\epsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right).$

$$\therefore W_4 = qV = \boxed{\frac{q^2}{4\pi\epsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)}.$$

(b) $W_1 = 0$, $W_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{-q^2}{a}\right)$; $W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{\sqrt{2}\,a}-\frac{q^2}{a}\right)$; $W_4 =$ (see (a)).

$$W_{\text{tot}} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}\left\{-1+\frac{1}{\sqrt{2}}-1-2+\frac{1}{\sqrt{2}}\right\} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}\left(-2+\frac{1}{\sqrt{2}}\right)}.$$

Problem

Find the energy of a uniformly charged spherical shell of total charge q and radius R.

Solution 1: Use Eq. 2.43, in the version appropriate to surface charges:

$$W = \frac{1}{2}\int \sigma V \, da.$$

Now, the potential at the surface of this sphere is $(1/4\pi\epsilon_0)q/R$ (a constant), so

$$W = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\int \sigma \, da = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}.$$

Solution 2: Use Eq. 2.45. Inside the sphere $\mathbf{E} = 0$; outside,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}, \quad \text{so} \quad E^2 = \frac{q^2}{(4\pi\epsilon_0)^2 r^4}.$$

Therefore,

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{\text{outside}}\left(\frac{q^2}{r^4}\right)(r^2\sin\theta\,dr\,d\theta\,d\phi) \\ &= \frac{1}{32\pi^2\epsilon_0}q^2 4\pi\int_R^\infty \frac{1}{r^2}\,dr = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}. \end{aligned}$$