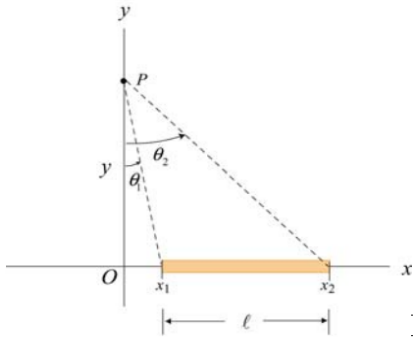
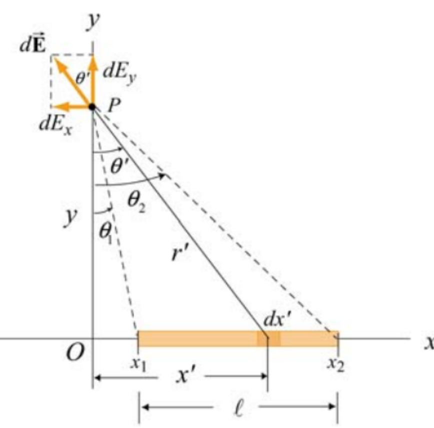
Q1. A non-conducting rod of length ℓ with a uniform charge density, λ, has a total charge Q. The rod is lying along the x-axis, as illustrated in the Figure. Compute the electric field at a point P, located at a distance y off the axis of the rod. **[5]**



Solution:

Consider a length element dx' on the rod, as shown in the Figure.

The charge carried by the element is $dq = \lambda\, dx'$



Figure

The electric field at P produced by this element is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r'^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}\left(-\sin\theta'\,\hat{\mathbf{i}} + \cos\theta'\,\hat{\mathbf{j}}\right)$$

Marks 1

where the unit vector $\hat{\mathbf{r}}$ has been written in Cartesian coordinates: $\hat{\mathbf{r}} = -\sin\theta'\,\hat{\mathbf{i}} + \cos\theta'\,\hat{\mathbf{j}}$. In the absence of symmetry, the field at P has both the x- and y-components. The x-component of the electric field is

$$dE_x = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}\sin\theta' = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}\frac{x'}{\sqrt{x'^2 + y^2}} = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda x'\, dx'}{(x'^2 + y^2)^{3/2}}$$

Marks 1

Integrating from $x' = x_1$ to $x' = x_2$, we have

$$E_x = -\frac{\lambda}{4\pi\varepsilon_0}\int_{x_1}^{x_2}\frac{x'\,dx'}{(x'^2+y^2)^{3/2}} = -\frac{\lambda}{4\pi\varepsilon_0}\frac{1}{2}\int_{x_1^2+y^2}^{x_2^2+y^2}\frac{du}{u^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0}u^{-1/2}\Bigg|_{x_1^2+y^2}^{x_2^2+y^2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x_2^2+y^2}} - \frac{1}{\sqrt{x_1^2+y^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0 y}\left[\frac{y}{\sqrt{x_2^2+y^2}} - \frac{y}{\sqrt{x_1^2+y^2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0 y}\left(\cos\theta_2 - \cos\theta_1\right)$$

Marks 1

Similarly, the y-component of the electric field due to the charge element is

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx'}{x'^2+y^2}\cos\theta' = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx'}{x'^2+y^2}\frac{y}{\sqrt{x'^2+y^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,y dx'}{(x'^2+y^2)^{3/2}}$$

Marks 1

Integrating over the entire length of the rod, we obtain

$$E_y = \frac{\lambda y}{4\pi\varepsilon_0}\int_{x_1}^{x_2}\frac{dx'}{(x'^2+y^2)^{3/2}} = \frac{\lambda y}{4\pi\varepsilon_0}\frac{1}{y^2}\int_{\theta_1}^{\theta_2}\cos\theta'\,d\theta' = \frac{\lambda}{4\pi\varepsilon_0 y}\left(\sin\theta_2 - \sin\theta_1\right)$$

Marks 1

Q2. Q. A sphere of radius R, centered at the origin, carries charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)sin\theta$$

where k is a constant and r and θ are the usual spherical coordinates. Find the mono and dipole terms in the expression of the potential for points on the z axis, far from the sphere. Will the value of the dipole term change if the origin is shifted? Explain. **[4+1]**

In multipole expansion, the monopole term is

$$V_{mono} = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(r)\,d\tau$$

Marks 0.5

The term $\int \rho(r)\,d\tau$ is the total charge Q.

$$\therefore\ Q = \int_\tau \rho(r)\,d\tau = kR\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta\,dr\,d\theta\,d\phi$$

Marks 1

$$= kR\int_0^{2\pi}d\phi\int_0^{\pi}\sin^2\theta\,d\theta\int_0^{R}(R-2r)\,dr$$

$$= kR\left(\int_0^{2\pi}d\phi\right)\left(\int_0^{\pi}\sin^2\theta\,d\theta\right)\underbrace{\left[Rr-r^2\right]_0^R}_{=0}$$

$$=0$$

$\therefore$ Total charge $Q=0$

So, $V_{mono} = \left(\frac{1}{4\pi\epsilon_0}\frac{1}{r}\right)0 = 0$

Marks 0.5

The dipole term,

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_\tau r\cos\theta\,\rho(r)\,d\tau$$

Marks 0.5

Now let us find the integration

$$\int_\tau r\cos\theta\,\rho(r)\,d\tau = kR\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r\cos\theta\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= kR\int_0^{2\pi}d\phi\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta\int_0^{R}r(R-2r)\,dr$$

Marks 1

$$= kR[2\pi]\underbrace{\left[\frac{\sin^3\theta}{3}\right]_0^{\pi}}_{=0}\int_0^{R}(Rr-2r^2)\,dr$$

$$=0$$

So the dipole contribution is also zero

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\times 0 = 0$$

Since mono and dipole do not exist, the potential will have contribution from higher order terms.

Marks 0.5

The dipole term in general change under the shift of origin, except when the total charge Q in the system is zero.

Marks 0.5

Since, here total charge $Q=0$, there will be no change in the value of dipole term.

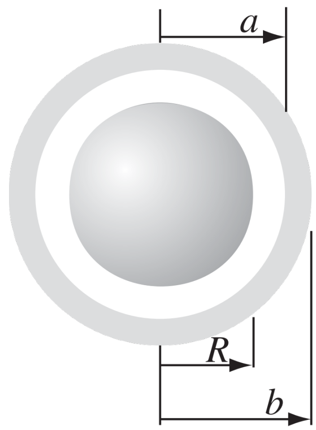
Marks 0.5

Q3. A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in **Figure** below). The shell carries no net charge.

(a) Find the surface charge density σ at *R*, at *a*, and at *b*. **[1.5]**

(b) Find the potential at the center, using infinity as the reference point. [**1**]

(c) Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as at infinity). Find capacitance of the system now? [**2.5**]



Solution

Marks 0.5 **Marks 0.5** **Marks 0.5**

(a) $$\boxed{\sigma_R = \frac{q}{4\pi R^2}\ ;\ \sigma_a = \frac{-q}{4\pi a^2}\ ;\ \sigma_b = \frac{q}{4\pi b^2}.}$$

(b) $V(0) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_b^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{b}+\frac{q}{R}-\frac{q}{a}\right).}$ **Marks 1**

Marks 0.5

(c) $\boxed{\sigma_b \to 0}$ (the charge "drains off"); $V(0) = -\int_\infty^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R}-\frac{q}{a}\right).}$ **Marks 1**

Capacitance of the system, C = q/V= $4\pi\varepsilon_0\left(\frac{Ra}{a-R}\right)$

Marks 1

Q4. (a) Consider a spherical charge distribution with a volume charge density

$$\rho(r) = \begin{cases} \rho_0 \cdot \dfrac{a}{r} & for\ 0 < r < a \\ 0 & for\ r > a \end{cases}$$

Show that the total electrostatic energy of the system is $Q^2/6\pi\varepsilon_0 a$, where Q is the total charge in the system. **[3]**

(b) A parallel plate air capacitor has the plates of area A each.

(i) Assuming the charge Q on it is kept constant, find the work done against the electrical forces to increase the plate separation from d_1 to d_2. **[1]**

(ii) Assuming that the potential difference V across it is maintained constant, find the change in energy of the system. **[1]**

Sol.

(a) Total charge in the system is

$$Q = \int_0^a \rho(r) \cdot 4\pi r^2 dr = 4\pi \rho_0 a \int_0^a r\, dr = 2\pi \rho_0 a^3$$

Marks 0.5

The electric field at any outside point can be obtained from Gauss's law as

$$\int \vec{E}_0 \cdot d\vec{S} = \frac{Q}{\varepsilon_0} \quad or \quad E_0 \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \quad or \quad E_0 = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Marks 0.5

For an internal point,

$$\int \vec{E}_i \cdot d\vec{S} = \frac{Q_r}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^r \rho(r) \cdot 4\pi r^2 dr = \frac{2\pi\rho_0 a r^2}{\varepsilon_0}$$

Marks 0.5

$$\Rightarrow E_i \cdot 4\pi r^2 = \frac{Q r^2}{\varepsilon_0 a^2} \quad or \quad E_i = \frac{Q}{4\pi\varepsilon_0 a^2}$$

Marks 0.5

The total energy in the system is

$$U = \int_0^a \frac{1}{2}\varepsilon_0 E_i^2 \cdot 4\pi r^2 dr + \int_a^\infty \frac{1}{2}\varepsilon_0 E_0^2 \cdot 4\pi r^2 dr$$

Marks 0.5

$$= 2\pi\varepsilon_0 \int_0^a \left(\frac{Q}{4\pi\varepsilon_0 a^2}\right)^2 r^2 dr + 2\pi\varepsilon_0 \int_a^\infty \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \cdot \frac{dr}{r^2}$$

Marks 0.5

$$= \frac{Q^2}{6\pi\varepsilon_0 a}$$

(b) The energy of the system is

$$U = \frac{Q^2}{2C} = \frac{Q^2 d}{2\varepsilon_0 A} \quad or \quad U = \frac{1}{2}CV^2 = \frac{\varepsilon_0 A V^2}{2d}$$

(i) When the charge is kept constant, the work done against the electrical forces is completely spent for increasing the energy of the system.

Thus, the work done in this case, would be

Marks 1

$$W = \Delta U = \frac{Q^2}{2\varepsilon_0 A}(d_2 - d_1)$$

(ii) When the potential difference is maintained constant, the change in energy of the system would be

Marks 1

$$\Delta U = \frac{\varepsilon_0 A V^2}{2}\left(\frac{1}{d_2} - \frac{1}{d_1}\right)$$

This is negative. To understand this we must consider the work done against the electrical forces as well as the energy supplied by the source, which maintains the potential difference constant.

Q5. (a) Show that the solution of Laplace's equation of electric potential V at any point $P(x, y, z)$ due to a point charge $+q$ placed at a distance z along the z-axis in a 3D case is given by

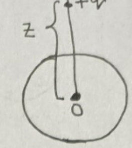
$$V(x, y, z) = V_{center} = \frac{1}{4\pi R^2} \oiint V da$$

where $P(x, y, z)$ refers to the center of a sphere of radius R and $'da'$ corresponds to a surface element of that sphere. Consider the situation when $z > R$ only. **[3]**

(b) A point charge $(-2q)$ and another point charge $(+q)$ is placed at a distance 'd' and '$3d$' from the origin above the $x - y$ plane. If the $x - y$ plane has an infinitely long grounded conductor $(V = 0)$, calculate the force on $+q$ charge. [Hint: You may apply the concept of first uniqueness theorem and results of a point charge in front of a grounded conductor (image charges).] **[2]**

a

Consider pt charge q at a distance z above the center of a sphere of radius R - find the potential V at center



Since at O, V satisfies Laplace's eqn

$\nabla^2 V = 0$, we can directly say

$$V(r) = \frac{1}{4\pi R^2} \oiint V da$$

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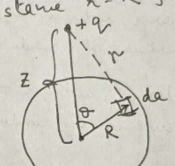
Consider elementary surface da at a distance $r = R$ from center - potential at da is



$$\frac{q}{4\pi\epsilon_0 r}$$

$$V(r) = \frac{1}{4\pi R^2} \iint \frac{q}{4\pi\epsilon_0} \frac{R^2 \sin\theta \, d\theta \, d\phi}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}}$$

Marks 1

$$= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \times R^2 \int_0^{2\pi} d\phi \left(\underbrace{\int_0^{\pi} \frac{\sin\theta \, d\theta}{(R^2+z^2-2Rz\cos\theta)^{1/2}}}_{I_\theta} \right)$$

$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$

$$I_\theta = \frac{1}{2Rz} \int \frac{d(R^2+z^2-2Rz\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{1/2}}$$

$$= \frac{1}{2Rz} \, 2 \, (R^2+z^2-2Rz\cos\theta)^{1/2} \Big|_{\theta=0}^{\pi} = \frac{1}{Rz}\left[\sqrt{(z+R)^2} - \sqrt{(z-R)^2}\right]$$

[Here $z > R$]

$$\cancel{= \frac{1}{Rz}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]}$$

$$= \frac{1}{Rz}\left[(z+R) - (z-R)\right] = \frac{1}{Rz} 2R = \frac{2}{z}$$

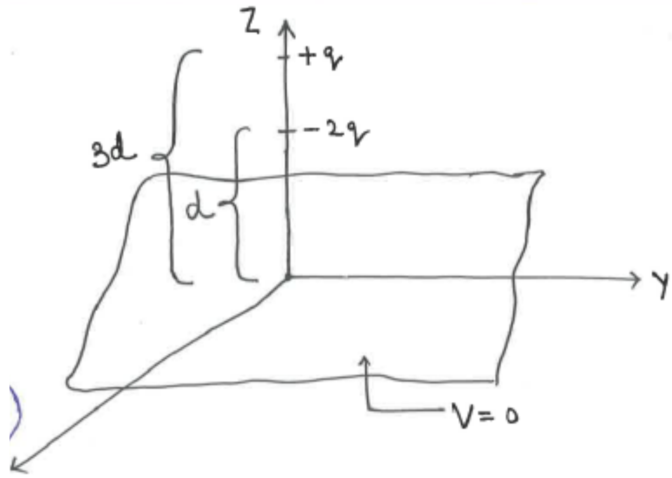
Marks 1

$$V(r) = \frac{1}{\cancel{4\pi R^2}} \frac{q}{4\pi\epsilon_0} \cancel{R^2} \, \cancel{2\pi} \times \left(\frac{2}{z}\right) = \frac{q}{4\pi\epsilon_0 z}$$

Info: When we calculate avg potential over the surface of sphere - we find it exactly equal to potential at center.

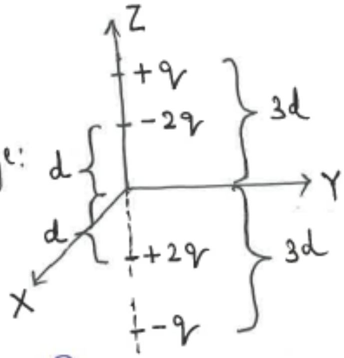
3

3(C) b



From first uniqueness theorem, we know that a point charge q kept in front of an infinitely large grounded conductor (at a distance d above it) induces an opposite charge (image charge) '$-q$' at a distance 'd' below the conductor.

The above problem reduces to formation of induced charges $+2q$ (at distance '$-d$') and $-q$ (at a distance '$-3d$')



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Net force on $+q$ charge:

$$\vec{F}_{+q} = \frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} - \frac{q}{(6d)^2}\right]\hat{z}$$

Marks 0.5

$$= -\frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z}$$

Marks 0.5