PHY 102 Introduction to Physics II

Spring Semester 2025

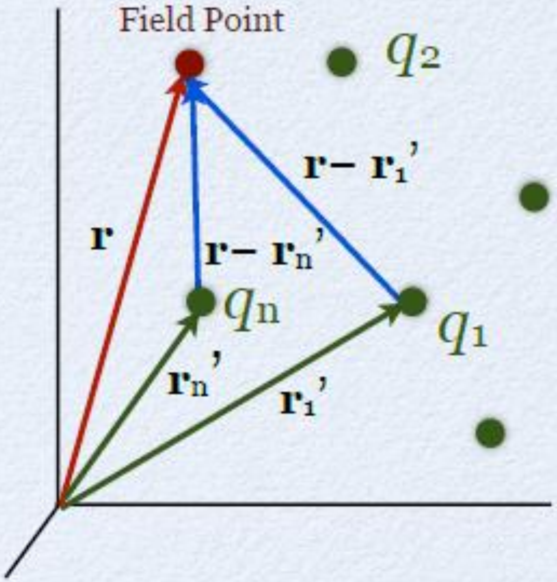
Lecture 9

Discrete Charge Distributions

Electric Field due to discrete charge distributions

We found that for a set of discrete charges q_1, q_2,…, q_n, located at positions $\mathbf{r}_1'$, $\mathbf{r}_2'$,…, $\mathbf{r}_n'$, the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n} q_j \frac{\mathbf{r}-\mathbf{r}'_j}{|\mathbf{r}-\mathbf{r}'_j|^3}$$

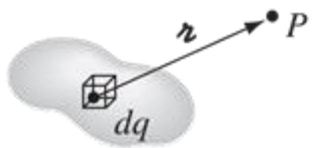


However, we may encounter a situation when the charge is continuously distributed over some region. For example, we can supply some electrons to an otherwise neutral metallic wire, thereby making it negatively charged. The distribution of charge in this case will be over the length of the wire.

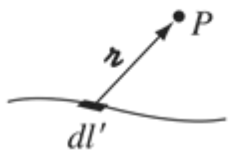
We should be clear that such a situation can arise when the length scale at which we are interested to observe the behavior of E-field is much large compared to the separation between the charge constituents (charged particles) and their "size".

Continuous Charge Distributions

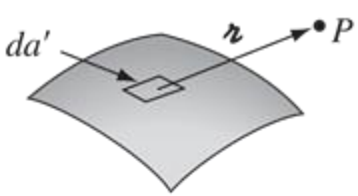
Electric Field due to continuous charge distributions



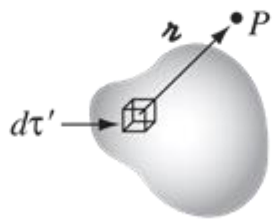
(a) Continuous distribution



(b) Line charge, λ



(c) Surface charge, σ



(d) Volume charge, ρ

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dq$$

$$dq \rightarrow \lambda\, dl' \sim \sigma\, da' \sim \rho\, d\tau'$$

$\lambda \rightarrow$ *Charge per unit length*

$\sigma \rightarrow$ *Charge per unit area*

$\rho \rightarrow$ *Charge per unit volume*

Thus the electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dl';$$

for a surface charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,da';$$

and for a volume charge,

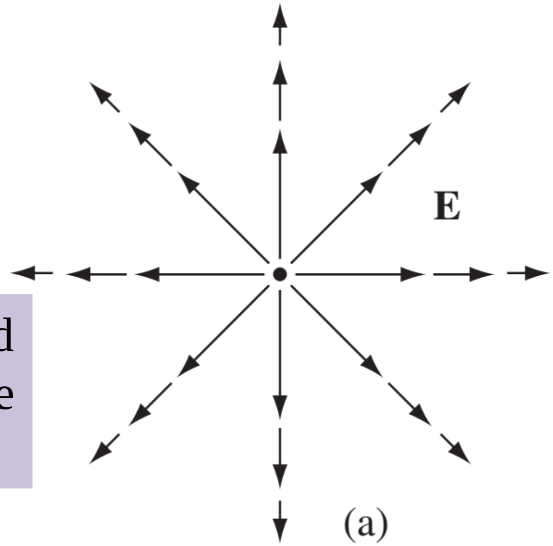
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

Electric Field

For a single point charge q, situated at the origin:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$

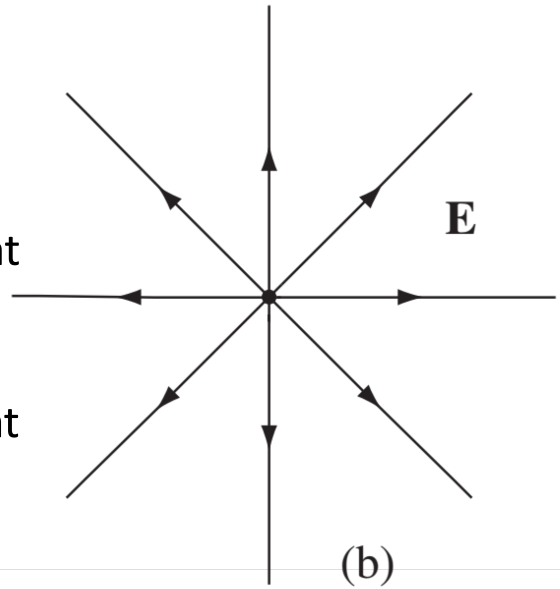
The magnitude of the field is indicated by the density of the field lines: it's strong near the center where the field lines are close together, and weak farther out, where they are relatively far apart.



(a)

Properties of Electric Field Lines

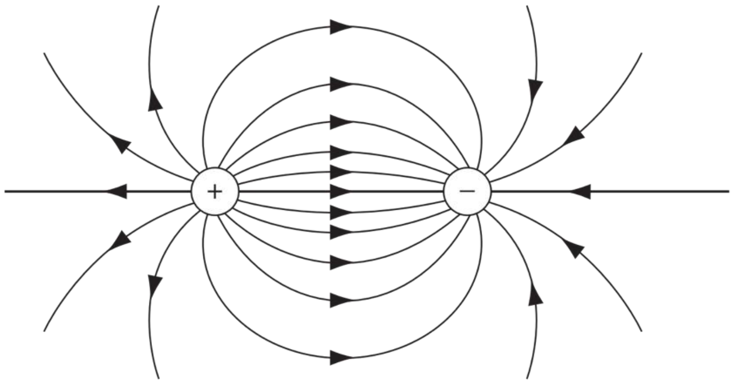
1. Starts from +Ve and ends with –Ve charge.
2. Two-field lines never intersect each other, which implies that the test charge cannot flow two paths simultaneously.
3. The lines of force originate and terminate at the surface at right angles to the surface.



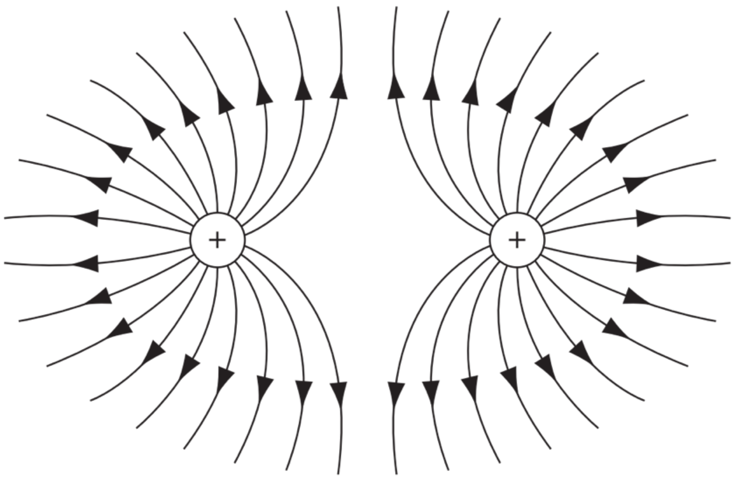
(b)

Electric Field

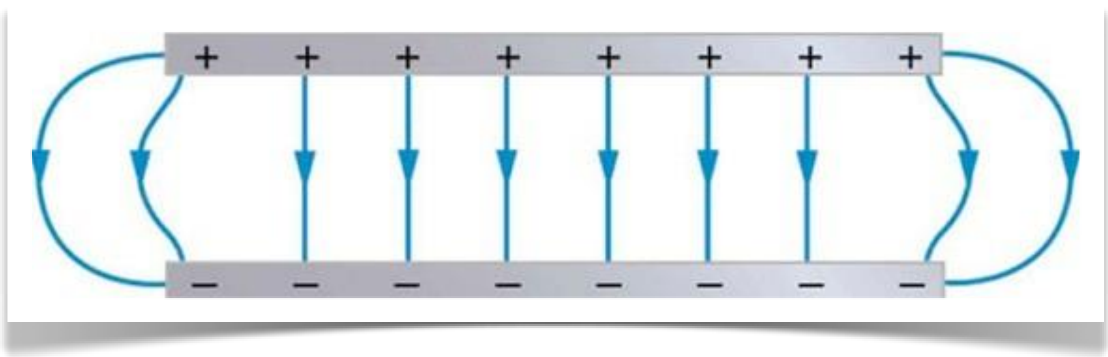
Electric Field



Electric field lines in a region (2D) with one positive and one negative charge.



Electric field lines in a region (2D) with two positive charges



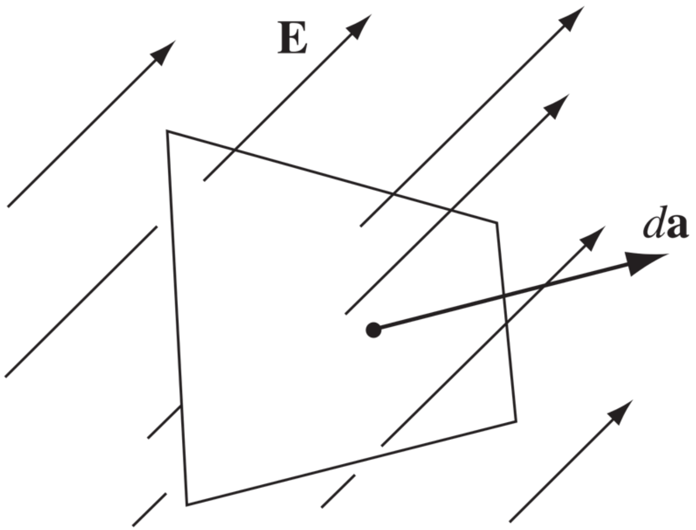
Two metal plates with equal, but opposite, excess charges. The electric field between them is uniform in strength and direction except near the edges.

Electric Field

Flux of Electric Field

Flux of **E** through a surface S

$$\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$$



Flux is a measure of "*number of field lines" passing through S* and is proportional to the *strength* of the field.

$\boldsymbol{E}.\boldsymbol{da}$ is proportional to the number of lines passing through the infinitesimal area ***da*.**

The dot product picks out the component of **da** along the direction of **E**. It is the *area in the plane perpendicular to **E*** that we have in mind when we say that the *density of field lines* is the *number per unit area*.

Electric Flux

It serves as the measure of number of representative field lines passing through the surface *S*.

Note that at each point in space we have corresponding **E** vector. Consequently we have infinite number of electric field lines. Since we can't draw infinite number of lines we only consider some of the field lines (and this defines our *sampling rate*) which are sufficient to give us the idea about the behavior of **E-**field in the given region.

For a given sampling rate, *the flux is proportional to the number of field lines drawn*

Electric Flux & Gauss's Law

The field strength is proportional to the density of field lines (the number per unit area) and hence $\mathbf{E}\cdot d\mathbf{S}$ is proportional to the number of lines passing through the infinitesimal area $d\mathbf{S}$.

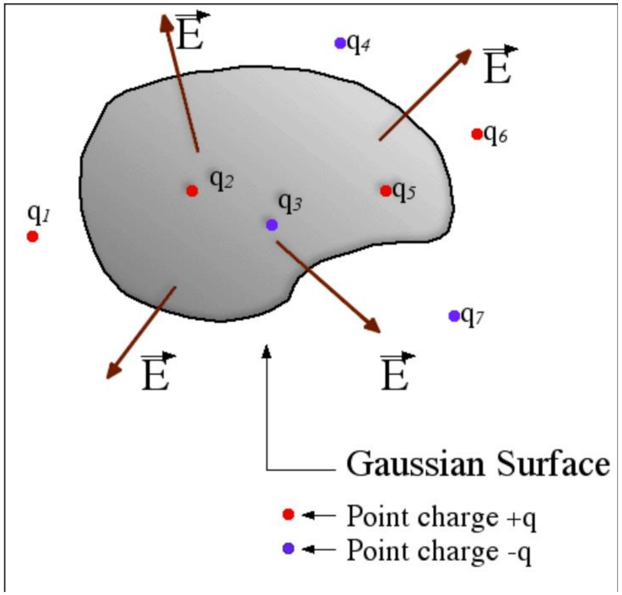
Thus for the surface S, flux is proportional to the number of representative lines passing through it, as already stated.

Gaussian Surface

An imaginary surface drawn around a charge is called a **Gaussian Surface**.
The flux cannot move out or in without passing through the surface.

Gauss's Law

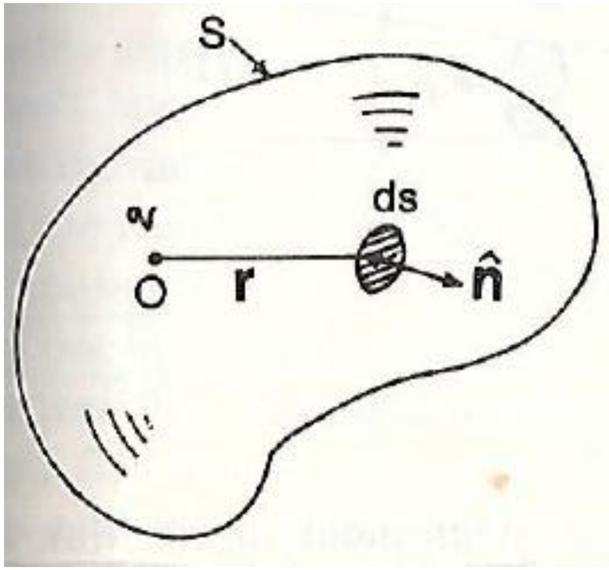
Gauss's law states that the total electric flux passing through any closed surface is equal to the total charge enclosed by that surface.



$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_o}(q_1 + q_2 + \ldots) = \left(\frac{1}{\epsilon_0}\right) Q_{enc}$$

Electric Field

Gauss's Law



Consider a point charge 'q' located at a point O inside a closed surface S. The charge produces an electric field ***E*** around it .

The flux of E through an element $\boldsymbol{dS} = \boldsymbol{\hat{n}}dS$

$$d\Phi = \boldsymbol{E}.\boldsymbol{\hat{n}}dS = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{\hat{r}}.\boldsymbol{\hat{n}}dS}{r^2} = \frac{q}{4\pi\epsilon_0}d\Omega$$

$$\frac{\boldsymbol{\hat{r}}.\boldsymbol{\hat{n}}dS}{r^2} = d\Omega$$ = solid angle subtended by dS at O

$$\Phi = \int d\Phi = \frac{q}{4\pi\epsilon_0}\int d\Omega$$, Total solid angle subtended by S at O is 4π

$$\Rightarrow \Phi = \frac{q}{\epsilon_0}$$

Electric Field

Gauss's Law

In the case of a point charge q at the origin, the flux of **E** through a spherical surface of radius **r** is

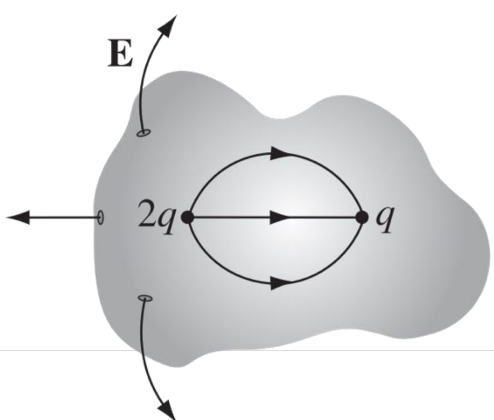
$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q$$

This suggests that the flux through any closed surface is a measure of the total charge inside.

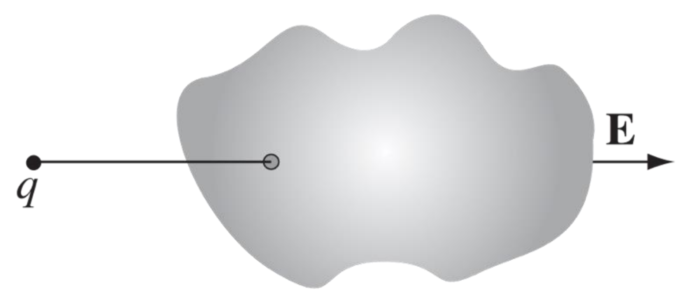
This is the essence of **Gauss's law.**

For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside

A charge outside the surface will contribute nothing to the total flux since its field lines pass in one side and out the other.



(a)



(b)

Electric Flux through the surface enclosing a bunch of charge

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i .$$

The flux through a surface that encloses them all is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i \right)$$

For any closed surface, then,

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},}$$

where Q_{enc} is the total charge enclosed within the surface.

Integral form of Gauss's law

Electric Field

Gauss's Law in differential form

Gauss's law is an integral equation, but we can easily turn it into a differential one, by applying the divergence theorem.

We have

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} \longrightarrow \quad \textbf{(1)}$$

"Divergence Theorem"

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau$$

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau.$$

Electric Field

Gauss's Law in differential form

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

Gauss's law in differential form

ELECTRIC FLUX DENSITY

For practical reasons, the electric field E is not usually considered the most useful flux in electrostatics, since the electric field intensity is dependent on the medium in which the charge is placed.

In the practical case, the **electric flux density (D)** is more useful, since this is independent of the medium.

$$\boxed{\mathbf{D} = \varepsilon_o \mathbf{E}}$$

The **electric flux density** is also called **electric displacement**

Unit: coulombs per square meter (C/m^2)

Electric Field

Gauss's Law in differential form

The Gauss's law in differential form

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

can be written as

$$\boxed{\nabla . \mathbf{D} = \rho}$$

The first form of **Maxwell's Equation**

Electric Field

Gauss's Law or Maxwell's Equation

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

$$\boxed{\nabla . \mathbf{D} = \rho}$$

Notes:

Gauss's law is an alternative statement of Coulomb's law; proper application of the divergence theorem to Coulomb's law results in Gauss's law

Gauss's law provides an easy means of finding E or D for symmetrical charge distributions such as a point charge, an infinite line charge, an infinite cylindrical surface charge, and a spherical distribution of charge.

A continuous charge distribution has rectangular symmetry if it depends only on x (or y or z), cylindrical symmetry if it depends only on ρ, or spherical symmetry if it depends only on r (independent of θ and Ø). It must be stressed that whether the charge distribution is symmetric or not, Gauss's law always holds.

optional

Examples

The Divergence of E

For a point charge at the origin

From Coulomb's law

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\mathbf{r}')\, d\tau'$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\mathbf{r}')\, d\tau'$$

We have

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

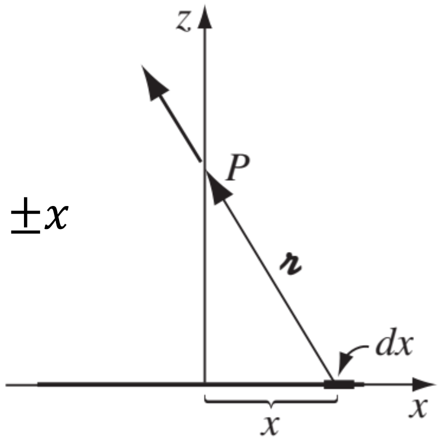
Thus

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0} \rho(\mathbf{r})$$

which is **Gauss's law in differential form**

Electric Field due to continuous charge distributions

P1. Find the electric field a distance z above the midpoint of a straight line segment of length 2L that carries a uniform line charge λ.

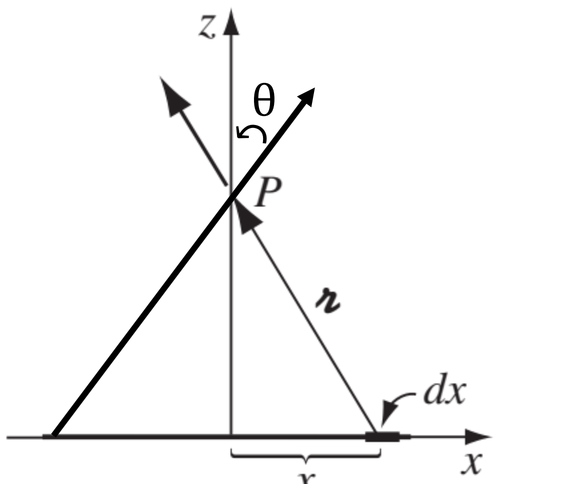


Method 1

The simplest method is to chop the line into symmetrically placed pairs at $\pm x$ and integrate $x: 0 \rightarrow l$.

$$\mathrm{dE} = \frac{\lambda\, dx}{4\pi\varepsilon_0 \mathcal{r}^2}$$

Effective field



$$d\vec{\mathrm{E}}_{eff} = 2\ \mathrm{dE} \cos\theta\ \hat{z}$$

$$= \frac{2\lambda\, dx}{4\pi\varepsilon_0 (x^2+z^2)^2}\left(\frac{z}{r}\right)\hat{z} = \frac{2\lambda\, dx\, z}{4\pi\varepsilon_0 (x^2+z^2)^{\frac{3}{2}}}$$

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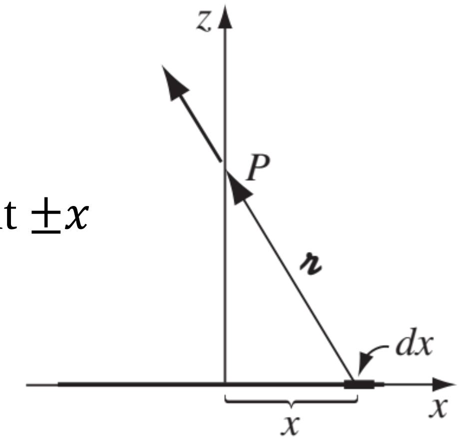

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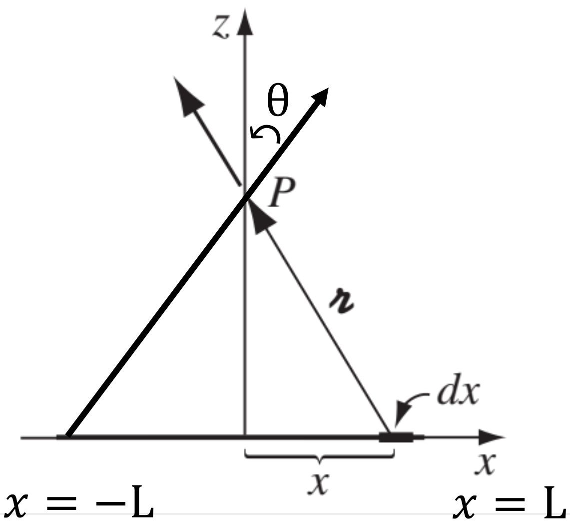

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Continuous Charge Distributions

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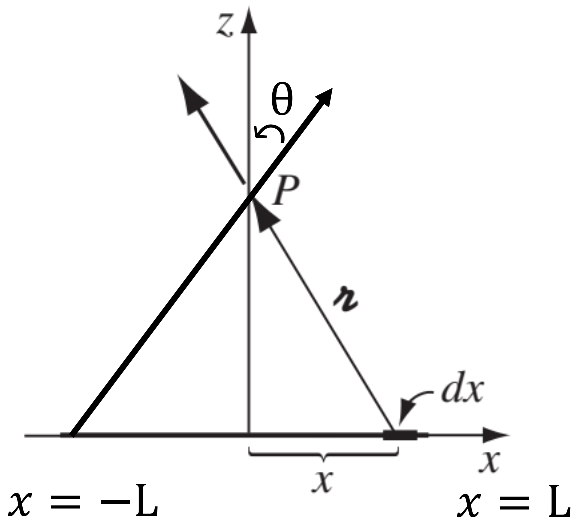

Method 1

$$\int d\vec{E}_{eff} = \int_0^L \frac{2\lambda\, dx\, z}{4\pi\varepsilon_0 (x^2+z^2)^{\frac{3}{2}}}$$

$$\vec{E}_{eff} = \frac{\lambda}{2\pi\varepsilon_0 z}\left(\frac{L}{\sqrt{L^2+z^2}}\right)\hat{z}$$

When $z \gg L$

$$\vec{E}_{eff} = \frac{\lambda L}{2\pi\varepsilon_0 z^2}\,\hat{z} = \frac{\lambda\,(2L)}{4\pi\varepsilon_0 z^2}\,\hat{z} = \frac{q}{4\pi\varepsilon_0 z^2}\,\hat{z}$$

$\because q = \lambda\,(2L)$

From far away, the line charge looks like a point charge.

Continuous Charge Distributions

Electric Field due to continuous charge distributions

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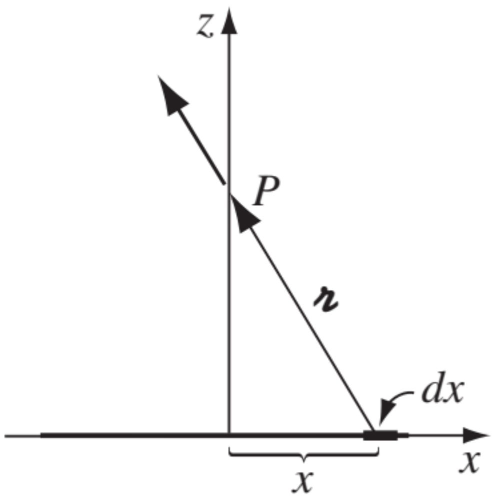

Method 2

$$\mathbf{r} = z\,\hat{\mathbf{z}}, \quad \mathbf{r}' = x\,\hat{\mathbf{x}}, \quad dl' = dx;$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}, \quad \mathscr{r} = \sqrt{z^2 + x^2}, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}.$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda}{z^2 + x^2} \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}\, dx$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \int_{-L}^{L} \frac{1}{(z^2 + x^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_{-L}^{L} \frac{x}{(z^2 + x^2)^{3/2}}\, dx \right]$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}\,\hat{\mathbf{z}}.$$

Electric Field due to continuous charge distributions

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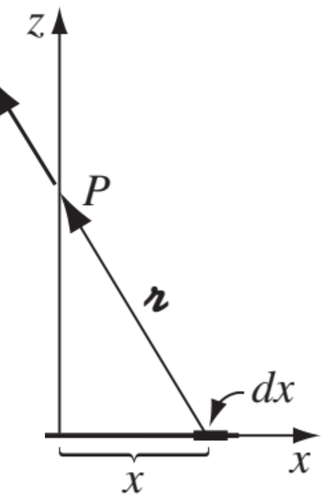


Method 2

When $z \gg L$

$$E \cong \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2},$$

which makes sense: From far away the line "looks" like a point charge $q = 2\lambda L$, so the field reduces to that of point charge $q/(4\pi\epsilon_0 z^2)$. In the limit $L \to \infty$, on the other hand, we obtain the field of an infinite straight wire:

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z};$$

or, more generally,

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s},$$

where s is the distance from the wire.

Continuous Charge Distributions

Electric Field (Field due to a hemisphere)

P5. A solid hemisphere has a radius of R and uniform charge density ρ. Find the electric field at the center.

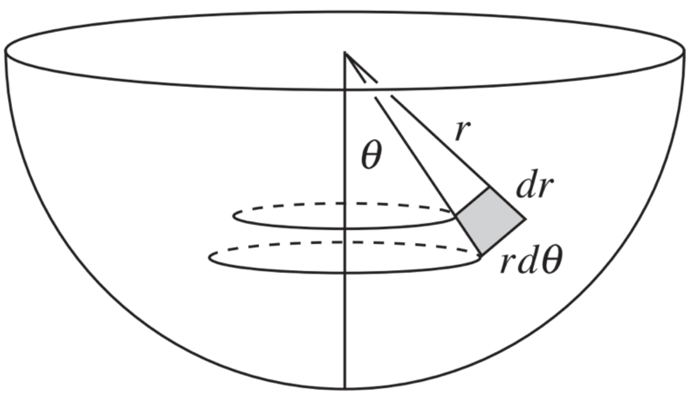
Solution

- Slice the hemisphere into rings around the symmetry axis.
- Find the electric field due to each ring, and then integrate over the rings to obtain the field due to the entire hemisphere.

The cross section of a ring is (essentially) a little rectangle with side lengths dr and r $d\theta$.

The cross-sectional area is thus r dr $d\theta$. The radius of the ring is r sin θ, so the volume is (r dr $d\theta$)(2πr sin θ).

The charge in the ring is therefore $\rho(2\pi r^2 \sin\theta \, dr \, d\theta)$. Equivalently, we can obtain this result by using the standard spherical-coordinate volume element, $r^2 \sin\theta \, dr \, d\theta \, d\phi$, and then integrating over ϕ to obtain the factor of 2π.



Continuous Charge Distributions

Electric Field (Field due to a hemisphere)

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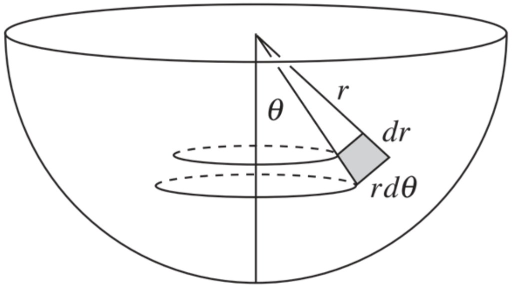
Solution



Consider a tiny piece of the ring, with charge dq. This piece creates an electric field at the center of the hemisphere that points diagonally upward (if ρ is positive) with magnitude $dq/4\pi\epsilon_0 r^2$. However, only the vertical component survives, because the horizontal component cancels with the horizontal component from the diametrically opposite charge dq on the ring. The vertical component involves a factor of $\cos\theta$. When we integrate over the whole ring, the dq simply integrates to the total charge we found above. The (vertical) electric field due to a given ring is therefore

$$dE_y = \frac{\rho(2\pi r^2 \sin\theta\, dr\, d\theta)}{4\pi\epsilon_0 r^2}\cos\theta = \frac{\rho \sin\theta\cos\theta\, dr\, d\theta}{2\epsilon_0}.$$

Integrating over r and θ to obtain the field due to the entire hemisphere gives

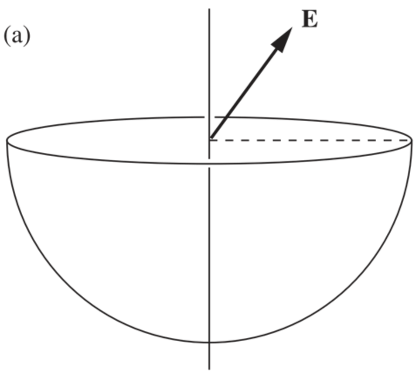
$$E_y = \int_0^R \int_0^{\pi/2} \frac{\rho\sin\theta\cos\theta\, dr\, d\theta}{2\epsilon_0} = \frac{\rho}{2\epsilon_0}\left(\int_0^R dr\right)\left(\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right) = \frac{\rho}{2\epsilon_0}\cdot R\cdot \frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{\rho R}{4\epsilon_0}$$

Electric Field (Field due to a hemisphere)

P4. A solid hemisphere has a radius of R and uniform charge density ρ. Find the electric field at the center.

Solution

(a)



The symmetry argument that explains why E must be vertical.

(b)