Tutorial 6 Solutions

PHY 101 Monsoon 2024

Q1. A particle of mass m is moving in a potential $V(x) = ax^3 - bx^2$. Initially the particle is at rest at a stable point. What minimum speed to be given to the particle so that it reaches unstable point. Plot the potential vs x curve.

Solution:

Solution –

for equilibrium point $\left.\frac{dv}{dx}\right|_{x=x_0} = 0$

$$3ax_0^2 - 2bx_0 = 0$$

$$\Rightarrow x_0 = 0, \frac{2b}{3a}$$

Now, $\left.\frac{d^2v}{dx^2}\right|_{x=x_0} = 6ax_0 - 2b$

$$\left.\frac{d^2v}{dx^2}\right|_{x=0} = -2b < 0 \text{ , unstable point}$$

$$\left.\frac{d^2v}{dx^2}\right|_{x=\frac{2b}{3a}} = 2b > 0 \text{ , Stable point}$$

To calculate speed let us apply conservation of energy

$$K.E + P.E \text{ at } \left(x = \frac{2b}{a}\right) = KE + PE \text{ at} (x=0)$$

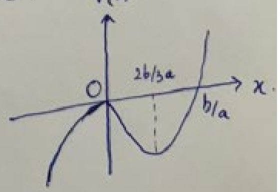
for minimum speed (u) at stable point particle will just reach unstable point & stop there

$$\frac{1}{2}mu^2 + V\left(x = \frac{2b}{3a}\right) = \frac{1}{2}m0^2 + V(x=0)$$

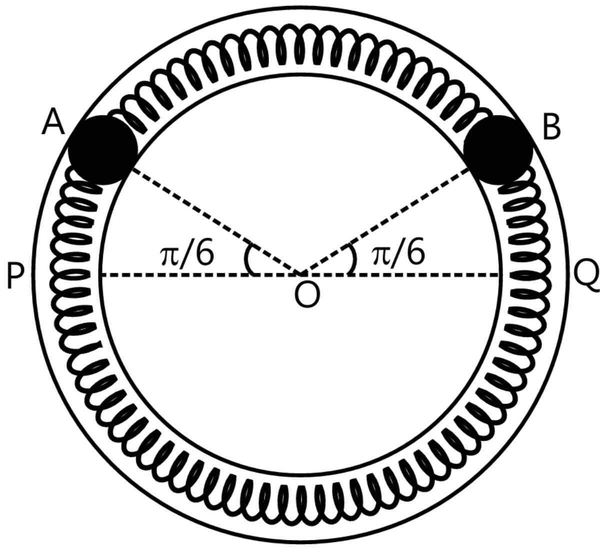
$$\frac{1}{2}mu^2 + a\left(\frac{2b}{3a}\right)^3 - b\left(\frac{2b}{3a}\right)^2 = 0 + 0$$

$$u^2 = -\frac{2}{m}\left(\frac{2b}{3a}\right)^2\left(\frac{2b}{3} - b\right)$$

$$\boxed{u = \sqrt{\frac{8b^3}{27ma^2}}}$$



Q2.Two identical balls A and B, each of mass 0.1 kg are attached to two identical massless springs. The spring mass system is constrained to move inside a rigid smooth pipe bent in the form of a circle as shown in figure. The pipe is fixed in a horizontal plane. The centers of the balls can move in a circle of radius 0.06 m. Each spring has a natural length 0.06π m and spring constant 0.1 N/m. Initially both the balls are displaced by angle $\pi/6$ radian with respect to the diameter PQ of the circle and released from rest.



(a) Calculate the frequency of oscillation of ball B.

(b) Find the speed of the ball A when A and B are at the two ends of diameter PQ.

(c) What is the total energy of the system.

Solution

Here the two balls connected by the springs are free to oscillate along the length of the springs, so the time period will depend on the reduced mass of the two-ball system.

(a) Restoring force on A or B $= k\Delta x + k\Delta x = 2k\Delta x.$

Where Δx is compression in the spring at one end? Effective force constant = 2k

Frequency $v = \frac{1}{2\pi}\sqrt{\frac{2k}{\mu}}$

Where μ is reduced mass of system.

$$\text{reduced mass.}\ \mu = \frac{mm}{m+m} = \frac{m}{2}$$

$$v = \frac{1}{2\pi}\sqrt{\frac{2k}{m/2}} = \frac{1}{3.14}\sqrt{\frac{0.1}{0.1}} = \frac{1}{3.14}s$$

(b) P and Q are equilibrium position. Balls A and B at P and Q have only kinetic energy and it is equal the potential energy at extreme positions.

Potential energy at extreme position

$$= \frac{1}{2}k(2\Delta x)^2 + \frac{1}{2}k(2\Delta x)^2 = 4k(\Delta x)^2$$

$$\text{Where}\ \Delta x = R x \frac{\pi}{6}$$

$$\Rightarrow \text{P.E.} = \frac{\pi^2 kR^2}{36} = \frac{(3.14)^2 x 0.1 x (0.06)^2}{36} \approx 3.94 x 10^{-4}\,J$$

When the balls A and B are at points P and Q respectively.

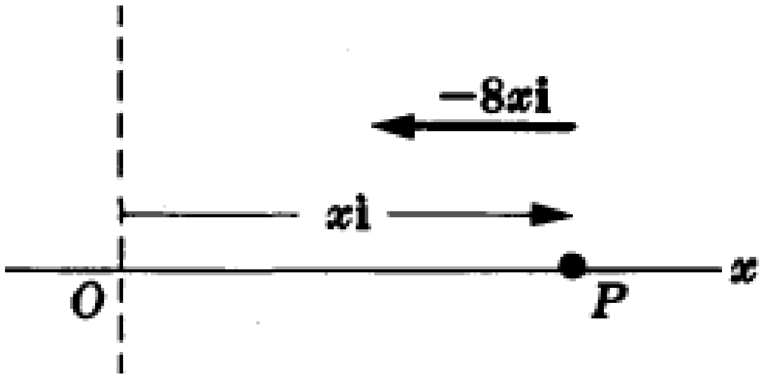
$$KE_{(A)} + KE_{(B)} = PE.;\ 2KE_{(A)} = P.E.$$

$$2x\frac{1}{2}mv^2 = 3.94\,x10^{-4}$$

$$\Rightarrow v = \left(\frac{3.94}{0.1}\right)^{\frac{1}{2}} x10^{-2} = 6.28x10^{-2} = 0.0628\,ms^{-1}$$

(c) Total potential and kinetic energy of the system is equal to total potential energy at the extreme position=$3.94x10^{-4}$J.

Q3.: A particle P of mass 2 unit moves along x-axis, as shown in figure, attracted toward origin O by a force whose magnitude is numerically equal to *8x*. If it is initially at rest at *x = 20* unit, find



(a) the differential equation of motion,

(b) the position of the particle at any time,

(c) the speed and velocity of the particle at any time, and

(d) the amplitude, period and frequency of the vibration.

Solutions:

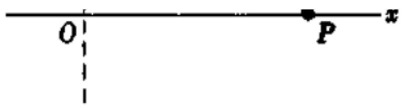


Fig. 4-7

(*a*) Let $\mathbf{r} = x\mathbf{i}$ be the position vector of P. The acceleration of P is $\frac{d^2}{dt^2}(x\mathbf{i}) = \frac{d^2x}{dt^2}\mathbf{i}$. The net force acting on P is $-8x\mathbf{i}$. Then by Newton's second law,

$$2\frac{d^2x}{dt^2}\mathbf{i} = -8x\mathbf{i} \quad \text{or} \quad \frac{d^2x}{dt^2} + 4x = 0 \tag{1}$$

which is the required differential equation of motion. The initial conditions are

$$x = 20, \; dx/dt = 0 \quad \text{at} \quad t = 0 \tag{2}$$

(*b*) The general solution of (*1*) is

$$x = A\cos 2t + B\sin 2t \tag{3}$$

When $t = 0$, $x = 20$ so that $A = 20$. Thus

$$x = 20\cos 2t + B\sin 2t \tag{4}$$

Then

$$dx/dt = -40\sin 2t + 2B\cos 2t \tag{5}$$

so that on putting $t = 0$, $dx/dt = 0$ we find $B = 0$. Thus (*3*) becomes

$$x = 20\cos 2t \tag{6}$$

which gives the position at any time.

(*c*) From (*6*) $dx/dt = -40\sin 2t$ which gives the speed at any time. The velocity is given by

$$\frac{dx}{dt}\mathbf{i} = -40\sin 2t\,\mathbf{i}$$

(*d*) Amplitude = 20. Period = $2\pi/2 = \pi$. Frequency = 1/period = $1/\pi$.

Q4.Potential energy function describing the interaction between two atoms of a diatomic molecule is $U(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$. For what value of distance r should the force acting between them will be zero. Given a = 1 and b = 2

Solution:

Potential energy U(r)=$\frac{a}{r^{12}} - \frac{b}{r^6}$.

Now, for equilibrium positions

$\frac{dU}{dr} = -F(r) = 0$

Now, putting the value of U

$$a(-12)r^{-13} - b(-6)r^{-7} = 0$$

$$\frac{2a}{r^{13}} = \frac{b}{r^7}$$

$$r^6 = \frac{2a}{b}$$

$$r = \left(\frac{2a}{b}\right)^{\frac{1}{6}}$$

$$r = 1 \quad \text{\{Putting a=1 , b=2\}}$$

Q5. Let ω > 0. A damped sinusoid $x(t) = Ae^{-at}\cos(\omega t)$ has "pseudo-period" $2\pi/\omega$. The pseudo-period, and hence ω, can be measured from the graph: it is twice the distance between successive zeros of x(t), which is always the same. Now what is the spacing between successive maxima of x(t)? Is it always the same, or does it differ from one successive pair of maxima to the next? Suppose that successive maxima of $x(t) = Ae^{-at}\cos(\omega t)$ occur at $t = t_0$ and $t = t_1$. What is the ratio $x(t_1)/x(t_0)$? (Hint: Compare $\cos(\omega t_0)$ and $\cos(\omega t_1)$.) Does this offer a means of determining the value of a from the graph?

Solution:

The extrema of $x(t) = Ae^{-at}\cos(\omega t)$ occur when $\dot{x}(t) = 0$, i.e., $-a\cos(\omega t) = \omega\sin(\omega t)$. The extrema are achieved at t where $\tan(\omega t) = -a/\omega$. Since minima and maxima of $x(t)$ are alternating, the maxima occur at every other such t, and the spacing between successive maxima is twice the period of $\tan(\omega t)$, which is just the pseudo-period $2\pi/\omega$. This is always the same since it doesn't depend on t or x.

As seen in Question 3, $t_1 - t_0 = 2\pi/\omega$. So $\cos(\omega t_1) = \cos(\omega t_0 + 2\pi) = \cos(\omega t_0)$ are the same, and the ratio is given by $x(t_1)/x(t_0) = e^{-a(t_1-t_0)}$. So a can be estimated graphically using the formula $a = \frac{\ln x(t_0) - \ln x(t_1)}{t_1 - t_0}$, where $(t_0, x(t_0))$ and $(t_1, x(t_1))$ are any two successive maxima.

Q6. For what value of b does $\ddot{x} + b\dot{x} + x = 0$ exhibit critical damping? For this value of b, what is the solution x_1 with $x_1(0) = 1$, $\dot{x}_1(0) = 0$? What is the solution x_2 with $x_2(0) = 0$, $\dot{x}_2(0) = 1$? What is the solution such that x with $x(0) = 2$, $\dot{x}(0) = 3$?

Solution:

Homogeneous second order linear equations with constant coefficients

The exponential e^{rt} is a solution of $m\ddot{x} + b\dot{x} + kx = 0$ (where m, b, and k are real constants, and $m \neq 0$) exactly when r is a root of the *characteristic polynomial* $p(s) = ms^2 + bs + k$.

(1) *Overdamped*: Roots real and distinct: the general solution is given by linear combinations of these two exponentials.

(2) *Underdamped*: Roots not real: they are $-\frac{b}{2m} \pm \omega_d i$ where $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$. The corresponding exponential solutions are complex conjugates of each other. To get basic real solutions take the real and imaginary parts of either one (again solutions since $\operatorname{Re} z$ and $\operatorname{Im} z$ are linear combinations of z and $\bar{z}$). The result is $x_1 = e^{-bt/2m}\cos(\omega_d t)$ and $x_2 = e^{-bt/2m}\sin(\omega_d t)$. ω_d is called the *damped circular frequency*. The general real solution is thus given by real linear combinations of these two, or, what is the same, $e^{-bt/2m}$ times the general sinusoid of circular frequency ω_d: $Ae^{-bt/2m}\cos(\omega_d t - \phi)$.

(3) *Critically damped*: Roots equal (and hence real, $r = -b/2m$). Then there are not enough exponential solutions and the general solution is $(a + ct)e^{-bt/2m}$.

The characteristic polynomial is $p(s) = s^2 + bs + 1$. For the system to be critically damped, the characteristic polynomial must be a perfect square, i.e., $p(s)$ must equal $(s-k)^2$ for some k. Multiplying and comparing gives $b = -2k$ and $k^2 = 1$. Therefore, $b = \pm 2$. When $b = -2$, e^t is a solution, and it exhibits exponential growth instead of damping, so we reject that value of b. So the system is critically damped when $b = 2$.

For this value of b, the general solution is $x(t) = (C_0 + C_1 t)e^{-t}$. The corresponding initial conditions for each solution are $x(0) = C_0$ and $\dot{x}(0) = -C_0 + C_1$. By using the given initial conditions to solve for the constants, we see that the normalized pair of solutions is $x_1 = e^{-t} + te^{-t}$ and $x_2 = te^{-t}$.

We can use these normalized solutions to read off the solution that satisfies any given inital condition. In particular, the solution with $x(0) = 2$ and $\dot{x}(0) = 3$ is

$$x = x(0) \cdot x_1 + \dot{x}(0) \cdot x_2 = 2x_1 + 3x_2 = 2e^{-t} + 5te^{-t}.$$