SOLUTIONS DAMPED Oscillations

QUESTION 1

(a) In the equation of motion,

$$m\ddot{x} + \gamma\,\dot{x} + m\omega_0^2\,x \;=\; 0 \qquad\qquad \textbf{\textit{or}} \qquad\qquad \ddot{x} + \frac{\gamma}{m}\,\dot{x} + \omega_0^2\,x \;=\; 0$$

$m\,\ddot{x}$ is the **net force**
$-\gamma\,\dot{x}$ is the **damping force**
$-m\omega_0^2\,x$ is the **natural restoring force**

or

$\ddot{x}$ is the **net acceleration**
$-(\gamma/m)\,\dot{x}$ is the **acceleration due to the damping force**
$-\omega_0^2\,x$ is the **acceleration due to the natural restoring force**

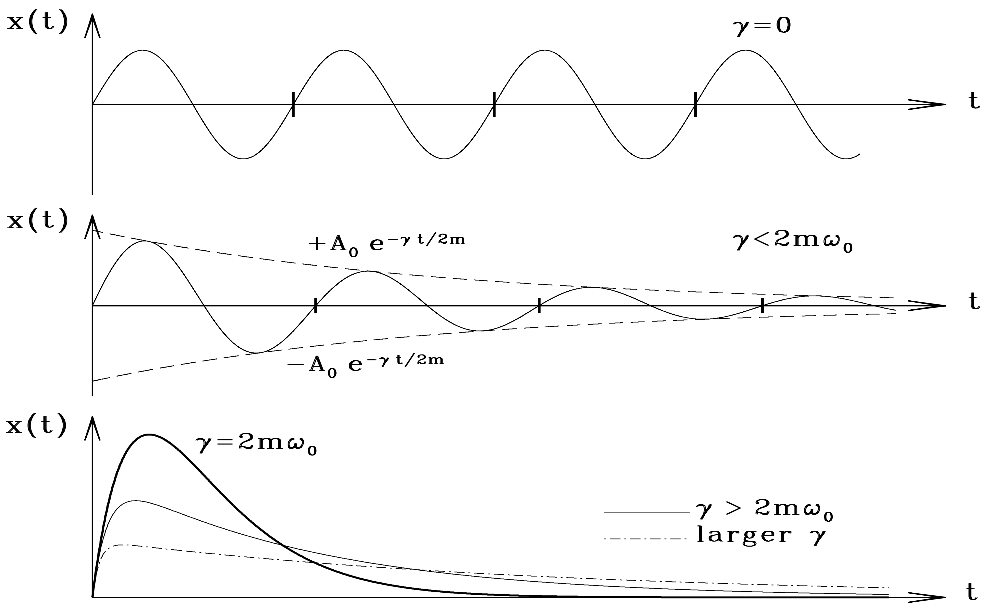
x is the displacement from equilibrium
m is the mass of the oscillator
γ is the damping constant
ω_0 is the natural angular frequency

(b) **Solution (i) is for critical damping**, which occurs when $\boldsymbol{\gamma = 2m\omega_0}$. This provides for the fastest return to equilibrium.

Solution (ii) is for underdamping (or light damping), when $\boldsymbol{\gamma < 2m\omega_0}$. This entails oscillation about equilibrium with an exponentially decaying amplitude.

Solution (iii) is for overdamping (or heavy damping), when $\boldsymbol{\gamma > 2m\omega_0}$. This gives non-oscillatory, aperiodic motion with a monotonic approach to equilibrium at late times, which is slower than for critical damping.

Sketches at the top of next page.



QUESTION 2

The equation of motion for a damped oscillator is **in general**

$$m\ddot{x} + \gamma\,\dot{x} + m\omega_0^2\,x \;=\; 0 \qquad\qquad \textbf{\textit{or}} \qquad\qquad \ddot{x} + \frac{\gamma}{m}\,\dot{x} + \omega_0^2\,x \;=\; 0$$

In this case, given that

$$2\,\ddot{x} + 12\,\dot{x} + 50\,x \;=\; 0$$

and told that $m = 2$ kg, then essentially by inspection:

(a) The damping constant is

$$\boldsymbol{\gamma = 12\ \ \mathrm{kg\ s^{-1}}}$$

and

$$m\omega_0^2 \;=\; 50 \qquad \Longrightarrow \qquad \boldsymbol{\omega_0 = \sqrt{50/2} = 5\ \mathrm{s^{-1}}}$$

(b) Here $2m\omega_0 = 2 \times (2\ \mathrm{kg}) \times (5\ \mathrm{s^{-1}}) = 20\ \mathrm{kg\ s^{-1}}$. This is greater than $\gamma = 12\ \mathrm{kg\ s^{-1}}$, so the system is **underdamped**. Thus the motion is **still oscillatory and periodic**. The damped angular frequency gives the period:

$$\omega \;=\; \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}} \;=\; \sqrt{5^2 - \frac{12^2}{4 \times (2^2)}} \;=\; 4\ \mathrm{s^{-1}} \qquad \Longrightarrow \qquad \boldsymbol{T \;=\; 2\pi/\omega \;=\; \pi/2\ \ \mathrm{s}}$$

(c) Fastest return to equilibrium is for critical damping, which requires

$$\boldsymbol{\gamma = 2m\omega_0 = 20\ \mathrm{kg\ s^{-1}}}$$

The equation of motion then would be

$$\mathbf{2\,\ddot{x} \;+\; 20\,\dot{x} \;+\; 50\,x \;=\; 0} \qquad \textit{or} \qquad \mathbf{\ddot{x} \;+\; 10\,\dot{x} \;+\; 25\,x \;=\; 0}$$

QUESTION 3

(a) The pendulum has a length of $L = 0.60$ m, and so the natural angular frequency of oscillations is

$$\omega_0 \;=\; \sqrt{g/L} \;=\; \sqrt{(9.81 \text{ m s}^{-2})/(0.60 \text{ m})} \;=\; 4.0435 \;\; \text{s}^{-1}$$

Given $\gamma = 0.016$ kg s^{-1} and $m = 0.010$ kg,

$$\mathbf{2m\omega_0 \;=\; 0.0809 \text{ kg s}^{-1}} \;>\; \gamma$$

and therefore the pendulum is underdamped. The damped angular frequency is

$$\omega \;=\; \sqrt{\omega_0^2 - \gamma^2/(4m^2)} \;=\; \mathbf{3.9636 \text{ s}^{-1}} \;\simeq\; \mathbf{0.98\,\omega_0}$$

and hence the period is

$$T \;=\; 2\pi/\omega \;=\; \mathbf{1.585 \;\; sec} \;\simeq\; \mathbf{1.02\,T_0} \qquad [T_0 = 2\pi/\omega_0 = 1.554 \;\; \text{s}]$$

(b) The time-dependent amplitude of underdamped oscillations is

$$A(t) \;\equiv\; A_0\, e^{-\gamma\, t/(2m)}$$

To find when the amplitude falls by a factor of 1000, set $A(t)/A_0 = 1/1000$, so

$$A(t)/A_0 \;=\; e^{-\gamma\, t/(2m)} \;=\; 0.001 \quad\Longrightarrow\quad t \;=\; \ln(0.001)/\left[-\gamma/(2m)\right] \;=\; \mathbf{8.63 \;\; sec}$$

Then, the mechanical energy is proportional to the square of amplitude, $E_{\text{tot}}(t) \propto A(t)^2$, so if the amplitude decreases by a factor of 10^3 then the **energy decreases by a factor of $\mathbf{10^6}$**.

(c) For the pendulum to be critically damped, it must have $\gamma = 2m\omega_0$. But $\omega_0 = \sqrt{g/L}$ for a simple pendulum. Given $m = 0.010$ kg, $\gamma = 0.016$ kg s^{-1}, and $g = 9.81$ m s^{-2} :

$$\boldsymbol{\gamma \;=\; 2m\omega_0} \quad\Longrightarrow\quad \gamma = 2m\sqrt{g/L} \quad\Longrightarrow\quad L = 4m^2 g/\gamma^2$$
$$\Longrightarrow\quad \mathbf{\boldsymbol{L} = 15.3 \;\; m}$$
