Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 13

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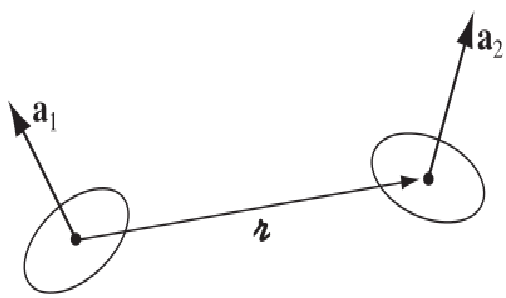
1. Two tiny wire loops, with areas $\mathbf{a_1}$ and $\mathbf{a_2}$, are situated as shown in the figure below.

(a) Find their mutual inductance.

[Hint: Treat them as magnetic dipoles]

(b) Suppose a current of magnitude I_1 is flowing in loop 1, and we propose to turn on a current of magnitude I_2 in loop 2. How much work must be done, against the mutually induced emf, to maintain the current magnitude I_1 flowing in loop 1?



Solution:

(a) $\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3} I_1[3(\mathbf{a}_1\cdot\hat{\boldsymbol{\mathscr{r}}})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}_1]$, since $\mathbf{m}_1 = I_1\mathbf{a}_1$. The flux through loop 2 is then

$$\Phi_2 = \mathbf{B}_1\cdot\mathbf{a}_2 = \frac{\mu_0}{4\pi}\frac{1}{\mathscr{r}^3} I_1[3(\mathbf{a}_1\cdot\hat{\boldsymbol{\mathscr{r}}})(\mathbf{a}_2\cdot\hat{\boldsymbol{\mathscr{r}}}) - \mathbf{a}_1\cdot\mathbf{a}_2] = MI_1. \quad \boxed{M = \frac{\mu_0}{4\pi\mathscr{r}^3}[3(\mathbf{a_1}\cdot\hat{\boldsymbol{\mathscr{r}}})(\mathbf{a_2}\cdot\hat{\boldsymbol{\mathscr{r}}}) - \mathbf{a}_1\cdot\mathbf{a}_2].}$$

(b) $\mathcal{E}_1 = -M\frac{dI_2}{dt}, \quad \left.\frac{dW}{dt}\right|_1 = -\mathcal{E}_1 I_1 = MI_1\frac{dI_2}{dt}.$

This is the work done per unit time against the mutual emf in loop 1 — hence the minus sign.
Since I_1 is maintained at a constant value, we have
$W_1 = M I_1 I_2$, where I_2 is the final current in loop 2. The required work done is

$$W = \frac{\mu_0}{4\pi\mathscr{r}^3}[3(\mathbf{m}_1\cdot\hat{\boldsymbol{\mathscr{r}}})(\mathbf{m}_2\cdot\hat{\boldsymbol{\mathscr{r}}}) - \mathbf{m}_1\cdot\mathbf{m}_2].$$

2. A sphere of radius a is so magnetized that its magnetization at any inside point (x,y,z) with respect to its Centre as origin is given by

$$\vec{M} = a_1 x^2 \hat{\imath} + (a_2 y^2 + b_2)\hat{\jmath}$$

Find the magnetization current densities.

Sol The volume density of magnetization current is

$$\vec{J}_m = \vec{\nabla} \times \vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 x^2 & a_2 y^2 + b_2 & 0 \end{vmatrix} = 0$$

The surface density of magnetization is

$$\vec{K}_m = \vec{M} \times \hat{n}$$

where $\hat{n}$ is the outward unit vector normal to the surface of the sphere. The equation of the surface of the sphere is

$$\psi(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$$

$$\therefore \hat{n} = \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|} = \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\therefore \vec{K}_m = \left[a_1 x^2 \hat{i} + (a_2 y^2 + b_2)\hat{j}\right] \times \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}\,\frac{z}{a}(a_2 y^2 + b_2) - \hat{j}\,\frac{z}{a} a_1 x^2 + \hat{k}\left[\frac{y}{a} \cdot a_1 x^2 - \frac{x}{a}(a_2 y^2 + b_2)\right]$$

3. Suppose a long cylinder of radius a carries a magnetization $\vec{M} = Kr^2\hat{\theta}$, where K is a constant, r is the distance from the axis and $\hat{\theta}$ is the usual unit vector in (r, θ, z) cylindrical coordinate system. Find the magnetic field due to $\vec{M}$ both inside and outside the cylinder.

Sol Magnetization volume current density

$$\vec{J}_m = \vec{\nabla} \times \vec{M} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r.Kr^2 & 0 \end{vmatrix}$$

$$= \hat{z}\,\frac{1}{r}\frac{\partial}{\partial r}(r.Kr^2) = \hat{z}\,3Kr$$

Magnetization surface current density

$$\vec{K}_m = \vec{M} \times \hat{n} = Ka^2\hat{\theta}\times\hat{r} = -Ka^2\hat{z}$$

Total volume current is

$$I_v = \int_0^a J_m \cdot 2\pi r\,dr = 6\pi K\int_0^a r^2\,dr = 2\pi K a^3$$

Total surface current is

$$I_s = -Ka^2 . 2\pi a = -2\pi K a^3$$

∴ Total magnetization current is Zero.

Now according an Amperian loop in the form of a circle of radius $r<a$, we can write from Ampere's law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0\int_0^r J_m \cdot 2\pi r\,dr$$

$$\Rightarrow B . 2\pi r = \mu_0 2\pi K r^3$$

$$\Rightarrow B = \mu_0 K r^2$$

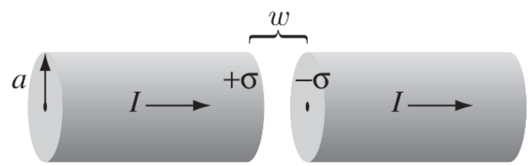
Vectorially, $\vec{B} = \mu_0 K r^2 \hat{\theta}$ for $r<a$ (inside the cylinder)

For $r>a$, we get

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = 0$$

∴ $B = 0$ for $r>a$ (outside the cylinder)

4. A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in the figure . Find the magnetic field in the gap, at a distance $s < a$ from the axis.



The displacement current density

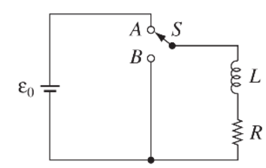
$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2}\,\hat{\mathbf{z}}.$$

Drawing an "amperian loop" at radius s,

$$\oint \mathbf{B} \cdot dl = B \cdot 2\pi s = \mu_0 I_{d_{\text{enc}}} = \mu_0 \frac{I}{\pi a^2} \cdot \pi s^2 = \mu_0 I \frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 I s^2}{2\pi s a^2}; \quad \boxed{\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

5. Suppose the circuit in the figure has been connected for a long time when suddenly, at time t = 0, switch S is thrown from A to B, bypassing the battery.

(a) What is the current at any subsequent time t?
(b) What is the total energy delivered to the resistor?
(c) Show that this is equal to the energy originally stored in the inductor.



Initial current: $I_0 = \mathcal{E}_0/R$.

So $-L\dfrac{dI}{dt} = IR \Rightarrow \dfrac{dI}{dt} = -\dfrac{R}{L}I \Rightarrow I = I_0 e^{-Rt/L}$, or $\boxed{I(t) = \dfrac{\mathcal{E}_0}{R}e^{-Rt/L}.}$

$$P = I^2 R = (\mathcal{E}_0/R)^2\, e^{-2Rt/L} R = \frac{\mathcal{E}_0^2}{R} e^{-2Rt/L} = \frac{dW}{dt}.$$

$$W = \frac{\mathcal{E}_0^2}{R}\int_0^\infty e^{-2Rt/L}\,dt = \frac{\mathcal{E}_0^2}{R}\left(-\frac{L}{2R}e^{-2Rt/L}\right)\Bigg|_0^\infty = \frac{\mathcal{E}_0^2}{R}(0 + L/2R) = \boxed{\frac{1}{2}L\,(\mathcal{E}_0/R)^2}$$

$$\underline{\underline{W_0 = \tfrac{1}{2}LI_0^2 = \tfrac{1}{2}L\,(\mathcal{E}_0/R)^2\,.}} \checkmark$$