PHY101: Introduction to Physics I

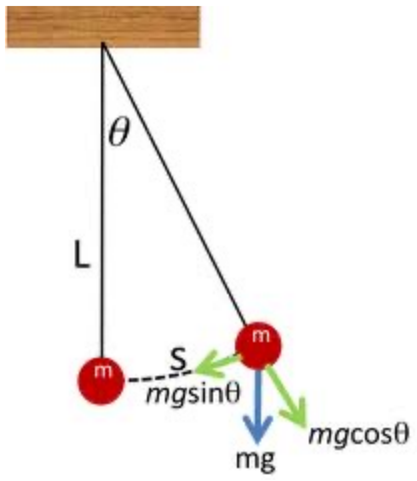
Monsoon Semester 2024
Lecture 14

Department of Physics, School of Natural Sciences,
Shiv Nadar Institution of Eminence, Delhi NCR

Dr. Binson Babu (*binson.babu@snu.edu.in*)

Previous Lecture

Restoring forces
Inverse square forces

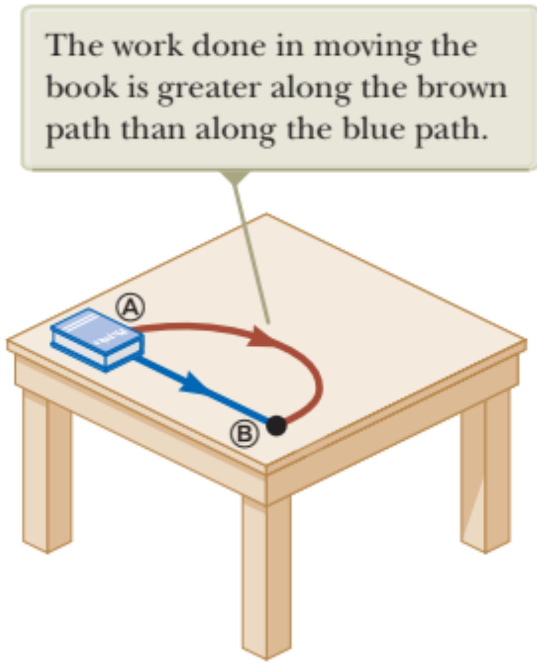


This Lecture

Work Energy theorem

Path dependent work

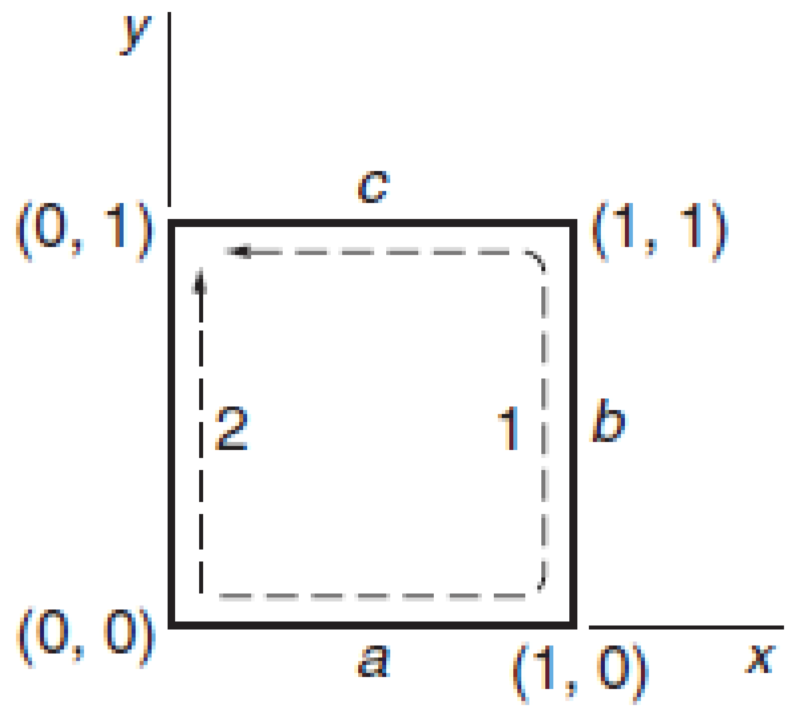
Work done under frictional force!



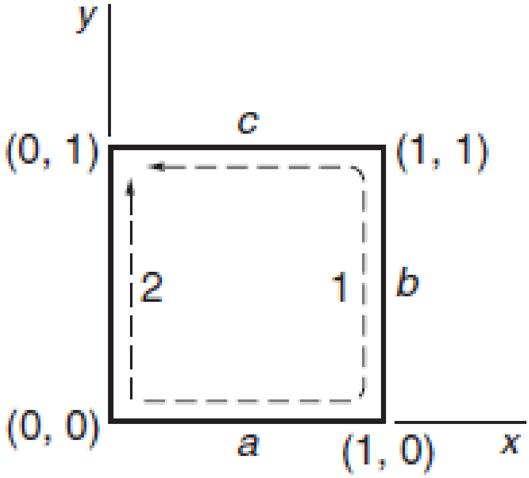
$$W_{ba} = -\oint_{\mathbf{r}_a}^{\mathbf{r}_b} f\, dS$$
$$= -fS,$$

S is total length of path

Question: Let $\boldsymbol{F} = A(xy\hat{\imath} + y^2\hat{\jmath})$. Show that work done under this force in taking an object from (0,0) to (0,1), first along path 1 and then along path 2, as shown in the figure, is path dependent.



$$\boldsymbol{F} = A(xy\hat{\imath} + y^2\hat{\jmath})$$



$$\oint_1 \mathbf{F} \cdot d\mathbf{r} = \int_a \mathbf{F} \cdot d\mathbf{r} + \int_b \mathbf{F} \cdot d\mathbf{r} + \int_c \mathbf{F} \cdot d\mathbf{r}$$

Along segment a, $d\mathbf{r} = dx\,\hat{\mathbf{i}}$, $\mathbf{F} \cdot d\mathbf{r} = F_x\,dx = Axy\,dx$. Since $y = 0$ along the line of this integration, $\int_a \mathbf{F} \cdot d\mathbf{r} = 0$. For path b,

$$\int_b \mathbf{F} \cdot d\mathbf{r} = A \int_{x=1,y=0}^{x=1,y=1} y^2\,dy = \frac{A}{3},$$

while for path c, where $y = 1$,

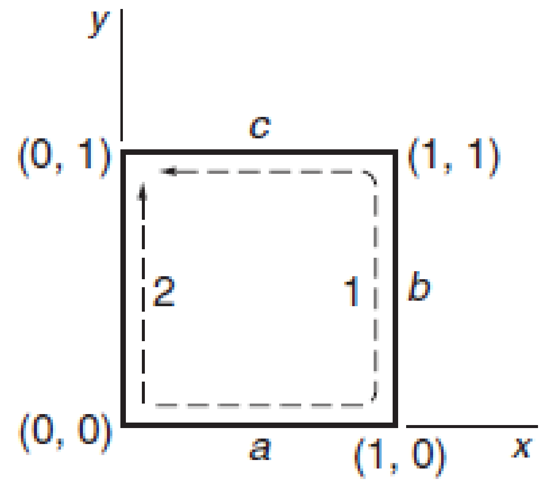
$$\int_c \mathbf{F} \cdot d\mathbf{r} = A \int_{x=1,y=1}^{x=0,y=1} xy\,dx = A \int_1^0 x\,dx = -\frac{A}{2}.$$

Thus

$$\oint_1 \mathbf{F} \cdot d\mathbf{r} = \frac{A}{3} - \frac{A}{2}$$
$$= -\frac{A}{6}.$$

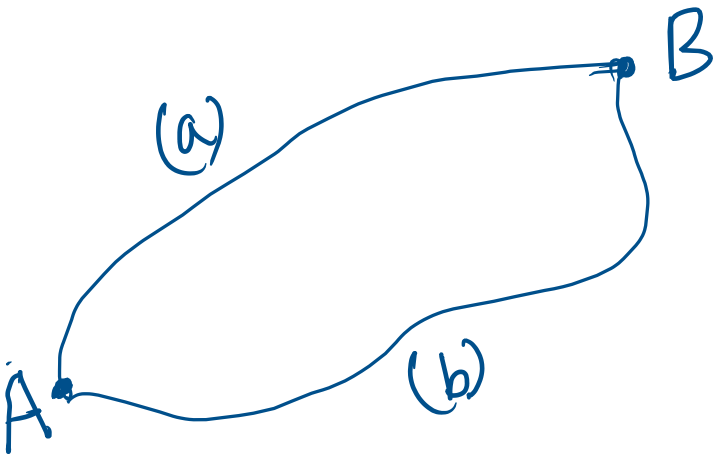
Along path 2 we have

$$\oint_2 \mathbf{F} \cdot d\mathbf{r} = A \int_{x=0,y=0}^{x=0,y=1} y^2\, dy$$
$$= \frac{A}{3}$$
$$\neq \oint_1 \mathbf{F} \cdot d\mathbf{r}.$$



Path independent work $\Longrightarrow$ Conservative force

Consider a particle of mass *m* moving in a force field $\vec{F}(\vec{r})$ that depends only on the position $\vec{r}$.



$$W_a = \int_a \vec{F}.\overrightarrow{dr}$$

$$W_b = \int_b \vec{F}.\overrightarrow{dr}$$

If for arbitrary paths (a) and (b) always $\boldsymbol{W_a = W_b}$ we name the integral path-independent and the force field $\vec{F}(\vec{r})$ conservative.

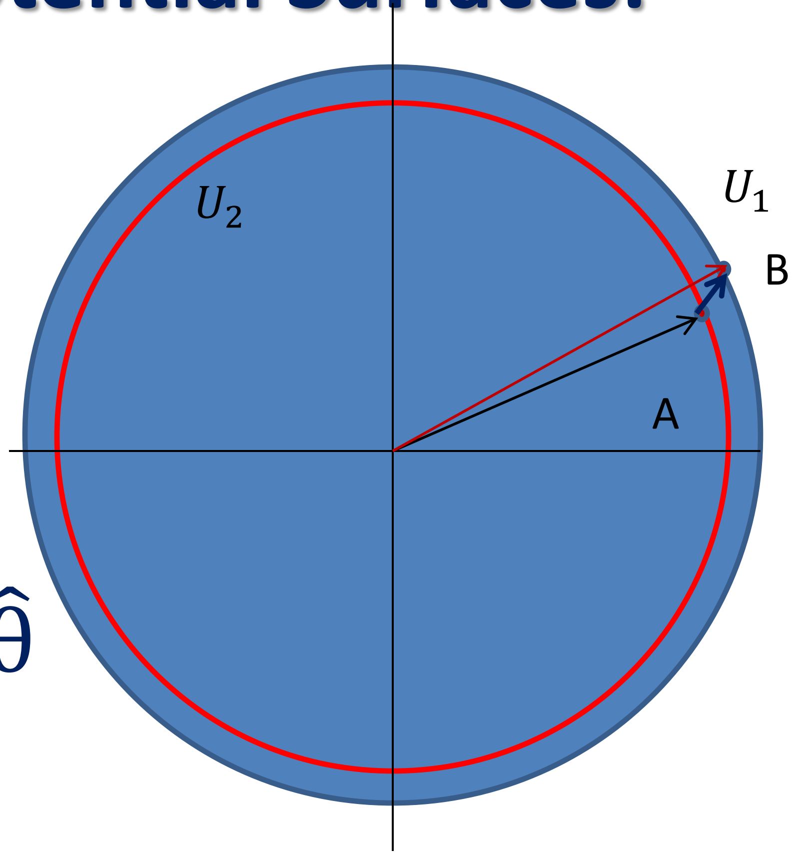
Equipotential Surfaces:

$$W_{AB} = F.ds$$

$$F = |F|\hat{r}$$

$$\mathrm{ds} = dr\hat{r} + rd\theta\hat{\theta}$$

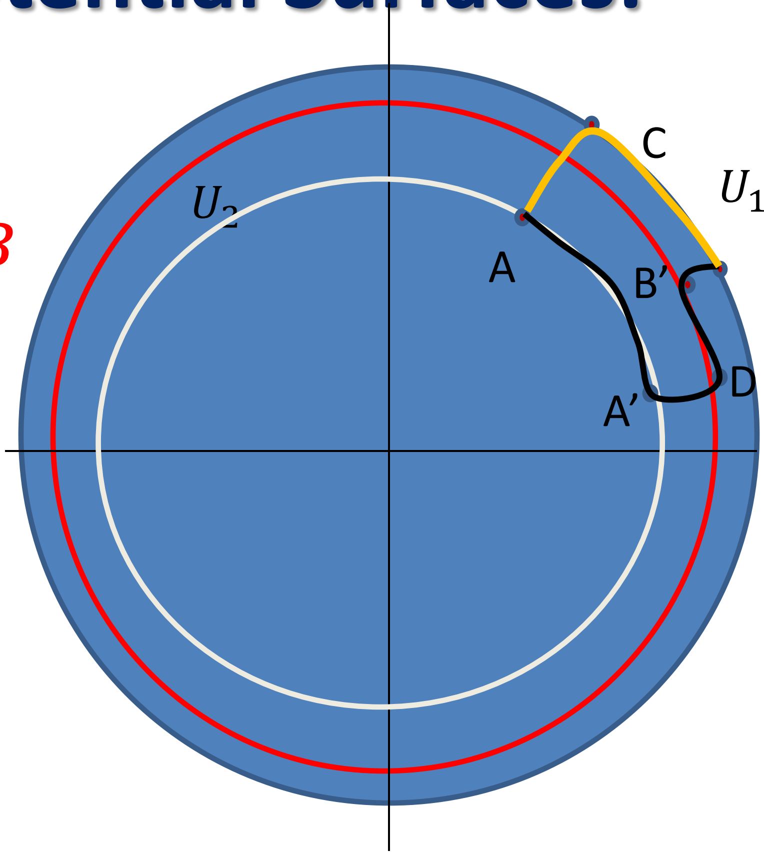
$$W_{AB} = |F|dr$$



Work is done only if you move along the radius vector

Equipotential Surfaces:

$$W_{AA'DB'B} = W_{AcB}$$



Work is done only if you move along the radius vector

Work is independent of the path

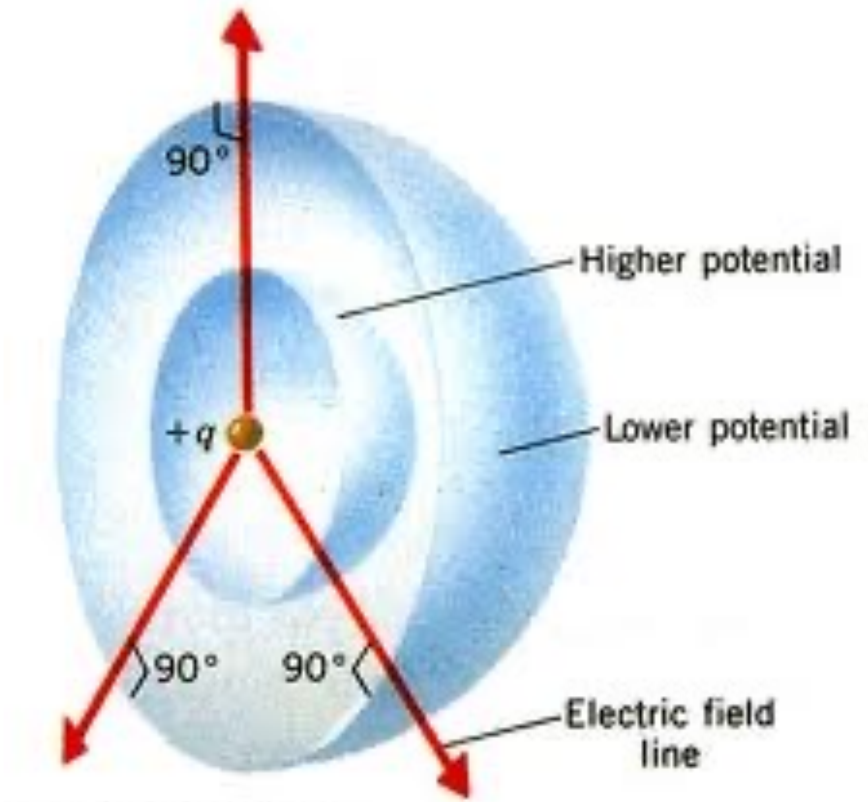
Equipotential Surfaces:

Assume any of the forces at a distance we studied

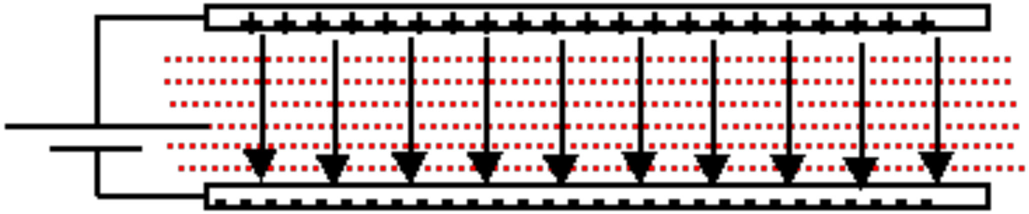
and a region where the force is effective (Field of F)

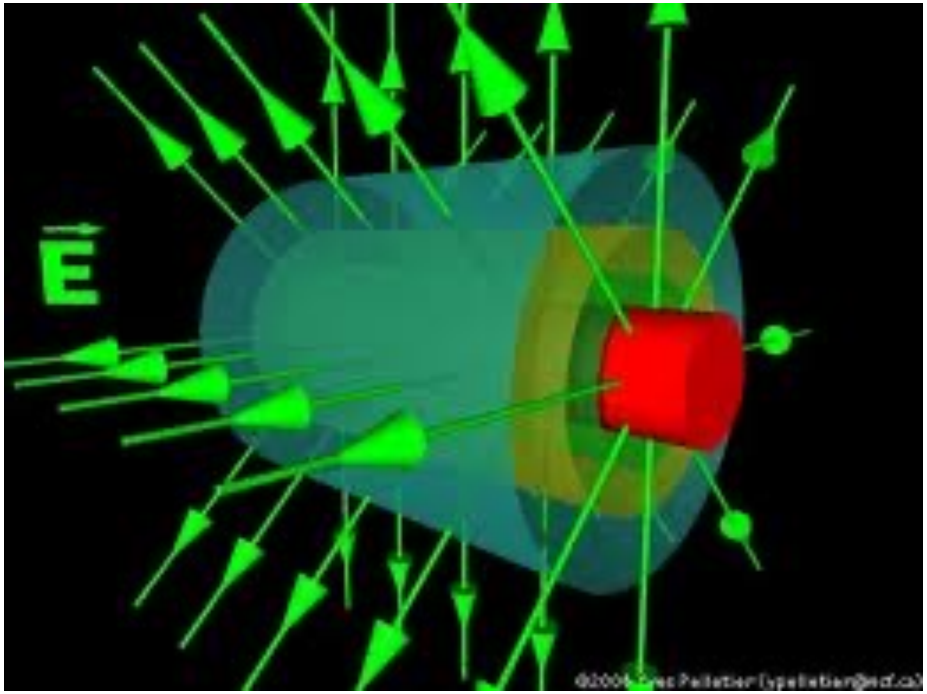
Can you draw a surface in the field where force is always Perpendicular to the surface:

Electric Field due to a charge

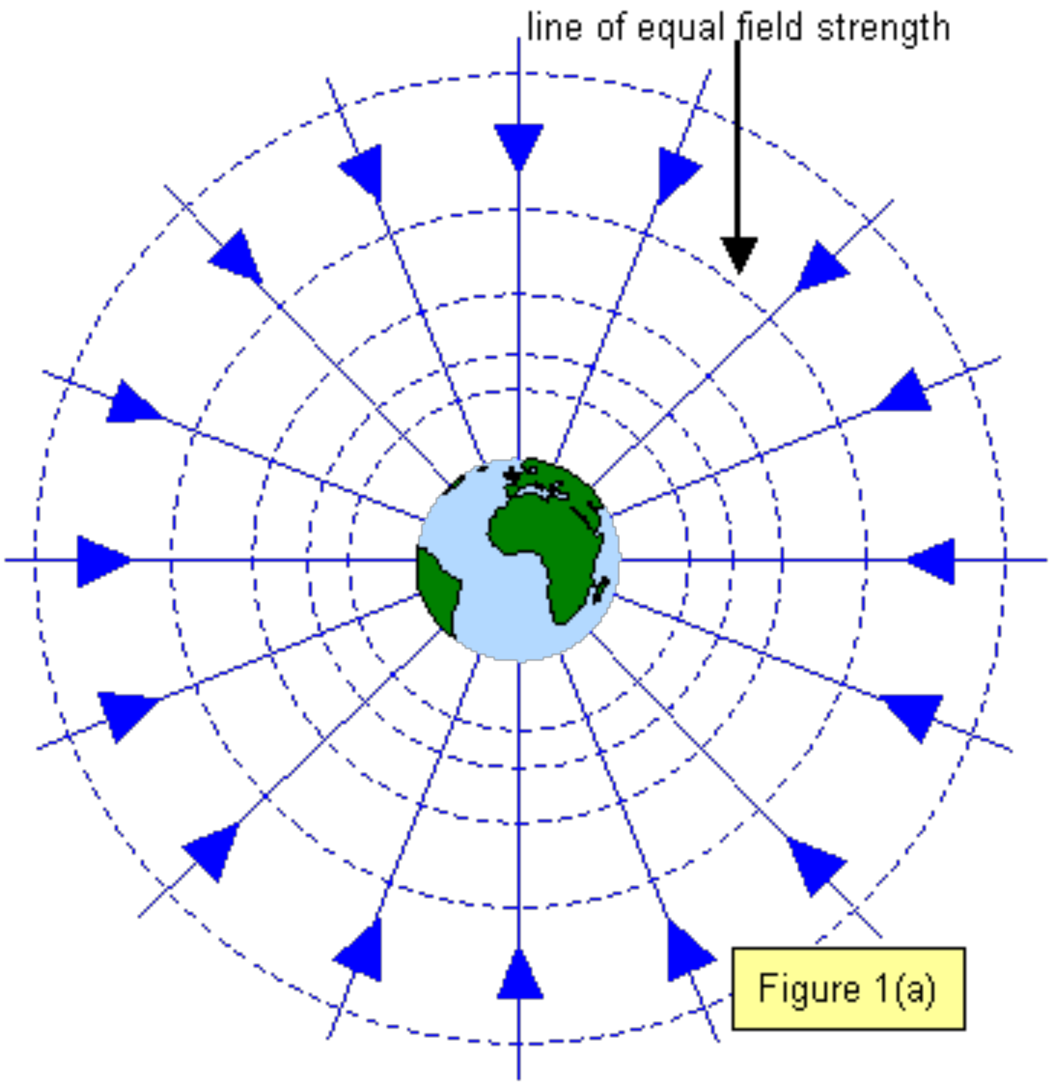


Equipotential Surfaces:

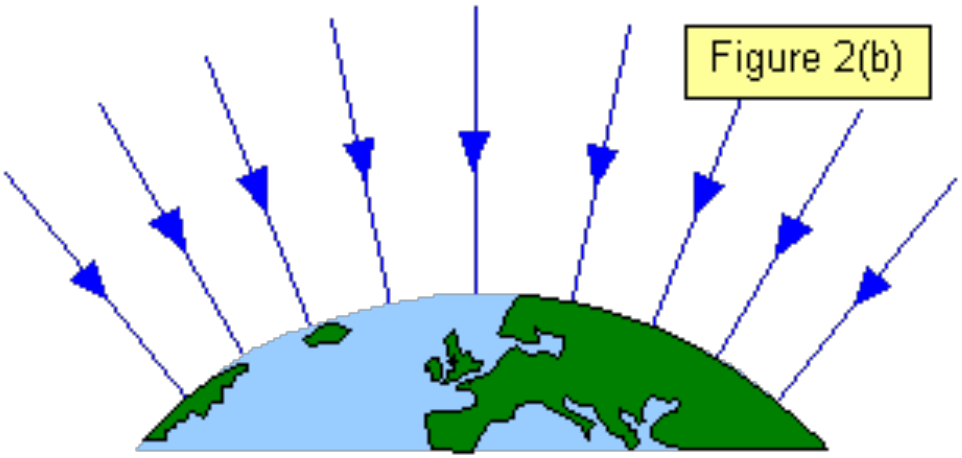


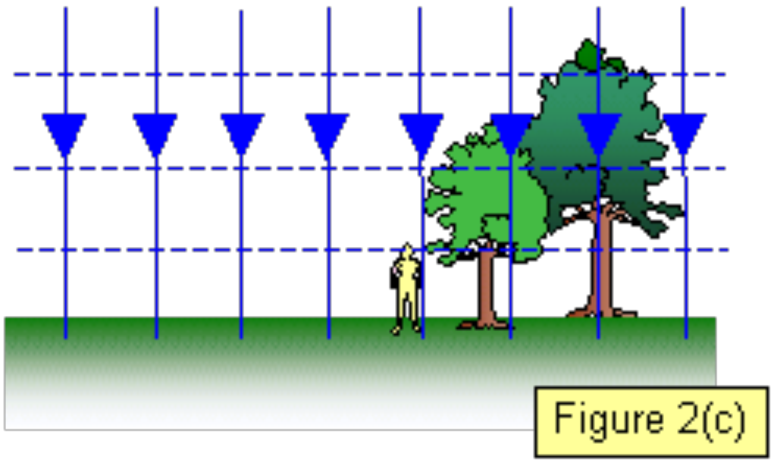


Equipotential Surfaces:



Equipotential Surfaces:





What is the work done by the force of the field if a mass move On one of the Equipotential Surface ?

Zero

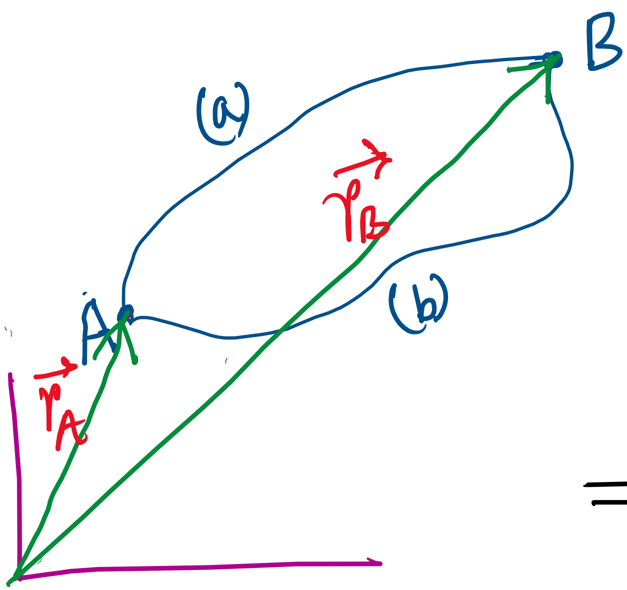
WHY ?

$$W = \int_A^B F.ds$$

$$\cos\frac{\text{Л}}{2} = 0$$

Potential energy function

Work done by a conservative force depends only on the end points, not on the path between them.



$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}\cdot\vec{dr} = \text{function of }(\vec{r}_B) - \text{function of }(\vec{r}_A)$$

$$\Rightarrow \int_{\vec{r}_A}^{\vec{r}_B} \vec{F}\cdot\vec{dr} = -U(\vec{r}_B) + U(\vec{r}_A)$$

$$= -U_B + U_A$$

$\boldsymbol{U(\vec{r})}$ is called **the potential energy function.**

$$W = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot \vec{dr} = -U(\vec{r}_B) + U(\vec{r}_A)$$

$$\Rightarrow \quad W = \int_A^B \vec{F} \cdot \vec{dr} = -(U_B - U_A) = -\Delta U$$

- ✓ Work done by the force on the body to move it from point A to B equals the difference of the potential energies in these two points.

- ✓ The **negative sign** provides the convention that work done against a force field increases potential energy ($W < 0, \Delta U > 0$).

  On the other hand, work done by the force field decreases potential energy ($W > 0, \Delta U < 0$).

When a ball is thrown upward:

Work is done against gravity, $W < 0$ ($W = -mgh$, see last class):

→ potential energy increases.

When the ball falls, work is done by gravity, $W > 0$.

→ potential energy decreases.

Mechanical energy

Work-energy theorem:

$$W = K_B - K_A = -U_B + U_A$$

$$\Longrightarrow \quad K_A + U_A = K_B + U_B$$

LHS depends only on the position and velocity at *A*.
RHS depends only on the position and velocity at *B*.

$$\longrightarrow \quad \boxed{K_A + U_A = K_B + U_B = E \text{ (constant)}}$$

E is called the total **mechanical energy** of the particle.

It remains constant i.e. independent of the position of the particle ------> **Energy (mechanical) is conserved**.

Now you see why forces for which work done are path independent, are called **conservative**.

Points to note

$$U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{r}$$

- ✓ Only the **difference in potential energy is defined**, not the potential energy itself. The value of total energy E of any particle is arbitrary to within an additive constant.
- ✓ Conservation of mechanical energy is derived directly from Newton's laws. **It is just a special case of more general law of energy conservation**.

Reserved Slides

Example 1: Uniform force field

Take a particle moving from $\vec{r}_A$ to $\vec{r}_B$. The work done is:

$$W = \int_A^B \vec{F}.\overrightarrow{dr} = F\hat{n}\,.\int_A^B \overrightarrow{dr} = F\,\hat{n}\,.(\vec{r}_A - \vec{r}_B)$$

For a constant force, the work is path independent .

$$\Longrightarrow U_B - U_A = -W$$

In case of **uniform gravitational field**, $\overrightarrow{F} = -mg\hat{k}$.
The change in potential energy is:

$$U_B - U_A = -\int_{z_A}^{z_B} (-mg)\,dz = mg\,(z_B - z_A)$$

Considering $U = 0$ on ground ($z = 0$), $U(h) = mgh$.

A mass is thrown upward with a velocity $v_o = v_{ox}\hat{\imath} + v_{oy}\hat{\jmath} + v_{oz}\hat{k}$. The speed at height can be found easily:

$$K_0 + U_0 = K(h) + U(h)$$

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv(h)^2 + mgh$$

$$v(h) = \sqrt{v_0^2 - 2gh}$$

Same result can be obtained directly from Newton's 2nd law.

$\vec{F} = m\vec{a}$, but need to solve three equations for three components!

The source of the kinetic energy of the particle is the work that has been done in throwing it. When it reach the highest point, it has the ***potential*** to possess kinetic energy, but it does not do so until it is allowed to fall. Hence, we call the energy storage mechanism before the particle is released **potential energy**.

Example 2: Central force

$$\vec{F} = f(r)\hat{r}.$$

Change in potential energy:

$$U_B - U_A = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}.\overrightarrow{dr} = -\int_{r_A}^{r_B} f(r)dr$$

For an **inverse square force**: $f(r) = \dfrac{A}{r^2}$ (gravitational, Coulomb)

$$U_B - U_A = -\int_{r_A}^{r_B} \frac{A}{r^2}dr = \frac{A}{r_B} - \frac{A}{r_A}$$

General potential energy function: $U(r) = \dfrac{A}{r} + \left(U_A - \dfrac{A}{r_A}\right) = \dfrac{A}{r} + \mathrm{C}$

Or, $U(r) = \dfrac{A}{r}$

Taking $C = 0$, which corresponds to $U(\infty) = 0$.

The spring force is also a central force:

$$\vec{F} = -k(r - r_0)\hat{r}$$

$$U(r) - U(r_0) = -\int_{r_0}^{r} (-k)(r - r_0)dr = \frac{1}{2}k(r - r_0)^2$$

$$U(r) = \frac{1}{2}k(r - r_0)^2 + U(r_0)$$
$$= \frac{1}{2}k(r - r_0)^2$$

Choosing potential energy to be zero at equilibrium.