PHY101: Introduction to Physics I

Monsoon Semester 2024

Lecture 18

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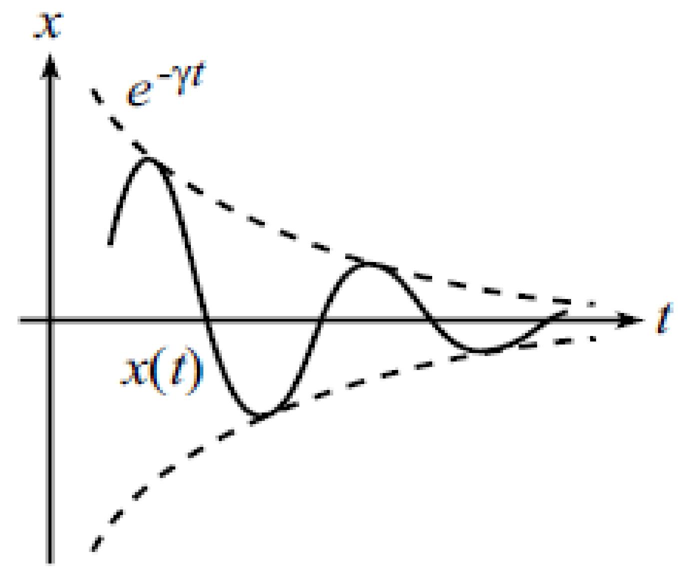
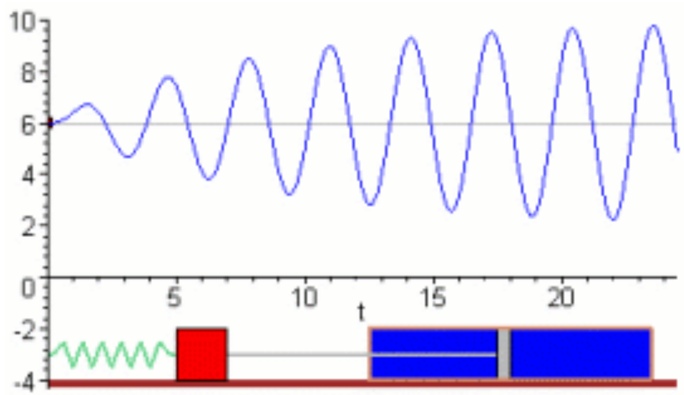
Previous Lecture

Damped harmonic motion

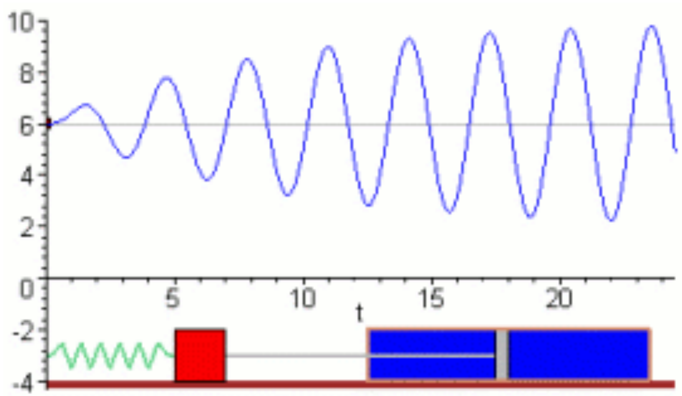


This Lecture

Driven harmonic motion



Forced (or Driven) vibration



Restoring force + resistive force + driven force

Forced vibrations are produced in the air inside the box and intensity of sound increases

Applications of forced vibration

1. **Structural dynamics**: To study the response of structures such as bridges, buildings, and towers to wind and earthquake forces.

2. **Vibration testing**: To test the structural integrity of products such as aircraft, automobiles, and electronic devices.

3. **Manufacturing**: It can shake any loose particles or contaminants on a surface and ensure that a surface or material is homogeneous.

4. **Energy Harvesting**: Vibration energy harvesting devices use forced vibration to generate electrical energy. Piezoelectric and electromagnetic materials are used to convert mechanical energy into electrical energy.

5. **Medical field**: Vibration therapy is used to help treat conditions such as osteoarthritis and improve muscle strength and flexibility.

6. **Music**: Forced vibration is used in instruments with strings.

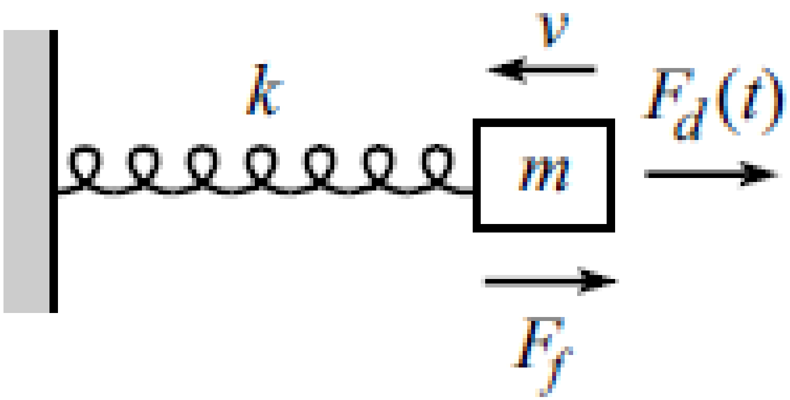
Driven harmonic motion

⇒ **System under restoring, resistive and driven forces.**

The external driving force should be periodic!

Let's apply a driving force,

$$F_d(t) = F_d \cos \omega_d t$$



The differential equation,

$$F(x, \dot{x}, t) = -b\dot{x} - kx + F_d \cos \omega_d t$$

$$\boxed{F \equiv F_d/m}$$

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega^2 x = F \cos \omega_d t$$

$$= \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)$$

$$\boxed{\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2} \end{aligned}}$$

Here, **ω is the natural frequency** of oscillation of the spring.

ω_d is the frequency of applied force.

damping parameter (2γ = b/m)

Solution of the differential equation

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega^2 x = \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)$$

Step 1: Taking first part of right hand side (RHS)

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega^2 x = \frac{F}{2} e^{i\omega_d t}$$

Trial solution, $x(t) = Ae^{i\omega_d t}$

$$\boldsymbol{A} = \left(\frac{F/2}{-\omega_d^2 + 2i\gamma\omega_d + \omega^2}\right)$$

Follow steps as before

Step 2: Taking second part of right hand side (RHS)

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = \frac{F}{2} e^{-i\omega_d t}$$

Trial solution, $x(t) = Be^{-i\omega_d t}$

$$\mathbf{B} = \left(\frac{F/2}{-\omega_d^2 - 2i\gamma\omega_d + \omega^2}\right)$$

The particular solution, $x_p(t) = Ae^{i\omega_d t} + Be^{-i\omega_d t}$

$$x_p(t) = \left(\frac{F/2}{-\omega_d^2 + 2i\gamma\omega_d + \omega^2}\right)e^{i\omega_d t} + \left(\frac{F/2}{-\omega_d^2 - 2i\gamma\omega_d + \omega^2}\right)e^{-i\omega_d t}.$$

Simplify to,

$$x_p(t) = \left(\frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\cos\omega_d t + \left(\frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\sin\omega_d t.$$

$$\equiv \quad \frac{F}{R}\cos(\omega_d t - \phi)$$

Note: Particular solution given above is not a complete solution.
A more detailed mathematical solution is beyond the scope of this course.

$$\boxed{R \equiv \sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}}$$

$$\boxed{x_p(t) \quad = \quad \frac{F}{R}\cos(\omega_d t - \phi)}$$

$$x_p(t) = \left(\frac{F/2}{\omega_0^2-\omega_d^2+2i\gamma\omega_d}\right)e^{i\omega_d t} + \frac{F/2}{(\omega_0^2-\omega_d^2-2i\gamma\omega_d)}e^{-i\omega_d t}$$

$$= \frac{F/2\,(\omega_0^2-\omega_d^2-2i\gamma\omega_d)(\cos(\omega_d t)+i\sin\omega_d t) + F/2\,(\omega_0^2-\omega_d^2+2i\gamma\omega_d)(\cos(\omega_d t)-i\sin\omega_d t)}{(\omega_0^2-\omega_d^2)^2+4\gamma^2\omega_d^2}$$

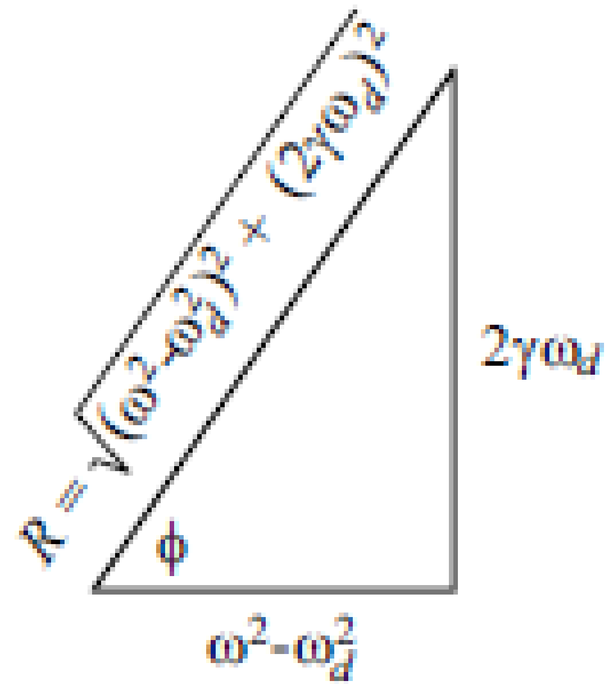
$$= \frac{\begin{aligned}&F/2\,(\omega_0^2-\omega_d^2)\cos\omega_d t - F/2\,\cancel{2i\gamma\omega_d\cos\omega_d t} + \\ &i\,F/2\,\cancel{(\omega_0^2-\omega_d^2)}\sin\omega_d t - F/2\,2i\gamma\omega_d\, i\sin\omega_d t \\ &+ F/2\,(\omega_0^2-\omega_d^2)\cos\omega_d t - F/2\, i\cancel{(\omega_0^2-\omega_d^2)}\sin\omega_d t \\ &+ F/2\,\cancel{2i\gamma\omega_d\cos\omega_d t} - F/2\,2i\gamma\omega_d\, i\sin\omega_d t\end{aligned}}{(\omega_0^2-\omega_d^2)+4\gamma^2\omega_d^2}$$

$$= \frac{F(\omega_0^2-\omega_d^2)\cos\omega_d t + 2F\gamma\omega_d\sin\omega_d t}{(\omega_0^2-\omega_d^2)+4\gamma^2\omega_d^2}$$

$$= \frac{F(\omega_0^2-\omega_d^2)\cos\omega_d t}{(\omega_0^2-\omega_d^2)+4\gamma^2\omega_d^2} + \frac{2F\gamma\omega_d\sin\omega_d t}{(\omega_0^2-\omega_d^2)+4\gamma^2\omega_d^2}$$

$$= \frac{F}{R}\cos\omega_d t\cos\phi + \frac{F}{R}\sin\omega_d t\sin\phi$$

$$x_p(t) = \cancel{\cos}\cdot F/R\,\cos(\omega_d t - \phi)$$



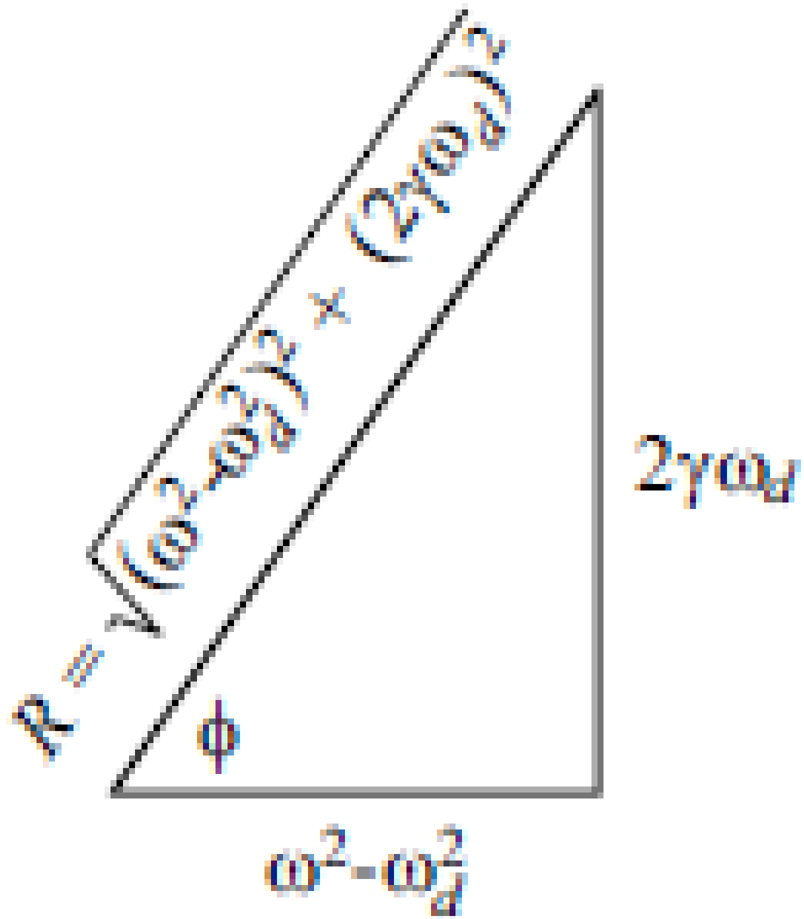
The Phase

$$\cos\phi = \frac{\omega^2 - \omega_d^2}{R}, \qquad \sin\phi = \frac{2\gamma\omega_d}{R} \qquad \Longrightarrow \qquad \tan\phi = \frac{2\gamma\omega_d}{\omega^2 - \omega_d^2}$$

The Amplitude

$$\frac{F}{R} = \frac{\boldsymbol{F}}{\sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}}$$

Triangle describes the relation among all the parameters.

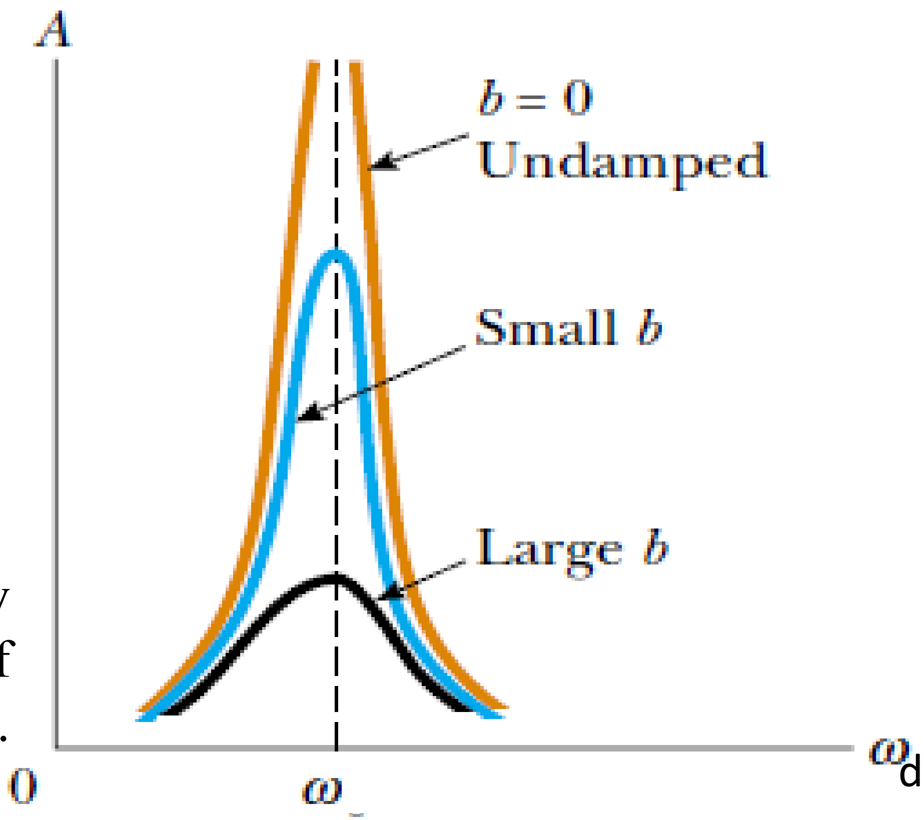


Resonance

$$\frac{F}{R} = \frac{F}{\sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}}$$

=> At values of ω_d less than the ω, the amplitude increases with ω_d.

⇒ if damping of the system is very small, at resonance, the sharpness of amplitude is maximum for $\omega_d = \omega$. **[Homework/Tutorial]**



=> At values of ω_d more than ω, the amplitude decreases.

=> The nature depends on **damping parameter** ($2\gamma = b/m$).

=> For no damping, the applied driving force at resonance frequency, will provide infinite amplitude.

Properties of driven harmonic motion

- The reason for **large-amplitude oscillations** at the resonance frequency is that **energy is being transferred** to the system under the most favourable conditions.

- In the absence of a damping force ($b = 0$), we see that **the steady-state amplitude approaches infinity as ω approaches ω_0**.

- In other words, if there are no losses in the system and we continue to drive an initially motionless oscillator with **a periodic force that is in phase with the velocity**, the amplitude of motion builds without limit.

- This limitless building does not occur in practice because **some damping is always present in reality**.

Risks factors associated with the driven vibration (Engineering applications)

1. **Damage to the system**: High amplitude vibrations can lead to fatigue failures and other types of damage.

2. **Noise and vibration**: Forced vibrations can generate noise and vibration that can disrupt people and equipment.

3. **Extra cost**: The excitation may incur an extra cost, both in terms of equipment and labour.

4. **The complexity of the system**: Different loading conditions can make the testing and analysis more complex and difficult to interpret.

5. **Inaccurate results**: It may give inaccurate results for the system's behaviour, as it may not simulate the real loading conditions of the system.

6. **Safety hazards**: It can create safety hazards, particularly if the system is not properly secured or the excitation source is not controlled properly.