Tutorial-3 Solutions

Q1: A bead moves along the spoke of a wheel at constant speed ‘u’ m/s. The wheel rotates with uniform angular velocity $\dot{\theta}$ = ω radians per second about an axis fixed in space. At t = 0 the spoke is along the x axis and the bead is at the origin. Find the velocity of the bead at time t in both polar and cartesian coordinates.

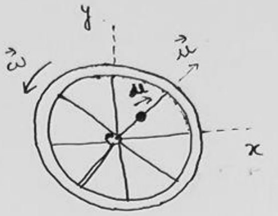
Solution :

In polar coordinates :

$r = ut$

$\dot{r} = u$

and $\dot{\theta} = \omega$, $\theta = \omega t$



Hence,

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$= u\hat{r} + ut\omega\hat{\theta}$$

To specify velocity completely, we need to know the direction of $\hat{r}$ and $\hat{\theta}$. This is obtained from $\vec{r} = (r, \theta) = (ut, \omega t)$.

In polar coordinates $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

and $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$.

$$\therefore\ \vec{v} = u\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right] + ut\omega\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right]$$

$$= u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} - ut\omega\sin\theta\,\hat{i} + ut\omega\cos\theta\,\hat{j}$$

$$= i\,(u\cos\theta - ut\omega\sin\theta) + \hat{j}\,(u\sin\theta + ut\omega\cos\theta).$$

$$\vec{v} = \hat{i}\,(u\cos\omega t - ut\omega\sin\omega t) + \hat{j}\,(u\sin\omega t + ut\omega\cos\omega t).$$

Q2 Consider a particle which feels the angular acceleration of the form $a_\theta = 3\dot{r}\dot{\theta}$. Show that $\dot{r} = \sqrt{Ar^4 + B}$ where A and B are constants.

Given, $a_r = 0$ and $a_\theta = 3\dot{r}\dot{\theta}$

We know that $a_r = (\ddot{r} - r\dot{\theta}^2)$

and $a_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta})$

So, $\ddot{r} - r\dot{\theta}^2 = 0 \quad -(1)$

& $2\dot{r}\dot{\theta} + r\ddot{\theta} = 3\dot{r}\dot{\theta} \quad -(2)$

From (2) we get

$$r\ddot{\theta} = \dot{r}\dot{\theta}$$

$$\Rightarrow \frac{\ddot{\theta}}{\dot{\theta}} = \frac{\dot{r}}{r} \Rightarrow d(\ln\dot{\theta}) = d(\ln r)$$

Integrate: $\int d(\ln\dot{\theta}) = \int d(\ln r)$

$$\Rightarrow \ln\dot{\theta} = \ln r + \underbrace{\ln c}_{\text{constant of integration}}$$

or $\ln\dot{\theta} = \ln(cr)$

$$\Rightarrow \dot{\theta} = cr \quad -(3)$$

From (1), $\ddot{r} = r\dot{\theta}^2 = r(c^2 r^2)$ {using (3)}

$$\Rightarrow \ddot{r} = c^2 r^3$$

Multiply both sides by $2\dot{r}$

$$\Rightarrow 2\dot{r}\ddot{r} = 2c^2 r^3 \dot{r} \Rightarrow d(\dot{r}^2) = \frac{2c^2}{4} d(r^4)$$

Integrate, $\int d(\dot{r}^2) = \frac{c^2}{2}\int d(r^4)$

$$\Rightarrow \dot{r}^2 = Ar^4 + B$$

or $\dot{r} = \sqrt{Ar^4 + B}$

{ $A = c^2/2$ & B in constant of integration }

Q3 A particle moves so that its position vector is given by $\boldsymbol{r} = Cos\omega t\hat{\imath} + Sin\omega t\hat{\jmath}$ where ω is a constant. Show that (a) the velocity **v** of the particle is perpendicular to **r.** (b)the acceleration **a** is directed towards the origin and has a magnitude proportional to the origin **, (c) r** X **v** = Constant vector.

(*a*) $\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\omega \sin \omega t \, \mathbf{i} + \omega \cos \omega t \, \mathbf{j}$

Then $\mathbf{r} \cdot \mathbf{v} = [\cos \omega t \, \mathbf{i} + \sin \omega t \, \mathbf{j}] \cdot [-\omega \sin \omega t \, \mathbf{i} + \omega \cos \omega t \, \mathbf{j}]$

$= (\cos \omega t)(-\omega \sin \omega t) + (\sin \omega t)(\omega \cos \omega t) = 0$

and **r** and **v** are perpendicular.

(*b*) $\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = -\omega^2 \cos \omega t \, \mathbf{i} - \omega^2 \sin \omega t \, \mathbf{j}$

$= -\omega^2 [\cos \omega t \, \mathbf{i} + \sin \omega t \, \mathbf{j}] = -\omega^2 \mathbf{r}$

Then the acceleration is opposite to the direction of **r**, i.e. it is directed toward the origin. Its magnitude is proportional to $|\mathbf{r}|$ which is the distance from the origin.

(*c*) $\mathbf{r} \times \mathbf{v} = [\cos \omega t \, \mathbf{i} + \sin \omega t \, \mathbf{j}] \times [-\omega \sin \omega t \, \mathbf{i} + \omega \cos \omega t \, \mathbf{j}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix} = \omega(\cos^2 \omega t + \sin^2 \omega t)\mathbf{k} = \omega \mathbf{k}, \text{ a constant vector.}$$

Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the center of the circle, is the *centripetal acceleration*.

Q4 In planar polar co-ordinates, an object's position at time t is given as (r, θ) $= (e^t \, meter, \sqrt{8} \, t \, radian)$

(a) Find radial velocity, tangential velocity and speed of particle at t=0

(b) Find radial acceleration, tangential acceleration and magnitude of acceleration at t=0

Solution →

$$(r, \theta) = (e^t, \sqrt{8}\,t)$$

$$r = e^t$$
$$\theta = \sqrt{8}\,t$$

(a) Radial velocity $v_r = \dot{r}$

$$\Rightarrow \dot{r} = e^t$$

at $t = 0$, $\dot{r} = e^0 = 1$ m/sec

Tangential velocity $v_\theta = r\dot{\theta}$

$$\Rightarrow r\dot{\theta} = e^t \frac{d}{dt}(\sqrt{8}\,t)$$
$$= e^t \times \sqrt{8} = \sqrt{8}\,e^t$$

at $t = 0$,
$$v_\theta = \sqrt{8}$$

$$\text{Speed} = |v| = \sqrt{v_r^2 + v_\theta^2}$$
$$= \sqrt{1 + 8}$$
$$= 9$$
$$= 3 \text{ m/sec}$$

(b) Radial acceleration

$$a_r = \ddot{r} - \dot{\theta}^2 r$$
$$= \underbrace{\frac{d}{dt}(\dot{r})}_{e^t} - (\sqrt{8})^2 e^t$$
$$= e^t - 8e^t$$
$$= -7e^t$$

at $t = 0$, $a_r = -7$ m/sec^2

Tangential Accelaration.

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$\Rightarrow \theta = \sqrt{8}t \Rightarrow \ddot{\theta} = 0$

$\Rightarrow a_\theta = 2e^t\sqrt{8}$

at $t=0$, $a_\theta = 2\sqrt{8}$.

Magnitude of accelaration.

$$|a| = [a_r^2 + a_\theta^2]^{1/2}$$

$$= \sqrt{(-7)^2 + (4\times8)}$$

$$= \sqrt{49+32}$$

$$= \sqrt{81}$$

$$= 9 m/sec^2.$$