PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 35

Electromagnetic Waves in Vacuum

Electromagnetic waves in vacuum

Consider Maxwell's equations in region of space where there is no charge or current (vacuum+no source):

$$\text{(i) } \nabla \cdot \mathbf{E} = 0, \qquad \text{(ii) } \nabla \cdot \mathbf{B} = 0,$$

$$\text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(iv) } \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Let us apply curl to (iii):

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Here, on the LHS we used the identity $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, while on the RHS we interchanged the order of curl and time-derivative operations. Now using (i)and (iv), we have

$$\nabla(0) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Electromagnetic waves in vacuum

$$\text{(i) } \nabla \cdot \mathbf{E} = 0, \qquad \text{(ii) } \nabla \cdot \mathbf{B} = 0,$$

$$\text{(iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(iv) } \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Similarly, applying curl to (iv):

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$$

Again, on the LHS we used the identity $\nabla\times(\nabla\times\mathbf{F})=\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, while on the RHS we interchanged the order of curl and time-derivative operations. Now using (ii)and (iii), we have

$$\nabla(0) - \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Electromagnetic waves in vacuum

Thus we have decoupled the E and B and obtained separate differential equations for them. Note that earlier we had four first order coupled (vector) differential equations, now we have two second order decoupled (vector) differential equations:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Each of these vector equations incorporates three scalar equations, one for each cartesian component (E_x, E_y, E_z, and similarly B_x, B_y, B_z). There, each cartesian component of **E** and **B** satisfies the three-dimensional wave equation:

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

i.e.,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial f}{\partial^2 t^2}$$

with wave propagation speed given by $$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

Electromagnetic waves in vacuum, Speed of Light

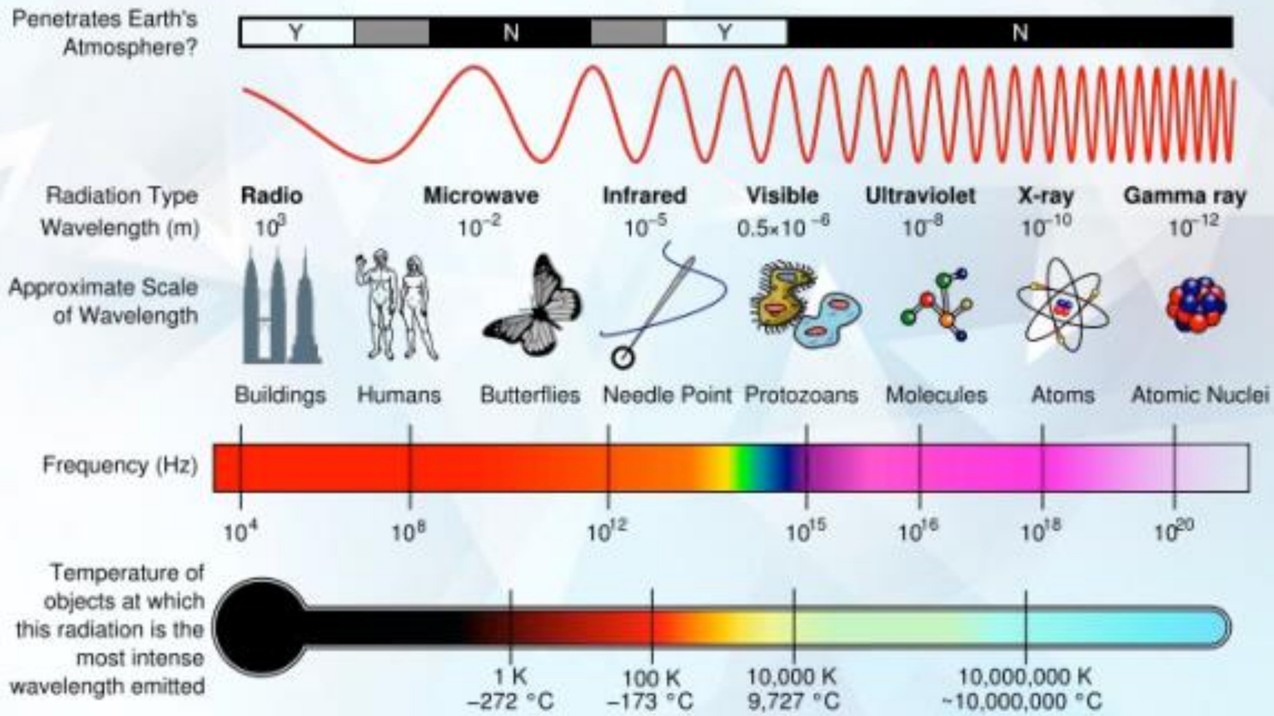
Therefore, Maxwell's equations imply that vacuum supports the propagation of electromagnetic waves, which travel at a speed

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299792458 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}.$$

This happens to be precisely the speed of light c in vacuum.

The Electromagnetic Spectrum

We will confine our attention to sinusoidal waves of frequency. Since different frequencies in the visible range correspond to different colors, such waves (**single frequency**) are called **monochromatic**.



Check this out- http://www.chromoscope.net/

The Electromagnetic Spectrum

The Electromagnetic Spectrum

Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range

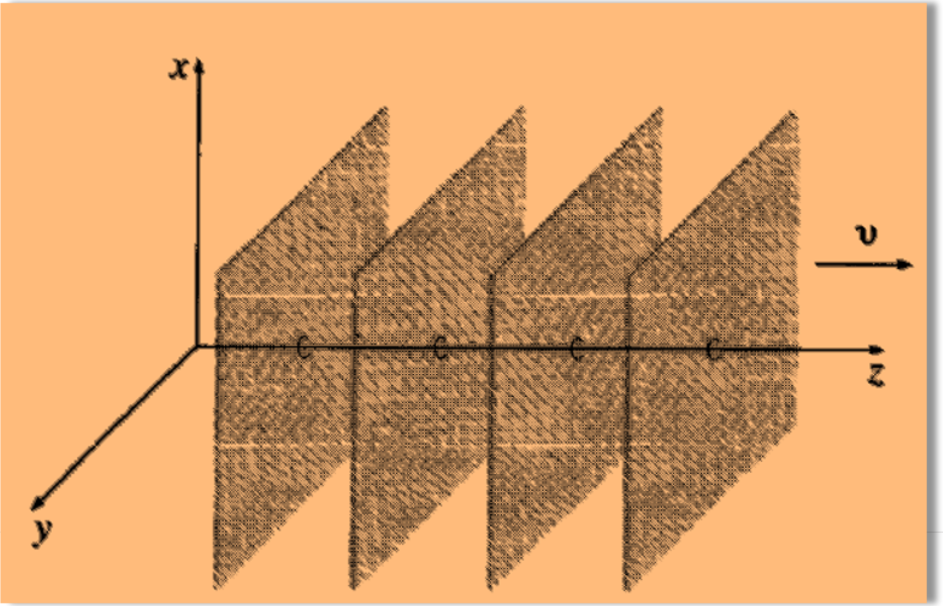
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

Electromagnetic waves should obey wave equations

$$\nabla^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \nabla^2 \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

$$\widetilde{\boldsymbol{E}}(z,t) = \widetilde{\boldsymbol{E}_0} e^{i(kz-\omega t)} \qquad \tilde{B}(z,t) = \widetilde{\boldsymbol{B}_0} e^{i(kz-\omega t)}$$

Assume plane waves for ***E*** and *B*: they are uniform over every plane perpendicular to the direction of propagation (z-axis)



What are the constraints on $\boldsymbol{E}$ and $\boldsymbol{B}$ that Maxwell's equations impose?

$$\widetilde{\boldsymbol{E}}(z,t) = \widetilde{\boldsymbol{E}_0} e^{i(kz-\omega t)}, \qquad \tilde{B}(z,t) = \widetilde{\boldsymbol{B}_0} e^{i(kz-\omega t)}$$

From $\boldsymbol{\nabla}.\boldsymbol{E} = 0$ and $\boldsymbol{\nabla}.\boldsymbol{B} = 0$

$$(\widetilde{E_0})_z = (\widetilde{B_0})_z = 0$$

Electromagnetic waves are transverse- the electric and magnetic field are perpendicular to the direction of propagation

From $\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$

$$-k\left(\widetilde{E_0}\right)_y = \omega(\widetilde{B_0})_x \qquad k\left(\widetilde{E_0}\right)_x = \omega(\widetilde{B_0})_y$$

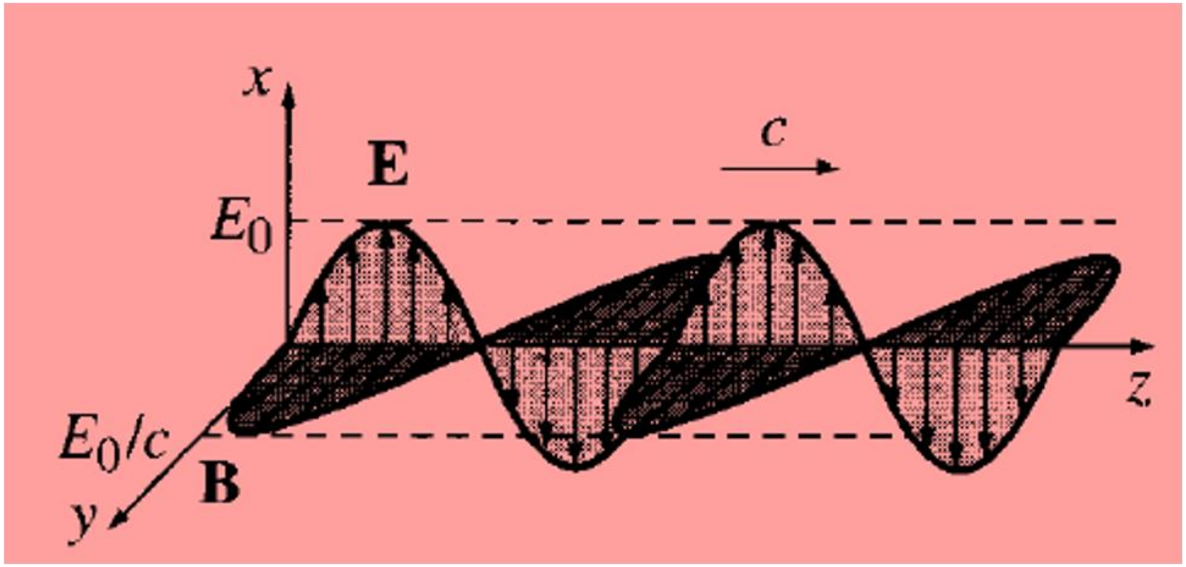
Evidently *E and B are in phase and mutually perpendicular*.

$$\boxed{\widetilde{B_0} = \frac{k}{\omega}\left(\hat{z}\times\widetilde{E_0}\right)} \qquad B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$$

If ***E*** points in the x-direction and ***B*** in y-direction

$$\tilde{\boldsymbol{E}}(z,t) = \widetilde{\boldsymbol{E}_0} e^{i(kz-\omega t)} \hat{\boldsymbol{x}} \qquad \tilde{\boldsymbol{B}}(z,t) = \widetilde{\boldsymbol{B}_0} e^{i(kz-\omega t)} \hat{\boldsymbol{y}}$$

$$\tilde{\boldsymbol{E}}(z,t) = E_0 \cos(kz - \omega t + \delta) \hat{\boldsymbol{x}} \qquad \tilde{\boldsymbol{B}}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta) \hat{\boldsymbol{y}}$$



The wave, as a whole, is said to be polarized in the x-direction (by convention, the direction of ***E***)

Energy and Momentum in Electromagnetic Waves

The energy per unit volume (energy density) stored in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right).$$

In the case of monochromatic plane wave

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2$$
$$\Rightarrow \frac{1}{\mu_0} B^2 = \epsilon_0 E^2$$

Thus in the energy density the contribution of electric and magnetic fields are same:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta).$$
$$u = \frac{1}{\mu_0} B^2 = \frac{1}{\mu_0} B_0^2 \cos^2(kz - \omega t + \delta).$$

Energy and Momentum in Electromagnetic Waves

As the wave travels, it carries energy along with it. The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector:

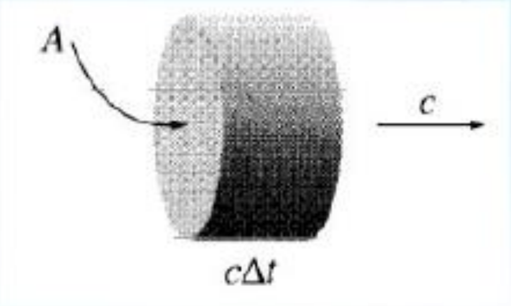
$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}).$$

For monochromatic plane waves propagating in the z direction,

$$\mathbf{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = cu\,\hat{\mathbf{z}}.$$

Notice that S is the energy density (u) times the velocity of the waves ($c\mathbf{z}$) − as it should be.

For in a time Δt, a length $c\Delta t$ passes through area A, carrying with it an energy ($uAc\Delta t$). The energy per unit time, per unit area, transported by the wave is therefore cu.



Energy and Momentum in Electromagnetic Waves

Electromagnetic fields not only carry energy, they also carry momentum. In fact, we have

$$\wp = \frac{1}{c^2}\mathbf{S}.$$

(The symbol $\wp$ used for the momentum is 'Weierstrass p'.)

For monochromatic plane waves, then,

$$\wp = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = \frac{1}{c}u\,\hat{\mathbf{z}}.$$

In the case of light, the wavelength is so short (~5×10^{-7}), and the period so brief (~10^{-15}), that any macroscopic measurement will encompass many cycles. Typically, therefore, we are not interested in fluctuating cosine-squared term in energy, and momentum densities. We just want the average value. The average of cosine-squared over a complete cycle is 1/2. Therefore, we obtain

Average energy density:

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2.$$

Average momentum:

$$\langle \wp \rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{\mathbf{z}}.$$

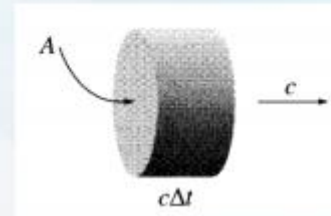
Average Poynting vector:

$$\langle \mathbf{S} \rangle = \frac{1}{2}c\epsilon_0 E_0^2\,\hat{\mathbf{z}}.$$

Intensity and Radiation Pressure

The average power per unit area transported by an electromagnetic wave is called intensity:

$$I = \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2.$$



When light falls on a perfect absorber it delivers its momentum to the surface. In a time Δt the momentum transfer is (Fig. 9.12) $\Delta \mathbf{p} = \langle \wp \rangle A c\, \Delta t$, so the **radiation pressure** (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}. \tag{9.64}$$

On a perfect reflector the pressure is twice as great, because the momentum switches direction, instead of simply being absorbed.

Watch this- https://youtu.be/ADNpCcKo_jw