PHY101: Introduction to Physics I

Monsoon Semester 2024

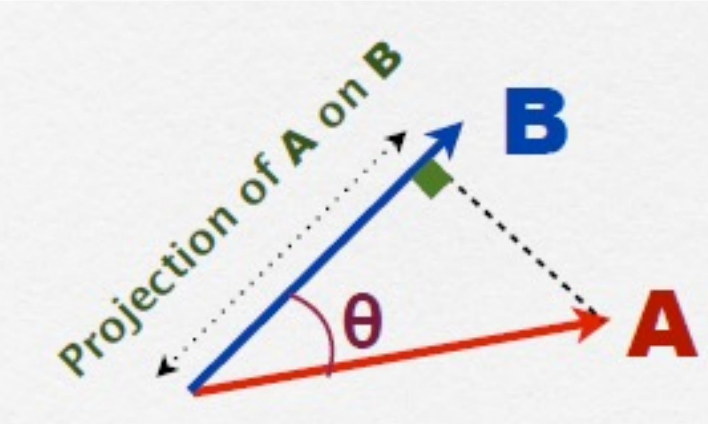
Lecture 6

Department of Physics, School of Natural Sciences,
Shiv Nadar Institution of Eminence, Delhi NCR

Previous Lecture

Properties of vectors

Geometrical addition, multiplication of vectors etc.



This Lecture

Dot product , cross product etc.

Vectors

Properties

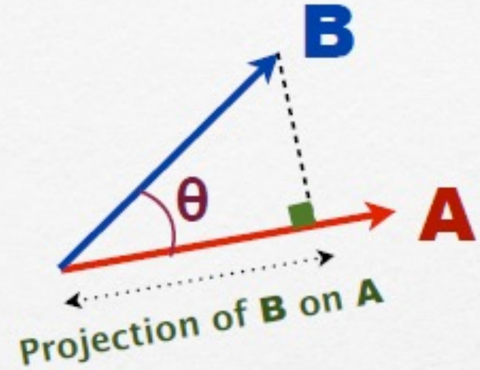
Scalar Multiplication (Dot Product)

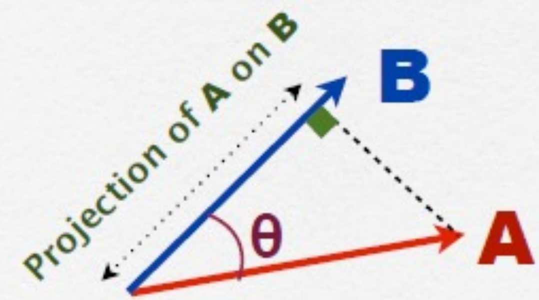
$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$$

$$= A\ (\text{Projection of } \mathbf{B} \text{ on } \mathbf{A})$$

$$= B\ (\text{Projection of } \mathbf{A} \text{ on } \mathbf{B})$$

$$= A\,B\cos\theta$$





Special cases:

$\theta = 0 \rightarrow \mathbf{A}\cdot\mathbf{B} = A\,B$

$\theta = \pi/2 \rightarrow \mathbf{A}\cdot\mathbf{B} = 0$

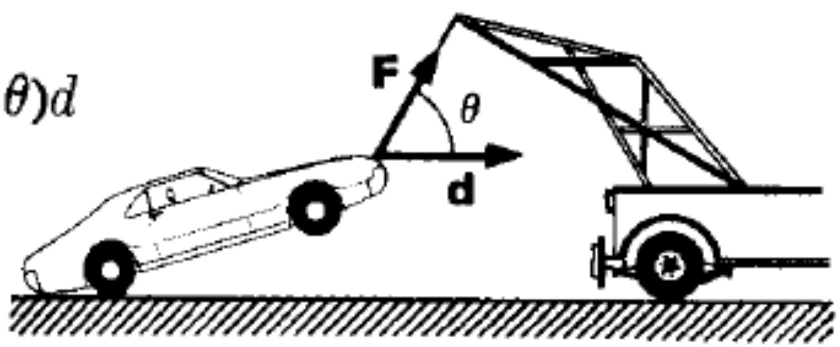
$\theta = \pi \rightarrow \mathbf{A}\cdot\mathbf{B} = -\,A\,B$

Physical Example:

Work done= Scalar product of Force and displacement

$W=\mathbf{F}\cdot\mathbf{d}$

$$W = (F\cos\theta)d$$



Ref: 1) An Introduction to Mechanics” by Kleppner and Kolenkow (McGrawHill, 1973)

Properties

Scalar Multiplication (Dot Product)

Properties of Dot Product

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

Commutative law

$$c(\vec{A}.\vec{B}) = (c\vec{A}).\vec{B} = \vec{A}.(c\vec{B})$$

Associative law

$$(\vec{A} + \vec{B}).\vec{C} = \vec{A}.\vec{C} + \vec{B}.\vec{C}$$

Distributive law

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Properties

Scalar Multiplication (Dot Product)

Properties of Dot Product

Vector Decomposition and the Dot Product

- Dot product of the unit vector $\hat{\imath}$ with itself is unity

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \cos(0) = 1$$

since the unit vector has magnitude $|\hat{\mathbf{i}}| = 1$ and $\cos(0) = 1$.

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

- Dot product of the unit vector $\hat{\imath}$ with the unit vector $\hat{\jmath}$ is zero because the two unit vectors are perpendicular to each other.

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \cos(\pi/2) = 0 \qquad \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

$\hat{\imath}, \hat{\jmath}, \hat{k}$

Ref: 1) MASSACHUSETTS INSTITUTE OF TECHNOLOGY Department of Physics, Review A: Vector Analysis

Vectors

Vector Decomposition and the Dot Product

Dot product of the two vectors

Let $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$ $\qquad$ $\vec{\mathbf{B}} = B_x\hat{\mathbf{i}}$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \cdot B_x\hat{\mathbf{i}}$$

$$= A_x\hat{\mathbf{i}} \cdot B_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \cdot B_x\hat{\mathbf{i}} + A_z\hat{\mathbf{k}} \cdot B_x\hat{\mathbf{i}}$$

$$= A_x B_x(\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}) + A_y B_x(\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}) + A_z B_x(\hat{\mathbf{k}} \cdot \hat{\mathbf{i}})$$

$$= A_x B_x$$

Let $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$ $\qquad$ $\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$

What is the dot product of these two arbitrary vectors ??

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z$$

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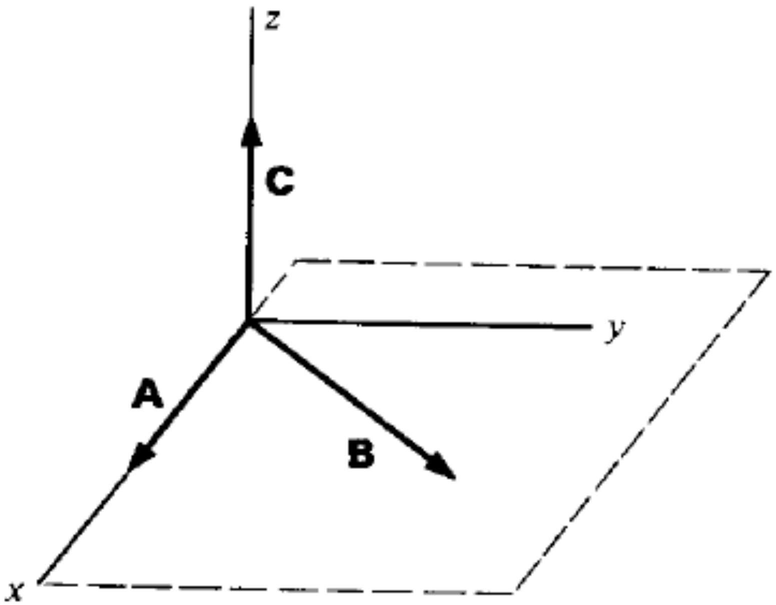
Vectors

Properties

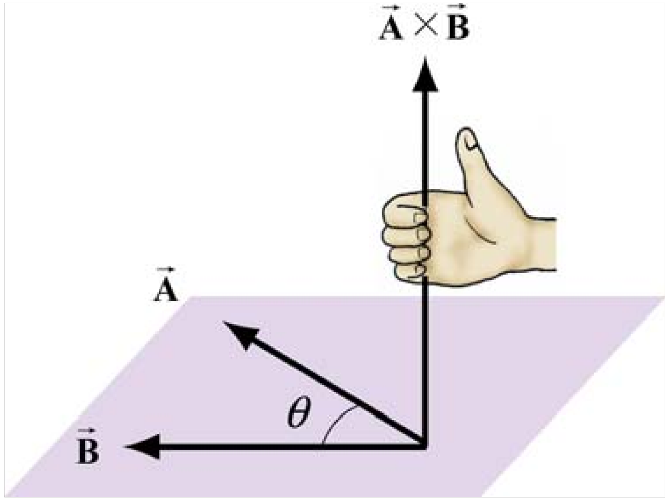
Vector Multiplication (Cross Product)

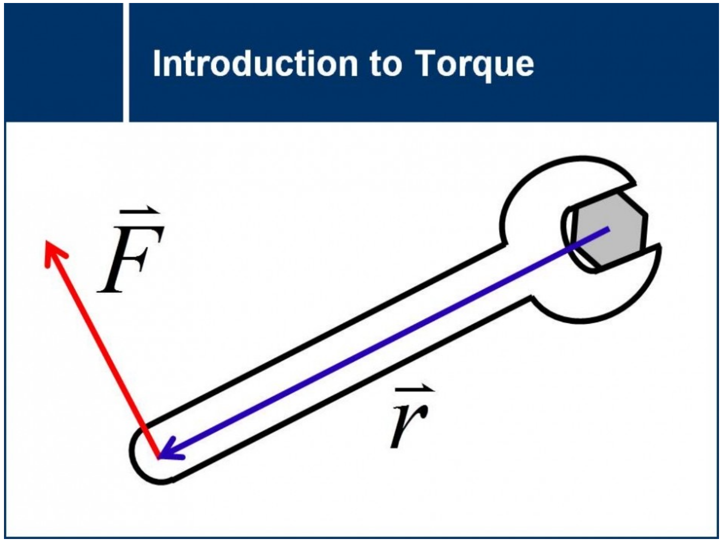


$$\vec{C} = \vec{A} \times \vec{B}$$

$$|\vec{C}| = |\vec{A}|\,|\vec{B}|\sin\theta$$

Right-hand Rule for the Direction of Cross Product





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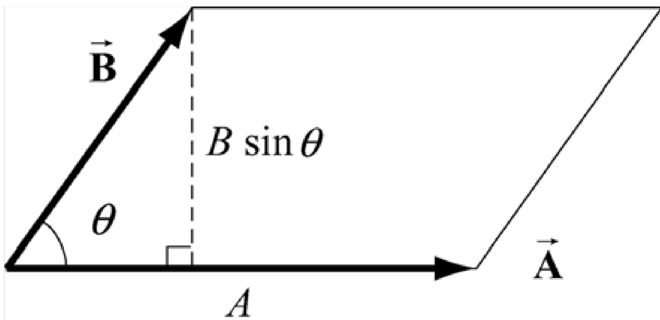
Vectors

Properties

Vector Multiplication (Cross Product)

Geometric interpretation to the magnitude of the cross product

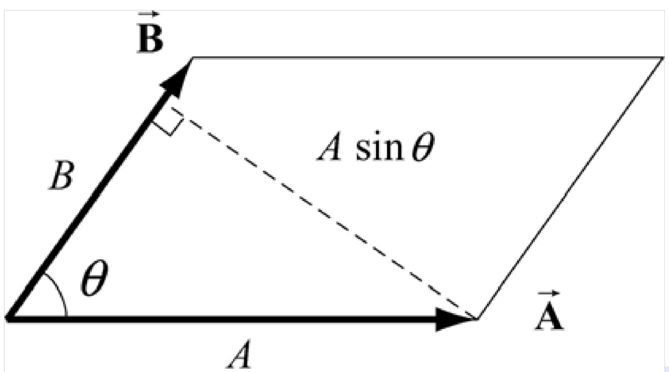


The vectors $\vec{A}$ and $\vec{B}$ form a parallelogram

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = A\left(B\sin\theta\right)$$

B sinθ is the projection of the vector $\vec{B}$ in the direction perpendicular to the vector $\vec{A}$

The area of the parallelogram equals the height times the base, which is the magnitude of the cross product.



$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (A\sin\theta)B$$

A sinθ is the projection of the vector $\vec{A}$ in the direction perpendicular to the vector $\vec{B}$

Two different representations of the height and base of a parallelogram

Ref: 1) MASSACHUSETTS INSTITUTE OF TECHNOLOGY Department of Physics, Review A: Vector Analysis

Properties

Vector Multiplication (Cross Product)

Properties of Cross Product

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$ Anti-commutative law

$$\mathrm{c}(\vec{A} \times \vec{B}) = (c\vec{A}) \times \vec{B} = \vec{A} \times (c\vec{B})$$ Associative law
(while c is a constant)

$$(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$$ Distributive law

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Vectors

Vector Decomposition and the Cross Product

Magnitude of the cross product of the unit vector $\hat{\imath}$ with $\hat{\jmath}$

$\hat{\imath}, \hat{\jmath}, \hat{k}$

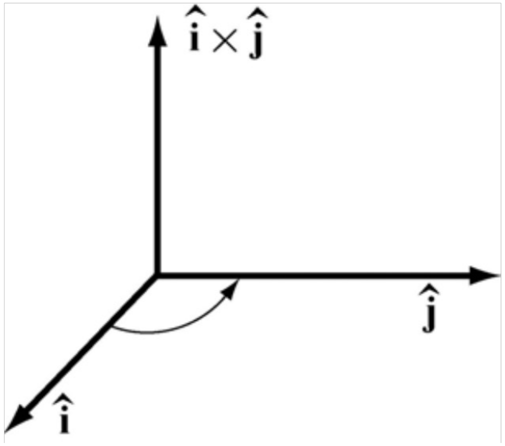
$$|\hat{\mathbf{i}} \times \hat{\mathbf{j}}| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \sin\left(\frac{\pi}{2}\right) = 1$$

since the unit vector has magnitude $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = 1$ and $\sin(\pi/2) = 1$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$



The cross product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, $(\sin(0) = 0)$,

$$|\hat{\mathbf{i}} \times \hat{\mathbf{i}}| = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \sin(0) = 0 \quad \left|\hat{\mathbf{j}} \times \hat{\mathbf{j}}\right| = 0, \quad \left|\hat{\mathbf{k}} \times \hat{\mathbf{k}}\right| = 0$$

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Vectors

Exercise: Vector Decomposition and the Cross Product

Let $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$ $\quad$ $\vec{\mathbf{B}} = B_x\hat{\mathbf{i}}$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times B_x\hat{\mathbf{i}}$$

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_x\hat{\mathbf{i}} \times B_x\hat{\mathbf{i}}) + (A_y\hat{\mathbf{j}} \times B_x\hat{\mathbf{i}}) + (A_z\hat{\mathbf{k}} \times B_x\hat{\mathbf{i}}) \\ &= A_xB_x(\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_yB_x(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_zB_x(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) \\ &= -A_yB_x\hat{\mathbf{k}} + A_zB_x\hat{\mathbf{j}}\end{aligned}$$

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Vectors

Vector Decomposition and the Cross Product

cross product of arbitrary two vectors

Let $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$ $\qquad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$

What is the cross product of these two arbitrary vectors ??

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{(Determinant)}$$

Ref: 1) An Introduction to Mechanics” by Kleppner and Kolenkow (McGrawHill, 1973)

Properties

Vector Multiplication (Cross Product)

Example 1:

For instance, if $\mathbf{A} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{B} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -1 \\ 4 & 1 & 3 \end{vmatrix}$$

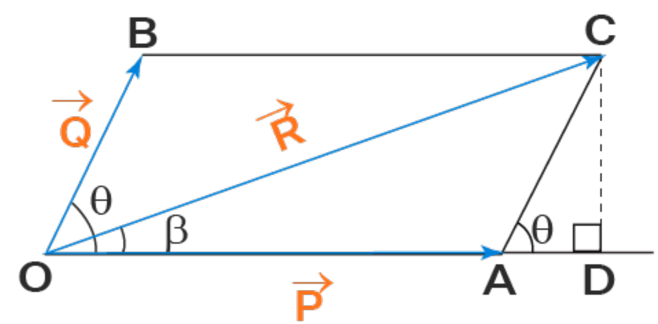
$$= 10\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 11\hat{\mathbf{k}}.$$

Ref: 1) An Introduction to Mechanics” by Kleppner and Kolenkow (McGrawHill, 1973)

Vectors

Question: When will the magnitude of the resultant of two vectors with equal magnitude added vectorially be equal to (i) √2 and (ii) √3 times the magnitude of each ?

Hints: Parallelogram law



|R| = √(P² + Q² + 2PQ cos θ)

β = tan⁻¹[(Q sin θ)/(P + Q cos θ)]

$$R^2 = P^2 + P^2 + 2P.P.\cos\theta$$
$$= 2P^2(1 + \cos\theta)$$
$$= 4P^2\cos^2\theta/2$$

Hence, $R = 2P\cos\theta/2$

(i) $\sqrt{2}P = 2P\ \cos\frac{\theta}{2} \quad \rightarrow \quad \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \quad \rightarrow \quad \theta = 90^0$

(ii) $\sqrt{3}P = 2P\ \cos\frac{\theta}{2} \quad \rightarrow \quad \cos\frac{\theta}{2} = \frac{\sqrt{3}}{2} \quad \rightarrow \quad \theta = 60^0$

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Next Lecture

Tangent vector,
Finding unit vectors etc.

Reserved slides

Vectors

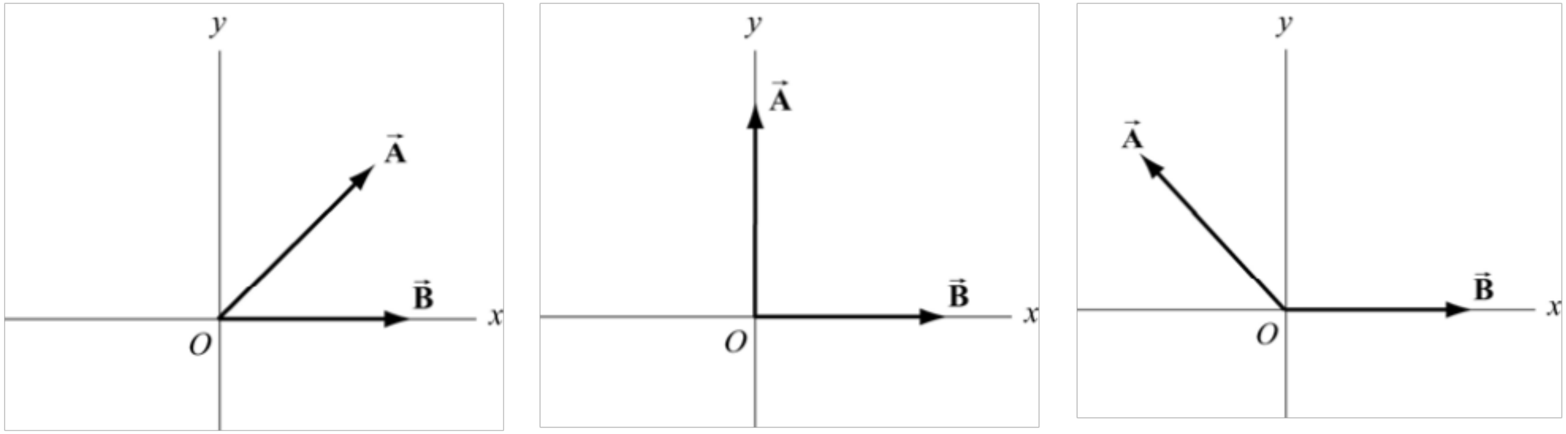
Properties

Scalar Multiplication (Dot Product)

Properties of Dot Product

Vector Decomposition and the Dot Product

Dot product of the two vectors



Dot product is **Positive** **Zero** **Negative**

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