PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 26

Introduction to magnetic vector potential

Vector Potential

$\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ permits us to introduce a scalar potential V in electrostatics

$$\boldsymbol{E} = -\boldsymbol{\nabla} V$$

Similarly, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ invites the introduction of vector potential $\boldsymbol{A}$ in magnetostatics

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

Divergence of curl of a vector is always zero, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$

Unit of **A:** Tm

Ampere's law:

$$\boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A} = \mu_0 \mathbf{J} \qquad (1)$$

Vector Potential

Just as you can add to V, any function whose gradient is zero (i.e constant) without altering ***E***, similarly you can add to ***A***, any function whose curl vanishes (i.e. gradient of any scalar) with no effect on ***B***.

Choose $\boldsymbol{A} = \boldsymbol{A_0} + \boldsymbol{\nabla}\lambda$ λ is any scalar

$\boldsymbol{A_0}$ is the original (old) vector function

$\boldsymbol{A}$ is the new vector function

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda \quad (2)$$

We can always choose λ in such a way to eliminate the divergence of A, so that

$$\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0 = f(r) \text{ (say)}$$

$$\boxed{\nabla \cdot \mathbf{A} = 0.}$$

This is similar to **Poisson equation**

In other words, given a vector potential, we can always choose to work with another vector potential that gives the same field as the original one, but that has zero divergence.

Introduction to magnetic vector potential

Vector Potential

Similarity to Poisson equation

$$\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

with $\nabla \cdot \mathbf{A}_0$ in place of ρ/ϵ_0 as the "source."

if $\nabla \cdot \mathbf{A}_0$ goes to zero at infinity

if ρ goes to zero at infinity

whose solutions are

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \mathbf{A}_0}{r}\, d\tau'$$

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r}\, d\tau'$$

$$\boxed{\nabla \cdot \boldsymbol{A} = 0}$$

Coulomb's Gauge

from (1)

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

This again is nothing but **Poisson's equation**

Introduction to magnetic vector potential

Vector Potential

Poisson's Equation

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

In Cartesian coordinates,

$$\nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$$

$$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = -\mu_0 J_z$$

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'.}$$

(Volume currents)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}}\, dl'$$

(Line currents)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}}\, da'$$

(Surface currents)

Summary (so far)

For time-independent fields, we can perform calculations more simply using the electric scalar and magnetic vector potentials. The potentials obey Poisson's equation in the presence of sources:

$$\nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon}, \qquad \nabla^2 \vec{A}(\vec{r}) = -\mu \vec{J}(\vec{r})$$

The physics is invariant under gauge transformations of the scalar and vector potentials:

$$\phi \mapsto \phi + \phi_0, \qquad \vec{A} \mapsto \vec{A} + \nabla \psi_0$$

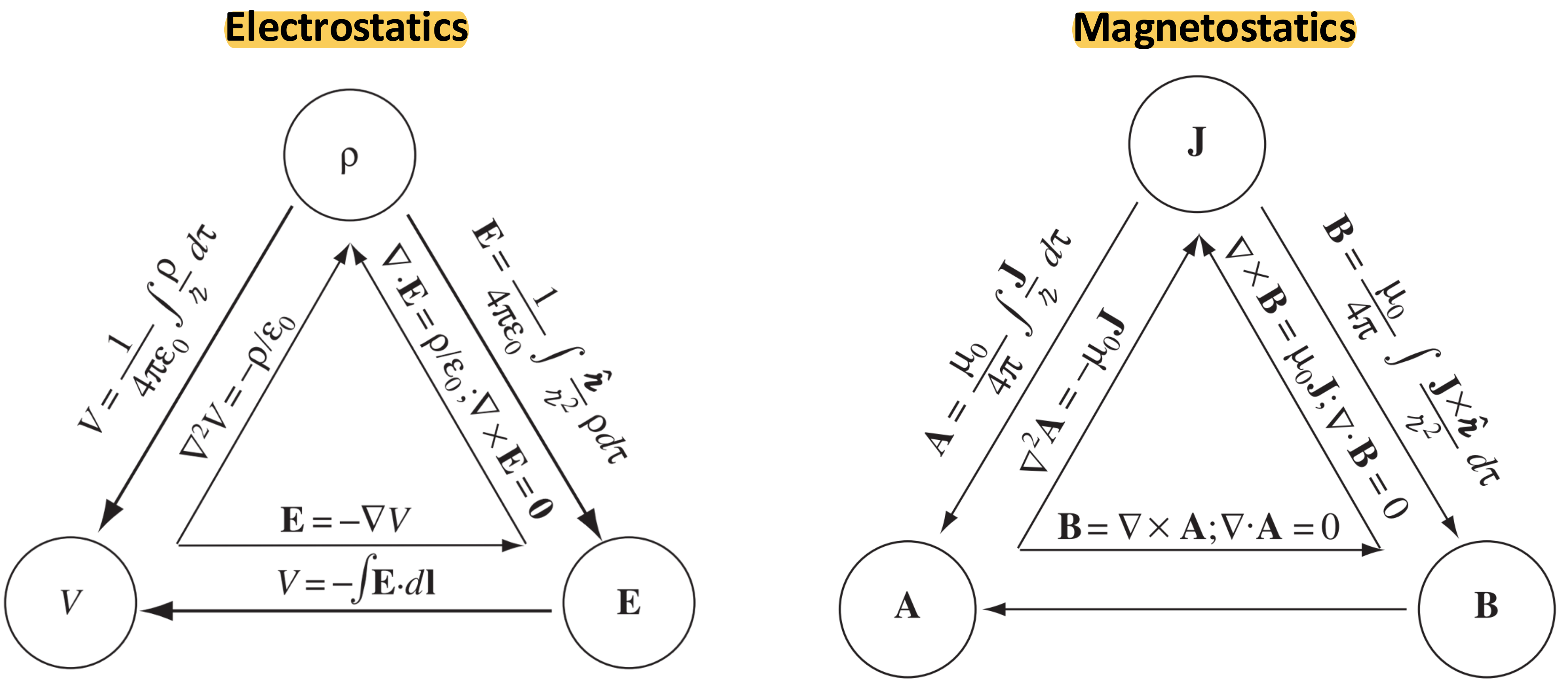
where ϕ_0 is a constant, and ψ_0 is any scalar field. One possible choice of gauge is such that:

$$\phi(|\vec{r}| = \infty) = 0, \qquad \nabla \cdot \vec{A} = 0$$

The potentials can be calculated directly from the sources:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, dV', \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, dV'$$

Boundary Conditions



Isn't it similar to electrostatics? Are ***B*** and ***A*** continuous over a current carrying surface?

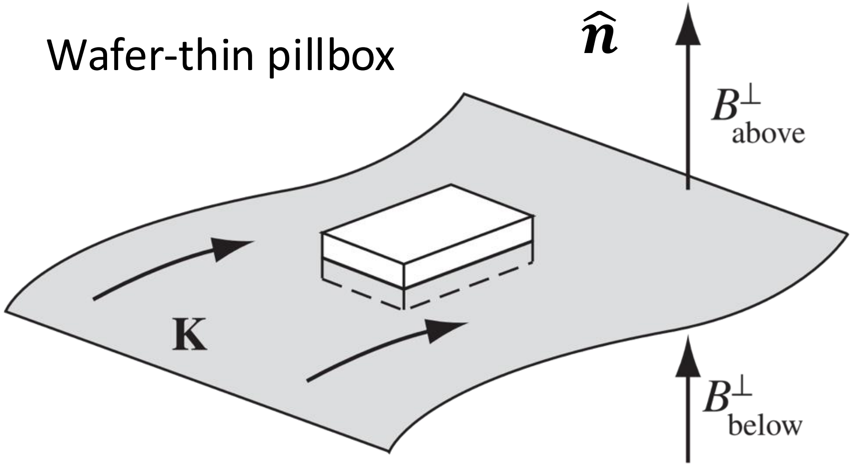
Boundary Conditions

Just as the electric field suffers discontinuity at a surface charge, similarly magnetic fields and potentials suffers discontinuity at surface currents

$$\oint \boldsymbol{B}.\boldsymbol{dA} = 0$$

$$\boxed{B_{\perp}^{above} = B_{\perp}^{below}}$$

Wafer-thin pillbox

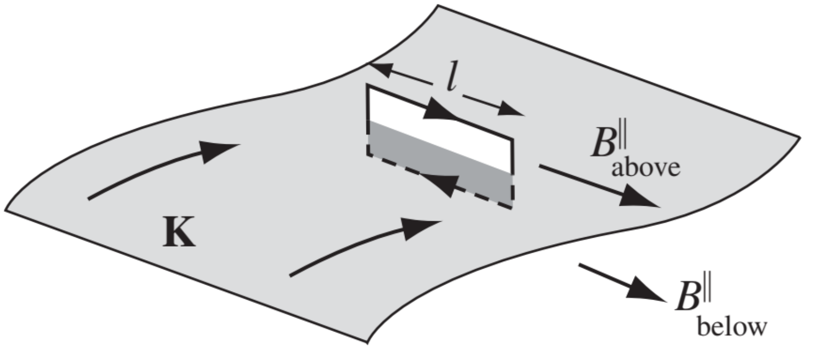


Normal components of ***B*** are continuous at surface of current density ***K***

$$\oint \boldsymbol{B}.\boldsymbol{dl} = \mu_0 I_{enc}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$



Tangential components of ***B*** are discontinuous at surface currents ***K***

Boundary Conditions

Thus the component of B that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 K$

Combining

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}),$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing "upward."

Boundary Conditions

Like the scalar potential in electrostatics, vector potential is continuous across any boundary

Scalar Potential

$$V_B - V_A = -\int_{a}^{b} \mathbf{E} \cdot d\mathbf{l}$$

$$V_B - V_A = 0$$

$$\nabla V_B - \nabla V_A = -(\mathbf{E_B} - \mathbf{E_A}) = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}.$$

$$\frac{\partial V_B}{\partial n} - \frac{\partial V_A}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

Vector Potential

$$\boxed{\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}}$$

Can be proved from the relations $\boldsymbol{\nabla}.\boldsymbol{A} = 0$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

But derivative of **A** inherits the discontinuity of **B**

$$\boxed{\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.}$$