PHY101: Introduction to Physics I

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Lecture 29

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Previous Lecture

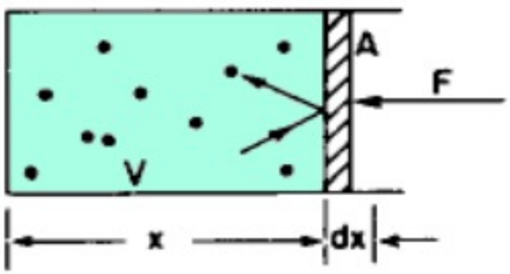
Kinetic theory of gases

This Lecture

Kinetic theory of gases - continued
Photon gas

Kinetic theory of gases

- Consider a box with a **frictionless piston** filled with some gas.
- We are interested in finding out **the force** on the piston due to the particles (atoms/molecules) constituting the gas.



- This force, however, is not localized at a single point but rather distributed over the **entire area of the piston**.
- A convenient way to measure it would be to talk about force per unit area, i.e., **Pressure** :

$$P = \frac{F}{A}$$

Kinetic theory of gases

- Consider a particle which has a **mass *m* and velocity $\boldsymbol{v}$**. If the x-component of the velocity is v_x , then when the atom hits the piston (elastic collision), **this component gets reversed**.
- The change in momentum is

$$\triangle P = m\left((-v_x) - (v_x)\right)$$

(-ve sign represents the loss in momentum)

- **Momentum** delivered to the piston because of this single collision
. $= 2mv_x$
- For simplicity let us assume that **all the atoms have the same velocity**. (We will generalize this to the case of unequal velocities soon).
- Let us consider a small time interval Δt. In this interval, **only the particles which lie within the distance $\boldsymbol{v_x \Delta t}$ from the wall will be able to hit the wall**. Others won't be able to reach the wall in Δt.

Kinetic theory of gases

- If A is the area of the piston, then **the particles which lie within the volume $\boldsymbol{Av_x\Delta t}$** will be able to hit the piston.
- If ***n* is the number of particles per unit volume**:

$$n = \frac{N}{V} ,$$

- The **number of particles that hit the wall** in time Δt is:

$$nAv_x\Delta t$$

- Thus, **total momentum** imparted to the piston in this interval is

$$= (nAv_x\Delta t)(2mv_x)$$

Kinetic theory of gases

- The **force** on the piston is therefore:

$$F = \frac{(nAv_x \Delta t)(2mv_x)}{\Delta t} = 2nmv_x^2 A$$

(The result does not change if we take the limit $\Delta t \to 0$).

- Hence, the **pressure** is:

$$P = \frac{F}{A} = 2mnv_x^2 \, .$$

- Now let us generalize to arbitrary velocities for the particles. However, we are considering identical particles, so masses are same for all.

- For that we need to replace v_x^2 by the **average velocity** in the x-direction. So,

$$v_x^2 \to \frac{1}{2} \left\langle v_x^2 \right\rangle$$

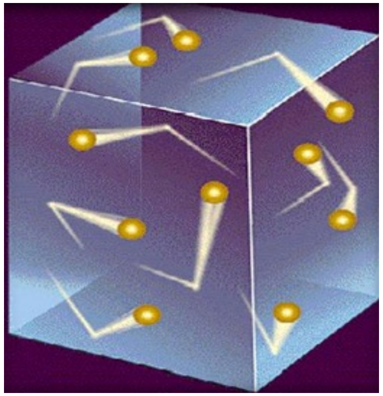
Kinetic theory of gases

- The factor of half must be introduced because $\langle v_x^2 \rangle$ counts contribution from both v_x and $-v_x$, whereas we are focusing on v_x only.
- Thus,

$$P = 2mnv_x^2 \to mn\langle v_x^2 \rangle$$

- But now there is nothing special about the x-direction, we might as well consider **y and z directions**. Since there is no preferred direction for the particles, for the averages we must have:

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle \ .$$



- Now if v^2 is the velocity squared of the particles (in general different for all), then $v^2 = v_x^2 + v_y^2 + v_z^2$.

Kinetic theory of gases

- We can write

$$P = \frac{1}{3} nm \left\langle v^2 \right\rangle = \frac{2}{3} n \left\langle \frac{1}{2} mv^2 \right\rangle .$$

- Clearly $\left\langle \frac{1}{2} mv^2 \right\rangle$ represents the **average kinetic energy** for the particles.
- Now using $n = N/V$, we obtain:

$$PV = \frac{2}{3} N \left\langle \frac{1}{2} mv^2 \right\rangle .$$

This is truly a remarkable relation. It relates the average of microscopic property, to the macroscopic observable**, the pressure exerted by the gas on the piston/wall**.

Kinetic theory of gases

- For a **monatomic gas**, e.g. Helium or Argon, i.e. molecules with just single atom in them, it is reasonable to assume that there is no other internal motion (rotation, vibration).

- Thus, the kinetic energy as obtained in the previous slide will represent the total energy. We will represent it by *U*, the total internal energy of the gas.

- Hence, we have

$$U = N \left\langle \frac{1}{2} m v^2 \right\rangle \, ,$$

resulting

$$PV = \frac{2}{3} U \, .$$

Kinetic theory of gases

- We might have a situation for a gas with complex molecules, then we need to consider the contributions from internal motion such as **rotation, vibration** etc.

- Therefore, for generality, we write

$$PV = (\gamma - 1)U \qquad (1).$$

- For a monatomic gas like Helium we have $\gamma = \frac{5}{3}$, resulting $PV = (2/3)U$.

- Taking the differential of equation (1) we get:

$$PdV + VdP = (\gamma - 1)dU \qquad (2).$$

Kinetic theory of gases

- Let us now examine the **compression of the gas** when we apply force on the piston.

- If we assume that the process is **adiabatic**: No heat energy is added or removed. The change in internal energy is then:

$$dU = -F dx = -\frac{F}{A}(A dx) = -P dV \,.$$

- From equation (2) we get

$$P dV + V dP = -(\gamma - 1) P dV \,.$$

- Rearranging we obtain

$$\gamma \frac{dV}{V} + \frac{dP}{P} = 0 \,.$$

- Assuming that γ is a constant, as it is for a monatomic gas, we can integrate the above equation to get:

$$\gamma \ln V + \ln P = \ln C \, ,$$

where ln C is the constant of integration.

- Exponentiating both sides we obtain

$$PV^{\gamma} = C \quad (\text{constant}) \, .$$

Photon Gas

- Consider a photon gas. We will avoid talking in terms of mass in this case as we are dealing with a relativistic system, and it has a very different behavior in the relativistic domain.

- However, $F = dp/dt$ still holds. Redoing the analysis and working with *p* we arrive at

$$P = 2np_x v_x \, .$$

- Introducing the averaged quantities and considering the three directions we obtain:

$$P = \frac{1}{3} n \left\langle \vec{p} \cdot \vec{v} \right\rangle$$

Or,

$$PV = \frac{1}{3} N \left\langle \vec{p} \cdot \vec{v} \right\rangle$$

Photon Gas

- The momentum $\vec{p}$ and $\vec{v}$ are in the same direction. Hence, $\vec{p} \cdot \vec{v} = pv$.

- Now for a photon $v = c$, **the speed of light**. Thus $\vec{p} \cdot \vec{v} = pc$.

- Special theory of relativity tells that pc for a photon is actually its total energy *E*.

- Thus, the internal energy of the photon gas is:

$$N \langle \vec{p} \cdot \vec{v} \rangle = NE = U \, .$$

- Finally, we get $$PV = \frac{1}{3} U \, .$$

Photon Gas

- Comparing with $PV = (\gamma - 1)U$, we find:

$$\gamma = \frac{4}{3}$$

- Therefore, the photon gas obeys (radiation in a box):

$$PV^{\gamma} = \text{constant}\,.$$

- Thus, we know about the behaviour of the radiation. This can be applied to radiation of hot stars!