Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 9

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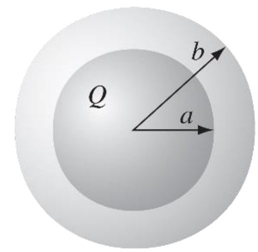
1. A spherical conductor of radius **a** carries a charge Q as shown in the figure. It is surrounded by linear dielectric material of susceptibility χ_e out to radius **b**. Find the energy of this configuration.



Q1)

$$\mathbf{D} = \left\{ \begin{array}{ll} 0, & (r<a) \\ \dfrac{Q}{4\pi r^2}\,\hat{\mathbf{r}}, & (r>a) \end{array} \right\}, \quad \mathbf{E} = \left\{ \begin{array}{ll} 0, & (r<a) \\ \dfrac{Q}{4\pi \epsilon r^2}\,\hat{\mathbf{r}}, & (a<r<b) \\ \dfrac{Q}{4\pi \epsilon_0 r^2}\,\hat{\mathbf{r}}, & (r>b) \end{array} \right\}.$$

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau = \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi \left\{ \frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2 dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}dr \right\} = \frac{Q^2}{8\pi}\left\{ \frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty \right\}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left\{ \frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right) + \frac{1}{b} \right\} = \boxed{\frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right).}$$

2. A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization :

$$\vec{P}(\vec{r}) = \frac{k}{r}\,\hat{r}$$

where k is a constant and r is the distance from the center (see **Fig. 1** below). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods :

(a) Locate all the *bound charge*, and use the original Gauss's law to calculate the field it produces.

(b) Find $\vec{D}$ from the Gauss's Law analogue for the Displacement, and then get $\vec{E}$ from the defining equation for $\vec{D}$. [Notice that the *second method is much faster*, and it avoids any explicit reference to the bound charges.]

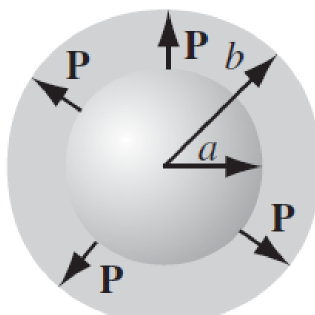


Fig. 1

Solution 2:

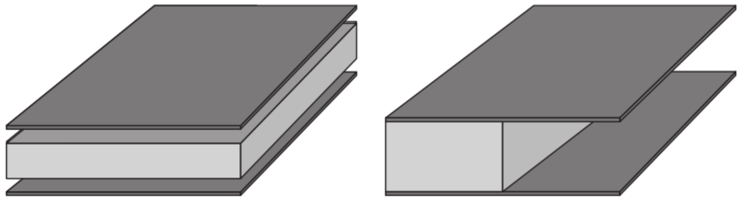
Working in Sphercial Polar Coordinates, we get :

(a) $\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{Bmatrix} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r=b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r=a). \end{Bmatrix}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\hat{\mathbf{r}}$. For $r<a$, $Q_{\text{enc}}=0$, so $\boxed{\mathbf{E}=\mathbf{0}.}$ For $r>b$, $Q_{\text{enc}}=0$ (Prob. 4.14), so $\boxed{\mathbf{E}=\mathbf{0}.}$ For $a<r<b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right) 4\pi\bar{r}^2 d\bar{r} = -4\pi ka - 4\pi k(r-a) = -4\pi kr$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = \mathbf{0}$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \mathbf{0} \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so $\boxed{\mathbf{E}=\mathbf{0} \text{ (for } r<a \text{ and } r>b);}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a<r<b).}$

3. Suppose we half-fill a parallel-plate capacitor in two ways, as shown in the two figures. By what fraction is the capacitance increased when the material is distributed as shown in each case? For a given potential difference V between the plates, find E, D, and P, in each region, and the free and bound charge on all surfaces.



(a) (b)

With no dielectric, $C_0 = A\epsilon_0/d$

In configuration (a), with $+\sigma$ on upper plate, $-\sigma$ on lower, $D = \sigma$ between the plates. $E = \sigma/\epsilon_0$ (in air) and $E = \sigma/\epsilon$ (in dielectric). So $V = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Qd}{2\epsilon_0 A}\left(1 + \frac{\epsilon_0}{\epsilon}\right)$.

$$C_a = \frac{Q}{V} = \frac{\epsilon_0 A}{d}\left(\frac{2}{1+1/\epsilon_r}\right) \Longrightarrow \boxed{\frac{C_a}{C_0} = \frac{2\epsilon_r}{1+\epsilon_r}.}$$

In configuration (b), with potential difference V: $E = V/d$, so $\sigma = \epsilon_0 E = \epsilon_0 V/d$ (in air). $P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ (in dielectric), so $\sigma_b = -\epsilon_0\chi_e V/d$ (at top surface of dielectric). $\sigma_{\text{tot}} = \epsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \epsilon_0\chi_e V/d$, so $\sigma_f = \epsilon_0 V(1+\chi_e)/d = \epsilon_0\epsilon_r V/d$ (on top plate above dielectric).

$$\Longrightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right). \boxed{\frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}.}$$

[Which is greater? $\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1+\epsilon_r)^2 - 4\epsilon_r}{2(1+\epsilon_r)} = \frac{1+2\epsilon_r+4\epsilon_r^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0$. So $C_b > C_a$.]

4. A point charge q is placed in a medium whose permittivity $\mathcal{E}$ changes with the distance r from q as $\mathcal{E} = 1 + \frac{A}{r}$ where A is a constant. Show that the potential at any point is given by

$$\varphi(r) = \frac{q}{4\pi A}\ln\left(1 + \frac{A}{r}\right)$$

S1 Considering a spherical Gaussian surface with the charge at its centre, we can write from Gauss's law

$$\oint_S \vec{D}\cdot d\vec{s} = q$$

Because of spherical ~~syd~~ symmetry $\vec{D}$ is constant all over the Gaussian surface and is normal to it at every point. Therefore,

$$D\cdot 4\pi r^2 = q \;\Rightarrow\; D = \frac{q}{4\pi r^2}.$$

$$\therefore\; E = \frac{D}{\varepsilon} = \frac{q}{4\pi\varepsilon r^2} = \frac{q}{4\pi\left(1+\frac{A}{r}\right)r^2} = \frac{q}{4\pi r(r+A)}$$

$$= \frac{q}{4\pi A}\left[\frac{1}{r} - \frac{1}{r+A}\right]$$

$$\therefore\; \phi(r) = -\int \vec{E}\cdot d\vec{r} = -\int E\,dr$$

$$= -\frac{q}{4\pi A}\left[\ln r - \ln(r+A)\right] + c$$

$$= \frac{q}{4\pi A}\ln\left(1+\frac{A}{r}\right) + c$$

$c \rightarrow$ constant here

Since $\phi \rightarrow 0$ as $r \rightarrow \infty$, we get $c = 0$

$$\boxed{\therefore\; \phi(r) = \frac{q}{4\pi A}\ln\left(1+\frac{A}{r}\right)}$$