PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 25

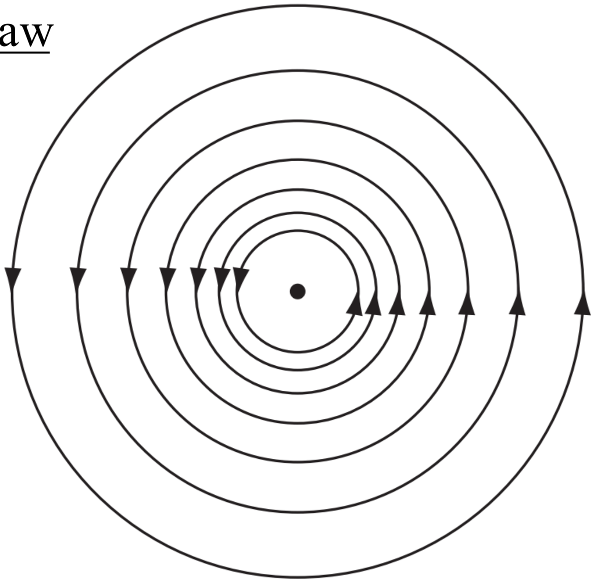
THE DIVERGENCE AND CURL OF B

Ampère's Law for steady currents of arbitrary shapes

THE DIVERGENCE AND CURL OF B

Straight-Line Currents

Magnetic field of a straight infinitely long wire: Ampere's Law



$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

What is the curl of such ***B***?

Consider line integral of B, around a *circular* path, of radius 's'

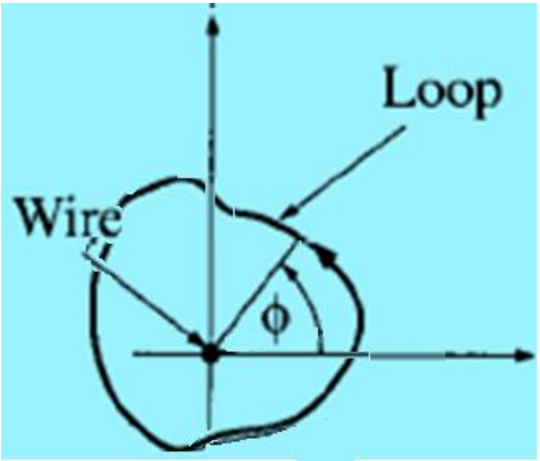
$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I$$

Direction of **B** is same as that of ***dl*** – hence, the dot product disappears

Line integral of ***B*** around a closed loop is proportional to *I*

Line integral of ***B*** is *independent* of 's'- as you go further away from the current carrying wire, ***B*** decreases at the same rate as 's' increases

For *non-circular* paths enclosing the current, the line integral of ***B*** is also proportional to I



$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

Use cylindrical co-ordinates ($\boldsymbol{s}, \boldsymbol{\varphi}, \boldsymbol{z}$)

$$\boldsymbol{dl} = ds\hat{\mathbf{s}} + sd\varphi\hat{\boldsymbol{\varphi}} + dz\hat{\mathbf{z}}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} s\, d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

Line integral of ***B*** around any closed path is $\mu_0 I$. This assumes that the <u>*loop encircles the current carrying wire* </u>only once.

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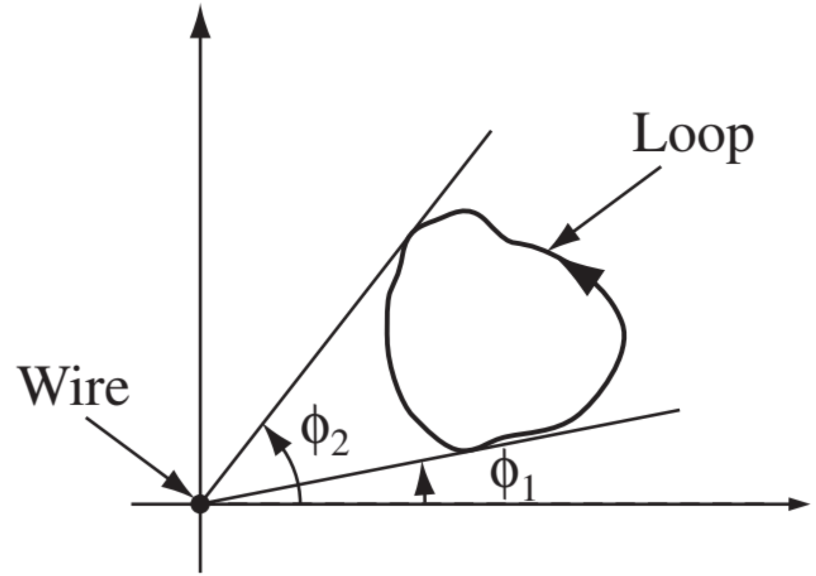
For *the wire outside the loop* , **Use cylindrical co-ordinates ($\boldsymbol{s}, \boldsymbol{\varphi}, \boldsymbol{z}$)**

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

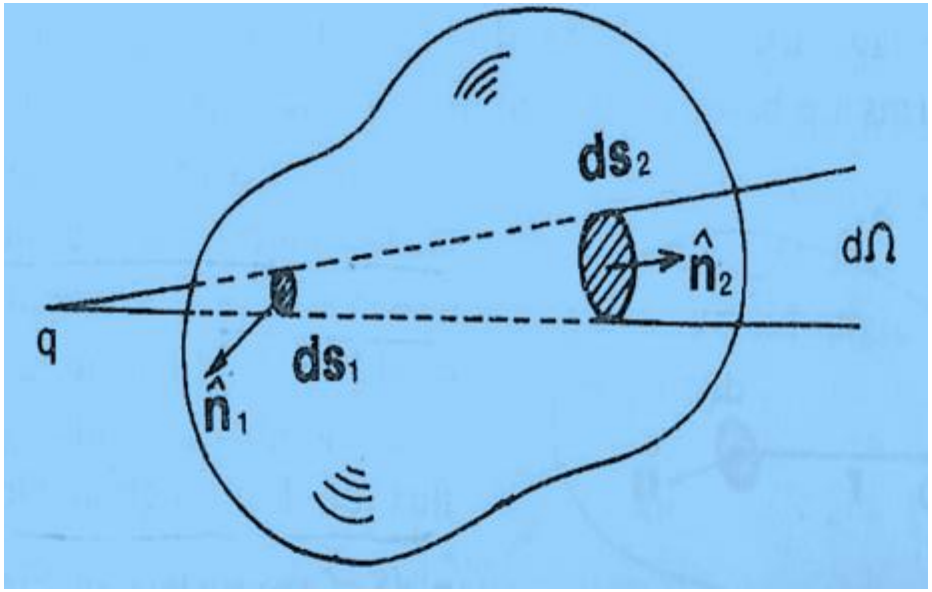
$$\boldsymbol{dl} = ds\hat{\mathbf{s}} + sd\varphi\hat{\boldsymbol{\varphi}} + dz\hat{\mathbf{z}}$$



$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s}\, s\, d\phi = \frac{\mu_0 I}{2\pi}\int_{\phi_1}^{\phi_2=\phi_1} d\phi = \frac{\mu_0 I}{2\pi}(\phi_1 - \phi_1) = 0$$

If the wire is outside the loop, φ would go from φ_1 to φ_2 and then back again from φ_2 to φ_1, so $\int d\varphi = 0$ and hence $\oint \boldsymbol{B}.\boldsymbol{dl} = 0$.

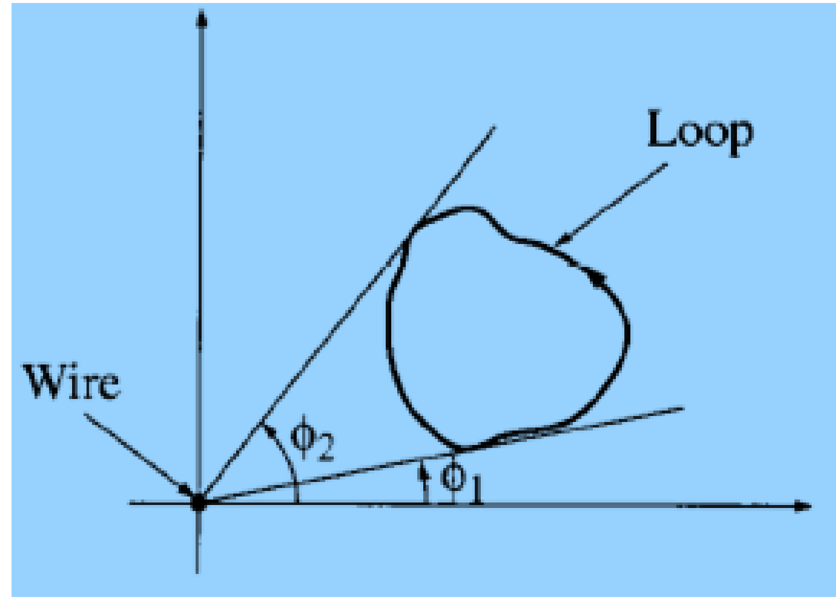
Electrostatics



Flux of electric field is zero if point charge is located **outside** the closed surface

$$\Phi = \iint_S \boldsymbol{E}.\hat{\boldsymbol{n}} dS = \frac{1}{\epsilon_0} Q_{enc}$$

Magnetostatics



Line integral of magnetic field is zero if the loop is **outside** the current carrying wire

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{enc}$$

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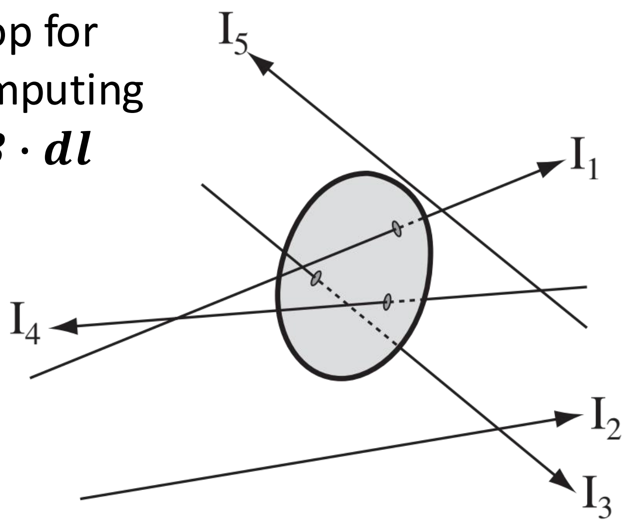
Suppose you have a bundle of straight wires. Each wire that passes through our loop contributes $\mu_0 I$ to the line integral of $\boldsymbol{B}$, and those outside contribute nothing.

Loop for computing $\oint \boldsymbol{B} \cdot \boldsymbol{dl}$



$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{enc}$$

(Integral form of Ampere's law)

$$I_{enc} = I_1 + I_3 + I_4$$

$$I_{enc} = \iint_S \boldsymbol{J}.\boldsymbol{da}$$

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{enc} = \mu_0 \iint \boldsymbol{J}.\boldsymbol{da}$$

$$\iint (\boldsymbol{\nabla} \times \mathrm{B}) \cdot \boldsymbol{da} = \mu_0 \iint \boldsymbol{J}.\boldsymbol{da}$$

(Stokes theorem)

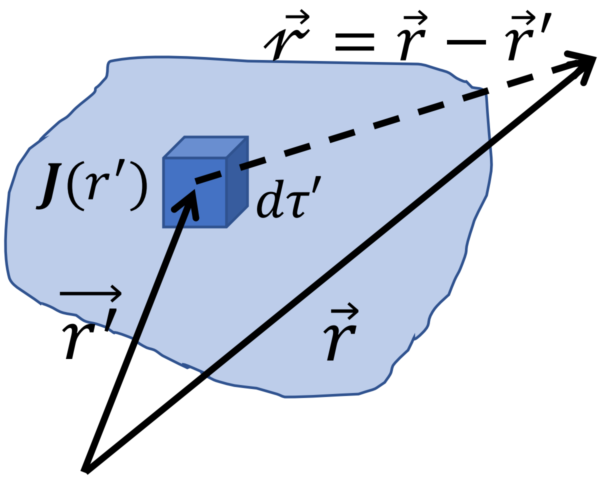
(Region of surface integration includes closed surface S bounding current carrying wires)

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}}$$

(Differential form of Ampere's law)

The divergence and curl of magnetic induction **B** with arbitrary shaped sources

Consider a *steady* current source of any *arbitrary shape*



Magnetic field at P

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

The Biot-Savart law for the general case of a volume current.

$\boldsymbol{B}$ is a function of r i.e (x, y, z)

$\boldsymbol{J}$ is a function of r' i.e (x', y', z')

Integration is w.r.t primed co-ordinates, divergence and curl of B are w.r.t unprimed co-ordinates

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

J does not depend on unprimed variables

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But $\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables

$$\nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = \mathbf{0}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0.}$$

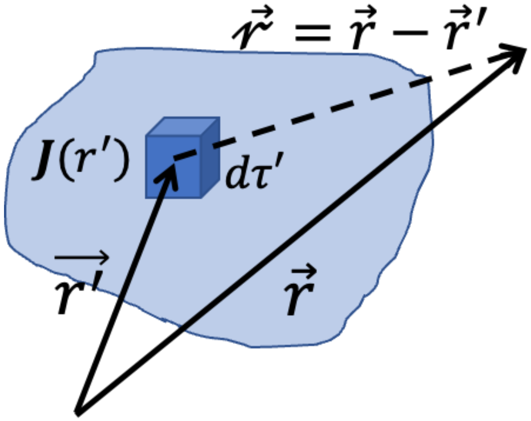
The divergence of the magnetic field is zero

Curl of **B**

Apply vector triple product

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau' \qquad \text{------------ (a)}$$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$



$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}')\, d\tau' = \mu_0 \mathbf{J}(\mathbf{r}),$$

Check that second term of Eq. (a) is zero

$$-(\mathbf{J}\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = (\mathbf{J}\cdot\nabla')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad (\boldsymbol{\nabla} = -\boldsymbol{\nabla}')$$

For steady currents
divergence of J = 0

The x component, in particular, is

$$(\mathbf{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right] - \left(\frac{x-x'}{\mathscr{r}^3}\right)(\nabla'\cdot\mathbf{J})$$

$$\int_{\mathcal{V}} \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]d\tau' = \oint_{\mathcal{S}} \frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\cdot d\mathbf{a}'$$

This integral is zero (as J = 0) if we increase the surface of integration to be infinity

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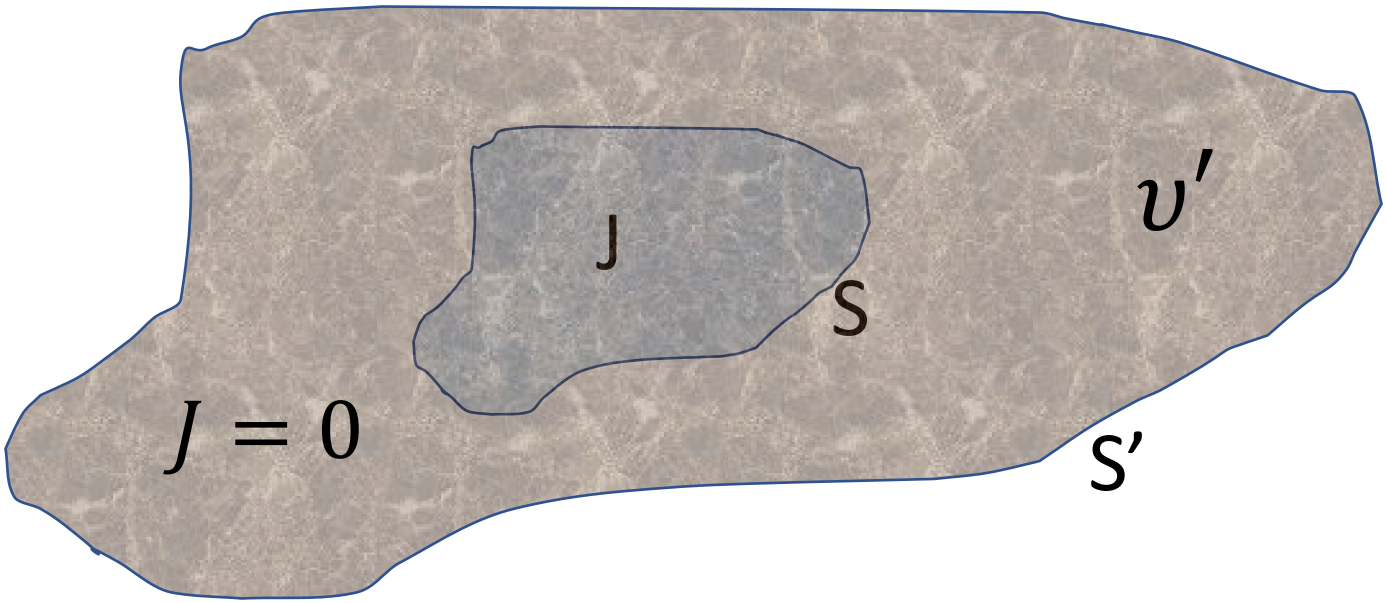
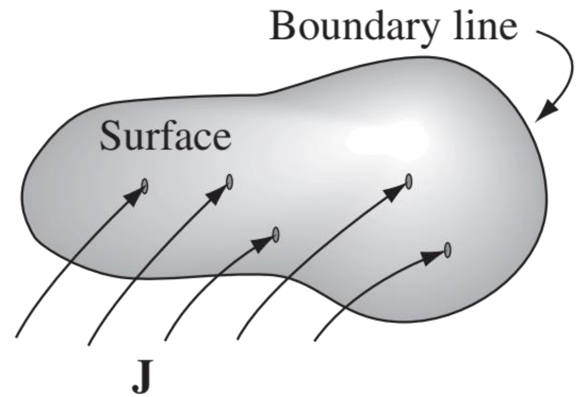
$$\int_{\mathcal{V}} \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3} \mathbf{J} \right] d\tau' = \oint_{\mathcal{S}} \frac{(x - x')}{\mathscr{r}^3} \mathbf{J} \cdot d\mathbf{a}'$$



The region of integration can be made large enough, even to infinity- there is no harm because J = 0 inside S'. The essential point is that on the boundary (S') current is zero (all current is safely inside) and hence the surface integral vanishes.

Ampère's Law for steady currents of arbitrary shapes

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},}$$



$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.}$$

I_{enc} (the **current enclosed** by the **Amperian loop**)

[Direction of I_{enc} is positive if C is traversed anticlockwise]

Consequences of divergence-less character of B

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0$$

One of the four Maxwell's equations for the foundation of electromagnetic theory

It reflects the absence of magnetic monopoles (divergence of a vector field is a measure of its **source density**)- compare with $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$

Since ***B*** is always divergence-less, it implies that no volume element contains magnetic charges of *one sign* only.

Magnetic Flux – The surface integral of B

The flux of the vector field ***B*** over any surface S can be written as

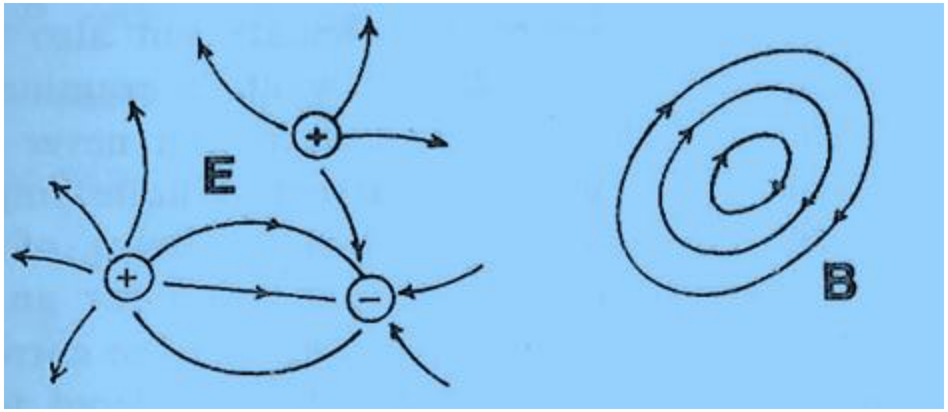
$$\Phi = \int_S \boldsymbol{B} \cdot \boldsymbol{dS} = \int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dS$$

For a closed surface S, we can use the divergence theorem to convert the surface integral into a volume integral

$$\Phi = \oint_S \boldsymbol{B} \cdot \boldsymbol{dS} = \oint_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} \boldsymbol{dS} \int \boldsymbol{\nabla} \cdot \boldsymbol{B} dV = 0$$

(Magnetic flux through a closed surface is zero)

Useful Comparison: E and B fields



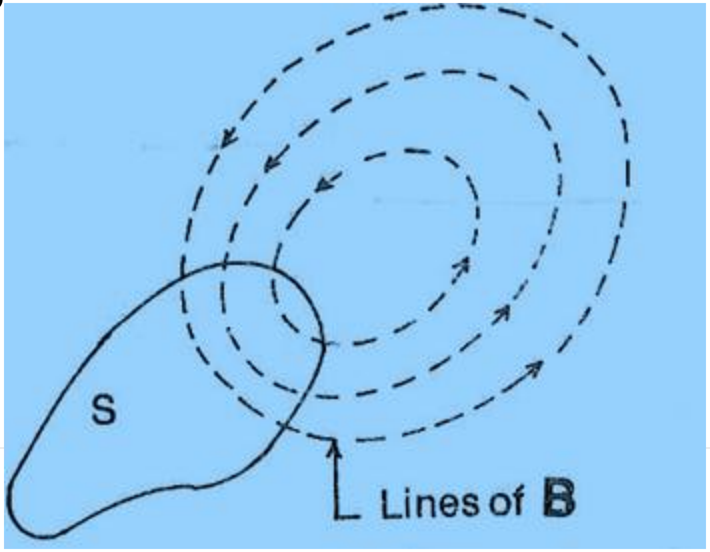
Electric field line has a definite *starting point* and a definite *termination point*. But lines representing a field of induction ***B*** have different properties- *they always form closed loops*. We cannot say where a line of field ***B*** starts or where it ends.

Why does magnetic field lines form closed loop?

Because the total flux of ***B*** over the closed surface is zero.

Through any closed surface S, the total incoming flux is exactly equal to the total outgoing flux

$$\Phi = \oint_S \boldsymbol{B} \cdot \boldsymbol{dS} = \boldsymbol{0}$$



ELECTROSTATICS AND MAGNETOSTATICS (MAXWELL'S EQUATIONS FOR STATIC FIELDS)

Maxwell's equations determine the field, if the source charge density ρ is given

Electrostatic field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0$$

Equivalent to Coulomb's law
+ Superposition

Maxwell's equations for electrostatics

+ Boundary condition: E → 0 far from all charges.

Magnetostatic field:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Equivalent to Biot-Savart law
+ Superposition

Maxwell's equations for magnetostatics.

+ Boundary condition: B → 0 far from all currents.

LORENTZ FORCE: $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

The most elegant formulation of electrostatics and magnetostatics