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PHY102: Introduction to Physics-II

Tutorial – 2

1. Find the directional derivative of $f(x,y) = -2xy - \frac{x^2}{2} - \frac{y^2}{2}$ at (-2,2) in the direction of $\frac{3\pi}{4}$.

Since the direction is given as an angle, the unit vector is:

$$u = \begin{bmatrix} cos\left(\frac{3\pi}{4}\right) \\ sin\left(\frac{3\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Computing the partial derivatives yields:

$$\frac{\delta}{\delta x}(-2xy - \frac{x^2}{2} - \frac{y^2}{2}) = -2y - x$$

and

$$\frac{\delta}{\delta y}(-2xy - \frac{x^2}{2} - \frac{y^2}{2}) = -2x - y$$

The directional derivative is then:

$$D_u\left(-2xy - \frac{x^2}{2} - \frac{y^2}{2}\right) = (-2y - x)\left(-\frac{\sqrt{2}}{2}\right) + (-2x - y)\left(\frac{2}{2}\right)$$

$$= \frac{2}{2}y - \frac{2}{2}x$$

At (-2, 2),

$$D_u f(-2,2) = \frac{\sqrt{2}}{2}(2) - \frac{2}{2}(-2)$$

$$= 2\sqrt{(2)}$$

2. Find the directional derivative of $f(x,y) = x^3e^{-y}$ at (3,2) in the direction of $\vec{v} = [\begin{smallmatrix}3\\4\end{smallmatrix}]$.

For this example, the direction is given as a vector, but not a unit vector. To find the unit vector, divide vector v by its magnitude:

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||} = \frac{\begin{bmatrix}3\\4\end{bmatrix}}{\sqrt{3^2+4^2}} = \frac{\begin{bmatrix}3\\4\end{bmatrix}}{5} = \begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\end{bmatrix}$$

We then compute the gradient as follows:

$$\nabla(x^3e^{-y}) = \begin{bmatrix}\frac{\delta}{\delta x}(x^3e^{-y})\\\frac{\delta}{\delta y}(x^3e^{-y})\end{bmatrix} = \begin{bmatrix}3x^2e^{-y}\\-x^3e^{-y}\end{bmatrix} = \begin{bmatrix}\frac{3x^2}{e^y}\\-\frac{x^3}{e^y}\end{bmatrix}$$

At (3, 2), $\nabla(3,2) = \begin{bmatrix}\frac{27}{e^2}\\-\frac{27}{e^2}\end{bmatrix}$. Thus:

$$D_u(3,2) = \nabla f(3,2)\hat{v} = \frac{27}{e^2}\cdot\frac{3}{5} - \frac{27}{e^2}\cdot\frac{4}{5} = -\frac{27}{5e^2} = -0.73$$

3. Compute the gradient of $f(x, y, z) = (x^2 + y^2 + z^2)^{-1}$.

Solution: By symmetry, it suffices to compute $\frac{\partial f}{\partial x}$, as $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are obtained through analogous computations. To compute $\frac{\partial f}{\partial x}$, we use the chain rule:

$\frac{\partial f}{\partial x} = -(x^2+y^2+z^2)^{-2}\frac{\partial}{\partial x}(x^2+y^2+z^2) = -2x\,(x^2+y^2+z^2)^{-2} = -2\,f^2\,x$

Finally, we get $\nabla f = -2\,f^2\,(x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$

4. The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east, of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and 1 mile east of South Hadley? In what direction is the slope steepest, at that point?
(To tutors: You may not need to do all the calculations in the class but provide hints)

The extrema of a multidimensional function occur where the function's gradient is equal to zero.

$$\begin{aligned}
\nabla h &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\right) h \\
&= \hat{\mathbf{x}}\frac{\partial h}{\partial x} + \hat{\mathbf{y}}\frac{\partial h}{\partial y} \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)] \\
&\qquad + \hat{\mathbf{y}}\frac{\partial}{\partial y}[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)] \\
&= \hat{\mathbf{x}}[10(2y - 6x - 18)] \\
&\qquad + \hat{\mathbf{y}}[10(2x - 8y + 28)]
\end{aligned}$$

The system of equations to solve is then

$$\left.\begin{aligned} 10(2y - 6x - 18) = 0 \\ 10(2x - 8y + 28) = 0 \end{aligned}\right\} \quad \Rightarrow \quad x = -2 \quad \text{and} \quad y = 3.$$

As a result, the extremum (or critical point) of $h(x, y)$ is $(-2, 3)$. Calculate the second derivatives of $h(x, y)$.

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}[10(2y - 6x - 18)] = 10(-6) = -60$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}[10(2x - 8y + 28)] = 10(-8) = -80$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial x}[10(2x - 8y + 28)] = 10(2) = 20$$

Apply the second derivative test to determine whether this extremum is a maximum, minimum, or saddle point.

$$D(-2, 3) = \begin{vmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{vmatrix}(-2, 3) = \frac{\partial^2 h}{\partial x^2}(-2, 3)\frac{\partial^2 h}{\partial y^2}(-2, 3) - \left[\frac{\partial^2 h}{\partial x \partial y}(-2, 3)\right]^2 = (-60)(-80) - (20)^2 = 4400$$

Since $D(-2,3) = 4400 > 0$ and $h_{xx}(-2,3) = -60 < 0$, the extremum at $(-2,3)$ is a maximum as expected. Therefore, the top of the hill is located 2 miles west and 3 miles north of South Hadley. Plug in $x = -2$ and $y = 3$ into $h(x,y)$ to find out how high the hill is.

$$h(-2,3) = 720$$

Therefore, the hill is 720 feet high. The direction of the steepest slope at any point is given by the gradient function.

$$\nabla h(x,y) = \hat{\mathbf{x}}[10(2y - 6x - 18)] + \hat{\mathbf{y}}[10(2x - 8y + 28)]$$

At a point 1 mile north and 1 mile east of South Hadley, the direction of the steepest slope is

$$\nabla h(1,1) = -220\hat{\mathbf{x}} + 220\hat{\mathbf{y}}.$$

Its magnitude tells how steep the slope is in feet per mile.

$$|\nabla h(1,1)| = \sqrt{(-220)^2 + (220)^2} = 220\sqrt{2} \approx 311$$

5. Compute the divergence and curl of the following fields

$$\vec{\boldsymbol{E}}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{\imath}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{\jmath}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{k}}$$

for $x^2 + y^2 + z^2 \neq 0$ (Coulomb electric field for a point charge)

$$\vec{\boldsymbol{B}}(x,y,z) = -\frac{y}{x^2+y^2}\hat{\boldsymbol{\imath}} + -\frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

for $x^2 + y^2 \neq 0$ (Magnetic field outside an infinite current-carrying wire)

$$\vec{\boldsymbol{A}}(x,y,z) = -y\hat{\boldsymbol{\imath}} + x\hat{\jmath}$$

(Vector potential for a uniform magnetic field)

(To tutors: You may not need to do all the calculations in the class but provide hints)

3 Sol^n (a) $\vec{E}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$

$= E_x(x,y,z)\,\hat{i} + E_y(x,y,z)\,\hat{j} + E_z(x,y,z)\,\hat{k}$

* Divergence :-

$$\vec{\nabla}\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right)$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

We have $\frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= \frac{(x^2+y^2+z^2)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3}$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly $\frac{\partial E_y}{\partial y} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}$

& $\frac{\partial E_z}{\partial z} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}$

$$\therefore \vec{\nabla}\cdot\vec{E} = \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{5/2}}(x^2+y^2+z^2)$$

$$= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}} = 0$$

Actually $\vec{\nabla}\cdot\vec{E} = 4\pi\,\delta^3(\vec{r})$ (for $\vec{r} \neq 0$)

This problem will be revisited after introducing Dirac-delta function.

* Curl :-

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

We have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = \frac{\partial}{\partial y}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] - \frac{\partial}{\partial z}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{3}{2} \cdot \frac{2y \cdot z}{(x^2+y^2+z^2)^{5/2}} + \frac{3}{2} \cdot \frac{2z \cdot y}{(x^2+y^2+z^2)^{5/2}} = 0$$

Similarly, $\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$.

$$\therefore \quad \vec{\nabla} \times \vec{E} = 0$$

(b) $\vec{B}(x,y,z) = \frac{-y}{(x^2+y^2)}\hat{i} + \frac{x}{(x^2+y^2)}\hat{j} \equiv B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

(with $B_z = 0$)

* Divergence :-

$$\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left[\frac{-y}{(x^2+y^2)}\right] + \frac{\partial}{\partial y}\left[\frac{x}{(x^2+y^2)}\right] + \frac{\partial}{\partial z}(0)$$

$$= \frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

* Curl :-

$$\vec{\nabla}\times\vec{B} = \hat{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{j}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

We have

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right) = 0$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(0) = 0$$

&

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2 + x^2-y^2}{(x^2+y^2)^2} = 0 \quad \text{for } \vec{r} \neq 0.$$

Actually, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ with $\vec{J}$ in this case involves the Dirac delta function $\delta(x^2+y^2)$ signifying that the current source lies along z-axis.

(c) $\vec{A}(x,y,z) = -y\hat{i} + x\hat{j} \equiv A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$(A_z = 0)$

* Divergence :-

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$

$$= 0 + 0 = 0$$

* Curl :-

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

We have, $\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x) = 0$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0) = 0$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2.$$

$$\therefore \ \vec{\nabla} \times \vec{A} = 2\hat{k}$$