PHY 102 Introduction to Physics II

Spring Semester 2025

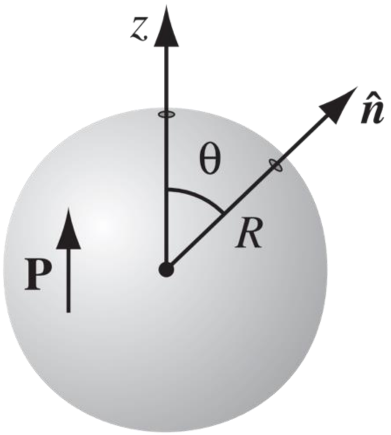
Lecture 21

Polarized Sphere

Polarized Sphere

I. Electric field produced by a **uniformly** polarized sphere of radius 'R'

Assume **P** is along the z-axis. What are the bound charges in this sphere?



$\sigma_b = \boldsymbol{P}.\hat{\boldsymbol{n}} = Pcos\theta$ $\qquad$ $\rho_b = -\boldsymbol{\nabla}.\boldsymbol{P} = 0$ $\qquad$ $\boldsymbol{P}$ is uniform

Problem reduces to finding the electric field due to $\sigma_b = \ Pcos\theta$ plastered on the surface of the sphere

By solving Laplace's equation, it can be shown

$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}.$$

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\[2ex] \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$

$\mathbf{p} = \mathbf{P} * \textbf{volume}$

Polarized Sphere

Use spherical polar co-ordinates, $z = rcos\theta$

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$

Since $r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$(r \le R)$: $$V(r,\theta) = \frac{P}{3\epsilon_0} z = -Ez$$

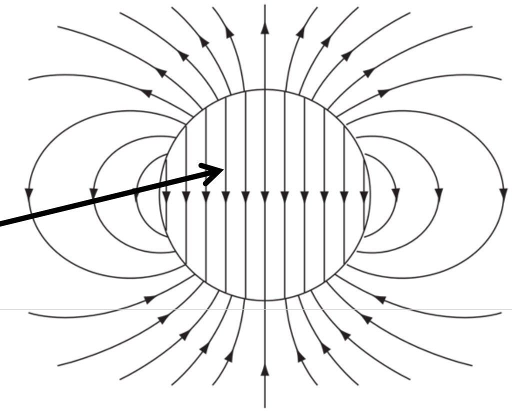
$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}}$$

$$\boxed{\boldsymbol{E} = \boldsymbol{E}_{in} = -\frac{\boldsymbol{P}}{3\epsilon_0}}$$

$(r \ge R)$: $$V(r,\theta) = \frac{pR^3cos\theta}{\frac{4}{3}\pi R^3 3\epsilon_0 r^2} = \frac{pcos\theta}{4\pi\epsilon_0 r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

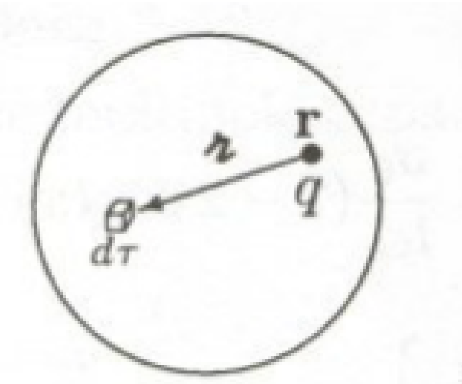
This is exactly the potential due to a dipole placed at the origin. We already know the field of such a dipole

$$\boldsymbol{E}_{in} = -\frac{\boldsymbol{P}}{3\epsilon_0}$$



Polarized Sphere

II. Show that the *average field* inside the sphere of radius 'R' due to *all charge within the sphere* is given by $\boldsymbol{E}_{ave} = \frac{-\boldsymbol{p}}{4\pi\epsilon_0 R^3}$



Consider a single point charge 'q' at a point **r**

Average field due to single charge 'q' at **r**

$$\boldsymbol{E}_{ave,q} = \frac{1}{\frac{4}{3}\pi R^3}\int \boldsymbol{E}\,d\tau \qquad \boldsymbol{E} = \frac{q}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

$$\boldsymbol{E}_{ave,q} = \frac{1}{\frac{4}{3}\pi R^3}\int \frac{q\,d\tau}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

q = source point
$d\tau$ = field point
$\hat{\boldsymbol{\mathscr{r}}}$ = vector from q to $d\tau$

Similarly, field at ***r*** due to uniform charge ρ over the sphere

$$\boldsymbol{E}_{\rho} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\,d\tau}{\mathscr{r}^2}\hat{\mathscr{r}}$$

Here $\hat{\mathscr{r}}$ points from $d\tau$ to r and hence carries an opposite sign. Hence with $\rho = \frac{-q}{\frac{4}{3}\pi R^3}$, it can be proved that $\boldsymbol{E}_{ave,q} = \boldsymbol{E}_{\boldsymbol{\rho}}$

Polarized Sphere

Now we know field inside the sphere having uniform ρ is $\frac{\rho r}{3\epsilon_0}$, 'r' is any radial distance from the center of sphere.

$$\boldsymbol{E}_{\boldsymbol{ave,q}} = \boldsymbol{E}_{\boldsymbol{\rho}}$$

$$\rho = \frac{-q}{\frac{4}{3}\pi R^3}$$

$$= \frac{\rho \boldsymbol{r}}{3\epsilon_0} = \frac{-\boldsymbol{p}}{4\pi\epsilon_0 R^3} \qquad \boldsymbol{p} = q\boldsymbol{r}$$

If there are many charges instead of just one charge inside the sphere, then

$$\boldsymbol{E}_{\boldsymbol{avg}} = \frac{-\boldsymbol{p}_{\boldsymbol{tot}}}{4\pi\epsilon_0 R^3}$$

$\boldsymbol{p}_{tot}$= sum of individual dipole moments.

Polarized Sphere

Significance

$$\mathbf{E}_{ave} = \frac{-\mathbf{p}}{4\pi\epsilon_0 R^3}$$

Substitute $\mathbf{P} = \frac{\mathbf{p}}{\frac{4}{3}\pi R^3}$

$$\mathbf{E}_{ave} = \frac{-\mathbf{P}}{3\epsilon_0}$$

[Field inside a uniformly polarized sphere]

I. Electric field produced by a **uniformly** polarized sphere of radius 'R'

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II. *Average field* inside the sphere of radius 'R' due to *all charge within the sphere*

The average field over any sphere (*due to charges inside*) is same as the field at the center of a *uniformly polarized sphere* with the same dipole moment

This means that no matter how crazy the actual *microscopic charge configuration*, we can replace it by a nice *smooth distribution of perfect dipoles*

Electric Displacement

We found that polarization leads to the accumulation of bound charges in a dielectric. Therefore within it, the total field is the resultant of contribution from the bound charges, and the field due to other sources (free charge).

$\rho_b = -\nabla . P$

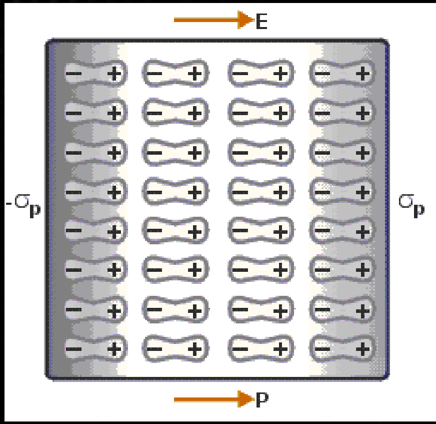
Total charge density within the dielectric, $\rho = \rho_b + \rho_f$.

Gauss's law therefore gives

Any charge that is not a result of polarization

$\sigma_b = P . \hat{n}$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f .$$



Since, $\rho_b = -\nabla \cdot \mathbf{P}$, we obtain

$$\epsilon_0 \nabla \cdot \mathbf{E} = -\nabla \cdot \mathbf{P} + \rho_f$$

$$\Rightarrow \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

The expression in parenthesis is referred to as the **electric displacement**:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} .$$

Gauss′s Law in Dielectric

Using the expression on previous slide we can write

$$\nabla \cdot \mathbf{D} = \rho_f$$

This is the differential form of Gauss's law in the presence of dielectric.

Correspondingly, the integral form is

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{f,\mathrm{enc}}.$$

Here $Q_{f,\mathrm{enc}}$ denotes the total free charge enclosed in the volume.

Gauss's Law in Dielectric

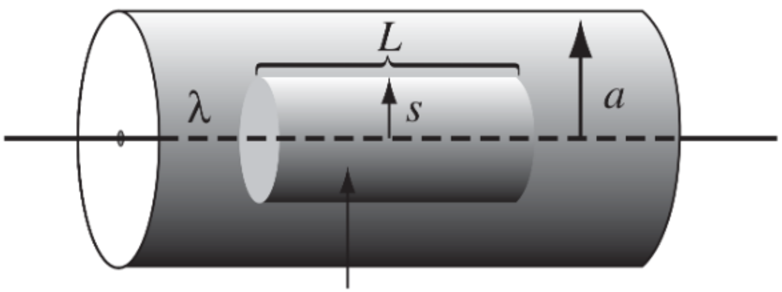
Advice: When you are asked to compute the electric displacement, first look for symmetry. If the problem exhibits spherical, cylindrical or planar symmetry, then you can get D from usual Gauss's law methods

$$\oint \boldsymbol{D}.\boldsymbol{da} = Q_{f_{enc}}$$

If the requisite symmetry is absent, you'll have to think of another approach, and in particular you must not assume that **D** is exclusively by the free charge

Gauss's Law in Dielectric

P:1 A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a (Fig. 4.17). Find the electric displacement.



Gaussian surface

Solution

Drawing a cylindrical Gaussian surface, of radius s and length L,

and apply $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

$$D(2\pi s L) = \lambda L$$

Gauss's Law in Dielectric

Therefore,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}.$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know $\mathbf{P}$.

A deceptive Parallel

Since the equation

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{f,\text{enc}}.$$

gives the appearance as if **D** is essentially the **E** replaced, so everything about **D** should be similar to **E**. However, this is not true. For instance

$$\nabla \times \mathbf{D} = \nabla \times (\epsilon_0 \mathbf{E} + \mathbf{P}) = \epsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P}$$

$$\Rightarrow \nabla \times \mathbf{D} = \nabla \times \mathbf{P} \neq 0,$$

in general. Therefore we don't have a Coulomb's law for **D**:

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho_f(\mathbf{r}') \, d\tau'.$$

Note that for E, **∇×E**=0, so only divergence is sufficient to determine it with proper boundary condition, which is equivalent to writing the Coulomb's law for it.

Boundary Condition for D

For **D** we have the following discontinuities in the perpendicular and parallel components, as we move from side A of an interface to side B:

$$D_B^{\perp} - D_A^{\perp} = \sigma_f,$$

$$D_B^{\parallel} - D_A^{\parallel} = P_B^{\parallel} - P_A^{\parallel}.$$

This should be compared with the corresponding relations for **E**:

$$E_B^{\perp} - E_A^{\perp} = \frac{\sigma}{\epsilon_0}.$$

$$E_B^{\parallel} - E_A^{\parallel} = 0.$$

Therefore in the case of **D** the parallel component is also discontinuous. This is a consequence of the fact that

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}.$$

Electric Susceptibility

We learnt that polarization of a dielectric results from applied electric field. For many substances, the polarization is proportional to the field, provided **E** is not too strong (linear regime):

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}.$$

The dimensionless constant of proportionality χe, is called the susceptibility. Its value depends on the microscopic structure of the concerned substance (and also on external conditions such as temperature).

Materials which obey above equation are called linear dielectrics.

Permittivity

In linear media, we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon_0(1+\chi_e)\mathbf{E}.$$

So **D** also turns out to be proportional to **E**:

$$\mathbf{D} = \epsilon\,\mathbf{E},$$

where

$$\epsilon = \epsilon_0(1+\chi_e),$$

is referred to as the **permittivity** of the given dielectric medium.

In vacuum there's nothing to polarize, so $\chi_e=0$ and hence $\epsilon=\epsilon_0$. This is why ϵ_0 is called the permittivity of the free space (vacuum).

Relative Permittivity/Dielectric constant

We found that

$$\epsilon = \epsilon_0(1 + \chi_e).$$

The ratio ϵ/ϵ_0 is called the relative permittivity of the dielectric constant of the given dielectric:

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e.$$

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7
Neon	1.00013	Salt	5.9
Hydrogen	1.00025	Silicon	11.8
Argon	1.00052	Methanol	33.0
Air (dry)	1.00054	Water	80.1
Nitrogen	1.00055	Ice (-30° C)	99
Water vapor (100° C)	1.00587	$KTaNbO_3$ (0° C)	34,000