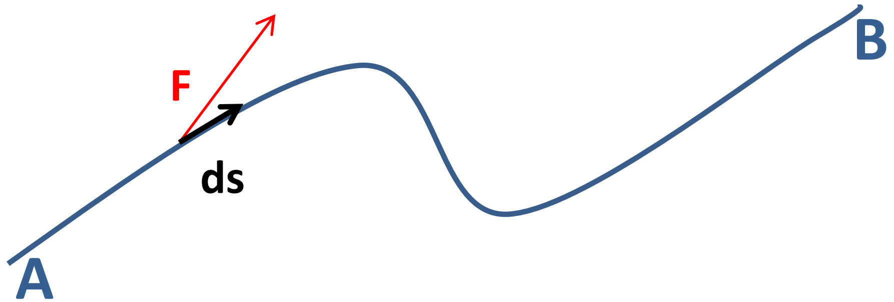
PHY101: Introduction to Physics I

Monsoon Semester 2024
Lecture 15

Department of Physics, School of Natural Sciences,
Shiv Nadar Institution of Eminence, Delhi NCR

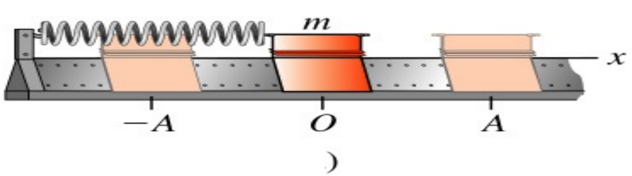
Previous Lecture

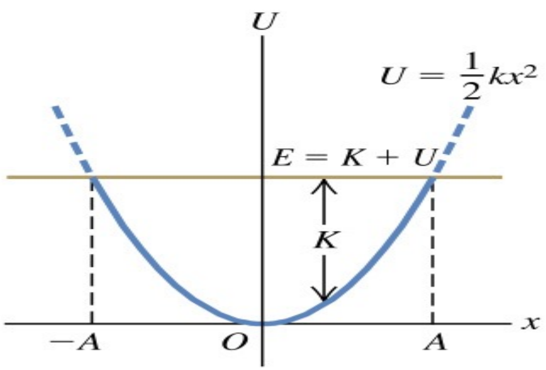
Work
Work energy theorem



This Lecture

Potential energy
Energy diagram





Energy in Simple Harmonic Motion

$$E_{\text{Total}} = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2.$$

The motion of the block on a spring in SHM is defined by the position $x(t) = A\cos(\omega t + \varphi)$ with a velocity of $v(t) = -A\omega\sin(\omega t + \varphi)$. Using these equations, the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ and $\omega = \sqrt{\frac{k}{m}}$, we can find the total energy of the system:

$$\begin{aligned}
E_{\text{Total}} &= \frac{1}{2}kA^2\cos^2(\omega t + \varphi) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \varphi) \\
&= \frac{1}{2}kA^2\cos^2(\omega t + \varphi) + \frac{1}{2}mA^2\left(\frac{k}{m}\right)\sin^2(\omega t + \varphi) \\
&= \frac{1}{2}kA^2\cos^2(\omega t + \varphi) + \frac{1}{2}kA^2\sin^2(\omega t + \varphi) \\
&= \frac{1}{2}kA^2\left(\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)\right) \\
&= \frac{1}{2}kA^2.
\end{aligned}$$

Energy in Simple Harmonic Motion

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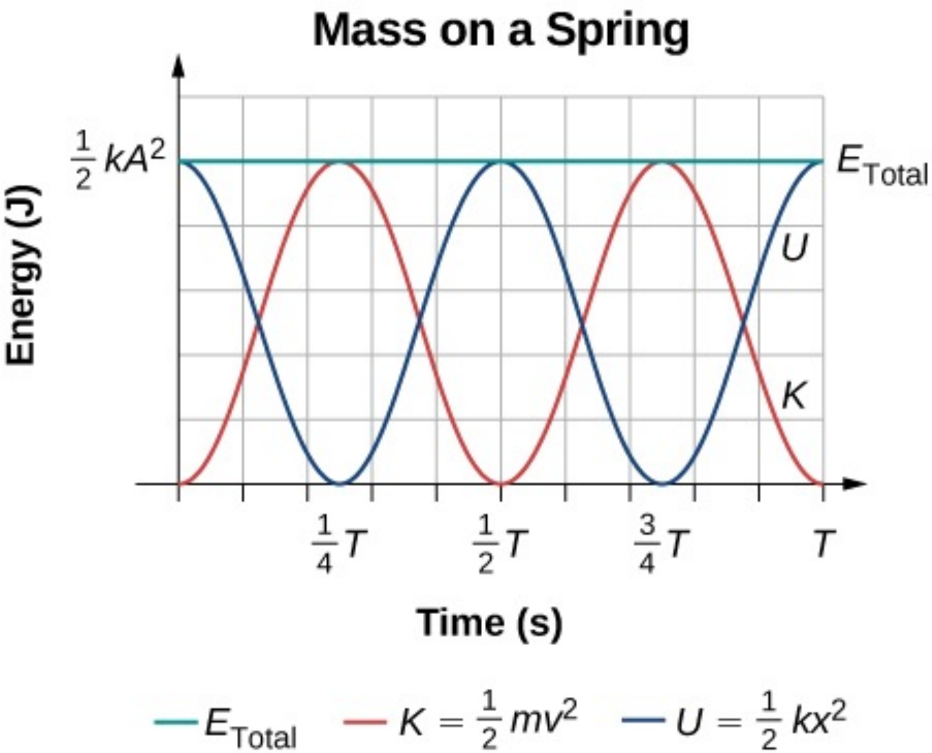
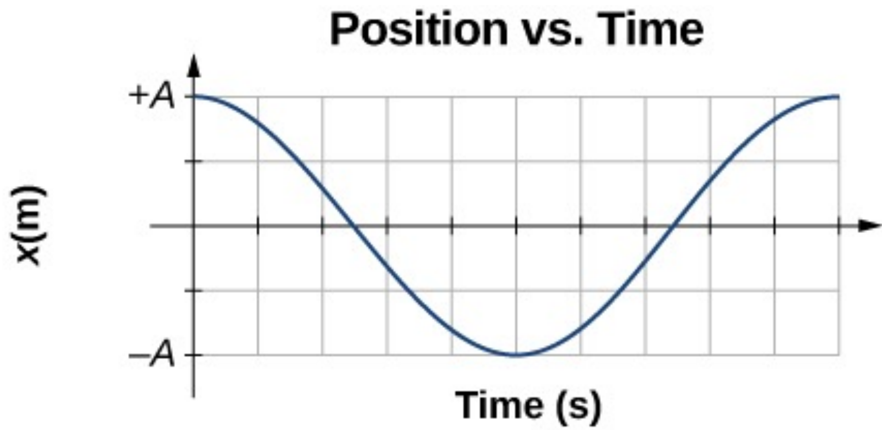
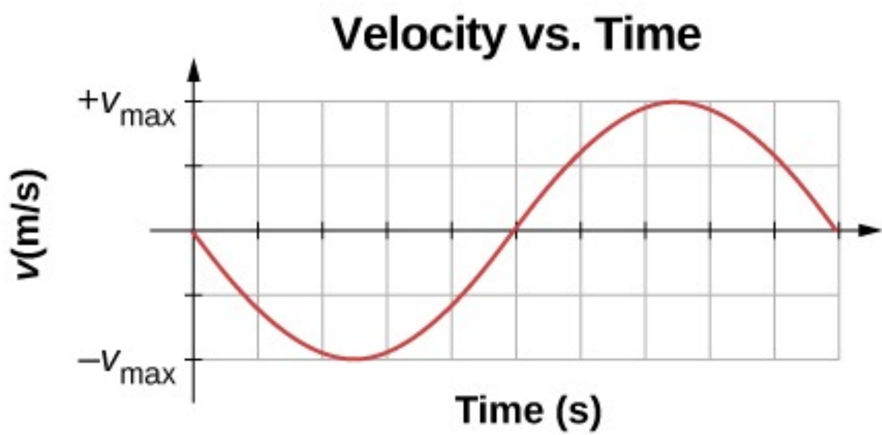






Force from potential energy

Consider a one-dimensional system:

$$U_B - U_A = -\int_{x_A}^{x_B} F(x)dx$$

When the system moves from x to $x + dx$:

$$dU = -F\ dx$$

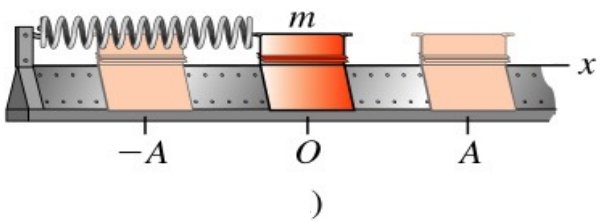
$$\Rightarrow \quad \boxed{F = -\frac{dU}{dx}}$$

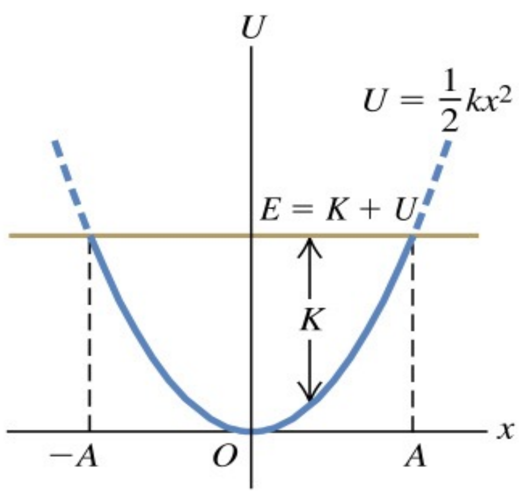
In three dimensions : $\vec{F} = -\frac{\partial U}{\partial x}\hat{\imath} - \frac{\partial U}{\partial y}\hat{\jmath} - \frac{\partial U}{\partial z}\hat{k} = -\vec{\nabla}U$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Energy diagrams

- In situations where a particle moves in one-dimension only under influence of a single conservative force it is very useful to study the graph of the potential energy as a **function** of position **U**(x)
- At any point on a graph of **U**(x), the **force** can be calculated as the negative of the **slope of the potential energy** function $\mathbf{F_x = - dU/dx}$



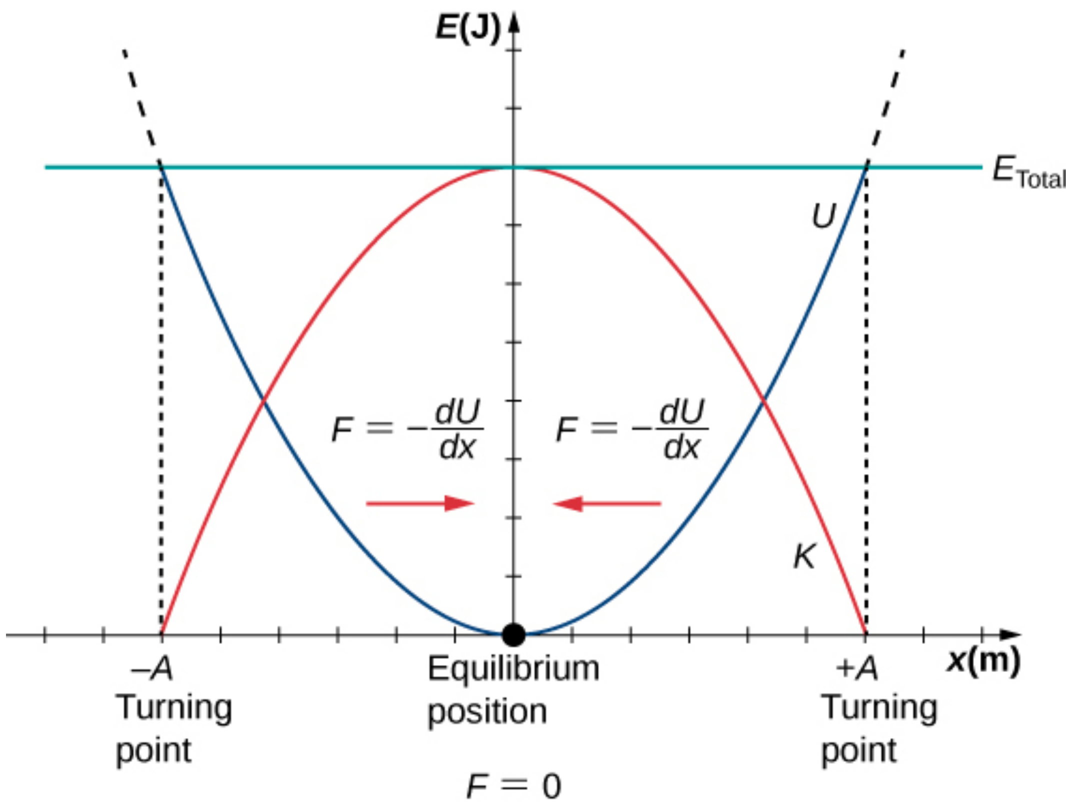


Example: Glider on an air track

- Spring exerts a force $\mathbf{F_x}$=-kx
- Potential energy function **U**(x)
- Limits of the motion are the points where **U** curve intersects the horizontal line representing the total mechanical energy **E.**

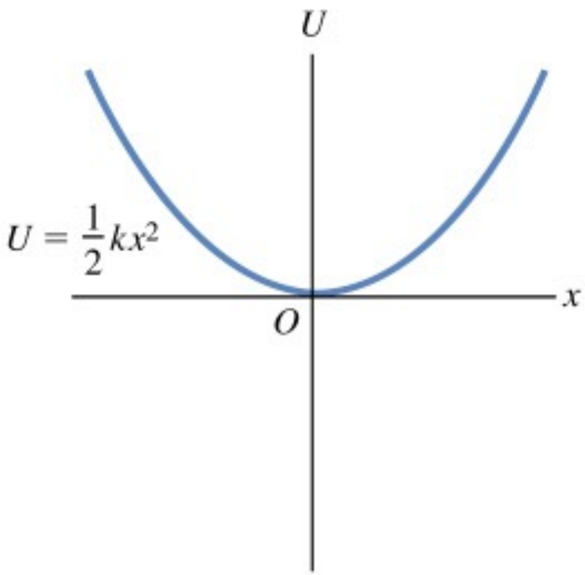
Oscillations About an Equilibrium Position

$$E_{\text{Total}} = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2.$$

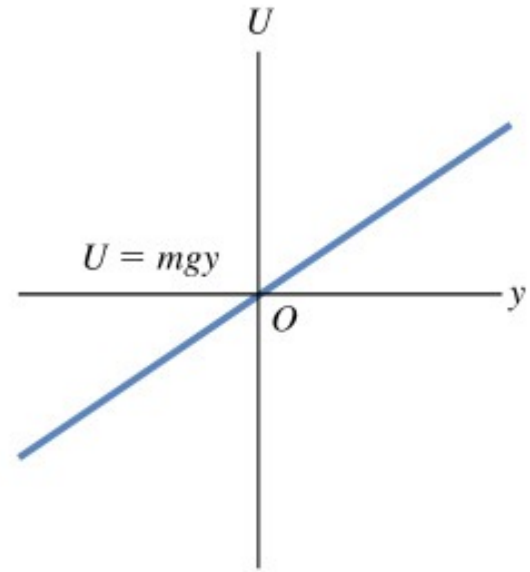


Force is positive when **x<0**, negative when **x>0**, and equal to zero when **x=0**.

U_x $vs.$ x

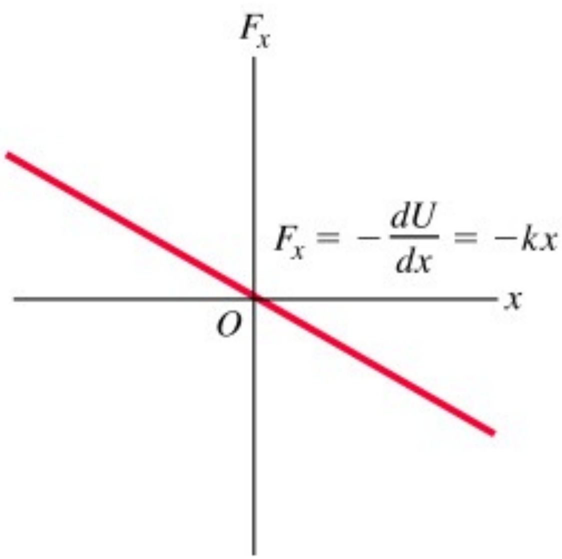


U
$U = \frac{1}{2}kx^2$
x
O

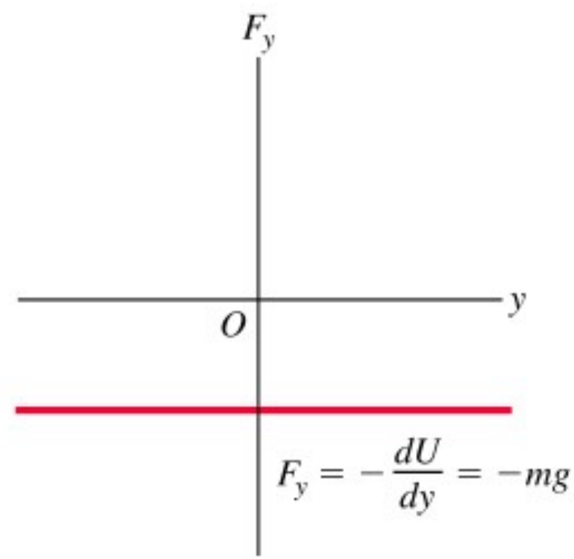


U
$U = mgy$
y
O

F_x $vs.$ x

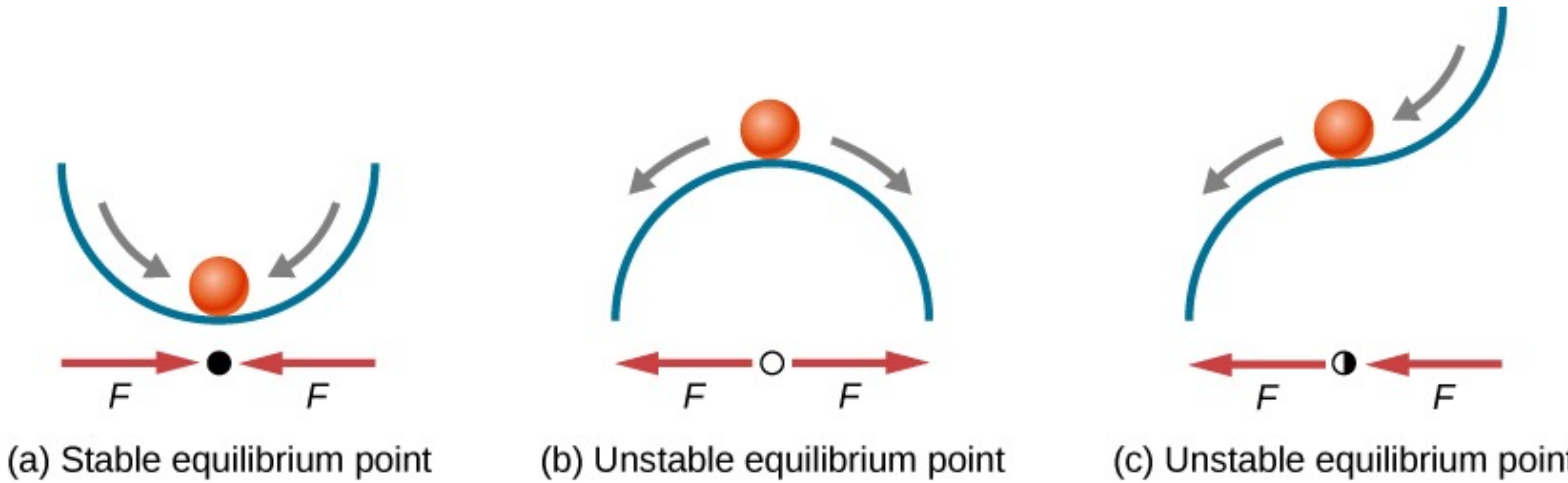


F_x
$F_x = -\frac{dU}{dx} = -kx$
x
O



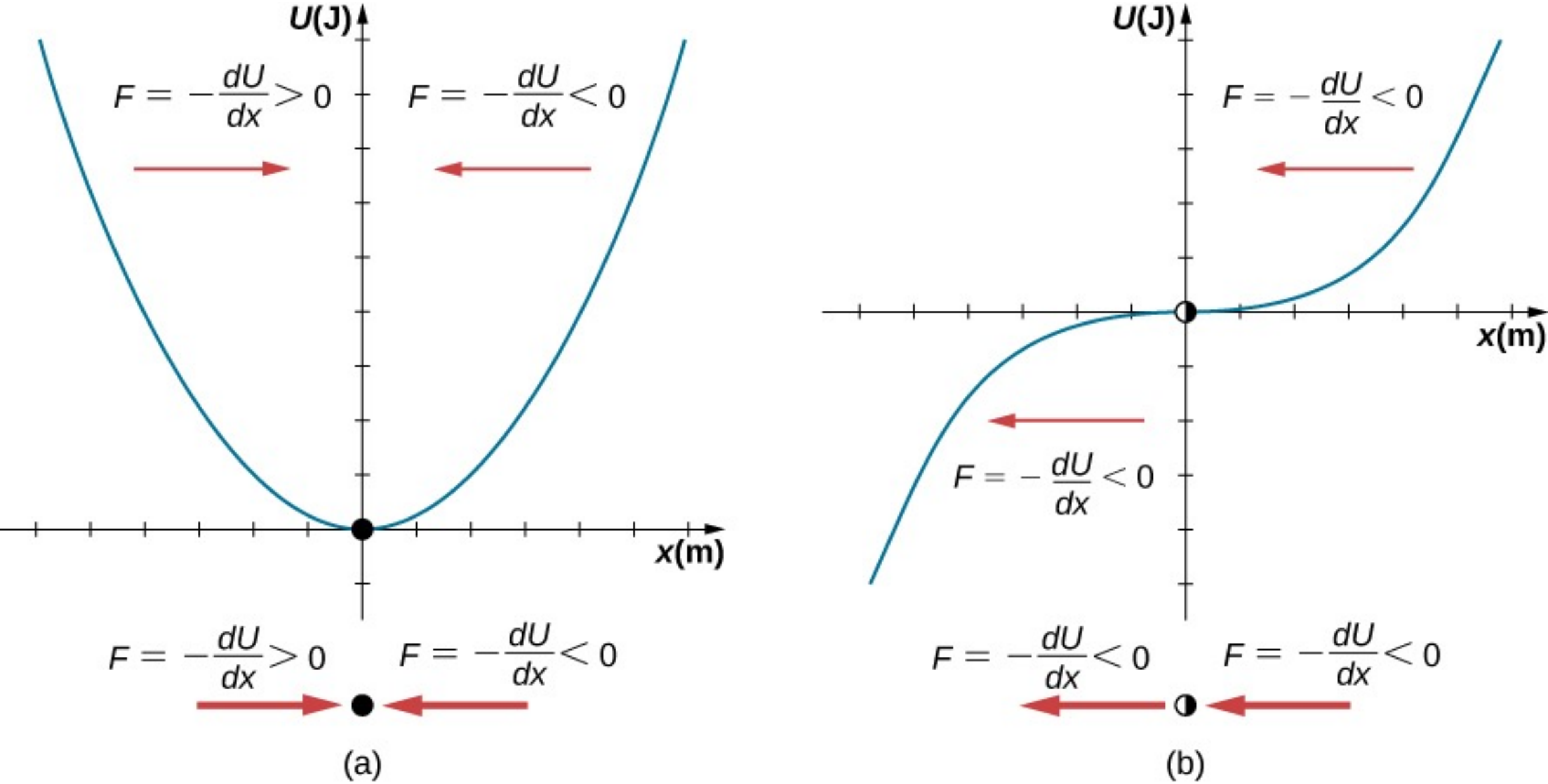
F_y
y
O
$F_y = -\frac{dU}{dy} = -mg$

Equilibrium Conditions



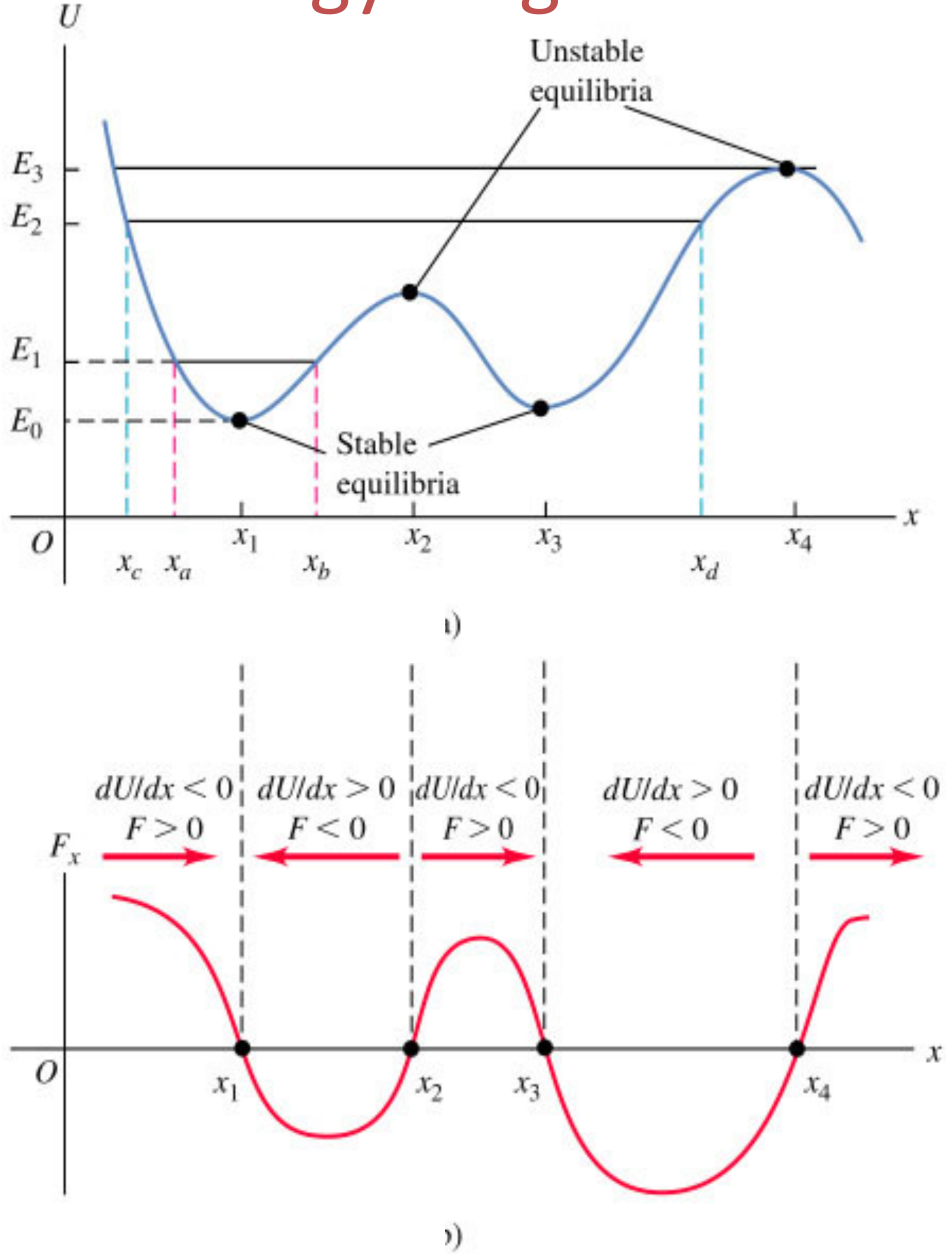
Examples of equilibrium points. (a) Stable equilibrium point; (b) unstable equilibrium point; (c) unstable equilibrium point (sometimes referred to as a half-stable equilibrium point).

Equilibrium Conditions

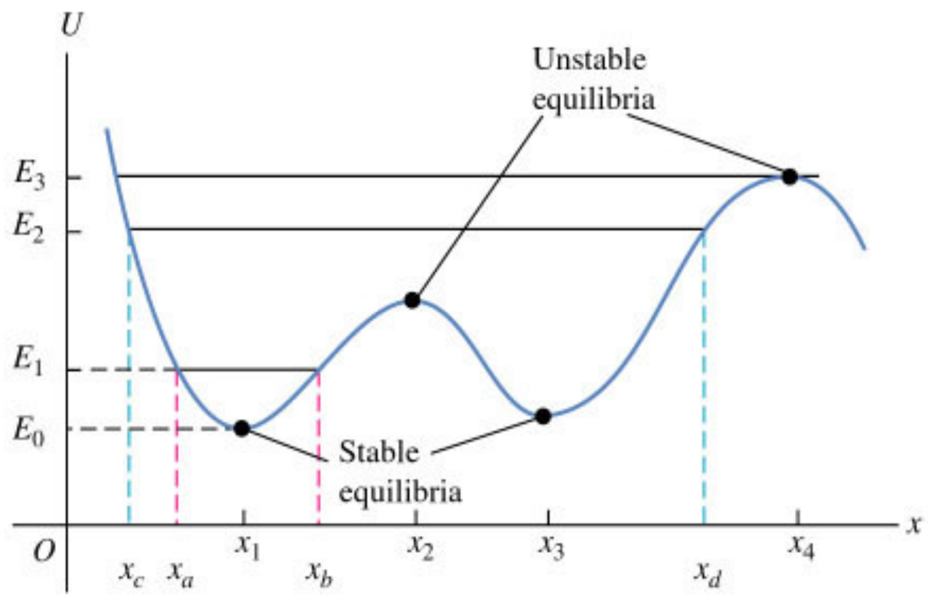


Two examples of a potential energy function. The force at a position is equal to the negative of the slope of the graph at that position. (a) A potential energy function with a stable equilibrium point. (b) A potential energy function with an unstable equilibrium point. This point is sometimes called half-stable because the force on one side points toward the fixed point.

Energy diagrams



Finding stable and unstable positions in Energy Diagram



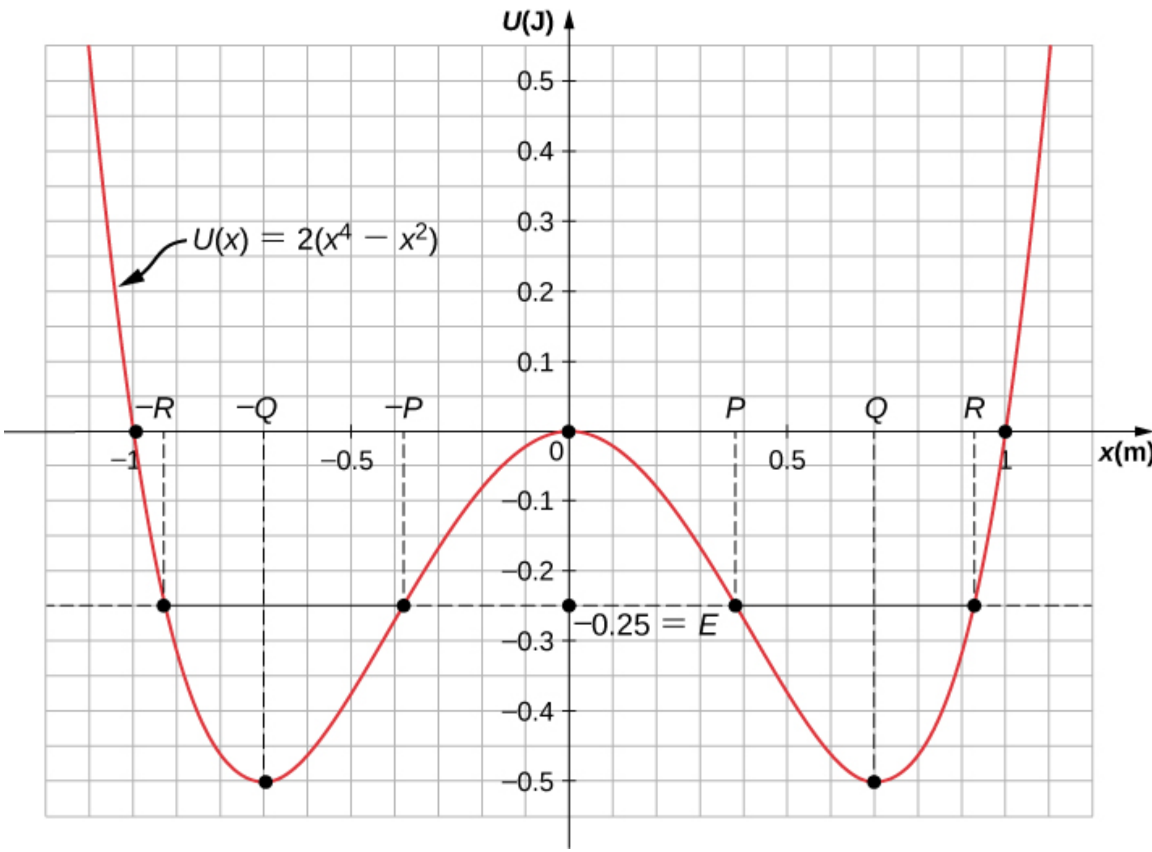
equilibrium points — slope $\frac{dU}{dx} = 0$

Unstable equilibrium points (Maxima) — $\frac{d^2U}{d^2x} < 0$

Stable equilibrium points (Minima) — $\frac{d^2U}{d^2x} > 0$

Quartic and Quadratic Potential Energy Diagram

The potential energy for a particle undergoing one-dimensional motion along the *x*-axis is $U(x) = 2(x^4 - x^2)$, where *U* is in joules and *x* is in meters. The particle is not subject to any non-conservative forces and its mechanical energy is constant at $E = -0.25\,\mathrm{J}$. (a) Is the motion of the particle confined to any regions on the *x*-axis, and if so, what are they? (b) Are there any equilibrium points, and if so, where are they and are they stable or unstable?



equilibrium points

slope $\frac{dU}{dx} = 0$

$8x^3 - 4x = 0$

x=0 and $x = \pm x_Q$, where

$x_Q = 1/\sqrt{2} = 0.707$ (meters)

$$\frac{d^2U}{d^2x} = 24x^2 - 4$$

x=0, Negative : Maxima: Unstable position

$x = \pm x_Q$ Positive: Minima: Stable Position

https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/8-4-potential-energy-diagrams-and-stability/#CNX_UPhysics_08_04_PE2blWell

Home work: A particle is in motion under the potential

$$U(x) = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right] \text{ where } U_0 > 0 \text{ and } a>0$$

Find the equilibrium position of the particle. Justify whether your answer gives a stable or unstable equilibrium.

Home work: A particle is in motion under the potential

$$U(x) = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right] \text{ where } U_0 > 0 \text{ and } a>0$$

Find the equilibrium position of the particle. Justify whether your answer gives a stable or unstable equilibrium.

Equilibrium positions correspond to minima and maxima of $U(x)$.

At the locations of minima and maxima $\frac{dU}{dx} = 0.$

$$\frac{dU}{dx} = U_0\left[12\left(\frac{a}{x}\right)^{11}\left(-\frac{a}{x^2}\right) - 12\left(\frac{a}{x}\right)^{5}\left(-\frac{a}{x^2}\right)\right]$$

$$= -\frac{12U_0}{a}\left[\left(\frac{a}{x}\right)^{13} - \left(\frac{a}{x}\right)^{7}\right] = 0$$

$$\xrightarrow{\text{yields}} \; x = a$$

To find whether $x = a$ is a maximum or minimum we need to evaluate $\left.\frac{d^2U}{dx^2}\right|_{x=a}$.

$$\frac{dU}{dx} = -\frac{12U_0}{a}\left[\left(\frac{a}{x}\right)^{13} - \left(\frac{a}{x}\right)^{7}\right]$$

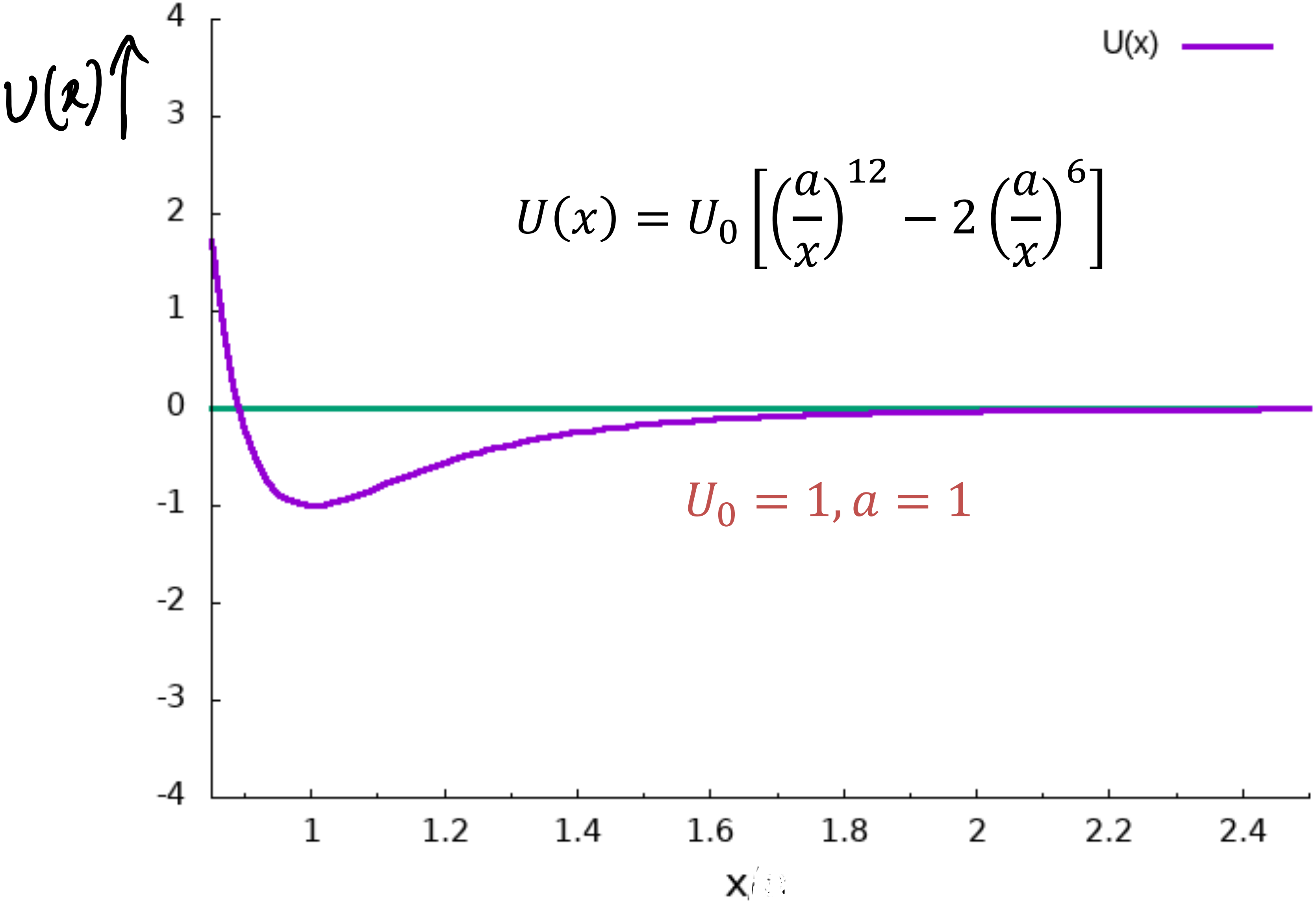
$$\frac{d^2U}{dx^2} = -\frac{12U_0}{a}\left[13\left(\frac{a}{x}\right)^{12}\left(-\frac{a}{x^2}\right) - 7\left(\frac{a}{x}\right)^{6}\left(-\frac{a}{x^2}\right)\right]$$

$$= \frac{12U_0}{a^2}\left[13\left(\frac{a}{x}\right)^{14} - 7\left(\frac{a}{x}\right)^{8}\right]$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=a} = \frac{72U_0}{a^2} > 0$$

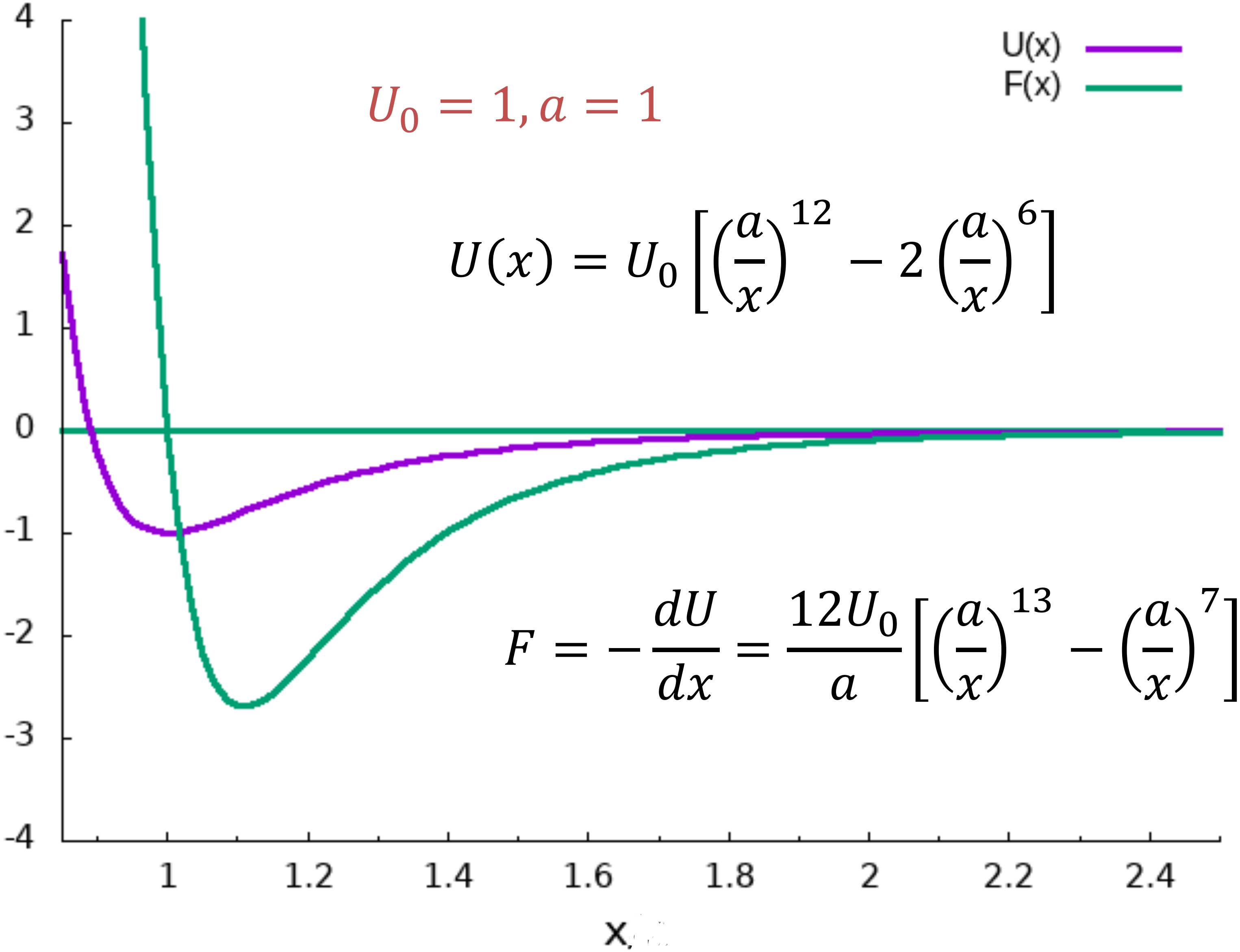
→ $x = a$ indeed the minimum → **Stable equilibrium**

Lennard-Jones Potential



$$U(x) = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right]$$

$$U_0 = 1, a = 1$$



U(x)
F(x)
$U_0 = 1, a = 1$
$U(x) = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right]$
$F = -\frac{dU}{dx} = \frac{12U_0}{a}\left[\left(\frac{a}{x}\right)^{13} - \left(\frac{a}{x}\right)^{7}\right]$
4
3
2
1
0
-1
-2
-3
-4
1
1.2
1.4
1.6
1.8
2
2.2
2.4
x,