PHY 102 Introduction to Physics II

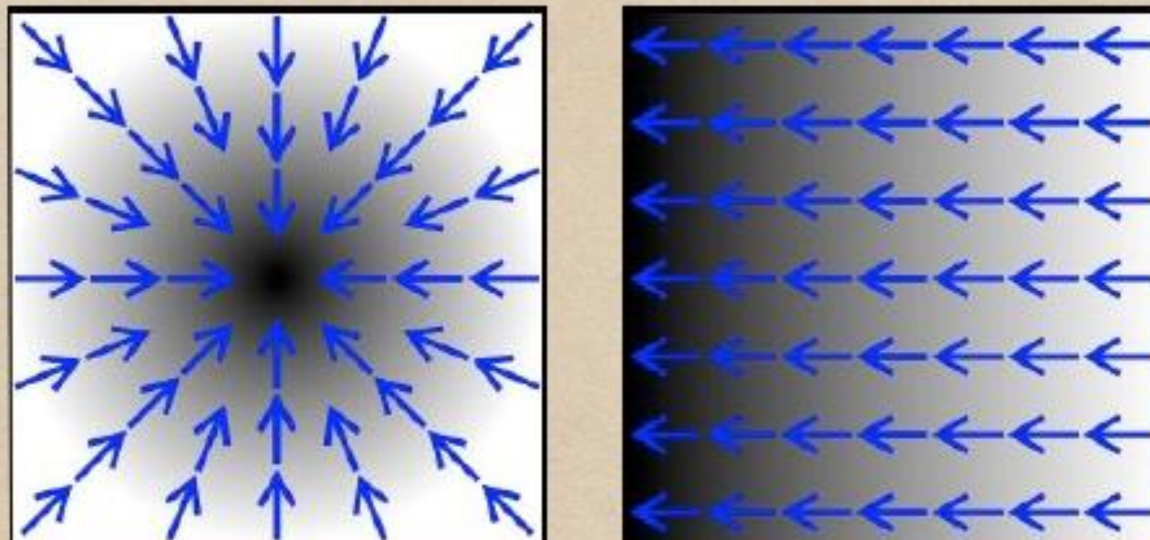
Spring Semester 2025

Lecture 2

Geometrical Interpretation of the Gradient

Derivatives of Field

Geometrical Interpretation of the Gradient



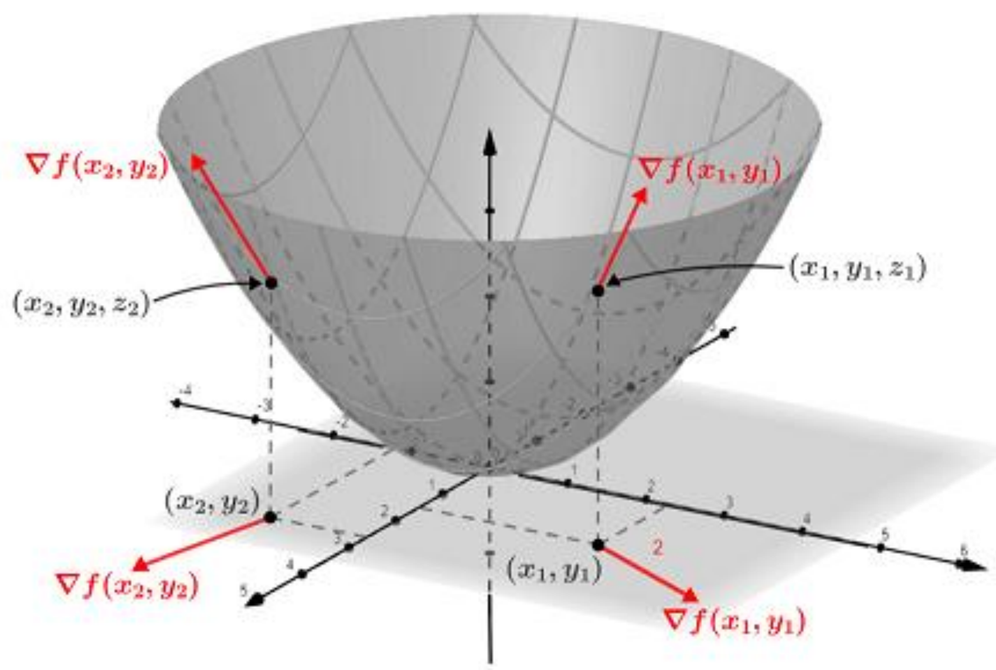
In the above two images, the values of the function are represented in black and white, black representing higher values, and its corresponding gradient is represented by blue arrows. (Source: Wikipedia)

Derivatives of Field

Geometrical Interpretation of the Gradient

z = f(x, y) = $x^2 + y^2$

Paraboloid opening upward along the z-axis whose vertex is at the origin



The gradient is

$$\nabla f(x,y) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f(x,y)$$

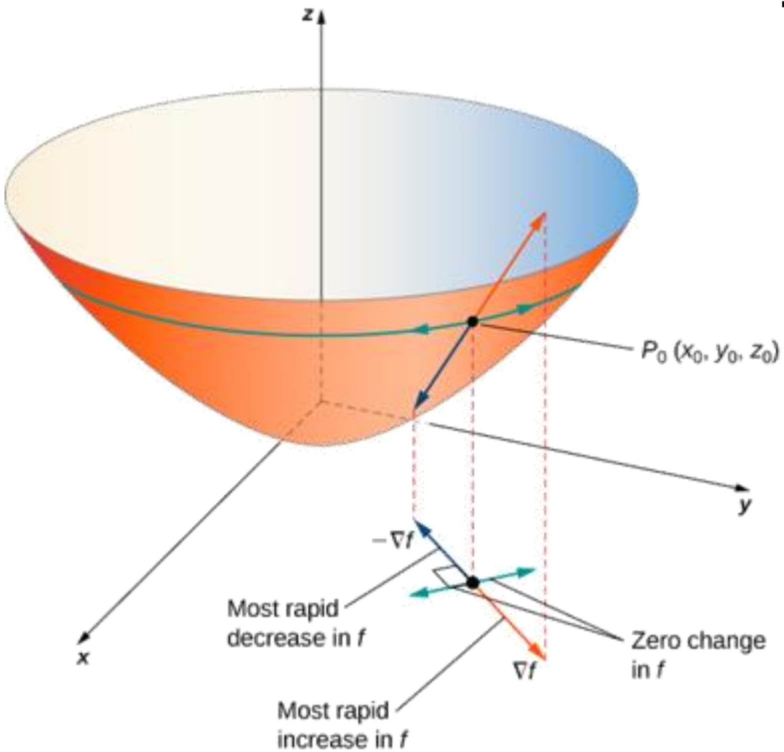
$$= \hat{i}\frac{\partial f(x,y)}{\partial x} + \hat{j}\frac{\partial f(x,y)}{\partial y} = 2x\,\hat{i} + 2y\,\hat{j}$$

- The magnitude of the gradient represents the slope along the tangent to the surface.
- The direction of the gradient points in the direction of the greatest rate of increase of the function.

Geometrical Interpretation of the Gradient

z = f(x, y) = $x^2 + y^2$

Paraboloid opening upward along the z-axis whose vertex is at the origin



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$$\nabla f(x,y) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f(x,y)$$

$$= \hat{i}\frac{\partial f(x,y)}{\partial x} + \hat{j}\frac{\partial f(x,y)}{\partial y} = 2x\,\hat{i} + 2y\,\hat{j}$$

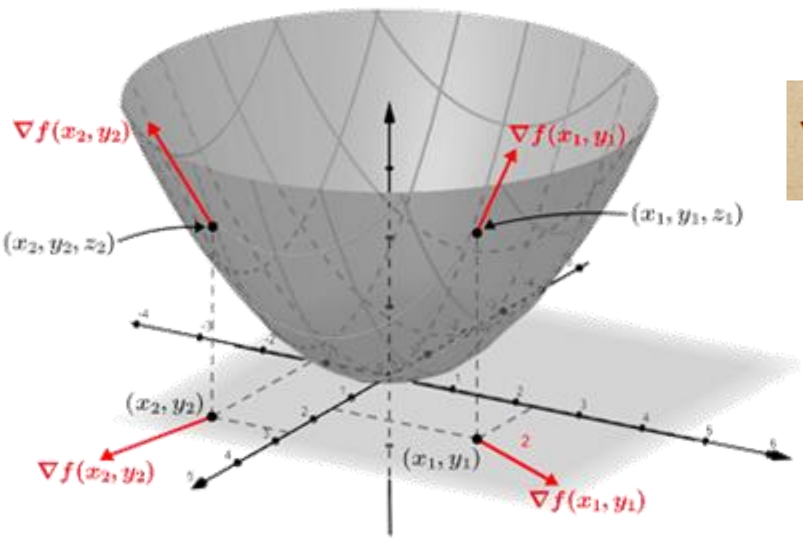
At point (x=1,y=0), $\nabla f = 2\hat{\imath} \Rightarrow$ The maximum change in f will occur if we move along $\hat{\imath}$ direction i.e., along the x-axis. The magnitude of rate of increase is *2.*

At point (x=1,y=1), $\nabla f = 2\hat{\imath} + 2\hat{\jmath} \Rightarrow$ The maximum change in f will occur if we move along in the direction of *2(i+j),* i.e., in a direction making 45° (anticlockwise) with the x-axis.

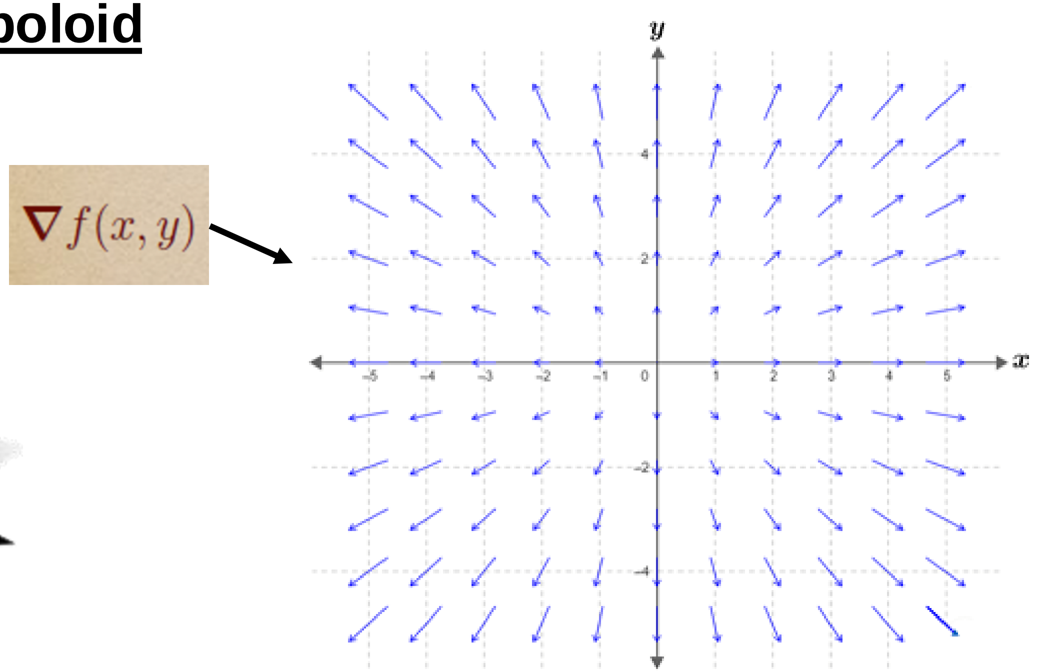
Derivatives of Field

Geometrical Interpretation of the Gradient

$z = f(x, y) = x^2 + y^2$ **Paraboloid**



Gradient vector field



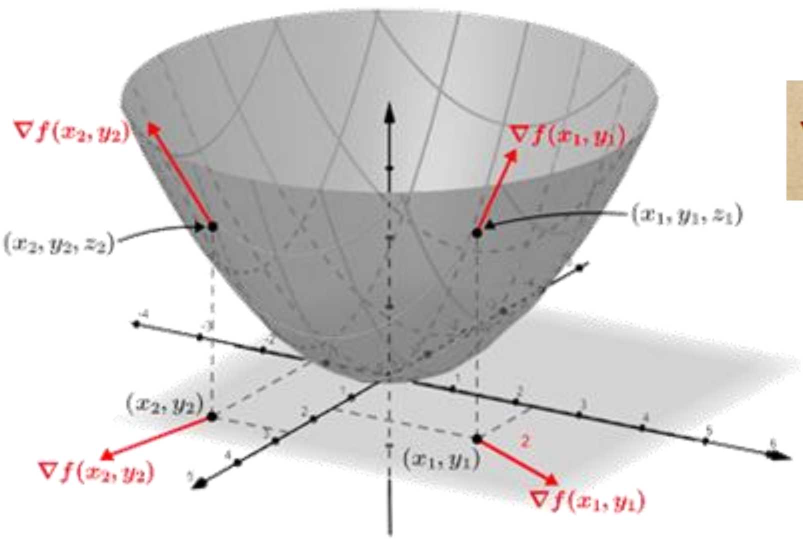
The gradient vector field gives a **2D** view of the direction of greatest increase for a 3D figure.

The maximum rate of change is achieved if we move radially outward.

The vectors lengthen as they move away from the origin, confirming that the surface of the paraboloid gets steeper the further from the origin it gets.

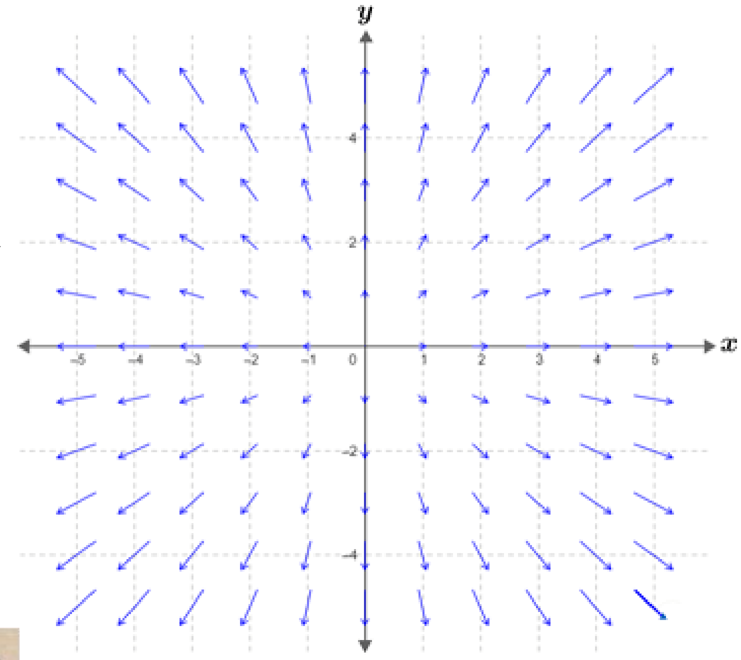
Geometrical Interpretation of the Gradient

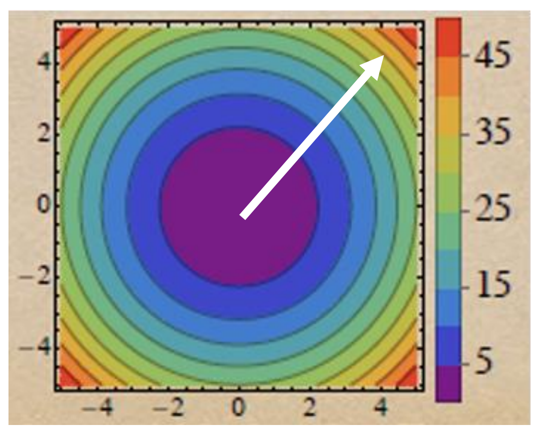
z = f(x, y) = $x^2 + y^2$ **Paraboloid**



$\nabla f(x, y)$

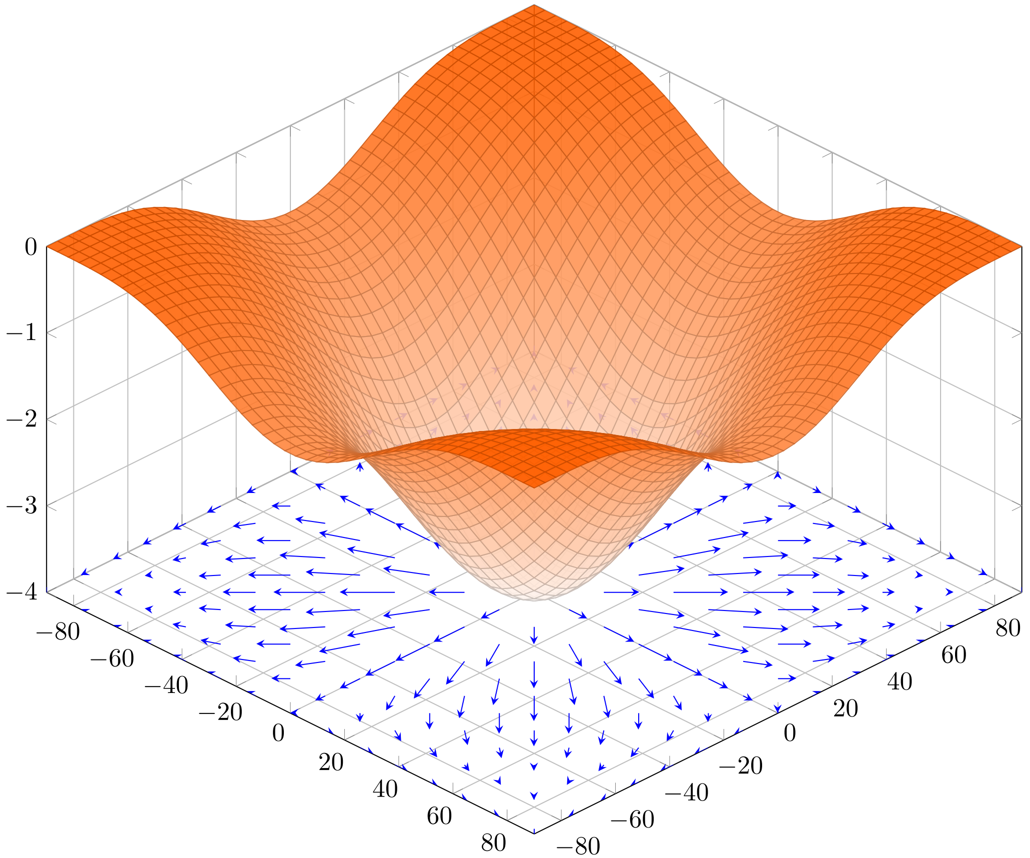
Gradient vector field





Contours (loci of x,y) for several f (fixed z) values. The direction of gradient will be perpendicular to the contour curves (circles in this case).

Geometrical Interpretation of the Gradient

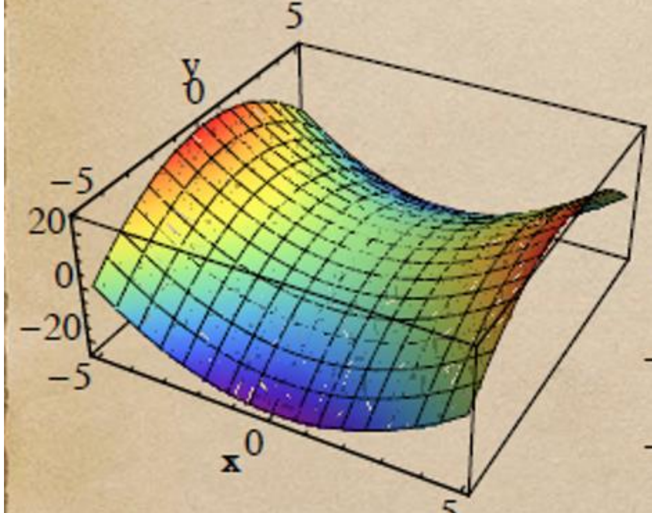


Derivatives of Field

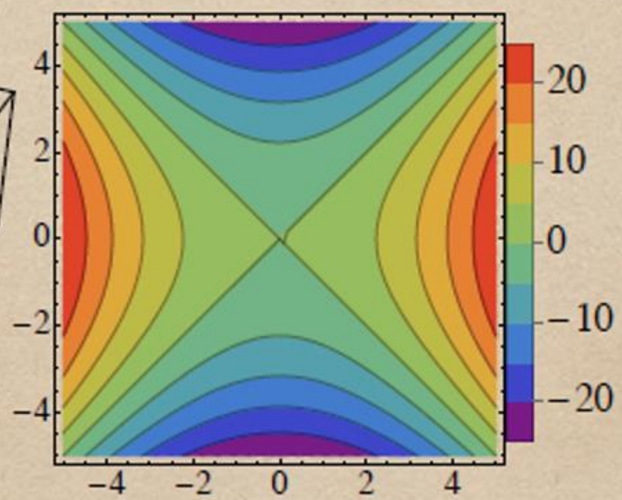
Geometrical Interpretation of the Gradient

Consider $f(x,y) = x^2 - y^2$

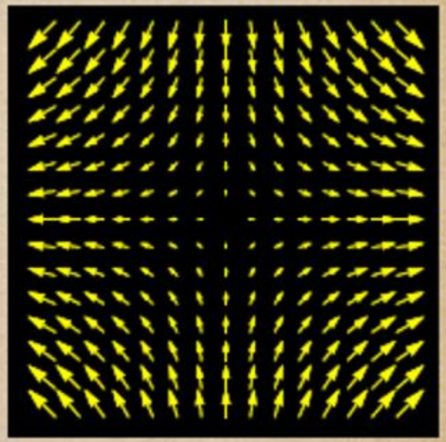
Gradient in this case is $\nabla f(x,y) = 2x\,\hat{i} - 2y\,\hat{j}$



The function $f(x,y)=x^2-y^2$



Contours (loci of x,y) for several f (fixed z) values



Gradient vectors

Derivatives of Field

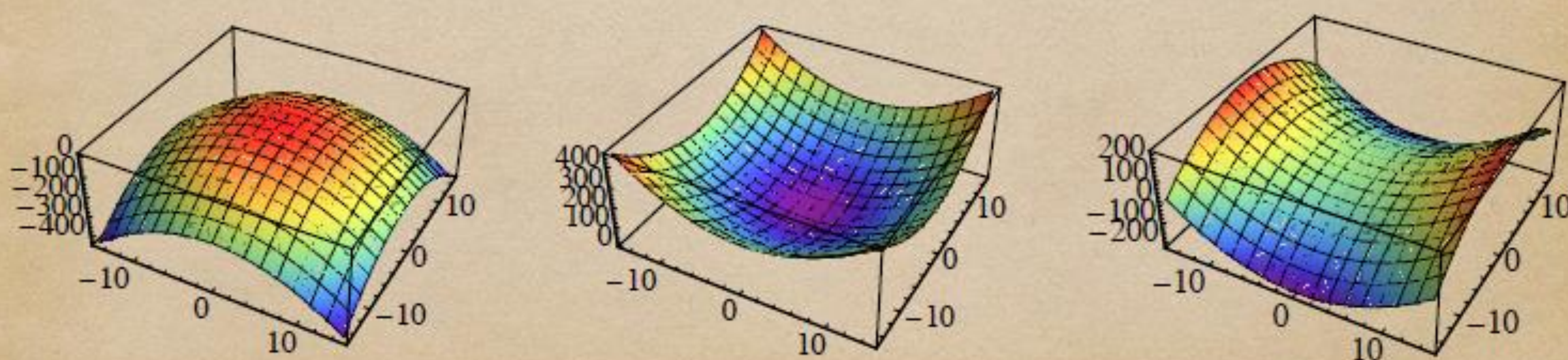
Geometrical Interpretation of the Gradient

What if $\nabla f(x, y, z) = 0$ at some point (x_0, y_0, z_0)?

Since $df = (\nabla f) \cdot d\mathbf{r}$, it would mean that for small displacement (dx, dy, dz) around (x_0, y_0, z_0), $df = 0$.

Thus (x_0, y_0, z_0) would be a stationary point of the function $f(x, y, z)$. It can be a maximum, a minimum or a saddle point.

The following figures depict these scenarios for a two-dimensional case:



(This is analogous to maximum, minimum and inflection in the one-dimensional case.)

Del (or Nabla) operator

The expression

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

can be thought of as the **Del** (or **Nabla**) operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

acting on $f \equiv f(x,y,z)$. The del operator ∇ acts on the scalar function f and returns a vector quantity.

Del (or Nabla) operator

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

which means, of course,

$$\nabla_x = \frac{\partial}{\partial x}, \qquad \nabla_y = \frac{\partial}{\partial y}, \qquad \nabla_z = \frac{\partial}{\partial z}.$$

- We cannot treat $\boldsymbol{\nabla}$ itself as a vector in the usual sense. Unless it acts on a function, it is without any specific meaning.
- However, it does mimic the behavior of an ordinary vector.

Gradient of a function from directional derivative

Say any function $f(x, y)$ denote 'height' of a mountain range at $\boldsymbol{a}(x, y)$

$\frac{\partial f}{\partial x}$ ⇨ Slope (rate of increase in f) in 'positive' x direction

$\frac{\partial f}{\partial y}$ ⇨ Slope (rate of increase in f) in 'positive' y direction

Generalize the partial derivative to calculate the slope along *any direction*- the result is called directional derivative

Definition

$$\boxed{D_u f(\boldsymbol{a}) = \lim_{h\to 0} \frac{f(\boldsymbol{a} + h\boldsymbol{u}) - f(\boldsymbol{a})}{h}}$$

Directional derivative of function f at $\boldsymbol{a}(x, y)$ in the direction of $\boldsymbol{u}$

http://mathinsight.org/directional_derivative_gradient_introduction

$$D_u f(\boldsymbol{a}) = \lim_{h \to 0} \frac{f(\boldsymbol{a} + h\boldsymbol{u}) - f(\boldsymbol{a})}{h}$$

$D_u f(\boldsymbol{a})$ is the slope of $f(x, y)$ when standing at point $\boldsymbol{a}(x, y)$ and facing the direction given by $\boldsymbol{u}$

In most cases, there is one direction '$\boldsymbol{u}$', where $D_u f(\boldsymbol{a})$ is largest. Let's call this ***direction of maximal slope 'm'***

Gradient (∇f) is a vector that points along the direction of 'm' and whose magnitude is $D_m f(\boldsymbol{a})$

Magnitude of $\boldsymbol{\nabla} f$

$$|\boldsymbol{\nabla} f(\mathrm{a})| = D_m f(\boldsymbol{a})$$

Direction of $\boldsymbol{\nabla} f(\boldsymbol{a})$

$$\boldsymbol{m} = \frac{\boldsymbol{\nabla} f(\boldsymbol{a})}{|\boldsymbol{\nabla} f(\mathrm{a})|}$$

$$D_u f(\boldsymbol{a}) = \boldsymbol{\nabla} f . \hat{u}$$

$$= |\boldsymbol{\nabla} f||\hat{u}|cos\theta$$

$$= |\boldsymbol{\nabla} f|cos\theta$$

$\theta = 0;$ $D_u f(\boldsymbol{a})$ and $\boldsymbol{\nabla} f$ are in same direction

$$D_u f(\boldsymbol{a}) = |\boldsymbol{\nabla} f|$$

$\theta = \pi;$ $D_u f(\boldsymbol{a})$ and $\boldsymbol{\nabla} f$ are in opposite directions

$$D_u f(\boldsymbol{a}) = -|\boldsymbol{\nabla} f|$$

Problems

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

Divergence of vector fields, its interpretation and visualization of vector fields

$$\boldsymbol{\nabla} \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\ \frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\ , \boldsymbol{Del\ Operator}$$

$$\boldsymbol{Operator\ \equiv (Mathematical\ Operation)}$$
$$\boldsymbol{Operate\ on\ some\ thing\ to\ produce\ an\ other\ function}$$

$$\boldsymbol{For\ example}$$
$$\boldsymbol{Del\ Operator\ operates\ on\ a\ scalar\ field\ \ to\ produce\ gradient\ of\ the\ scalar\ field}$$

$$\boldsymbol{Gradient\ of\ scalar\ field\ T(x,y,z)} = \boldsymbol{\nabla} T$$

$$\boldsymbol{\nabla} \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\,\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\ ,\boldsymbol{Del\ Operator}$$

How will you apply $\boldsymbol{\nabla}$ on a vector field ?

Let $\quad \boldsymbol{V} = V_1\hat{x} + V_2\hat{y} + V_3\hat{z}$

$(\boldsymbol{i})$ $\boldsymbol{\nabla \cdot V}$ $\boldsymbol{Divergence\ \ of\ V}$

$(\boldsymbol{ii})$ $\boldsymbol{\nabla \times V}$ $\boldsymbol{Curl\ \ of\ V}$

$$\boldsymbol{\nabla \cdot V} = \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$\boldsymbol{\nabla \times V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{x} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\hat{y} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{z}$$

Divergence

The divergence of a vector function V is defined as

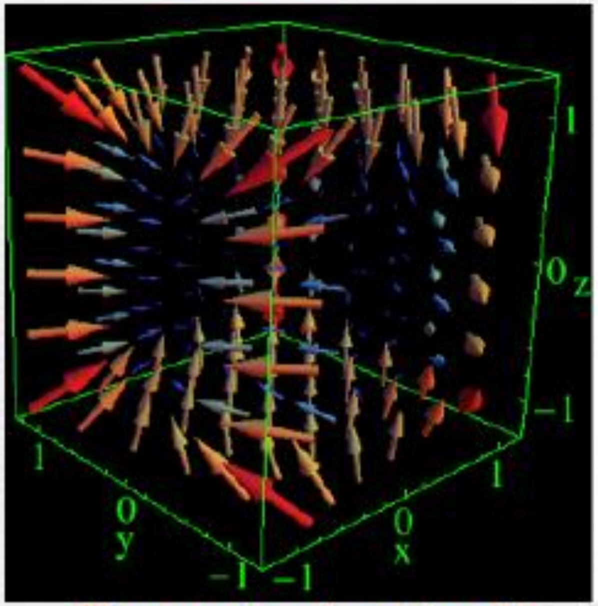
$$\nabla \cdot \mathbf{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot (\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

For example, consider

$$\mathbf{V} \equiv \mathbf{V}(x, y, z) = \hat{i}\, x^2 y + \hat{j}\, xy - \hat{k}\, z^3$$

then $\nabla \cdot \mathbf{V} = 2xy + x - 3z^2$

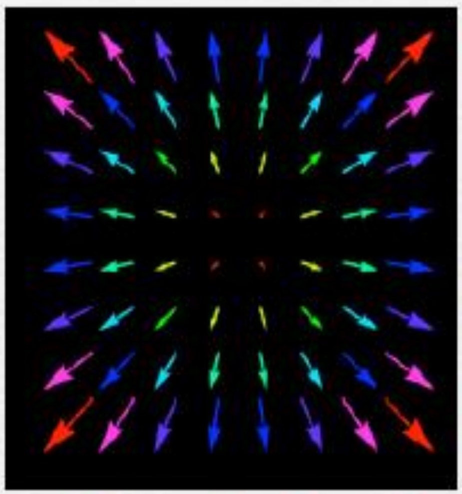


The vector function **V**

Note that $\mathbf{V}$ is a vector-valued function (vector field). At each point (x,y,z) , there's a vector associated with it.

Geometrical interpretation of Divergence

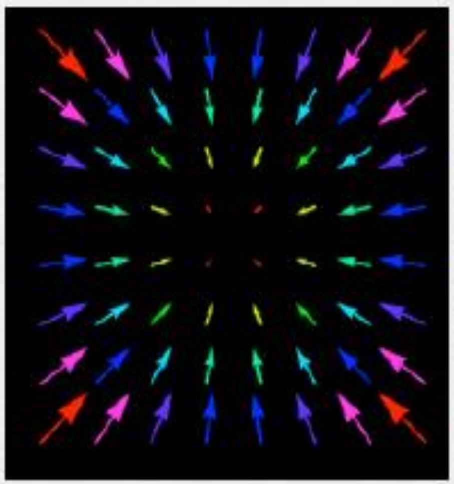
Divergence of a vector function $\mathbf{V}$ serves as the measure of how much the vector spreads out (diverges) from the concerned point.



$\mathrm{V}(x,y) = \hat{\imath}\, x + \hat{\jmath}\, y$

→ Positive divergence

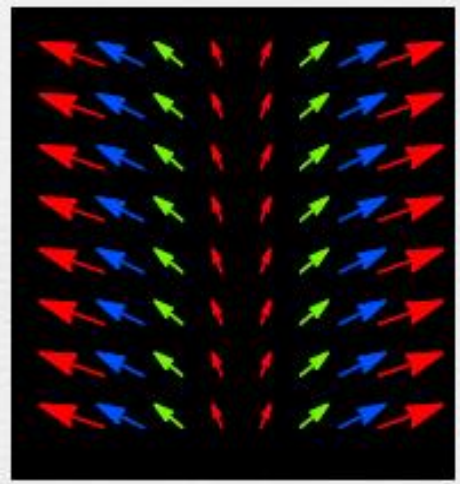
$(\nabla \cdot \mathbf{V} = 2)$



$\mathrm{V}(x,y) = -\hat{\imath}\, x - \hat{\jmath}\, y$

→ Negative divergence

$(\nabla \cdot \mathbf{V} = -2)$



$\mathrm{V}(x,y) = \hat{\imath}\, x + \hat{\jmath}$

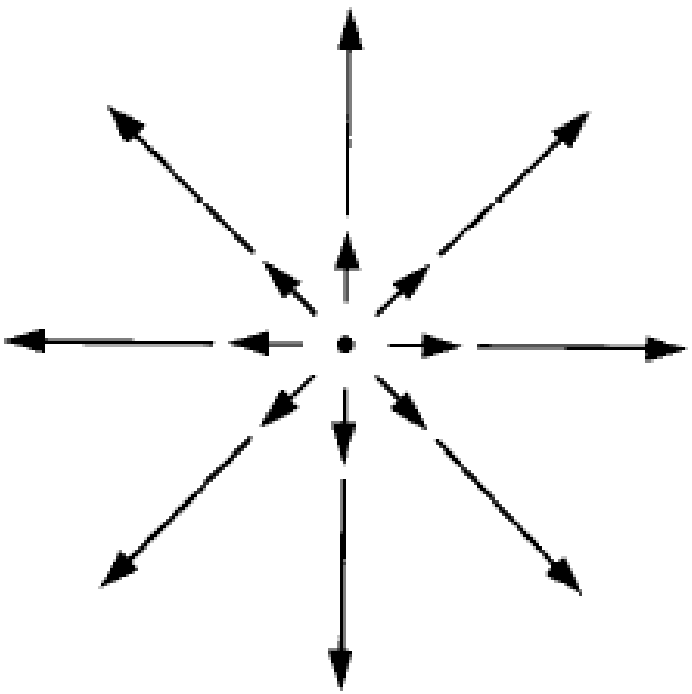
→ Positive divergence

$(\nabla \cdot \mathbf{V} = 1)$

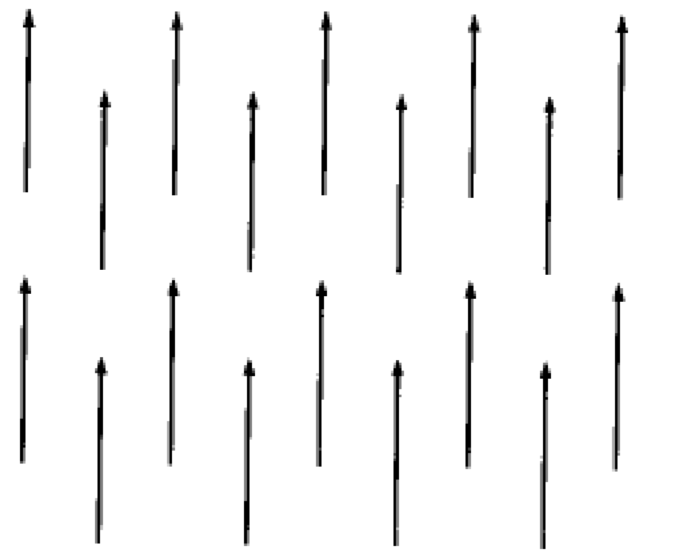
$\mathrm{V}(x,y) = \hat{\imath} + \hat{\jmath}$

→ No divergence

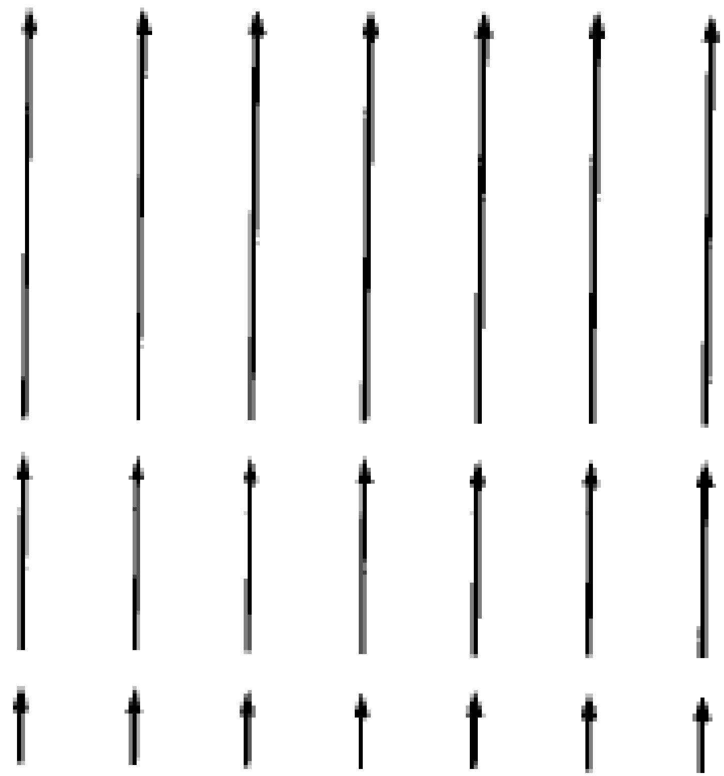
$(\nabla \cdot \mathbf{V} = 0)$



Vector function having large positive divergence



Vector function having zero divergence



Vector function having positive divergence

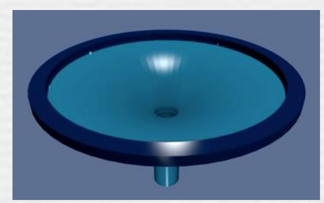
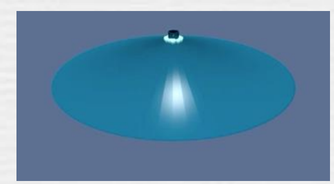
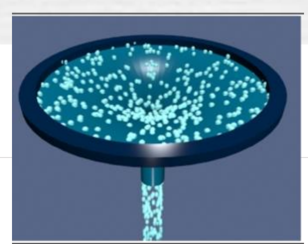
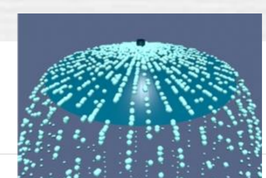
Geometrical interpretation of Divergence

In this example, the vector function is the water flow $\mathbf{W}$.

Let us examine the behavior of flow on the surface of the basin.

- Positive divergence means that the field is "flowing" out of a region (source/faucet).
- Negative divergence means that the field is "flowing" into a region (sink/ drain).
- Zero divergence signifies that the amount that "flows" in must be equal to the amount that flows out.

Negative divergence | Zero divergence | Positive divergence

Divergence of a vector function

P1. Consider the example of a vector field $\vec{A} = \hat{\boldsymbol{r}} r^n$

Find out the divergence of the vector field, i.e. $\nabla \cdot \vec{A}$

$$\nabla \cdot \vec{A} = (2 + n) r^{n-1}$$

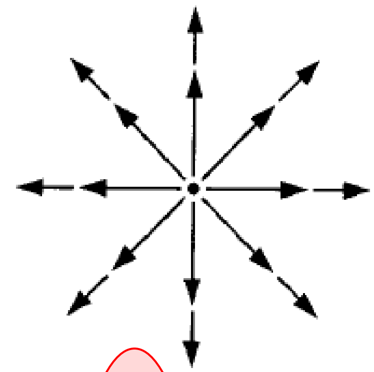
Prove it!

n	3	2	1	0	-1	-2	-3	-4
$\vec{A}$	$\hat{\boldsymbol{r}} r^3$	$\hat{\boldsymbol{r}} r^2$	$\boldsymbol{r}$	$\hat{\boldsymbol{r}}$	$\frac{\hat{\boldsymbol{r}}}{r}$	$\frac{\hat{\boldsymbol{r}}}{r^2}$	$\frac{\hat{\boldsymbol{r}}}{r^3}$	$\frac{\hat{\boldsymbol{r}}}{r^4}$
$\nabla \cdot \vec{A}$	$5r^2$	4r	3	$\frac{2}{r}$	$\frac{1}{r^2}$	0	$-\frac{1}{r^4}$	$-\frac{2}{r^5}$

Divergence of a vector function

$$\vec{A} = \hat{\boldsymbol{r}}r^n$$

$\vec{A}$ is *a divergent* vector field for all values of '*n*'



An example of divergent field

$$\vec{A} = \frac{\hat{\boldsymbol{r}}}{r^2}$$

n	3	2	1	0	-1	-2	-3	-4
$\vec{A}$	$\hat{\boldsymbol{r}}r^3$	$\hat{\boldsymbol{r}}r^2$	$\boldsymbol{r}$	$\hat{\boldsymbol{r}}$	$\frac{\hat{\boldsymbol{r}}}{r}$	$\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}$	$\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^3}$	$\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^4}$
$\nabla \cdot \vec{A}$	$5r^2$	4r	3	$\frac{2}{r}$	$\frac{1}{r^2}$	0	$-\frac{1}{r^4}$	$-\frac{2}{r^5}$

$\nabla \cdot \vec{A}$ increasing with r

$\nabla \cdot \vec{A}$ decreasing with r