PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 4

Line, Surface and Volume Integration of vectors

Second derivatives using the ∇ operator

We saw that the gradient, divergence and curl involve first derivatives resulting from the del operator. We can apply the del operator once more in the following five ways to obtain second-derivative expressions:

Given the gradient, ∇f, of a scalar function f we can use the divergence and curl operations, leading to

- Divergence of a gradient: $\nabla \cdot (\nabla f)$
- Curl of a gradient: $\nabla \times (\nabla f)$

Given the divergence, $\nabla \cdot \mathbf{V}$, of a vector function $\mathbf{V}$ we can use the gradient operation to obtain

- Gradient of a divergence: $\nabla(\nabla \cdot \mathbf{V})$

Given the curl, $\nabla \times \mathbf{V}$, of a vector function $\mathbf{V}$ we can use the divergence and curl operations, resulting in

- Divergence of a curl: $\nabla \cdot (\nabla \times \mathbf{V})$
- Curl of a curl: $\nabla \times (\nabla \times \mathbf{V})$

Second derivatives using the ∇ operator

- **Divergence of a gradient:** $\nabla\cdot(\nabla f)$

We have
$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \boldsymbol{\nabla}^2 f$$

where we defined the **Laplacian** or the **Laplace operator**
$$\boldsymbol{\nabla}^2 \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

As we can see Laplacian of a scalar function gives another scalar function.

We can also consider Laplacian of a vector function, yielding another vector function
$$\boldsymbol{\nabla}^2\mathbf{V} = \boldsymbol{\nabla}^2(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)$$
$$= \hat{i}(\boldsymbol{\nabla}^2 V_x) + \hat{j}(\boldsymbol{\nabla}^2 V_y) + \hat{k}(\boldsymbol{\nabla}^2 V_z)$$

Second derivatives using the ∇ operator

- **Curl of a gradient:** $\nabla\times(\nabla f)$

A little algebra reveals that

$$\nabla\times(\nabla f) = 0$$

i.e., curl of a gradient is always zero.

Therefore, the gradient of a scalar function is irrotational or curl-free or curl-less.

Second derivatives using the ∇ operator

- **Gradient of a divergence:** $\nabla(\nabla\cdot\mathbf{V})$

Gradient of a divergence, $\nabla(\nabla\cdot\mathbf{V})$, for some reason seldom occurs in physics, and it has not been given any special name of its own—it's just the gradient of the divergence.

(-D.J. Griffiths)

Note that $\nabla(\nabla\cdot\mathbf{V})$ is not the same as the Laplacian of a vector:

$$\nabla(\nabla\cdot\mathbf{V}) \neq (\nabla\cdot\nabla)\mathbf{V} = \nabla^2\mathbf{V}$$

Second derivatives using the ∇ operator

- **Divergence of a curl:** $\nabla\cdot(\nabla\times\mathbf{V})$

It can be shown using a little exercise that that divergence of a curl is always zero, viz.

$$\nabla\cdot(\nabla\times\mathbf{V}) = 0$$

Thus divergence of a curl is solenoidal or incompressible or divergence free.

Second derivatives using the ∇ operator

- **Curl of a curl:** $\nabla\times(\nabla\times\mathbf{V})$

It turns out that

$$\nabla\times(\nabla\times\mathbf{V})=\nabla\,(\nabla\cdot\mathbf{V})-\nabla^2\mathbf{V}$$

= Gradient of divergence − Laplacian

Therefore it is just a combination of quantities we already discussed.

Some Remarks

- Using the basic properties of the operator ∇, as we did for finding the second order derivatives, we can go on defining higher order derivatives. However, it turns out that second derivative suffices for practically all physical applications.

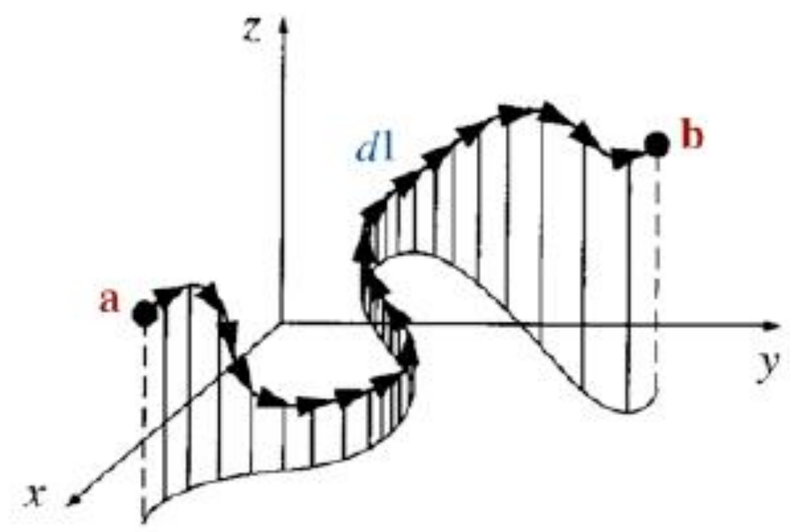
- Next, we move on to deal with some important kinds of integrals which we encounter in Electrodynamics and also in other branches.

- We already familiarized ourselves with some important aspects of Differential Calculus. We will now familiarize ourselves with some important aspects of Integral Calculus. Needless to say, these are intimately related.

Line Integral

Consider a vector function $\mathbf{V}(\mathbf{r}) \equiv \mathbf{V}(x,y,z)$. Then line integral of $\mathbf{V}$ along the path joining $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{V} \cdot d\mathbf{l}$$



Here $d\mathbf{l}$ represents the infinitesimal displacement along the path joining $\mathbf{a}$ and $\mathbf{b}$. In cartesian coordinate system, $d\mathbf{l} = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}$.

Line Integral

- In general the line integral will depend critically on the actual path taken between the two points .
- However, there's a special class of vector functions (conservative vector functions) for which the line integral depends only on the end points of the path. As a consequence if we consider any closed path, the line integral vanishes

$$\oint \mathbf{V}_c \cdot d\mathbf{l} = 0$$

(The notation $\mathbf{V}_c$ has been used above to emphasize that $\mathbf{V}$ here is conservative.)

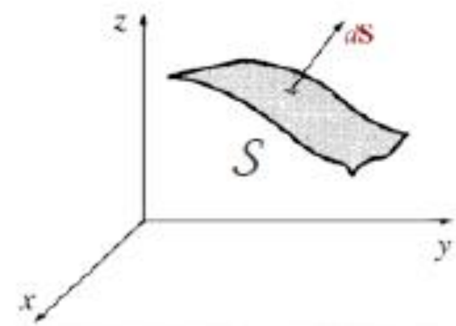
- The above relation is equivalent to the condition that the conservative vector function $\mathbf{V}_c$ is expressible as the gradient of some scalar function, i.e.,

$$\mathbf{V}_c(\mathbf{r}) = \boldsymbol{\nabla}\Phi(\mathbf{r})$$

Surface Integral

The surface integral associated with a vector function $\mathbf{V}(\mathbf{r})$ is given by

$$\int_{\mathcal{S}} \mathbf{V}(\mathbf{r}) \cdot d\mathbf{S} = \int_{\mathcal{S}} \mathbf{V}(\mathbf{r}) \cdot \hat{\mathbf{n}} \, dS$$



Here $d\mathbf{S}$ represents an infinitesimal area, with direction perpendicular to the surface. $\mathbf{n}$ is the unit vector in the direction perpendicular to the surface. Note that there are two directions perpendicular to any surface. The integral is over the entire surface of interest ($\mathcal{S}$)

If the surface is closed we represent the integral as

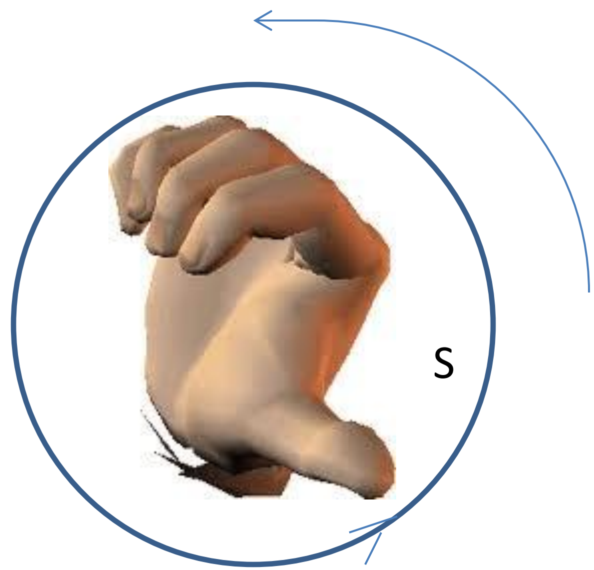
$$\oint_{\mathcal{S}} \mathbf{V}(\mathbf{r}) \cdot d\mathbf{S} = \oint_{\mathcal{S}} \mathbf{V}(\mathbf{r}) \cdot \hat{\mathbf{n}} \, dS$$

In this case, conventionally, "outward" is taken as the positive direction.

Surface Integral :

When we treat surface as a vector .
We assign its area as magnitude and its direction
Normal to the surface toward you.

In the figure, we show a surface S , we put our right
Hand on the surface , thumb is telling the direction
Of the normal .



Open Surface :

Surface which is having boundary. Example disk, bowl, Square, rectangle etc.

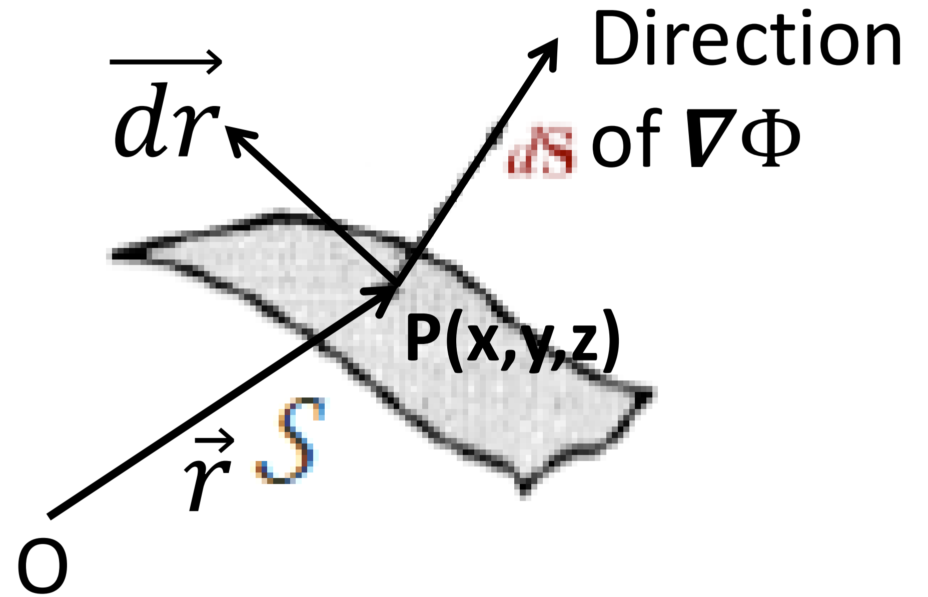
Closed Surface :

No boundary. Example box , tumbler covered with lid.

To find the *normal vector of any surface* (in double integrals), remember that

$\nabla\Phi$ is a vector normal to any surface $\Phi(x, y, z) = c$, 'c' is a constant

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is the position vector to any point P (x,y,z) on the surface



$d\boldsymbol{r} = \boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz$ lies in the tangent plane to the surface at P

$\boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{c}$

$$\Phi(x, y, z) = c$$

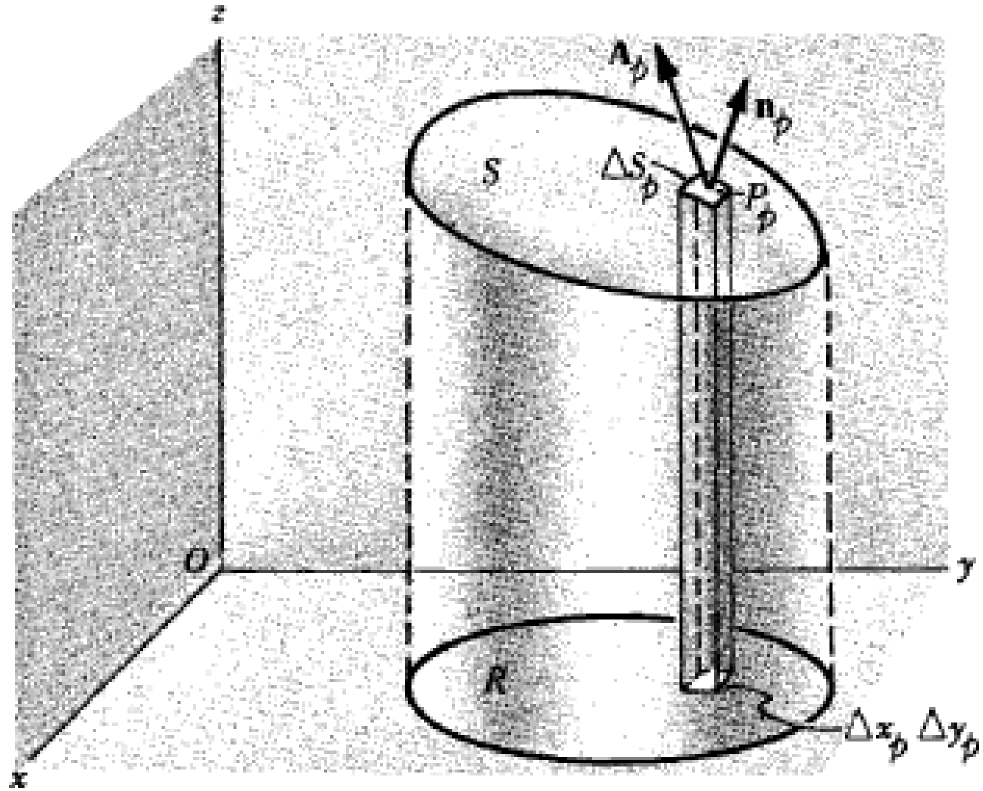
$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = 0 \text{ [since } \Phi(x, y, z) = c]$$

$$= \left(\boldsymbol{i}\frac{\partial\Phi}{\partial x} + \boldsymbol{j}\frac{\partial\Phi}{\partial y} + \boldsymbol{k}\frac{\partial\Phi}{\partial z}\right).(\boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz) = \boldsymbol{\nabla}\Phi.\boldsymbol{dr} = 0$$

$\boldsymbol{\nabla}\Phi$ represents a vector perpendicular to the surface $\Phi(x, y, z) = c$

Definition of surface integral

$$\iint_S \vec{A}.\hat{n}dS$$



Subdivide the area S into M elements of area ΔS_p where p = 1,2,3,….., M. Choose any point P_p within ΔS_p whose co-ordinates are (x_p,y_p,z_p). Define $\mathbf{A}(x_p,y_p,z_p) = \mathbf{A_p}$. Let $\mathbf{n_p}$ be the positive unit normal to ΔS_p at P

Form the sum: $\sum_{p=1}^{M} \boldsymbol{A_p}.\boldsymbol{n_p}\Delta S_p$

($\boldsymbol{A_p}.\boldsymbol{n_p}$ is the normal component of $\mathbf{A_p}$ at P_p)

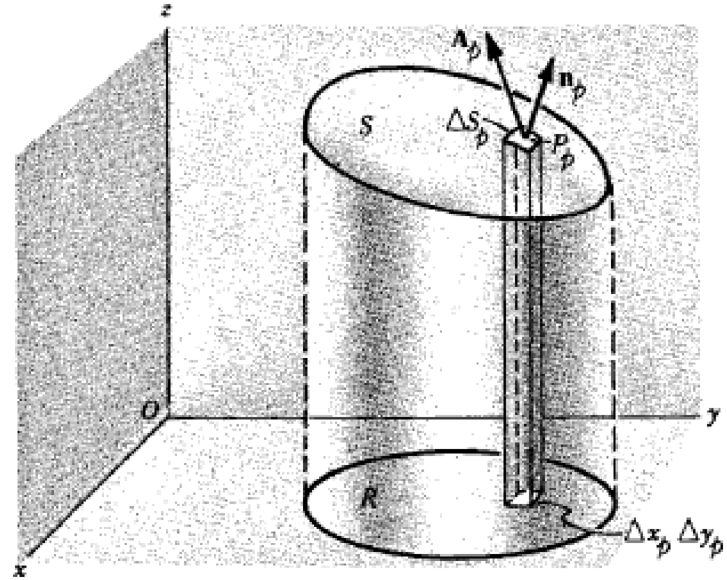
$$\iint_S \boldsymbol{A}.\boldsymbol{n}dS = \lim_{M\to\infty} \sum_{p=1}^{M} \boldsymbol{A_p}.\boldsymbol{n_p}\Delta S_p$$

Take the limit of this sum $M \to \infty$ in such a way that the largest dimension of each ΔS_p approaches zero. The limit, if exists, is called the surface integral of normal component of **A** over S , $\iint_S \boldsymbol{A}.\boldsymbol{n}dS$

Suppose that the surface has a projection R on the x-y plane.

The projection of ΔS_p (a vector) on the x-y plane is

$|\boldsymbol{n_p}\Delta S_p.\boldsymbol{k}|$ or $|\boldsymbol{n_p}.\boldsymbol{k}|\Delta S_p$



which is equal to $\Delta x_p \Delta y_p$ so that $\Delta S_p = \dfrac{\Delta x_p \Delta y_p}{|\boldsymbol{n_p}.\boldsymbol{k}|}$

$$\iint_S \boldsymbol{A}.\boldsymbol{n}dS = \lim_{M\to\infty}\sum_{p=1}^{M}\boldsymbol{A_p}.\boldsymbol{n_p}\Delta S_p = \lim_{M\to\infty}\sum_{p=1}^{M}\boldsymbol{A_p}.\boldsymbol{n_p}\frac{\Delta x_p \Delta y_p}{|\boldsymbol{n_p}.\boldsymbol{k}|} = \iint_R \boldsymbol{A}.\boldsymbol{n}\frac{dxdy}{\boldsymbol{n}.\boldsymbol{k}}$$

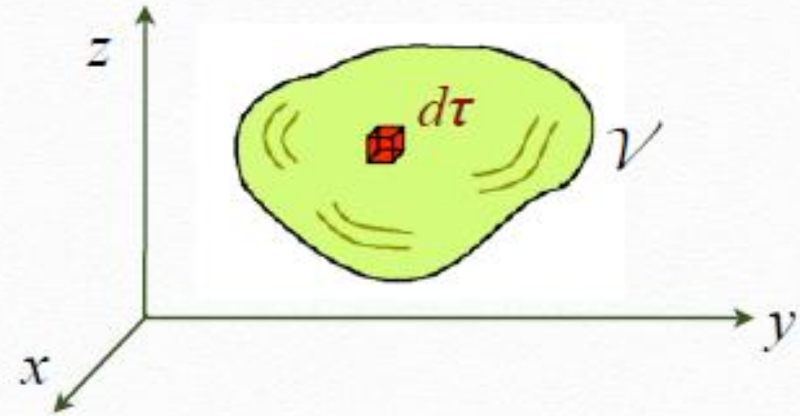
(by fundamental theorem of integral calculus)

$$\iint_S \boldsymbol{A}.\boldsymbol{n}dS = \iint_R \boldsymbol{A}.\boldsymbol{n}\frac{dxdy}{\boldsymbol{n}.\boldsymbol{k}}$$

Volume Integral

The volume integral associated with a scalar function $f(\mathbf{r}) \equiv f(x,y,z)$ is given by

$$\int_{\mathcal{V}} f(\mathbf{r})d\tau$$



Here $d\boldsymbol{\tau}$ represents an infinitesimal volume element. In cartesian coordinates $d\boldsymbol{\tau} = dxdydz$. The integral is over some volume of interest $\mathcal{V}$.

We may also consider the volume integral of a vector function in the following way:

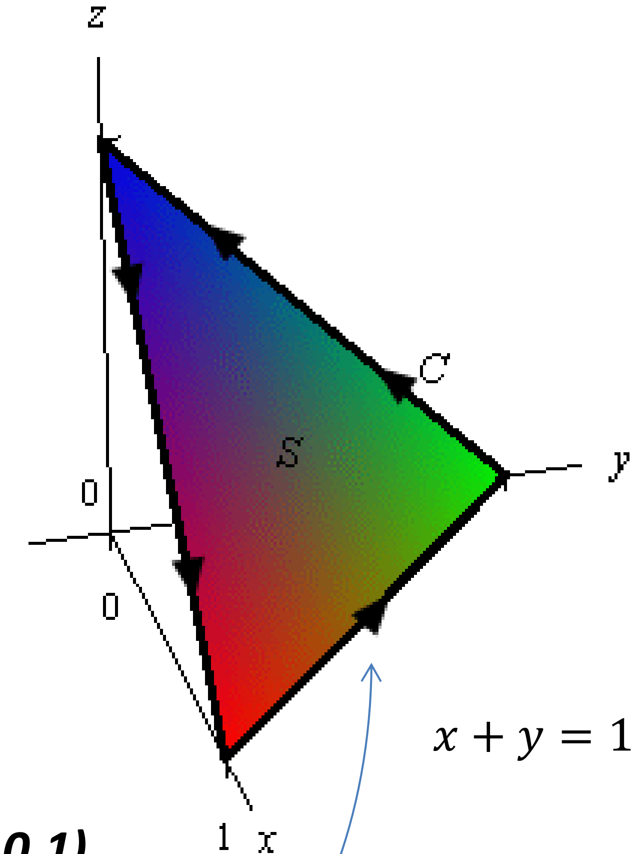
$$\int_{\mathcal{V}} \mathbf{V}\,d\tau = \int_{\mathcal{V}} (\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)d\tau$$
$$= \hat{i}\int_{\mathcal{V}} V_x d\tau + \hat{j}\int_{\mathcal{V}} V_y d\tau + \hat{k}\int_{\mathcal{V}} V_z d\tau$$

Examples

Optional

Line Integral

$$\int_C \vec{F} \cdot d\vec{r}$$



$$\vec{F} = z^2\,\vec{i} + y^2\,\vec{j} + x\vec{k}$$

$x + y = 1$

and C is the triangle with vertices (1,0,0) ,(0,1,0), (0,0,1).

$\Rightarrow$ *along this line*

The plane S has the equation $\boldsymbol{x + y + z = 1}$.

$$\overrightarrow{\boldsymbol{dr}} = \boldsymbol{dx\,\hat{\imath} + dy\hat{\jmath} + dz\,\hat{k}}$$

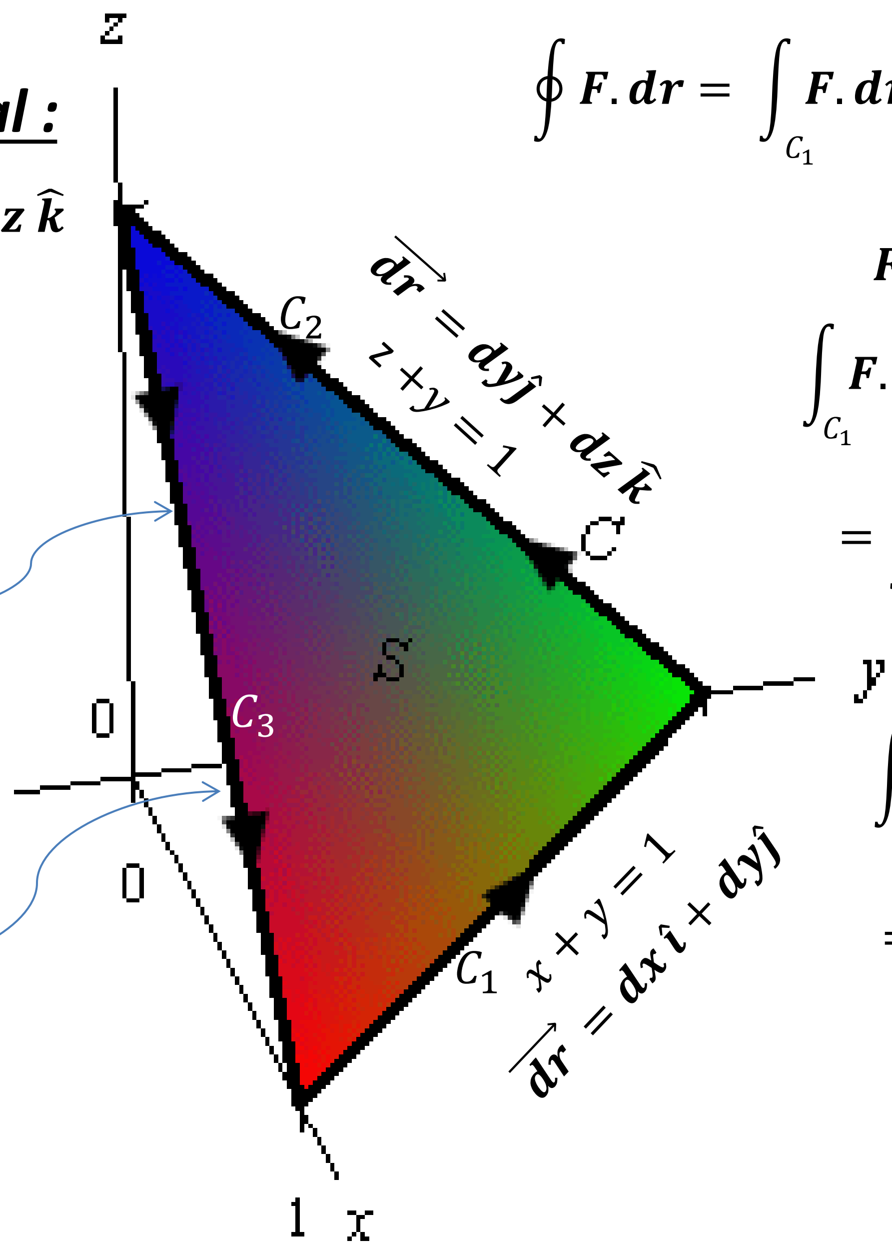
$$\overrightarrow{\boldsymbol{dr}} = \boldsymbol{dx\,\hat{\imath} + dy\hat{\jmath}}$$

Can you tell me what is the equation of the line on the xy plane ?

Line Integral :

$$\overrightarrow{dr} = dx\,\hat{\imath} + dy\hat{\jmath} + dz\,\hat{k}$$

$$\oint \boldsymbol{F}.\boldsymbol{dr} = \int_{C_1} \boldsymbol{F}.\boldsymbol{dr} + \int_{C_2} \boldsymbol{F}.\boldsymbol{dr} + \int_{C_3} \boldsymbol{F}.\boldsymbol{dr}$$



$$\overrightarrow{dr} = dx\,\hat{\imath} + dz\,\hat{k}$$

$$x + z = 1$$

$$\boldsymbol{F} = \boldsymbol{z^2}\,\hat{\boldsymbol{\imath}} + \boldsymbol{y^2}\hat{\boldsymbol{\jmath}} + \boldsymbol{x}\hat{\boldsymbol{k}}$$

$$\int_{C_1} \boldsymbol{F}.\boldsymbol{dr} = \int (z^2 dx + y^2 dy)$$

$$= \int_0^1 y^2 dy = \left|\frac{y^3}{3}\right|^1 = \frac{1}{3}$$

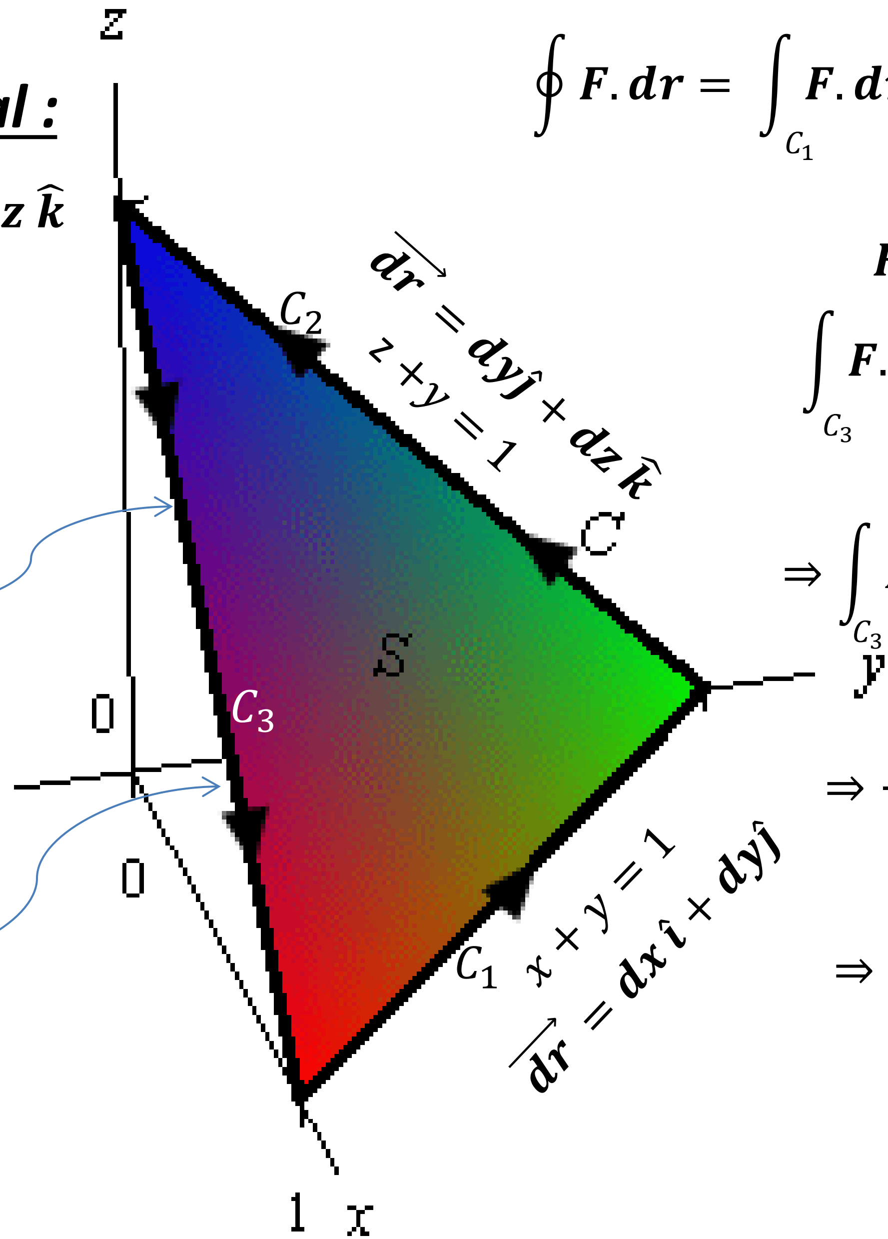
$$\int_{C_2} \boldsymbol{F}.\boldsymbol{dr} = \int (x dz + y^2 dy)$$

$$= \int_1^0 y^2 dy = \left|\frac{y^3}{3}\right|_1^0 = -\frac{1}{3}$$

Line Integral :

$$\oint \boldsymbol{F}.\boldsymbol{dr} = \int_{C_1} \boldsymbol{F}.\boldsymbol{dr} + \int_{C_2} \boldsymbol{F}.\boldsymbol{dr} + \int_{C_3} \boldsymbol{F}.\boldsymbol{dr}$$

$$\overrightarrow{\boldsymbol{dr}} = \boldsymbol{dx}\,\hat{\boldsymbol{\imath}} + \boldsymbol{dy}\hat{\boldsymbol{\jmath}} + \boldsymbol{dz}\,\widehat{\boldsymbol{k}}$$



$$\boldsymbol{F} = \boldsymbol{z^2}\,\hat{\boldsymbol{\imath}} + \boldsymbol{y^2}\hat{\boldsymbol{\jmath}} + \boldsymbol{x}\widehat{\boldsymbol{k}}$$

$$\int_{C_3} \boldsymbol{F}.\boldsymbol{dr} = \int (z^2dx + xdz)$$

$$dx + dz = 0$$

$$\Rightarrow \int_{C_3} \boldsymbol{F}.\boldsymbol{dr} = \int -(z^2dz + xdx)$$

$$\overrightarrow{\boldsymbol{dr}} = \boldsymbol{dx}\,\hat{\boldsymbol{\imath}} + \boldsymbol{dz}\,\widehat{\boldsymbol{k}}$$

$$x + z = 1$$

$$\Rightarrow -\int_1^0 z^2dz = -\left|\frac{z^3}{3}\right|_1^0 = \frac{1}{3}$$

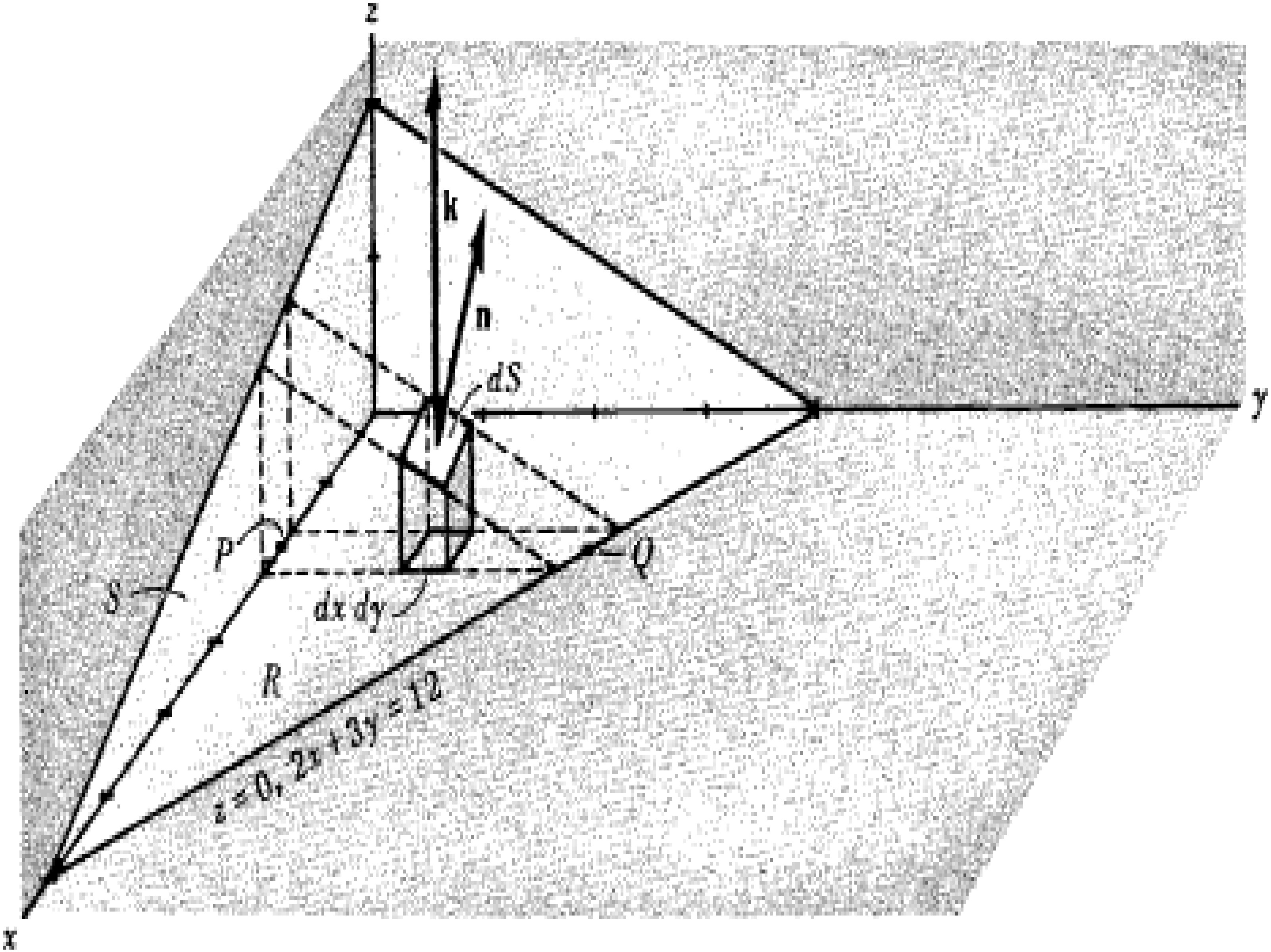
$$\Rightarrow -\int_0^1 xdx = -\left|\frac{x^2}{2}\right|_0^1 = -\frac{1}{2}$$

$$\oint \boldsymbol{F}.\boldsymbol{dr} = \int_{C_1} \boldsymbol{F}.\boldsymbol{dr} + \int_{C_2} \boldsymbol{F}.\boldsymbol{dr} + \int_{C_3} \boldsymbol{F}.\boldsymbol{dr} = \frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

Surface Integral

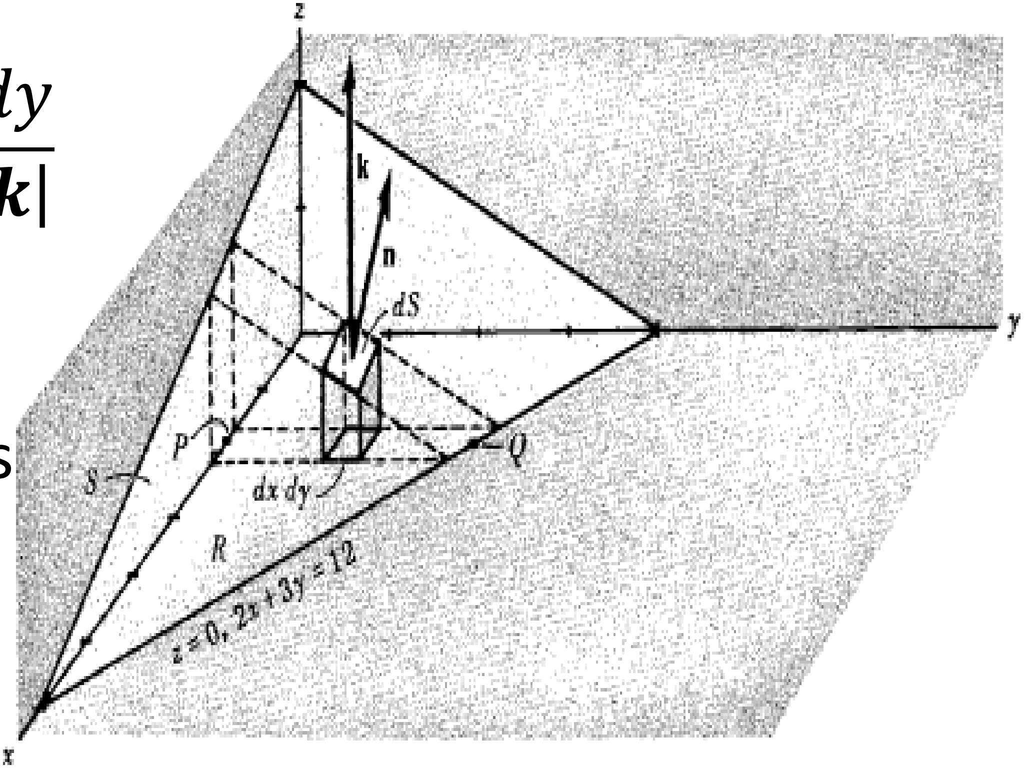
Example: 1

Evaluate $\iint_S \vec{\boldsymbol{A}}.\widehat{\boldsymbol{n}}dS$ where $\vec{\boldsymbol{A}} = 18z\hat{\boldsymbol{\imath}} - 12\hat{\boldsymbol{\jmath}} + 3y\widehat{\boldsymbol{k}}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.



z
y
x
k
n
dS
P
Q
S
R
dx dy
z = 0, 2x + 3y = 12

$$\iint_S \boldsymbol{A}.\boldsymbol{n}dS = \iint_R \boldsymbol{A}.\boldsymbol{n}\frac{dxdy}{|\boldsymbol{n}.\boldsymbol{k}|}$$



To obtain **n**:

A vector perpendicular to the surface $2x + 3y + 6z = 12$ is given by $\boldsymbol{\nabla}(2x + 3y + 6z) = 2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$

Unit normal to any point of S is

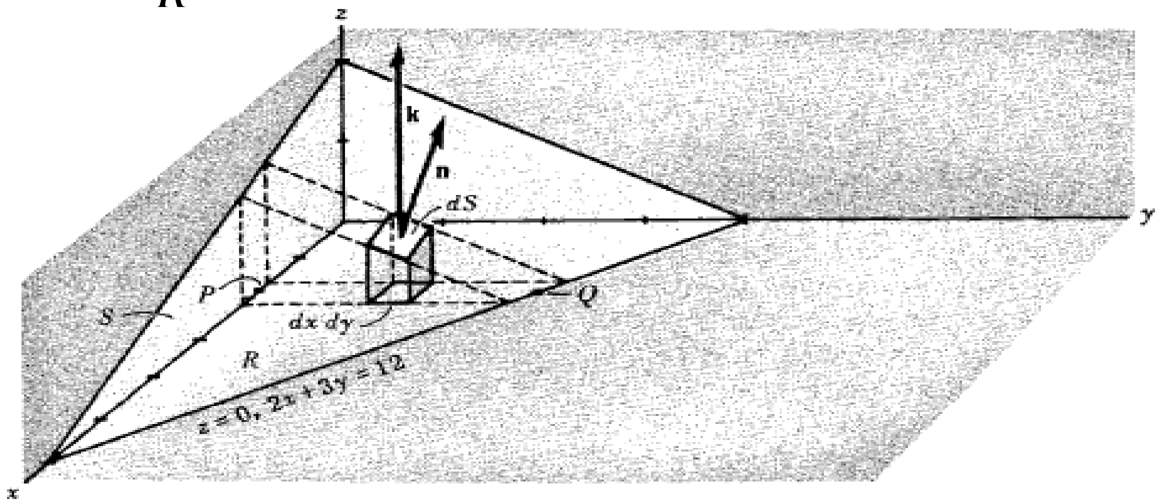
$$\boldsymbol{n} = \frac{2i + 3j + 6k}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7}\boldsymbol{i} + \frac{3}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k} \qquad \boldsymbol{n}.\boldsymbol{k} = \frac{6}{7} \qquad \frac{dxdy}{|\boldsymbol{n}.\boldsymbol{k}|} = \frac{7}{6}dxdy$$

Also $\mathbf{A}\cdot\mathbf{n} = (18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k})\cdot(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}) = \frac{36z - 36 + 18y}{7} = \frac{36 - 12x}{7}$,

using the fact that $z = \frac{12 - 2x - 3y}{6}$ from the equation of S. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_R (\frac{36 - 12x}{7})\frac{7}{6}\,dx\,dy = \iint_R (6 - 2x)\,dx\,dy$$

$$\iint_S \boldsymbol{A}.\boldsymbol{n}dS = \iint_R (6-2x)dxdy$$



To evaluate the double integral over R, keep x fixed and integrate with respect to y from y = 0 to $y = \frac{12-2x}{3}$; then integrate w.r.t x from x = 0 to x = 6. In this manner, *R is completely recovered*.

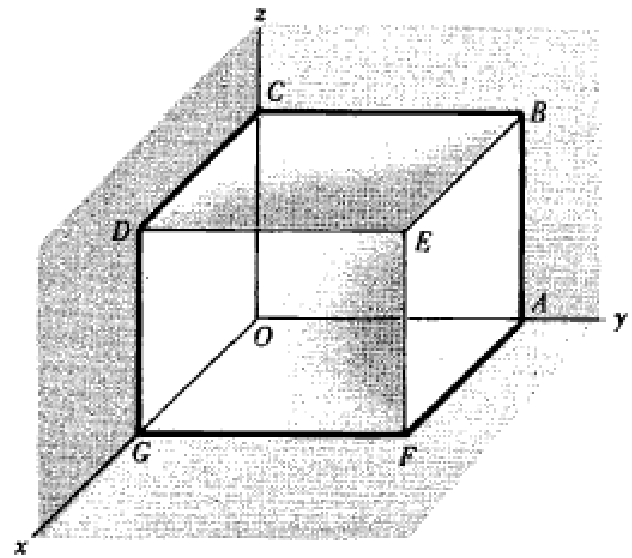
The integral becomes

$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)\,dy\,dx = \int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx = 24$$

Example 2

If $\boldsymbol{F} = 4xz\hat{\boldsymbol{\imath}} - y^2\hat{\boldsymbol{\jmath}} + yz\widehat{\boldsymbol{k}}$, evaluate $\iint \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS$ where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0, z = 1$.

Face DEFG: $\boldsymbol{n}$=**i**, x = 1. Then



$$\iint_{DEFG} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS = \int_0^1 \int_0^1 (4z\boldsymbol{i} - y^2\boldsymbol{j} + yz\boldsymbol{k}) \cdot \boldsymbol{i} dydz$$

$$= \int_0^1 \int_0^1 4zdydz = 2 = \int_{z=0}^1 4zdz \int_{y=0}^1 dy$$

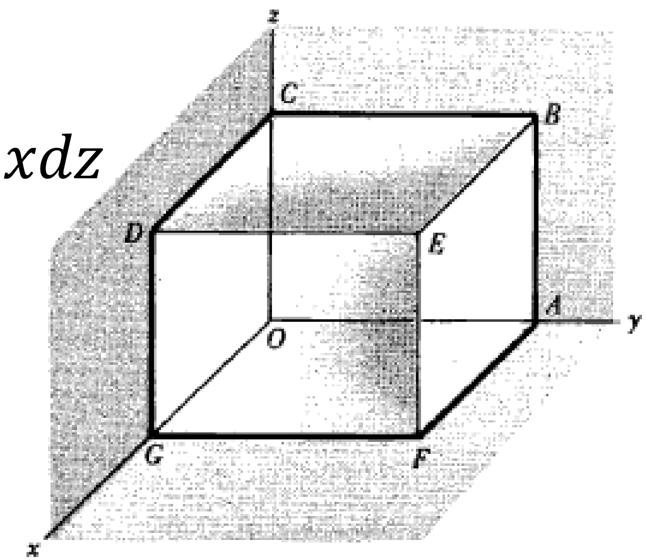
Face ABCO: $\boldsymbol{n}$ = -$\boldsymbol{i}$, x = 0. Then

$$\iint_{ABCO} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS = \int_0^1 \int_0^1 (-y^2\boldsymbol{j} + yz\boldsymbol{k}) \cdot (-\boldsymbol{i}) dydz = 0$$

Face ABEF: **n** = **j**, y= 1. Then

$$\iint_{ABEF} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS = \int_0^1 \int_0^1 (4xz\boldsymbol{i} - \boldsymbol{j} + z\boldsymbol{k}) \cdot \boldsymbol{j} dxdz$$

$$= \int_0^1 \int_0^1 -dxdz = -1$$



Face OGDC: **n** = **-j**, y= 0. Then

$$\iint_{OGDC} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS = \int_0^1 \int_0^1 (4xz\boldsymbol{i}) \cdot (-\boldsymbol{j}) dxdz = 0$$

Face BCDE: **n** = **k**, z = 1. Then

$$\iint_{BCDE} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS = \int_0^1 \int_0^1 (4x\boldsymbol{i} - y^2\boldsymbol{j} + \boldsymbol{y}\boldsymbol{k}) \cdot (\boldsymbol{k}) dxdy = \int_0^1 \int_0^1 ydxdy = \frac{1}{2}$$

Face AFGO: **n** = **-k**, z= 0. Then,

$$\iint_{AFGO} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}} dS = \int_0^1 \int_0^1 y^2\boldsymbol{j} \cdot (-\boldsymbol{k}) dxdy = 0$$

Adding, we get

$$\iint F \cdot \hat{n} dS = \frac{3}{2}$$

Surface Integral :

$$\boldsymbol{E}(x,y,z), is\ a\ vector\ field,$$

$$\int_{\boldsymbol{S}} \boldsymbol{E}\cdot\overrightarrow{\boldsymbol{ds}}$$

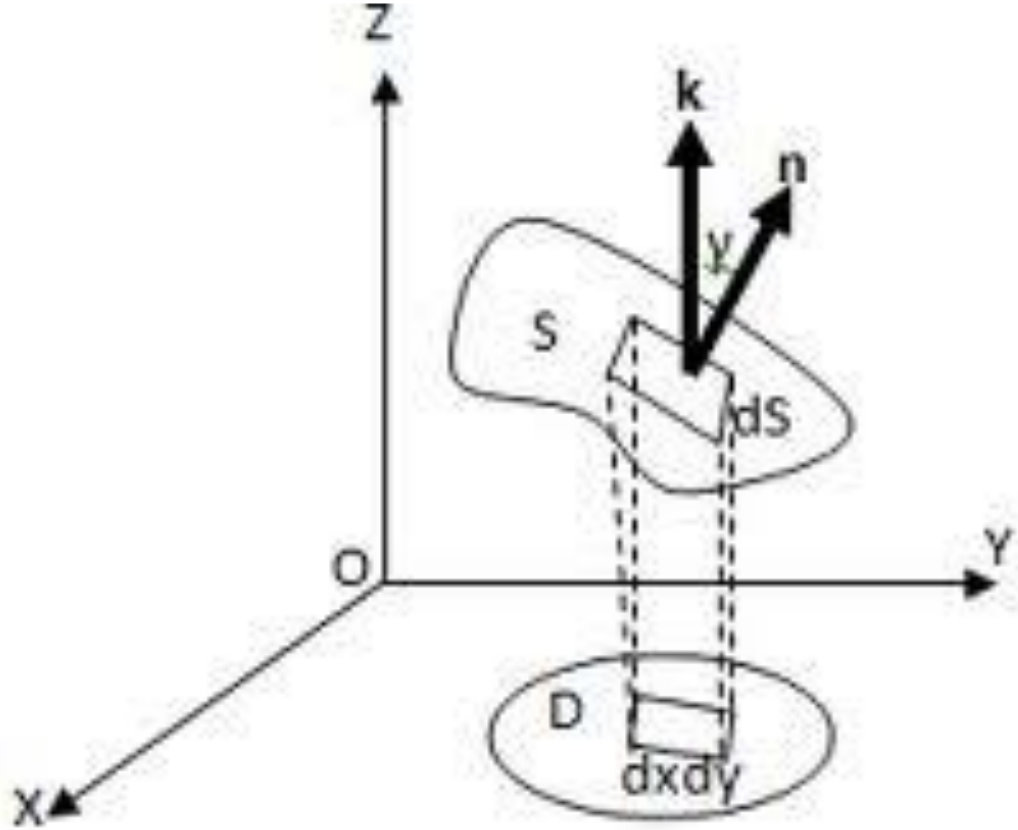
$$S(x,y,z)\ is\ a\ surface$$

$$\overrightarrow{\boldsymbol{ds}} = ds\,\widehat{\boldsymbol{n}} = ds\ \frac{\boldsymbol{\nabla}S}{|\boldsymbol{\nabla}S|}$$

$$\overrightarrow{\boldsymbol{ds}} = \frac{dxdy\widehat{\boldsymbol{n}}}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{k}}} = \frac{dxdy\boldsymbol{\nabla}S}{\boldsymbol{\nabla}S\cdot\widehat{\boldsymbol{k}}}$$

$$\overrightarrow{\boldsymbol{ds}} = \frac{dxdz\widehat{\boldsymbol{n}}}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{j}}} = \frac{dxdz\boldsymbol{\nabla}S}{\boldsymbol{\nabla}S\cdot\widehat{\boldsymbol{j}}}$$

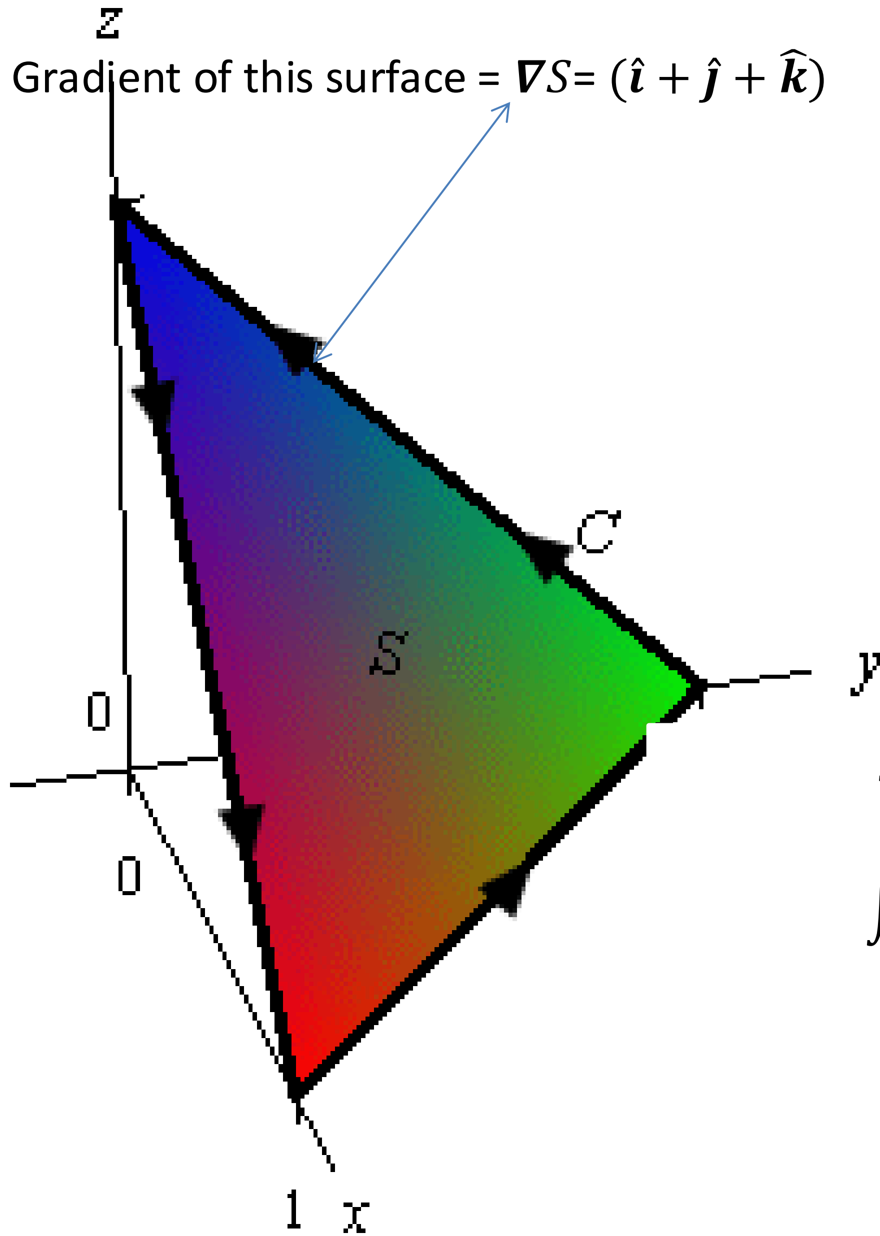
$$\overrightarrow{\boldsymbol{ds}} = \frac{dydz\widehat{\boldsymbol{n}}}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{\imath}}} = \frac{dydz\boldsymbol{\nabla}S}{\boldsymbol{\nabla}S\cdot\widehat{\boldsymbol{\imath}}}$$



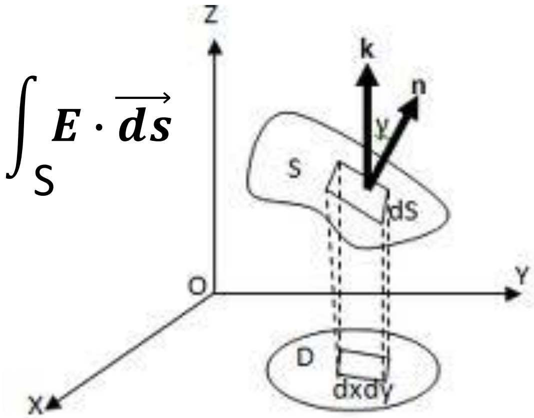
Example: The plane S has the equation $x + y + z = 1$.

$$E = (2z - 1)\,\hat{\jmath}$$

Gradient of this surface = $\nabla S = (\hat{\imath} + \hat{\jmath} + \hat{k})$



$$\int_S E \cdot \overrightarrow{ds}$$



$$\overrightarrow{ds} = \frac{dxdy\hat{n}}{\hat{n}\cdot\hat{k}} = \frac{dxdy\nabla S}{\nabla S\cdot\hat{k}} = \frac{dxdy(\hat{\imath}+\hat{\jmath}+\hat{k})}{(\hat{\imath}+\hat{\jmath}+\hat{k})\cdot\hat{k}}$$

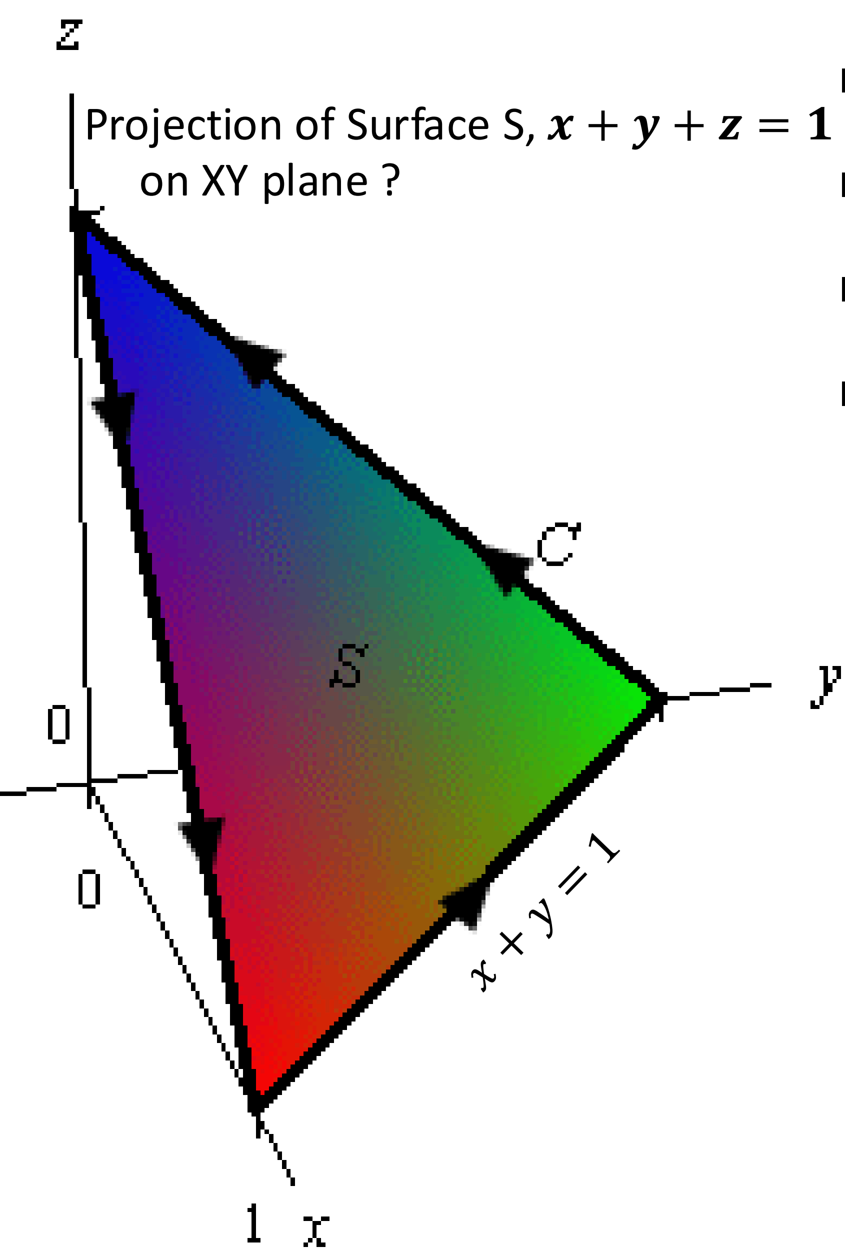
$$\int_S E \cdot \overrightarrow{ds} = \int (2z-1)\,\hat{\jmath}\cdot\overrightarrow{ds}$$

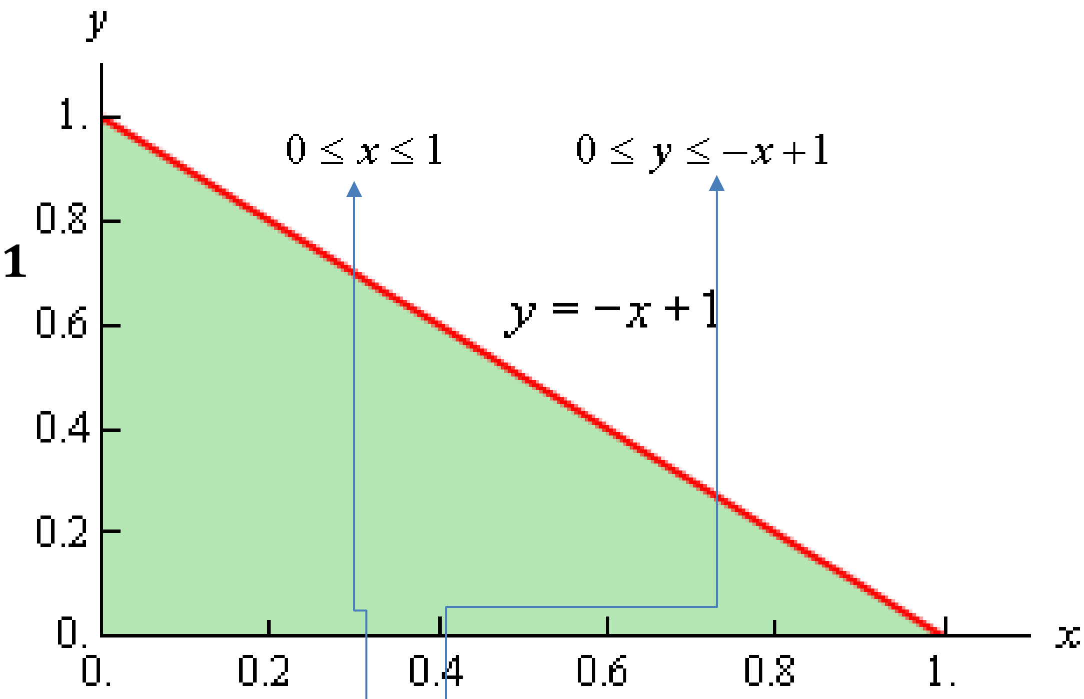
$$\int \boxed{E \cdot \overrightarrow{ds}} = \iint_D (2z-1)\,\vec{j}\bullet\left(\vec{i}+\vec{j}+\vec{k}\right)dA$$

$$= \int_0^1\int_0^{-x+1} 2(1-x-y)-1\,dy\,dx$$

Surface Integral :

Projection of Surface S, $\boldsymbol{x+y+z=1}$
on XY plane ?

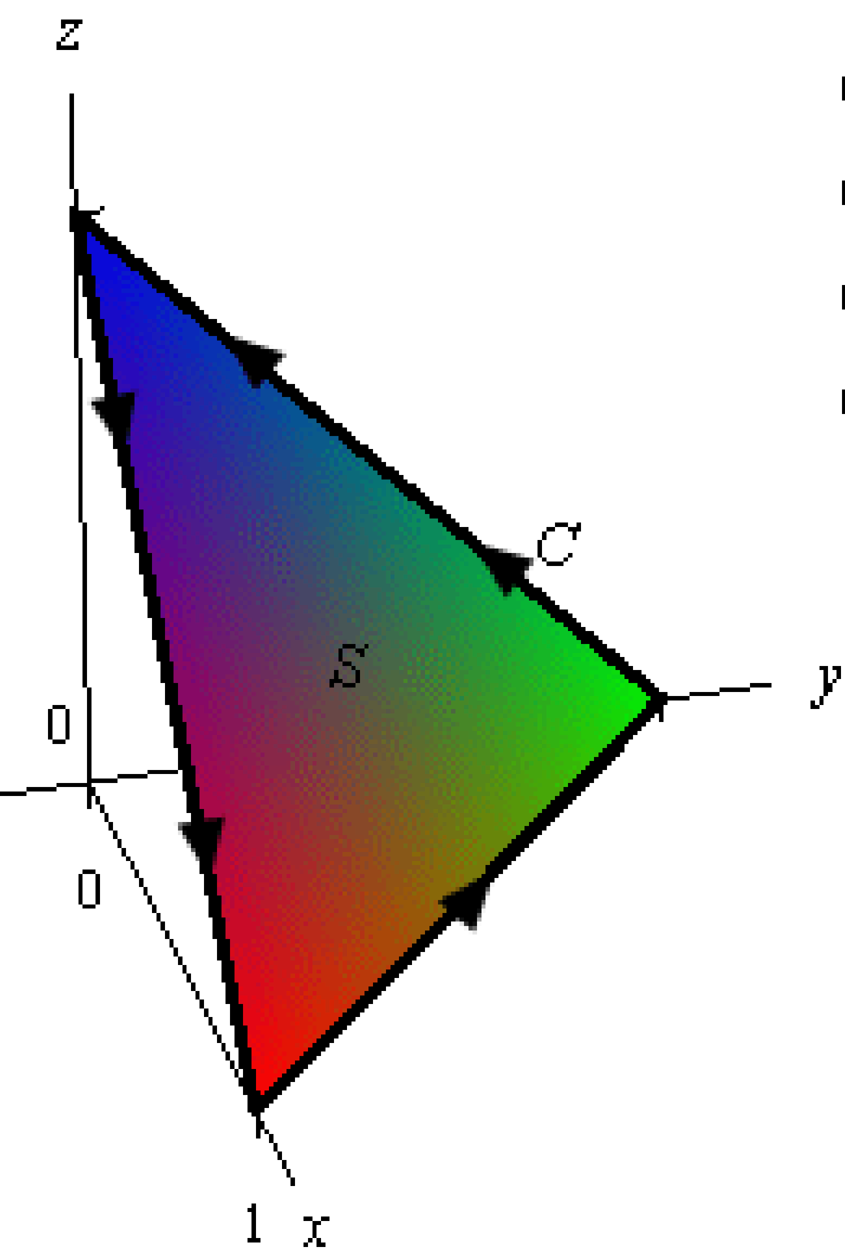


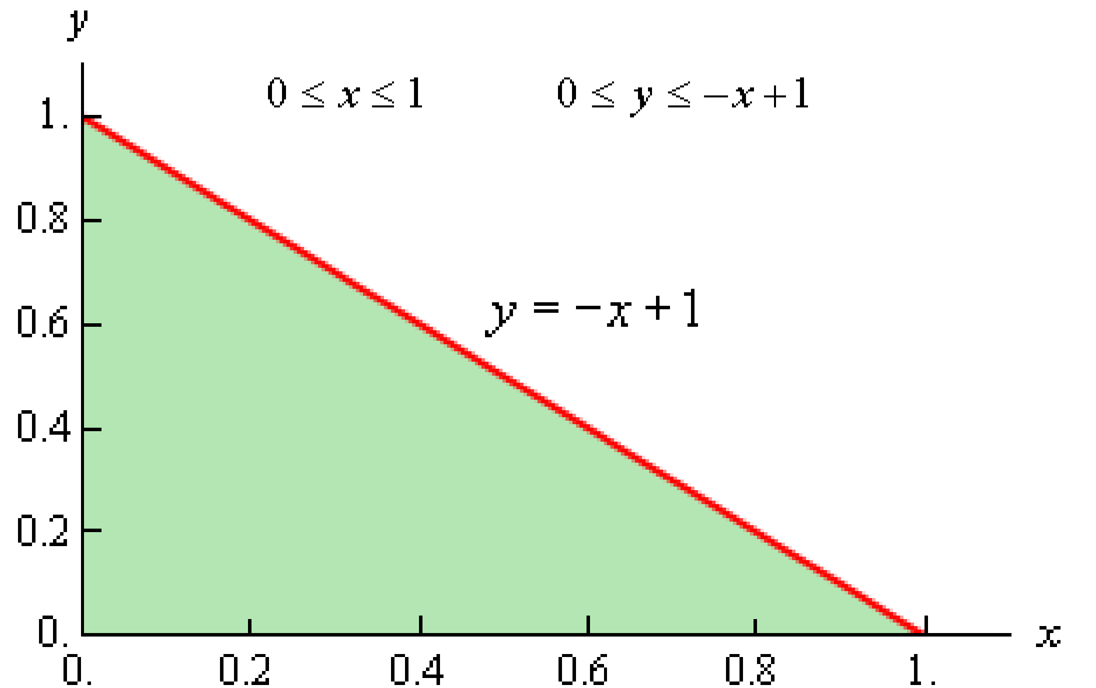


$$\int \boldsymbol{E}\cdot\overrightarrow{\boldsymbol{ds}} = \iint_D (2z-1)\vec{j}\cdot(\vec{i}+\vec{j}+\vec{k})\,dA$$

$$= \int_0^1 \int_0^{-x+1} 2(1-x-y)-1\,dy\,dx$$

Surface Integral :





$$= \int_0^1 \int_0^{-x+1} 1 - 2x - 2y \, dy \, dx$$

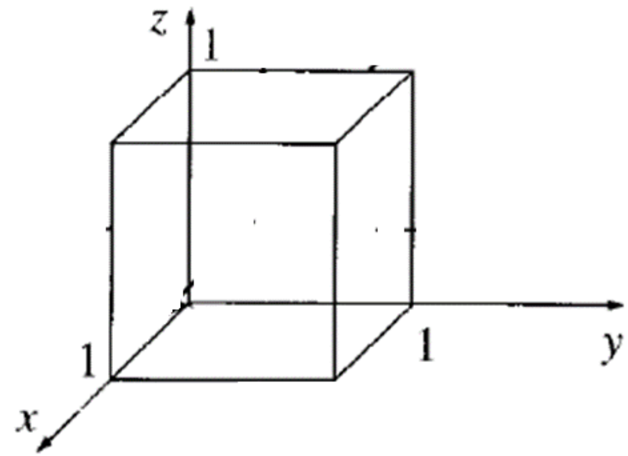
$$= \int_0^1 \left(y - 2xy - y^2 \right) \Big|_0^{-x+1} dx$$

$$= \int_0^1 x^2 - x \, dx$$

$$= \left(\frac{1}{3} x^3 - \frac{1}{2} x^2 \right) \Big|_0^1$$

$$= -\frac{1}{6}$$

Example of volume integral



$$\int_{\mathcal{V}} 2(x+y)\,d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz,$$

$$\int_0^1 (x+y)\,dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2}+y)\,dy = 1, \quad \int_0^1 1\,dz = 1.$$