Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 3

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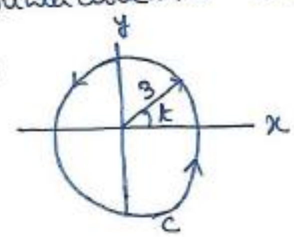
1. Find the work done in moving a particle around a circle C in the x-y plane, if the circle has a center at origin and radius 3 and if the force field is given by

$$\boldsymbol{F} = (2x - y + z)\hat{\boldsymbol{\imath}} + (x + y - z^2)\hat{\boldsymbol{\jmath}} + (3x - 2y + 4z)\hat{\boldsymbol{k}}$$

Ans. In the plane $z = 0$, $\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$ and $\vec{dr} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$ so that the work done is

$$\int_C \vec{F}\cdot\vec{dr} = \int_C \left[(2x-y)\,dx + (x+y)\,dy\right].$$

We can choose C as the path traversed in a circle in a counterclockwise sense. Choose the parametric eqs of the circle as $x = 3\cos t$, $y = 3\sin t$ where t varies from 0 to 2π.



$$W = \int_C \vec{F}\cdot\vec{dr} = \int_{t=0}^{2\pi} \Big[(2(3\cos t) - 3\sin t)(-3\sin t)\,dt + (3\cos t + 3\sin t)(3\cos t)\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt = 9t - \frac{9}{2}\sin^2 t \Big|_0^{2\pi} = \underline{18\pi}$$

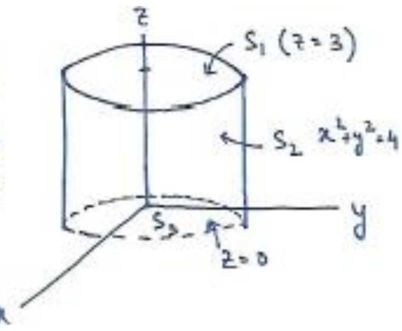
Note that if C ~~were~~ was traversed in clockwise direction, the value of the integral would have been -18π.

Note also that instead of using parametric eqs, you could have substituted $y = +\sqrt{3-x^2}$ and take the limit of x from $+3$ to -3 (covering upper ~~semi~~ semicircle) and then again substitute $y = -\sqrt{3-x^2}$ and ~~substitute~~ vary x from -3 to $+3$ (covering lower semicircle).

Both are equivalent

2. Find out the flux of vector field $\boldsymbol{A} = 4x\hat{\boldsymbol{\imath}} - 2y^2\hat{\boldsymbol{\jmath}} + z^2\hat{\boldsymbol{k}}$ taken over the region bounded by $x^2 + y^2 = 4$ and $z = 0$, and $z = 3$, that is a ***closed cylinder*** with a circular top and base of radius 2. [Hint: The flux of a vector field through a closed surface S is defined as $\iint_S \boldsymbol{F}.\boldsymbol{dS}$]

2.



Flux is given by the surface integral of the vector function $\vec{A}$ through surfaces S_1, S_2 and S_3

$$\iint_S \vec{A}\cdot d\vec{S} = \iint_{S_1} \vec{A}\cdot d\vec{S} + \iint_{S_2} \vec{A}\cdot d\vec{S} + \iint_{S_3} \vec{A}\cdot d\vec{S}$$

<u>On S_1</u> (z = 3)

$\hat{n} = \hat{k}$ (unit vector of area normal)

$$\iint_{S_1} \vec{A}\cdot d\vec{S} = \iint_{S_1} \vec{A}\cdot\hat{n}\, dS = \iint_{S_1} z^2\, dS \quad [\because \vec{A}\cdot\hat{n} = \vec{A}\cdot\hat{k} = z^2]$$

$$= 3^2 \iint_{S_1} dS \quad [\because z = 3] \quad = 9.(\pi . 2^2) = 36\pi$$

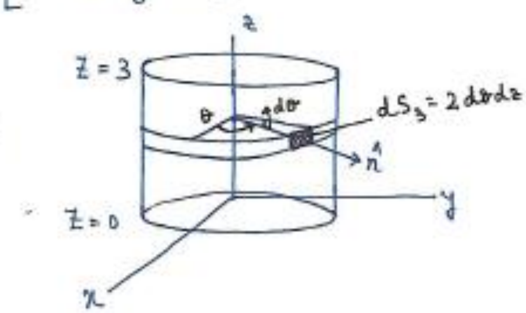
<u>On S_2</u>: We have to evaluate $\iint_{S_2} \vec{A}\cdot d\vec{S}_2 = \iint_{S_2} \vec{A}\cdot\hat{n}\, dS_2$

A perpendicular to $x^2+y^2=4$ has direction $\vec{\nabla}(x^2+y^2-4)$
$= 2x\hat{i} + 2y\hat{j}$

Unit normal perpendicular to surface of $x^2+y^2=4$ is $\dfrac{2x\hat{i} + 2y\hat{j}}{\sqrt{4(x^2+y^2)}} = \dfrac{x}{2}\hat{i} + \dfrac{y}{2}\hat{j}$ $[\because x^2+y^2 = 2^2]$

$\therefore \vec{A}\cdot\hat{n} = (2x^2 - y^3)$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\, dS = \iint_{S_3} (2x^2 - y^3)\, dS$$



To solve the integral, it is useful to realise the elementary surface dS_3 on S_3 – it is clear that in polar co-ordinates (r, θ), $dS_3 = r\,d\theta\,dz = 2\,d\theta\,dz$

Substitute $x = r\cos\theta = 2\cos\theta$

$y = r\sin\theta = 2\sin\theta$

$$\therefore \iint_{S_3} \vec{A}\cdot\hat{n}\,dS = \iint_{\theta\ z}\left[2\,(4\cos^2\theta) - 8\sin^3\theta\right].\,2\,d\theta\,dz$$

$$= 16\iint_{\theta\ z}(\cos^2\theta - \sin^3\theta)\,d\theta\,dz$$

Here limits of integration are from $z = 0$ to 3 and $\theta = 0$ to 2π.
For double integrals, it is important to judge which integral should be carried first. For e.g., in this case, since the z-integration returns a constant, we should proceed with first with z-integration. Above integral can be written as,

$$16\int_{\theta}(\cos^2\theta - \sin^3\theta)\,d\theta\int_{z=0}^{3}dz$$

$$= 16\times 3\int_{\theta=0}^{2\pi}(\cos^2\theta - \sin^3\theta)\,d\theta$$

Using standard integral formulas for ~~double~~ squares and cubes of angle of trigonometric functions, i.e.

$$\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x$$

$$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x$$

$$\int 2\cos^2\theta = \int(1 + \cos 2\theta)\,.$$

The above integral turns out to be $\underline{48\pi}$

For surface S_3 $(z = 0)$

$\hat{n} = -\hat{k}$ $\quad \vec{A}\cdot\hat{n} = -z^2 = 0$ $[\because z = 0]$. $\therefore \iint_{S_3}\vec{A}\cdot n\,dS = 0$

The net flux is

$$\iint_S \vec{A}\cdot dS = \iint_{S_1}\vec{A}\cdot dS + \iint_{S_2}\vec{A}\cdot d\vec{S} + \iint_{S_3}\vec{A}\cdot d\vec{S}$$

$$= 36\pi + 48\pi + 0$$

$$= \underline{84\pi} \quad (\text{Answer})$$

3. Evaluate the volume integral

$$\int_0^1 \int_0^{z^2} \int_0^3 y\cos\left(z^5\right)\,dx\,dy\,dz$$

Solution:

We need to integrate following the given order and recall that we start with the "inside" integral and work our way out.

First, here is the x integration:

$$\begin{aligned}\int_0^1 \int_0^{z^2} \int_0^3 y\cos\left(z^5\right)\,dx\,dy\,dz &= \int_0^1 \int_0^{z^2} \left(y\cos\left(z^5\right)x\right)\Big|_0^3\,dy\,dz \\ &= \int_0^1 \int_0^{z^2} 3y\cos\left(z^5\right)\,dy\,dz\end{aligned}$$

Next, we perform the y-integral:

$$\begin{aligned}\int_0^1 \int_0^{z^2} \int_0^3 y\cos\left(z^5\right)\,dx\,dy\,dz &= \int_0^1 \left(\frac{3}{2}y^2\cos\left(z^5\right)\right)\Big|_0^{z^2}\,dz \\ &= \int_0^1 \frac{3}{2}z^4\cos\left(z^5\right)\,dz\end{aligned}$$

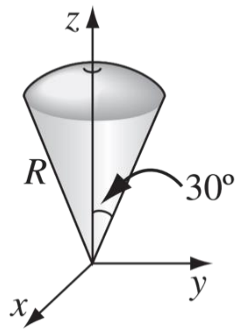
Finally, we perform the z-integral:

$$\int_0^1 \int_0^{z^2} \int_0^3 y\cos\left(z^5\right)\,dx\,dy\,dz = \left(\frac{3}{10}\sin\left(z^5\right)\right)\Big|_0^1 = \boxed{\frac{3}{10}\sin(1) = 0.2524}$$

4. Verify the divergence theorem for the function

$$\mathbf{v} = r^2 sin\theta\,\hat{\mathbf{r}} + 4r^2 cos\theta\,\hat{\boldsymbol{\theta}} + r^2 tan\theta\,\hat{\emptyset}$$

using the volume of the "ice-cream cone" shown below2 (the top surface is spherical, with radius R and centered at the origin).



Solution

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2 \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, 4r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^2\tan\theta\right) \\
&= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2\left(\cos^2\theta - \sin^2\theta\right) = \frac{4r}{\sin\theta}\left(\sin^2\theta + \cos^2\theta - \sin^2\theta\right) \\
&= 4r\frac{\cos^2\theta}{\sin\theta}.
\end{aligned}$$

$$\begin{aligned}
\int(\boldsymbol{\nabla}\cdot\mathbf{v})\,d\tau &= \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \int_0^R 4r^3\,dr\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]\Bigg|_0^{\pi/6} \\
&= 2\pi R^4\left(\frac{\pi}{12}+\frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{6}\left(\pi + 3\frac{\sqrt{3}}{2}\right) = \boxed{\tfrac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right).}
\end{aligned}$$

Surface coinsists of two parts:

(1) *The ice cream:* $r = R$; $\phi : 0 \to 2\pi$; $\theta : 0 \to \pi/6$; $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $\mathbf{v}\cdot d\mathbf{a} = \left(R^2\sin\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right) = R^4\sin^2\theta\,d\theta\,d\phi$.

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/6}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} = 2\pi R^4\left(\frac{\pi}{12}-\frac{1}{4}\sin 60^\circ\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right.$$

(2) *The cone:* $\theta = \frac{\pi}{6}$; $\phi : 0 \to 2\pi$; $r : 0 \to R$; $d\mathbf{a} = r\sin\theta\,d\phi\,dr\,\hat{\boldsymbol{\theta}} = \frac{\sqrt{3}}{2}r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\,r^3\,dr\,d\phi$

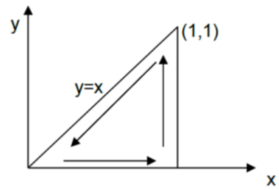
$$\int\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\int_0^R r^3\,dr\int_0^{2\pi}d\phi = \sqrt{3}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi R^4.$$

Therefore $\int\mathbf{v}\cdot d\mathbf{a} = \frac{\pi R^4}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3}\right) = \frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right)$. ✓.

5. An incompressible, steady velocity field is given by,

$$\vec{V} = (x^2y - xy^2)\hat{i} + (\frac{y^3}{3} - xy^2)\hat{j}$$

For the plane shown below, show that the circulation around the boundary is equal to the surface integral of the curl of the velocity field over the surface (Verification of Stokes' Theorem).



(3) (5)

$$\vec{V} = (x^2y - xy^2)\hat{i} + (\frac{y^3}{3} - xy^2)\hat{j}$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2y - xy^2) & \frac{y^3}{3} - xy^2 & 0 \end{vmatrix}$$

$$= \hat{k}\,(-y^2 - x^2 + 2xy) = -(x-y)^2\,\hat{k}$$

Statement $\int \vec{\nabla}\times\vec{V}\cdot d\vec{s} = \oint_c \vec{V}\cdot d\vec{l}$

$$\int \vec{\nabla}\times\vec{V}\cdot d\vec{s} = -\int (x-y)^2\,dx\,dy = -\int x^2\,dx\,dy - \int y^2\,dx\,dy + \int 2xy\,dy\,dx$$

$$= -\int x^2\,dx\,dy = -\int_0^1\int_0^x x^2\,dy\,dx - \int_0^1\int_0^x y^2\,dy\,dx + \int_0^1\int_0^x 2xy\,dy\,dx$$

$$= -\int_0^1 x^2y\Big|_0^x dx - \int_0^1 \frac{y^3}{3}\Big|_0^x dx + \int_0^1 2x\frac{y^2}{2}\Big|_0^x dx$$

$$= -\int_0^1 \cancel{x^3}\,dx - \int_0^1 \frac{x^3}{3}\,dx + \int_0^1 \cancel{x^3}\,dx$$

$$= -\frac{1}{3}\times\frac{x^4}{4}\Big|_0^1 = -\frac{1}{12}$$

$$\int \vec{V}\cdot\vec{dl} = \int_{C_1} \vec{V}\cdot dx\,\hat{i} + \int_{C_2} \vec{V}\cdot dy\,\hat{j} + \int_{C_3} \vec{V}\cdot(\hat{i}+\hat{j})\,dy$$

$$= \int (x^2y - y^2x)\,dx + \int^1 (\frac{y^3}{3} - xy^2)\,dy + \int_1^0 (x^2y - xy^2)\,dy + \int_1^0 (\frac{y^3}{3} - xy^2)\,dy$$

$$= 0 + \int_0^1 \left(\frac{y^3}{3} - y\,y^2\right) dy + \int_1^0 \cancel{(y^3 - y^3)}\, dy$$

C_2

$$+ \int_1^0 \left(\frac{y^3}{3} - y\,y^2\right) dy$$

$$= -\int_0^1 y^2\,dy - \int_1^0 y^3\,dy$$

$$= -\left|\frac{y^3}{3}\right|_0^1 - \left|\frac{y^4}{4}\right|_1^0$$

$$= -\frac{1}{3} + \frac{1}{4} = \frac{-4+3}{12} = -\frac{1}{12}$$