PRACTICE PROBLEMS

PHY 101

Q1. If **r_1**=2i-j+k , **r_2**=i+3j+2k, **r_3**=-2i+j-3k, **r_4**=3i+2j+5k then find scalars such that **r_4**=a**r_1**+b**r_2**+c**r_3**.

We require $$\begin{aligned} 3\mathbf{i}+2\mathbf{j}+5\mathbf{k} &= a(2\mathbf{i}-\mathbf{j}+\mathbf{k}) + b(\mathbf{i}+3\mathbf{j}-2\mathbf{k}) + c(-2\mathbf{i}+\mathbf{j}-3\mathbf{k}) \\ &= (2a+b-2c)\,\mathbf{i} + (-a+3b+c)\,\mathbf{j} + (a-2b-3c)\,\mathbf{k}. \end{aligned}$$

Since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are non-coplanar we have by Problem 15,

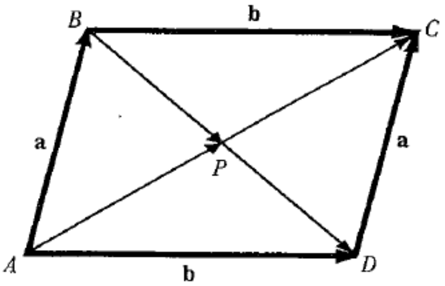
$$2a+b-2c = 3, \qquad -a+3b+c = 2, \qquad a-2b-3c = 5.$$

Solving, $a=-2,\ b=1,\ c=-3$ and $\mathbf{r}_4 = -2\mathbf{r}_1 + \mathbf{r}_2 - 3\mathbf{r}_3$.

The vector $\mathbf{r}_4$ is said to be *linearly dependent* on $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; in other words $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ constitute a *linearly dependent* set of vectors. On the other hand any three (or fewer) of these vectors are *linearly independent.*

In general the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ are called linearly dependent if we can find a set of scalars, $a, b, c, \ldots,$ not all zero, so that $a\mathbf{A}+b\mathbf{B}+c\mathbf{C}+\ldots = \mathbf{0}$, otherwise they are linearly independent.

Q2. Prove that diagonals of a parallelogram bisect each other using properties of vectors.



Let $ABCD$ be the given parallelogram with diagonals intersecting at P.

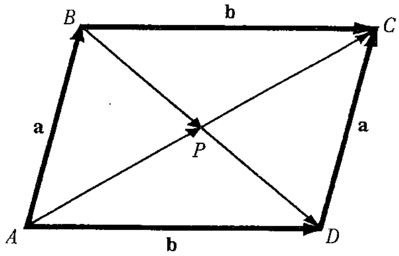
Since $\mathbf{BD}+\mathbf{a} = \mathbf{b}$, $\mathbf{BD} = \mathbf{b}-\mathbf{a}$. Then $\mathbf{BP} = x(\mathbf{b}-\mathbf{a})$.

Since $\mathbf{AC} = \mathbf{a}+\mathbf{b}$, $\mathbf{AP} = y(\mathbf{a}+\mathbf{b})$.

But $\mathbf{AB} = \mathbf{AP} + \mathbf{PB} = \mathbf{AP} - \mathbf{BP}$,

i.e. $\mathbf{a} = y(\mathbf{a}+\mathbf{b}) - x(\mathbf{b}-\mathbf{a}) = (x+y)\mathbf{a} + (y-x)\mathbf{b}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are non-collinear we have by Problem 13, $x+y = 1$ and $y-x = 0$, i.e. $x = y = \frac{1}{2}$ and P is the mid-point of both diagonals.



Q3. Q3. Given a scalar field defined by $\varphi(x,y,z) = 3x^2z - xy^3 + 5$. Find φ at the points

(a) (0,0,0) , (1,-2,2) , (-1,-2,-3)

(b) If $\vec{V}(x,y,z) = \vec{\nabla}\varphi(x,y,z) = \frac{\delta}{\delta x}(\varphi)\hat{\imath} + \frac{\delta}{\delta y}(\varphi)\hat{\jmath} + \frac{\delta}{\delta z}(\varphi)\hat{k}$,where $\frac{\delta}{\delta a}$ is the partial derivative wrt the variable a, then find the value of $\vec{V}(x,y,z)$ at the same points given in (a)

$(a)\ (0,0,0),\quad (b)\ (1,-2,2)\quad (c)\ (-1,-2,-3).$

$$(a)\quad \phi(0,0,0) = 3(0)^2(0) - (0)(0)^3 + 5 = 0 - 0 + 5 = 5$$

$$(b)\quad \phi(1,-2,2) = 3(1)^2(2) - (1)(-2)^3 + 5 = 6 + 8 + 5 = 19$$

$$(c)\quad \phi(-1,-2,-3) = 3(-1)^2(-3) - (-1)(-2)^3 + 5 = -9 - 8 + 5 = -12$$

(b) $\vec{\nabla}\varphi(x,y,z) = (6xz - y^3)\hat{\imath} - 3y^2x\hat{\jmath}$

$$\vec{\nabla}\varphi(0,0,0) = 0$$

$$\vec{\nabla}\varphi(1,-2,2) = 4\hat{\imath} - 12\hat{\jmath}$$

$$\vec{\nabla}\varphi(-1,-2,-3) = 26\hat{\imath} + 12\hat{\jmath}$$

Q4. Find the angle vector $\vec{A} = 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$ makes with the coordinate axes.

Let α, β, γ be the angles which $\mathbf{A}$ makes with the positive x, y, z axes respectively.

$$\mathbf{A}\cdot\mathbf{i} = (A)(1)\cos\alpha = \sqrt{(3)^2+(-6)^2+(2)^2}\ \cos\alpha = 7\cos\alpha$$

$$\mathbf{A}\cdot\mathbf{i} = (3\mathbf{i}-6\mathbf{j}+2\mathbf{k})\cdot\mathbf{i} = 3\mathbf{i}\cdot\mathbf{i} - 6\mathbf{j}\cdot\mathbf{i} + 2\mathbf{k}\cdot\mathbf{i} = 3$$

Then $\cos\alpha = 3/7 = 0.4286$, and $\alpha = 64.6^\circ$ approximately.

Similarly, $\cos\beta = -6/7$, $\beta = 149^\circ$ and $\cos\gamma = 2/7$, $\gamma = 73.4^\circ$.

The cosines of α, β, and γ are called the *direction cosines* of $\mathbf{A}$. (See Prob. 27, Chap. 1).

Q5.

If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, find (a) $\mathbf{A}\times\mathbf{B}$, (b) $\mathbf{B}\times\mathbf{A}$, (c) $(\mathbf{A}+\mathbf{B})\times(\mathbf{A}-\mathbf{B})$.

Find the unit vector in the direction of each vectors in (a),(b)and (c)

If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, find (a) $\mathbf{A}\times\mathbf{B}$, (b) $\mathbf{B}\times\mathbf{A}$, (c) $(\mathbf{A}+\mathbf{B})\times(\mathbf{A}-\mathbf{B})$.

$$(a)\quad \mathbf{A}\times\mathbf{B} = (2\mathbf{i}-3\mathbf{j}-\mathbf{k})\times(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -3 & -1 \\ 4 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$$

Another Method.

$$(2\mathbf{i}-3\mathbf{j}-\mathbf{k})\times(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = 2\mathbf{i}\times(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) - 3\mathbf{j}\times(\mathbf{i}+4\mathbf{j}-2\mathbf{k}) - \mathbf{k}\times(\mathbf{i}+4\mathbf{j}-2\mathbf{k})$$

$$= 2\mathbf{i}\times\mathbf{i} + 8\mathbf{i}\times\mathbf{j} - 4\mathbf{i}\times\mathbf{k} - 3\mathbf{j}\times\mathbf{i} - 12\mathbf{j}\times\mathbf{j} + 6\mathbf{j}\times\mathbf{k} - \mathbf{k}\times\mathbf{i} - 4\mathbf{k}\times\mathbf{j} + 2\mathbf{k}\times\mathbf{k}$$

$$= \mathbf{0} + 8\mathbf{k} + 4\mathbf{j} + 3\mathbf{k} - \mathbf{0} + 6\mathbf{i} - \mathbf{j} + 4\mathbf{i} + \mathbf{0} = 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$$

(b) $\mathbf{B}\times\mathbf{A} = (\mathbf{i}+4\mathbf{j}-2\mathbf{k})\times(2\mathbf{i}-3\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -2 \\ 2 & -3 & -1 \end{vmatrix}$

$$= \mathbf{i}\begin{vmatrix} 4 & -2 \\ -3 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -10\mathbf{i} - 3\mathbf{j} - 11\mathbf{k}.$$

Comparing with (a), $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$. Note that this is equivalent to the theorem: If two rows c a determinant are interchanged, the determinant changes sign.

(c) $\mathbf{A}+\mathbf{B} = (2\mathbf{i}-3\mathbf{j}-\mathbf{k}) + (\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = 3\mathbf{i}+\mathbf{j}-3\mathbf{k}$

$\mathbf{A}-\mathbf{B} = (2\mathbf{i}-3\mathbf{j}-\mathbf{k}) - (\mathbf{i}+4\mathbf{j}-2\mathbf{k}) = \mathbf{i}-7\mathbf{j}+\mathbf{k}$

Then $(\mathbf{A}+\mathbf{B})\times(\mathbf{A}-\mathbf{B}) = (3\mathbf{i}+\mathbf{j}-3\mathbf{k})\times(\mathbf{i}-7\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -3 \\ 1 & -7 & 1 \end{vmatrix}$

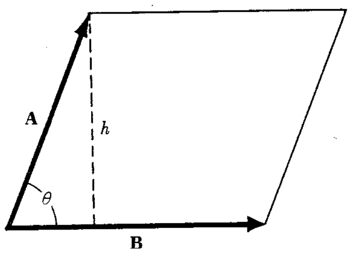
$$= \mathbf{i}\begin{vmatrix} 1 & -3 \\ -7 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ 1 & -7 \end{vmatrix} = -20\mathbf{i} - 6\mathbf{j} - 22\mathbf{k}.$$

Another Method.

$$(\mathbf{A}+\mathbf{B})\times(\mathbf{A}-\mathbf{B}) = \mathbf{A}\times(\mathbf{A}-\mathbf{B}) + \mathbf{B}\times(\mathbf{A}-\mathbf{B})$$
$$= \mathbf{A}\times\mathbf{A} - \mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{A} - \mathbf{B}\times\mathbf{B} = \mathbf{0} - \mathbf{A}\times\mathbf{B} - \mathbf{A}\times\mathbf{B} - \mathbf{0} = -2\mathbf{A}\times\mathbf{B}$$
$$= -2(10\mathbf{i}+3\mathbf{j}+11\mathbf{k}) = -20\mathbf{i} - 6\mathbf{j} - 22\mathbf{k}, \quad \text{using } (a).$$

Q6. (a)Prove that area of a parallelogram of sides $\vec{A}$ and $\vec{B}$ is $|\vec{A}\times\vec{B}|$

(b)Similarly Prove that area of a triangle of sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}|\vec{A}\times\vec{B}|$



(c) Find the area of the triangle having vertices at $P(1,3,2)$, $Q(2,-1,1)$, $R(-1,2,3)$.

Solution

$$\text{Area of parallelogram} = h\,|\mathbf{B}| = |\mathbf{A}|\sin\theta\,|\mathbf{B}| = |\mathbf{A}\times\mathbf{B}|.$$

Note that the area of the triangle with sides **A** and **B** $= \frac{1}{2}|\mathbf{A}\times\mathbf{B}|$.

(c)

$\mathbf{PQ} = (2-1)\mathbf{i} + (-1-3)\mathbf{j} + (1-2)\mathbf{k} = \mathbf{i} - 4\mathbf{j} - \mathbf{k}$

$\mathbf{PR} = (-1-1)\mathbf{i} + (2-3)\mathbf{j} + (3-2)\mathbf{k} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\text{area of triangle} = \tfrac{1}{2}\left|\mathbf{PQ}\times\mathbf{PR}\right| = \tfrac{1}{2}\left|(\mathbf{i}-4\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}-\mathbf{j}+\mathbf{k})\right|$$

$$= \tfrac{1}{2}\left|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -4 & -1\\ -2 & -1 & 1\end{vmatrix}\right| = \tfrac{1}{2}\left|-5\mathbf{i}+\mathbf{j}-9\mathbf{k}\right| = \tfrac{1}{2}\sqrt{(-5)^2+(1)^2+(-9)^2} = \tfrac{1}{2}\sqrt{107}.$$

Q7. Use the method of dimensions to obtain the form of the dependance of the lift force per unit wingspan on an aircraft wing of width (in the direction of motion) L, moving with velocity v through the air density ρ, on the parameters L, v, ρ.

Let us call the lift per unit wingspan Φ, and write

$$\Phi = kL^\alpha v^\beta \rho^\gamma,$$

where k, α, β and γ are dimensionless constants. Since the dimensions of force are $\mathrm{MLT^{-2}}$, the dimensions of Φ are $\mathrm{MT^{-2}}$. Thus

$$\mathrm{MT^{-2} = L^\alpha L^\beta T^{-\beta} M^\gamma L^{-3\gamma}}.$$

So by equating the terms in M, $\gamma = 1$.
By equating the terms in T, $-\beta = -2$ so $\beta = 2$.
By equating the terms in L, $\alpha + \beta - 3\gamma = 0$, therefore $\alpha = 1$.
Thus we may write

$$\Phi = kLv^2\rho.$$

Q8. Speed of waves v on a string depend on its mass m, length l, force τ by the equation

$$v = m^a l^b \tau^c$$

Find the values of a,b and c using dimensional analysis. Write down the final form of the equation.

Solution:

(Q2) Speed of waves v on a string ... m, length l and tension (Force) τ by the eqn.

$$v = m^a l^b \tau^c$$

Find the values of a, b and c using dimensional analysis. Write down the final form of the equation for v.

$[v]$ velocity $= \frac{\text{distance}}{\text{Time}} = \frac{[L]}{[T]} = [L][T]^{-1}$

$[\tau]$ Force $=$ Mass $\times$ acceleration

$\Rightarrow \quad = [M][L][T]^{-2}$

$[l] = [L]$

$[m] = [M]$

equating both sides of the equation

$$[L][T]^{-1} = [M]^a [L]^b \left([M][L][T]^{-2}\right)^c$$

$$= [M]^a [L]^b [M]^c [L]^c [T]^{-2c}$$

$$= [M]^{a+b} [L]^{b+c} [T]^{-2c}$$

We get the equations

$\Rightarrow$ $a + b = 0$ — (1)

$b + c = 1$ — (2)

$-2c = -1 \Rightarrow \boxed{c = \frac{1}{2}}$ — (3)

using (3) in (2)

$\Rightarrow b + \frac{1}{2} = 1$

$\Rightarrow \boxed{b = \frac{1}{2}}$ — (4)

using (4) and (1)

$a + \frac{1}{2} = 0$

$\Rightarrow \boxed{a = -\frac{1}{2}}$ — (5)

$\Rightarrow \boxed{v = m^{-1/2} l^{1/2} \tau^{1/2}} \Rightarrow$ Final equation.

$\Rightarrow \boxed{v = \sqrt{\frac{l\tau}{m}}}$

$x = r\cos\theta,\ y = r\sin\theta$

$r = 8 + 8\cos\theta$

$\sqrt{x^2+y^2} = 8\left(1 + \frac{x}{r}\right) \Rightarrow r = \frac{8(r+x)}{r}$

$\Rightarrow r^2 = 8(r + x)$

$\Rightarrow x^2 + y^2 = 8\left(\sqrt{x^2+y^2} + x\right)$

Q9. Convert (-1, -1) into polar coordinates.

Let us first get r

$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$

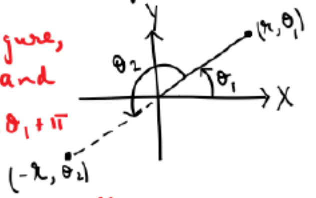
Now let's get θ,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{-1}\right)$$

$$= \tan^{-1}\tan(\pi/4) = \pi/4$$

But this value of θ is not the correct answer as it belongs to the first quadrant whereas the point in question $(-1,-1)$ corresponds to the third quadrant.

From adjacent figure, we know (r, θ_1) and $(-r, \theta_2)$ where $\theta_2 = \theta_1 + \pi$ represent same point in a polar graph.



So, correct value of θ will be $\pi/4 + \pi = \frac{5\pi}{4}$ which lies in third quadrant

Answer $\left(\sqrt{2}, \frac{5\pi}{4}\right)$

Answer: Magnitude of vector A = Magnitude of vector B = 10 units

Angle between vector A and the positive x-axis = 30^0
Angle between vector B and the positive y-axis = 45^0

First, let's represent vectors A and B in component form:
Vector A (A_x, A_y):
A_x = Magnitude of A * cos (angle with x-axis)
$A_x = 10 * \cos(30^0) = 10 * \sqrt{3}/2 = 5\sqrt{3}$
A_y = Magnitude of A * sin (angle with x-axis)
$A_y = 10 * \sin(30^0) = 10 * 1/2 = 5$
So, vector A can be represented as $A = 5\sqrt{3}i + 5j$
Vector B (B_x, B_y):
B_x = Magnitude of B * cos (angle with y-axis)
$B_x = 10 * \cos(45^0) = 10 * 1/\sqrt{2} = 5\sqrt{2}$
B_y = Magnitude of B * sin (angle with y-axis)
$B_y = 10 * \sin(45^0) = 10 * 1/\sqrt{2} = 5\sqrt{2}$
So, vector B can be represented as $B = 5\sqrt{2}i + 5\sqrt{2}j$

Now, let's calculate the dot product (A · B):
A · B = (5√3i + 5j) · (5√2i + 5√2j)
Using the dot product formula, A · B = (A_x * B_x) + (A_y * B_y):
A · B = (5√3 * 5√2) + (5 * 5√2)
A · B = (25√6) + (25√2)
A · B = 25(√6 + √2) units

Now, let's calculate the cross-product (A × B):
The cross product of two vectors in 2D is always a scalar, and its magnitude can be calculated as:
|A × B| = |A| * |B| * sin(θ)
Where θ is the angle between vectors A and B, which is 90 degrees in this case because they are perpendicular. Also, |A| = 10 and |B| = 10.
|A × B| = 10 * 10 * sin (90^0) = 100 * 1 = 100 units

Q.11 A particle sliding along a radial groove in a rotating turntable has polar coordinates at time t given by r = ct, θ = Ωt, where c and Ω are positive constants. Find the velocity and acceleration vectors of the particle at time t and find the speed of the particle at time t. Deduce that, for t > 0, the angle between the velocity and acceleration vectors is always acute.

If a particle is moving in a plane and has polar coordinates r, θ at time t, then its velocity and acceleration vectors are given by

$$\boldsymbol{v} = \dot{r}\widehat{\boldsymbol{r}} + (r\dot{\theta})\widehat{\boldsymbol{\theta}},$$
$$\boldsymbol{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\widehat{\boldsymbol{\theta}}.$$

In this problem,

$$r = ct, \quad \dot{r} = c, \quad \ddot{r} = 0,$$

and

$$\theta = \Omega t, \quad \dot{\theta} = \Omega, \quad \ddot{\theta} = 0.$$

Using these in the expressions for the velocity and acceleration, we get

$$\boldsymbol{v} = c\widehat{\boldsymbol{r}} + (ct)\Omega\widehat{\boldsymbol{\theta}} = c\left(\widehat{\boldsymbol{r}} + \Omega t\,\widehat{\boldsymbol{\theta}}\right)$$

and

$$\boldsymbol{a} = \left(0 - (ct)\Omega^2\right)\widehat{\boldsymbol{r}} + (0 + 2c\Omega)\,\widehat{\boldsymbol{\theta}} = c\Omega\left(-\Omega t\,\widehat{\boldsymbol{r}} + 2\widehat{\boldsymbol{\theta}}\right).$$

The *speed* of the particle at time t is thus given by $|\boldsymbol{v}| = c\left(1 + \Omega^2 t^2\right)^{1/2}$

To find the angle between $\boldsymbol{v}$ and $\boldsymbol{a}$, consider

$$\begin{aligned}\boldsymbol{v} \cdot \boldsymbol{a} &= c^2\Omega(-\Omega t + 2\Omega t) = c^2\Omega^2 t \\ &> 0\end{aligned}$$

for $t > 0$. Hence, for $t > 0$, the angle between $\boldsymbol{v}$ and $\boldsymbol{a}$ is acute. ■

Q.12 A light rope fixed at one end of a wooden clamp on the ground passes over a tree branch and hangs on the other side. It makes an angle of 30° with the ground. A man weighing (60 kg) wants to climb up the rope. The wooden clamp can come out of the ground if an upward force greater than 360 N is applied to it. Find the maximum acceleration in the upward direction with which the man can climb safely. Neglect friction at the tree branch. Take g = 10 m/s^2.

Ans: Let T be the tension in the rope.

The upward force on the clamp is T sin 30° = T/2.

The maximum tension that will not detach the clamp from the ground is, therefore, given by

T/2 = 360 N

Or, T = 720 N.

If the acceleration of the man in the upward direction is a, the equation of motion of the man is

T– 600 N= (60 kg) a

The maximum acceleration of the man for safe climbing is, therefore

a = (720 N – 600 N)/60kg = 2m/s2

Q.13 A cricketer can throw a ball to a maximum horizontal distance of 100 m. How high above the ground can the cricketer throw the same ball?

Ans:

Maximum horizontal distance, R=100m

The cricketer will only be able to throw the ball to the maximum horizontal distance when the angle of projection 45 degree

The maximum horizontal range for a projection velocity v is given by the relation:

$R_{max}=u^2/g$

$100=u^2/g$

The ball will achieve the maximum height when it is thrown vertically upward. For such motion, the final velocity v is zero at the maximum height H.

Acceleration, a= -g

Using the third equation of motion:

$$v^2 - u^2 = 2aH$$

$$v^2 - u^2 = -2gH$$

$$u^2 = 2gH$$

$$H = \frac{u^2}{2g}$$

$$H = \frac{100}{2} = 50m$$

Q14 Consider an automobile moving along a straight horizontal road with a speed v_0. If the coefficient of static friction between the tires and the road is μ_s, what is the shortest distance in which the automobile can be stopped?

Soln:

Q1. If we assume the motion is along the positive x-direction and the frictional force f_s is constant, we have uniformly decelerated motion of the automobile of weight W.

Using final speed $v = 0$ in $v^2 = {v_0}^2 + 2ax$

We obtain $x = -{v_0}^2/2a$ ....(1)

where negative sign indicates that a points in the negative x-direction.

Applying 2nd law of motion to the x-component of the motion

$-f_s = ma = (W/g)a$

That gives $a = -g(f_s/W)$ ....(2)

From the y-components we obtain

$N - W = 0 \quad \rightarrow N = W$

Hence, $\mu_s = \frac{f_s}{N} = f_s/W$ and thus from Eq. 2 we found

$$a = -\mu_s g$$

Then from Eq. (1) we can find the distance of stopping which is

$$x = -\frac{{v_0}^2}{2a} = \frac{{v_0}^2}{2g\mu_s}$$

Q.15 A small body was launched up an inclined plane set at an angle θ= 150 against the horizontal. Find the coefficient of friction if the time of the ascent of the body is β = 2 times less than the time of its descent.

Ans:

Q3. **Case 1:** When the body is launched up:

Let k be the coefficient of friction and s is the distance traversed along the incline at time t_1.

Retarding force on the block is

$$mg \sin \alpha + k\, mg \cos \alpha$$

Hence, the retardation is

$$a_1 = g(\sin \alpha + k \cos \alpha)$$

The kinematic equation along the incline

$S = ½\, a_1 t_1^2 \ \rightarrow \ t_1 = \sqrt{\frac{2s}{a_1}}$ ....(1)

Case 2: When the body comes down:

The net force on the body is

$$mg \sin \alpha - k\, mg \cos \alpha$$

Hence, its acceleration is

$$a_2 = g(\sin \alpha - k \cos \alpha)$$

Let, t_2 is the time required to traverse distance s along the incline

The kinematic equation along the incline

$S = ½\, a_2 t_2^2 \ \rightarrow \ t_2 = \sqrt{\frac{2s}{a_2}}$ ....(2)

From Eqs. (1) and (2)

$$\frac{t_1}{t_2} = \sqrt{\frac{a_2}{a_1}} \quad \rightarrow \quad \left(\frac{t_1}{t_2}\right)^2 = \frac{a_2}{a_1}$$

$$\left(\frac{1}{\beta}\right)^2 = \frac{a_2}{a_1} = \frac{g(\sin\alpha - k\cos\alpha)}{g(\sin\alpha + k\cos\alpha)} = \frac{(\sin\alpha - k\cos\alpha)}{(\sin\alpha + k\cos\alpha)}$$

$$\sin\alpha + k\cos\alpha = \beta^2 \sin\alpha - \beta^2 \cos\alpha$$

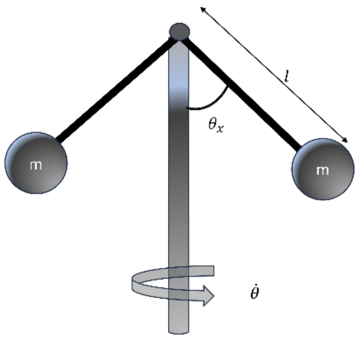
$$(\beta^2 + 1)k\cos\alpha = (\beta^2 - 1)\sin\alpha$$

$$k = \frac{(\beta^2 - 1)}{(\beta^2 + 1)} \tan\alpha$$

Since $\alpha = 15^0$ and $\beta = 2$

$$k = 0.16$$

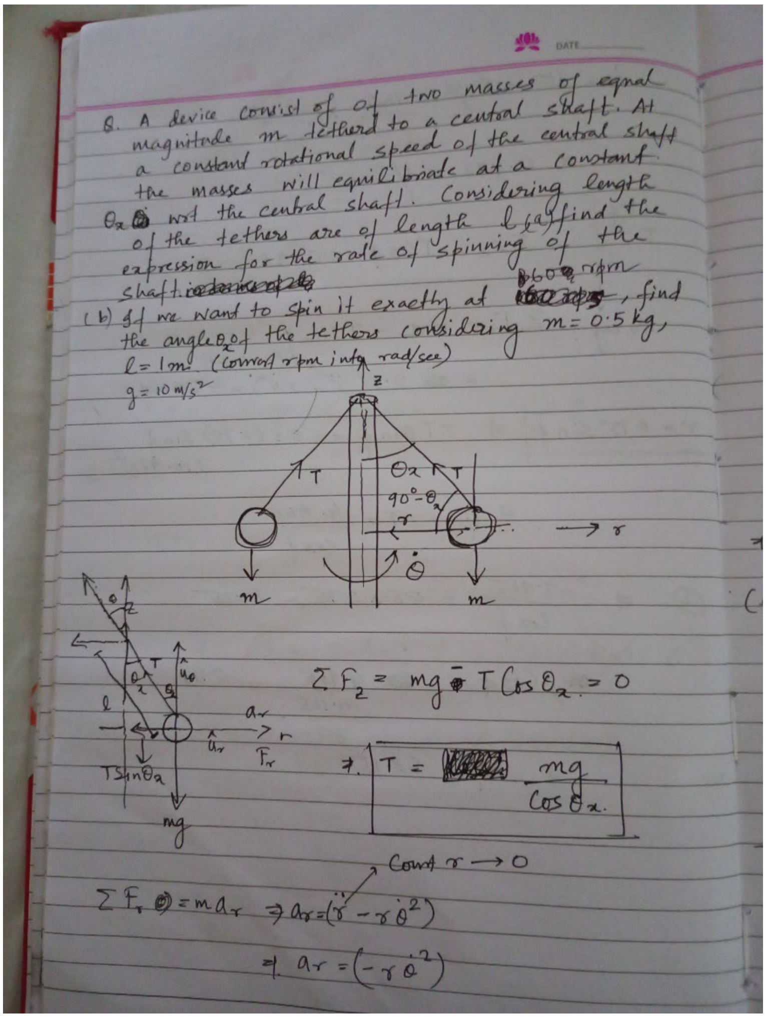
Q.16 **A device consists of mass of equal magnitude m tethered to a central shaft as shown in the figure. At a constant rotational speed of the central shaft the masses will be at a constant angle θ_x wrt to the central shaft. Considering length of the tethers are l and acceleration due to gravity g.**



(a) **Rate of spinning of the shaft is** $\omega = \dot{\theta} = \sqrt{\frac{g}{l\, Cos\theta_x}}$

(b) **If we want to spin it exactly at 60 rpm, what will be the angle** θ_x **if m=0.5kg and** l**=1m**

Solution:



DATE

Q. A device consist of of two masses of equal magnitude m tetherd to a central shaft. At a constant rotational speed of the central shaft the masses will equilibriate at a constant θ_x wrt the central shaft. Considering length of the tethers are of length l (a) find the expression for the rate of spinning of the shaft.

(b) If we want to spin it exactly at 60 rpm, find the angle θ_x of the tethers considering $m = 0.5$ kg, $l = 1$ m. (convert rpm into rad/sec)

$g = 10\ m/s^2$

$$\Sigma F_z = mg - T\cos\theta_x = 0$$

$$\Rightarrow \boxed{T = \frac{mg}{\cos\theta_x}}$$

Const $r \rightarrow 0$

$$\Sigma F_r = m a_r \Rightarrow a_r = (\ddot{r} - r\dot{\theta}^2)$$

$$\Rightarrow a_r = (-r\dot{\theta}^2)$$

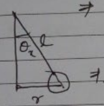
T goes opposite to $\vec{F}_r$

$\Rightarrow -T\sin\theta_x = m\,a_r$

$\Rightarrow -\frac{\cancel{m}g\sin\theta_x}{\cos\theta_x} = \cancel{m}\times(-r\dot{\theta}^2)$

$\Rightarrow \cancel{+}g\tan\theta_x = \cancel{+}r\dot{\theta}^2$

$\Rightarrow g\frac{\sin\theta_x}{\cos\theta_x} = l\,\cancel{\sin\theta_x}\,\dot{\theta}^2$



$\Rightarrow \dot{\theta}^2 = \frac{g}{l\cos\theta_x}$

$\frac{r}{l} = \sin\theta_x$ $\Rightarrow$ $\boxed{\dot{\theta} = \sqrt{\frac{g}{l\cos\theta_x}}}$

$\Rightarrow r = l\sin\theta_x$

(b) ~~3600 rpm = 3600 Revolution / 60x60 sec~~ $1\,rpm = \frac{1\ revolution}{1\ min\ (60\,sec)}$

~~⇒ 1 Rev/sec = 2π rad/sec~~

~~⇒ 3600 rpm = 3600×2π rad / 60×60 sec~~

$\dot{\theta} = 2\pi$ rad/sec.

We need to find θ_x

~~$\Rightarrow 2\pi = \sqrt{\frac{g}{l\cos\theta_x}}$~~

(b) $1\,rpm = \frac{1\ revolution}{1\ min}$ $\frac{360}{2\pi}$

$\Rightarrow 1\,rpm = \frac{1\ revolution}{60\ sec.}$

$\Rightarrow 60\,rpm = \frac{60\ revolution}{60\ sec.}$

$\Rightarrow 60\,rpm = 1$ revolution/sec.

1 revolution $\cong 2\pi$ rad.

$\Rightarrow 60\,rpm = 2\pi$ rad/sec $= \dot{\theta}$

We need to find θ_x

$\Rightarrow 2\pi = \sqrt{\frac{g}{l\cos\theta_x}}$

$\Rightarrow \left(\frac{4\pi^2 l}{g}\right) = \frac{1}{\cos\theta_x}$

$\Rightarrow \cos\theta_x = \frac{g}{4\pi^2 l}$

$\Rightarrow \theta_x = \cos^{-1}\left[\frac{10}{4\pi^2\times 1}\right]$

$\Rightarrow \theta_x = 1.314$ rad $[\because 2\pi\ rad = 360^\circ]$
$= 75.35^\circ$

Q.17 Considering the identities:

$$(a)\quad \frac{d}{du}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\cdot\mathbf{B}, \qquad (b)\quad \frac{d}{du}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du}\times\mathbf{B}$$

If $\mathbf{A} = 5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ and $\mathbf{B} = \sin t\ \mathbf{i} - \cos t\ \mathbf{j}$, find (a) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$, (b) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B})$, (c) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A})$.

Sol:

$$(a)\quad \frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{B}$$

$$= (5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k})\cdot(\cos t\ \mathbf{i} + \sin t\ \mathbf{j}) + (10t\,\mathbf{i} + \mathbf{j} - 3t^2\mathbf{k})\cdot(\sin t\ \mathbf{i} - \cos t\ \mathbf{j})$$

$$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t = (5t^2-1)\cos t + 11t\sin t$$

Another Method. $\mathbf{A}\cdot\mathbf{B} = 5t^2\sin t - t\cos t$. Then

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d}{dt}(5t^2\sin t - t\cos t) = 5t^2\cos t + 10t\sin t + t\sin t - \cos t$$
$$= (5t^2-1)\cos t + 11t\sin t$$

$$(b)\quad \frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= [t^3\sin t\ \mathbf{i} - t^3\cos t\ \mathbf{j} + (5t^2\sin t - t\cos t)\mathbf{k}]$$
$$+ [-3t^2\cos t\ \mathbf{i} - 3t^2\sin t\ \mathbf{j} + (-10t\cos t - \sin t)\mathbf{k}]$$
$$= (t^3\sin t - 3t^2\cos t)\mathbf{i} - (t^3\cos t + 3t^2\sin t)\mathbf{j} + (5t^2\sin t - \sin t - 11t\cos t)\mathbf{k}$$

Another Method.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = -t^3\cos t\ \mathbf{i} - t^3\sin t\ \mathbf{j} + (-5t^2\cos t - t\sin t)\mathbf{k}$$

Then $\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = (t^3\sin t - 3t^2\cos t)\mathbf{i} - (t^3\cos t + 3t^2\sin t)\mathbf{j} + (5t^2\sin t - 11t\cos t - \sin t)\mathbf{k}$

$$(c)\quad \frac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\frac{d\mathbf{A}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{A} = 2\mathbf{A}\cdot\frac{d\mathbf{A}}{dt}$$

$$= 2(5t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k})\cdot(10t\,\mathbf{i} + \mathbf{j} - 3t^2\mathbf{k}) = 100t^3 + 2t + 6t^5$$

Another Method. $\mathbf{A}\cdot\mathbf{A} = (5t^2)^2 + (t)^2 + (-t^3)^2 = 25t^4 + t^2 + t^6$

Then $\frac{d}{dt}(25t^4 + t^2 + t^6) = 100t^3 + 2t + 6t^5$.

Q.18 Acceleration of a particle at any time $t \geq 0$ is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = 12Cos\,(2t)\hat{\imath} + 8Sin(2t)\hat{\jmath} + 16t\hat{k}$$

If $\vec{v} = 0$ and $\vec{r} = 0$ at t=0(initial conditions), calculate the value of $\vec{v}$ and $\vec{r}$ at any time t.

(hint: compute the constant of integration using the initial conditions given in the problem)

Soln:

The acceleration of a particle at any time $t \geqq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

Integrating, $\mathbf{v} = \mathbf{i}\int 12\cos 2t\,dt + \mathbf{j}\int -8\sin 2t\,dt + \mathbf{k}\int 16t\,dt$

$= 6\sin 2t\,\mathbf{i} + 4\cos 2t\,\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}_1$

Putting $\mathbf{v}=\mathbf{0}$ when $t=0$, we find $\mathbf{0} = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c}_1$ and $\mathbf{c}_1 = -4\mathbf{j}$.

Then $\mathbf{v} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k}$

so that $\frac{d\mathbf{r}}{dt} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k}$.

Integrating, $\mathbf{r} = \mathbf{i}\int 6\sin 2t\,dt + \mathbf{j}\int (4\cos 2t - 4)\,dt + \mathbf{k}\int 8t^2\,dt$

$= -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + \mathbf{c}_2$

Putting $\mathbf{r}=\mathbf{0}$ when $t=0$, $\mathbf{0} = -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{c}_2$ and $\mathbf{c}_2 = 3\mathbf{i}$.

Then $\mathbf{r} = (3 - 3\cos 2t)\mathbf{i} + (2\sin 2t - 4t)\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k}$.

Q.19 If $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$ evaluate work done $\int \vec{F}.\overrightarrow{dr}$ along the curve C in xy plane given by the equation $y = 2x^2$ in the limit (0,0) to (1,2)

Soln:

Since the integration is performed in the xy plane ($z=0$), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$
$$= \int_C 3xy\,dx - y^2\,dy$$

First Method. Let $x=t$ in $y=2x^2$. Then the parametric equations of C are $x=t,\ y=2t^2$. Points (0,0) and (1,2) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} 3(t)(2t^2)\,dt - (2t^2)^2\,d(2t^2) = \int_{t=0}^{1} (6t^3 - 16t^5)\,dt = -\frac{7}{6}$$

Second Method. Substitute $y=2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2) = \int_{x=0}^{1} (6x^3 - 16x^5)\,dx = -\frac{7}{6}$$

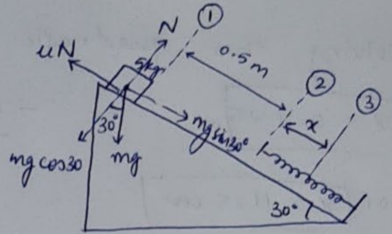
Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been 7/6 instead of $-7/6$.

Q.20 A block of mass 5 kg resting on a 30 degree inclined plane is released. The block after travelling a distance of 0.5 m along the inclined plane hits a spring of stiffness 15N/cm. Find the maximum compression of the spring. Assume the coefficient of friction between the block and inclined plane as 0.2.

solution:

given- $m = 5kg$, $\mu = 0.2$

$k = 15\ N/cm = 15 \frac{N}{10^{-2} m} = 1500\ N/m$



Position 1: $V_1 = 0$ $T_1 = 0$

2: $V_2 = ?$ $T_2 = ?$

3: $V_3 = 0$ $T_3 = 0$

⊛ Apply work-Energy Principle b/w position ① & ②, i.e

$W_{12} = T_2 - T_1$ —— ⓐ

and for that we need corresponding forces that are acting on the system.

∴ $N = mg\cos 30 = 5 \times 9.8 \times \cos 30° = 42.48N$

$f = \mu N = 0.2 \times 42.48\ N = 8.496\ N.$

∴ eq^n ⓐ gives us. (if F & displacement are in same dirc^n then + & vice versa.)

$(+mg\sin 30°) \cdot (0.5) - (\mu N) \cdot 0.5 = \frac{1}{2} \times m V_2^2 - 0$

$\Rightarrow 5 \times 9.8 \times \sin 30° \times 0.5 - 8.496 \times 0.5 = \frac{1}{2} \times 5 \times V_2^2$

$\Rightarrow 24.53 \times 0.5 - 4.248 = \frac{5}{2} V_2^2$

$\Rightarrow V_2^2 = 3.2068 \Rightarrow \boxed{V_2 = 1.79\ ms^{-1}}$ This is the velocity by which block hits the spring.

Now, we have $V_2 = 1.79\ ms^{-1}$ so $T_2 = \frac{1}{2} m V_2^2$.

⊛ Applying work-Energy Principle b/w position ② & ③.

also as the block hits the spring it starts to compress it say after x-distance we have max. compression.

so the workdone due to spring force we have will be given as.

$W_{sp} = \frac{1}{2} k (0^2 - x^2) = -\frac{1500}{2} x^2 = -750 x^2\ J$

∴ $W_{23} = T_3 - T_2$

$+(mg\sin 30) \cdot x - \mu N x - 750x^2 = 0 - \frac{1}{2} \times 5 \times (1.79)^2.$

$\Rightarrow 24.53x - 8.496x - 750x^2 = -8.01$