PHY101: Introduction to Physics I

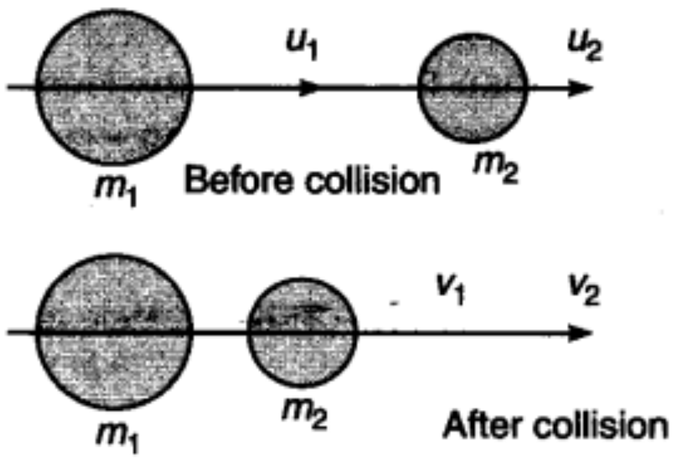
Monsoon Semester 2024

Lecture 22

Department of Physics, School of Natural Sciences,
Shiv Nadar Institution of Eminence, Delhi NCR

Previous Lecture

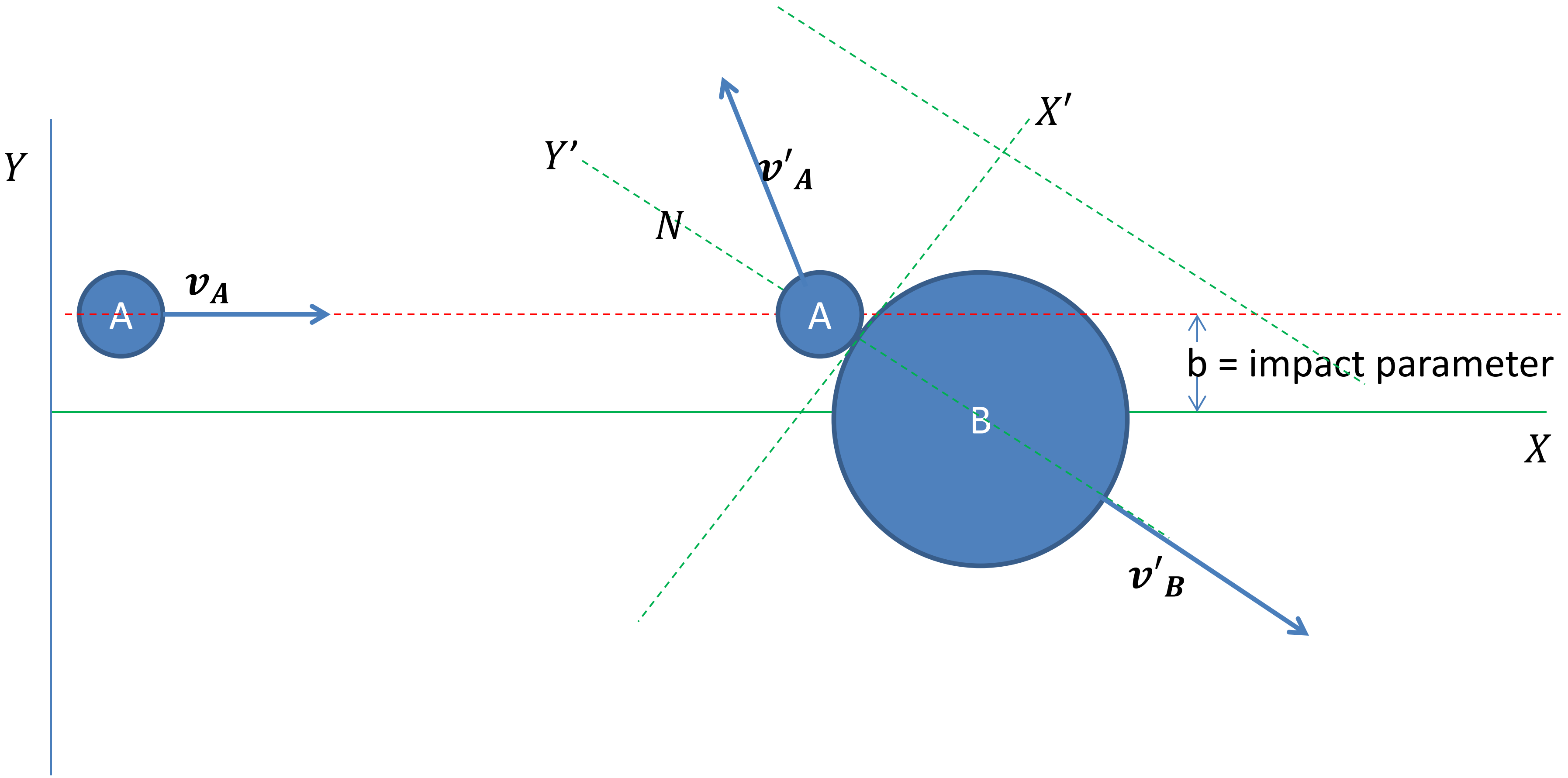
Collison in 1D



This Lecture

Collison in 2D

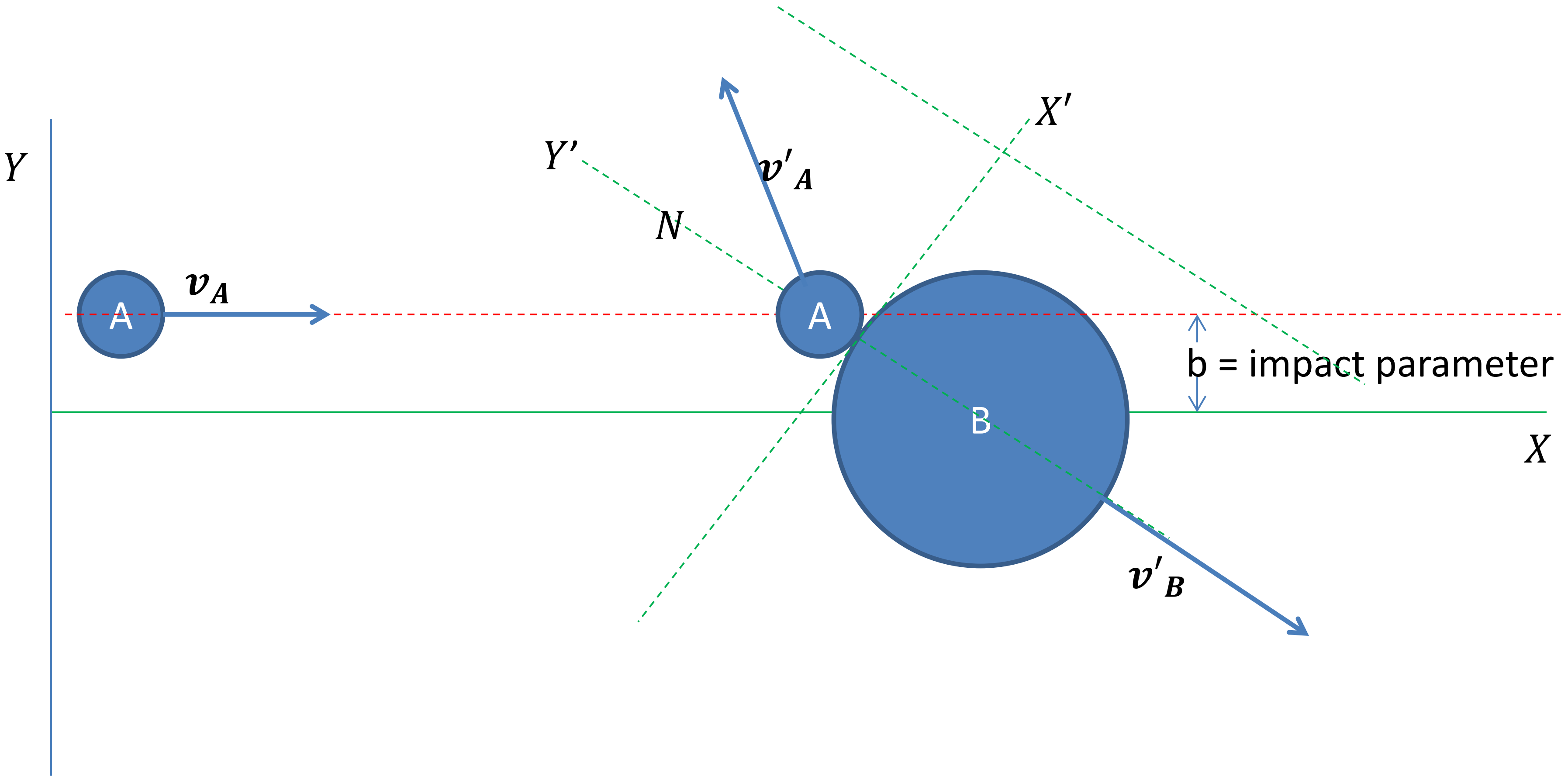
Collision in 2D



As shown in this figure, we assume the Center of Mass of both the balls lie in the XY plane, but the trajectory of the balls do not pass through the center of the balls. Now you see the balls will not confine in the line as it was in the previous case.

Perpendicular distance of the trajectory from the center of mass of the ball is called **Impact Parameter.**

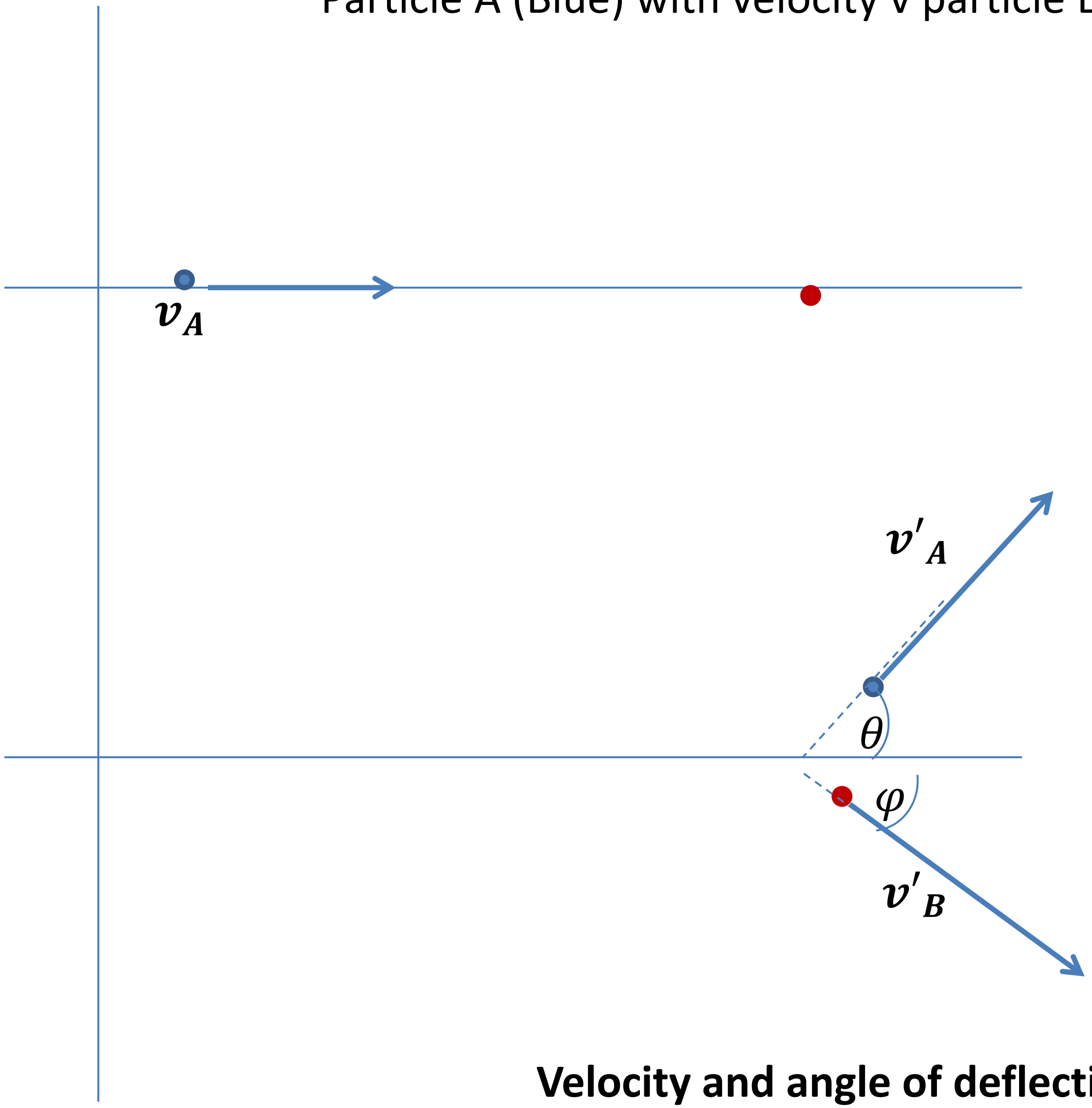
Collision in 2D



When the balls are being treated as the point particles the detail of this impact parameter is not given. In a typical problem, the information of initial velocity and deflection from the previous direction of velocity are given.

Collision in 2D

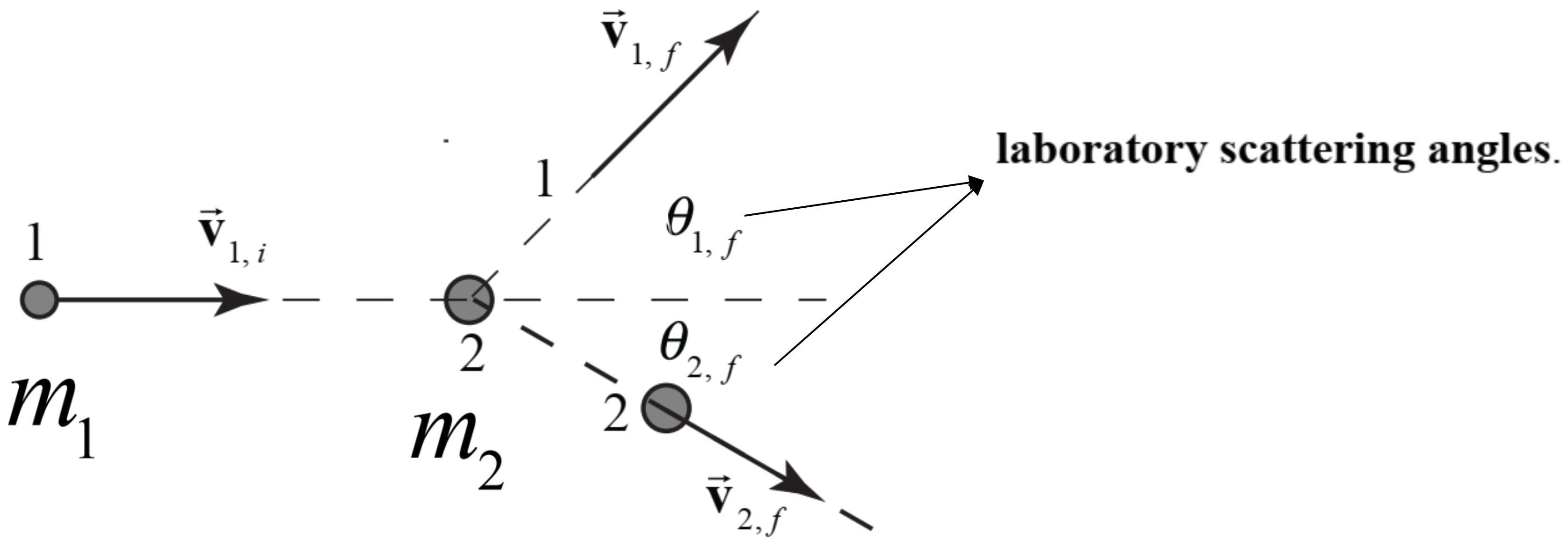
Particle A (Blue) with velocity v particle B(red) stationary



Velocity and angle of deflection are given.

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



For an elastic collision in two dimensions we have three conservation equations

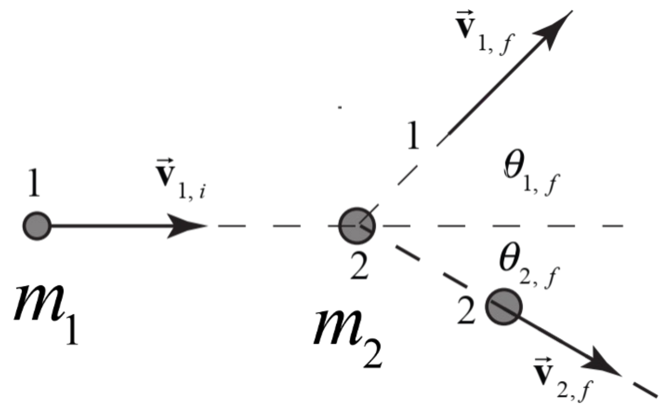
C1. Conservation of x-component of momentum

C2. Conservation of y-componentof momentum

C3. Conservation of kinetic energy

Collision in 2D

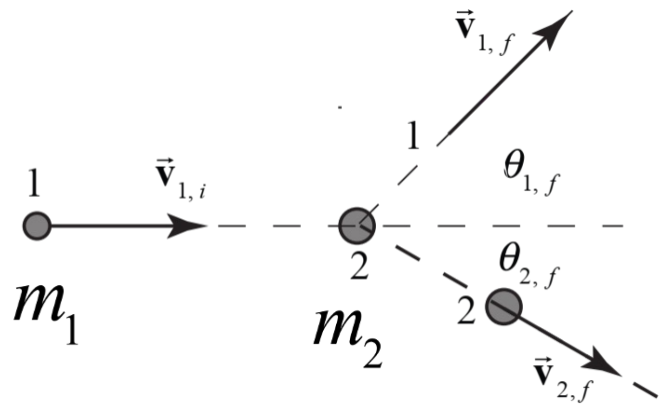
Two-dimensional Elastic Collision in Laboratory Reference Frame



Generally the initial velocity $\vec{\mathbf{v}}_{1,i}$ of particle 1 is known and we would like to determine the final velocities $\vec{\mathbf{v}}_{1,f}$ and $\vec{\mathbf{v}}_{2,f}$, which requires finding the magnitudes and directions of each of these vectors, $v_{1,f}$, $v_{2,f}$, $\theta_{1,f}$, and $\theta_{2,f}$.

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



The components of the total momentum $\vec{\mathbf{p}}_i^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,i} + m_2\vec{\mathbf{v}}_{2,i}$ in the initial state are given by

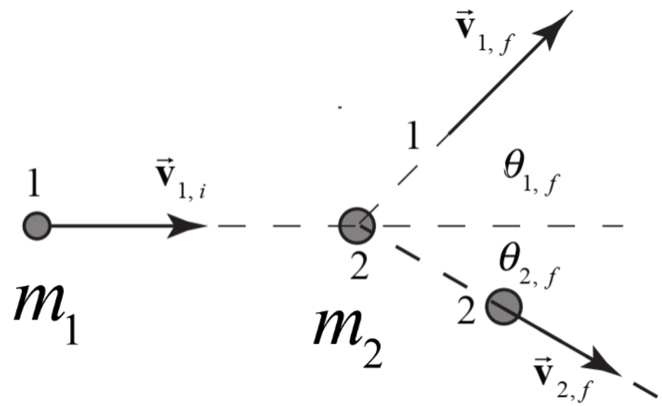
$$p_{x,i}^{\text{sys}} = m_1 v_{1,i}$$
$$p_{y,i}^{\text{sys}} = 0.$$

The components of the momentum $\vec{\mathbf{p}}_f^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f}$ in the final state are given by

$$p_{x,f}^{\text{sys}} = m_1\, v_{1,f} \cos\theta_{1,f} + m_2\, v_{2,f} \cos\theta_{2,f}$$
$$p_{y,f}^{\text{sys}} = m_1\, v_{1,f} \sin\theta_{1,f} - m_2\, v_{2,f} \sin\theta_{2,f}.$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



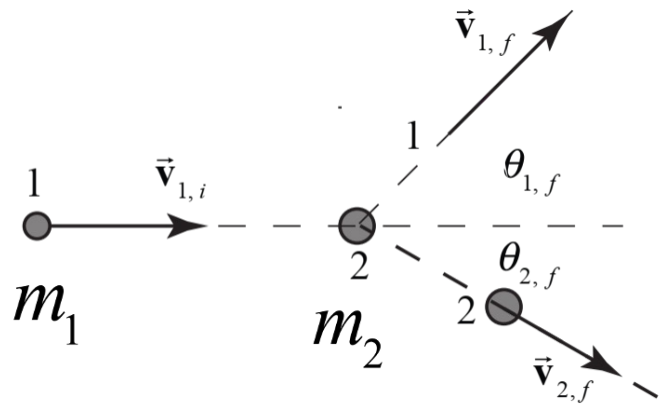
There are no any external forces acting on the system, so each component of the total momentum remains constant during the collision,

$$p_{x,i}^{\text{sys}} = p_{x,f}^{\text{sys}}$$
$$p_{y,i}^{\text{sys}} = p_{y,f}^{\text{sys}} .$$

$$m_1 v_{1,i} = m_1 v_{1,f} \cos\theta_{1,f} + m_2 v_{2,f} \cos\theta_{2,f},$$
$$0 = m_1 v_{1,f} \sin\theta_{1,f} - m_2 v_{2,f} \sin\theta_{2,f} \ .$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



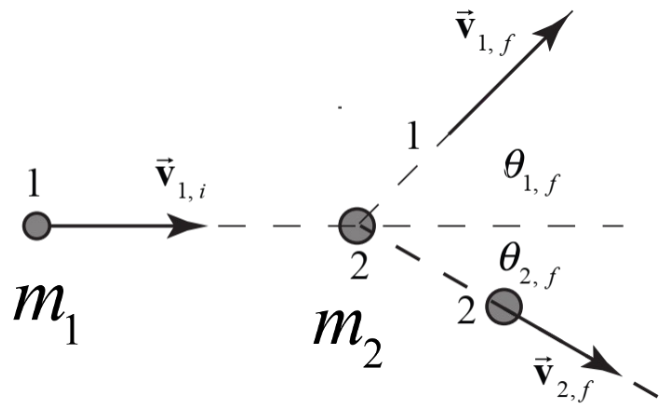
The collision is elastic and therefore the system kinetic energy of is constant

$$K_i^{\text{sys}} = K_f^{\text{sys}}.$$

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



$$m_1 v_{1,i} = m_1 v_{1,f} \cos\theta_{1,f} + m_2 v_{2,f} \cos\theta_{2,f},$$
$$0 = m_1 v_{1,f} \sin\theta_{1,f} - m_2 v_{2,f} \sin\theta_{2,f} \; .$$

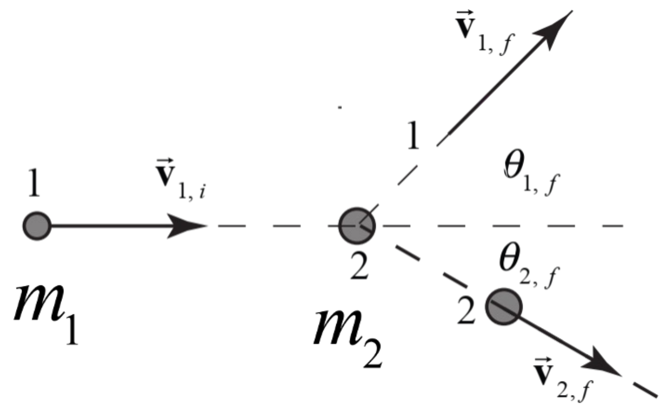
Rewrite the above equation as

$$\boxed{\begin{aligned} m_2 v_{2,f} \cos\theta_{2,f} &= m_1 (v_{1,i} - v_{1,f} \cos\theta_{1,f}) \\ m_2 v_{2,f} \sin\theta_{2,f} &= m_1 v_{1,f} \sin\theta_{1,f} \end{aligned}} \longrightarrow \textbf{(1)}$$

Squaring and adding, we get

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame

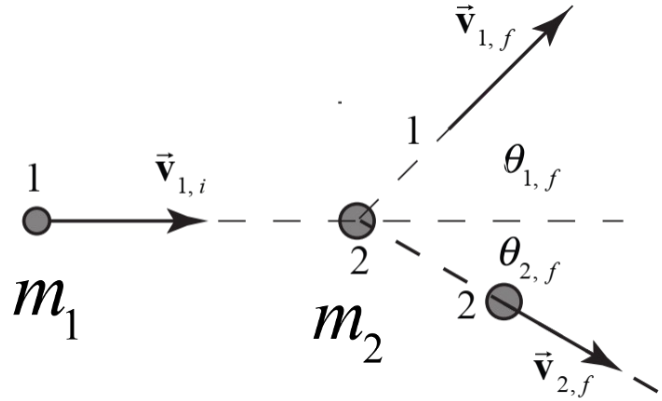


$$\boxed{v_{2,f}^2 = \frac{m_1^2}{m_2^2}(v_{1,i}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2)}$$

Substitute the above relation in the KE equation $\frac{1}{2}m_1v_{1,i}^2 = \frac{1}{2}m_1v_{1,f}^2 + \frac{1}{2}m_2v_{2,f}^2$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



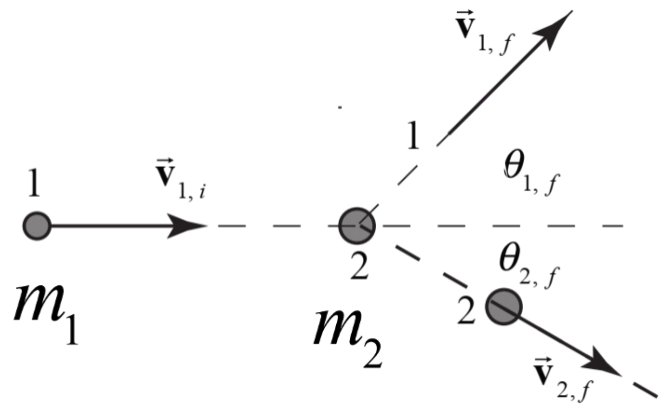
$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}\frac{m_1^2}{m_2}(v_{1,i}^2 - 2v_{1,i}\,v_{1,f}\cos\theta_{1,f} + v_{1,f}^2)$$

$$0 = \left(1 + \frac{m_1}{m_2}\right)v_{1,f}^2 - \frac{m_1}{m_2}2v_{1,i}\,v_{1,f}\cos\theta_{1,f} - \left(1 - \frac{m_1}{m_2}\right)v_{1,i}^2$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



$$0=\left(1+\frac{m_1}{m_2}\right)v_{1,f}^2-\frac{m_1}{m_2}2v_{1,i}\,v_{1,f}\cos\theta_{1,f}-\left(1-\frac{m_1}{m_2}\right)v_{1,i}^2$$

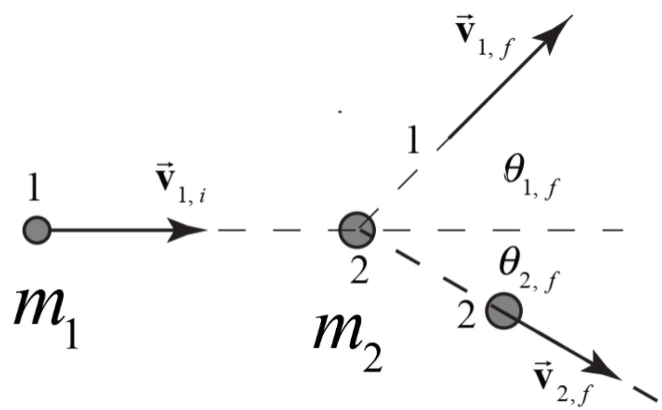
Let $\alpha = m_1 / m_2$

$$0=(1+\alpha)v_{1,f}^2-2\alpha v_{1,i}\,v_{1,f}\cos\theta_{1,f}-(1-\alpha)v_{1,i}^2$$

This is a quadratic equation in $v_{1,f}$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame

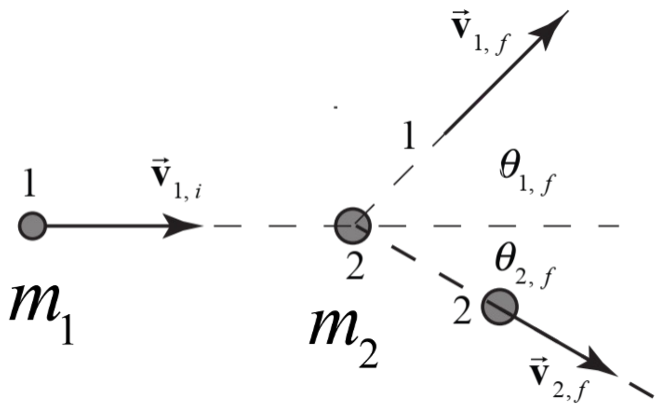


The solution to this quadratic equation is given by

$$v_{1,f} = \frac{\alpha v_{1,i}\cos\theta_{1,f} \pm \left(\alpha^2 v_{1,i}^2 \cos^2\theta_{1,f} + (1-\alpha)v_{1,i}^2\right)^{1/2}}{(1+\alpha)} \longrightarrow \textbf{(2)}$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



From (1)

$$m_2 v_{2,f} \sin\theta_{2,f} = m_1 v_{1,f} \sin\theta_{1,f}$$

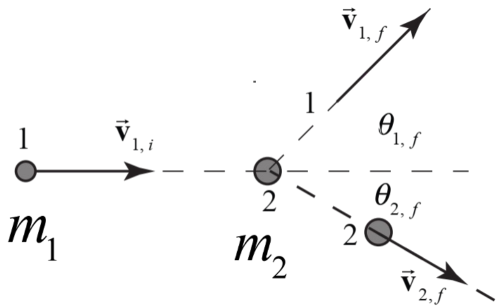
$$m_2 v_{2,f} \cos\theta_{2,f} = m_1 (v_{1,i} - v_{1,f} \cos\theta_{1,f}).$$

Dividing, we get

$$\frac{v_{2,f} \sin\theta_{2,f}}{v_{2,f} \cos\theta_{2,f}} = \frac{v_{1,f} \sin\theta_{1,f}}{v_{1,i} - v_{1,f} \cos\theta_{1,f}}$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



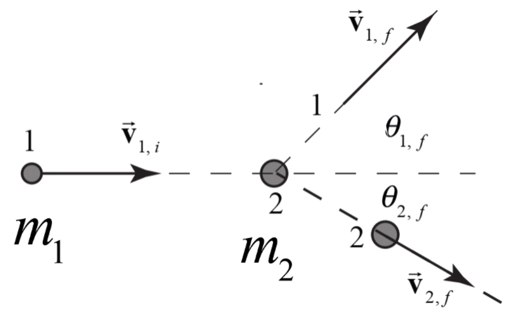
$$\boxed{\tan\theta_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}} \longrightarrow \quad \textbf{(2)}$$

The relationship between the scattering angles in **(2)** is independent of the masses of the colliding particles. Thus the scattering angle for particle 2 is

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right)$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



Now from

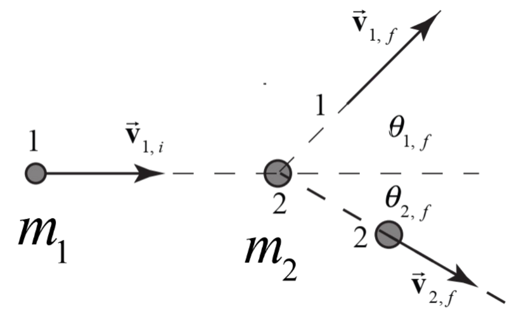
$$m_2 v_{2,f} \sin\theta_{2,f} = m_1 v_{1,f} \sin\theta_{1,f}$$

$$\boxed{v_{2,f} = \frac{\alpha\, v_{1,f} \sin\theta_{1,f}}{\sin\theta_{2,f}}}$$

$$\alpha = m_1 / m_2$$

Collision in 2D

Two-dimensional Elastic Collision in Laboratory Reference Frame



$v_{1,f}$, $v_{2,f}$, $\theta_{1,f}$, and $\theta_{2,f}$

$$\boxed{v_{1,f} = \frac{\alpha v_{1,i}\cos\theta_{1,f} \pm \left(\alpha^2 v_{1,i}^2 \cos^2\theta_{1,f} + (1-\alpha)v_{1,i}^2\right)^{1/2}}{(1+\alpha)}}$$

$$\alpha = m_1 / m_2$$

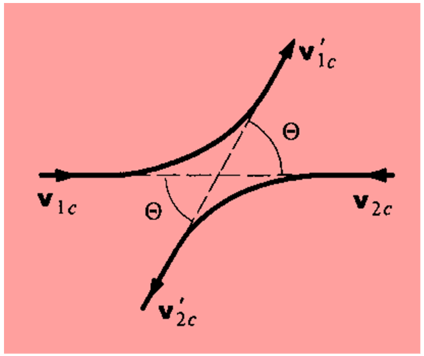
$$\boxed{v_{2,f} = \frac{\alpha\, v_{1,f}\sin\theta_{1,f}}{\sin\theta_{2,f}}}$$

$$\boxed{\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right)}$$

Collision in 2D

Two-dimensional Elastic Collision in the Center of Mass Frame

- The situation in COM system is much simpler.
- The initial and final velocities in COM system determine a plane which is referred to as the **PLANE OF SCATTERING**



- Each particle is deflected through the same scattering angle Θ in this plane.
- Θ is decided by the nature of the interaction between the two particles. If one (Θ or interaction) is known, we can get an idea about the other.

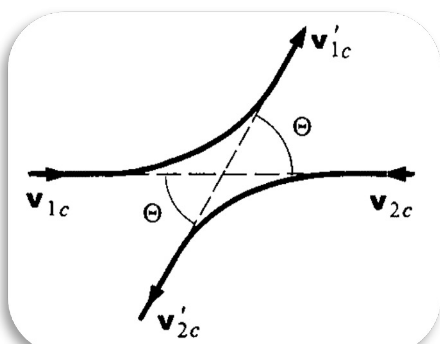
Collision in 2D

Two-dimensional Elastic Collision in the Center of Mass Frame

- For the COM system energy conservation gives

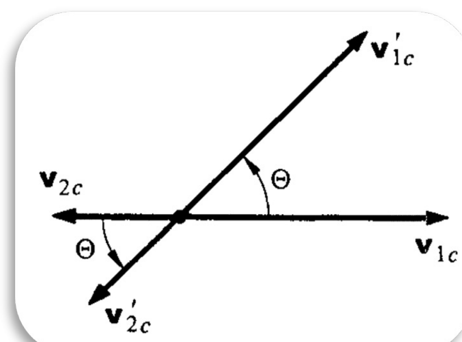
$$\frac{1}{2}m_1v_{1c}^2+\frac{1}{2}m_2v_{2c}^2=\frac{1}{2}m_1v_{1c}'^2+\frac{1}{2}m_2v_{2c}'^2$$

- Total momentum is zero in COM system, thus



$$m_1v_{1c}-m_2v_{2c}=0$$
$$m_1v_{1c}'-m_2v_{2c}'=0$$

Considering opposite direction of velocities.

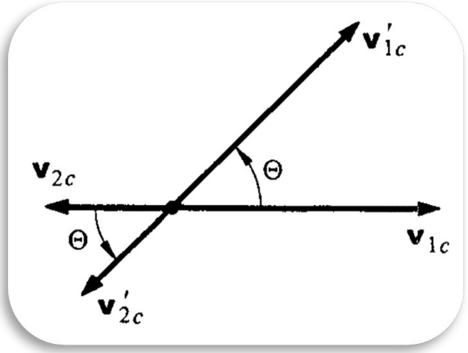


Collision in 2D

Two-dimensional Elastic Collision in the Center of Mass Frame

Eliminating v_{2c} and v'_{2c} from KE equation using the momentum relations we obtain

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1c}^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1c}'^{\,2}$$



which gives $\boldsymbol{v_{1c} = v'_{1c}}$.

Eliminating v_{1c} and v'_{1c} from KE equation using the momentum relations we obtain

$$\frac{1}{2}\left(\frac{m_2^2}{m_1} + m_2\right) v_{2c}^2 = \frac{1}{2}\left(\frac{m_2^2}{m_1} + m_2\right) v_{2c}'^{\,2}$$

which gives $\boldsymbol{v_{2c} = v'_{2c}}$.

Thus in an elastic collision the speed of each particle is same before and after the collision in the COM system: **The velocity vectors simply rotate in the scattering plane by the center-of-mass scattering angle Θ .**