PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 34

Poynting's theorem

Electromagnetic Waves

Suppose we have some charge and current configurations, which produces fields ′ ′ and ′ ′ at a time ‘ ’. In the next instance, the charges move around a bit. How much work, , is done by the electromagnetic forces acting on these charges?

Work done on charge at location .

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For a volume , rate at which work is done

$$\frac{}{} = \int_{\mathcal{V}} \ .$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \quad \mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{E}\cdot\mathbf{J}=\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})-\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

We now use the identity

$$\nabla\cdot(\mathbf{E}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{B})$$

$$\Rightarrow \mathbf{E}\cdot(\nabla\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{E})-\nabla\cdot(\mathbf{E}\times\mathbf{B}).$$

From $\boldsymbol{\nabla}\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t}$

$$\mathbf{E}\cdot(\nabla\times\mathbf{B})=-\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}-\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(B^2)$$ and $$\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

$$\Rightarrow \mathbf{E}\cdot\mathbf{J}=-\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{E}\cdot\mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$\dfrac{dW}{dt} = \int_{\mathcal{V}} \boldsymbol{E}.\boldsymbol{J}\,d\tau$ and using Gauss divergence theorem.

$$\Rightarrow \frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

$u = \dfrac{1}{2}\left(\epsilon_0 E^2 + \dfrac{B^2}{\mu_0}\right)$ Energy per unit volume stored in electromagnetic fields.

$\boldsymbol{S} = \dfrac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$ Energy per unit area per unit time, transported out of $\mathcal{V}$ across its surface (Poynting vector).

Poynting's theorem

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u\,d\tau - \int_{S} \boldsymbol{S}.\boldsymbol{da}$$

Work done by the electromagnetic forces on the charges in a volume $\boldsymbol{\mathcal{V}}$**,** equals the decrease in energy stored in the fields in that volume less the energy that flows out of the surface of $\boldsymbol{\mathcal{V}}$.

In regions of empty space where there are no charges (but finite fields from elsewhere)

$$\frac{dW}{dt} = 0 \qquad \int \frac{\partial u}{\partial t} d\tau = -\oint_S \boldsymbol{S}.\boldsymbol{da} = -\int_v \boldsymbol{\nabla}.\boldsymbol{S}\,d\tau$$

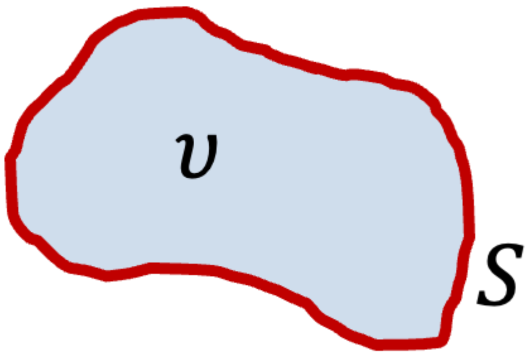
$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla}.\boldsymbol{S}$$

Equation of continuity for 'u'

Law of conservation of electromagnetic energy

The continuity equation (charges and currents)

If charge in some region changes, then exactly that amount of charge must have passed in or out through the surface



Let total charge in the volume be 'Q'

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau$$

Current flowing out of the boundary $-\oint J.da$

$$\frac{dQ}{dt} = -\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$$

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau$$

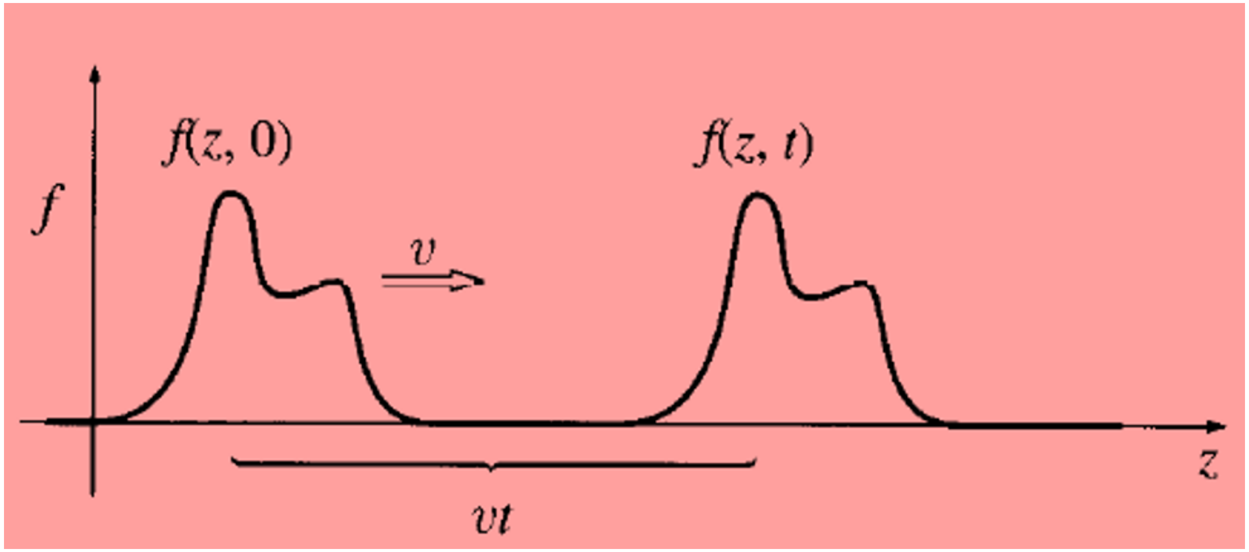
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

Equation of continuity serves as a constraint on the sources (ρ and J). They can't be just any odd functions- they have to respect conservation of charge.

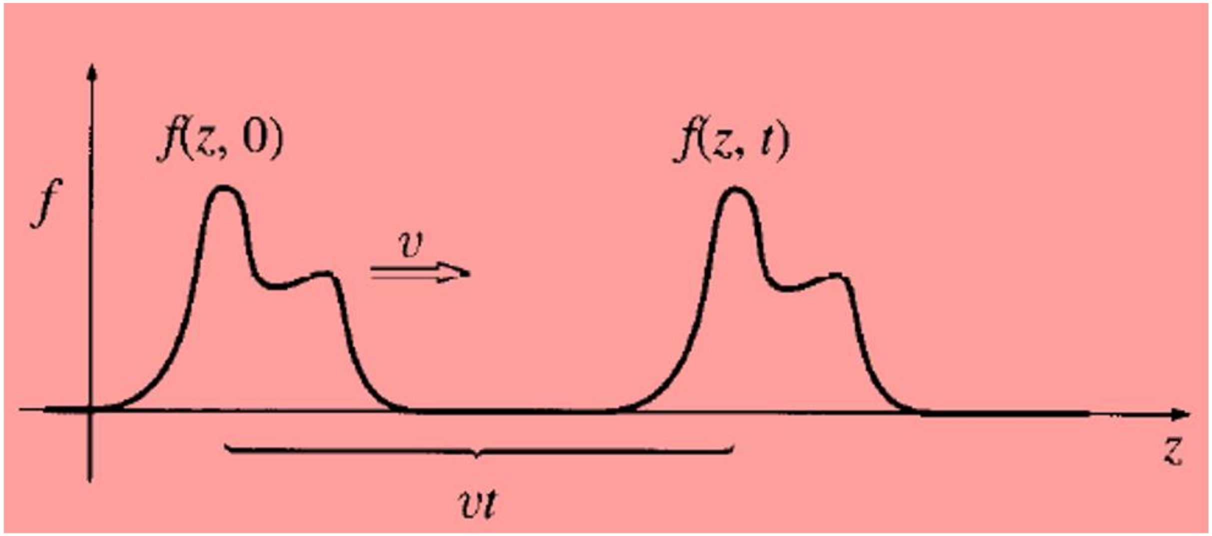
What is a wave?

A wave is a disturbance of a continuous medium that propagates with a *fixed shape* at a *constant velocity*

Consider a wave of fixed shape moving with a constant velocity



How would you represent such a wave mathematically?



Given the initial shape of the string $g(z) \equiv f(z,0)$, what is the subsequent form $f(z,t)$?

Displacement at a point 'z', at a later time 't' is equal to the displacement a distance 'vt' to the left (i.e at $z - vt$) at time $t = 0$

$$f(z,t) = f(z - vt, 0) \equiv g(z - vt)$$

Essence of wave motion: function $f(z,t)$ that might have depended on z and t in any old way, in fact, depends on them in a very special combination; $z - vt$. When this is true, $f(z,t)$ represents a wave of fixed shape traveling in $+z$ direction at speed 'v'

The classical wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Why is it a wave equation?

Because it ($f(z,t)$) admits all functions of form $(z - vt)$ as its solutions

$$f(z,t) = f(z - vt, 0) \equiv g(z - vt) = g(u)$$

$$u \equiv z - vt$$

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}, \quad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$$

$$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$$

$$\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

The classical wave equation

Question: Which expression represents travelling wave .

$$f_1(z,t) = Ae^{-b(z-vt)^2}$$

$$f_4(z,t) = Ae^{-b(bz^2+vt)}$$

$$f_2(z,t) = A\sin[b(z-vt)]$$

$$f_5(z,t) = A\sin(bz)\cos(bvt)^3$$

$$f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

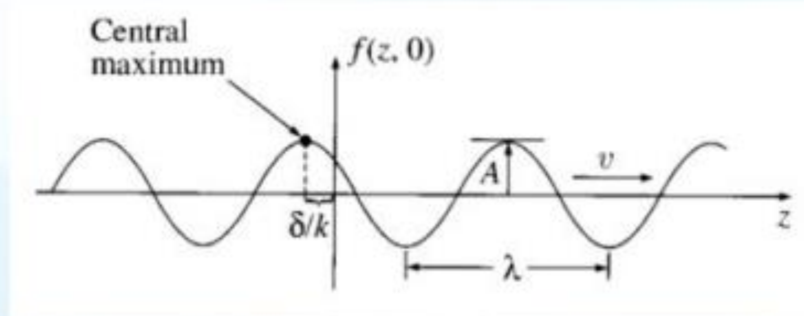
f_4, f_5 does not represent waves

f_1, f_2, f_3 represent waves

Sinusoidal waves

Sinusoidal waves have a special place among all possible wave forms. Mathematically, a general sinusoidal wave can be given as

$$f(z,t) = A\cos[k(z - vt) + \delta]$$



A is the amplitude of the wave (it is positive, and represents the maximum displacement from equilibrium).

The argument of cosine is called the phase, and δ is the phase constant. It is clear that we can add any integer multiple of 2π to δ without changing $f(z,t)$. Usually we use the values in the range $0 \le \delta < 2\pi$.

k is the wave number. It relates to the wavelength λ as $\lambda = \dfrac{2\pi}{k}$.

A full cycle is achieved in one time period, $T = \dfrac{2\pi}{kv}$.

The linear frequency (number of oscillations per unit time) is $\nu = \dfrac{1}{T} = \dfrac{kv}{2\pi} = \dfrac{v}{\lambda}$.

We will be also using the angular frequency, $\omega = 2\pi\nu = kv$

Sinusoidal waves

In terms of ω the wave expression can be written as

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

A sinusoidal oscillation of wave number k and angular frequency ω traveling to the left would be

$$f(z,t) = A\cos(kz + \omega t - \delta)$$

But since cosine is an even function we may write the above as

$$f(z,t) = A\cos(kz + \omega t - \delta) = A\cos(-kz - \omega t + \delta)$$

Comparing with the equation at the top we find that the same effect is obtained by switching sign of *k* to negative.

Sinusoidal Waves

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$ with solutions $f(z,t) = g(z - vt)$

Represents a wave equation traveling in +z direction with velocity 'v'

$h(z + vt)$ would also be the solution of $f(z,t)$: this represents a wave traveling in –z directions with velocity 'v'

<u>Most general solution of wave equation</u>

$$f(z,t) = g(z - vt) + h(z + vt)$$

(sum of wave moving to the left and to the right)

Wave equation is thus linear- sum of two solutions is itself a solution.

Complex notation

In view of the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we have

$$\cos\theta = \mathrm{Re}[e^{i\theta}],$$

where Re represents the real part. Therefore, we may write our wave function as

$$f(z,t) = A\cos(kz - \omega t + \delta) = \mathrm{Re}[Ae^{i(kz-\omega t+\delta)}].$$

With this idea, we can introduce a complex wave function

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)},$$

where $\tilde{A} \equiv Ae^{i\delta}$ is the complex amplitude. Note that the phase constant $e^{i\delta}$ has been absorbed in $\tilde{A}$.

The actual wave is therefore the real part of this 'complex wave':

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)].$$

The advantage of going to this complex form is that manipulation of exponentials is much simpler than that of sine or cosine.

Electromagnetic waves in vacuum

Consider Maxwell's equations in region of space where there is no charge or current (vacuum+no source):

$$\text{(i)}\ \nabla \cdot \mathbf{E} = 0, \qquad \text{(ii)}\ \nabla \cdot \mathbf{B} = 0,$$

$$\text{(iii)}\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(iv)}\ \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Let us apply curl to (iii):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Here, on the LHS we used the identity $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, while on the RHS we interchanged the order of curl and time-derivative operations. Now using (i)and (iv), we have

$$\nabla(0) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Electromagnetic waves in vacuum

$$\text{(i) } \nabla \cdot \mathbf{E} = 0, \qquad \text{(ii) } \nabla \cdot \mathbf{B} = 0,$$

$$\text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(iv) } \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Similarly, applying curl to (iv):

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$$

Again, on the LHS we used the identity $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, while on the RHS we interchanged the order of curl and time-derivative operations. Now using (ii)and (iii), we have

$$\nabla(0) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Electromagnetic waves in vacuum

Thus we have decoupled the E and B and obtained separate differential equations for them. Note that earlier we had four first order coupled (vector) differential equations, now we have two second order decoupled (vector) differential equations:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Each of these vector equations incorporates three scalar equations, one for each cartesian component (E_x, E_y, E_z, and similarly B_x, B_y, B_z). There, each cartesian component of **E** and **B** satisfies the three-dimensional wave equation:

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

i.e.,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial f}{\partial^2 t^2}$$

with wave propagation speed given by $v = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}.$

Electromagnetic waves in vacuum, Speed of Light

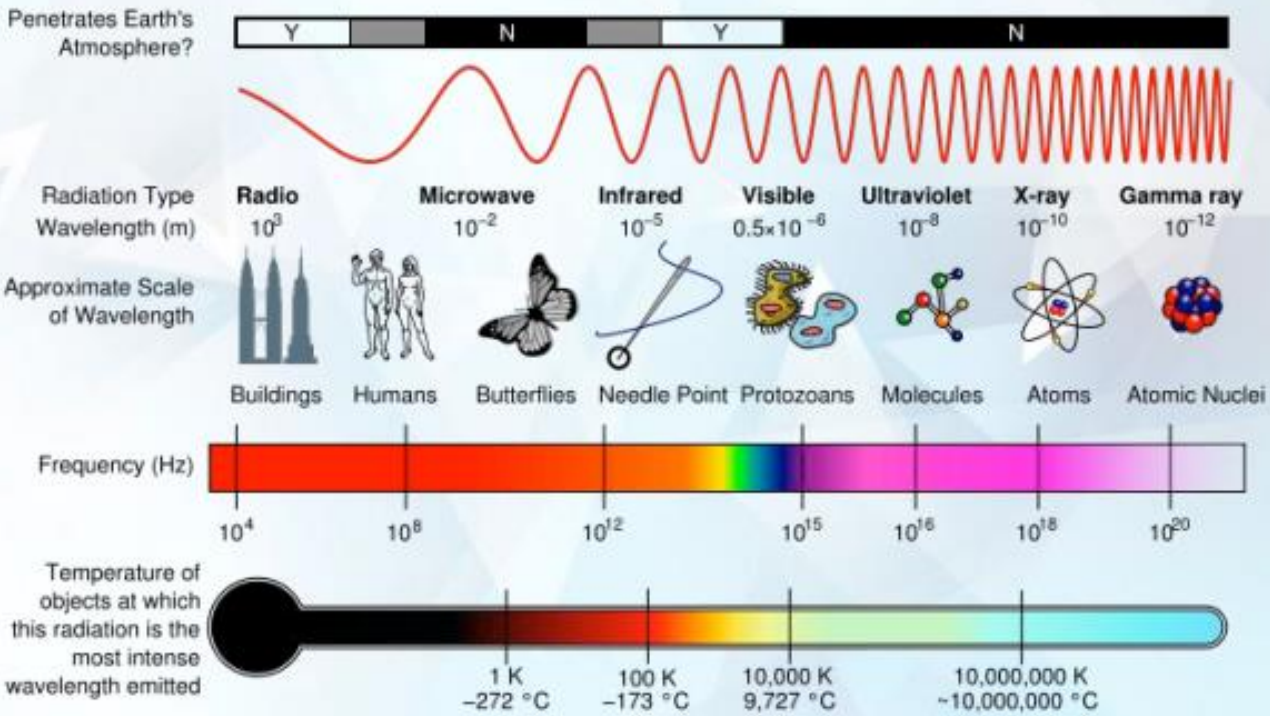
Therefore, Maxwell's equations imply that vacuum supports the propagation of electromagnetic waves, which travel at a speed

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299792458 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}.$$

This happens to be precisely the speed of light c in vacuum.

The Electromagnetic Spectrum

We will confine our attention to sinusoidal waves of frequency. Since different frequencies in the visible range correspond to different colors, such waves (**single frequency**) are called **monochromatic**.



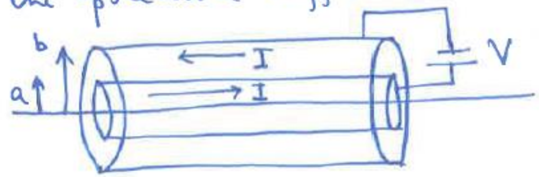
Check this out- http://www.chromoscope.net/

The Electromagnetic Spectrum

The Electromagnetic Spectrum		
Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

A long coaxial cable carries current I (flowing down the surface through inner cylinder 'a' and coming back along outer cylinder (b) — Find the power (energy per unit time) transported down the cables if V is the potential difference between the 2 cylinders.



The current ~~flows~~ distributed on the surface flows down the inner cylinder and comes back through the surface of outer cylinder. Both $\vec{E}$ and $\vec{B}$ are non-zero only in the region between the 2 cylinders, $a < s < b$

Energy per unit time transported down the cables = power

$$= \frac{1}{\mu_0} \oint_S (\vec{E} \times \vec{B}) \cdot d\vec{a} = \oint \vec{S} \cdot d\vec{a}$$

We know $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$ $\quad \underline{I = \lambda v}$ $\quad \underline{a < s < b}$ $\quad \vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

$$P = \frac{1}{\cancel{\mu_0}} \int_{s=a}^{b} \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{\cancel{\mu_0} I}{\cancel{2\pi}} \frac{1}{s^2} \cancel{2\pi} s \, ds = \frac{\mu_0 I}{2\pi\epsilon_0} \lambda \int_a^b \frac{ds}{s} = \frac{\lambda I}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$$\text{But } V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l} = -\int_b^a \frac{\lambda}{2\pi\epsilon_0 s} ds = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$$\boxed{P = V \times I}$$