PHY101: Introduction to Physics I

Monsoon Semester 2024

Lecture 26

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Previous Lecture

Moment of Inertia
Parallel axis theorem

This Lecture

Dynamics of pure rotation
Rotational kinetic energy

Dynamics of pure rotation

We will consider only **pure rotation**, where the axis of rotation is at rest.

Consider a body rotating with angular velocity ω about the *z*-axis.
We already found that the z-component of angular momentum is:

$$\boldsymbol{L_z = I\ \omega,}$$

Where I is the moment of inertial about the *z*-axis.

Now $\vec{L}$ is decided by the net external torque $\vec{\tau}$ on the system.
Therefore,

$$\tau_z = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt}$$

The last step followed since I is fixed for a rigid object (and we are working in nonrelativistic domain).

Dynamics of pure rotation

Now $d\omega/dt$ measures the change in angular velocity with time and therefore can be called as angular acceleration.

Denoting it by α we get,

$$\tau_z = I\alpha$$

Thus for a given I, the angular acceleration is decided by the external torque.

This equation is reminiscent of

$$\overrightarrow{F} = m\overrightarrow{a}$$

Rotational kinetic energy

For a rigid body undergoing pure rotation (no translation), the total kinetic energy is:

$$K = \sum_j K_j = \sum_j \left(\frac{1}{2} m_j v_j^2\right)$$
$$= \sum_j \left(\frac{1}{2} m_j s_j^2 \omega^2\right) = \frac{1}{2}\left(\sum_j m_j s_j^2\right)\omega^2.$$

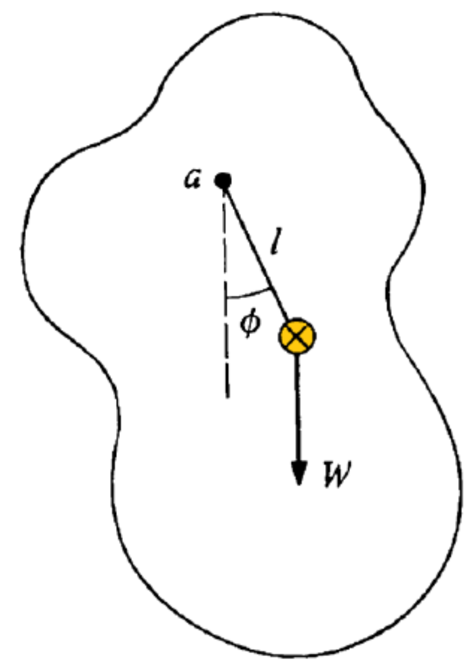
Since, $\sum_j m_j s_j^2 = I$, we obtain

$$K = \frac{1}{2} I \omega^2 .$$

The Physical Pendulum

Consider a physical pendulum: an object of any shape hanging about a fixed pivot.

Let *M* be the mass of the object, and distance of the Center of Mass from the pivot is l.



Let axis of rotation be through the pivot. The object can rotate freely about the axis.

The only torque on the system is because of the gravity. If I_a be the moment of inertia about the pivot, then

$$-l\,W\,\sin\phi = I_a\,\ddot{\phi}.$$

If ϕ is small (Small angle approximation), then **sin ϕ ≈ ϕ** , and therefore we obtain,

$$I_a\,\ddot{\phi} + l\,W\phi = 0 \Rightarrow \ddot{\phi} + \left(\frac{l\,Mg}{I_a}\right)\phi = 0.$$

The Physical Pendulum

$$\ddot{\phi} + \left(\frac{l\,Mg}{I_a}\right)\phi = 0$$

This is the equation of simple harmonic motion. The general solution is:

$$\phi = A\cos(\omega t) + B\sin(\omega t),$$

where $\omega = \sqrt{\frac{Mlg}{I_a}}$ is the angular frequency of small oscillations.

Radius of gyration

The **radius of Gyration** is an imaginary radial distance from the axis of rotation to a point where, if the total mass were concentrated, it would have the same moment of inertia as the actual mass distribution.

If an object of mass *M* has moment of inertia I_0 about the center of mass, then the radius of gyration is given by:

$$I_0 = M\kappa^2 \qquad \text{or,} \quad \kappa = \sqrt{\frac{I_0}{M}}$$

Now using parallel axis theorem, the moment of intertia I_a of the object about an axis distant l from the center of mass and parallel is:

$$I_a = I_0 + M\,l^2 = M(\kappa^2 + l^2).$$

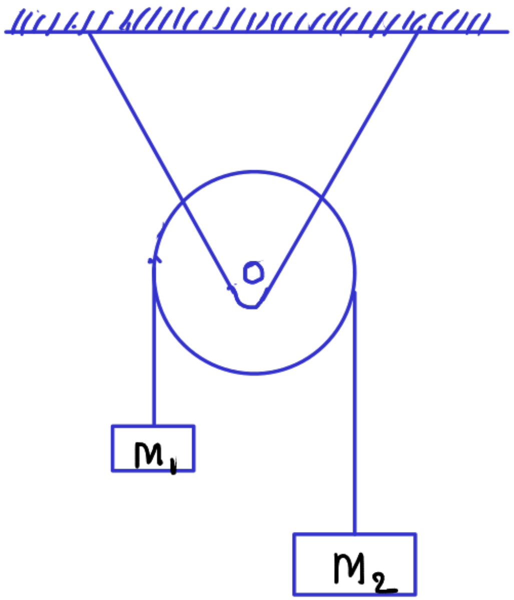
In case of physical pendulum this gives:

$$\omega = \sqrt{\frac{M\,l\,g}{I_a}} = \sqrt{\frac{l\,g}{\kappa^2+l^2}}.$$

$$\omega = \sqrt{\frac{M\,l\,g}{I_a}} = \sqrt{\frac{l\,g}{\kappa^2 + l^2}}\,.$$

This simplifies to $\omega = \sqrt{g/l}$ for $\kappa = 0$, which is the case of a simple pendulum.

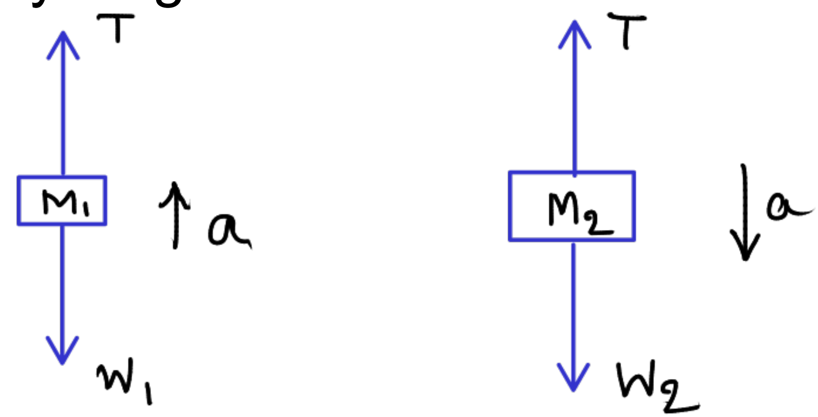
Example: Atwood's Machine



Let us consider the **ideal case** where the pulley is assumed to be static and massless, the string is also massless and there is no friction.

We want to calculate the acceleration of the masses.

The free body diagrams for the masses are:

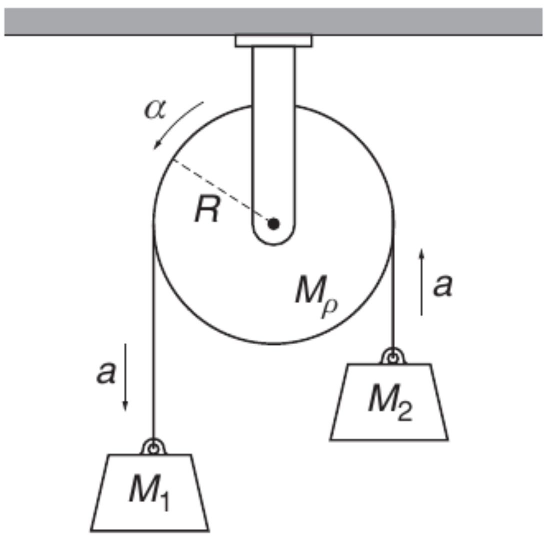


Note that for the ideal case the tension acting on both the masses are the same.

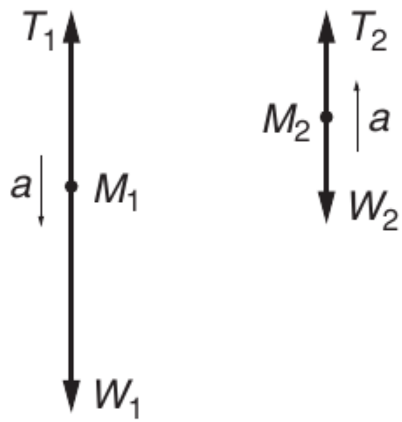
The equations of motion are: $T - W_1 = M_1 a, \qquad W_2 - T = M_2 a$

Solving we get: $$a = \frac{W_2 - W_1}{M_1 + M_2} = \frac{M_2 - M_1}{M_1 + M_2} g$$

Now consider the case where the pulley has **mass M_p** and can undergo free rotation. The string is still massless and we neglect friction.



The free body diagrams for the masses are similar to the ideal case:

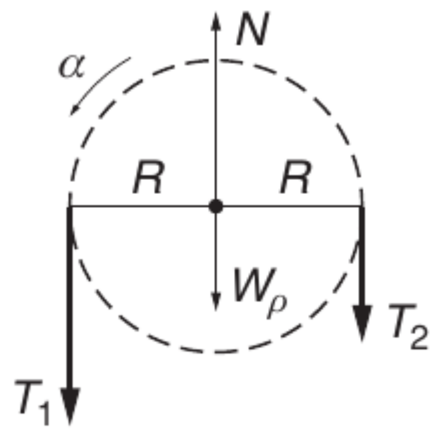


Note here $T_1 \neq T_2$

The equations of motion of the masses are:

$$
\begin{aligned}
W_1 - T_1 &= M_1 a \\
T_2 - W_2 &= M_2 a
\end{aligned}
\qquad (1)
$$

The free body diagram and the equation of motion for the pulley are given below:



$$N - T_1 - T_2 - W_p = 0 \qquad (2)$$

For net vertical force on the pulley.
N is the force on the pulley.

$$\tau = T_1 R - T_2 R = I\alpha \qquad (3)$$

For net torque on the pulley.
I is the moment of inertia of the pulley and α is its angular acceleration.

Assuming that the rope does not slip, the velocity of the rope is the same as the velocity of a point on the surface of the wheel. So, we have:

$$v = \omega R \quad \text{and} \quad a = \alpha R \qquad (4)$$

Eliminating T_1, T_2 , and α from eqns (1), (2), and (3) and using (4) we get:

$$
\begin{aligned}
W_1 - W_2 - (T_1 - T_2) &= (M_1 + M_2)a \\
T_1 - T_2 = \frac{I\alpha}{R} &= \frac{Ia}{R^2} \\
W_1 - W_2 - \frac{Ia}{R^2} &= (M_1 + M_2)a.
\end{aligned} \tag{5}
$$

If the pulley is a simple uniform disk, we have:

$$I = \frac{1}{2} M_p R^2$$

Plugging it in equation (5) we finally obtain:

$$a = \frac{(M_1 - M_2)g}{M_1 + M_2 + M_p/2}.$$

When compared with ideal system we see that the inertial mass has increased, but not by M_p but $M_p/2$.