PHY 102 Introduction to Physics II

Spring Semester 2025

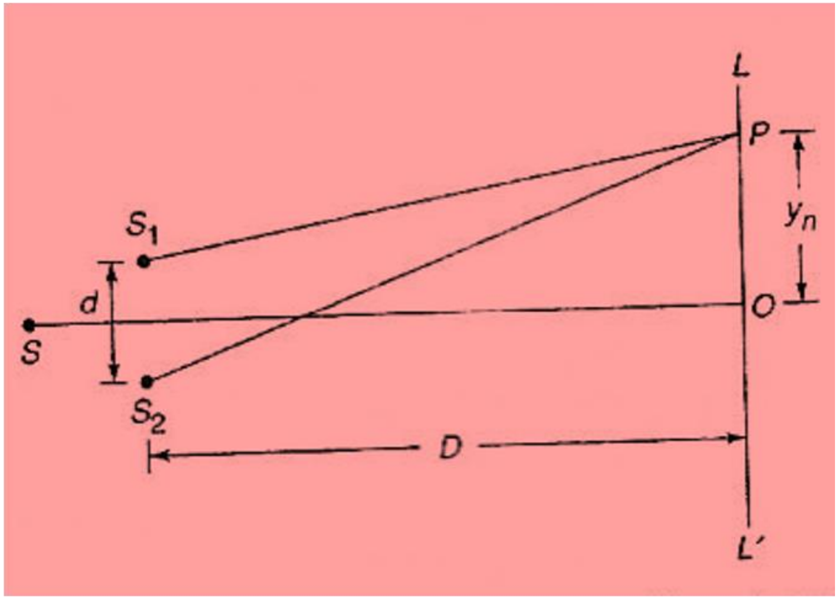
Lecture 37

Intensity Distribution

Let $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ be the electric fields produced at the point P by S1 and S2.

$$\left.\begin{aligned}\mathbf{E}_1 &= \hat{\mathbf{i}} E_{01} \cos\left(\frac{2\pi}{\lambda} S_1P - \omega t\right)\\ \mathbf{E}_2 &= \hat{\mathbf{i}} E_{02} \cos\left(\frac{2\pi}{\lambda} S_2P - \omega t\right)\end{aligned}\right\}$$

(If S_1P and S_2P are large compared to $S_1S_2 = d$)



Resultant field

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2\\ &= \hat{\mathbf{i}}\left[E_{01}\cos\left(\frac{2\pi}{\lambda} S_1P - \omega t\right)\right.\\ &\qquad \left.+ E_{02}\cos\left(\frac{2\pi}{\lambda} S_2P - \omega t\right)\right]\end{aligned}$$

Intensity $I = KE^2$

$$\begin{aligned}I = K\Bigg[& E_{01}^2 \cos^2\left(\frac{2\pi}{\lambda} S_1P - \omega t\right) +\\ & E_{02}^2 \cos^2\left(\frac{2\pi}{\lambda} S_2P - \omega t\right) +\\ & E_{01}E_{02}\left\{\cos\left[\frac{2\pi}{\lambda}(S_2P - S_1P)\right] +\right.\\ & \left.\cos\left[2\omega t - \frac{2\pi}{\lambda}(S_2P + S_1P)\right]\right\}\Bigg]\end{aligned}$$

Intensity Distribution

For an optical beam, the frequency is very large ($\omega = 10^{15}/s$) and *all the terms depending on ωt* will vary rapidly (10^{15} times in a sec). We have to take average values of various quantities

$$\langle \cos^2(\omega t - \theta) \rangle \approx \frac{1}{2} \qquad \langle \cos(2\omega t - \varphi) \rangle = 0$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} cos(\delta) \qquad \delta = \frac{2\pi}{\lambda}(S_2P - S_1P) \qquad I_1 = \frac{1}{2} K E_{01}^2$$

(a) $I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 \quad I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$

(Maximum and minimum values of $cos\delta$ are +1 and -1)

Maximum intensity occurs when $\delta = 2n\pi \qquad n = 0,1,2$

$$S_2P - S_1P = n\lambda$$

Minimum intensity occurs when $\delta = (2n+1)\pi$

$n = 0,1,2$

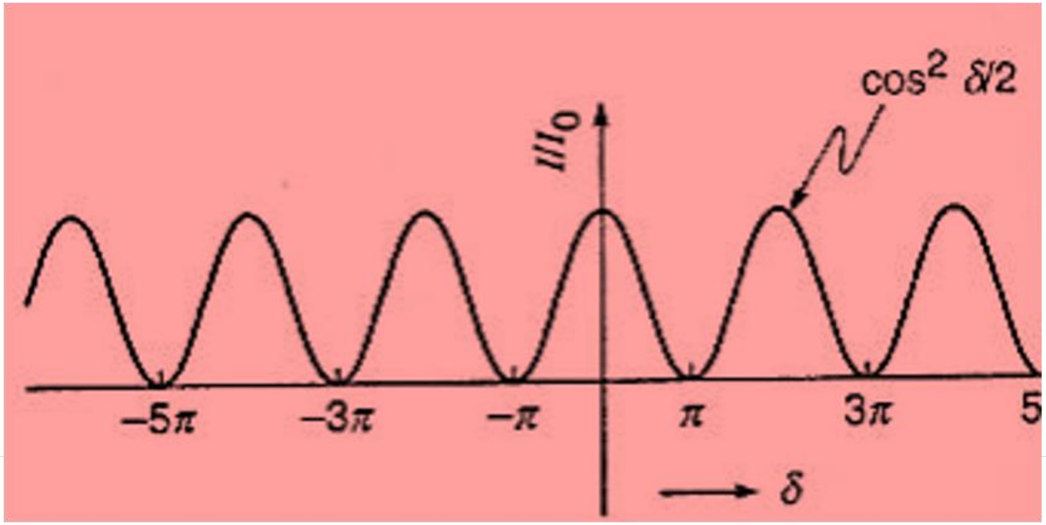
$$S_2P - S_1P = (n + \frac{1}{2})\lambda$$

(b) If the holes S1 and S2 are illuminated by different light sources, then phase difference δ would vary with time and $\langle cos\delta \rangle = 0$

$$I = I_1 + I_2$$

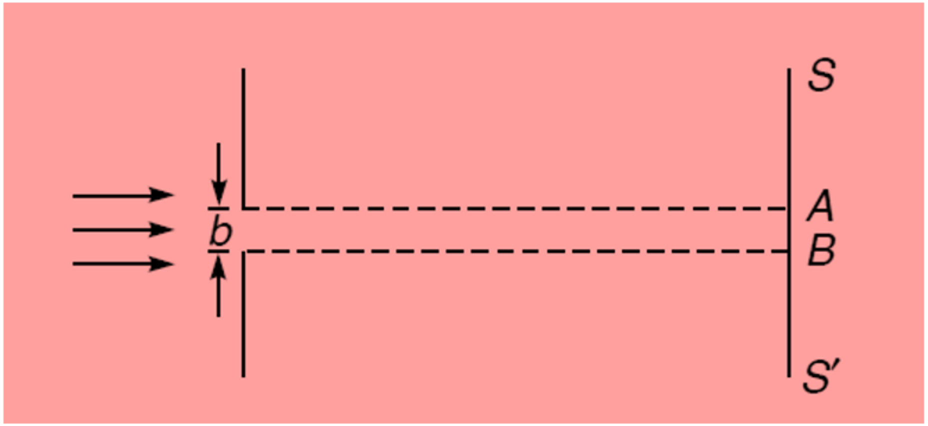
For two incoherent sources, the resultant intensity is the sum of intensities produced by each source and no interference pattern is observed

(c) $I_1 \simeq I_2 \simeq I_0$ $\qquad I = 2I_0 + 2I_0 cos\delta = 4I_0 cos^2\frac{\delta}{2}$



Intensity Distribution

Consider a plane wave incident on a long narrow slit of width $'b'$



According to geometrical optics, region AB of the screen should be illuminated and the remaining portion (geometrical shadow) should be dark

In reality, two things occur:

- Light intensity in the region AB is not uniform
- There is some intensity in the geometrical shadow
- If the width is made smaller, larger amounts of energy enters the geometrical shadow

The spreading-out of a wave when it passes through a narrow opening is usually referred to as ***diffraction***

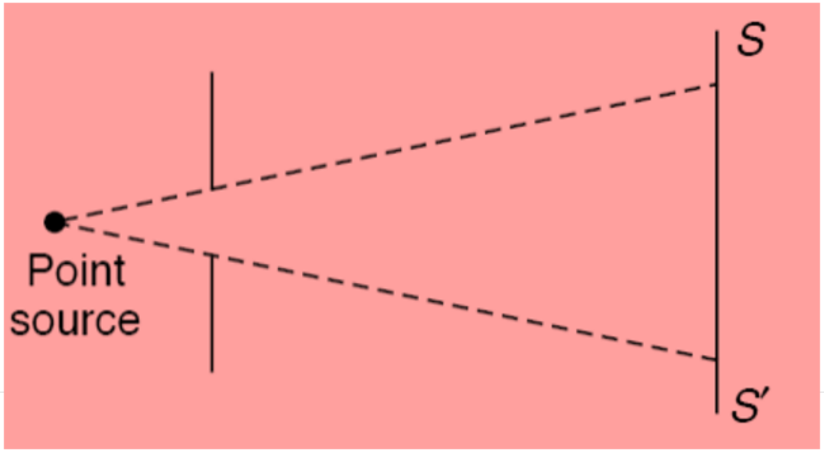
Diffraction is very closely related to interference

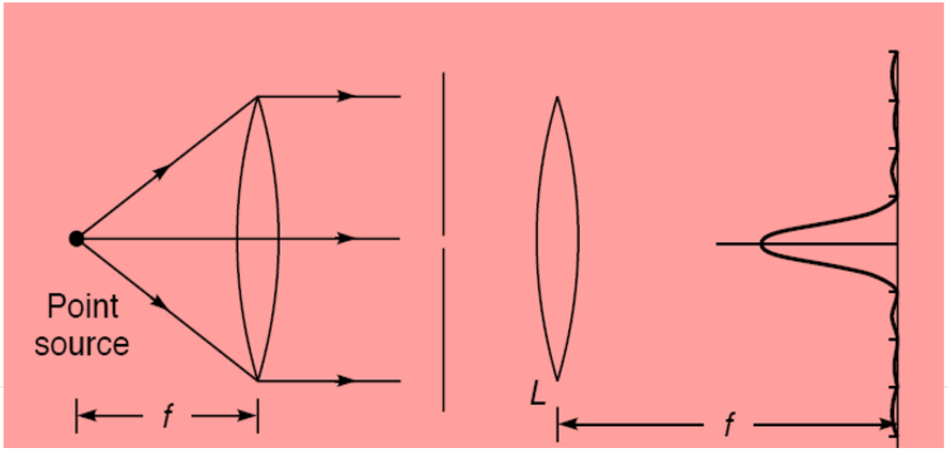
Fresnel diffraction

Source of light and screen are at finite distances from the diffracting aperture

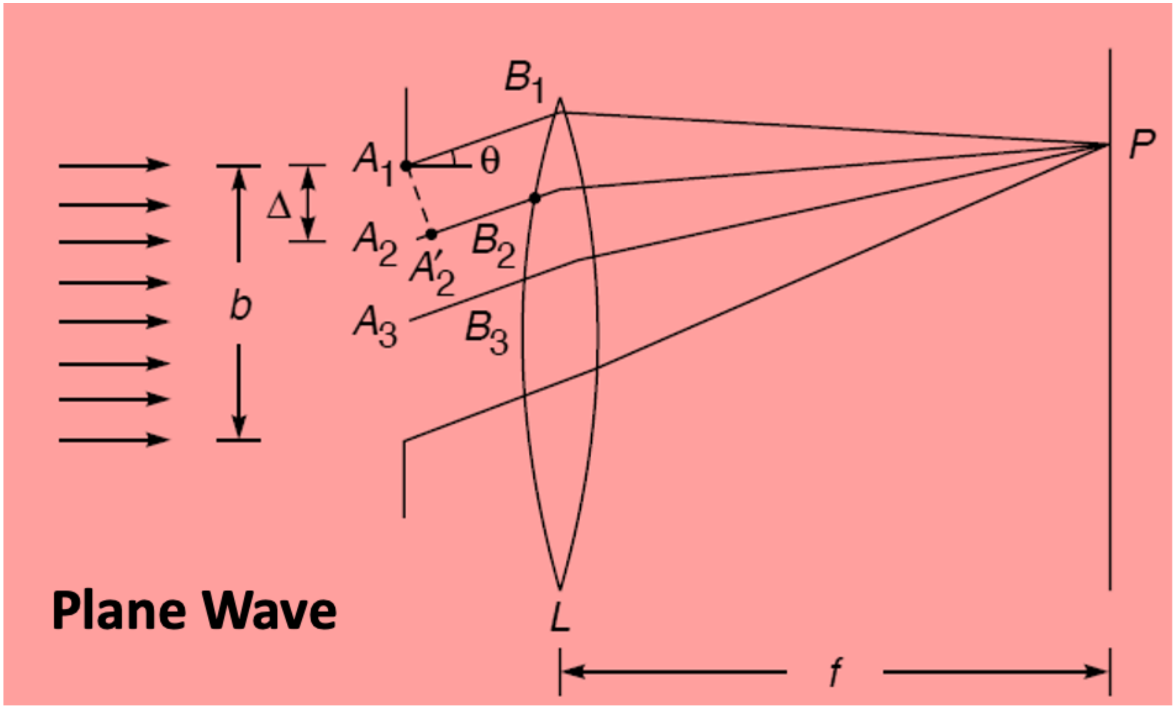
Fraunhofer diffraction

Source of light and screen are at infinite distances from the diffracting aperture





Single– slit diffraction pattern

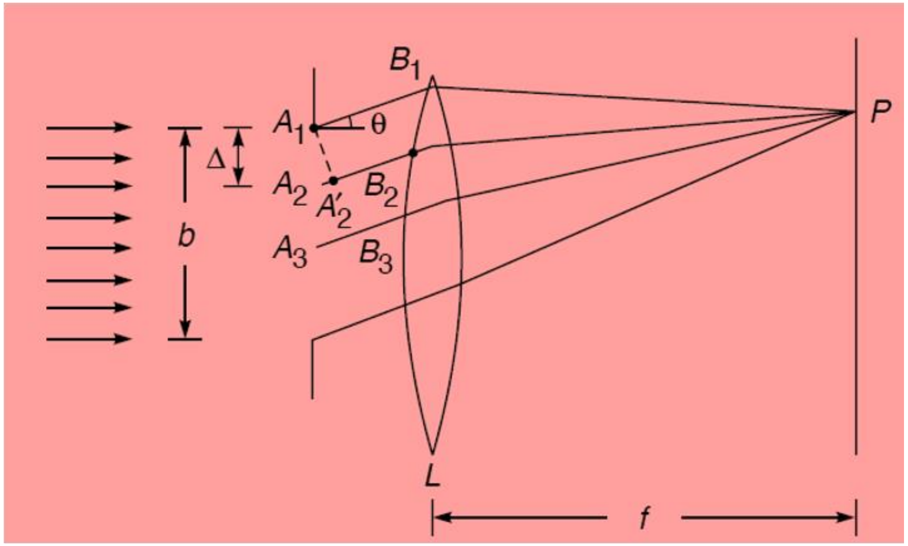


Assume that the slit consist of large number of equally spaced point sources (A1, A2, A3)- each point on the slit is a source of Huygen's secondary wavelets which interfere with wavelets emanating from other points

We will calculate the *intensity distribution* on the screen which is at the focal plane of the lens

Single– slit diffraction pattern

$$b = (n-1)\Delta$$



n is the number of point sources
Calculate the resultant field at P due to these 'n' sources

Because of slightly different path lengths of A_1B_1P and A_2B_2P, *field produced by A1 will be different from A2*. The path difference is $A_2A_2' = \Delta\, sin\theta$ $\qquad \phi = \dfrac{2\pi}{\lambda}\Delta\, sin\theta$

If the field at P due to A1 is $a\ cos\omega t$, then field due to A2 is $a\cos(\omega t - \phi)$

$$E = a[\cos \omega t + \cos(\omega t - \phi) + \cdots + \cos[(\omega t - (n-1)\phi)]$$

$$\cos \omega t + \cos(\omega t - \phi) + \cdots + \cos[\omega t - (n-1)\phi]$$
$$= \frac{\sin(n\phi/2)}{\sin(\phi/2)} \cos\left[\omega t - \frac{1}{2}(n-1)\phi\right]$$

Single- slit diffraction pattern

$$E = E_0 \cos\left[\omega t - \tfrac{1}{2}(n-1)\phi\right]$$

$$E_\theta = a \frac{\sin(n\phi/2)}{\sin(\phi/2)}$$

(Field amplitude)

In the limit $n \to \infty$, $\Delta \to 0$ so that $n\Delta \to b$ (i.e continuous distribution of point sources)

$$\frac{n\phi}{2} = \frac{\pi}{\lambda} n\Delta \sin\theta \to \frac{\pi}{\lambda} b \sin\theta$$

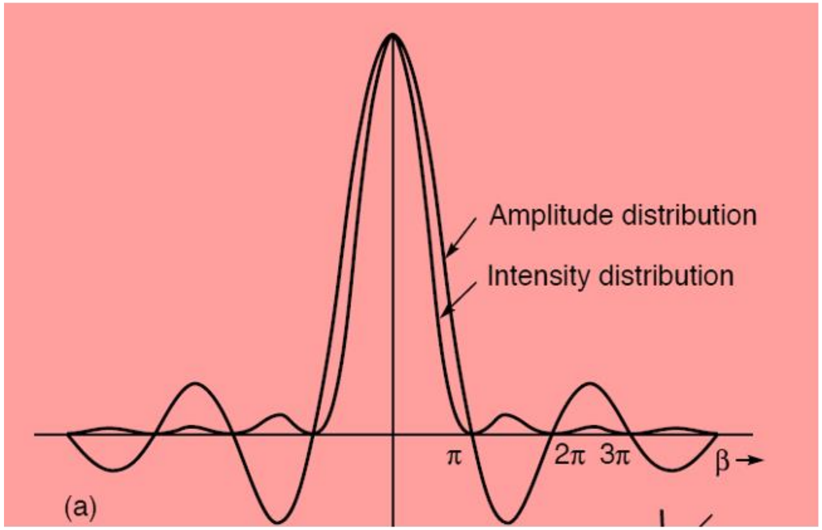
$$\phi = \frac{2\pi}{\lambda}\Delta \sin\theta = \frac{2\pi}{\lambda}\frac{b\sin\theta}{n}$$

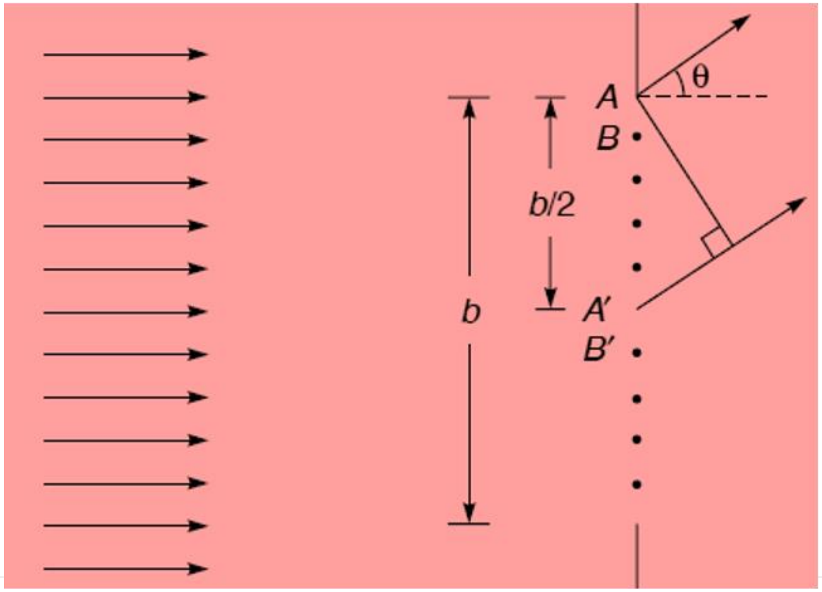
$$E_\theta \approx \frac{a\sin(n\phi/2)}{\phi/2} = na\frac{\sin(\pi b\sin\theta/\lambda)}{(\pi b\sin\theta/\lambda)} = A\frac{\sin\beta}{\beta}$$

$$E = A\frac{\sin\beta}{\beta}\cos(\omega t - \beta)$$

$$I = I_0\frac{\sin^2\beta}{\beta^2}$$

I_0 is the intensity at $\theta = 0$





Minima

$$\beta = m\pi \qquad m \neq 0$$

$$b \sin\theta = m\lambda$$

$$m = \pm 1, \pm 2, \pm 3, \dots\dots$$

The slit is divided into two halves for deriving the condition of first minimum (m = 1)

Path difference between A and A'

$$\left({}^{b}/_{2}\right) sin\theta = {}^{\lambda}/_{2} = \pi$$

Disturbance from upper half of slit and lower half of slit would cancel which corresponds to minima

Maxima for single slit diffraction

$$\frac{dI}{d\beta} = I_0\left(\frac{2\sin\beta\cos\beta}{\beta^2} - \frac{2\sin^2\beta}{\beta^3}\right) = 0$$

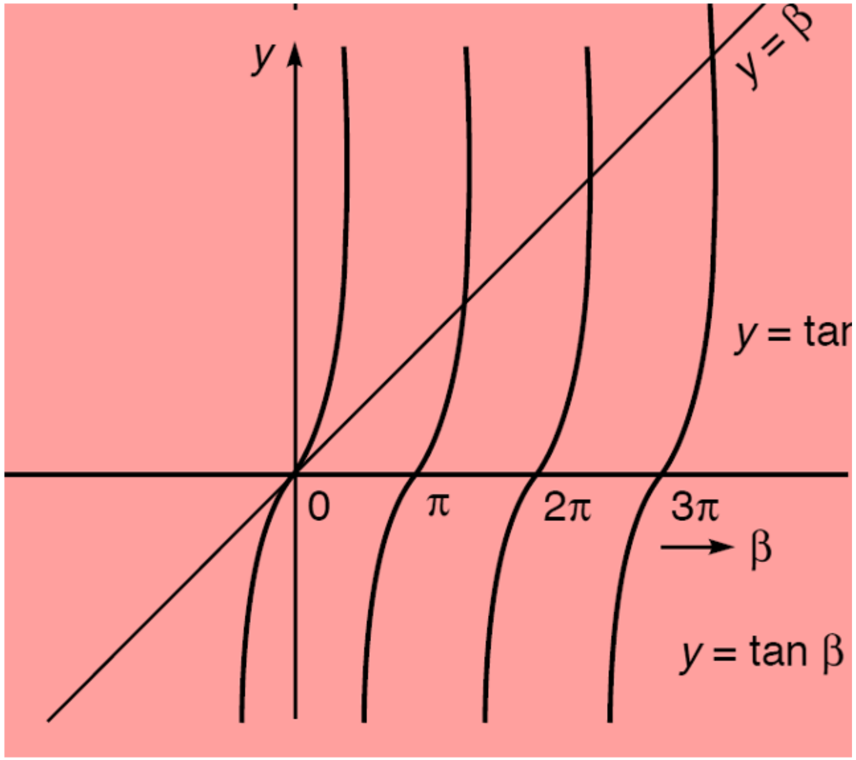
$$\sin\beta\,(\beta - \tan\beta) = 0$$

$\tan\beta = \beta \qquad \beta = 0$

(principal maximum)

$\beta = 1.43\pi$ (first maxima)

$\beta = 2.46\pi$ (second maxima)



Problem 1

A parallel beam of light is incident normally on a narrow slit of width 0.2 mm. The Fraunhofer diffraction pattern is observed on a screen which is placed at the focal plane of a convex lens whose focal length is 20 cm. Calculate the distance between the first two minima and the first two maxima on the screen. Assume that $\lambda = 5 \times 10^{-5}$ cm and that the lens is placed very close to the slit.

$$\frac{\lambda}{b} = \frac{5\times10^{-5}}{2\times10^{-2}} = 2.5\times10^{-3}$$

Now, the conditions for diffraction minima are given by $\sin\theta = m\lambda/b$. We assume θ to be small (measured in radians) so that we may write $\sin\theta \approx \theta$ (an assumption which will be justified by subsequent calculations); thus, on substituting the value of λ/b, we get

$$\theta \simeq 2.5\times10^{-3} \text{ and } 5\times10^{-3} \text{ rad}$$

as the angles of diffraction corresponding to the first and second minima, respectively. Notice that since

$$\sin(2.5\times10^{-3}) = 2.4999973\times10^{-3}$$

the error in the approximation $\sin\theta \simeq \theta$ is about 1 part in 1 million! These minima will be separated by a distance $(5\times10^{-3} - 2.5\times10^{-3})\times 20 = 0.05$ cm on the focal plane of the lens. Similarly, the first and second maxima occur at

$$\beta = 1.43\pi \quad \text{and} \quad 2.46\pi$$

respectively. Thus

$$b\sin\theta = 1.43\lambda \quad \text{and} \quad 2.46\lambda$$

or

$$\sin\theta = 1.43\times2.5\times10^{-3} \quad \text{and} \quad 2.46\times2.5\times10^{-3}$$

Consequently, the maxima will be separated by the distance given by

$$(2.46 - 1.43)\times2.5\times10^{-3}\times20 \simeq 0.05 \text{ cm}$$

Solution: The interference minima will occur when Eq. (46) is satisfied, i.e., when

$$\sin\theta = \left(n+\frac{1}{2}\right)\frac{\lambda}{d} = 0.904\times 10^{-3}\left(n+\frac{1}{2}\right)$$

$$n = 0, 1, 2, \ldots$$

$$= 0.452\times 10^{-3}, 1.356\times 10^{-3}, 2.260\times 10^{-3},$$
$$3.164\times 10^{-3}, 4.068\times 10^{-3}, 4.972\times 10^{-3},$$
$$5.876\times 10^{-3}, 6.780\times 10^{-3}$$

Thus there will be 16 minima between the two first-order diffraction minima.

The angular separation between two interference maxima is approximately given by [see Eq. (47)]

$$\Delta\theta \simeq \frac{\lambda}{d} = 0.904\times 10^{-4}$$

Thus the fringe width is

$$15\times 12\times 2.54\times 0.904\times 10^{-4} \simeq 0.0413 \text{ cm}$$

[redacted] Consider a diffraction grating with 15,000 lines per inch. (a) Show that if we use a white light source, the second- and third-order spectra overlap. (b) What will be the angular separation of the D_1 and D_2 lines of sodium in the second-order spectra?

Solution: (a) The grating element is

$$d = \frac{2.54}{15{,}000} = 1.69 \times 10^{-4} \text{ cm}$$

Let θ_{mv} and θ_{mr} represent the angles of diffraction for the mth-order spectrum corresponding to the violet and red colors, respectively. Thus

$$\theta_{2v} = \sin^{-1}\left(\frac{2\times 4\times 10^{-5}}{1.69\times 10^{-4}}\right) \approx \sin^{-1} 0.473 \approx 28.2^\circ$$

$$\theta_{2r} = \sin^{-1}\left(\frac{2\times 7\times 10^{-5}}{1.69\times 10^{-4}}\right) \approx \sin^{-1} 0.828 \approx 55.90^\circ$$

and

$$\theta_{3v} = \sin^{-1}\left(\frac{3\times 4\times 10^{-5}}{1.69\times 10^{-4}}\right) \approx \sin^{-1} 0.710 \approx 45.23^\circ$$

where we have assumed the wavelengths of the violet and red colors to be 4×10^{-5} and 7×10^{-5} cm, respectively. Since $\theta_{2r} > \theta_{3v}$, the second- and third-order spectra will overlap. Further since $\sin \theta_{3r} > 1$, the third-order spectrum for the red color will not be observed.

(b) Since $d \sin \theta = m\lambda$, we have for small $\Delta\lambda$:

$$(d \cos \theta)\, \Delta\theta = m(\Delta\lambda)$$

or

$$\Delta\theta = \frac{m\Delta\lambda}{d\left\{1-(m\lambda/d)^2\right\}^{1/2}}$$

$$\simeq \frac{2\times 6\times 10^{-8}}{1.69\times 10^{-4}\left[1-\left(2\times 6\times 10^{-5}/1.69\times 10^{-4}\right)^2\right]^{1/2}}$$

$$\simeq 0.0010 \text{ rad} \simeq 3.44 \text{ minutes}$$

Thus, if we are using telescope of angular magnification 10, the two lines will appear to have an angular separation of 34.4 minutes.