PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 24

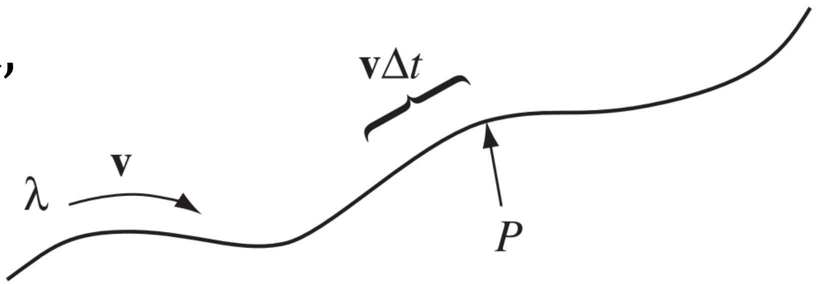
Line, Surface, and Volume Currents
Magnetic field due to steady currents

THE BIOT-SAVART LAW

Currents

A line charge traveling down a wire at speed v constitutes a current, ***I***

Let the mobile charge per unit length is λ,



Then a segment of length $v\Delta t$ will carry charge $\lambda v\Delta t$,

What current $(I\ )$does pass point P in a time interval Δt?

$$I = \frac{dq}{dt} = \frac{\lambda v\Delta t}{\Delta t} = \lambda v$$

Magnetic force on a segment (dl) of wire carrying current in presence of magnetic field $\boldsymbol{B}$.

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl$$

Currents

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\lambda\, dl$$

(Assuming the presence of a finite magnetic field in the region of the wire)

As the current in the wire is $I = \lambda v$, with $\boldsymbol{dl} = dl\hat{\boldsymbol{v}}$

$$\Rightarrow \boldsymbol{F_{mag}} = \lambda v \int (dl\hat{\boldsymbol{v}} \times \boldsymbol{B}) = I \int (\boldsymbol{dl} \times \boldsymbol{B})$$

With a vector quantity, $\boldsymbol{I} = \lambda \boldsymbol{v}$

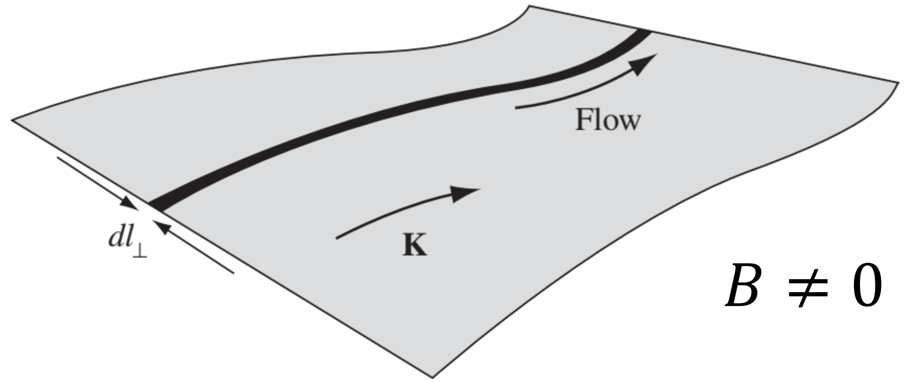
(Current $\boldsymbol{I}$ is constant in magnitude along the wire)

$$\boldsymbol{F_{mag}} = I \int (\boldsymbol{dl} \times \boldsymbol{B}) = \int (\boldsymbol{I} \times \boldsymbol{B})\, dl$$

Surface and Volume current density

When a charge flows over a surface, we describe it by surface current density, $\boldsymbol{K}$

Consider current flowing over a sheet- consider a ribbon of infinitesimal width $dl_\perp$ (running parallel to the flow). If current in this ribbon is $\boldsymbol{dI}$, surface current density,



$B \neq 0$

$$\mathbf{K} = \frac{\boldsymbol{dI}}{dl_\perp} \text{ (Current per unit width)}$$

In terms of surface charge density of mobile electrons (σ), $\boldsymbol{K} = \sigma \boldsymbol{v}$

$$\boldsymbol{dI} = \lambda \boldsymbol{v} = \frac{dq}{dl_{||}} \boldsymbol{v}$$

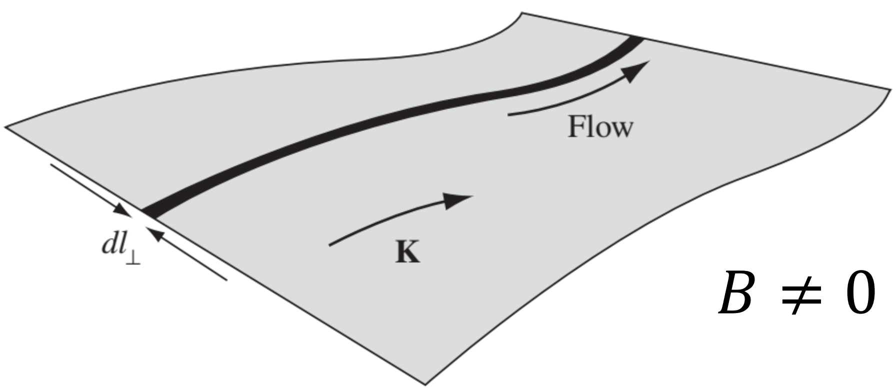
$$\Rightarrow \mathbf{K} = \frac{\boldsymbol{dI}}{dl_\perp} = \frac{dq}{dl_\perp dl_{||}} \boldsymbol{v} = \frac{dq}{da} \boldsymbol{v} = \sigma \boldsymbol{v}$$

Surface and Volume current density

The magnetic force on the surface current is given by

$$\boldsymbol{F_{mag}} = \int dq(\boldsymbol{v} \times \boldsymbol{B}) = \int \sigma(\boldsymbol{v} \times \boldsymbol{B})da = \int (\boldsymbol{K} \times \boldsymbol{B})da$$

Using $(dq = \sigma dA)$

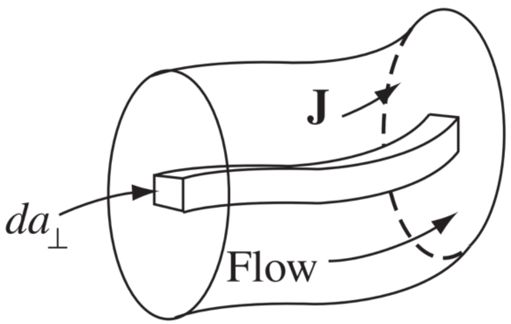


$B \neq 0$

(Just as ***E*** suffers a discontinuity at surface charge, ***B*** is also discontinuous at a surface current)

Surface and Volume current density

In 3D region, we use volume current density $\boldsymbol{J}$. Consider current flowing across the tube and consider an infinitesimal tube of cross-sectional area $da_{\perp}$ along the floor.



If current in this tube is $\boldsymbol{dI}$, volume current density is

$B \neq 0$

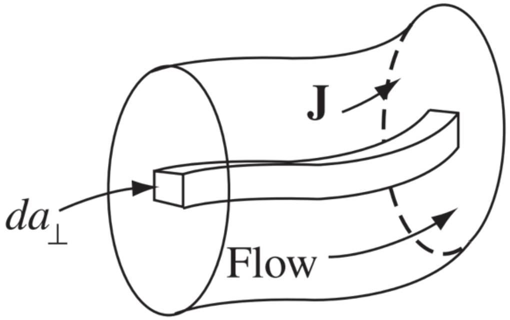
$$\mathbf{J} = \frac{\boldsymbol{dI}}{da_{\perp}} \quad \text{(Current per unit area)}$$

If volume charge density of mobile electrons is ρ, $\boldsymbol{J} = \rho\boldsymbol{v}$

$$\boldsymbol{dI} = \lambda\boldsymbol{v} = \frac{dq}{dl_{||}}\boldsymbol{v} \qquad \Rightarrow \boldsymbol{J} = \frac{\boldsymbol{dI}}{da_{\perp}} = \frac{dq}{da_{\perp}dl_{||}}\boldsymbol{v} = \frac{dq}{d\tau}\boldsymbol{v} = \rho\boldsymbol{v}$$

Surface and Volume current density

Magnetic force on a volume current is therefore



$$\boldsymbol{F}_{\boldsymbol{mag}} = \int dq(\boldsymbol{v} \times \boldsymbol{B}) = \int \rho(\boldsymbol{v} \times \boldsymbol{B})d\tau$$

$B \neq 0$

Where we have used $dq = \rho d\tau$ in the above equation.

In terms of volume current density $\boldsymbol{J} = \rho\boldsymbol{v}$,

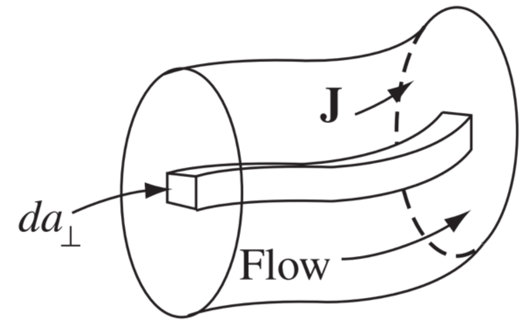
Magnetic force on a volume current is therefore

$$\Rightarrow \boldsymbol{F}_{\boldsymbol{mag}} = \int \rho(\boldsymbol{v} \times \boldsymbol{B})d\tau = \int (\boldsymbol{J} \times \boldsymbol{B})d\tau$$

Surface and Volume current density

Total current crossing a surface S

$$I = \int_S J \, da_\perp = \int_S \mathbf{J} \cdot d\mathbf{a}$$



The dot product neatly picks out the component of ***da*** along the current flow ***J***.

The charge per unit time leaving volume V is

$$I = \oint \boldsymbol{J}.\boldsymbol{da} = \int_V \boldsymbol{\nabla}.\boldsymbol{J} d\tau$$

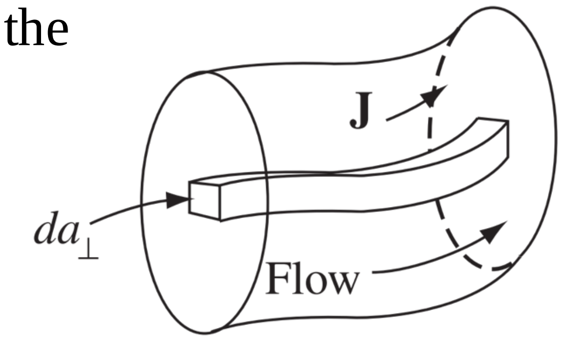
This is also the amount of charge (per unit time) crossing through S and entering υ

This is also the definition of current.

$$I = \sum_{i=1}^{n} (\ ) q_i \mathbf{v}_i \sim \int_{\text{line}} (\ ) \mathbf{I} \, dl \sim \int_{\text{surface}} (\ ) \mathbf{K} \, da \sim \int_{\text{volume}} (\ ) \mathbf{J} \, d\tau. \quad \Bigg| \quad q \sim \lambda \, dl \sim \sigma \, da \sim \rho \, d\tau$$

Equation of Continuity

Since the **charge is conserved**, whatever flows out through the surface must come at the expense of what remains inside:



$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

(The minus sign reflects the fact that an outward flow decreases the charge left in V.) Since this applies to any volume, we conclude that

Rate of decrease of this charge in a given closed region (in the absence of any source or sinks)

$$-\frac{d}{dt}\int \rho d\tau$$

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.}$$

$$\boxed{\boldsymbol{\nabla} . \boldsymbol{J} + \frac{\partial \rho}{\partial t} = \mathbf{0}}$$

This is the precise mathematical statement of local charge conservation; it is called the ***continuity equation***.

Steady Currents

Stationary charges	$\Rightarrow$	**constant electric fields: electrostatics.**
Steady currents	$\Rightarrow$	**constant magnetic fields: magnetostatics.**

Steady currents mean a continuous flow of charge that has been going on forever, without change and without charge piling up anywhere

Formally, electrostatics/magnetostatics is the regime where

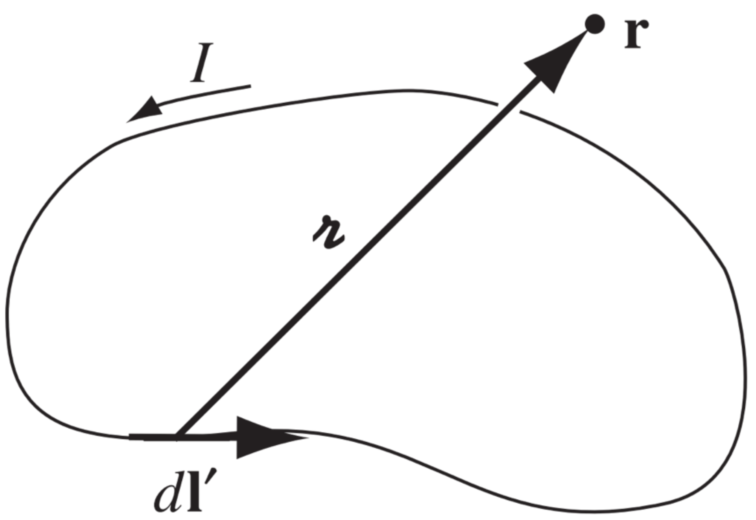
$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \boldsymbol{J}}{\partial t} = 0 \qquad \text{at all places and at all times}$$

$$\nabla \cdot \mathbf{J} = 0.$$

Magnetic Field of a steady current: Biot-Savart law

☆☆

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.}$$



$\boldsymbol{dl'}$=element of length along the wire

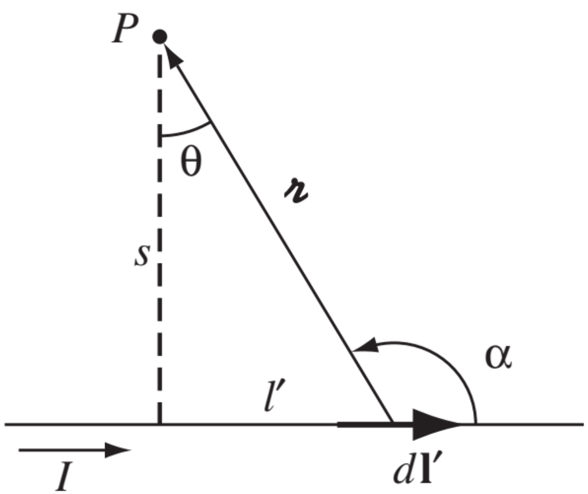
$\boldsymbol{\mathscr{r}}$ = vector pointing from source ($\boldsymbol{dl'}$) to field point r

μ_0 = permeability of free space

$= 4\pi \times 10^{-7} N/A^2$

Unit = $1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}$

P1 Find the magnetic field at a distance 's' from a long straight wire carrying a steady current *I*



Solution

Consider a line element dl' at a distance l' from origin

Field at P is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int \frac{\boldsymbol{dl'} \times \hat{\boldsymbol{r}}}{r^2}$$ (Biot-Savart law)

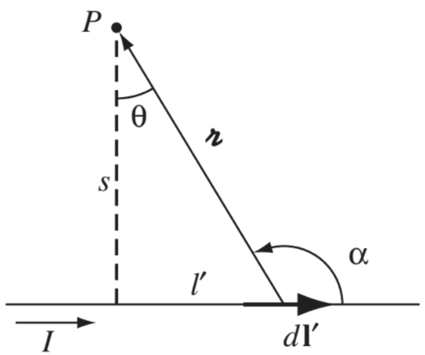
$(d\mathbf{l}' \times \hat{\mathbf{r}})$ points *out* of the page,

$$dl' \times \hat{\boldsymbol{r}} = dl' sin\alpha = dl' cos\theta$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int \frac{dl' cos\theta}{r^2}$$

THE BIOT – SAVART LAW

Choose variables from *dl′ to θ*

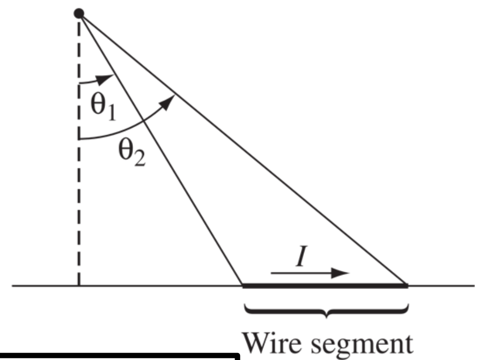


$$l' = s\, tan\theta$$

$$dl' = \frac{s}{\cos^2\theta}\, d\theta$$

Also $s = r\cos\theta$

$$\frac{1}{r^2} = \frac{cos^2\theta}{s^2}$$

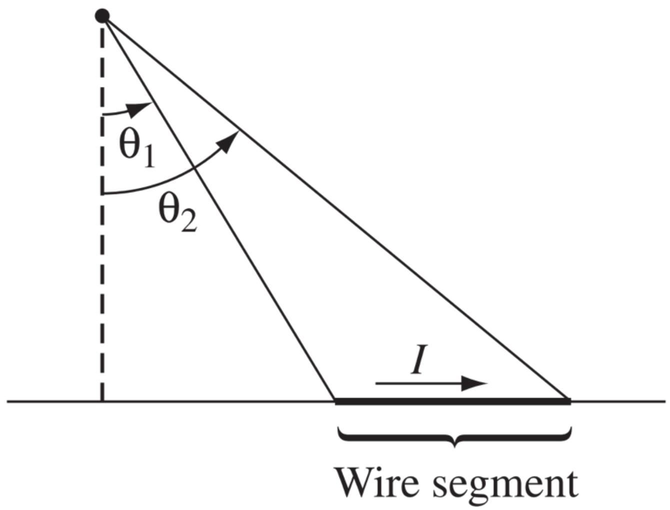


$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta$$

$$\boxed{B(r) = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2} cos\theta d\theta = \frac{\mu_0 I}{4\pi s}(sin\theta_2 - sin\theta_1)}$$

The field of any straight segment of wire, in terms of the initial and final angles θ_1 and θ_2

For a wire of infinite length



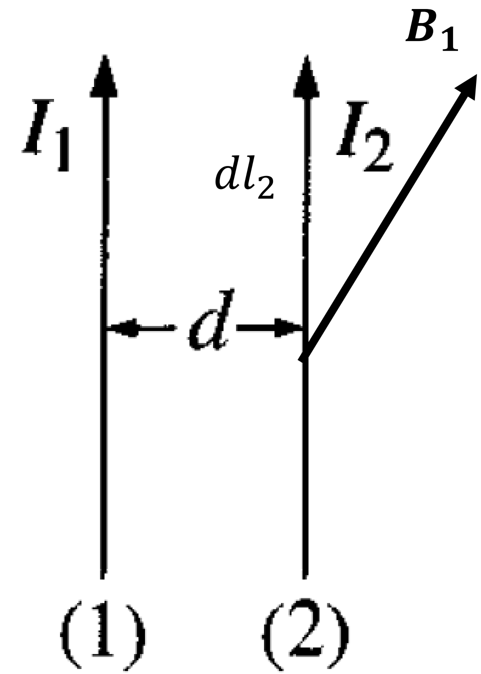
$$\theta_1 = -\frac{\pi}{2} \qquad \theta_2 = +\frac{\pi}{2}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

Here $\hat{\boldsymbol{\varphi}}$ is the unit vector along the azimuthal direction in spherical polar co-ordinates

When P is *above* the wire, B points *out-of-plane* of paper. When P is *below* the wire, B points *into* the plane of paper.

P2 Find the force of attraction between two long parallel wires kept at a distance 'd' apart carrying currents I_1 and I_2



Solution

Field at (2) due to (1) is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

This field acts into the plane of paper. Lorentz force law says that, due to this magnetic field B_1, the wire (2) would experience a force which would be directed towards wire (1).

F_{21}= force on wire (2) due to current in wire (1)

$$F_{21} = I_2 \int dl_2 \times B_1 = \frac{\mu_0 I_1 I_2}{2\pi d} \int dl_2$$

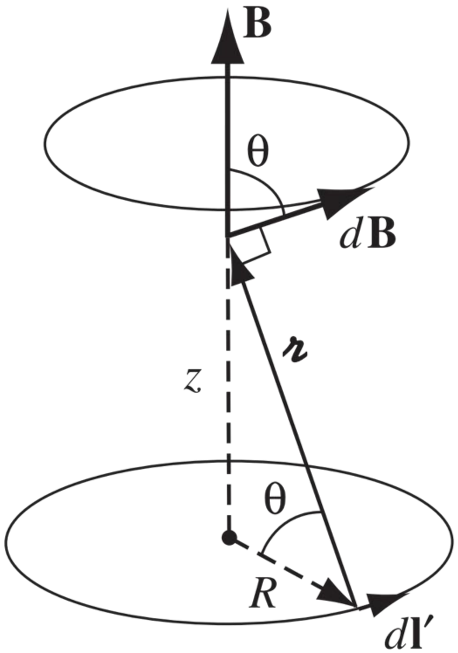
Force per unit length,

$$\boxed{f = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{\mu_0}{4\pi}\left(\frac{2I_1 I_2}{d}\right)}$$

Force of attraction is proportional to the product of currents and to $\frac{1}{d}$

Magnetic field due to current in a circular loop

P3 Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I



Solution

The field ***dB*** attributable to the segment ***dl'*** is shown (i.e along $\boldsymbol{dl}' \times \hat{\boldsymbol{r}}$)

As we integrate ***dl'*** around the loop, ***dB*** sweeps out cone. The horizontal components cancel, and the vertical components combine to give $B(z)$.

Note that dl' and r are perpendicular in this case; factor of $\cos(\theta)$ projects out the vertical component. Also r is constant.

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{r^2} \cos\theta$$

$$\boldsymbol{dB} = \frac{\mu_0 I}{4\pi} \frac{\boldsymbol{dl}' \times \hat{\boldsymbol{r}}}{r^2}$$

(dl' and r are perpendicular)

$$B(z) = \frac{\mu_0 I}{4\pi} \left(\frac{\cos\theta}{r^2} \right) 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

($\cos(\theta)$ projects out the vertical (z) component)

P4 Find the magnetic field at a point P on the axis of the solenoid (helical coil) consisting of 'n' turns per unit length wrapped around a cylindrical tube of radius 'a' and carrying current I. Express your answer in terms of θ_1 and θ_2 where these represent angle subtended by nearest and farthest turn (respectively) w.r.t to the field point on the axis.

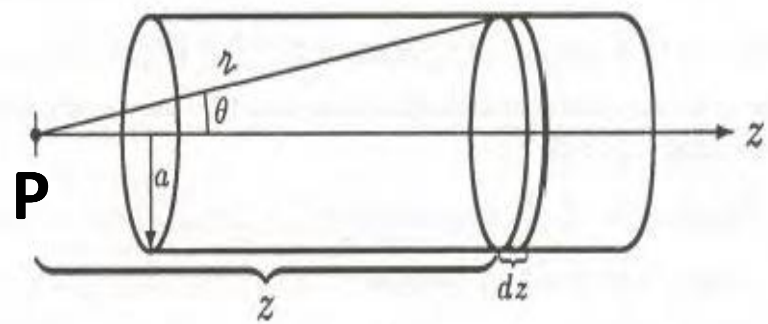
Solution

Use the equation for magnetic field due to a circular loop done previously

Use Eq. 5.38 for a ring of width dz, with $I \to nI\,dz$:

$$B = \frac{\mu_0 nI}{2} \int \frac{a^2}{(a^2+z^2)^{3/2}}\,dz. \text{ But } z = a\cot\theta,$$

$$\text{so } dz = -\frac{a}{\sin^2\theta}\,d\theta, \text{ and } \frac{1}{(a^2+z^2)^{3/2}} = \frac{\sin^3\theta}{a^3}.$$



So

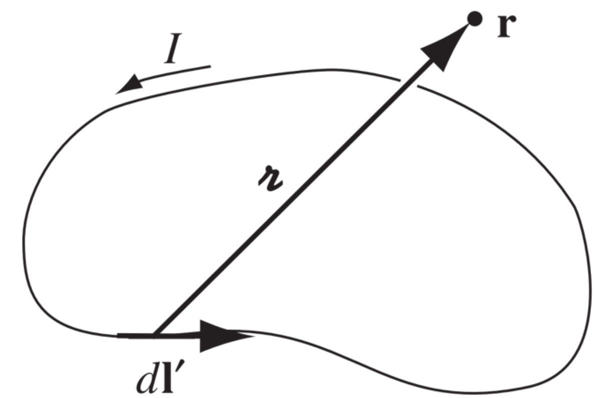
$$B = \frac{\mu_0 nI}{2} \int \frac{a^2 \sin^3\theta}{a^3 \sin^2\theta}(-a\,d\theta) = -\frac{\mu_0 nI}{2} \int \sin\theta\,d\theta = \frac{\mu_0 nI}{2} \cos\theta\Big|_{\theta_1}^{\theta_2} = \boxed{\frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1).}$$

For an infinite solenoid, $\theta_2 = 0$, $\theta_1 = \pi$, so $(\cos\theta_2 - \cos\theta_1) = 1 - (-1) = 2$, and $B = \boxed{\mu_0 nI.}$

Magnetic Field of a steady current: Biot-Savart law

Magnetic field of a steady current is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{I\boldsymbol{dl'}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl'$$



Expression of Biot-Savart law for surface and volume currents

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,da'$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$