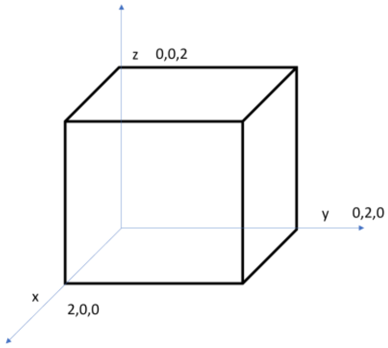
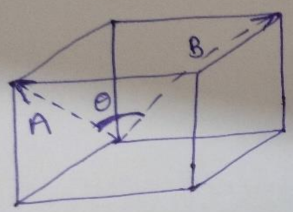
Tutorial 2 with solutions

Q1 Find the angle between the face diagonals of the given cube



Solⁿ :- As, the given side of the cube is 2. With one corner at the origin. The face diagonals A & B are



$$\vec{A} = 2\hat{x} + 0\hat{y} + 2\hat{z}$$

$$\vec{B} = 0\hat{x} + 2\hat{y} + 2\hat{z}$$

$$|\vec{A}| = \sqrt{4+0+4} = 2\sqrt{2}$$

$$|\vec{B}| = \sqrt{0+4+4} = 2\sqrt{2}$$

In abstract form :-

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

$$(2\hat{x} + 0\hat{y} + 2\hat{z}) \cdot (0\hat{x} + 2\hat{y} + 2\hat{z}) = 2\sqrt{2} \cdot 2\sqrt{2} \cos\theta$$

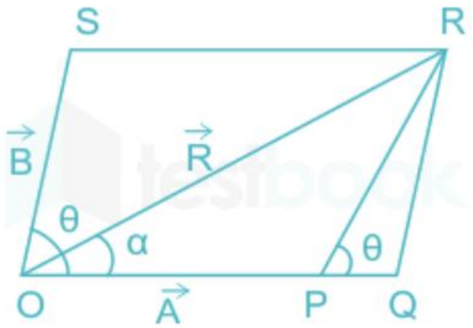
$$4 = 4 \times 2 \cos\theta$$

$$\cos\theta = 1/2$$

$$\boxed{\theta = 60^\circ}$$

Q2 A man uses a boat to cross the river if the velocity of the boat is 20 km/h having an angle of 60° the direction of river flow and the resultant velocity by which boat crosses the river is 25 km/h then what is the velocity by which river is flowing?

Solution: According to the parallelogram law of vector addition,



$R^2=A^2+B^2+2AB\cos\theta$

where A and B are the two vector quantity: θ = Angle between two vector quantity

V_B = 20 Km / h and V_{BR} =25 Km/h

Where V_B = Velocity of a boat; V_{BR} = Resultant velocity of boat and river.

V_R = Velocity of river

The angle between the velocity of the river and the velocity of the boat is θ= 90°

By Parallelogram Law of Vector Addition

$V_{BR}^2 = V_B^2 + V_R^2 +(2V_B V_R \cos\theta)$

$25^2 = 20^2 + V_R^2 + (2 * 20*V_R * \cos60°)$ $\cos60°=0.5$

$625 = 400 + V_R^2 + 20V_R$

Quadratic equation that we have to solve to get the value of V_R i.e velocity of River is

$V_R^2 + 20V_R - 225 =0$

$V_R = ½ [-20 \pm \sqrt{ \{400 – 4.1.(-225)\}}]$

i.e $-10 -5\sqrt{13}$ and $-10 +5\sqrt{13}$

value of $\sqrt{13} = 3.605$

therefore $V_R= -10 – (5*3.605) = -10 -18.02= -28.02$ and $V_R= -10 +(5*3.605)=-10 + 18.02 = 8.02$ km/h

Q3 Find the area of the triangle having vertices at P(1, 3, 2), Q(2, -1, 1), R(-1, 2, 3).

Solution:

$$\mathbf{PQ} = (2-1)\mathbf{i} + (-1-3)\mathbf{j} + (1-2)\mathbf{k} = \mathbf{i} - 4\mathbf{j} - \mathbf{k}$$
$$\mathbf{PR} = (-1-1)\mathbf{i} + (2-3)\mathbf{j} + (3-2)\mathbf{k} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\text{area of triangle} = \tfrac{1}{2}|\mathbf{PQ}\times\mathbf{PR}| = \tfrac{1}{2}|(\mathbf{i}-4\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}-\mathbf{j}+\mathbf{k})|$$
$$= \tfrac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}\right| = \tfrac{1}{2}|-5\mathbf{i}+\mathbf{j}-9\mathbf{k}| = \tfrac{1}{2}\sqrt{(-5)^2+(1)^2+(-9)^2} = \tfrac{1}{2}\sqrt{107}.$$

Q4 Two vectors A and B have equal magnitudes of 10 units. Vector A makes an angle of 30 degrees with the positive x-axis, while vector B makes an angle of 45 degrees with the positive y-axis. Calculate the dot product and cross product of vectors A and B.

Answer: Magnitude of vector A = Magnitude of vector B = 10 units

Angle between vector A and the positive x-axis = 30^0
Angle between vector B and the positive y-axis = 45^0

First, let's represent vectors A and B in component form:
Vector A (A_x, A_y):
A_x = Magnitude of A * cos (angle with x-axis)
$A_x = 10 * \cos(30^0) = 10 * \sqrt{3} / 2 = 5\sqrt{3}$
A_y = Magnitude of A * sin (angle with x-axis)
$A_y = 10 * \sin(30^0) = 10 * 1/2 = 5$
So, vector A can be represented as $A = 5\sqrt{3}i + 5j$
Vector B (B_x, B_y):
B_x = Magnitude of B * cos (angle with y-axis)
$B_x = 10 * \cos(45^0) = 10 * 1/\sqrt{2} = 5\sqrt{2}$
B_y = Magnitude of B * sin (angle with y-axis)
$B_y = 10 * \sin(45^0) = 10 * 1/\sqrt{2} = 5\sqrt{2}$
So, vector B can be represented as $B = 5\sqrt{2}i + 5\sqrt{2}j$

Now, let's calculate the dot product $(A \cdot B)$:
$A \cdot B = (5\sqrt{3}i + 5j) \cdot (5\sqrt{2}i + 5\sqrt{2}j)$
Using the dot product formula, $A \cdot B = (A_x * B_x) + (A_y * B_y)$:
$A \cdot B = (5\sqrt{3} * 5\sqrt{2}) + (5 * 5\sqrt{2})$
$A \cdot B = (25\sqrt{6}) + (25\sqrt{2})$
$A \cdot B = 25(\sqrt{6} + \sqrt{2})$ units

Now, let's calculate the cross-product $(A \times B)$:
The cross product of two vectors in 2D is always a scalar, and its magnitude can be calculated as:
$|A \times B| = |A| * |B| * \sin(\theta)$
Where θ is the angle between vectors A and B, which is 90 degrees in this case because they are perpendicular. Also, $|A| = 10$ and $|B| = 10$.
$|A \times B| = 10 * 10 * \sin(90^0) = 100 * 1 = 100$ units

Q5 Find the unit vector parallel to resultant vector of $\mathbf{r_1} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{r_2} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Solution

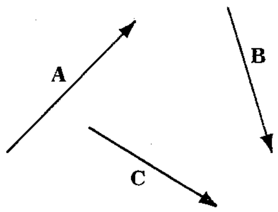
Resultant $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 = (2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

$R = |\mathbf{R}| = |3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}| = \sqrt{(3)^2 + (6)^2 + (-2)^2} = 7$.

Then a unit vector parallel to $\mathbf{R}$ is $\dfrac{\mathbf{R}}{R} = \dfrac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{7} = \dfrac{3}{7}\mathbf{i} + \dfrac{6}{7}\mathbf{j} - \dfrac{2}{7}\mathbf{k}$.

Check: $\left|\dfrac{3}{7}\mathbf{i} + \dfrac{6}{7}\mathbf{j} - \dfrac{2}{7}\mathbf{k}\right| = \sqrt{(\frac{3}{7})^2 + (\frac{6}{7})^2 + (-\frac{2}{7})^2} = 1$.

Q6 Given vectors **A**, **B** and **C** construct (a) **A**-**B**+2**C** and (b) 3**C**- 1/2 (2**A**-**B**).(use scale and only parallelly shift the vectors and the resultants)



Solution

(a)

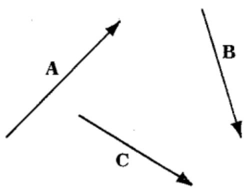


Fig. 1 (a)

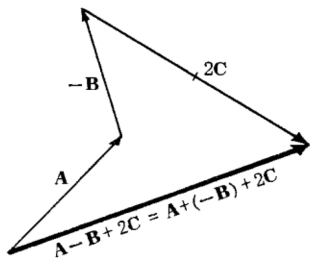


Fig. 2 (a)

(b)