PHY101: Introduction to Physics I

Monsoon Semester 2024

Lecture 23

Department of Physics, School of Natural Sciences,
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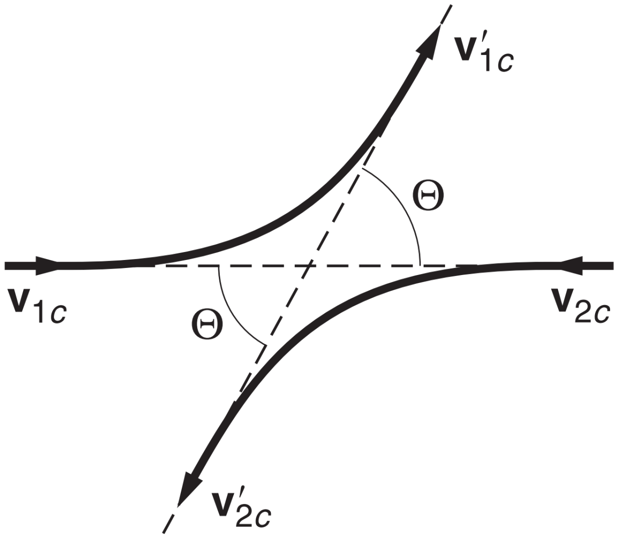
Previous Lecture

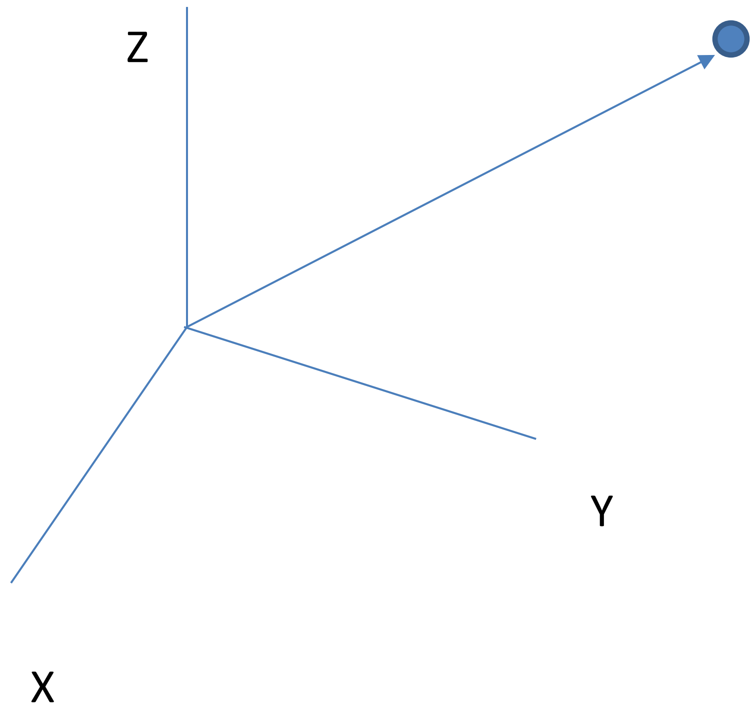
Collison in 2D



This Lecture

Reduced Mass
Angular Momentum
Rigid Body

How to deal with the problem of a single particle subjected to a force?

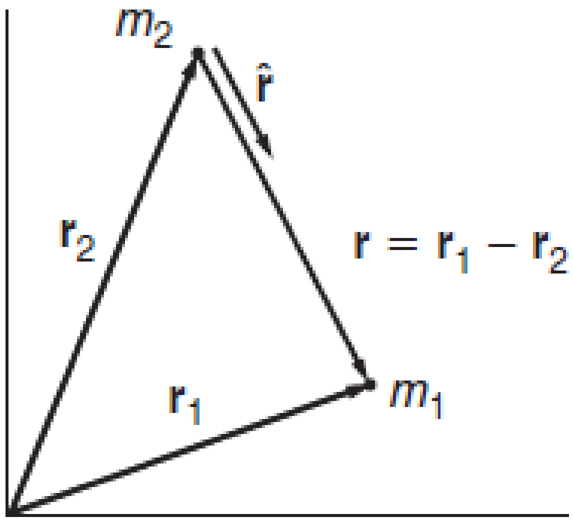


Newton's Second Law of Motion

$$\boldsymbol{F}(\vec{\boldsymbol{r}}) = m\,\ddot{\vec{\boldsymbol{r}}}$$

How to deal with the problem of two particles interacting with central force?

Force between the particle usually depends on the relative position.



$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|.$$

a central force $f(r)\hat{\mathbf{r}}$

two particles interacting under a central force $f(r)\hat{\mathbf{r}}$

The equations of motion are

$$m_1 \ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}}$$

$$m_2 \ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}$$

The equations of motion are

$$m_1\ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}}$$

$$m_2\ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}$$

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{\mathbf{r}}$$

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = f(r)\hat{\mathbf{r}}.$$

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}}, \quad \Rightarrow \quad \mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}$$

μ is the reduced mass

Our old good friend $\vec{\boldsymbol{F}} = m\,\vec{\boldsymbol{a}}$

How to deal with the problem of more than two particles interacting with central force?

For three-body problem

$$\ddot{\mathbf{r}}_1 = -Gm_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - Gm_3 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3},$$

$$\ddot{\mathbf{r}}_2 = -Gm_3 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} - Gm_1 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3},$$

$$\ddot{\mathbf{r}}_3 = -Gm_1 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} - Gm_2 \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3}.$$

There is no general analytical solution to the three-body problem given by simple algebraic expressions and integrals.

How to deal with the problem of more than two particles interacting with central force?

For n-body problem

n-body problem describes how *n* objects will move under one of the physical forces, such as gravity.

It is not solvable in terms of analytic functions.

Two special cases required special attention.

Rigid Body :

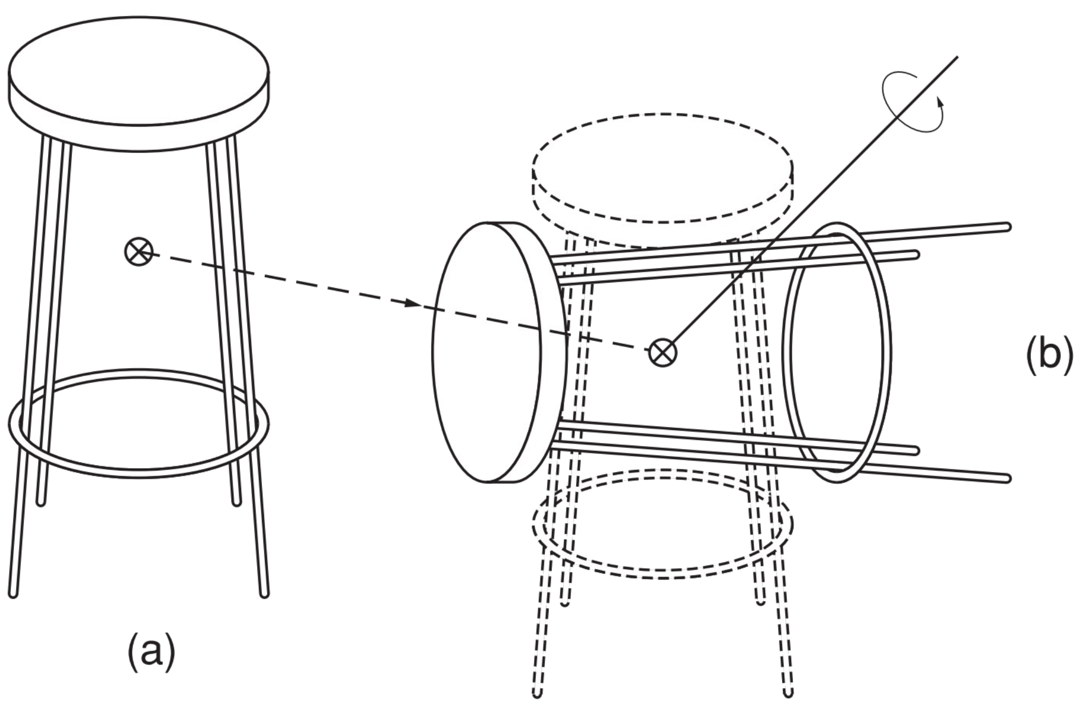
If all the particles are interacting very strongly, such that the relative positions are fixed. Then the problem is reduced to the problem of rigid body. Which can again be cast in the form of Newton's law.

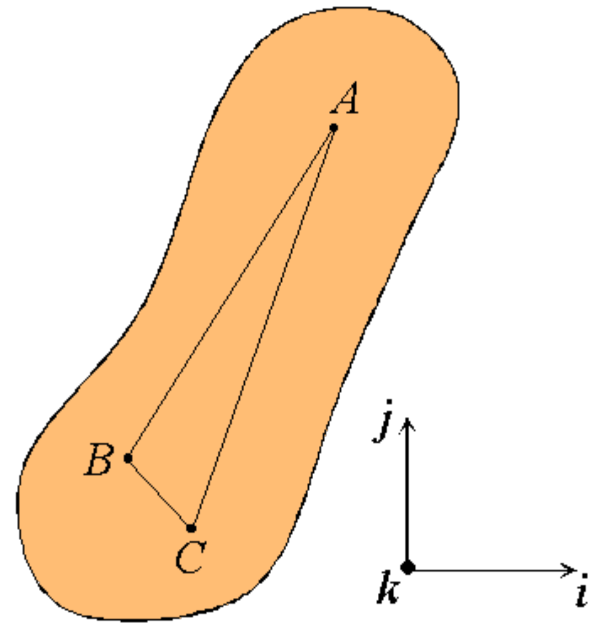
Ideal Gas :

In the second, particles are not interacting with each other ie, problem of ideal gas which we deal in thermodynamics.

Rigid Body

A rigid body is an idealization of a body that does not deform or change shape. Formally it is defined as a collection of particles with the property that the distance between particles remains unchanged during the course of motions of the body.



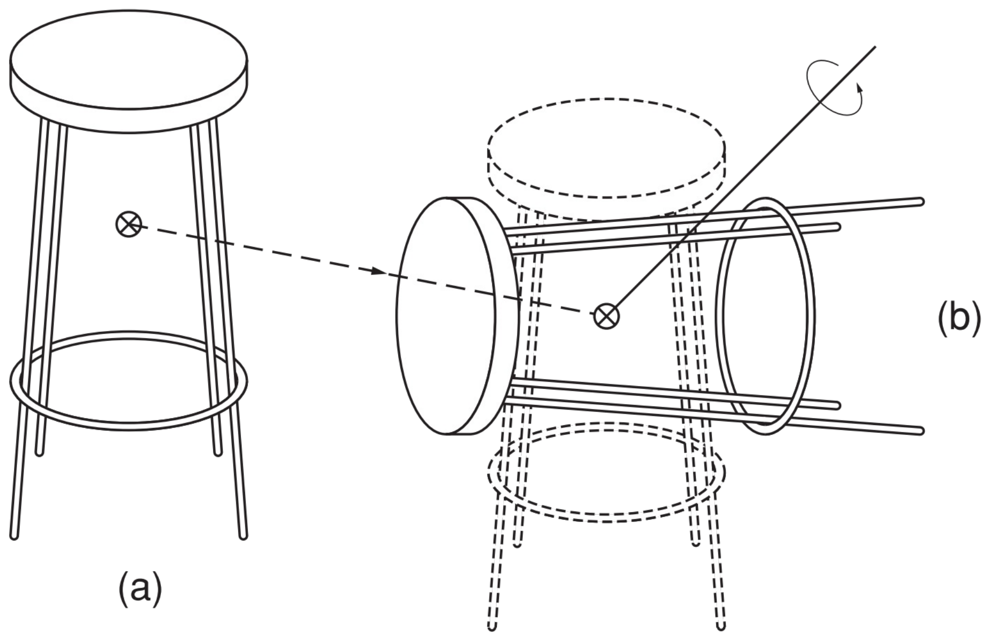


Rigid Body

Dynamics of a Rigid Body

It can be shown that any arbitrary displacement of a rigid body can be decomposed into two independent contributions:

i. Translation of the Center of Mass
ii. Rotation about the Center of Mass



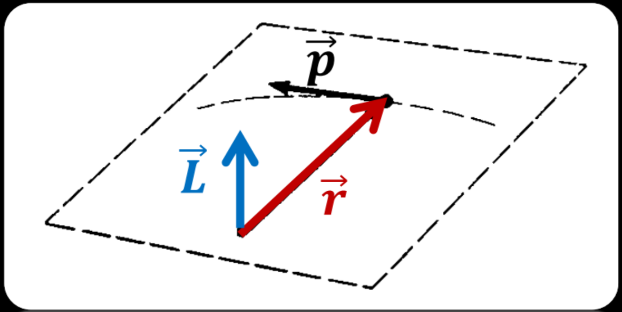
ROTATIONAL MOTION

ANGULAR MOMENTUM

Before we move onto the analysis of extended objects, let's begin with our good old one particle system.

This will be helpful in developing the concepts of Angular momentum and Torque etc.

Consider the particle with position $\vec{r}$ and momentum $\vec{p}$ with respect to some coordinate system. Then angular momentum, $\vec{L}$, of this particle is defined as



$$\vec{L} = \vec{r} \times \vec{p}$$

ROTATIONAL MOTION

ANGULAR MOMENTUM

In the general case, when and are not restricted to the XY-pane, we have

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} \text{ and } \vec{p} = p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k}$$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

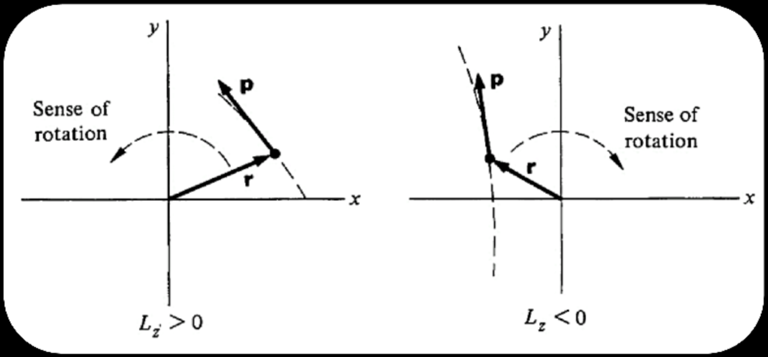
$$= \left(y\,p_z - z\,p_y\right)\hat{\imath} + (z\,p_x - x\,p_z)\hat{\jmath} + \left(x\,p_y - y\,p_x\right)\hat{k}$$

$$\equiv L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$$

ROTATIONAL MOTION

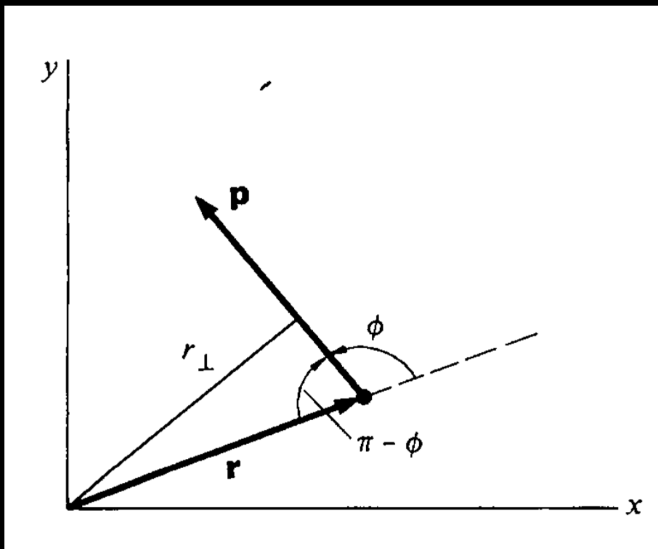
ANGULAR MOMENTUM

- $\vec{L}$ involves the cross (or vector) product of $\vec{r}$ and $\vec{p}$, thus it will lie perpendicular to the plane consisting $\vec{r}$ and $\vec{p}$.
- The direction can be determined by right- handed screw rule.
- If $\vec{r}$ and $\vec{p}$ lie in the XY plane, then $\vec{L}$ is in the Z-direction.
- $\vec{L}$ is in the positive Z-direction if the sense of rotation of the point about the origin is counterclockwise .
- $\vec{L}$ is in the negative Z-direction if the sense of rotation of the point about the origin is clockwise.



ROTATIONAL MOTION

ANGULAR MOMENTUM



If we consider the plane consisting $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{p}}$ as the XY plane, then

$$\vec{L} = r\,p\,\sin\phi\,\hat{k}$$

But, as clear from the figure,

$$r\sin\phi = r\sin(\pi-\phi) = r_\perp$$

Thus,

$$\vec{L} = r_\perp\,p\,\hat{k} = L_z\hat{k}$$

$r_\perp$ is the perpendicular distance between the origin and the line of $\vec{\boldsymbol{p}}$.

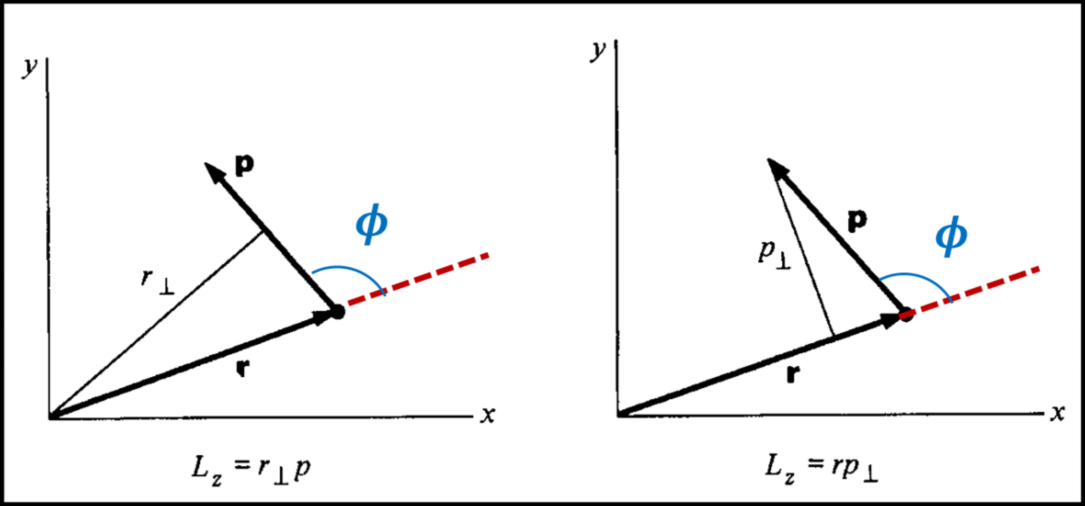
$L_z = r_\perp\,p$ is the z-component of $\vec{L}$. Clearly in this special case ($\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{p}}$ in the XY plane) the x- and y-components of $\vec{L}$ are zero.

ROTATIONAL MOTION

ANGULAR MOMENTUM

Instead of considering the perpendicular component of $\vec{\boldsymbol{r}}$ to evaluate L_z we can also consider the perpendicular component (perpendicular to $\vec{\boldsymbol{r}}$) $p_\perp = p\ \sin\phi$ of $\vec{\boldsymbol{p}}$. This gives $L_z = r\ p_\perp$.

The two approaches are sketched below:



ROTATIONAL MOTION

ANGULAR MOMENTUM

Instead of using the geometrical interpretation, the above results for L_z can also be obtained as follows:

Let us resolve $\vec{\boldsymbol{r}}$ into terms of two (independent) vectors,

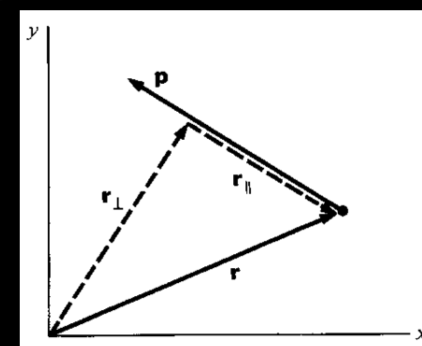
$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\perp} + \vec{\boldsymbol{r}}_{\parallel}$$

Here $\vec{\boldsymbol{r}}_{\perp}$ is perpendicular to $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{r}}_{\parallel}$ is parallel to $\vec{\boldsymbol{p}}$. Then

$$\vec{L} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}} = (\vec{\boldsymbol{r}}_{\perp} + \vec{\boldsymbol{r}}_{\parallel}) \times \vec{\boldsymbol{p}} = \vec{\boldsymbol{r}}_{\perp} \times \vec{\boldsymbol{p}} + \vec{\boldsymbol{r}}_{\parallel} \times \vec{\boldsymbol{p}}$$

$$= \left(r_{\perp}\, p \sin\frac{\pi}{2}\right) \hat{k} + (r_{\parallel}\, p \sin 0)\, \hat{k}$$

$$= \left(r_{\perp}\, p \sin\frac{\pi}{2}\right) \hat{k} \equiv L_z \hat{k}$$



giving $L_z = r_{\perp}\, p,$ as before. $L_z = r\, p_{\perp}$ can similarly be obtained by resolving $\vec{\boldsymbol{p}}$ in vectors perpendicular ($p_{\perp}$) and parallel ($p_{\parallel}$) to $\vec{\boldsymbol{r}}$.