PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 23

Introduction to Magnetostatics
Lorentz force law
Charged particle in a uniform magnetic field
Cyclotron motion
Helical motion
Cycloid Motion
Work done by magnetic force

Introduction to magnetostatics

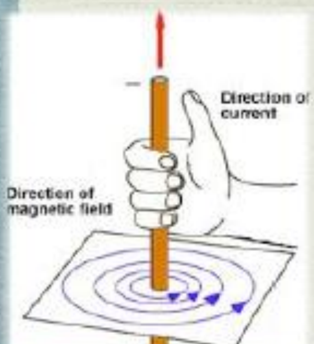
Lorentz Force and charges in the presence of electric and magnetic fields

Magnetic Field

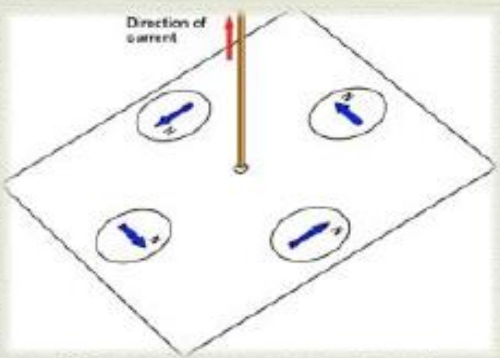
So far we restricted ourselves to electrostatics, in which we always considered the source charge at rest. We will now consider the effects emerging out of charges in motion.

While a stationary charge produces only an electric field **E** in the space around it, a moving charge generates, in addition, a magnetic field **B**.

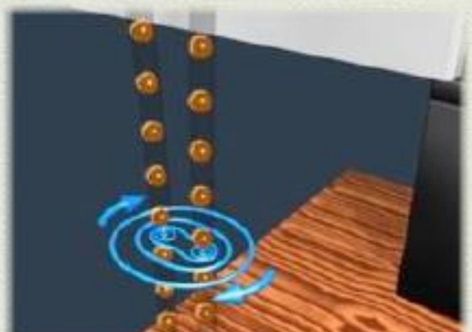
The behavior of **B** is quite different from that of **E**. For example, the magnetic field due to a long straight current carrying wire, does not point toward the wire, nor away from it, but rather it *circles* around the wire.



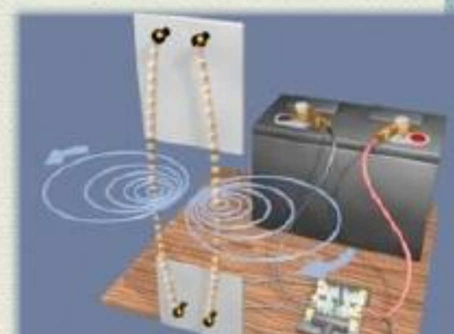
Right hand rule for magnetic field



Magnetic compasses show the direction of magnetic field

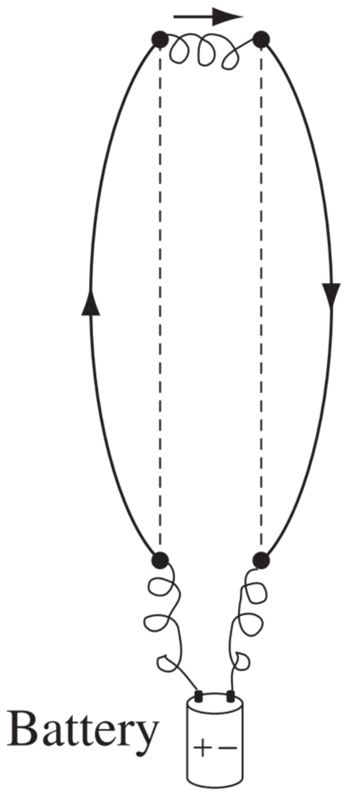


A visualization of the attraction of two wires carrying current in the same direction.

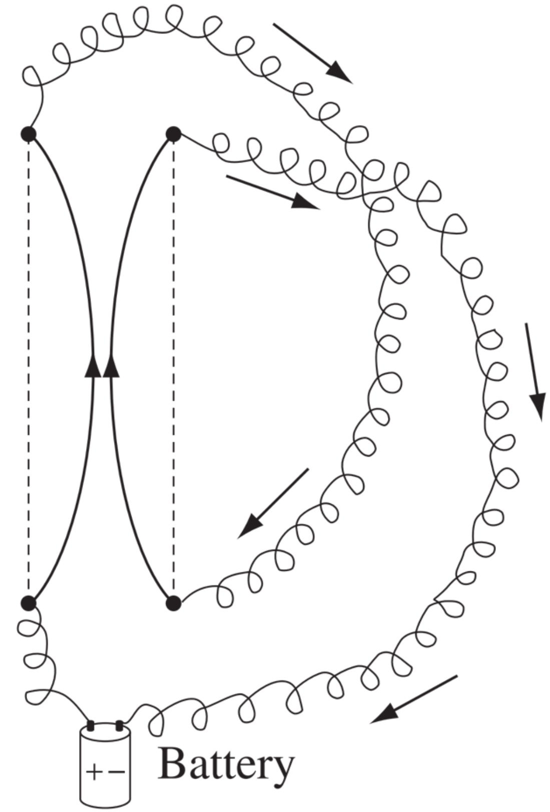


A visualization of the repulsion of two wires carrying current in the opposite direction.

Image Sources: https://www.princeton.edu/ssp/64-tiger-cub-1/64-electrical/theoryphysics-review/rhr.gif; http://www.school-for-champions.com/science/images/magnetic_field_moving_charges__compasses.gif; http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/magnetostatics/images/13-Parallel_Wires_320_f185.jpg; http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/magnetostatics/images/14-Series_320.jpg



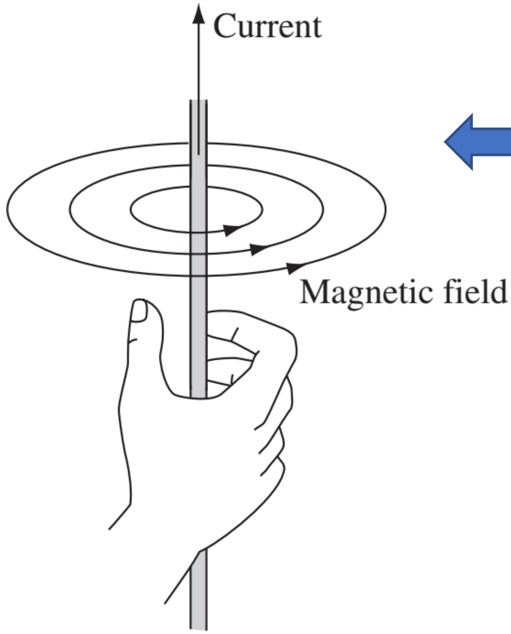
(a) Currents in opposite directions repel.



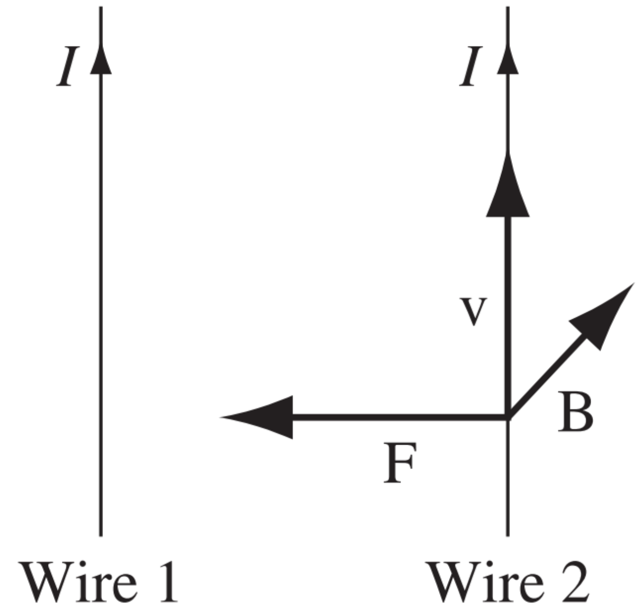
(b) Currents in same directions attract.

What force accounts for attraction of parallel currents and repulsion of antiparallel ones?

Magnetic Field



(Magnetic field direction given by right hand thumb rule)



B due to current in Wire 1

At the second wire, the magnetic field points into the page, the current is upward, and yet the resulting force is to the left.

Lorentz force law

The combination of directions is just right for a *cross-product*: the magnetic force on a charge Q, moving with a velocity ***v*** in a magnetic field ***B*** is

$$\boldsymbol{F}_{mag} = Q[\boldsymbol{v} \times \boldsymbol{B}]$$

Lorentz force law

In the presence of ***both electric and magnetic fields***, the net force on Q would be

$$\boldsymbol{F} \quad = Q[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}]$$

$$\boldsymbol{F} = \boldsymbol{F}_e + \boldsymbol{F}_{mag}$$

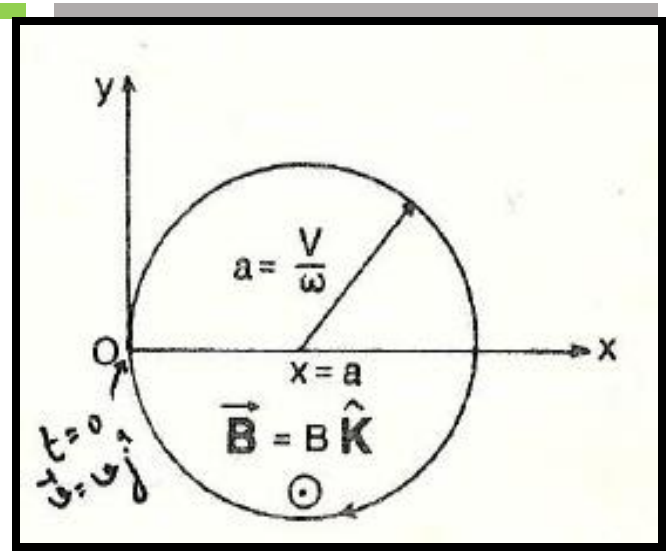
v = velocity of moving charge Q

$\boldsymbol{B}$ = magnetic field acting on Q

Lorentz force law leads to some bizarre particle trajectories of charged particles

Charged particle in a uniform magnetic field

A uniform magnetic field acts perpendicular to the plane of paper. A charged particle ‘q’ is *released from origin* with *initial velocity* $v\hat{\boldsymbol{j}}$. Find the trajectory of the particle.



Boundary conditions

$$\dot{x} = 0, \dot{y} = v$$

$$\boldsymbol{F} = q(v \times B) = q(\boldsymbol{v} \times B\hat{\boldsymbol{k}})$$

Force is always in x-o-y plane- particle cannot leave x-o-y plane

$$\mathbf{F} = m(\hat{\mathbf{i}}\ddot{x} + \hat{\mathbf{j}}\ddot{y}) = q(\hat{\mathbf{i}}\dot{x} + \hat{\mathbf{j}}\dot{y}) \times B\hat{\mathbf{k}}$$

$$\hat{\mathbf{i}}m\ddot{x} + \hat{\mathbf{j}}m\ddot{y} = \hat{\mathbf{i}}qB\dot{y} - \hat{\mathbf{j}}qB\dot{x}$$

$$\left.\begin{aligned}\ddot{x} &= \omega\dot{y} \doteq \\ \ddot{y} &= -\omega\dot{x}\end{aligned}\right\} \omega v$$

$$\omega = \frac{qB}{m}$$

Charged particle in a uniform magnetic field

$$\begin{aligned} \ddot{x} &= \omega \dot{y} = \\ \ddot{y} &= -\omega \dot{x} \end{aligned} \quad \omega \vartheta$$

These are a system of coupled differential equations. To decouple them, we have to differentiate again with respect to time.

Decouple- differentiate again w.r.t time (t)

$$\dddot{x} = \omega \ddot{y} = -\omega^2 \dot{x}$$

$$\dddot{y} = -\omega \ddot{x} = -\omega^2 \dot{y}$$

$$\frac{d^2}{dt^2}(\dot{x}) = -\omega^2 \dot{x}$$

$$\dot{x} = A \cos \omega t + C \sin \omega t,$$

Charged particle in a uniform magnetic field

$$\dot{x} = A \cos \omega t + C \sin \omega t,$$

A and C needs to be found from boundary conditions

At t = 0, $\dot{x} = 0$

$\boxed{A = 0}$

$$\dot{x} = C \sin \omega t$$

$$\ddot{x} = \omega C \cos \omega t$$

At t = 0 $\ddot{x} = \omega \dot{y} = \omega v = \omega C$ $\boxed{C = v}$

$$\dot{x} = v \sin \omega t$$
$$\dot{y} = v \cos \omega t$$

These are parametric equations of a circle

$$\dot{x}^2 + \dot{y}^2 = v^2$$

The velocity of the charged particle is constant in magnitude

Charged particle in a uniform magnetic field

$$x = \frac{v}{\omega}(1 - \cos \omega t)$$

$$y = \frac{v}{\omega} \sin \omega t$$

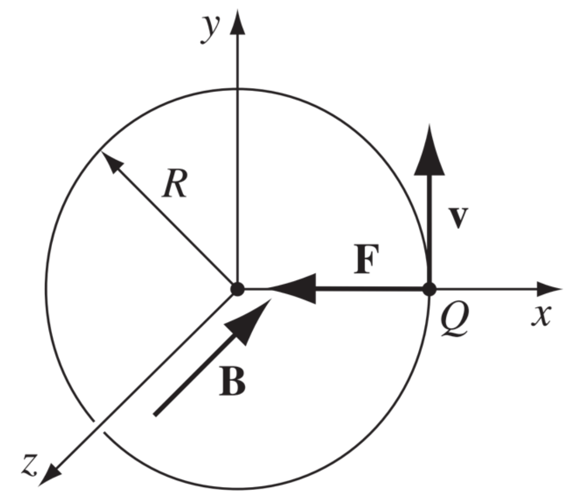
Parametric equations of a circle

$$\left(x - \frac{v}{\omega}\right)^2 + y^2 = \frac{v^2}{\omega^2}$$

Equation of a circle with center $(\frac{v}{\omega}, 0)$ and radius $\frac{v}{\omega} = a$

Cyclotron motion

The archtypical motion of a charged particle in a magnetic field is circular, with the magnetic force providing the centripetal acceleration.



A uniform magnetic field points into the page; if the charge Q moves counterclockwise, with speed v, around a circle of radius R, the magnetic force points inward, and has a fixed magnitude QvB—just right to sustain uniform circular motion:

Magnetic force QvB is the *centripetal force* (acting towards the center). The centrifugal force is $\frac{mv^2}{R}$

$$QvB = \frac{mv^2}{R}$$

$$mv = QRB = p \qquad \text{(Momentum)}$$

(Standard method to find *momenta of elementary particles*)

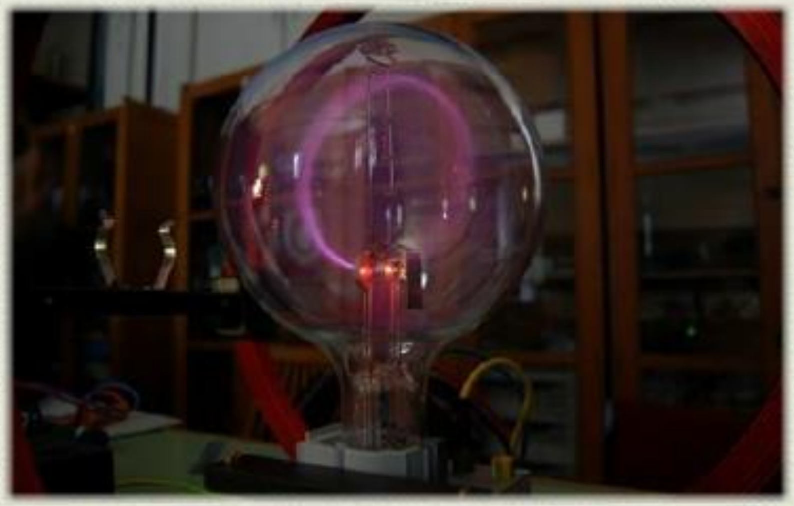
$$KE = \frac{p^2}{2m} = \frac{(QBR)^2}{2m}$$

Cyclotron motion

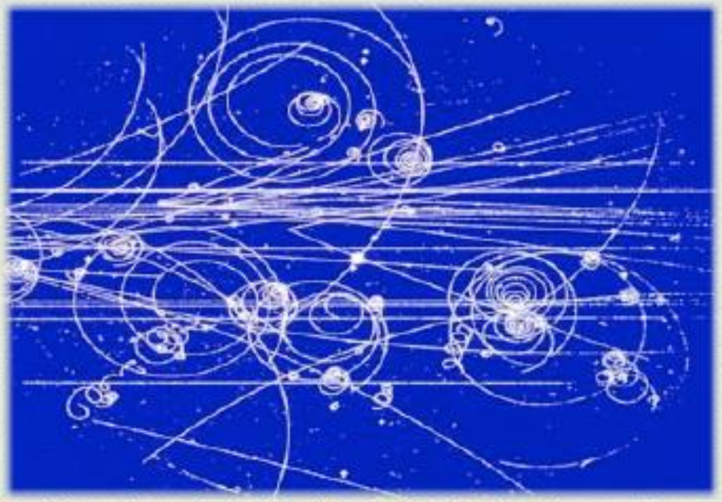
The cyclotron formula

$$p = mv = QBR,$$

suggests a simple experimental technique for finding momentum of a particle. It can be deduced by measuring the radius of circular trajectory of a particle of known charge sent through a region of known magnetic field. This is actually implemented in particle accelerators to determine the momenta of elementary particles.



Beam of electrons moving in a circle in accordance with the Lorentz force law. Visible light is emitted by excitation of gas atoms in the bulb.

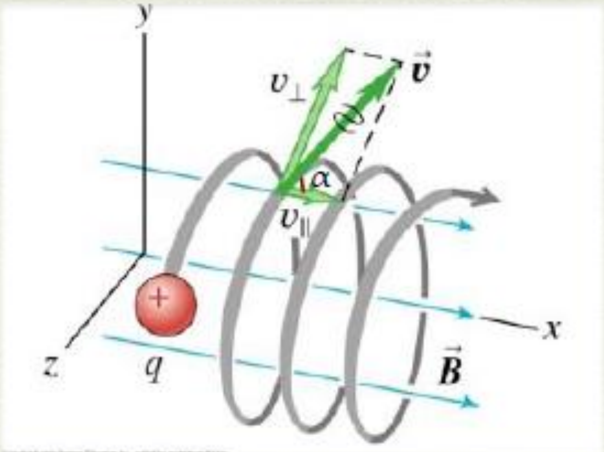


Particle tracks made visible with a bubble chamber immersed in a magnetic field. (A **bubble chamber** is a vessel filled with a superheated transparent liquid (*most often liquid hydrogen*) used to detect electrically charged particles moving through it.)

Image Sources: http://upload.wikimedia.org/wikipedia/commons/thumb/c/cf/Cyclotron_motion_wider_view.jpg/800px-Cyclotron_motion_wider_view.jpg; http://1.bp.blogspot.com/-zWYEQkz984o/T9TJnU4tTrI/AAAAAAAADkg/cW_kKJshXS0/s1600/A+bubble+chamber+7.jpg

Helical motion

If the charged particle has a velocity **v** which is not entirely perpendicular to the direction of **B**, then only the perpendicular (to the **B**) component, $v_\perp$, of the velocity will result in the Lorentz force. The motion of the particle associated with the parallel component, $v_\parallel$, will remain unaffected.



For example, in the adjoining figure, the magnetic field has a direction along X direction. The charged particle has a velocity **v** which is at a certain angle to the X-direction. This **v** can be resolved in component $v_\parallel$ which is parallel to the **B** field, i.e., along X direction, and $v_\perp$ which is perpendicular to the **B** field, i.e., lies in the YZ plane.

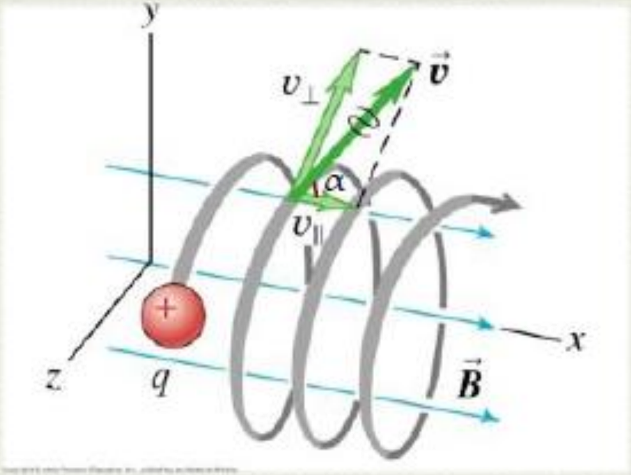
Lorentz formula still applies but with v replaced by $v_\perp$, and therefore radius of the circular component of the motion is

$$R = \frac{mv_\perp}{QB}.$$

Image Source: http://www.physics.miami.edu/~zuo/class/spr_05/supplement/Figure27_16.jpg

Helical motion

The component $v_\parallel$ does not enter in the Lorentz force formula and leads to a linear motion in the X direction. The full motion of the particle is the resultant of circular motion parallel to YZ plane and linear motion along the X direction, giving rise to a helical trajectory (a helix).



The **angular frequency ω** at which the charge revolves about the magnetic field is referred to as the cyclotron frequency and is given by

$$\omega = \frac{v_\perp}{R} = \frac{QB}{m}.$$

We can see that it is independent of the speed of the particle!

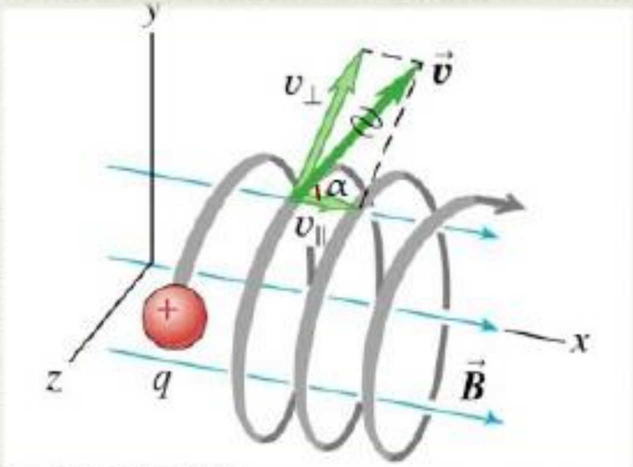
The corresponding linear frequency and time period are respectively

$$f = \frac{\omega}{2\pi} = \frac{v_\perp}{2\pi R} = \frac{QB}{2\pi m}, \qquad T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi m}{QB}.$$

Image Source: http://www.physics.miami.edu/~zuo/class/spr_05/supplement/Figure27_16.jpg

Helical motion

For the setup shown in the figure, if at t=0 we take the particle to be at (0,0,R), where $R = m\, v_\perp/(QB)$, then after time t, the angular displacement of the particular relative to Z direction would be $\omega t=(QB/m)t$ **clockwise**. Also, in this time the particle would have moved a distance $v_\parallel t$ along the X axis. Thus, we can write the trajectory (helical) of the charge in the following parametric form:



$$x = v_\parallel\, t\,,$$

$$y = \frac{mv_\perp}{QB}\sin\left(\frac{QB}{m}t\right),$$

$$z = \frac{mv_\perp}{QB}\cos\left(\frac{QB}{m}t\right).$$

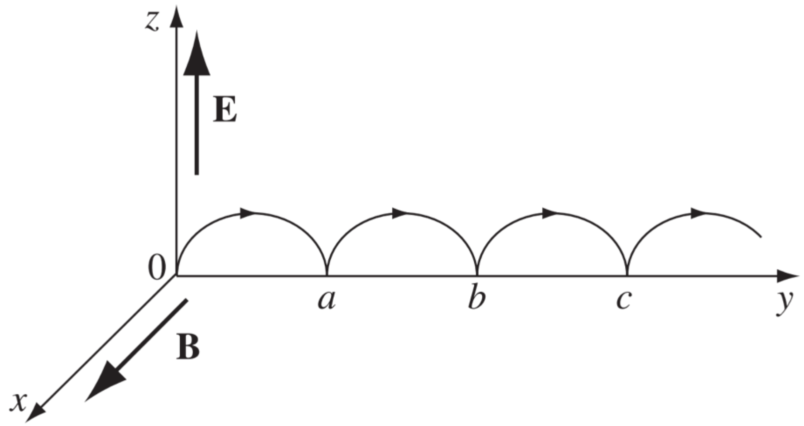
If the particle velocity **v** makes an angle α with the B field (X-direction in the present setup), then $v_\perp = v\sin\alpha$, and $v_\parallel = v\cos\alpha$.

Image Source: http://www.physics.miami.edu/~zuo/class/spr_05/supplement/Figure27_16.jpg

Cycloid Motion: (Qualitative Analysis)

Motion of a charged particle in both E and B (crossed fields)

Include a uniform electric field at right angles to the magnetic one. Suppose $\boldsymbol{B}$ points in x-direction, and $\boldsymbol{E}$ in the z-direction. A positive charge Q is released from the origin; what path will it follow?



Since **v** is zero at this instant, the magnetic force is zero. However, the electric field acts on the particle and accelerates it in the Z-direction. As soon as the particle attains a nonzero velocity, the magnetic force comes into action and pulls the charge around to the right. The faster the charge starts moving, the more stronger the F_{mag} gets, keeping its direction perpendicular to **v** (and **B**). Eventually it curves the particle back around towards the Y axis. At this point the charge is moving against the electrical force, therefore it begins to slow down which leads to a decrease in the magnetic force also. The electric force then dominates and brings the charge to rest at point a. The entire process is repeated again, and carries the particle over to point b, and so on.

What work is done by the magnetic force?

$$F_{mag} = Q(\boldsymbol{v} \times \boldsymbol{B})$$ (Lorentz force)

If a charge Q moves an amount $\boldsymbol{dl} = \boldsymbol{v}dt$, work done

$$dW_{mag} = \boldsymbol{F}_{mag} \cdot \boldsymbol{dl} = Q(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}dt = 0$$

$$(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} = (\boldsymbol{v} \times \boldsymbol{v}) \cdot \boldsymbol{B} = \boldsymbol{0}$$ (Dot and cross can always be interchanged)

Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down (for example, a *charged particle* moving with a *velocity 'v'* under a *magnetic field* normal to its plane of motion has a circular trajectory with *no change in its velocity*)

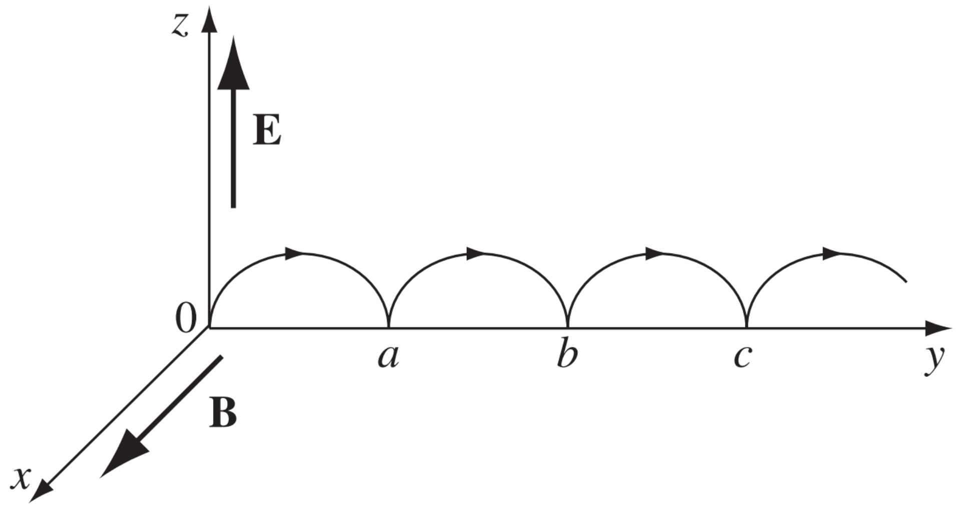
Magnetic forces do no work

Optional Slides

Cycloid Motion (Quantitative Analysis)

Motion of a charged particle in both E and B (crossed fields)

$$\boldsymbol{F} = q\boldsymbol{E} + q(\boldsymbol{v} \times \boldsymbol{B})$$
$$= qE\hat{\boldsymbol{z}} + q(\boldsymbol{v} \times \boldsymbol{B})$$
$$\boldsymbol{B} = B\hat{\boldsymbol{x}}$$



$\boldsymbol{F}$ must be normal to $\hat{\boldsymbol{x}}$, that is y-o-z plane, so the velocity should also be in y-o-z plane

There being no force in the x-direction, the position of the particle at any time t can be described by the vector (0, y(t), z(t))

Velocity vector $\boldsymbol{v} = \left(0, v_y, v_z\right) = (0, \dot{y}, \dot{z})$

where dots indicate time derivatives.

Cycloid Motion (Quantitative Analysis)

Thus

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}},$$

and hence, applying Newton's second law,

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}}) = m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}).$$

treating the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ components separately,

$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}.$$

$\boxed{\text{These are coupled differential equations}}$

Cycloid Motion (Quantitative Analysis)

Let

$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}.$$

$$\omega \equiv \frac{QB}{m}$$

This is the **cyclotron frequency**, at which the particle would revolve in the absence of any electric field.

Then the equations of motion take the form

$$\ddot{y} = \omega\dot{z}, \qquad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

Cycloid Motion (Quantitative Analysis)

$$\ddot{y} = \omega \dot{z}, \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

The second differential equation above (involving $\ddot{z}$) can be integrated once† (with respect to t) to give

$$\dot{z} = \omega(Et/B - y + C),$$

where ωC is the constant of integration (ω has been chosen as a factor for convenience; see below). We can now substitute this in the first differential equation above (involving $\ddot{y}$). This gives

$$\ddot{y} = \omega^2(Et/B - y + C) \Rightarrow \ddot{y} + \omega^2(y - Et/B - C) = 0.$$

Let us define a new variable, $u = y - Et/B - C$. Clearly $\ddot{u} = \ddot{y}$. Therefore, the above differential equation in y gets reduced to a simple-harmonic differential equation in u:

$$\ddot{u} + \omega^2 u = 0,$$

for which the general solution is

$$u(t) = C_1 \cos \omega t + C_2 \sin \omega t.$$

† *We can't perform the second integration with respect to t on the resultant expression, since it involves y(t), which we don't know yet.*

Cycloid Motion (Quantitative Analysis)

$$u = y - Et/B - C.$$

$$\dot{z} = \omega(Et/B - y + C)$$

Returning back to the y variable we have,

$$\begin{aligned} y(t) &= u(t) + Et/B + C \\ &= C_1 \cos\omega t + C_2 \sin\omega t + Et/B + C \\ &= C_1 \cos\omega t + C_2 \sin\omega t + Et/B + C_3, \end{aligned}$$

where we rechristened C to C_3. Plugging this in the differential equation for $\dot{z}$ obtained on the last page,

$$\begin{aligned} &\dot{z} = \omega(Et/B - y + C_3) \\ \Rightarrow\ &\dot{z} = \omega(Et/B - C_1 \cos\omega t - C_2 \sin\omega t - Et/B - C_3 + C_3) \\ \Rightarrow\ &\dot{z} = -\omega C_2 \sin\omega t - \omega C_1 \cos\omega t. \end{aligned}$$

This can now be integrated (with respect to t) to give

$$z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4.$$

Hence, we have the general solutions for $y(t)$ and $z(t)$, as claimed.

Cycloid Motion (Quantitative Analysis)

General solution

$$\left.\begin{aligned} y(t) &= C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3, \\ z(t) &= C_2 \cos\omega t - C_1 \sin\omega t + C_4. \end{aligned}\right\}$$

Applying the initial conditions:

* The particle started from the origin: $y(t{=}0)=z(t{=}0) = 0$,
* The particle started from rest: $\dot{y}(t{=}0) = \dot{z}(t{=}0) = 0$,

we obtain $C_1{=}0$, $C_2{=}{-}E/(\omega B)$, $C_3{=}0$, $C_4{=}E/(\omega B)$, and therefore we have

$$\left.\begin{aligned} y(t) &= \frac{E}{\omega B}(\omega t - \sin\omega t) \\ z(t) &= \frac{E}{\omega B}(1 - \cos\omega t) \end{aligned}\right\} \text{Parametric equations of } \underline{\textbf{cycloid}}$$

Cycloid Motion (Quantitative Analysis)

Let

$$R \equiv \frac{E}{\omega B}$$ (Radius of trajectory in crossed fields)

$$y = R\omega t - R\sin\omega t \Rightarrow y - R\omega t = -R\sin\omega t$$
$$z = R - R\cos\omega t \Rightarrow z - R = -R\cos\omega t.$$

Squaring and adding, and using trigonometric identity $\sin^2\omega t+\cos^2\omega t=1$, we obtain

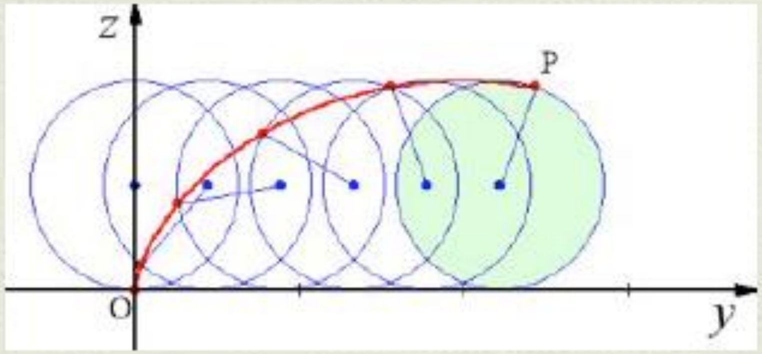
$$(y - R\omega t)^2 + (z - R)^2 = R^2$$

Cycloid Motion (Quantitative Analysis)

This represents a *circle*, of radius R, whose center $(R\omega t, R)$ travels in the Y-direction at a constant speed,

$$(y - R\omega t)^2 + (z - R)^2 = R^2.$$

The particle moves as if it were on a spot on the rim of a wheel, rolling down the y-axis at speed v. The curve generated by the particle in this way is, of course, a **cycloid**.



$$v = \omega R = \frac{E}{B}.$$

Note that the *overall* motion of the particle is *not* in the direction of **E**, but perpendicular to it.