PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 10

Applications of Gauss's Law

Gauss's law is always ***true***, but it is not always ***useful***.

Even if the charge distribution is not uniform or arbitrary shape (asymmetric) of the Gaussian surface, the flux of **E** is $\frac{q}{\varepsilon_0}$ will be always true. However, E would not have pointed in the same direction as da, and its magnitude would not have been constant over the surface.

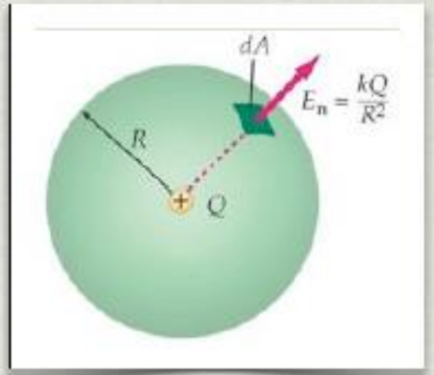
Symmetry is crucial to this application of Gauss's law.

Three kinds of symmetry works

Applications of Gauss's Law

Symmetries & Gauss's Law

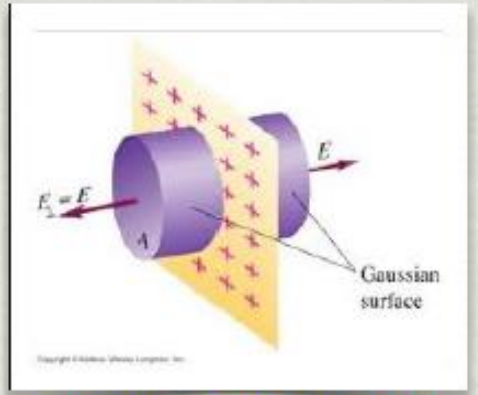
Gauss's law can be immensely powerful when one has some good symmetry in the problem:



Make your Gaussian surface a concentric sphere.

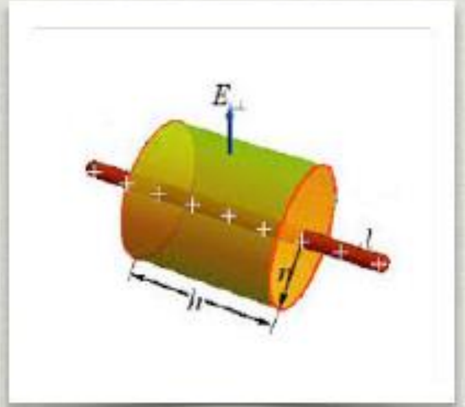
SPHERICAL SYMMETRY

Use a Gaussian "pillbox" that straddles the surface.



PLANE SYMMETRY

Make your Gaussian surface a coaxial cylinder.

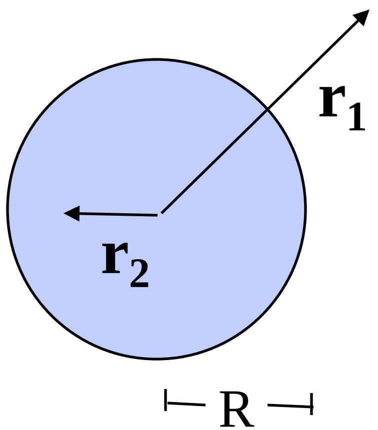


CYLINDRICAL SYMMETRY

Applications of Gauss's Law

Sphere of Charge Q

P1. A charge Q is uniformly distributed through a sphere of radius R. What is the electric field as a function of $\mathbf{r}$? Find $\mathbf{E}$ at $\mathbf{r}_1$ and $\mathbf{r}_2$.



Applications of Gauss's Law

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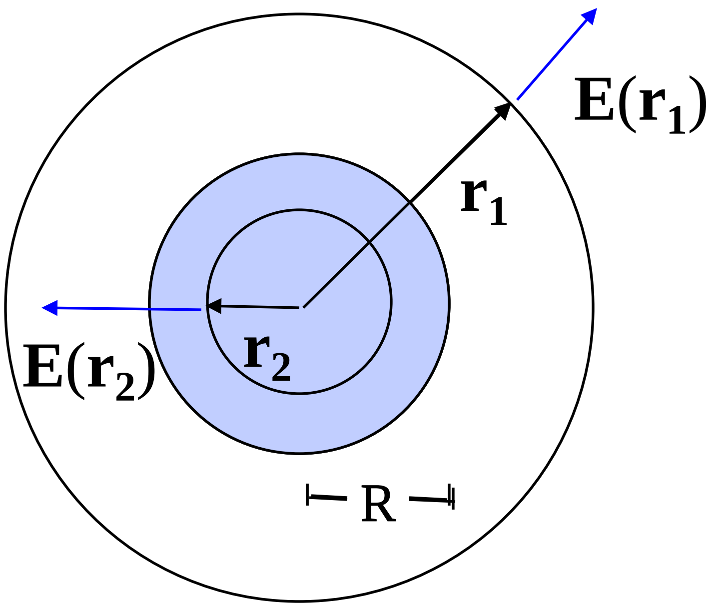


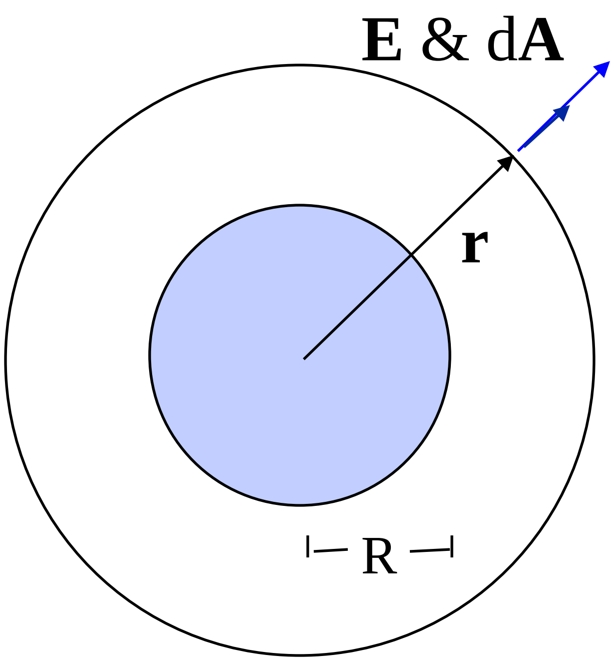
Use symmetry!

This is spherically symmetric. That means that $\mathbf{E}(\mathbf{r})$ is radially outward and that all points, at a given radius ($|\mathbf{r}|=r$), have the same magnitude of the field.

Applications of Gauss's Law

Sphere of Charge Q

First find **E**(**r**) at a point **outside** the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



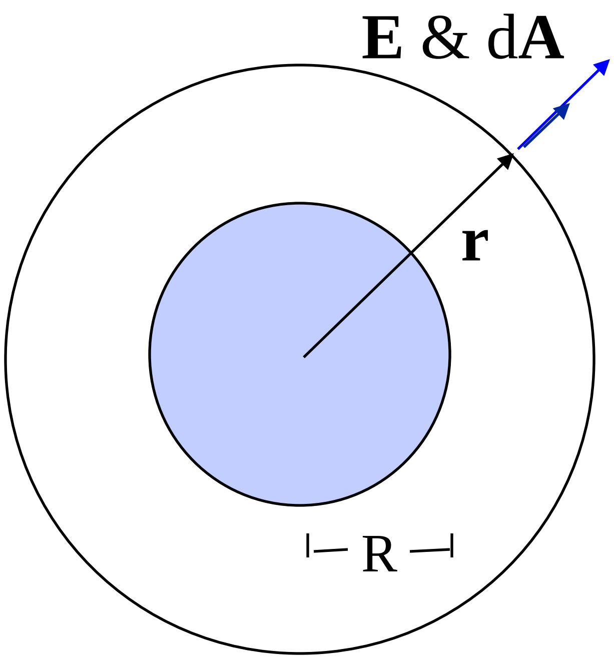
What is the enclosed charge? **Q**

What is the flux through this surface?

$$\Phi = \oint \vec{\mathbf{E}} \bullet d\vec{\mathbf{A}} = \oint E\, dA$$

$$= E \oint dA = EA = E(4\pi r^2)$$

Applications of Gauss's Law



Sphere of Charge Q

$$\Phi = \oint \vec{\mathbf{E}} \bullet d\vec{\mathbf{A}} = \oint E\, dA$$

$$= E\oint dA = EA = E(4\pi r^2)$$

$$\text{Gauss} \Rightarrow \quad \Phi = Q/\varepsilon_o$$

$$Q/\varepsilon_0 = \Phi = E(4\pi r^2)$$

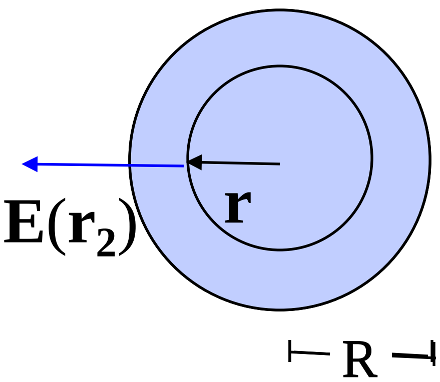
Exactly as though all the charge were at the origin! (for r>R)

So $$\boxed{\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}}$$

Applications of Gauss's Law

Sphere of Charge Q

E & d**A**



Next find **E**(**r**) at a point **inside** the sphere. Apply Gauss's law, using a little sphere of radius r as a Gaussian surface.

The little sphere has some fraction of the total charge. What fraction?

That's given by volume ratio:

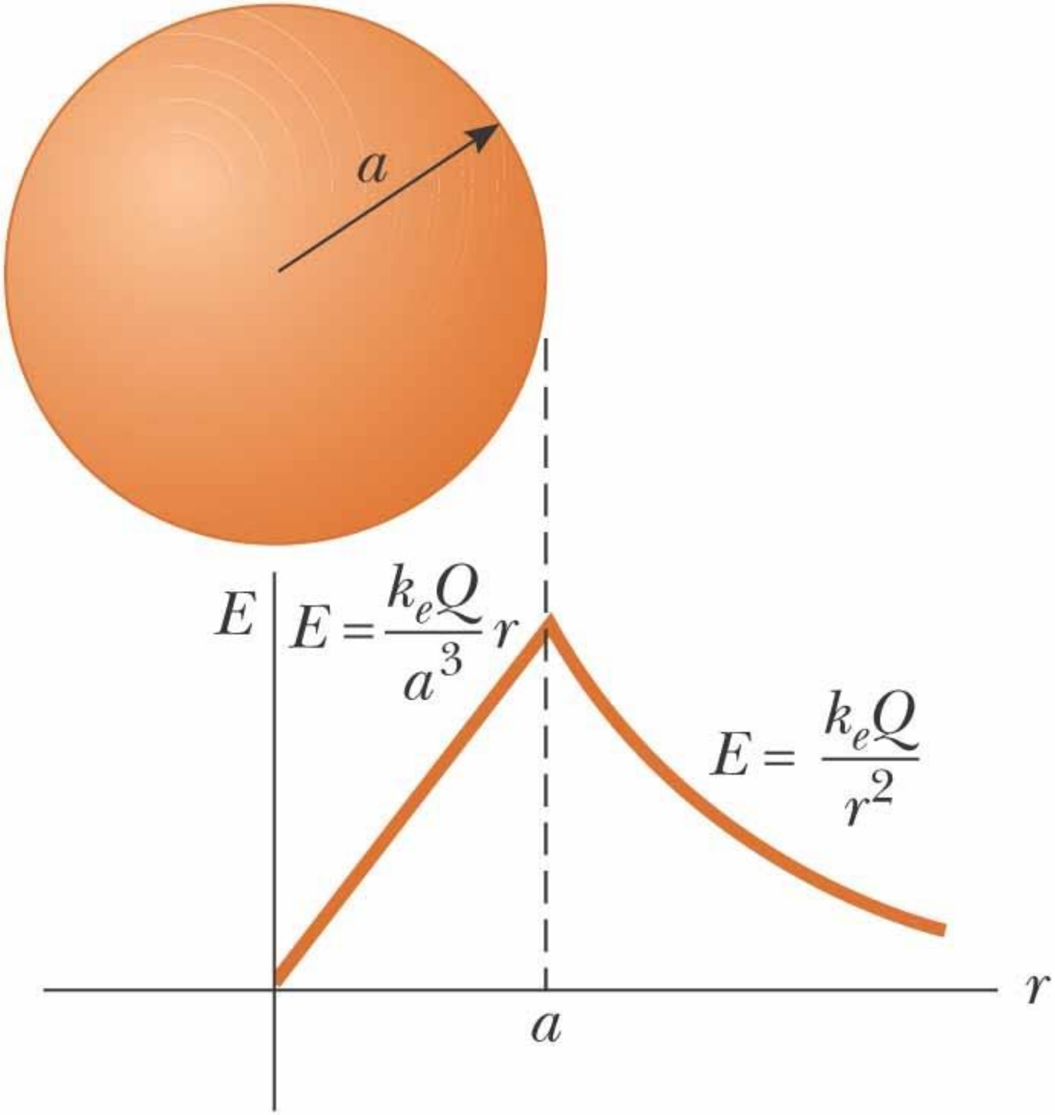
Again the flux is: $\Phi = EA = E(4\pi r^2)$

Setting $\Phi = Q_{enc}/\varepsilon_o$ gives $E = \dfrac{(r^3/R^3)Q}{4\pi\varepsilon_o r^2}$

For r<R

$$\boxed{\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{Q}{4\pi\varepsilon_o R^3}\, r\,\hat{\mathbf{r}}}$$

Applications of Gauss's Law

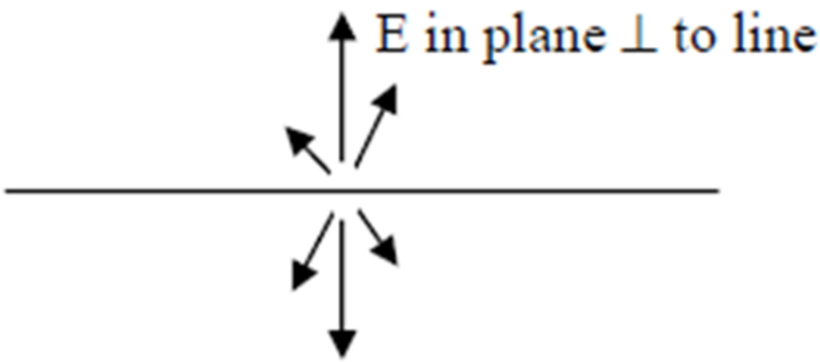


Application of Gauss's Law: Cylindrical Symmetry

A long straight wire carries a uniform linear charge density λ. Find the Electric field at a perpendicular distance r from the wire.

Solution:

By symmetry, **E** is in the cylindrically radial direction and E = E(r).



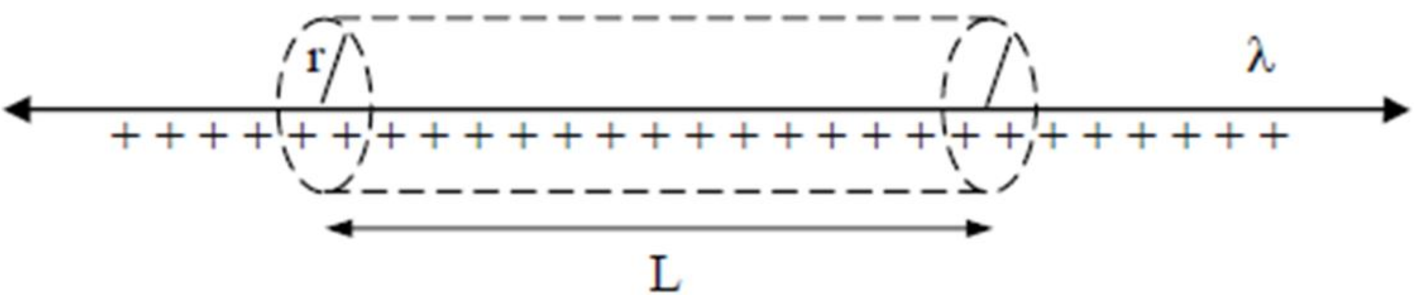
What is the enclosed charge? λL

What is the flux through this surface?

$$\Phi = \oint \vec{\mathbf{E}} \bullet d\vec{\mathbf{A}} = \oint E\, dA$$

$$= E \oint dA = EA = E(2\pi r L)$$

gaussian surface S, radius r, length L

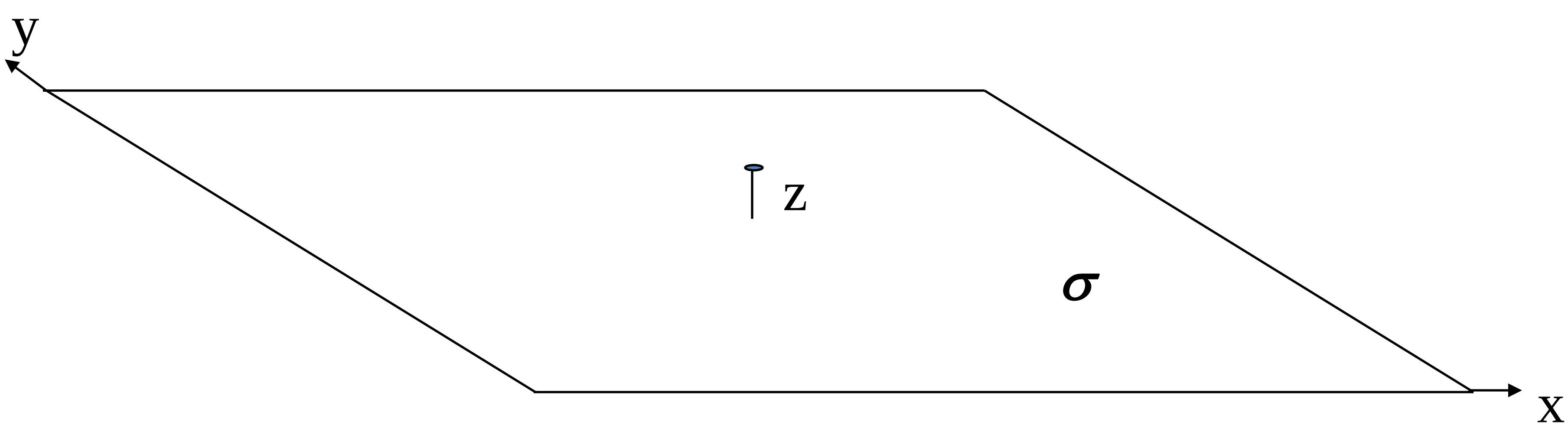


$$\Rightarrow E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Applications of Gauss's Law

Infinite charged plane

P4. Consider an infinite plane with a constant surface charge density σ (which is some number of Coulombs per square meter). What is **E** at a point located a distance z above the plane?



Applications of Gauss's Law

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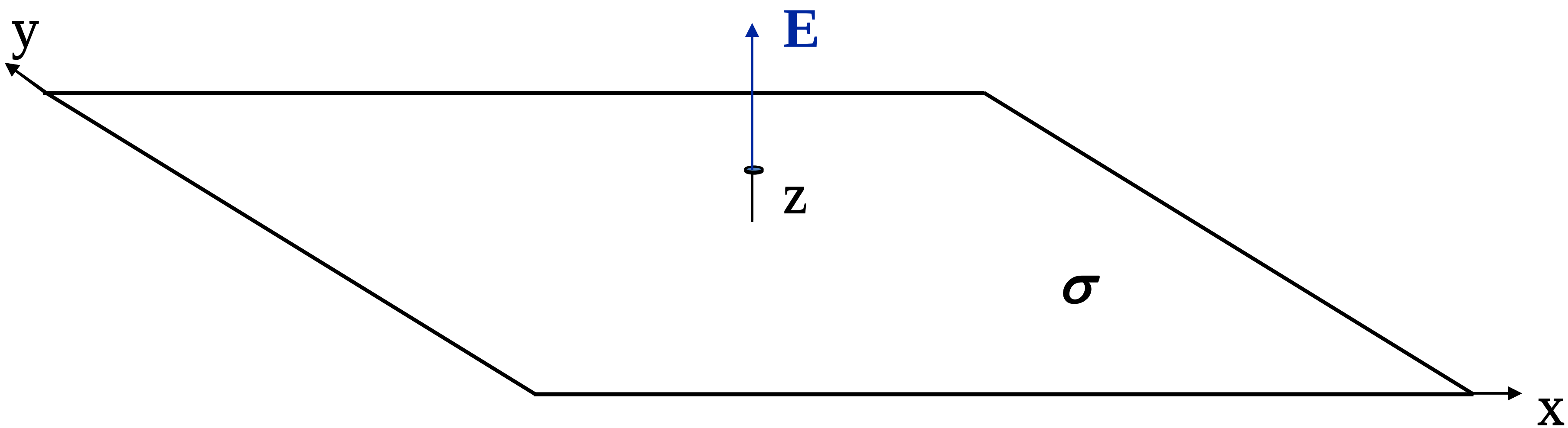


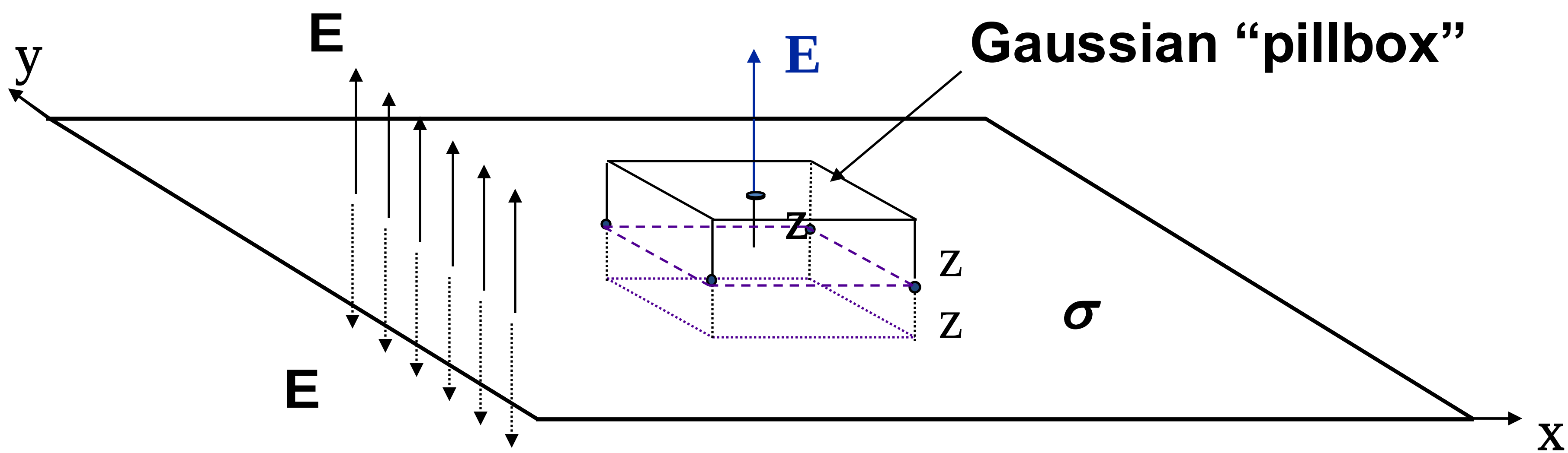
Use symmetry!

The electric field must point straight away from the plane (if $\sigma > 0$). Maybe the Magnitude of E depends on z, but the direction is fixed. And **E** is independent of x and y.

Applications of Gauss's Law

Infinite charged plane

So choose a Gaussian surface that is a "pillbox", which has its top above the plane, and its bottom below the plane, each a distance z from the plane. That way the observation point lies at the top.



Let the area of the top and bottom be A.

Total charge enclosed by box = Aσ

Outward flux through the top: EA

Outward flux through the bottom: EA

Outward flux through the sides:
E x (some area) x cos(90^0) = 0

So the total flux is: **2EA**

Applications of Gauss's Law

Infinite charged plane

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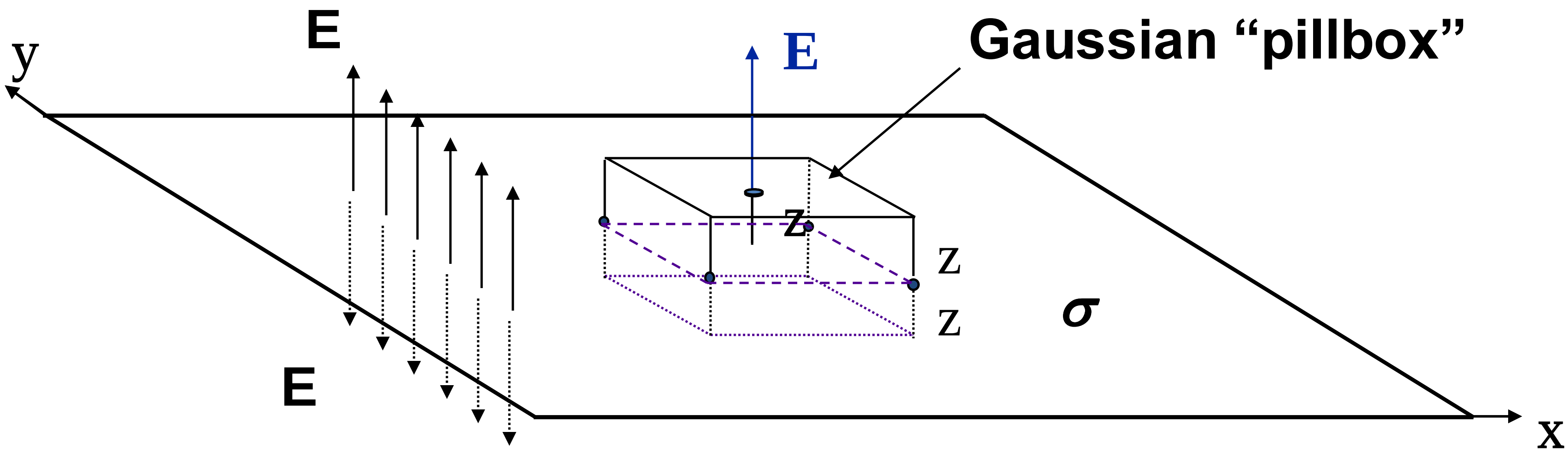


Gauss's law then says that $A\sigma/\varepsilon_0 = 2EA$ so that $\boxed{\mathbf{E}=\sigma/2\varepsilon_0\text{, outward.}}$

This is constant everywhere in each half-space!

Notice that the area A canceled: this is typical!

It seems surprising, at first, that the field of an infinite plane is *independent of how far away you are.* What about the $1/r^2$ in Coulomb's law?

The electric field of a sphere falls off like $1/r^2$;

The electric field of an infinite line falls off like $1/r$;

The electric field of an infinite plane does not fall off at all.

Well, the point is that as you move farther and farther away from the plane, more and more charge comes into your "field of view" (a cone shape extending out from your eye), and this compensates for the diminishing influence of any particular piece.

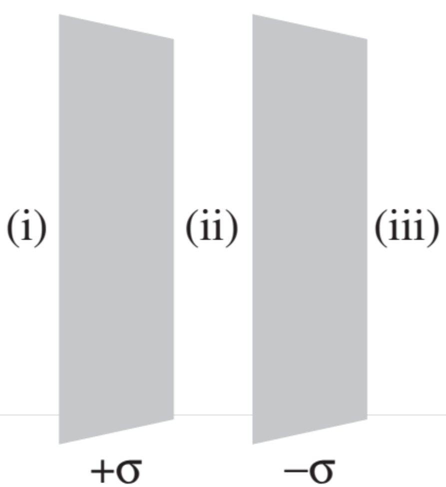
Electric Field

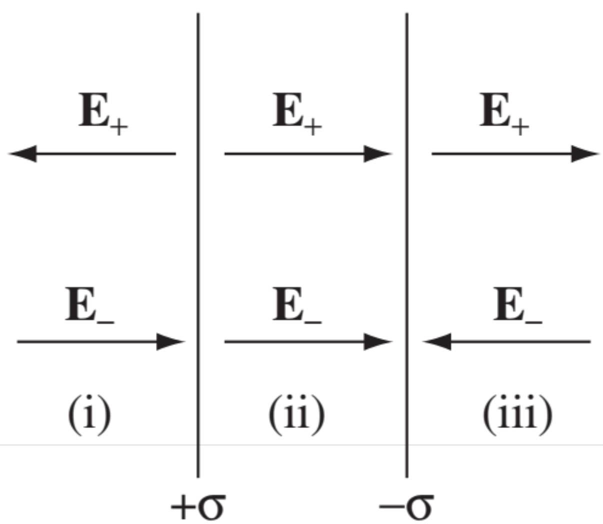
Applications of Gauss's Law

Although the direct use of Gauss's law to compute electric fields is limited to cases of **spherical**, **cylindrical**, and **planar symmetry**, we can put together combinations of objects possessing such symmetry, even though the arrangement as a whole is not symmetrical.

For example, invoking the principle of superposition, we could find the field in the vicinity of two uniformly charged parallel cylinders, or a sphere near an infinite charged plane.

P7. Two infinite parallel planes carry equal but opposite uniform charge densities ±σ (Fig. 2.23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both





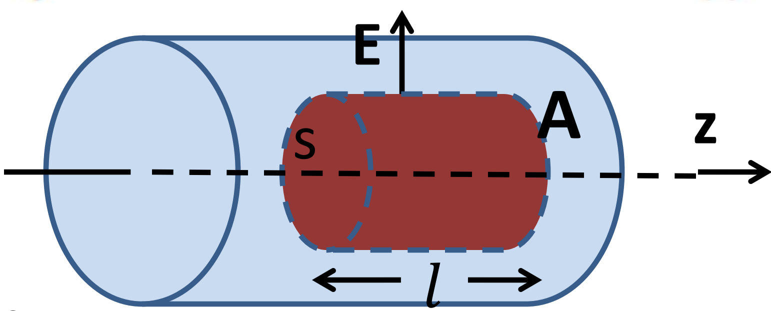
$$\mathbf{E}_{+} = \mathbf{E}_{-} = \frac{\sigma}{2\varepsilon_0}$$

The field between the plates is $\frac{\sigma}{\varepsilon_0}$, and points to the right; elsewhere it is zero

A long cylinder carries a charge density that is proportional to the distance from the axis $\rho = ks$, for some constant *k*. Find the *electric field inside this cylinder*.

Solution: By symmetry, **E** is in the cylindrically radial direction and E = E(s).

Draw a Gaussian cylinder of length *l* and radius *s*.



The enclosed charge is $Q_{enc} = \int \rho d\tau = \int (ks')(s'ds'd\varphi dz) = 2\pi kl \int_0^s s'^2 ds'$

(*z* is integrated from 0 to *l*. Prime on the integration variable is to distinguish it from the radius *s* of Gaussian surface)

$$\boldsymbol{\Phi} = \iint_{Two\ ends} \boldsymbol{E}.\boldsymbol{da} + \iint_{curved} \boldsymbol{E}.\boldsymbol{da} = \boldsymbol{0} + \iint_{A} \boldsymbol{E}.\boldsymbol{da}$$

At the *two ends* of the cylinder, **E** is perpendicular to **da**, so *flux is zero*

$$\iint_A \boldsymbol{E}.\boldsymbol{da} = \iint_A |\boldsymbol{E}| da = |\boldsymbol{E}| \iint_A da = |\boldsymbol{E}| 2\pi sl$$

On the *curved portion* of the Gaussian cylinder

From Gauss's Law $\boldsymbol{\Phi} = \frac{Q_{enc}}{\epsilon_0}$ $\quad \Rightarrow |\boldsymbol{E}| 2\pi sl = \frac{1}{\epsilon_0}\frac{2}{3}\pi kls^3$

$$\boxed{\Rightarrow \boldsymbol{E} = \frac{1}{3\epsilon_0} ks^2 \hat{\boldsymbol{s}}}$$