PHY101: Introduction to Physics I

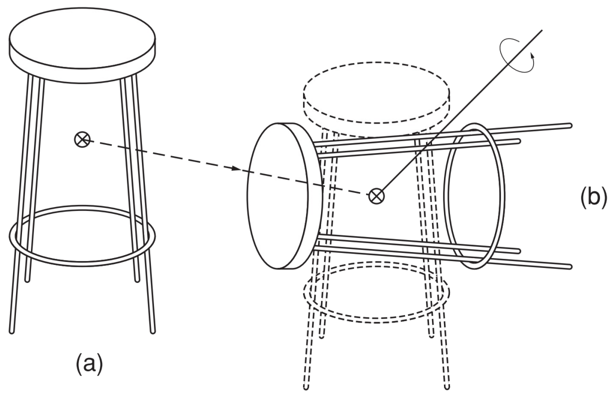
Monsoon Semester 2024

Lecture 24

Department of Physics, School of Natural Sciences,
Shiv Nadar Institution of Eminence, Delhi NCR

Previous Lecture

Reduced Mass
Angular Momentum
Rigid Body



This Lecture

Torque
Central force motion
Rigid Body

ROTATIONAL MOTION

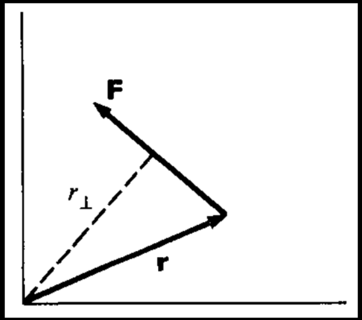
TORQUE

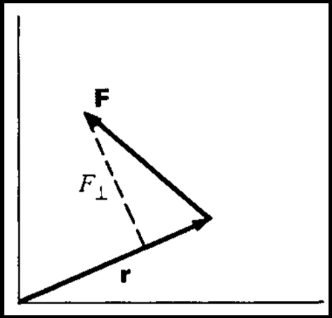
The torque, $\vec{\tau}$ due to a force $\vec{F}$ on a particle at position $\vec{r}$ is defined by

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

Since this structure is identical to $\vec{L} = \vec{r} \times \vec{p}$, the analysis done for the angular momentum applies in this case also. So, for example, we have

$$|\vec{\tau}| = |\vec{r}_\perp|\left|\vec{F}\right| \quad \text{or} \quad |\vec{\tau}| = |\vec{r}|\left|\vec{F}_\perp\right|$$





ROTATIONAL MOTION

TORQUE

We have

$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}.$

Therefore,

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(y\,F_z - z\,F_y\right)\hat{\imath} + (z\,F_x - x\,F_z)\hat{\jmath} + \left(x\,F_y - y\,F_x\right)\hat{k}$$

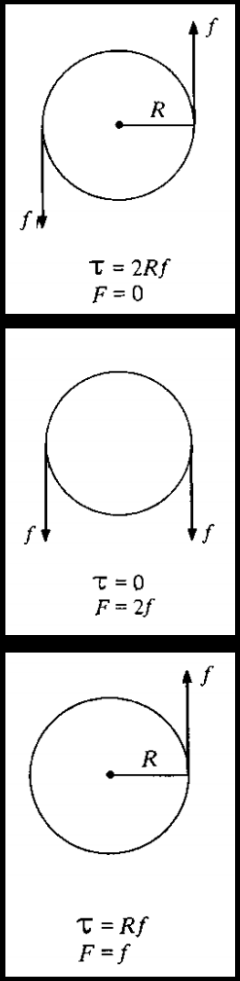
$$\equiv \tau_x\hat{\imath} + \tau_y\hat{\jmath} + \tau_z\hat{k}$$

ROTATIONAL MOTION

TORQUE

It's clear that $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$ and depends on the choice of the origin.

- There can be nonzero net torque on a system with zero net force.
- There can be zero net torque on a system with nonzero net force.
- There can be nonzero net torque on a system with nonzero net force.



ROTATIONAL MOTION

RELATION BETWEEN TORQUE AND ANGULAR MOMENTUM

Consider the time rate of change of angular momentum,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$= 0 + \vec{r} \times \vec{F} = \vec{\tau}$$

Thus,

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

i.e., Torque equals the rate of change of angular momentum.

ROTATIONAL MOTION

CONSERVATION OF ANGULAR MOMENTUM

If $\vec{\tau}=0$, $\frac{d\vec{L}}{dt}=0$, implying that $\vec{L}$ remains conserved in time. Thus we encounter the conservation of angular momentum.

We have obtained this result for a single particle. We will soon generalize this to a rigid body composed of many particles.

However, even with this one particle result we can make some powerful predictions. We consider such an example next.

CENTRAL FORCE MOTION AND LAWS OF EQUAL AREAS

Consider a particle moving under the influence of a central force,

$$\vec{F} = f(r)\,\hat{r}$$

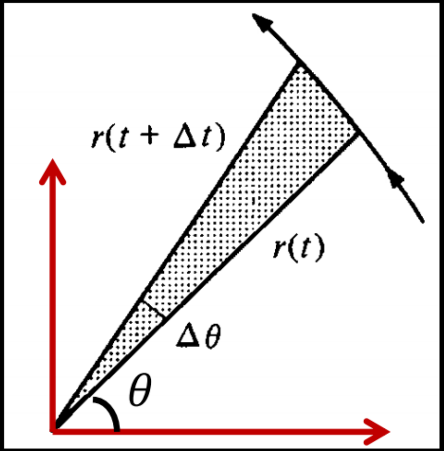
The torque on the particle about the origin is,

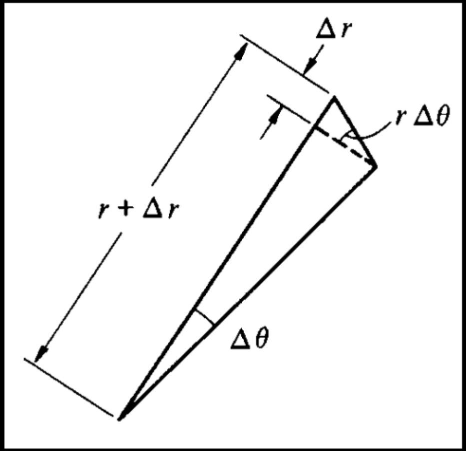
$$\vec{\tau} = \vec{r} \times \vec{F} = r\hat{r} \times f(r)\,\hat{r} = 0.$$

This implies that the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of the particle will remain conserved (both in magnitude and direction). Consequently, the motion must be restricted to a plane (say XY), else the direction of $\vec{L}$ will change with time.

CENTRAL FORCE MOTION AND LAWS OF EQUAL AREAS

Consider the positions of the particle at time instants t and Δt. In terms of polar coordinates these are (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$, respectively.





The small area ΔA swept during the small interval Δt is,

$$\Delta A \approx \frac{1}{2}(r + \Delta r)(r\Delta\theta) = \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\,\Delta\theta$$

The rate at which the area is swept out is,

$$\frac{dA}{dt} = \operatorname*{Lim}_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \operatorname*{Lim}_{\Delta t \to 0} \left(\frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t} + \frac{1}{2} r \frac{\Delta r\, \Delta \theta}{\Delta t} \right) = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

(Note that the second term does not contribute in the limit $\Delta t \to 0$, since $\frac{\Delta r\, \Delta \theta}{\Delta t} = \frac{\Delta r}{\Delta t} \frac{\Delta \theta}{\Delta t} \Delta t$. In the limit $\Delta t \to 0$, it behaves like $\dot{r}\, \dot{\theta}\, \Delta t$ and hence goes to zero).

Now in polar coordinates, the velocity of the particle is

$$\vec{v} = \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta}.$$

Its angular momentum is therefore,

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = (r\hat{r}) \times m \left(\frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta} \right) = mr^2 \frac{d\theta}{dt} \hat{k}$$

CENTRAL FORCE MOTION AND LAWS OF EQUAL AREAS

Thus we have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2m}\left(mr^2 \frac{d\theta}{dt}\right),$$

and

$$\vec{L} = mr^2 \frac{d\theta}{dt} \hat{k} \equiv L_z \hat{k} \, .$$

Comparing these two we obtain,

$$\frac{dA}{dt} = \frac{L_z}{2m} = \text{constant}.$$

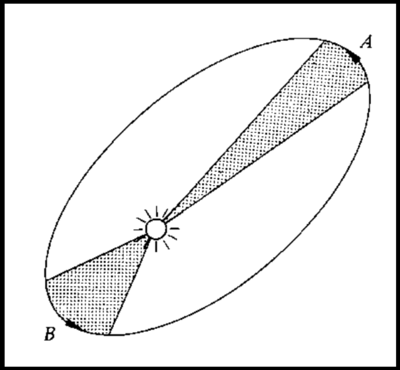
Since L_z is conserved for any central force, we conclude that $\frac{dA}{dt}$ remains constant in the motion under central force. In other words, the particle sweeps equal areas in equal intervals of time.

CENTRAL FORCE MOTION AND LAWS OF EQUAL AREAS

If we consider the motion of planets around the Sun, because the length scale of motion of the planets is quite large compared to the size of the planet, they can be treated as point objects.

Moreover, the planets move under the influence of central force, the Gravitational force,

$$\vec{F} = -\frac{G\, M_{Sun}\, M_{Planet}}{r^2}\hat{r}.$$

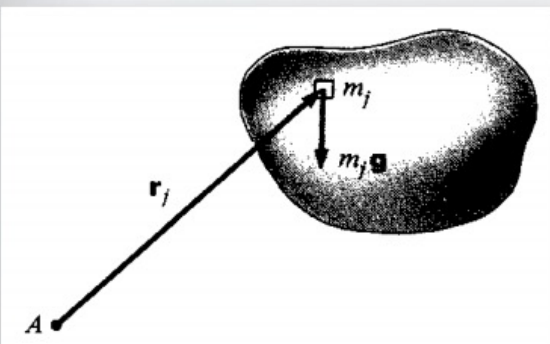


Thus, our result, the Law of Equal areas, hold for the planets. This is nothing but Kepler's Second Law of planetary motion.

(1) planets move in elliptical orbits with the Sun as a focus, (2) a planet covers the same area of space in the same amount of time no matter where it is in its orbit, and (3) a planet's orbital period is proportional to the size of its orbit.

Torque due to Gravity

Consider a rigid body of mass M in a uniform gravitational field $\vec{g}$. The body can be treated as a collection of very large number of particles.



Torque on the jth particle about the point A (some choice for the origin),

$$\vec{\tau}_j = \vec{r}_j \times m_j \vec{g}$$

The total torque is

$$\vec{\tau} = \sum_j \vec{\tau}_j = \sum_j \vec{r}_j \times m_j \vec{g} = \sum_j m_j \vec{r}_j \times \vec{g}$$

The last step follows since m_j is just a scalar.

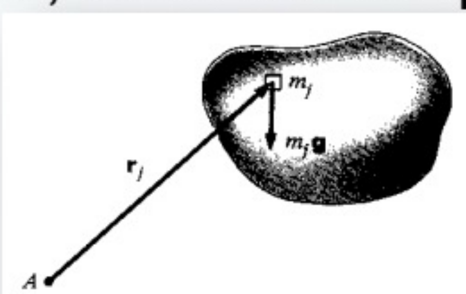
Torque due to Gravity

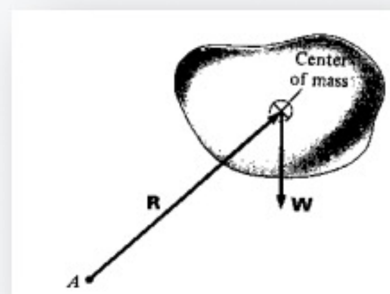
$$\vec{\tau} = \left(\frac{1}{M}\sum_j m_j \vec{r}_j\right) \times (M\vec{g})$$

$\frac{1}{M}\sum_j m_j \vec{r}_j$ is clearly the position of the center of mass (COM) and $M\vec{g}$ is the weight of the object. Thus

$$\vec{\tau} = \vec{R}_{COM} \times \overrightarrow{W} \ .$$

Therefore if the origin is chosen at the position of the COM, then net torque on the object is zero.





In order to balance an object in a uniform gravitational field, the pivot point must be at the COM.