PHY101: Introduction to Physics I

Monsoon Semester 2024
Lecture 27

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Previous Lecture

Dynamics of pure rotation
Rotational kinetic energy

This Lecture

Translation and rotation

Translation and rotation

Let us consider a rigid body composed of N particles with masses m_j ($j = 1, \dots, N$) and position vectors $\vec{r}_j$ measured relative to the origin of some choice of an inertial frame of reference.

The angular momentum of the *j*-th particle is:

$$\vec{L}_j = \vec{r}_j \times \vec{p}_j$$

$$= \vec{r}_j \times m_j \dot{\vec{r}}_j$$

The total angular momentum is:

$$\vec{L} = \sum_j^N \vec{L}_j = \sum_j^N \vec{r}_j \times m_j \dot{\vec{r}}_j$$

Translation and rotation

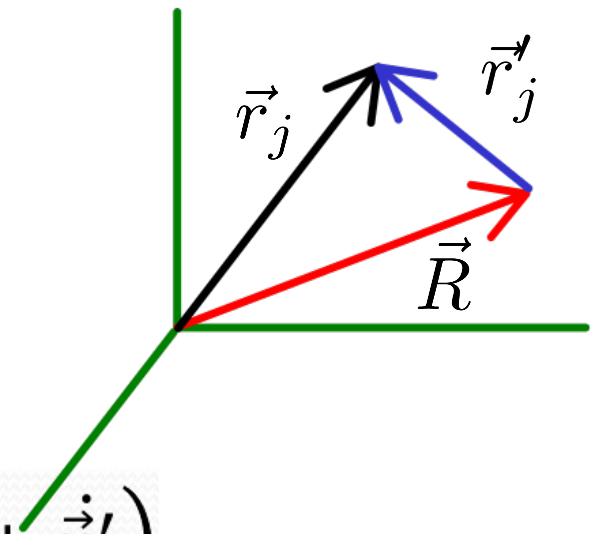
The position of the center of mass (COM) is given by:

$$\vec{R} = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j \,,$$

where *M* is the total mass of the body.

The position of the particles with respect to the COM is:

$$\vec{r}_j' = \vec{r}_j - \vec{R}.$$



Therefore,

$$\vec{L} = \sum_{j=1}^{N} \vec{r}_j \times m_j \dot{\vec{r}}_j$$

$$= \sum_{j=1}^{N} (\vec{R} + \vec{r}_j') \times m_j \left(\dot{\vec{R}} + \dot{\vec{r}}_j'\right)$$

Translation and rotation

$$\vec{L} = \sum_{j=1}^{N}(\vec{R} + \vec{r}_j') \times m_j\left(\dot{\vec{R}} + \dot{\vec{r}}_j'\right)$$
$$= \sum_{j=1}^{N}\left(\vec{R} \times m_j\dot{\vec{R}}\right) + \sum_{j=1}^{N}\left(\vec{r}_j' \times m_j\dot{\vec{R}}\right)$$
$$+ \sum_{j=1}^{N}(\vec{R} \times m_j\dot{\vec{r}}_j') + \sum_{j=1}^{N}(\vec{r}_j' \times m_j\dot{\vec{r}}_j')$$

Let us examine the terms one by one.

First term:

$$\sum_{j=1}^{N}\left(\vec{R} \times m_j\dot{\vec{R}}\right)$$
$$= \vec{R} \times \left(\sum_{j=1}^{N} m_j\right)\dot{\vec{R}} = \vec{R} \times M\dot{\vec{R}}$$
$$= \vec{R} \times M\vec{V} = \vec{R} \times \vec{P}$$

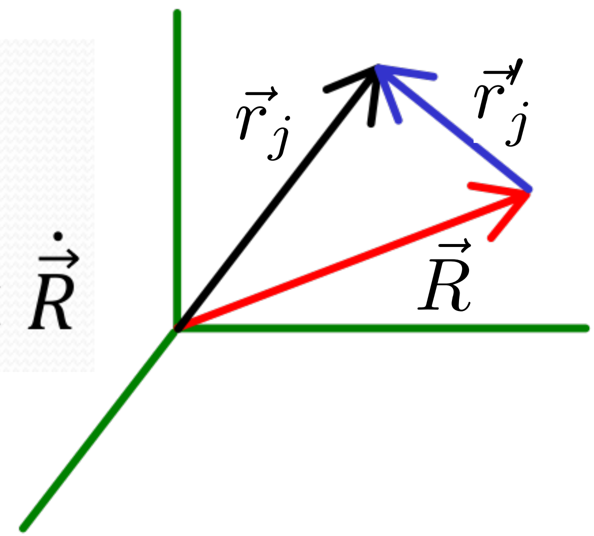
Here $\vec{V}$ and $\vec{P}$ represent the velocity and the momentum of the Center of Mass, respectively. Thus, the first term is nothing but the angular momentum of the center of mass.

Translation and rotation

Second term:

$$\sum_{j=1}^{N}\left(\vec{r}_j{}' \times m_j\dot{\vec{R}}\right)$$

$$= \sum_{j=1}^{N}\left(m_j\vec{r}_j{}' \times \dot{\vec{R}}\right) = \left(\sum_{j=1}^{N} m_j\vec{r}_j{}'\right) \times \dot{\vec{R}}$$



Now $\displaystyle\sum_{j=1}^{N} m_j\vec{r}_j{}'$

$$= \sum_{j=1}^{N} m_j(\vec{r}_j - \vec{R}) = \sum_{j=1}^{N} m_j\vec{r}_j - \left(\sum_{j=1}^{N} m_j\right)\vec{R}$$

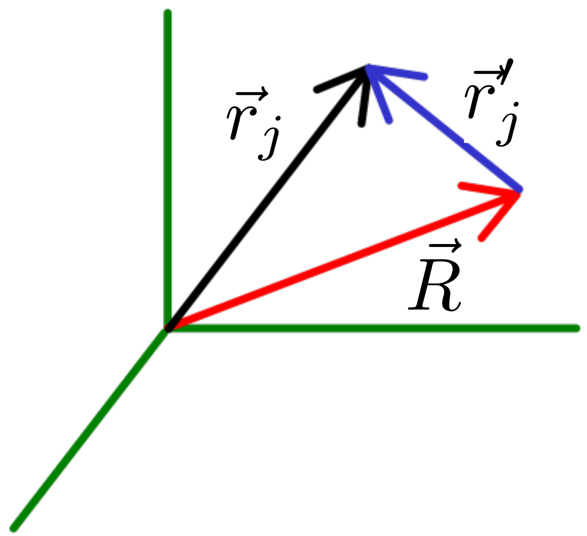
$$= M\vec{R} - M\vec{R} = 0$$

Hence, the second term vanishes.

Translation and rotation

Third term:

$$\sum_{j=1}^{N}\left(\vec{R}\times m_j\dot{\vec{r}}_j'\right)$$
$$=\vec{R}\times\sum_{j=1}^{N} m_j\dot{\vec{r}}_j'$$
$$=\vec{R}\times\frac{d}{dt}\left(\sum_{j=1}^{N} m_j\,\vec{r}_j'\right)$$
$$=\vec{R}\times\frac{d}{dt}(0)=0.$$

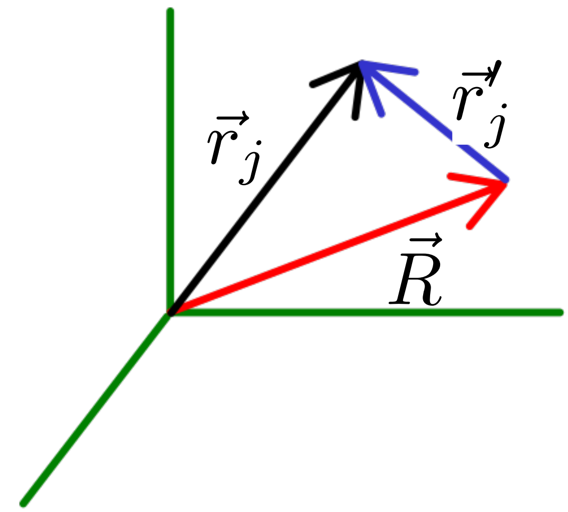


Since $\sum_{j=1}^{N} m_j\,\vec{r}_j' = 0$, as proved in the last slide.

Translation and rotation

Fourth term:

$$\sum_{j=1}^{N} \left(\vec{r}_j' \times m_j \dot{\vec{r}}_j'\right)$$
$$= \sum_{j=1}^{N} \left(\vec{r}_j' \times m_j \vec{v}_j'\right)$$
$$= \sum_{j=1}^{N} \left(\vec{r}_j' \times \vec{p}_j'\right)$$



Here $\vec{v}_j'$ and $\vec{p}_j'$ represent the velocity and momentum of the particle relative to the center of mass.

Therefore, this term represent the angular momentum of the rigid body with respect to the center of mass.

Collecting all the terms we obtain:

$$\vec{L} = \vec{R} \times \vec{P} + \sum_{j=1}^{N} \left(\vec{r}_j' \times \vec{p}_j'\right).$$

Translation and rotation

The *z*-component of the angular momentum is:

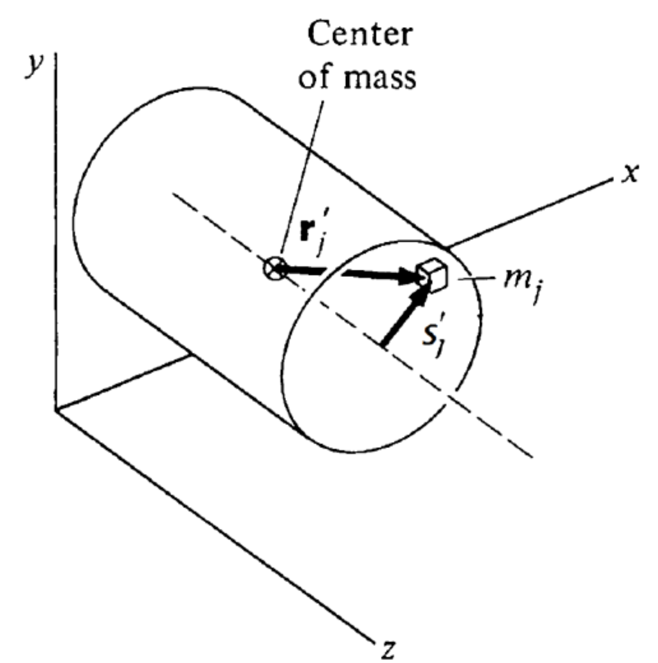
$$L_z = (\vec{R} \times \vec{P})_z + \left(\sum_{j=1}^{N} (\vec{r}_j' \times \vec{p}_j') \right)_z$$

If we fix the axis of rotation parallel to the z-axis, then

$$\left(\sum_{j=1}^{N} \vec{r}_j' \times \vec{p}_j'\right)_z$$

$$= \left(\sum_{j=1}^{N} \vec{r}_j' \times m_j \dot{\vec{r}}_j'\right)_z = \left(\sum_{j=1}^{N} \vec{s}_j' \times m_j \dot{\vec{s}}_j'\right)_z$$

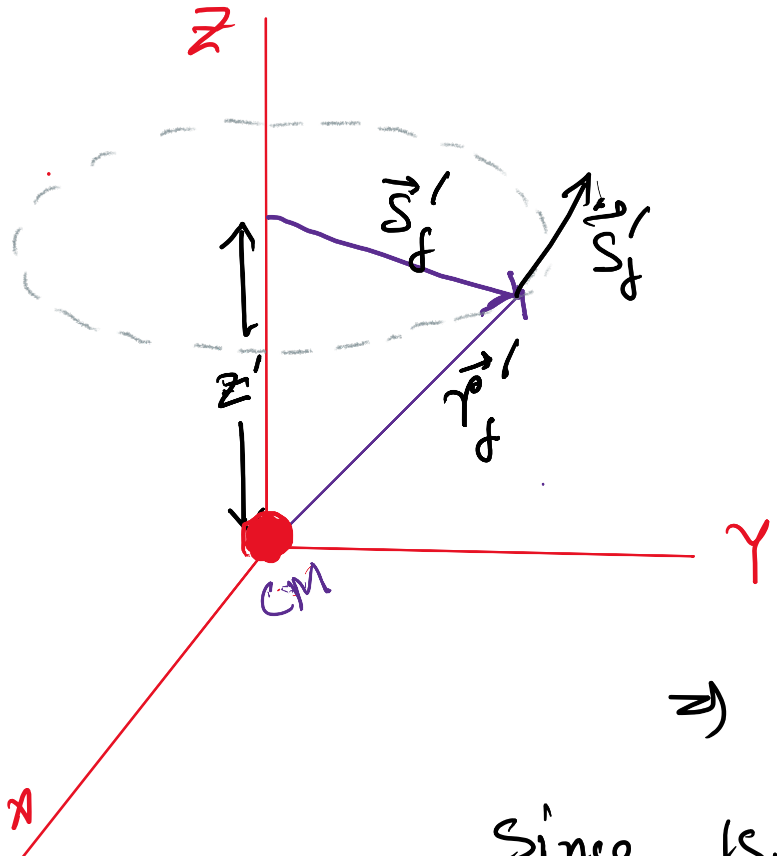
$$= \sum_{j=1}^{N} (s_j')(m_j\, \omega s_j') = \left(\sum_{j=1}^{N} m_j s_j'^2\right)\omega$$



(See the next slide for a rigorous proof)

$$= I_0\, \omega$$

Here I_0 is the moment of inertia of the body about the rotation axis.



We can write,

$$\vec{r}_j' = \vec{s}_j' + z'\hat{k}$$

$$\dot{\vec{r}}_j' = \dot{\vec{s}}_j' \quad (\text{since } z' \text{ is fixed})$$

$$\therefore \vec{r}_j' \times \dot{\vec{r}}_j' = (\vec{s}_j' + z'\hat{k}) \times \dot{\vec{s}}_j'$$

$\hat{k} \times \dot{\vec{s}}_j'$ can't have z-comp.

$$(\vec{r}_j' \times \dot{\vec{r}}_j')_z = (\vec{s}_j' \times \dot{\vec{s}}_j')_z$$

$$\Rightarrow (\vec{r}_j' \times m_j \dot{\vec{r}}_j')_z = (\vec{s}_j' \times m_j \dot{\vec{s}}_j')_z$$

Since, $|s_j'| = \text{constant}$ $\dot{\vec{s}}_j'$ is perpendicular to $\vec{s}_j'$

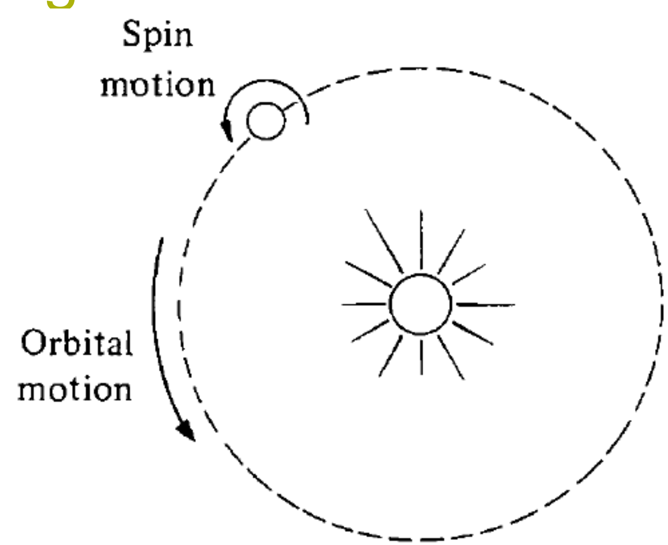
and $|\dot{\vec{s}}_j'| = \omega s_j'$

$$\Rightarrow (\vec{r}_j' \times m_j \dot{\vec{r}}_j')_z = \sum_j m_j s_j'^2 \omega$$

Translation and rotation

$$\therefore \quad L_z = I_0\omega + \left(\vec{R} \times \vec{P}\right)_z$$

- Hence, the angular momentum of a rigid body is the sum of the angular momentum about its center of mass and the angular momentum of the center of mass about the origin.
- The first is referred to as the spin orbital angular momentum and the second term is called the orbital angular momentum.
- The motion of the Earth is a is a very nice example for this.
- The daily rotation of the Earth about its axis gives rise to the earth's spin angular momentum, and its annual revolution the sun gives rise to the earth's angular momentum about the sun.



Translation and rotation

Let us again examine the expression for the total angular momentum:

$$\vec{L} = \vec{R} \times \vec{P} + \sum_{j=1}^{N} \left(\vec{r}_j' \times \vec{p}_j'\right).$$

Taking the time derivative we obtain:

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\left(\vec{R} \times \vec{P}\right) + \sum_{j=1}^{N} \frac{d}{dt}\left(\vec{r}_j' \times \vec{p}_j'\right)$$

$$= \frac{d\vec{R}}{dt} \times \vec{P} + \vec{R} \times \frac{d\vec{P}}{dt} + \sum_{j=1}^{N} \left(\frac{d\vec{r}_j'}{dt} \times \vec{p}_j' + \vec{r}_j' \times \frac{d\vec{p}_j'}{dt}\right)$$

$$= \vec{R} \times \frac{d\vec{P}}{dt} + \sum_{j=1}^{N} \vec{r}_j' \times \frac{d\vec{p}_j'}{dt}$$

Since $\frac{d\vec{R}}{dt} \times \vec{P} = 0 = \frac{d\vec{r}_j'}{dt} \times \vec{p}_j'$.

Translation and rotation

$$\frac{d}{dt}\vec{L} = \vec{R} \times \frac{d\vec{P}}{dt} + \sum_{j=1}^{N} \vec{r}_j{}' \times \frac{d\vec{p}_j{}'}{dt}$$

- $\dfrac{d}{dt}\vec{L} = \vec{\tau}$, the net torque on the system about the chosen origin.
- $\dfrac{d}{dt}\vec{P} = \vec{F}_{\text{ext}}$, the net external force on the system.
- $\sum_{j=1}^{N} \vec{r}_j{}' \times \dfrac{d\vec{p}_j{}'}{dt} = \displaystyle\sum_{j=1}^{N} \vec{r}_j{}' \times \vec{F}_j = \sum_{j=1}^{N} \vec{\tau}_j{}' = \vec{\tau}\,'$,

  the net torque in the center of mass frame.
- Thus,

$$\vec{\tau} = \vec{R} \times \vec{F}_{\text{ext}} + \vec{\tau}\,'$$

Total kinetic energy

$$E = \frac{1}{2}\sum m_j v_j^2 = \frac{1}{2}\sum m_j \vec{v_j} \,.\, \vec{v_j}$$

$$= \frac{1}{2}\sum m_j \left(\dot{\vec{r_j}}' + \vec{V}\right).\left(\dot{\vec{r_j}}' + \vec{V}\right).$$

(Since $\vec{r_j} = \vec{r_j}' + \vec{R}$, $\vec{v_j} = \dot{\vec{r_j}} = \dot{\vec{r_j}}' + \dot{\vec{R}}$, and $\vec{V} = \dot{\vec{R}}$)

$$= \frac{1}{2}\sum m_j \dot{r_j}'^2 \quad + \sum m_j \dot{\vec{r_j}}' \,.\, \vec{V} + \frac{1}{2}\sum m_j V^2$$

The 2nd term vanishes (see slides 7 and 8).

Let us consider rotation about z-axis. Then $\dot{r_j}' = \left|\dot{\vec{r_j}}'\right| = \omega s_j'$.

Thus, $E = \frac{1}{2}\sum m_j s_j'^2 \, \omega^2 + \frac{1}{2} M V^2 = \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M V^2$

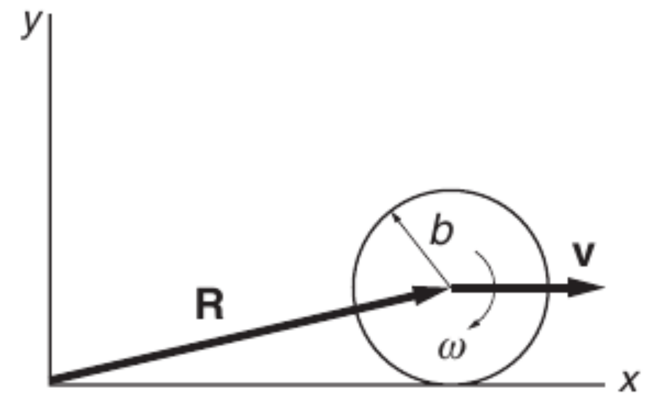
The 1st term is the kinetic energy of the spin and the 2nd term corresponds to the orbital motion of the center of mass.

Example: angular momentum of a rolling wheel

Calculate the angular momentum of a uniform wheel of mass ***M*** and radius ***b*** that rolls uniformly and without slipping.

We will be using the following equation:

$$L_z = I_0\omega + \left(\vec{R} \times \vec{P}\right)_z$$



The moment of inertia of the wheel about its center of mass is :

$$I_0 = \frac{1}{2}Mb^2$$

Its angular momentum about the center of mass is :

$$L_0 = -I_0\omega = -\frac{1}{2}Mb^2\omega$$

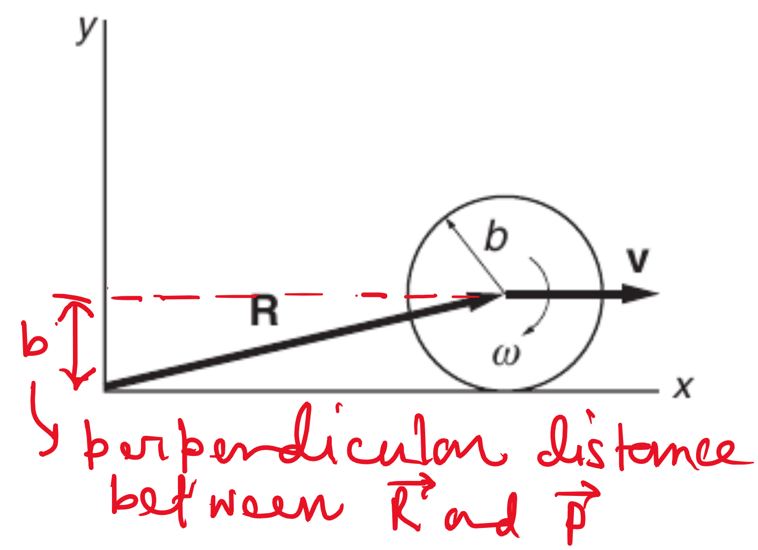
L_0 points in the negative z-direction, hence the minus sign.

Example: angular momentum of a rolling wheel

$$L_z = I_0\omega + \left(\vec{R} \times \vec{P}\right)_z$$

The angular momentum of the center of mass of the wheel w.r.t. to the origin is :

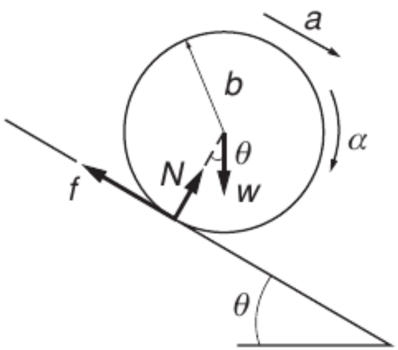
$$\left(\vec{R} \times \vec{P}\right)_z = -MbV$$
$$= -Mb^2\omega\,.$$



Since, $V = b\,\omega$ for a wheel that rolls without slipping.

$$\therefore \quad L_z = -\frac{1}{2}Mb^2\omega - Mb^2\omega = -\frac{3}{2}Mb^2\omega\,.$$

Example: Drum rolling down a plane



A uniform drum of radius *b* and mass *M* rolls without slipping down a plane inclined at angle *θ*. The moment of inertia of the drum around its axis is $I_0 = Mb^2/2$. Find the drum's acceleration along the plane.

The forces on the drum are shown in the diagram. f is the force of friction. The translation of the center of mass along the plane is given by:

$$\mathrm{W}\sin\theta - f = Ma \qquad (1)$$

For the rotation about the center of mass we have:

$$bf = I_0\alpha \qquad (2)$$

Note here the torque is in the negative z direction and the angular acceleration is in the clockwise direction.

Since the drum is rolling without slipping, we can write:

$$a = b\alpha \qquad (3)$$

Using (2) in (1) we can eliminate f as

$$\mathrm{W}\sin\theta - I_0\frac{\alpha}{b} = Ma\,.$$

Now using (3), $I_0 = Mb^2/2$, and $W = Mg$ in the above equation we get:

$$Mg\sin\theta - \frac{1}{2}Ma = Ma\,.$$

Or,

$$a = \frac{2}{3}g\sin\theta\,.$$