PHY 102 Introduction to Physics II

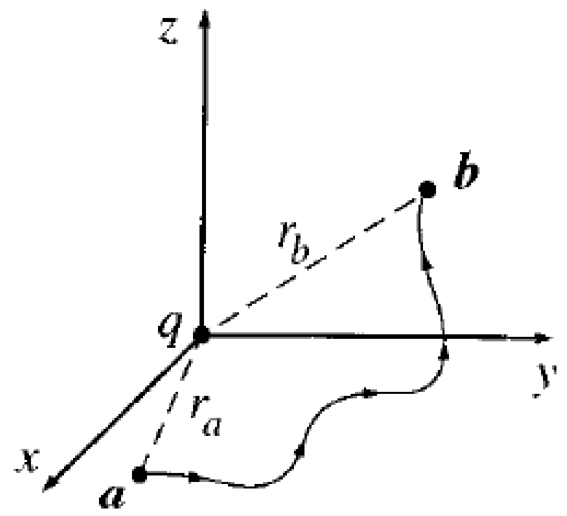
Spring Semester 2025

Lecture 11

Electric Potential

Potential from the line integral of Electric field

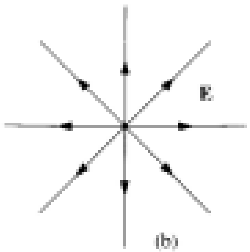
Let us calculate the line integral of $\boldsymbol{E}$ from a to b, $\int_a^b \boldsymbol{E}.\boldsymbol{dl}$



For a point charge at origin $\boldsymbol{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$

Spherical co-ordinates, $\boldsymbol{dl} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + rsin\theta d\varphi\hat{\boldsymbol{\varphi}}$

$$\boldsymbol{E}.\boldsymbol{dl} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$



$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

Potential from the line integral of Electric field

Therefore line integral of **E** around a closed path is evidently zero ($r_a = r_b$, in that case)

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0$$

By Stoke's theorem, $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$

Hence, $\boldsymbol{E} = -\boldsymbol{\nabla} V$

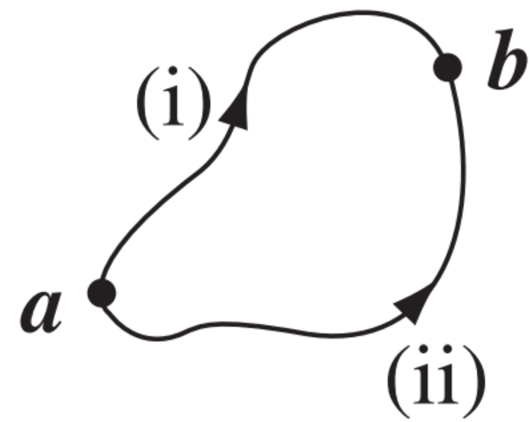
True for any static charge distribution due to *superposition principle of electric fields*

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = 0$$

Static electric fields have zero curl

Electric Potential

Since $\oint \boldsymbol{E}.\boldsymbol{dl} = 0$, line integral of **E** from point ‘a’ to point ‘b’ is *independent of path*



Because of path independence of line integral, we define a function

$$\boxed{V(\boldsymbol{r}) \equiv -\int_{\mathcal{O}}^{r} \boldsymbol{E}.\boldsymbol{dl}\,,}$$

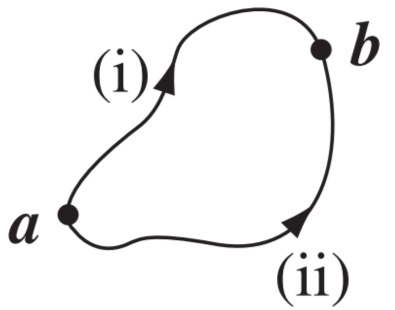
$\mathcal{O}$ is some standard reference point

V (***r***) depends only on the point **r**. It is called the **electric potential**. The line integral of **E** depends only on the initial and final points.

$$V(b) \equiv -\int_{\mathcal{O}}^{b} \boldsymbol{E}.\boldsymbol{dl}\,,\quad V(a) \equiv -\int_{\mathcal{O}}^{a} \boldsymbol{E}.\boldsymbol{dl}\,,$$

Electric Potential

The potential difference between two points **a** and **b** is



$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

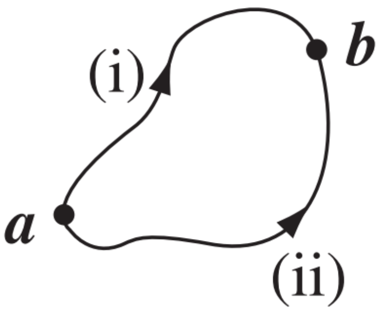
Now, the fundamental theorem for gradients states that

$$V(b) - V(a) = \int_a^b dV = \int_a^b \boldsymbol{\nabla} V . \boldsymbol{dl}$$

$$= -\int_a^b \boldsymbol{E} . \boldsymbol{dl} \qquad \boxed{\boldsymbol{E} = -\boldsymbol{\nabla} V}$$

Electric Potential

$$\boxed{\boldsymbol{E} = -\boldsymbol{\nabla} V}$$ is the differential form of



$$\boxed{V(\boldsymbol{r}) \equiv -\int_{\mathcal{O}}^{r} \boldsymbol{E} . \boldsymbol{dl} \, ,}$$

Electric Potential

Realization of electric potential in another way

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{(\infty)} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \rho(\mathbf{r}')\, d\tau'$$

Using the identity (*you should be able to prove this!*)

$$-\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

we obtain

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int_{(\infty)} \boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \rho(\mathbf{r}')\, d\tau'$$

Since $\boldsymbol{\nabla}$ involves the derivative with respect to unprimed coordinates, we can pull it out of the integral which involves primed coordinates:

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\left(\frac{1}{4\pi\epsilon_0} \int_{(\infty)} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau'\right) \equiv -\boldsymbol{\nabla} V(\mathbf{r})$$

Electric Potential

The scalar potential V

Here $V(\mathbf{r})$ is the scalar electric potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{(\infty)} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau'$$

If we have a system of discrete charges q_1, q_2,..., q_n, located at positions $\mathbf{r}_1$, $\mathbf{r}_2$,..., $\mathbf{r}_n$ respectively, then

$$\rho(\mathbf{r}') = \sum_{j=1}^{n} q_j\, \delta^3(\mathbf{r}' - \mathbf{r}_j)$$

which, when plugged in the expression for $V(\mathbf{r})$ gives, upon using the property of Dirac-delta function,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n} \frac{q_j}{|\mathbf{r}-\mathbf{r}_j|}$$

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\, d\tau = f(\mathbf{a})$$

Comments on Potential

- **Name**

We found that naturally associated with the concept of Electric field is the idea of Electric potential V.

A surface over which the potential is constant is called an **equipotential**.

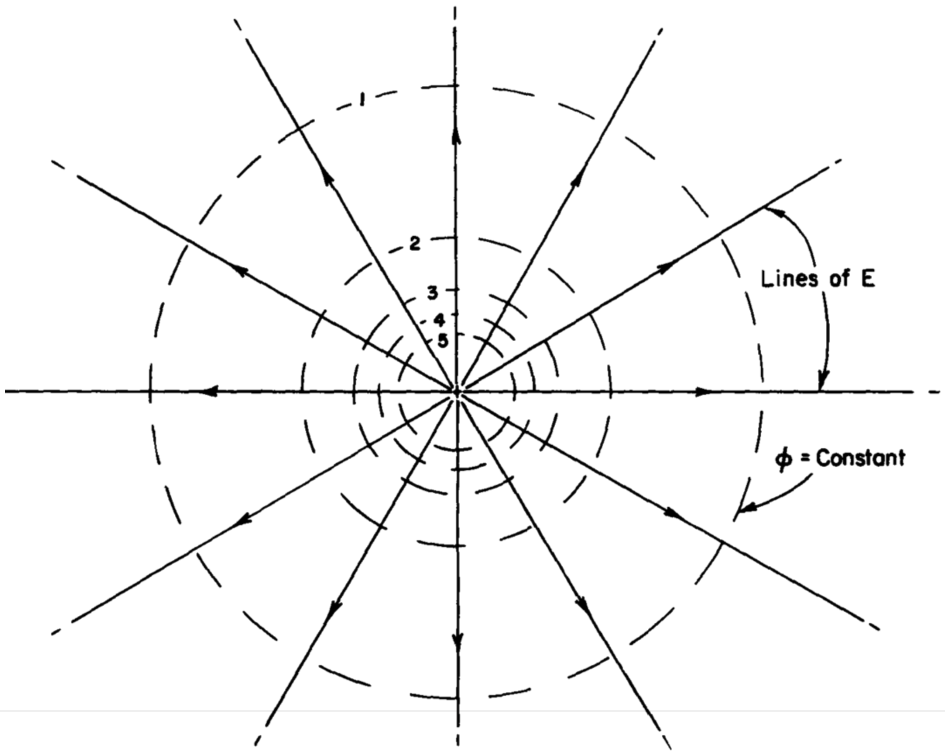
Since $\boldsymbol{E} = -\boldsymbol{\nabla} V$

Comments on Potential

- **Name**

Equipotential Surfaces

Field lines and equipotential lines of a point charge

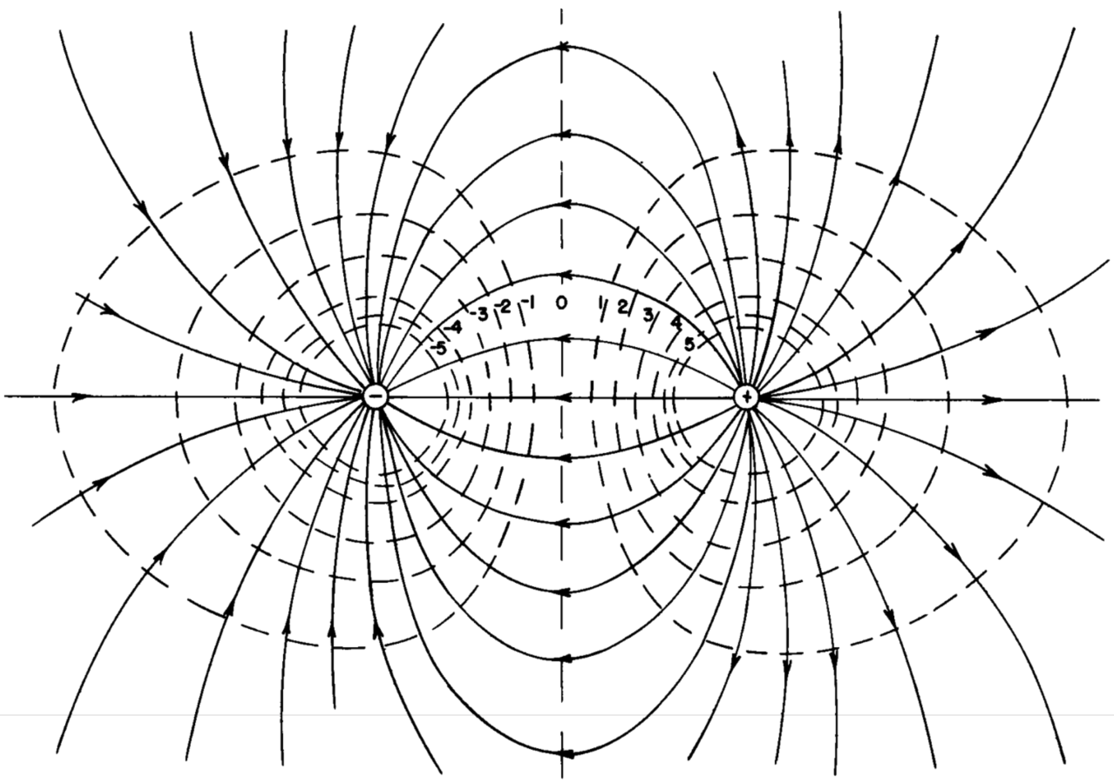


Comments on Potential

- **Name**

Equipotential Surfaces

Field lines and equipotential lines of two equal and opposite charges



Comments on Potential

- **Advantage of the potential formulation**

- The potential formulation is advantageous in the sense that V is a scalar quantity, and therefore its calculation, in general, will require less effort — we don't have to worry about directions.

- Once we know the potential V, the electric field can be easily calculated by taking the negative-gradient, i.e.,

$$\mathbf{E} = -\nabla V$$

- Thus the potential formulation reduces down a vector problem to a scalar one.

Comments on Potential

- **Ambiguity in the Electric Potential**

We already saw that there is an ambiguity in the definition of electric potential. It is unique up to addition of a constant. If we consider,

$$V_{\text{new}} = V_{\text{old}} + K$$

with K some constant (of appropriate dimension*), then

$$\mathbf{E}_{\text{new}} = -\boldsymbol{\nabla} V_{\text{new}} = -\boldsymbol{\nabla}(V_{\text{old}} + K) = -\boldsymbol{\nabla} V_{\text{old}} + 0 = \mathbf{E}_{\text{old}}$$

Thus the electric field remains unchanged.

**The SI unit of electric potential is* Joule/Coulomb (J/C) *or* Volt.

Comments on Potential

- **The reference point *O***

There is an essential ambiguity in the definition of potential, since the choice of reference point **O** was arbitrary.

Changing reference points amounts to adding a constant K to the potential:

$$V'(\mathbf{r}) = -\int_{\mathcal{O}'}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r}),$$

where *K* is the line integral of **E** from the old reference point ***O*** to the new one ***O’***

Comments on Potential

- **The reference point *O***

Thus, we have

$$V_{\text{new}}(\mathbf{r}) = K + V_{\text{old}}(\mathbf{r})$$

where $V_{\text{old}}(\mathbf{r})=V(\mathbf{r})$ refers to the potential defined with respect to the origin reference point $\mathcal{O}$.

We, therefore, conclude that changing the reference point amounts to shifting the potential by a constant amount, which does not change anything physically —The potential difference between two points remains the same. (*The potential difference remains invariant under the shift of the reference point.*):

$$V_{\text{new}}(\mathbf{r}_1) - V_{\text{new}}(\mathbf{r}_2) = (K + V_{\text{old}}(\mathbf{r}_1)) - (K + V_{\text{old}}(\mathbf{r}_2)) = V_{\text{old}}(\mathbf{r}_1) - V_{\text{old}}(\mathbf{r}_2)$$

Comments on Potential

- **Infinity as the Reference Point**

Usually we choose the reference point as infinity, where we set V to be zero. This is reasonable since we expect that as we move away from all sources, the potential should vanish (*This is true for 3-dimensional and higher dimensional space*). Thus under this convention, the potential at any point $\mathbf{r}$ will be given by (setting $\mathbf{b}=\mathbf{r}$, $\mathbf{a}=\mathcal{O}=\infty$ and $V(\infty)=0$),

$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \int_{\mathbf{r}}^{\infty} \mathbf{E}\cdot d\mathbf{l}$$

However, as we will see, this choice is not a good one when we have an infinitely extended source (*or if we are working in dimensions less than 3*).

Comments on Potential

- **Appropriate Reference Point**

Consider an infinite sheet of charge with surface density σ. The electric field is given by

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

Here **n** represent the unit normal to the sheet. Thus the potential at a distance r from the sheet in the direction of **n** (using infinity as the reference point):

$$V(\mathbf{r}) = -\int_{\infty}^{r} \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \cdot (dl\,\hat{\mathbf{n}}) = \frac{\sigma}{2\epsilon_0}(\infty - r).$$

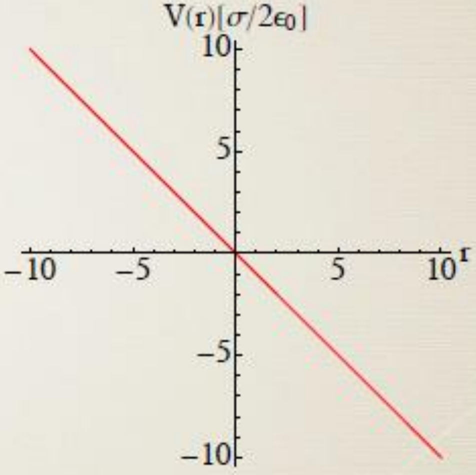
Now this result does not make much sense. Thus the choice $V(\infty)=0$ is not a good one for this problem.

Comments on Potential

- **Appropriate Reference Point**

Thus we need to choose some other suitable reference point for this problem: We can consider a position corresponding to some finite value of **r**. For example, we can choose **r**=0 as the reference point, so that $V(0)=0$. Thus

$$V(\mathbf{r}) = -\int_0^{\mathbf{r}} \frac{\sigma}{2\epsilon_0}\mathbf{n}\cdot dl\mathbf{n} = -\frac{\sigma}{2\epsilon_0}r.$$



This is meaningful now!

In "real life" we don't have infinite charge distributions, and therefore we can safely choose infinity as the reference point. However, for practical electrical circuits, the earth or ground potential is usually taken to be zero and everything is referenced to the earth.

Comments on Potential

- **Electric Potential for different Charge Systems**

☆ In 3-dimensional space, with the reference point set at infinity, for a system of discrete charges $q_1, q_2, \ldots, q_n$, located at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively, the electric potential at some position $\mathbf{r}$ is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n} \frac{q_j}{|\mathbf{r} - \mathbf{r}_j|}$$

☆ If, instead, we have a continuous volume charge distribution with density ρ then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

☆ Similarly, for a surface charge distribution (density: σ), or a line charge distribution (density: λ) we respectively have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS' \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dl'$$

Comments on Potential

- **Super position Principle**

☆ We have the superposition principle for the force acting on a test charge Q:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots$$

☆ We then have the superposition principle for electric field, obtained by dividing the above equation by Q:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots$$

☆ Finally, choosing an appropriate common reference point and considering the line integral of the above equation gives superposition principle for the electric potential:

$$V = V_1 + V_2 + V_3 + \cdots$$

Comments on Potential

- **Units of Potential**

In our units, force is measured in newtons and charge in coulombs, so electric fields are in newtons per coulomb. Accordingly, potential is newton-meters per coulomb, or joules per coulomb. A joule per coulomb is a **volt**.