PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 5

Fundamental theorems for gradients, divergence and curls

Fundamental Theorem for Gradients

Consider a scalar function $f\,(x,y,z)$. The change, df, in f associated with infinitesimal displacement $d\mathbf{l}$ along a given path is

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right) \\ &= \boldsymbol{\nabla} f \cdot d\mathbf{l} \end{aligned}$$

Therefore

$$\int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} f \cdot d\mathbf{l} = \int_{\mathbf{a}}^{\mathbf{b}} df = f(\mathbf{b}) - f(\mathbf{a})$$

Dependence only on the end points!

Fundamental Theorem for Gradients

Geometrical interpretation: Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up (LHS of the equation below), or you could place altimeters at the top and the bottom, and subtract the two readings (RHS of the equation below); you should get the same answer either way. (D. J. Griffiths)

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla f \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

Note again that the line integrals depend, in general, on the actual path $\mathcal{P}$ connecting $\mathbf{a}$ and $\mathbf{b}$. However, in the RHS of the equation above there is no reference to the path P, it just depends on the end points. Thus gradients have a very special property that their line integrals are path independent.

Fundamental Theorem for Gradients

- **Corollary 1**

$\int_{\mathbf{a}}^{\mathbf{b}} \nabla f \cdot d\mathbf{l}$ is independent of the path ($\boldsymbol{\mathcal{P}}$) taken between $\mathbf{a}$ and $\mathbf{b}$.

- **Corollary 2:**

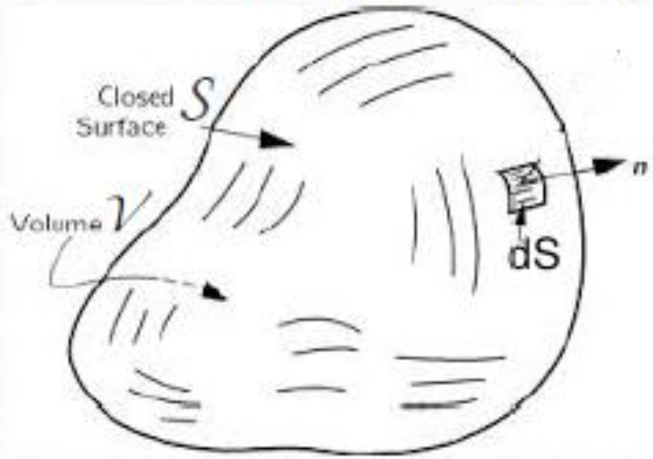
$\oint \nabla f \cdot d\mathbf{l} = 0$, since the beginning and end points are identical and hence $f(\mathbf{b}) - f(\mathbf{a}) = 0$.

For a conservative vector function $\mathbf{V}_c$, the line integral is zero, thus Corollary 2 implies that it is expressible as the gradient of some scalar function. This proves the assertion made earlier (See the Line Integral *slides in the beginning).*

Fundamental Theorem for Divergences

Consider a volume $\mathcal{V}$ enclosed by a surface $\mathcal{S}$ in a region of space where some vector function $\mathbf{V}$ exists. Then according to this theorem

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{V})\, d\tau = \oint_{\mathcal{S}} \mathbf{V} \cdot d\mathbf{S}$$



This theorem is referred to as **Gauss's theorem**, **Ostrogradsky's theorem** or simply **Divergence theorem.**

Thus we see that integral of a divergence over a volume equals the boundary term (the integral over the surface that bounds the volume).

Fundamental Theorem for Divergences

Geometrical interpretation: Let $\mathbf{V}$ represent the flow of an incompressible fluid, and consider lots of faucets inside some volume enclosed by a surface, pouring out the fluid. There are two ways we could determine how much fluid is being produced: (a) we could count up all the faucets, recording how much each puts out (LHS of the equation below), or (b) we could go around the boundary, measuring the flow at each point, and add it all up (RHS of the equation below). We will get the same answer either way. (D. J. Griffiths)

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{V}) d\tau = \oint_{\mathcal{S}} \mathbf{V} \cdot d\mathbf{S}$$

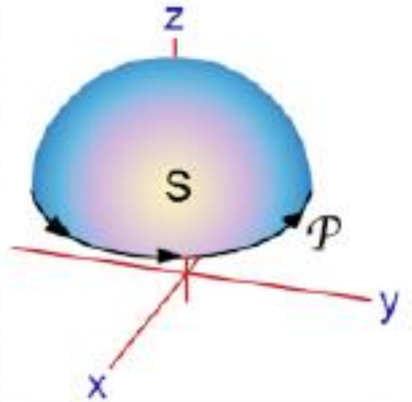
$$\downarrow \qquad\qquad \downarrow$$

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

Fundamental Theorem for Curls

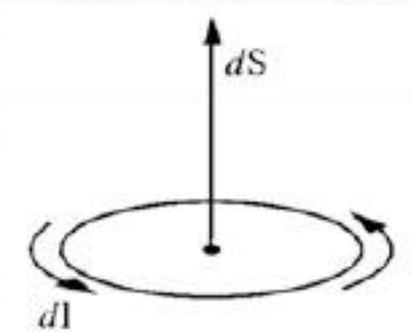
Consider a surface $\mathcal{S}$ enclosed by a path (contour) $\boldsymbol{\mathcal{P}}$. If there exists a vector function in this region, then the surface integral of curl of $\mathbf{V}$ over $\mathcal{S}$ is equal to the line integral of $\mathbf{V}$ around $\boldsymbol{\mathcal{P}}$:

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{V}) \cdot d\mathbf{S} = \oint_{\mathcal{P}} \mathbf{V} \cdot d\mathbf{l}$$



This theorem is referred to as the **Stokes' theorem**.

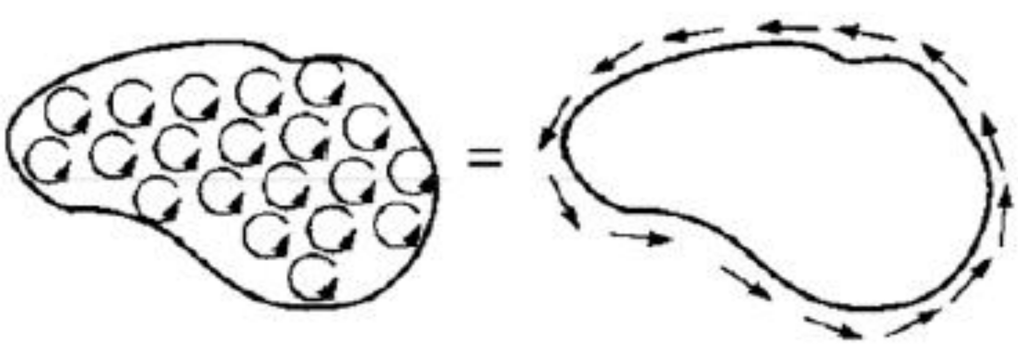
The ambiguity in the direction of $d\mathbf{S}$ can be fixed by using the right-hand rule: If the fingers point in the direction of the line integral, then the thumb fixes the direction of $d\mathbf{S}$.



Fundamental Theorem for Curls

Geometrical interpretation: Recall that the curl measures the "twist" of the vectors **V**; a region of high curl is a whirlpool— if you put a tiny paddle wheel there, it will rotate. Now, the integral of the curl over some surface represents the "total amount of whirl" (LHS of the equation below), and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (RHS of the equation below). (D. J. Griffiths)

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{V}) \cdot d\mathbf{S} = \oint_{\mathcal{P}} \mathbf{V} \cdot d\mathbf{l}$$



Fundamental Theorem for Curls

- **Corollary 1:**

$\int_S (\boldsymbol{\nabla} \times \mathbf{V}) \cdot d\mathbf{S}$ depends only on the boundary line, not on the particular surface used.

- **Corollary 2:**

$\oint (\boldsymbol{\nabla} \times \mathbf{V}) \cdot d\mathbf{S} = 0$ for any closed surface, since the boundary enclosing the surface shrinks to zero in this case.

Optional

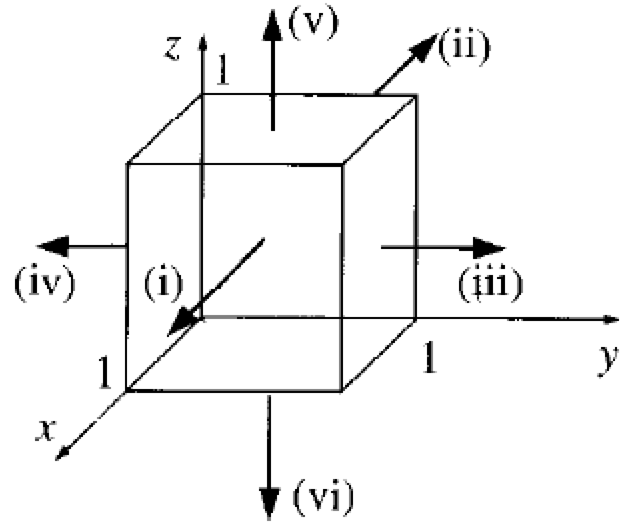
Examples

Example of Divergence theorem

Check the divergence theorem using the function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

and the unit cube situated at the origin



$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.$$

$$\nabla \cdot \mathbf{v} = 2(x + y),$$

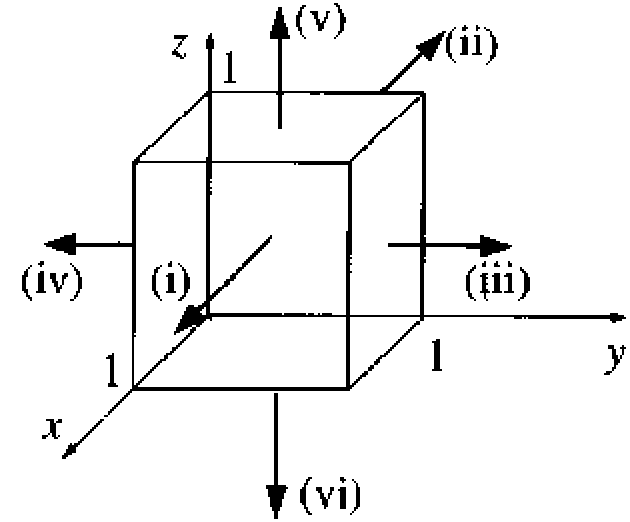
$$\int_{\mathcal{V}} 2(x+y)\, d\tau = 2\int_0^1 \int_0^1 \int_0^1 (x+y)\, dx\, dy\, dz,$$

$$\int_0^1 (x+y)\, dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2} + y)\, dy = 1, \quad \int_0^1 1\, dz = 1. \qquad \int_{\mathcal{V}} \nabla \cdot \mathbf{v}\, d\tau = 2.$$

Divergence theorem: Surface Integral

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

$$\oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} =$$



(i) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1\int_0^1 y^2 dy\,dz = \tfrac{1}{3}.$$

(ii) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1\int_0^1 y^2\,dy\,dz = -\tfrac{1}{3}.$$

(iii) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1\int_0^1 (2x + z^2)\,dx\,dz = \tfrac{4}{3}.$$

(iv) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1\int_0^1 z^2\,dx\,dz = -\tfrac{1}{3}.$$

(v) $$\int \mathbf{v}\cdot d\mathbf{a} = \int_0^1\int_0^1 2y\,dx\,dy = 1.$$

(vi) $$\int \mathbf{v}\cdot d\mathbf{a} = -\int_0^1\int_0^1 0\,dx\,dy = 0.$$

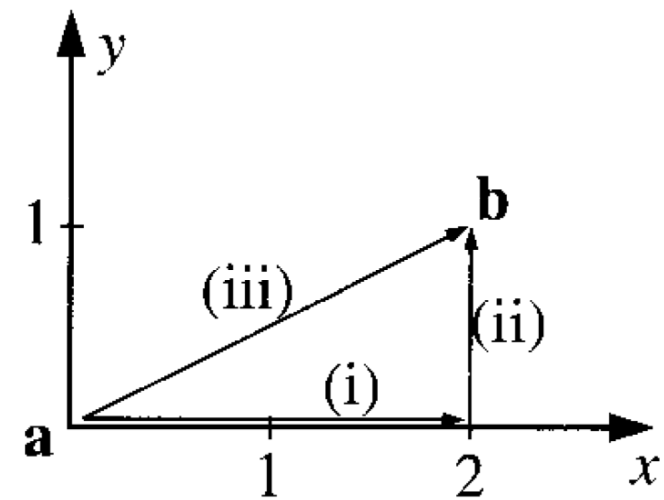
$$\oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a} = \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{1}{3} + 1 + 0 = 2,$$

$$\int_{\mathcal{V}} (\nabla\cdot\mathbf{v})\,d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a}.$$

Question: Fundamental theory of gradients

Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla f \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

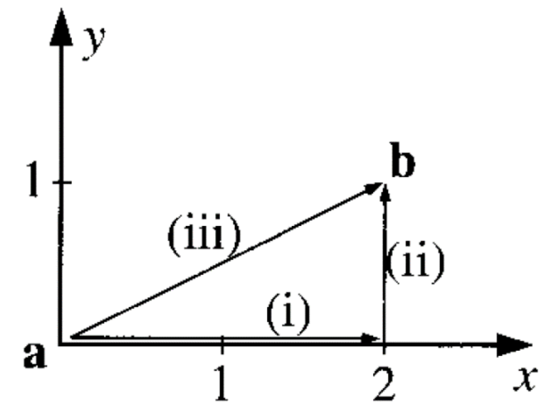


Solution: Although the integral is independent of path, we must *pick* a specific path in orde
to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig.). As always
$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$;

$$\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}.$$

$$T(\boldsymbol{b}) = 2, \quad T(\boldsymbol{a}) = 0 \Rightarrow T(\boldsymbol{b}) - T(\boldsymbol{a}) = 2$$

$$\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}.$$



(i) $y = 0$; $d\mathbf{l} = dx\,\hat{\mathbf{x}}$, $\nabla T \cdot d\mathbf{l} = y^2\,dx = 0$, so

$$\int_{\mathrm{i}} \nabla T \cdot d\mathbf{l} = 0.$$

(ii) $x = 2$; $d\mathbf{l} = dy\,\hat{\mathbf{y}}$, $\nabla T \cdot d\mathbf{l} = 2xy\,dy = 4y\,dy$, so

$$\int_{\mathrm{ii}} \nabla T \cdot d\mathbf{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2.$$

Evidently the total line integral is 2. Is this consistent with the fundamental theorem? Yes: $T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2$.

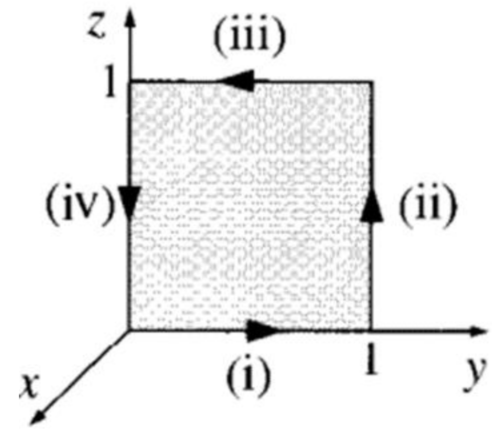
Now, just to convince you that the answer is independent of path, let me calculate the same integral along path iii (the straight line from **a** to **b**):

(iii) $y = \frac{1}{2}x$, $dy = \frac{1}{2}\,dx$, $\nabla T \cdot d\mathbf{l} = y^2\,dx + 2xy\,dy = \frac{3}{4}x^2\,dx$, so

$$\int_{\mathrm{iii}} \nabla T \cdot d\mathbf{l} = \int_0^2 \tfrac{3}{4}x^2\,dx = \tfrac{1}{4}x^3\Big|_0^2 = 2.$$

Example of Stokes' theorem

Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$. Check Stokes' theorem for the square surface shown in Fig.



Solution: Here

$$\int_{\mathcal{S}} (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \oint_{\mathcal{P}} \mathbf{V} \cdot d\mathbf{l}$$

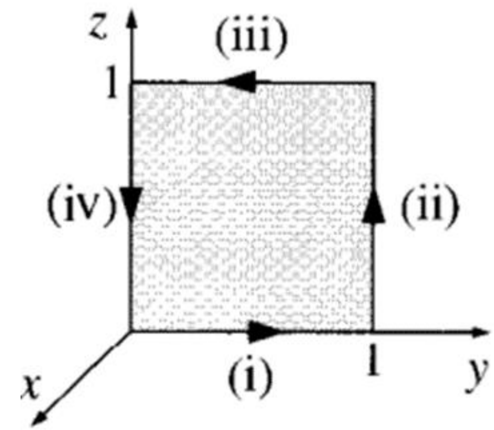
$$\nabla \times \mathbf{v} = (4z^2 - 2x)\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}} \quad \text{and} \quad d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}.$$

(In saying that $d\mathbf{a}$ points in the x direction, we are committing ourselves to a counterclockwise line integral. We could as well write $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}$, but then we would be obliged to go clockwise.) Since $x = 0$ for this surface,

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2\,dy\,dz = \tfrac{4}{3}.$$

Stokes' theorem: line integral part

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}.$$



$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2\, dy\, dz = \tfrac{4}{3}.$$

$$\oint \mathbf{v} \cdot d\mathbf{l} =$$

(i) $x = 0, \quad z = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\, dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2\, dy = 1,$

(ii) $x = 0, \quad y = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 4z^2\, dz, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2\, dz = \frac{4}{3},$

(iii) $x = 0, \quad z = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\, dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2\, dy = -1,$

(iv) $x = 0, \quad y = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 0, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0\, dz = 0.$

$$\Rightarrow \oint \mathbf{v} \cdot d\mathbf{l} = 1 + \tfrac{4}{3} - 1 + 0 = \tfrac{4}{3}. \quad \Rightarrow \quad \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}.$$