PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 17

Multipole Expansion: The Monopole Term

Multipole Expansion: The Dipole Term

Multipole Expansion

[redacted] we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{c_n}{r^{n+1}}$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{c_0}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^3} + \cdots \right].$$

This gives the **Multipole expansion** for the potential, i.e., $V(\mathbf{r})$ resolved into the contributions from monopole, dipole, quadrupole,... terms.

Multipole Expansion: The Monopole Term

Let us examine the first term in the expansion — the **Monopole term**

$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{c_0}{r}$$
$$= \frac{1}{4\pi\epsilon_0 r}\int (r')^0 P_0(\cos\theta')\rho(\mathbf{r}')\,d\tau'$$
$$= \frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')\,d\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{r},$$

which is exactly what we expected. This term gives the crudest approximation to the potential $V(\mathbf{r})$: The monopole approximation.

The multipole expansion is dominated (at large r) by the monopole term

Multipole Expansion: The Dipole Term

Now let us examine the second term in the expansion – the **Dipole term**. If the total charge in the distribution is zero, then dipole term (*unless, it also vanishes*) is the dominant term in the potential. We have,

$$\begin{aligned}
V_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{c_1}{r^2} = \frac{1}{4\pi\epsilon_0\, r^2}\int (r')^1 P_1(\cos\theta')\rho(\mathbf{r}')\, d\tau' \\
&= \frac{1}{4\pi\epsilon_0\, r^2}\int r'\cos\theta'\,\rho(\mathbf{r}')\, d\tau' = \frac{1}{4\pi\epsilon_0\, r^2}\int (\hat{\mathbf{r}}\cdot\mathbf{r}')\rho(\mathbf{r}')\, d\tau' \\
&= \frac{1}{4\pi\epsilon_0\, r^2}\hat{\mathbf{r}}\cdot\left(\int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2}.
\end{aligned}$$

We used above $r'\cos\theta' = |\hat{\mathbf{r}}||\mathbf{r}'|\cos\theta' = \hat{\mathbf{r}}\cdot\mathbf{r}'$. Also, we pulled out the dot-product operation outside the integral using the distributive property and observing that **r'** is the only vector left in the integrand once $\hat{\mathbf{r}}$ is outside the integral.

Multipole Expansion: The Dipole Term

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The quantity p used in the previous equation is the **dipole moment** of the distribution,

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau' .$$

The dipole moment is determined by the geometry (size, shape and density) of the charge distribution. It gives an idea about the effective separation (*polarity*) of the positive and negative charges in the given charge distribution.

If the charge distribution consists of discrete charges (say n of them located at $\mathbf{r}_1'$, $\mathbf{r}_2'$, ..., $\mathbf{r}_n'$), then

$$\rho(\mathbf{r}') = \sum_{j=1}^{n} q_j \delta^3(\mathbf{r}' - \mathbf{r}_j').$$

Therefore

$$\mathbf{p} = \int \mathbf{r}' \left(\sum_{j=1}^{n} q_j \delta^3(\mathbf{r}' - \mathbf{r}_j') \right) d\tau' = \sum_{j=1}^{n} q_j \int \mathbf{r}' \delta^3(\mathbf{r}' - \mathbf{r}_j')\, d\tau'$$

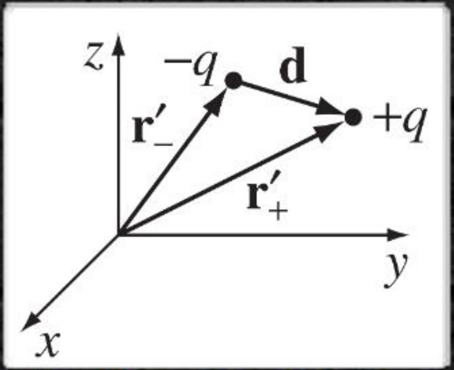
Multipole Expansion: The Dipole Term

The trivial integration over the Dirac-delta function then yields the dipole moment

$$\mathbf{p} = \sum_{j=1}^{n} q_j \mathbf{r}'_j$$

for a system of discrete charges.

For a physical dipole, we have n=2, with equal and opposite charges. Thus,



$$\mathbf{p} = (+q)(\mathbf{r}'_+) + (-q)(\mathbf{r}'_-) = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d},$$

where $\mathbf{d}$ is the vector from the negative charge to the positive one. Plugging this in the expression for V_{dip}, we obtain

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot(q\mathbf{d})}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2},$$

which is consistent with the expression obtained earlier [redacted]

Multipole Expansion: The Dipole Term

Notice, however, that this is only the approximate potential of the physical dipole—evidently, there are higher multipole contributions. Of course, as you go farther and farther away, V_{dip} becomes a better and better approximation, since the higher terms die off more rapidly with increasing **r**.

By the same token, at a fixed **r** the dipole approximation improves as you shrink the separation **d**.

To construct a perfect (point) dipole whose potential is given exactly

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}. \qquad V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$

you'd have to let **d** approach zero.

A *physical* dipole becomes a *pure* dipole, then, in the rather artificial limit $d \to 0$, $q \to \infty$, with the product $qd = p$ held fixed.

Multipole Expansion: The Dipole Term

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}.$$

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$

When someone uses the word “dipole,” you can’t always tell whether they mean a physical dipole (with finite separation between the charges) or an ideal (point) dipole. If in doubt, assume that d is small enough (compared to r) that you can safely apply the above equation.

Dipole moments are *vectors*, and they add accordingly: if you have two dipoles, $\mathbf{p}_1$ and $\mathbf{p}_2$, the total dipole moment is $\mathbf{p}_1 + \mathbf{p}_2$. For instance, with four charges at the corners of a square, as shown in Fig. 3.30, the net dipole moment is zero. You can see this by combining the charges in pairs (vertically, $\downarrow + \uparrow = 0$, or horizontally, $\rightarrow + \leftarrow = 0$) or by adding up the four contributions individually,

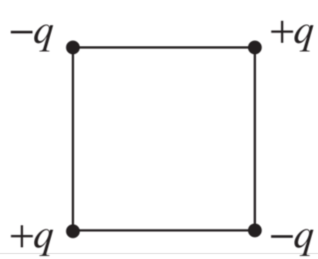


FIGURE 3.30

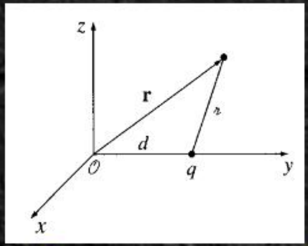
Multipole Expansion: Choice of Origin

A point charge q kept at the origin ($\mathbf{r}'=0$) constitutes a *pure* monopole, and there is only one term in the multipole expansion—the monopole term:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

However, if the charge is moved away from the origin ($\mathbf{r}'\neq 0$), it's no longer a pure monopole. The dipole moment formula tells us that it has a nonzero dipole moment $\mathbf{p} = q\,\mathbf{r}'$ (and similarly other terms in the multipole expansion). This is consequence of the fact that now the exact potential at point $\mathbf{r}$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}'|},$$



which when expanded in powers of $(1/r)$ has not only the monopole term (the expression at the top), but also other contributions.

Therefore, shifting the origin (or equivalently moving the charge(s)) can radically alter the multipole expansion.

Multipole Expansion: Choice of Origin

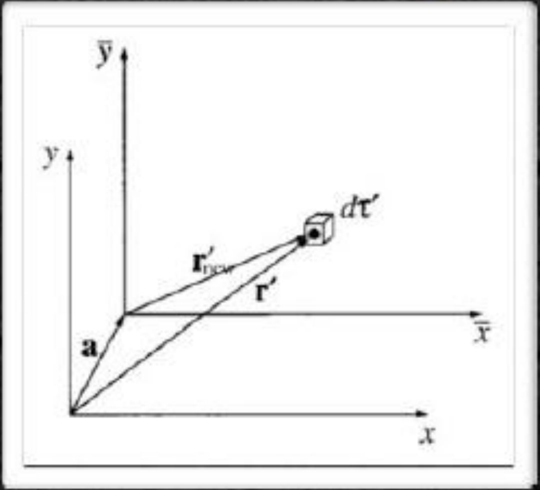
The monopole term, $V_{\text{mono}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$, remains unaffected by the shift of origin, since the **monopole moment** (the total charge) Q is not changed (*it's an invariant under the spatial translation*).

The dipole term, $V_{\text{dip}}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2}$, does, in general, change under the origin shift, **except when the total charge Q in the system is zero**. This can be easily seen as follows:

Suppose we shift the origin by a constant vector **a**, then the new dipole moment is

$$\mathbf{p}_{\text{new}} = \int \mathbf{r}'_{\text{new}}\,\rho(\mathbf{r}')\,d\tau' = \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau'$$

$$= \int \mathbf{r}'\,\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' = \mathbf{p}_{\text{old}} - Q\mathbf{a}.$$

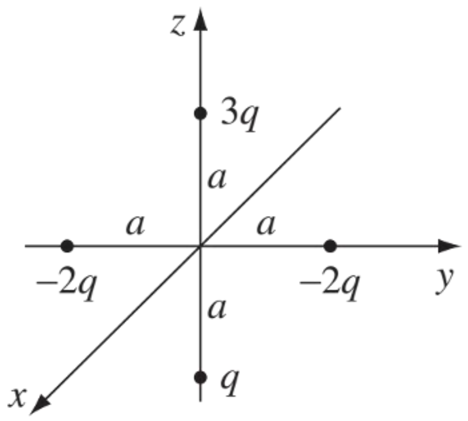


Therefore if Q=0, then the dipole moment remains unchanged under the origin shift (e.g., in the case of a *physical* dipole).

Electric Dipole

P 1

Four particles (one of charge q, one of charge $3q$, and two of charge $-2q$) are placed as shown in Fig. 3.31, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)



$\mathbf{p} = (3qa - qa)\,\hat{\mathbf{z}} + (-2qa - 2q(-a))\,\hat{\mathbf{y}} = 2qa\,\hat{\mathbf{z}}$. Therefore

$$V \cong \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

and $\mathbf{p}\cdot\hat{\mathbf{r}} = 2qa\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 2qa\cos\theta$, so

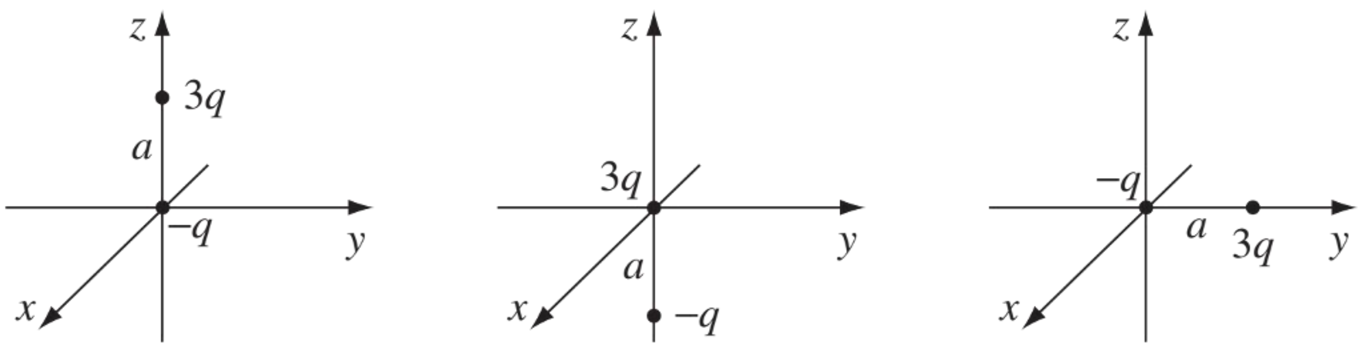
$$V \cong \boxed{\frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}.} \quad \text{(Dipole.)}$$

Electric Dipole

P 2

Two point charges, $3q$ and $-q$, are separated by a distance a. For each of the arrangements in Fig. 3.35, find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both the monopole and dipole contributions).

Fig. 3.35



(a) (b) (c)

(a) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right] = \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]}.$

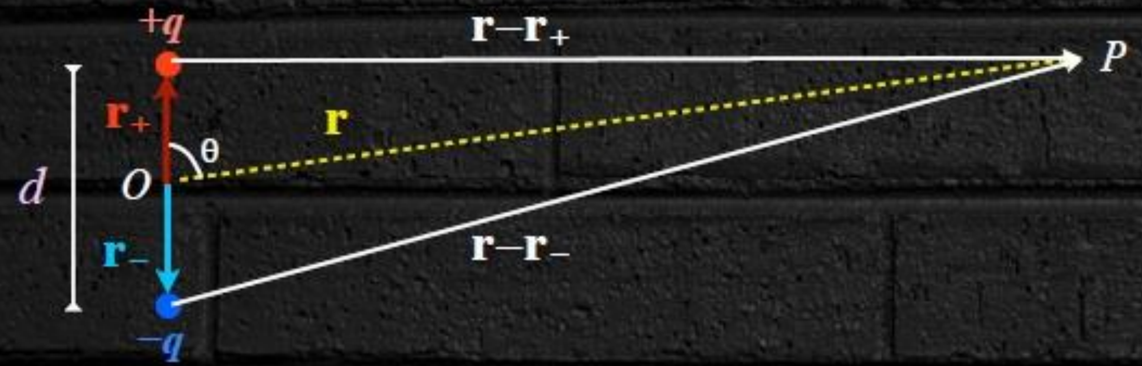
(b) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{qa\,\hat{\mathbf{z}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]}.$

(c) (i) $Q = \boxed{2q,}$ (ii) $\mathbf{p} = \boxed{3qa\,\hat{\mathbf{y}},}$ (iii) $V \cong \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\phi}{r^2}\right]}$

$\hat{\mathbf{y}}\cdot\hat{\mathbf{r}} = \sin\theta\sin\phi)$

Potential due to a Physical Electric Dipole

Before we consider an arbitrary charge distribution, let us calculate the approximate potential because of a physical electric dipole at a far away point. A physical electric dipole consists of two equal and opposite charges ($+q$ and $-q$) separated by a distance d (the dipole length).



$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{q\,d\cos\theta}{r^2}.$$

This gives the approximate potential due to a dipole at a large distance away. We can see that it decays as $1/r^2$.

Electric Field of a Physical Electric Dipole

We now calculate the electric field due to a dipole. For the sake of simplicity, let us fix our coordinate system such the dipole moment **p** lies at the origin and points in the z-direction. With such a choice we have forced azimuthal symmetry in the problem (independence of the azimuthal angle ϕ). The potential at a point (r,θ) is given by

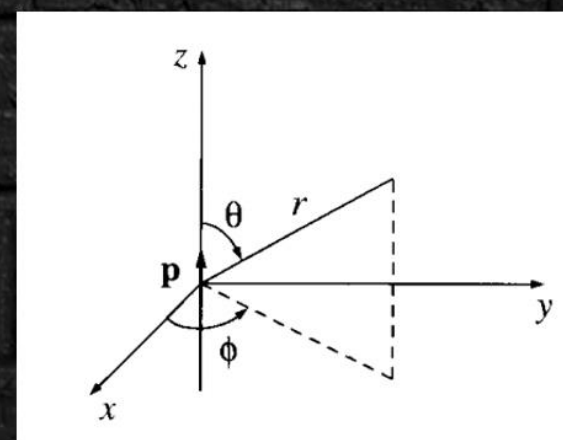
$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}.$$

The electric field can be calculated by taking the negative gradient of the electric potential (*look for the gradient expression in spherical coordinates; Griffiths' book* [redacted]

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0.$$



Electric Field of a Dipole

Combining the components together, we obtain the expression for the electric field as

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

This result makes explicit reference to a particular coordinate system (spherical) and assumes a particular orientation for **p** (along z-direction). It can be recast in the following coordinate-free form:

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right].$$

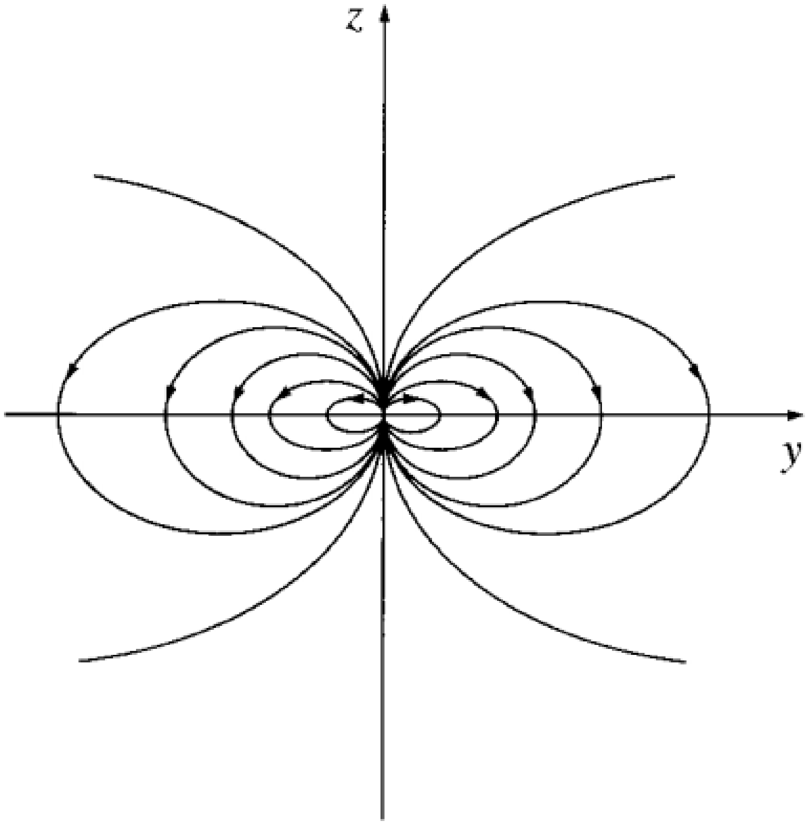
The choice of **p** lying along the z axis, $\mathbf{p} = p\,\hat{\mathbf{z}} = p\,(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$ in this expression does return back the expression at the top. (*We have used here the unit vector along z-direction expressed in terms of the unit vectors of spherical coordinate system;* [redacted])

$$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = p\cos\theta\,\hat{\mathbf{r}} - p\sin\theta\,\hat{\boldsymbol{\theta}}$$

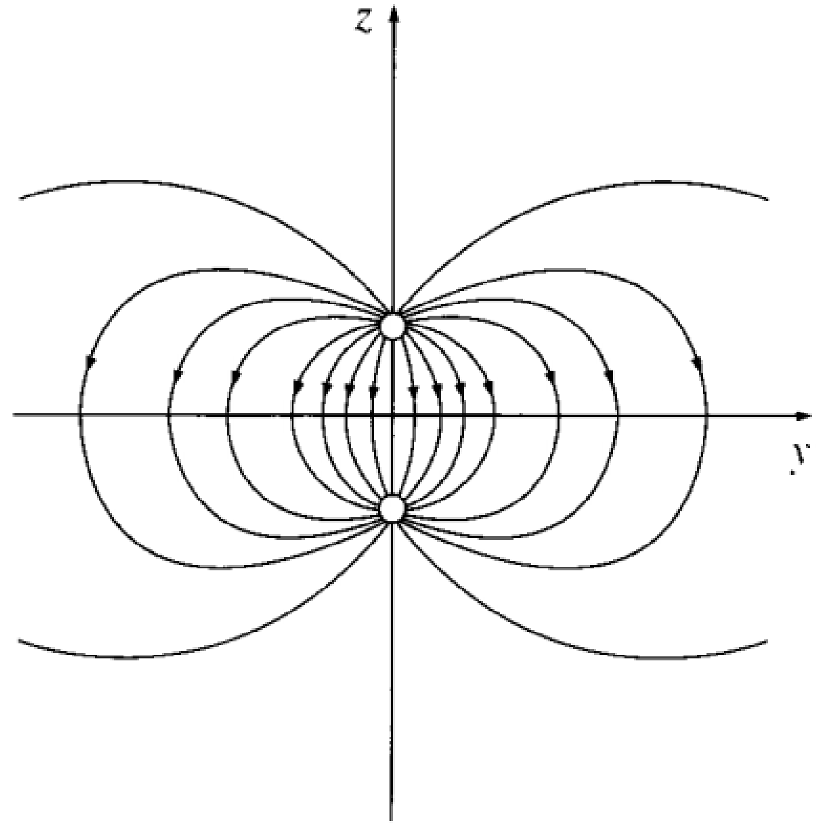
So $3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}} = 2p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}.$

Electric Field of a Dipole

r>>d



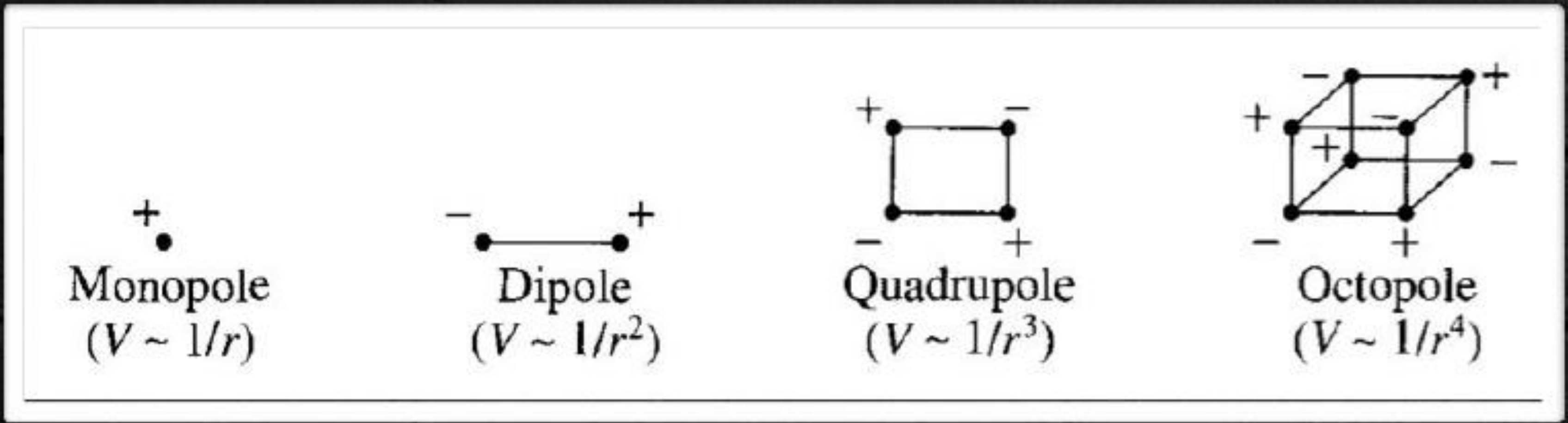
(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Potential due to a Physical Electric Dipole

Thus we found that potential* due to an electric monopole goes as 1/r. For an electric dipole, the potential* goes as $1/r^2$. Incidentally, if we put a pair of equal and opposite dipoles to make a quadrupole, the potential* has the dependence $1/r^3$. Similarly, for back-to-back quadrupoles (an octopole) the potential* goes like $1/r^4$; and so on.



Electric Dipole

P 1

A "pure" dipole p is situated at the origin, pointing in the z direction.

(a) What is the force on a point charge q at $(a, 0, 0)$ (Cartesian coordinates)?

(b) What is the force on q at $(0, 0, a)$?

(c) How much work does it take to move q from $(a, 0, 0)$ to $(0, 0, a)$?

$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

$$V_{\text{dip}}(r, \theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}.$$

(a) This point is at $r = a$, $\theta = \frac{\pi}{2}$, $\phi = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\epsilon_0 a^3}(-\hat{\mathbf{z}})$; $\mathbf{F} = q\mathbf{E} = \boxed{-\frac{pq}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{z}}.}$

(b) Here $r = a$, $\theta = 0$, so $\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}(2\hat{\mathbf{r}}) = \frac{2p}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{z}}.$ $\boxed{\mathbf{F} = \frac{2pq}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{z}}.}$

(c) $W = q\,[V(0,0,a) - V(a,0,0)] = \frac{qp}{4\pi\epsilon_0 a^2}\left[\cos(0) - \cos\left(\frac{\pi}{2}\right)\right] = \boxed{\frac{pq}{4\pi\epsilon_0 a^2}}.$