# PHY 102 Introduction to Physics II

## Spring Semester 2025

# Lecture 32

***MAXWELL'S EQUATIONS***

**Electrodynamics Before Maxwell**

$$
\begin{array}{llll}
\text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho & & \text{(Gauss's law)}, \\
\text{(ii)} & \nabla \cdot \mathbf{B} = 0 & & \text{(no name)}, \\
\text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & & \text{(Faraday's law)}, \\
\text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} & & \text{(Ampère's law)}.
\end{array}
$$

## Electrodynamics Before Maxwell

If you apply the divergence

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B})$$

Left and right side are **zero**.

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J})$$

Left side is zero, but the right side is non-zero in the case of non-steady current.

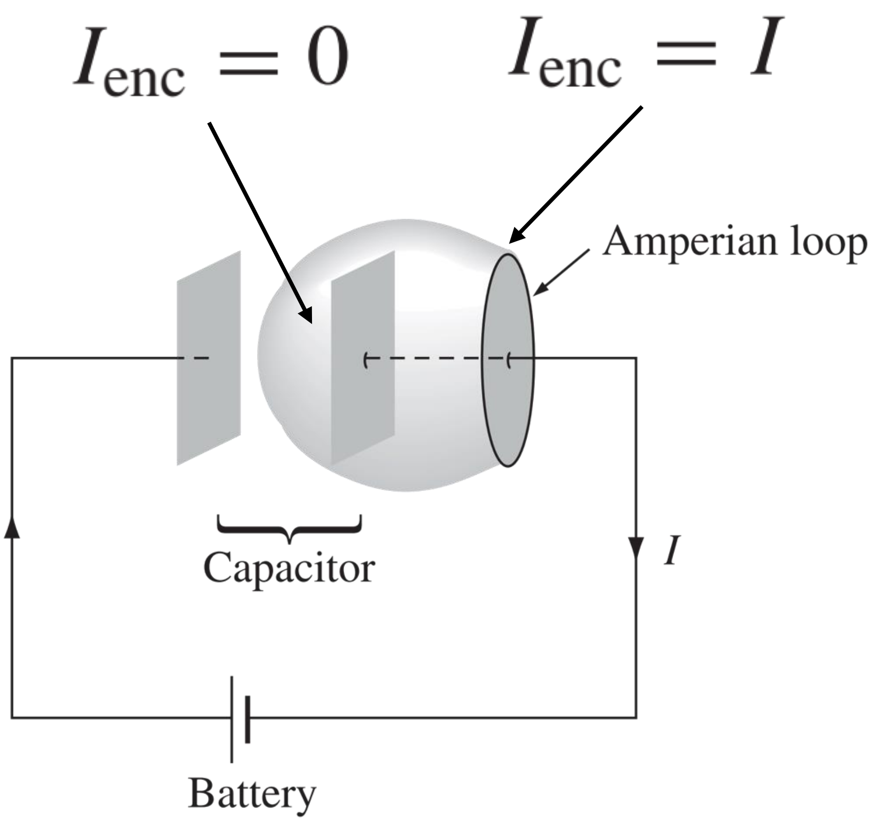
***Beyond magnetostatics, Ampère's law cannot be right***

## Failure of Ampère's law in non-steady currents.

Suppose we're in the process of charging up a capacitor

In the integral form, Ampère's law reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$




## How Maxwell Fixed Ampère's Law?

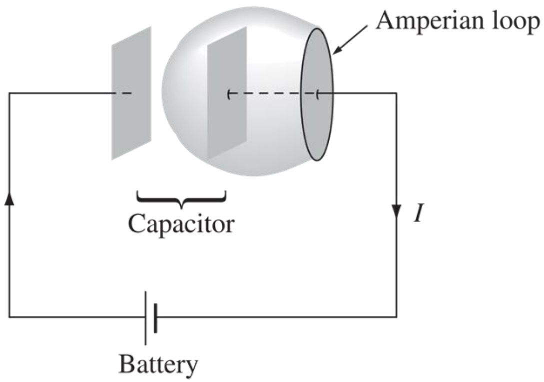
The problem is on the right side of

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J})$$

which should be zero, but isn't

## How Maxwell Fixed Ampère's Law?

From continuity equation and Gauss's law




$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

If we were to combine $\epsilon_0(\partial \mathbf{E}/\partial t)$ with $\mathbf{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.}$$

For magnetostatics, **E** is constant

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

## How Maxwell Fixed Ampère's Law?

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.}$$

**A changing electric field induces a magnetic field.**

Maxwell called his extra term the **displacement current**:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

The real confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

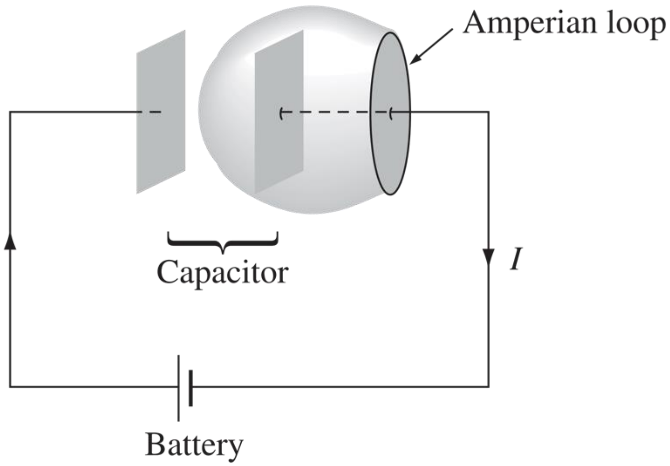
## Ampere's Law with Maxwell's correction

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Integral form:

$$\oint_L \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$$

## How displacement current resolves the paradox of the charging capacitor




If the capacitor plates are very close together, then the electric field between them is

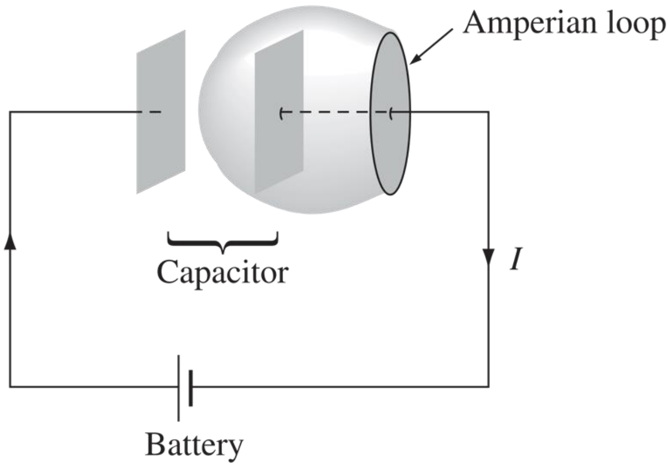
$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A}$$

where $Q$ is the charge on the plate and $A$ is its area.

Thus, between the plates

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A}I$$

# How displacement current resolves the paradox of the charging capacitor




From the integral form of Ampere's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left( \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{a}$$

If we choose the flat surface, then **E = 0** and $I_{enc}$ = I.

On the balloon-shaped surface, then $I_{enc}$ = 0, but

$$\int (\partial \mathbf{E}/\partial t) \cdot d\mathbf{a} = I/\epsilon_0$$

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A} \frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

So we get the same answer for either surface, though in the first case it comes from the **conduction current**, and in the second from the **displacement current**.

## Maxwell's Equation

Gauss's Law: $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ $\qquad \oint_S \vec{E} \cdot d\vec{s} = \dfrac{Q_{enc}}{\varepsilon_0}$

Gauss's Law for magnetism: $\nabla \cdot \vec{B} = 0$ $\qquad \oint_S \vec{B} \cdot d\vec{s} = 0$

Faraday's Law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\qquad \oint_L \vec{E} \cdot d\vec{l} = -\dfrac{\partial \Phi_B}{\partial t}$

Ampère's Law: $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ $\qquad \oint_L \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \dfrac{\partial \Phi_E}{\partial t}$

Ampère's law with Maxwell's correction

# MAXWELL'S EQUATIONS

**Maxwell's equations**

I. $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$ (Flux of $\boldsymbol{E}$ through a closed surface) $=$ (Charge inside)$/\epsilon_0$

II. $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ (Line integral of $\boldsymbol{E}$ around a loop) $= -\dfrac{d}{dt}$(Flux of $\boldsymbol{B}$ through the loop)

III. $\nabla \cdot \boldsymbol{B} = 0$ (Flux of $\boldsymbol{B}$ through a closed surface) $= 0$

IV. $c^2 \nabla \times \boldsymbol{B} = \dfrac{\boldsymbol{j}}{\epsilon_0} + \dfrac{\partial \boldsymbol{E}}{\partial t}$ $c^2$(Integral of $\boldsymbol{B}$ around a loop) $=$ (Current through the loop)$/\epsilon_0$

$+\dfrac{\partial}{\partial t}$(Flux of $\boldsymbol{E}$ through the loop)

[**Conservation of charge**

$\nabla \cdot \boldsymbol{j} = -\dfrac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) $= -\dfrac{\partial}{\partial t}$(Charge inside)]

**Force law**

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

**Law of motion**

$$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}, \quad \text{where} \quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$$

(Newton's law, with Einstein's modification)

**Gravitation**

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\boldsymbol{e}_r$$