Spring 2023 **PHY102: Introduction to Physics-II** **03/03/2023**

Midterm Exam (Full Marks = 25)

Department of Physics **Shiv Nadar Institution of Eminence**

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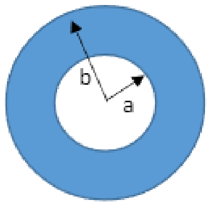
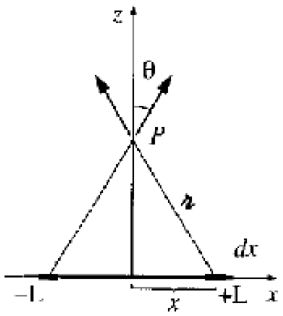
*Answer **all** questions. Marks are indicated in square brackets. Remember to indicate vector quantities with appropriate notation wherever possible. Step marking will be given for correct steps only.*

1. (a) Find the electric field a distance 'z' above the midpoint of a straight line segment of length 2L that carries the uniform line charge λ. **[3]**

Evaluate the electric field expression if $z \gg L$ and $L \rightarrow \infty$ and comment on the results. **[3]**

(b) A thick spherical shell (as shown in the figure below) carries charge density $\rho = \frac{k}{r^2}$, estimate the electric field in the three regions i) r<a, ii) a<r<b, iii) r>b and also plot the electric field as a function of r. **[1+1+1+1=4]**





$$\mathbf{r} = z\,\hat{\mathbf{z}}, \quad \mathbf{r}' = x\,\hat{\mathbf{x}}, \quad dl' = dx;$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}, \quad \mathscr{r} = \sqrt{z^2 + x^2}, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}.$$

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{\lambda}{z^2 + x^2} \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}\, dx \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \int_{-L}^{L} \frac{1}{(z^2 + x^2)^{3/2}}\, dx - \hat{\mathbf{x}} \int_{-L}^{L} \frac{x}{(z^2 + x^2)^{3/2}}\, dx \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\,\hat{\mathbf{z}} \left(\frac{x}{z^2\sqrt{z^2 + x^2}} \right)\Bigg|_{-L}^{L} - \hat{\mathbf{x}} \left(-\frac{1}{\sqrt{z^2 + x^2}} \right)\Bigg|_{-L}^{L} \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}}\, \hat{\mathbf{z}}.
\end{aligned}$$

For points far from the line ($z \gg L$), this result simplifies:

$$E \cong \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2},$$

which makes sense: From far away the line "looks" like a point charge $q = 2\lambda L$, so the field reduces to that of point charge $q/(4\pi\epsilon_0 z^2)$. In the limit $L \to \infty$, on the other hand, we obtain the field of an infinite straight wire:

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z};$$

or, more generally,

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s},$$

where s is the distance from the wire.

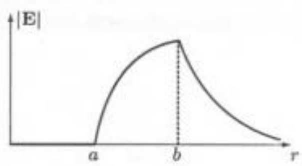
(i) $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = 0.}$

(ii) $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int \frac{k}{\bar{r}^2}\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\ phi$

$$= \frac{4\pi k}{\epsilon_0}\int_a^r d\bar{r} = \frac{4\pi k}{\epsilon_0}(r-a) \therefore \boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.}$$

(iii) $E(4\pi r^2) = \frac{4\pi k}{\epsilon_0}\int_a^b d\bar{r} = \frac{4\pi k}{\epsilon_0}(b-a)$, so

$$\boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.}$$



2. a) A long, straight wire is surrounded by a hollow metal cylinder whose axis coincides with that of the wire. The wire has a charge per unit length of λ, and the cylinder has a net charge per unit length of 2λ. From this information, use Gauss's law to find (i) the charge per unit length on the inner and outer surface of the cylinder and (ii) the electric field outside the cylinder at a distance r from the axis. **[1 + 2 = 3]**

b) The plates of a parallel plate capacitor, having area A, are maintained at constant potential difference V. If the initial separation between the plates is d, find the work done in increasing the separation of plates to $2d$. **[2]**

4)

(i) The inner wire will induce $(-\lambda L)$ charge on the inner side of the cylinder.

1/2

This, in turn, will give $+\lambda L$ charge on the outer surface of the cylinder.

The net $2\lambda L$ charge will always resides on the outer surface.

1/2

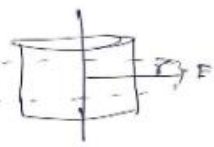
Therefore, the charge/length on the inner surface $= \lambda$

the charge/length on the outer surface $= \lambda + 2\lambda = 3\lambda$

(ii) From Gauss law

$$\oint E \cdot ds = q/\epsilon_0$$

$$\Rightarrow E(2\pi r L) = \frac{\lambda L + 2\lambda L}{\epsilon_0}$$



$$\Rightarrow E = \frac{3\lambda}{2\pi\epsilon_0 r}$$

direction is along $\hat{r}$

2

5) To charge up a capacitor, work done

1

$$W = \int_0^Q V dq = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 \text{ as } Q = CV$$

Now for parallel plate capacitor; $C = \frac{\epsilon_0 A}{d}$

$\therefore$ work done for 1st case $= \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)V^2 = W_1$

work done for 2nd case $= \frac{1}{2}\left(\frac{\epsilon_0 A}{2d}\right)V^2 = W_2$

$\therefore$ Extra work $= W_1 - W_2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2 - \frac{1}{4}\frac{\epsilon_0 A}{d}V^2$

$$\boxed{\delta W = \frac{1}{4}\frac{\epsilon_0 A}{d}V^2}$$

1

3. The electrostatic energy associated with a volume charge density ρ is given by

$W = \frac{1}{2}\iiint \rho V d\tau$, where V represents the electrostatic potential.

Taking the case of a uniformly charged *solid sphere* of radius R, show that first the electrostatic potential inside the sphere is given by

$$V_{in}(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

Here Q is the total charge.

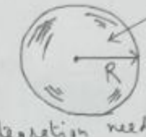
Hence prove that the total electrostatic energy of the sphere is given by

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

[4+3]

Answer

$$W = \frac{1}{2}\iiint \rho V d\tau$$



We have to calculate V_{in}, that is the potential inside the sphere. Integration needs to be carried out within the entire volume of sphere

Gauss's law gives us

$$E_{in} = \frac{\rho r}{3\epsilon_0} \quad (r < R) \quad = \frac{Q r}{4\pi\epsilon_0 R^3} \qquad \left[\because \rho = \frac{Q}{\frac{4}{3}\pi R^3}\right] \text{ — (1)}$$

$$E_{out} = \frac{\rho R^3}{3\epsilon_0 r^2} \quad (r > R) \quad = \frac{Q}{4\pi\epsilon_0 r^2}$$

Definition of potential, $V(r) = -\int_{\infty}^{r} \vec{E}\cdot\vec{dr}$

$$V_{in}(r) = -\int_{\infty}^{r} \vec{E}\cdot\vec{dr} = -\int_{\infty}^{R} \vec{E}_{out}\cdot\vec{dr} - \int_{R}^{r} \vec{E}_{in}\cdot\vec{dr} \text{ — (2)}$$

Substitute eqn (1) in eqn (2)

$$V_{in}(r) = -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{R}\frac{dr}{r^2} - \frac{Q}{4\pi\epsilon_0 R^3}\int_{R}^{r}\frac{dr}{R^3}$$

1

After performing integration

$$V_{in}(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

$$W = \frac{1}{2}\iiint \rho V_{in}(r)\, d\tau$$

$$= \frac{1}{2}\,\rho\,\frac{Q}{4\pi\epsilon_0}\,\frac{1}{2R}\left[\int_0^R \left(3 - \frac{r^2}{R^2}\right) r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\right]$$

[$\because d\tau = r^2\, dr \sin\theta\, d\theta\, d\phi$ in spherical symmetry]

$$= \frac{\rho}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)\left(\frac{4\pi}{2R}\right)\int_0^R \left(3 - \frac{r^2}{R^2}\right) r^2\, dr$$

$$= \left(\frac{\rho Q}{5\epsilon_0}\right) R^2 = \frac{3Q^2}{20\pi\epsilon_0 R} \qquad \left[\because \rho = \frac{Q}{\frac{4}{3}\pi R^3}\right]$$

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Q4. (a) In spherical coordinates, $V = 0\,V$ for $r = 0.1\,m$ and $V = 100\,V$ for $r = 2.0\,m$. Assuming free space between these concentric spherical shells, find $\vec{E}$. [4]

(b) The region between two concentric right circular cylinders contains a **uniform charge density** ρ. Use Poisson's equation to find V. [3]

Cylindrical Coordinates.

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

so that Laplace's equation is

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical Coordinates.

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

and

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

so that Laplace's equation is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Since V is not a function of θ or ϕ, Laplace's equation reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)=0$$

Integrating gives

$$r^2\frac{dV}{dr}=A$$

and a second integration gives

$$V=\frac{-A}{r}+B$$

The boundary conditions give

$$0=\frac{-A}{0.10}+B \qquad \text{and} \qquad 100=\frac{-A}{2.00}+B$$

whence $A = 10.53\ \mathrm{V\cdot m}$, $B = 105.3\ \mathrm{V}$. Then

$$V=\frac{-10.53}{r}+105.3 \quad (\mathrm{V})$$

$$\mathbf{E}=-\nabla V=-\frac{dV}{dr}\mathbf{a}_r=-\frac{10.53}{r^2}\mathbf{a}_r \quad (\mathrm{V/m})$$

1.(b)

Neglecting fringing, Poisson's equation reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\frac{\rho}{\epsilon}$$

$$\frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\frac{\rho r}{\epsilon}$$

Integrating,

$$r\frac{dV}{dr}=-\frac{\rho r^2}{2\epsilon}+A$$

$$\frac{dV}{dr}=-\frac{\rho r}{2\epsilon}+\frac{A}{r}$$

$$V=-\frac{\rho r^2}{4\epsilon}+A\ln r+B$$