PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 8

VECTOR FIELDS: THE HELMHOLTZ THEOREM

- The laws of electromagnetism are expressed in terms of electric (**E**) and magnetic (**B**) fields, using differential equations.
- Therefore these laws involve the divergence and curl operations.
- A natural question arises from this formulation: If we know that divergence and curl of **E** and **B**, how much information do we have about them?

VECTOR FIELDS: THE HELMHOLTZ THEOREM

In general, if we have a vector field say **F**, and we know its divergence and curl, viz.,

$$\nabla \cdot \mathbf{F} = D, \qquad \nabla \times \mathbf{F} = \mathbf{C}$$

then can we determine the **F** itself?

In other words, to what extent is a vector function determined by its divergence and curl?

VECTOR FIELDS: THE HELMHOLTZ THEOREM

It turns out that, if along with the information about the curl and divergence of vector field **F**, we also provide the information about the boundary conditions then **F** is uniquely determined.

For example, in electrodynamics we typically require that the field should vanish (become 0) "at infinity", i.e., far-far away from all sources (charges).

This uniqueness of the field F, once we know its divergence, curl, along with appropriate boundary conditions, follows from the **Helmholtz theorem**.

CURL-LESS FIELD AND SCALAR POTENTIAL

- If the curl of a vector field (F) vanishes everywhere, then we can express it as a gradient of a scalar function, which we refer to as scalar potential:

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V$$

- The negative sign is because of conventional reason.
- We are free to add any constant (say c) to V, i.e., $V \to V + c$ does not change anything, since

$$\nabla(V + c) = \nabla V + \nabla c = \nabla V + 0 = \nabla V$$

- Thus scalar potential is not unique.

CURL-LESS FIELD AND SCALAR POTENTIAL

We have the following equivalent conditions for a curl-less or irrotational vector field $\mathbf{F}$:

(a) $\boldsymbol{\nabla} \times \mathbf{F} = 0$ everywhere.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path between the end points.

(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

(d) $\mathbf{F} = -\boldsymbol{\nabla} V$, i.e., $\mathbf{F}$ is gradient of some scalar.

Potential field

DIVERGENCE-LESS FIELD AND VECTOR POTENTIAL

- If the divergence of a vector field (**F**) vanishes everywhere, then we can express it as a curl of some vector function **A**, which is referred to as vector potential.

$$\boldsymbol{\nabla}\cdot\mathbf{F}=0\Longleftrightarrow\mathbf{F}=\boldsymbol{\nabla}\times\mathbf{A}$$

- Note that we are free to add gradient of some scalar function (say χ) to **A**, i.e., $\mathbf{A}\rightarrow\mathbf{A}+\boldsymbol{\nabla}\chi$ without changing **F**, since curl of gradient gives zero:

$$\boldsymbol{\nabla}\times(\mathbf{A}+\boldsymbol{\nabla}\chi)=(\boldsymbol{\nabla}\times\mathbf{A})+(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi)=\boldsymbol{\nabla}\times\mathbf{A}+0=\boldsymbol{\nabla}\times\mathbf{A}$$

- Thus vector potential is not unique. This liberty in the choice of **A** leads to gauge-freedom in the formulation of electrodynamics.

DIVERGENCE-LESS FIELD AND VECTOR POTENTIAL

We have the following equivalent conditions for a divergence-less or solenoidal field $\mathbf{F}$:

(a) $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ everywhere.

(b) $\int_{\mathrm{a}}^{\mathrm{b}} \mathbf{F} \cdot d\mathbf{S}$ is independent of surface, for any given boundary line.

(c) $\oint \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed surface.

(d) $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$, i.e., $\mathbf{F}$ is curl of some vector (vector potential).

HELMHOLTZ DECOMPOSITION

- A very interesting result about any general (differentiable) vector field $\mathbf{F}$ is that it can always be written as the sum of a gradient of a scalar and curl of a vector field:

$$\mathbf{F} = -\boldsymbol{\nabla} V + \boldsymbol{\nabla} \times \mathbf{A}$$

- This is referred to as the Helmholtz decomposition.

HELMHOLTZ DECOMPOSITION

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

(Irrotational component) Curl (or rotation) free

Divergence free (Rotational component)

$$\nabla \times (-\nabla V) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Helmholtz's theorem states that any single-valued and continuous vector field **F** which vanishes at infinity can be represented as the sum of the gradient of a certain scalar function (V) and the curl of a certain vector function (**A**) whose divergence is equal to zero.

Classical Electrodynamics

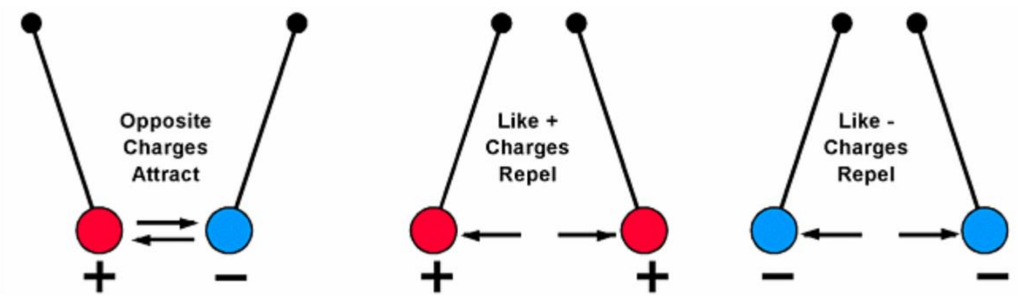
The Field Formulation of Electrodynamics

Electric charge - is the physical property of matter that makes it interact with other electrically charged matter (SI Unit: Coulomb, C).

Properties of Charge

1) Two types of Charges

There are two types of electric charges: positive charges and negative charges.



"The extraordinary fact is that plus and minus charges occur in exactly equal amounts, to fantastic precision, in bulk matter, so that their effects are almost completely neutralized. Were it not for this, we would be subjected to enormous forces: a potato would explode violently if the cancellation were imperfect by as little as **one part in 10^{10}.**"

Ref:- Richard Feymann "Lectures on Physics, volume 2", David J Griffiths, "Introduction to Electrodynamics"

The Field Formulation of Electrodynamics

Electric charge - is the physical property of matter that makes it interact with other electrically charged matter (SI Unit: Coulomb, C).

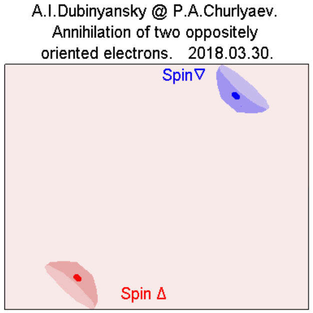
Properties of Charge

2) Charge is conserved

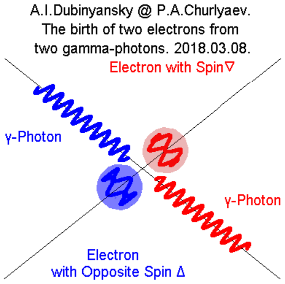
The total charge in an isolated system never changes.

Within the system charged particles may vanish ("annihilate") or reappear ("pair production", but they always do so in pairs of equal and opposite charge.

Anihilation



Pair production



- Total charge of the universe is fixed for all time.
- The total electric charge of an isolated system is a relativistically invariant number.

(A charge whose magnitude is found to be Q in one frame of reference is also Q in all other frames)

Ref:- Richard Feymann "Lectures on Physics, volume 2", David J Griffiths, "Introduction to Electrodynamics"

The Field Formulation of Electrodynamics

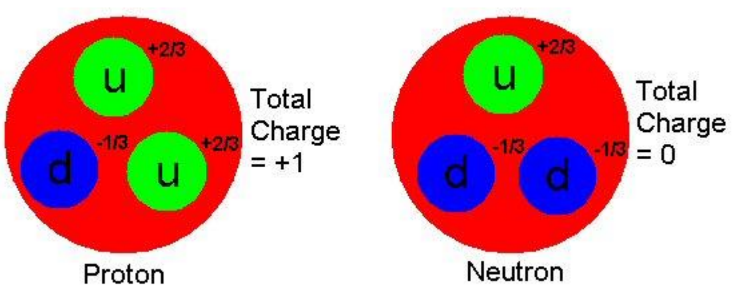
Electric charge - is the physical property of matter that makes it interact with other electrically charged matter (SI Unit: Coulomb, C).

Properties of Charge

3) Charge is quantized

Electric charge comes only in discrete lumps—integer multiples of the basic unit of charge

$Q = \pm ne$ $\qquad$ $e \rightarrow Charge\ of\ a\ single\ electron\ (1.602 \times 10^{-19} C)$



Free quarks do not appear to exist in nature, and in any event, this does not alter the fact that charge is quantized; it merely reduces the size of the basic unit.

Ref:- Richard Feymann "Lectures on Physics, volume 2", David J Griffiths, "Introduction to Electrodynamics"

The Field Formulation of Electrodynamics

Methods of Charging

Charging by Friction

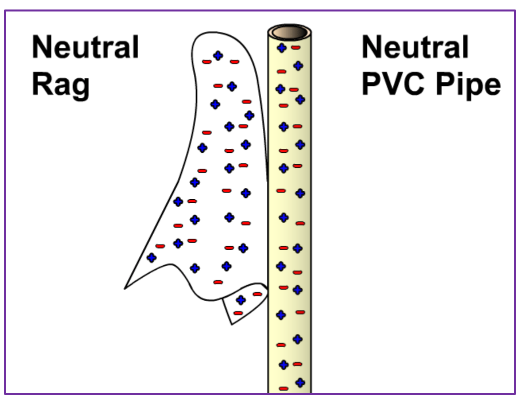
The process in which a body gets charged when it is rubbed against other body.

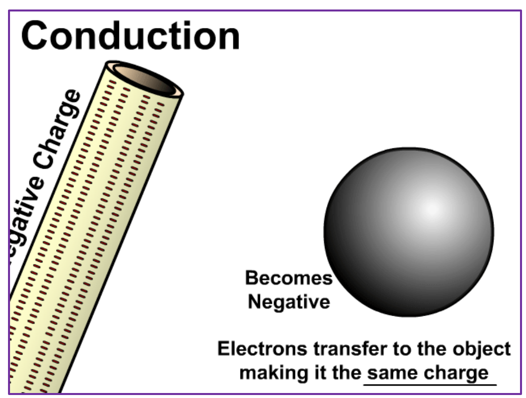
Charging by Conduction

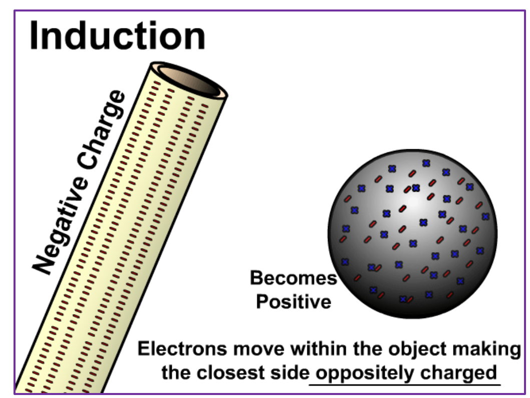
The process in which a body gets charged by making contact with a charged object.

Charging by Induction

The process in which a body gets charged by a charged object without contact.







Ref:- https://stickmanphysics.com/

Electrostatics

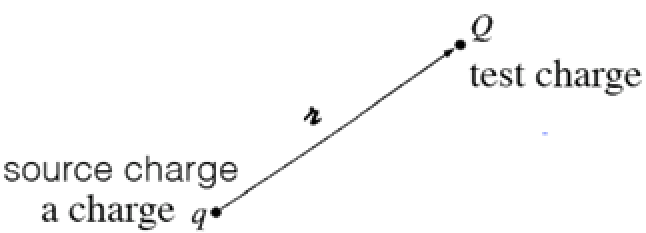
Charges at Rest

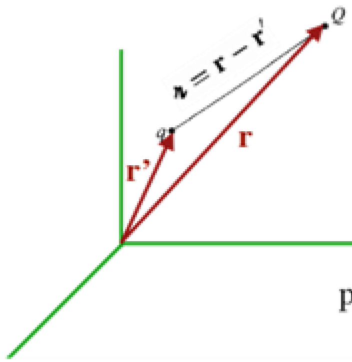
Fundamental problem of electrodynamics

Coulomb's Law

Electric Force experienced by the charged objects is described by the Coulomb's law.

Coulomb's law is a law of physics describing the electrostatic interaction between the electrically charged particles.

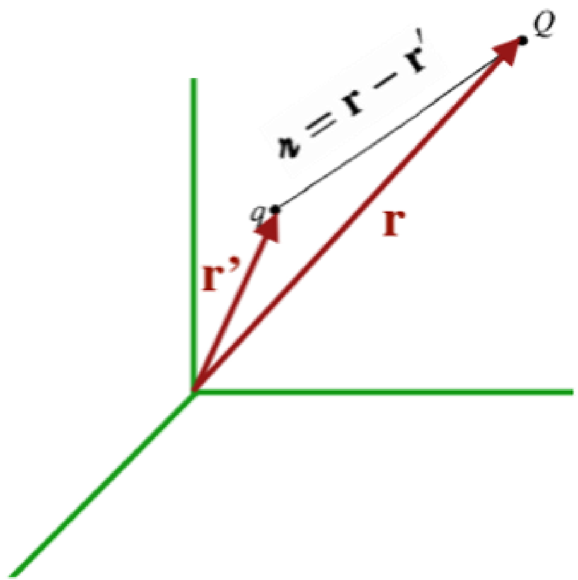




$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.}$$

permittivity of free space (vacuum), $\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N \cdot m}^2}$.

Coulomb's Law



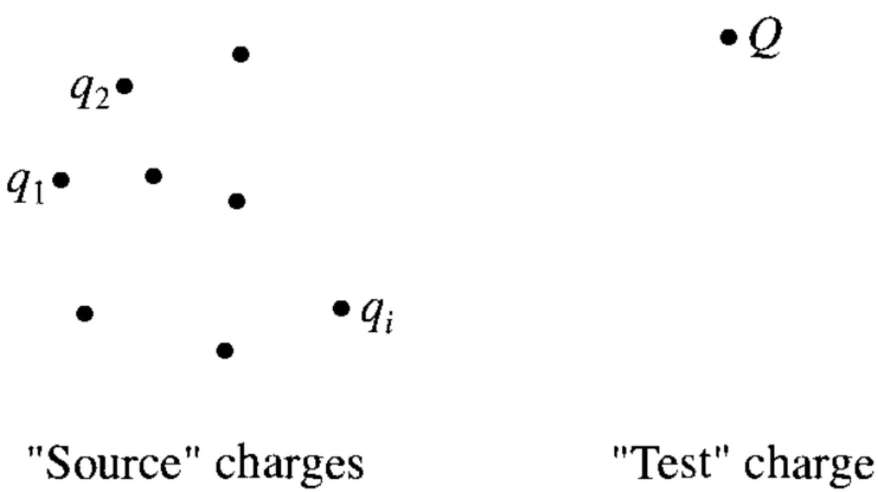
Force on Q due to q: $\mathbf{F}_{Qq} = F\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$ Force on q due to Q: $\mathbf{F}_{qQ} = F\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|}$

We can see that $\mathbf{F}_{Qq} = -\mathbf{F}_{qQ}$, which is consistent with the Newton's third law. Note that $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$ is the unit vector in the direction of $\mathbf{r}-\mathbf{r}'$.

Ref:- *Richard Feymann "Lectures on Physics, volume 2"*

"Principle of Superposition"

It was also shown experimentally that the **total force** produced on one small charged body by a number of other small charged bodies placed around, is the **vector sum** of the individual two-body forces of Coulomb, i.e.,



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n$$

"Principle of Superposition"

Vector sum is commutative as well as associative!

F depends not only on 'r' but also on *velocity* and *acceleration* of charge q (only during motion)

Ref:- *Richard Feymann "Lectures on Physics, volume 2"*

Fundamental problem of electrodynamics

Total Force

Coulomb's law and **the principle of superposition** constitute the physical input for **electrostatics**.

As per previous example, if we have several point charges say $q_1, q_2, \ldots, q_n$, at distances $\mathscr{r}_1, \mathscr{r}_2, \ldots, \mathscr{r}_n$ from Q, the total force on Q can then be given by:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1\hat{\boldsymbol{\mathscr{r}}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\boldsymbol{\mathscr{r}}}_2}{\mathscr{r}_2^2} + \frac{q_3\hat{\boldsymbol{\mathscr{r}}}_3}{\mathscr{r}_3^2} + \ldots\right),$$

$$\boxed{\mathbf{F} = Q\mathbf{E},}$$

where, $$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i.$$ Electric Field=F/*Q*

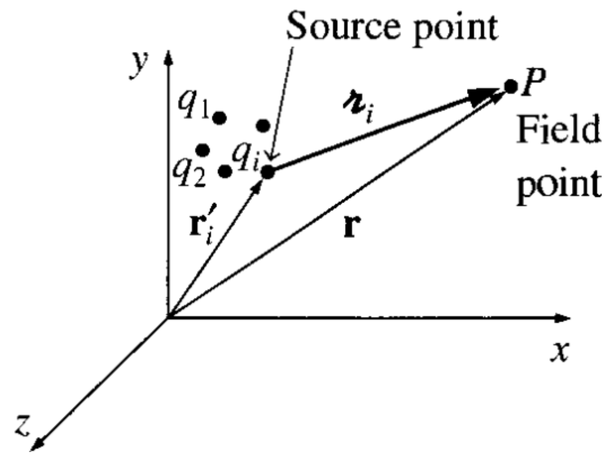
Force on charge Q is charge Q times the *electric field* produced by all different charges at P.

Ref:- *Richard Feymann "Lectures on Physics, volume 2"*

Electric Field

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q}$$

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$



- $\mathbf{E}(\mathbf{r})$ is the net electric field at the position '**r**' because of all source charges. It depends on the separation vector $\mathscr{r}_i$ (separation between the source point and the field point (P), the point where we want to measure the field) and makes no reference to the test charge Q.

- The **electric field** is a **vector quantity** that varies from point to point and is determined by the configuration of source charges.

- Physically, the electric field can essentially be described as the **force per unit charge** (i.e., E=F/Q)that would be exerted on a test charge, if we place one at P.

__Ref:-__ Richard Feymann "Lectures on Physics, volume 2"

Optional

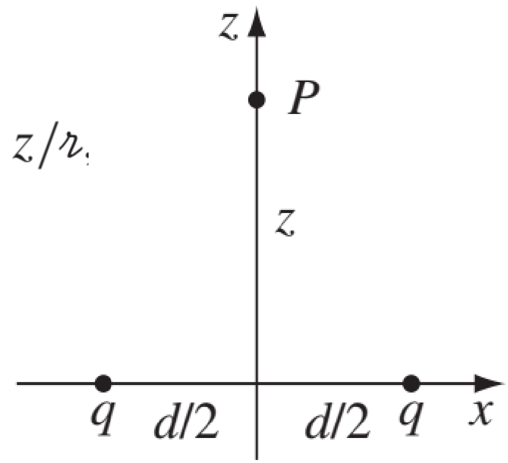
Examples

P1. Find the electric field a distance **z** above the midpoint between two equal charges (**q**), a distance **d** apart

Solution

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta.$$

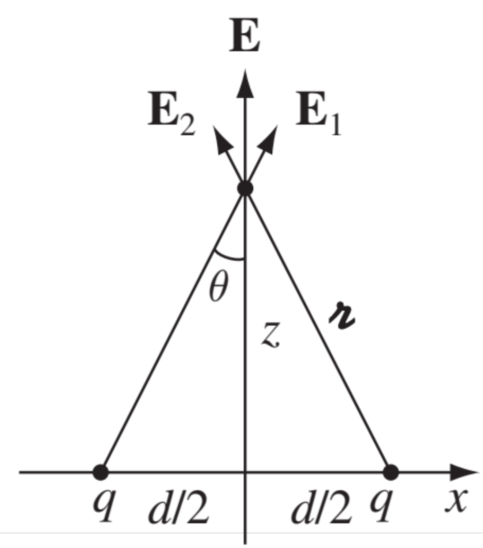
$$\mathscr{r} = \sqrt{z^2 + (d/2)^2} \text{ and } \cos\theta = z/\mathscr{r},$$



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + (d/2)^2\right]^{3/2}}\,\hat{\mathbf{z}}.$$

When $z \gg d$

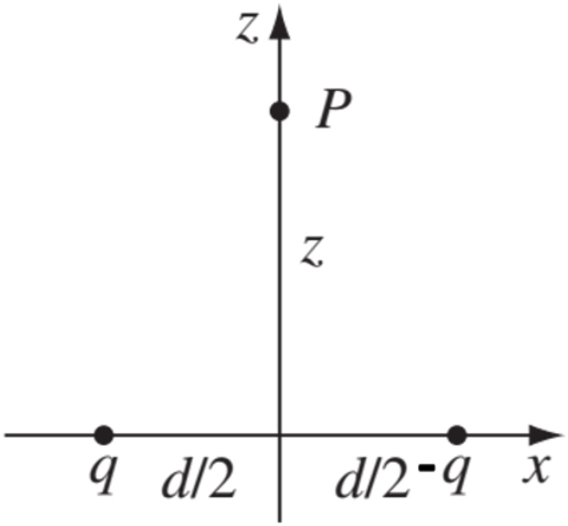
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\,\hat{\mathbf{z}}.$$



Electric field due to a single charge **2q**

Electric Field

P2. Find the electric field a distance **z** above the midpoint between two equal and opposite charges (**q**), a distance **d** apart



__Solution__

This time the "vertical" components cancel, leaving

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\mathscr{r}^2} \sin\theta \, \hat{\mathbf{x}}, \text{ or}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{qd}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \hat{\mathbf{x}}.}$$

When $z \gg d$

$$\mathbf{E} \approx \frac{1}{4\pi\epsilon_0} \frac{qd}{z^3} \hat{\mathbf{x}}.$$