Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 7

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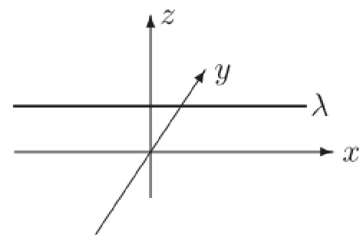
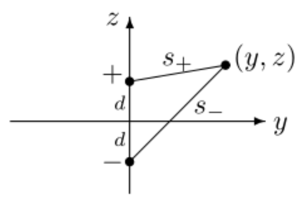
1. A uniform line charge, with λ denoting the charge per unit length, is placed on an infinite straight wire, at a distance d above a grounded conducting plane. Assume that the wire runs parallel to the x-axis and directly above it, and the conducting plane lies along the xy-plane.
(a) Find the potential in the region above the plane.
(b) Find the charge density σ induced on the conducting plane.



Solution:

(a) Method of Images: Uniform line charges, λ and -λ

$$V(y,z) = \frac{2\lambda}{4\pi\epsilon_0}\ln(s_-/s_+) = \frac{\lambda}{4\pi\epsilon_0}\ln(s_-^2/s_+^2)$$
$$= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}}$$



(b)

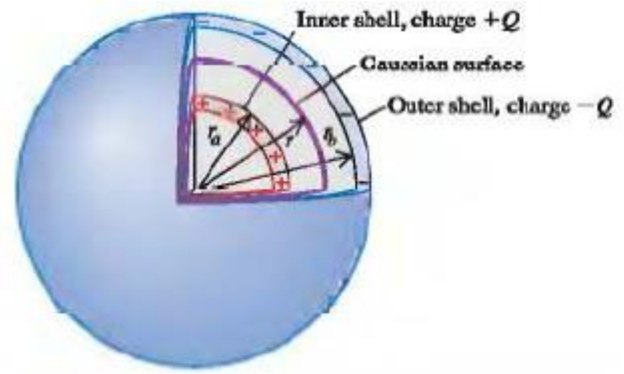
$$\sigma = -\epsilon_0\frac{\partial V}{\partial n}. \text{ Here } \frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}, \text{ evaluated at } z=0.$$

$$\sigma(y) = -\epsilon_0\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{1}{y^2+(z+d)^2}2(z+d) - \frac{1}{y^2+(z-d)^2}2(z-d)\right\}\Bigg|_{z=0}$$
$$= -\frac{2\lambda}{4\pi}\left\{\frac{d}{y^2+d^2} - \frac{-d}{y^2+d^2}\right\} = \boxed{-\frac{\lambda d}{\pi(y^2+d^2)}.}$$

Check: Total charge induced on a strip of width l parallel to the y axis:

$$q_{\text{ind}} = -\frac{l\lambda d}{\pi}\int_{-\infty}^{\infty}\frac{1}{y^2+d^2}\,dy = -\frac{l\lambda d}{\pi}\left[\frac{1}{d}\tan^{-1}\left(\frac{y}{d}\right)\right]\Bigg|_{-\infty}^{\infty} = -\frac{l\lambda d}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$
$$= -\lambda l. \quad \text{Therefore } \lambda_{\text{ind}} = -\lambda, \text{ as it should be.}$$

2. Two concentric spherical conducting shells are separated by vacuum. The inner shell has total charge $+Q$ and outer radius r_a, and the outer shell has charge $-Q$ and inner radius r_b (see Figure). (The inner shell is attached to the outer shell by thin insulating rods that have negligible effect on the capacitance.) Find the capacitance of this spherical capacitor. Also, find the electric potential energy stored in the capacitor (a) by using the capacitance and (b) by integrating the electric field energy density.



(a) Hence

SET UP: We use Gauss's law to find the electric field between the spherical conductors. From this value we determine the potential difference V_{ab} between the two conductors; we then use Eq. (24.1) to find the capacitance $C = Q/V_{ab}$.

EXECUTE: Using the same procedure as in Example 22.5 (Section 22.4), we take as our Gaussian surface a sphere with radius r between the two spheres and concentric with them. Gauss's law, Eq. (22.8), states that the electric flux through this surface is equal to the total charge enclosed within the surface, divided by ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

By symmetry, $\vec{E}$ is constant in magnitude and parallel to $d\vec{A}$ at every point on this surface, so the integral in Gauss's law is equal

to $(E)(4\pi r^2)$. The total charge enclosed is $Q_{encl} = Q$, so we have

$$(E)(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

The electric field between the spheres is just that due to the charge on the inner sphere; the outer sphere has no effect. We found in Example 22.5 that the charge on a conducting sphere produces zero field *inside* the sphere, which also tells us that the outer conductor makes no contribution to the field between the conductors.

The above expression for E is the same as that for a point charge Q, so the expression for the potential can also be taken to be the same as for a point charge, $V = Q/4\pi\epsilon_0 r$. Hence the potential of the inner (positive) conductor at $r = r_a$ with respect to that of the outer (negative) conductor at $r = r_b$ is

$$V_{ab} = V_a - V_b = \frac{Q}{4\pi\epsilon_0 r_a} - \frac{Q}{4\pi\epsilon_0 r_b}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) = \frac{Q}{4\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

Finally, the capacitance is

$$C = \frac{Q}{V_{ab}} = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$$

the energy stored in this capacitor is

$$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b}$$

(b) The electric field in the volume between the two conducting spheres has magnitude $E = Q/4\pi\epsilon_0 r^2$. The electric field is zero inside the inner sphere and is also zero outside the inner surface of the outer sphere, because a Gaussian surface with radius $r < r_a$ or $r > r_b$ encloses zero net charge. Hence the energy density is nonzero only in the space between the spheres ($r_a < r < r_b$). In this region,

$$u = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 = \frac{Q^2}{32\pi^2\epsilon_0 r^4}$$

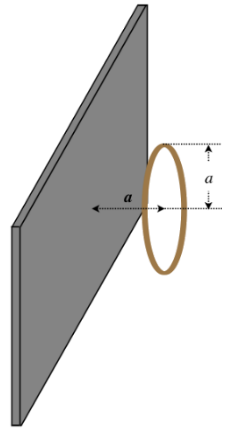
The energy density is *not* uniform; it decreases rapidly with increasing distance from the center of the capacitor. To find the total electric-field energy, we integrate u (the energy per unit volume) over the volume between the inner and outer conducting spheres. Dividing this volume up into spherical shells of radius r, surface area $4\pi r^2$, thickness dr, and volume $dV = 4\pi r^2\, dr$, we have

$$\begin{aligned} U &= \int u\, dV = \int_{r_a}^{r_b}\left(\frac{Q^2}{32\pi^2\epsilon_0 r^4}\right)4\pi r^2\, dr \\ &= \frac{Q^2}{8\pi\epsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0}\left(-\frac{1}{r_b} + \frac{1}{r_a}\right) \\ &= \frac{Q^2}{8\pi\epsilon_0}\frac{r_b - r_a}{r_a r_b} \end{aligned}$$

3. A thin circular plastic ring carries a net charge that is uniformly distributed throughout the ring with a linear density of λ. This ring is positioned parallel to a neutrally charged infinite conducting plane such that its distance from the plane equals the radius of the ring (Figure 1). Let a charged point particle having a charge Q which is equal and opposite to the net charge of the ring is placed at the center of the ring. Find the magnitude and direction of the net force on this particle in terms of λ?

Hint: The magnitude of the electric field on the axis of the ring is

$$E(x) = \frac{\lambda\, ax}{2\varepsilon_0 (a^2 + x^2)^{\frac{3}{2}}}$$



Solution

We can use the method of images to replace the conductor with image charges placed symmetrically on the opposite side of the conducting surface. We need images for both the point charge and the ring. The plastic ring will not contribute to the force on the point particle, because the particle is at its center, where the electric field of the ring vanishes. So the only two contributors to the force are the image point charge and the image ring. Start by determining the amount of charge on the particle in terms of the given linear density:

$$Q = \lambda l = 2\pi a\ \lambda$$

The force between the point particle and its image is just the Coulomb force (with a separation of $r = 2a$*), and it is attractive (toward the plane), since the image charge has the opposite sign:*

$$F_1 = \frac{Q^2}{4\pi\epsilon_o (2a)^2} = \frac{\pi\lambda^2}{4\epsilon_o}$$

The force on the point charge by the image ring is the product of Q *and the field of the image ring, and it is repulsive (the charge in the image ring has the opposite sign of the real ring, which is the same sign as the point charge). The image ring is a distance* $x = 2a$ *from the point charge, so:*

$$F_2 = QE_{image\ ring} = (2\pi a\ \lambda)\frac{\lambda a\,(2a)}{2\epsilon_o \left(a^2 + (2a)^2\right)^{\frac{3}{2}}} = \frac{2\pi\lambda^2}{5\sqrt{5}\epsilon_o}$$

These forces are in opposite directions, so the net force on the point charge is:

$$F = F_1 - F_2 = \boxed{\left(\frac{1}{4} - \frac{2}{5\sqrt{5}}\right)\frac{\pi\lambda^2}{\epsilon_o}}$$

The fact that this number is positive means that $F_1 > F_2$*, so the net force is toward the conductor.*

4. Consider three point charges located at the vertices of an equilateral triangle of side 'a' as shown in the figures below. Calculate the dipole moment of this charge configuration using the result $\boldsymbol{p} = \Sigma\, q_j \boldsymbol{r_j}$, where $\boldsymbol{r_j}$ refers to the position of the charge q_j. Consider the following situations:
(a) The coordinate system is chosen in such a way that the origin coincides with the position of the charge q on the left (figure (a)).
(b) The coordinate system is chosen in such a way that the origin coincides with the position of charge q on the right (figure(b)).
Do the answers match?

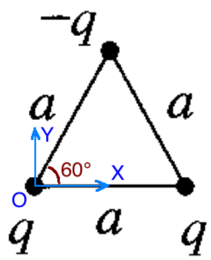


Figure (a)

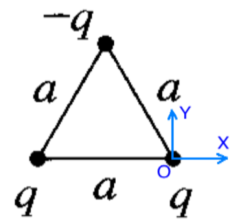
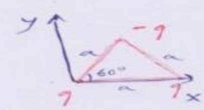


Figure (b)

Soln (a) $\vec{p} = \sum_j q_j \vec{r}_j$

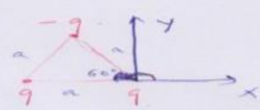


$\Rightarrow \vec{p} = q(0) + q(a\hat{i}) - q(a\cos 60\hat{i} + a\sin 60°\hat{j})$

or, $\vec{p} = qa\,\hat{i} - qa\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$

or, $\vec{p} = qa\left(\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)$.

(b) $\vec{p} = q(-a\hat{i}) + q(0) - q(-a\cos 60°\hat{i} + a\sin 60°\hat{j})$



or, $\vec{p} = qa(-\hat{i}) - qa\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$

$= qa\left(-\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)$.

Clearly the answers in (a) and (b) don't match. Dipole moment of a charge configuration depends on the choice of origin (in general).

However, if total charge in the system is zero, then $\vec{p}$ is independent of the choice of origin. This will be verified in problem 5.

5. A charge distribution has the charge density given by $\rho = Q\{\boldsymbol{\delta}(\boldsymbol{x} - \boldsymbol{x_0}) - \boldsymbol{\delta}(\boldsymbol{x} + \boldsymbol{x_0})$. For this charge distribution find the electric field at ($2x_0$,0, 0), considering only monopole and dipolar contributions.

Solution: Potential $V(r) = \frac{1}{4\pi\varepsilon_0}\left[\int_{-a}^{a} \frac{\rho(x')}{x} dx' + \int_{-a}^{a} \frac{\rho(x')}{x^2} x' dx' + \int_{-a}^{a} \frac{\rho(x')}{x^3} x'^2 dx' + \ldots\right]$

First term, total charge

$$Q_T = \int \rho(x')dx' = Q\int_{-x_0}^{x_0} \delta(x' - x_0)dx' - Q\int_{-x_0}^{x_0} \delta(x' + x_0)dx' = Q - Q = 0$$

Second term, dipole moment

$$p = \int x'\rho(x')dx' = Q\int_{-x_0}^{x_0} x'\delta(x' - x_0)dx' - Q\int_{-x_0}^{x_0} x'\delta(x' + x_0)dx' = Qx_0 - Q \times -x_0 = 2Qx_0$$

$$V = \frac{2Qx_0}{4\pi\varepsilon_0 x^2} \Rightarrow \vec{E} = -\frac{\partial V}{\partial x}\hat{x} = \frac{4Qx_0}{4\pi\varepsilon_0 x^3}\hat{x} = \frac{4Qx_0}{4\pi\varepsilon_0 (2x_0)^3}\hat{x} = \frac{Q}{8\pi\varepsilon_0 x_0^2}\hat{x}$$