PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 12

BOUNDARY CONDITIONS IN ELECTRIC FIELD AND POTENTIAL

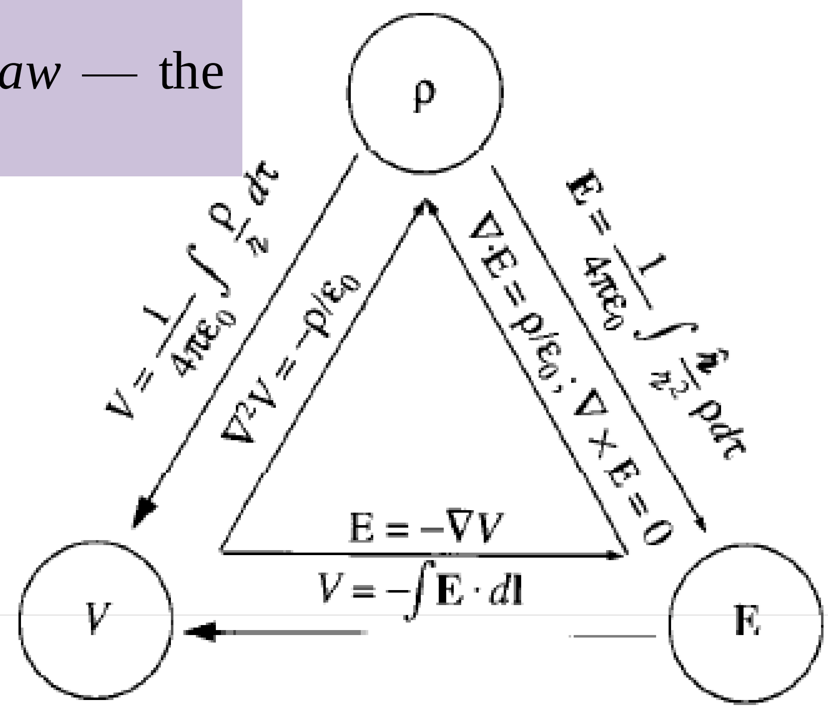
BOUNDARY CONDITIONS IN E and V

Typical electrostatic problem: $\rho \longrightarrow$ given, $\mathbf{E} \longrightarrow$?

Unless the problem is symmetrical and allows you to find **E** by Gauss's law, potential (V) is calculated as an intermediate step prior to calculation of **E**

3 fundamental quantities in electrostatics: ρ, **E** and V. There are six formulas interrelating them.
We began only with (i) *principle of superposition* — a broad general rule applying to all electromagnetic forces, and (ii) *Coulomb's law* — the fundamental law of electrostatics

Second order differential equations in V and its solution in integral form.

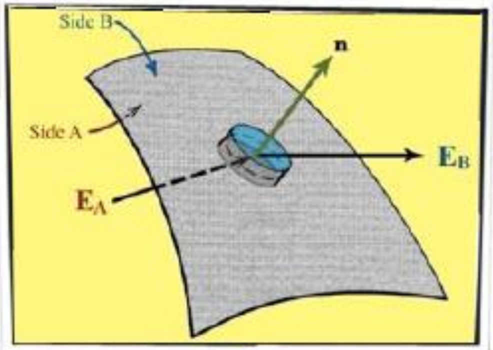


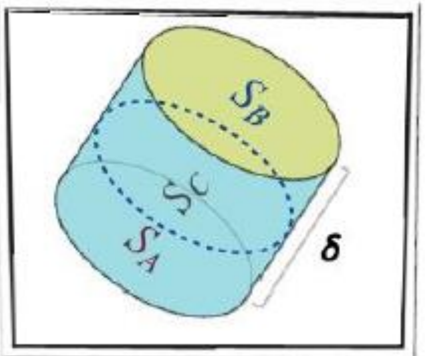
Discontinuity in the perpendicular component of the electric field

Consider an arbitrary surface with surface charge density $\sigma \equiv \sigma(\mathbf{r})$.

Let the electric fields on the two sides (say A and B) of the surface be $\mathbf{E_A}$ and $\mathbf{E_B}$ respectively.

We are interested in finding the continuity or discontinuity in the electric field as we move across the surface from side A to side B.





Consider a tiny Gaussian pillbox (cylindrical) extending barely over the edge on either direction. Let length of this Gaussian cylinder be $\boldsymbol{\delta}$ and area of the plane surfaces S_A, S_B be ΔS (S_A lies on the side A, and S_B on the side B). Also, let the curved surface of the cylinder be S_C. $\mathbf{n}$ is the unit normal to S_A and S_B pointing from side A to B (i.e., the local unit normal vector to the surface).

Discontinuity in the perpendicular component of the electric field

According to the Gauss's law,

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

If we consider $\delta \to 0$, the contribution in the surface integral in the above equation comes only from the surfaces S_A and S_B, since S_C vanishes in this limit. This limit ($\delta \to 0$) assures the examination of the behavior of $\mathbf{E}$ as we barely cross the surface. The total charge enclosed by the Gaussian cylinder is $Q_{enc} = \sigma\, \Delta S$. Therefore, we obtain

$$\oint_{S_A} \mathbf{E} \cdot d\mathbf{S} + \oint_{S_B} \mathbf{E} \cdot dS = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$\Rightarrow \int_{S_A} \mathbf{E_A} \cdot (-\hat{\mathbf{n}}) dS + \int_{S_B} \mathbf{E_B} \cdot \hat{\mathbf{n}}\, dS = \frac{\sigma \Delta S}{\epsilon_0}$$

Now $\mathbf{E}_A \cdot \hat{\mathbf{n}} = E_A^{\perp}$ is the perpendicular (to the surface) component of $\mathbf{E_A}$ and similarly $\mathbf{E}_B \cdot \hat{\mathbf{n}} = E_B^{\perp}$.

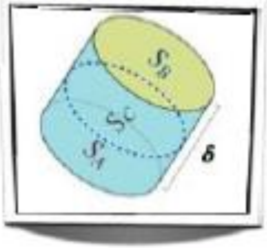
Discontinuity in the perpendicular component of the electric field

Hence, we have

$$-\int_{S_A} E_A^{\perp} dS + \int_{S_B} E_B^{\perp} dS = \frac{\sigma \Delta S}{\epsilon_0}$$

Now considering ΔS sufficiently small, so that $E_A^{\perp}$ and $E_B^{\perp}$ are effectively constant over S_A and S_B respectively, we can pull them outside the respective integrals, i.e.,

$$-E_A^{\perp} \int_{S_A} dS + E_B^{\perp} \int_{S_B} dS = \frac{\sigma \Delta S}{\epsilon_0}$$

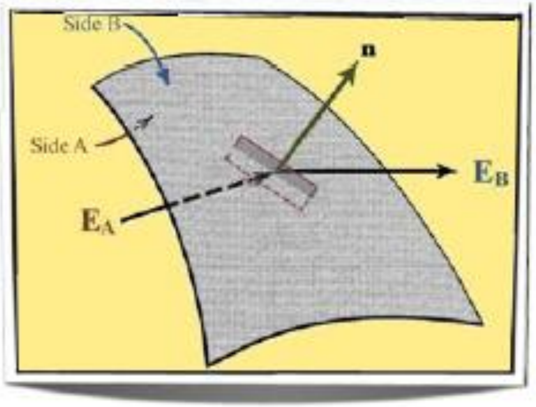


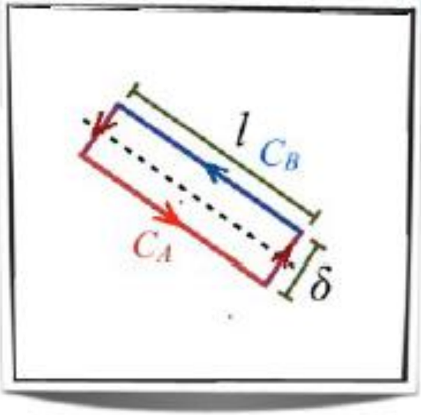
The remaining surface integrals over S_A and S_B yield just the value ΔS which cancels that on the right hand side of the equation. As a consequence, we have the following discontinuity relation for the perpendicular component of the electric field:

$$E_B^{\perp} - E_A^{\perp} = \frac{\sigma}{\epsilon_0}$$

Continuity of the parallel component of the electric field

To examine the behavior of the parallel component of the **E**-field, we consider a tiny rectangular loop, with sides l (parallel to the local surface) and $\boldsymbol{\delta}$ (perpendicular to the local surface).





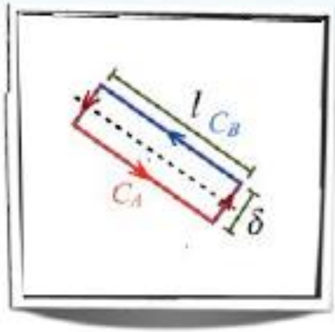
We know that line integral of **E**-field vanishes for any closed loop (*electrostatics*). In the present case this should hold for our tiny rectangular loop,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \ .$$

Continuity of the parallel component of the electric field

In the limit $\delta \to 0$, the contribution to the line integral comes only from the sides C_A and C_B (both of length l), since the sides perpendicular to the local surface vanish in this limit. As a consequence,

$$\int_{C_A} \mathbf{E}_A \cdot d\mathbf{l} + \int_{C_B} \mathbf{E}_B \cdot d\mathbf{l} = 0 \; .$$



Now considering l sufficiently small so that the electric field remains effectively constant on C_A and C_B respectively, and using the fact that the dot product picks up the parallel components $E_A^{\|}$ and $E_B^{\|}$, we obtain

$$E_A^{\|}\, l - E_B^{\|}\, l = 0 \; .$$

The negative sign appears since the directions traversed along C_A and C_B are opposite to each other.

Thus, we are led to the conclusion that the parallel component of the electric field does not change across a surface, i.e., it remains continuous:

$$E_B^{\|} - E_A^{\|} = 0.$$

Discontinuity in the electric field

The two results for the perpendicular and parallel components can be combined in one, giving a net discontinuity relation for the **E** field:

$$\mathbf{E_B} - \mathbf{E_A} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

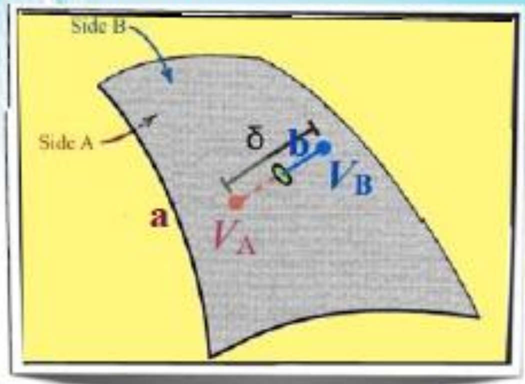
Note that if we take the dot product of the above relation with the local unit normal **n**, we recover the relation for the perpendicular component, since $\mathbf{E}_A \cdot \hat{\mathbf{n}} = E_A^{\perp}$, $\mathbf{E}_B \cdot \hat{\mathbf{n}} = E_B^{\perp}$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$.

On the other hand, if we consider the magnitude of the cross product of the above relation with **n** (*or equivalently the dot product with the unit tangent vector* **t**), we obtain the relation for the parallel component. This is because $|\mathbf{E}_A \times \hat{\mathbf{n}}| = E_A^{\parallel}$, $|\mathbf{E}_B \times \hat{\mathbf{n}}| = E_B^{\parallel}$ and $|\hat{\mathbf{n}} \times \hat{\mathbf{n}}| = 0$.

We also notice that if $\boldsymbol{\sigma} = \mathbf{0}$, then both the normal and tangential components of **E**, and hence **E** itself, become continuous.

Continuity of electric potential

We now find out the continuity/discontinuity in the electric potential. To this end, we consider two points **a** and **b** lying on the two sides of the surface and separated by a small distance δ.



If V_A and V_B be the respective electric potentials at the points **a** and **b**, then

$$V_B - V_A = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}\,.$$

As we consider the limit $\delta \to 0$, i.e., as we shrink the separation between the two points (but still keeping them on the different sides), the line integral on the right side of the above equation vanishes, and therefore we obtain the continuity relation for the electric potential:

$$V_B - V_A = 0.$$

Discontinuity in the gradient of electric potential

We found that the electric potential remains the same as we cross a surface. However, since **E** is the negative gradient of V and as we concluded, **E** is discontinuous across a surface. Therefore, the gradient of V inherits this discontinuity, and we have

$$\nabla V_B - \nabla V_A = -(\mathbf{E_B} - \mathbf{E_A}) = -\frac{\sigma}{\epsilon_0}\hat{\mathrm{n}}.$$

The above equation, upon taking the dot product with **n** on both sides, can also be written as

$$\frac{\partial V_B}{\partial n} - \frac{\partial V_A}{\partial n} = -\frac{\sigma}{\epsilon_0}.$$

Here $\partial V/\partial \mathrm{n} \equiv (\nabla V) \cdot \mathbf{n}$ represents the normal derivative of V, i.e., the rate of change in the direction perpendicular to the surface (*Recall the definition of directional derivative*)