PHY101: Introduction to Physics I

Monsoon Semester 2024

Lecture 7

Department of Physics, School of Natural Sciences,
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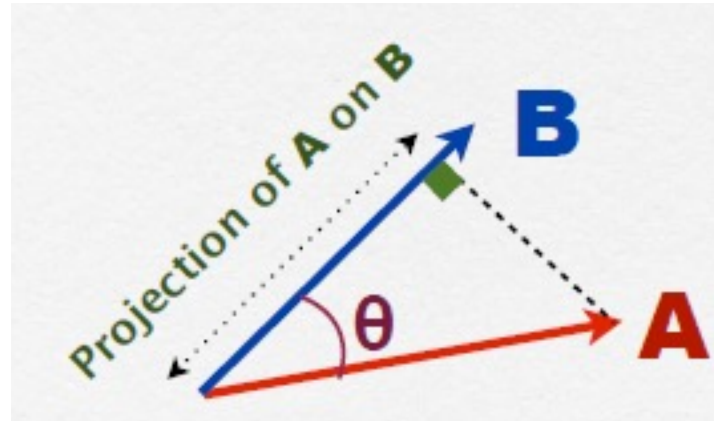
<u>Previous Lecture</u>

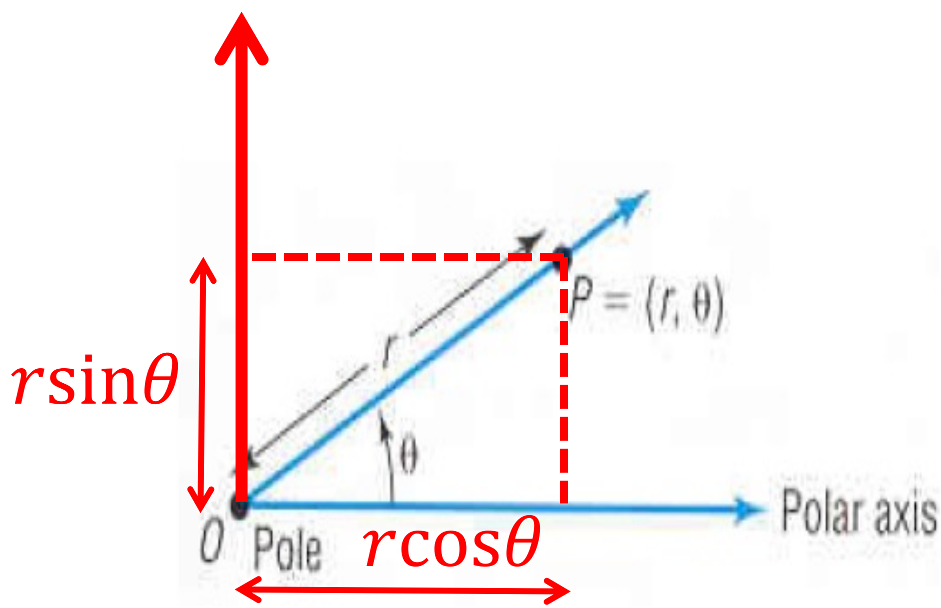
<u>Dot product , cross product etc.</u>



<u>This Lecture</u>

Derivative of vectors,

Finding Unit vectors etc.



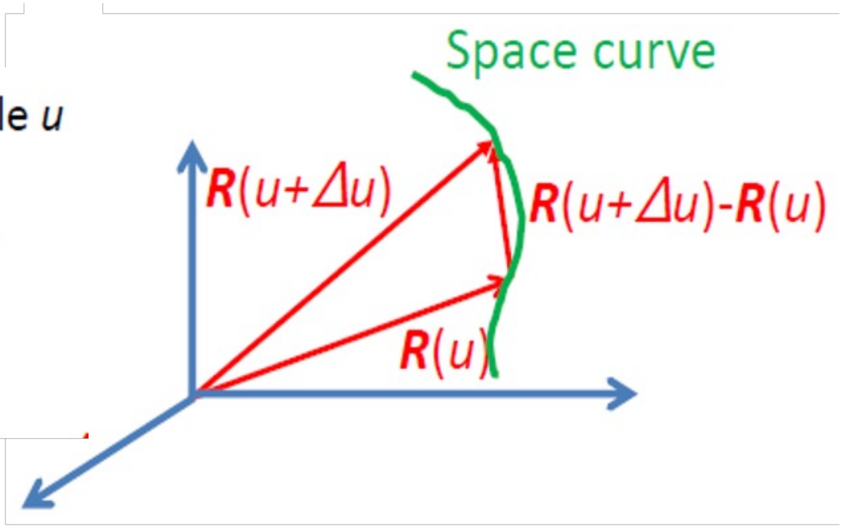
Vectors

Ordinary derivatives of vectors (Mathematical tool)

Let $\boldsymbol{R}(u)$ be a vector depending on single scalar variable u

$$\frac{\Delta \boldsymbol{R}}{\Delta u} = \frac{\boldsymbol{R}(u+\Delta u) - \boldsymbol{R}(u)}{\Delta u}$$

where Δu denotes an increment in u



Tangent Vector

Hence, ordinary derivative of R(u) with respect to u is

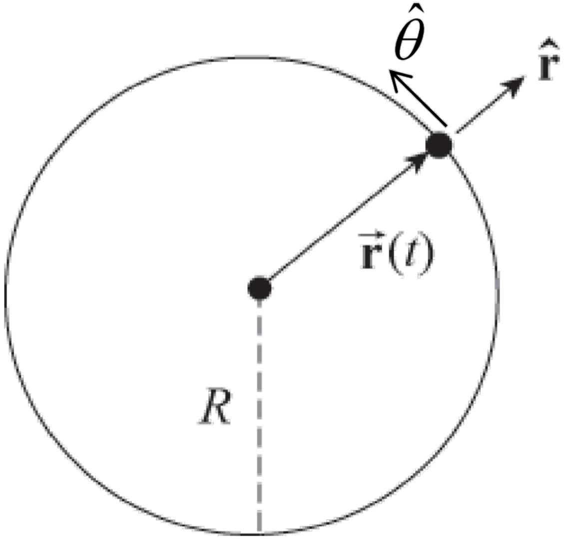
$$\frac{d\boldsymbol{R}}{du} = Lim_{\Delta u \to 0} \frac{\Delta \boldsymbol{R}}{\Delta u} = Lim_{\Delta u \to 0} \frac{\boldsymbol{R}(u+\Delta u) - \boldsymbol{R}(u)}{\Delta u}$$

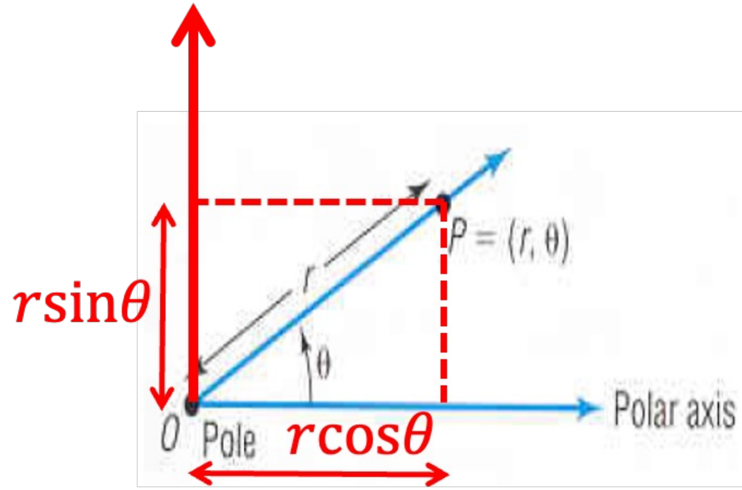
Properties of the Derivative of Vector (Mathematical tool)

Let **r** and **u** be differentiable vector-valued functions of *t*, let *f* be a differentiable real-valued function of *t*, and let *c* be a scalar.

i.	$\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$		Scalar multiple
ii.	$\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$		Sum and difference
iii.	$\frac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$		Scalar product
iv.	$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}(t) + \mathbf{r}(t) \cdot \mathbf{u}'(t)$		Dot product
v.	$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t)$		Cross product
vi.	$\frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t)) \cdot f'(t)$		Chain rule
vii.	If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.		

Unit vectors in plane polar coordinate system





The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

By coordinate transformations: $x = r\cos\theta$

$y = r\sin\theta$

In Cartesian coordinate: $\vec{r} = x\hat{i} + y\hat{j}$ ➡ $\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$

Find $\hat{r}$ and $\hat{\theta}$ in polar coordinate

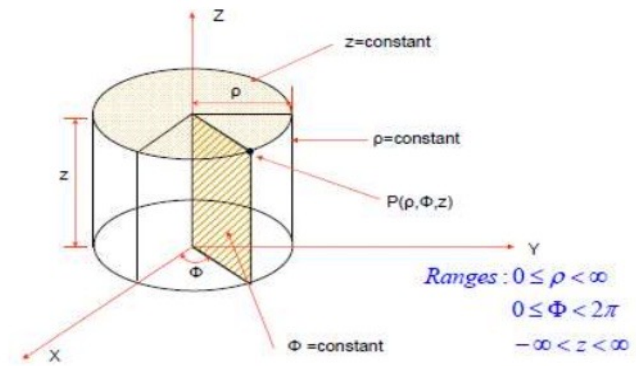
Unit vectors only depend on θ

$$\hat{r} = \frac{\partial\vec{r}/\partial r}{|\partial\vec{r}/\partial r|} = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

Unit vectors in cylindrical polar coordinate system

Cylindrical Coordinate Systems



$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \rho\, cos\varphi\, \hat{\imath} + \rho\, sin\varphi\, \hat{\jmath} + z\hat{k}$$

Tangent vector:

$$\frac{\partial}{\partial\rho}\vec{\boldsymbol{r}} = cos\varphi\, \hat{\imath} + sin\varphi\, \hat{\jmath}$$

Magnitude of the tangent vector,

$$\left|\frac{\partial}{\partial\rho}\vec{\boldsymbol{r}}\right| = \sqrt{cos^2\varphi + sin^2\varphi} = 1$$

Unit vector:

$$\hat{\rho} = \frac{\partial}{\partial\rho}\vec{\boldsymbol{r}} \Big/ \left|\frac{\partial}{\partial\rho}\vec{\boldsymbol{r}}\right| = cos\varphi\, \hat{\imath} + sin\varphi\, \hat{\jmath}$$

$$\frac{\partial}{\partial\varphi}\vec{\boldsymbol{r}} = -\rho\, \sin\varphi\, \hat{\imath} + \rho\, cos\varphi\, \hat{\jmath}$$

Magnitude of the tangent vector,

$$\left|\frac{\partial}{\partial\varphi}\vec{\boldsymbol{r}}\right| = \rho$$

Unit vector:

$$\hat{\varphi} = \frac{\partial}{\partial\varphi}\vec{\boldsymbol{r}} \Big/ \left|\frac{\partial}{\partial\varphi}\vec{\boldsymbol{r}}\right| = -\sin\varphi\, \hat{\imath} + cos\varphi\, \hat{\jmath}$$

$$\frac{\partial}{\partial z}\vec{\boldsymbol{r}} = \hat{k}$$

$$\left|\frac{\partial}{\partial z}\vec{\boldsymbol{r}}\right| = 1$$

$$\hat{k} = \hat{k}$$

Unit vectors in spherical polar coordinate system

Homework: Find unit vectors in a *spherical polar coordinate system*.

***Hint**: Follow the same mathematical approach as a cylindrical polar coordinate*

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Spherical to Cartesian

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Solution: Next tutorial

Velocity and acceleration of typical systems

Velocity in Cartesian coordinate system

Average velocity in 1D

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

Instantaneous velocity in 1D

$$v = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

$$v = \frac{dx}{dt},$$

Vectorial approach

$$\boldsymbol{v} = \frac{d\mathbf{r}}{dt} = \frac{d(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})}{dt}.$$

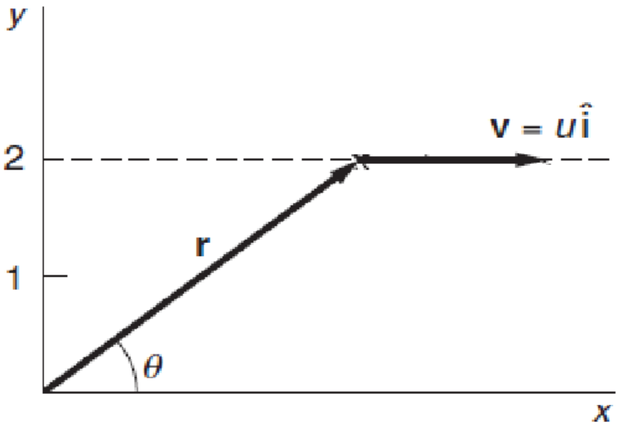
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\,\hat{\mathbf{i}} + \frac{dv_y}{dt}\,\hat{\mathbf{j}} + \frac{dv_z}{dt}\,\hat{\mathbf{k}}$$

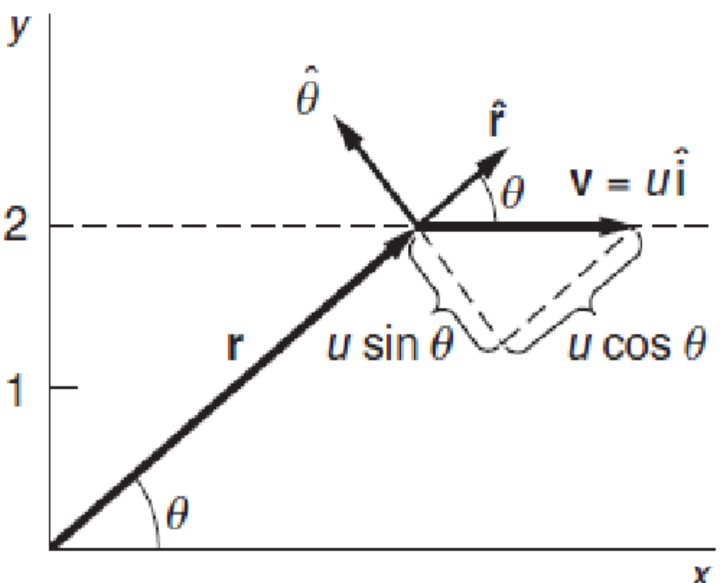
$$= \frac{d^2\mathbf{r}}{dt^2}.$$

Ref: 1) https://www.cuemath.com/calculus/triangle-law-of-vector-addition/

Velocity in polar coordinate system for straight line motion

Consider a particle moving with constant velocity $\mathbf{v} = u\hat{\mathrm{i}}$ along the line $y = 2$. Describe $\mathbf{v}$ in plane polar coordinate:





$$\mathbf{v} = u\hat{\mathbf{i}}$$

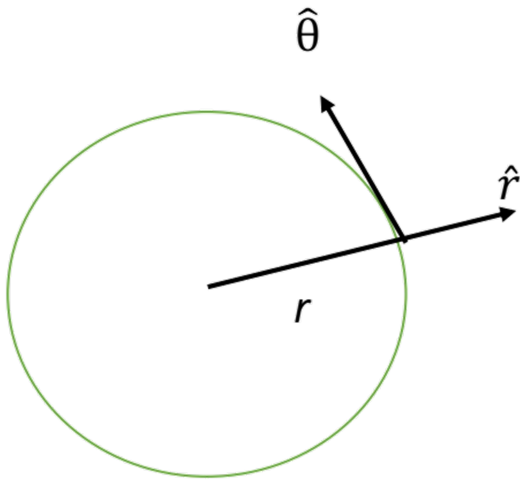
$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}.$$

$$v_r = u\cos\theta$$

$$v_\theta = -u\sin\theta$$

$$\mathbf{v} = u\cos\theta\,\hat{\mathbf{r}} - u\sin\theta\,\hat{\boldsymbol{\theta}}.$$

Velocity in plane polar coordinate system for circular motion



The position vector $\vec{r}$ in polar coordinate is given by : $\vec{r} = r\hat{r}$

$$\hat{r} = cos\theta\ \hat{\imath} + sin\theta\ \hat{\jmath}$$

$$\hat{\theta} = -sin\theta\ \hat{\imath} + cos\theta\ \hat{\jmath}$$

Step 1: Find expression of the followings (change of unit vectors with time):

(1) $\frac{\partial}{\partial \boldsymbol{t}}\hat{\boldsymbol{r}}$ **(2)** $\frac{\partial}{\partial \boldsymbol{t}}\hat{\theta}$

$$\frac{\partial}{\partial \boldsymbol{t}}\hat{\boldsymbol{r}} = \dot{\hat{r}} = \dot{\theta}\left(-\sin\theta\hat{i} + \cos\theta\,\hat{j}\right) = \dot{\theta}\hat{\theta}$$

$$\frac{\partial}{\partial \boldsymbol{t}}\hat{\theta} = \dot{\hat{\theta}} = \dot{\theta}\left(-\cos\theta\hat{i} - \sin\theta\,\hat{j}\right) = -\dot{\theta}\hat{r}$$

Contd..

Step 2: Find expression of the velocity components

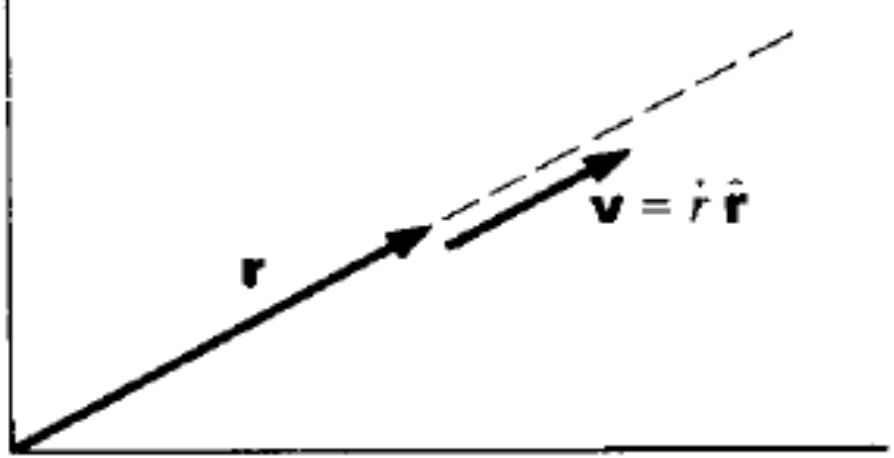
$$\mathbf{v} = \frac{d}{dt} r\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}.$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

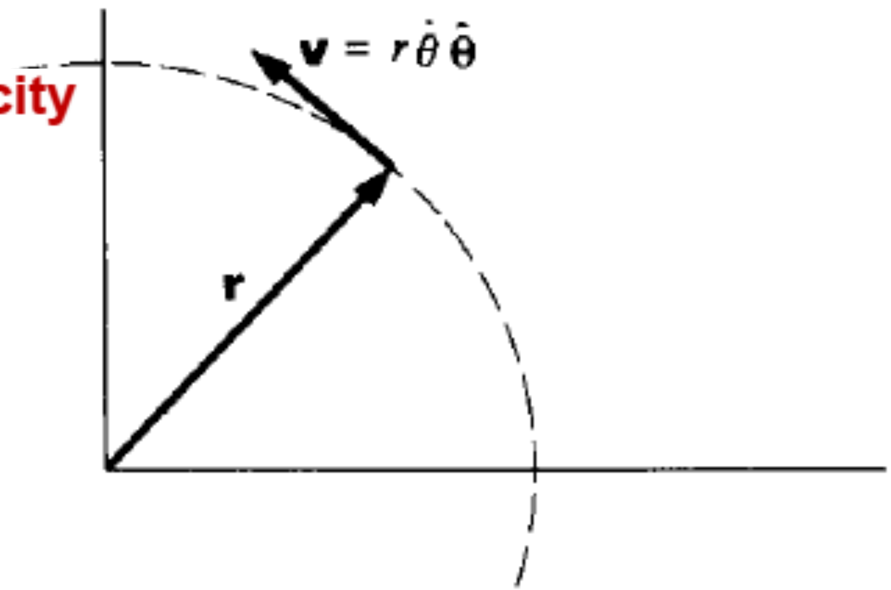
$$\boxed{\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta}\hat{\boldsymbol{\theta}} \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\hat{\mathbf{r}}.\end{aligned}}$$

Radial velocity

tangential velocity



Case 1

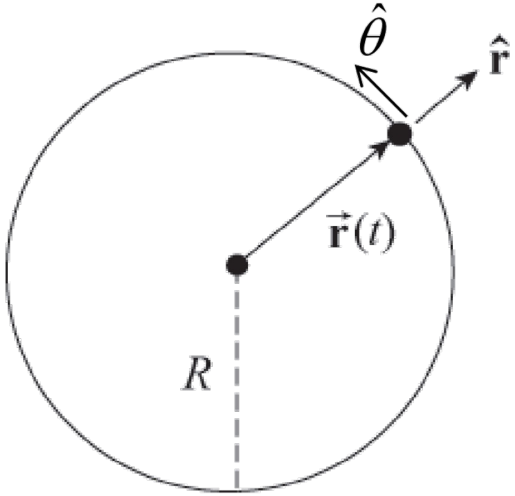


Case 2

Contd..

Uniform Circular Motion

$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}} \qquad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$



Since $\dot{r} = \dfrac{dR}{dt} = 0$ and $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\omega\hat{\boldsymbol{\theta}}(t)$$

Since $\vec{v}$ is along $\hat{\theta}$ it must be perpendicular to the radius vector $\vec{r}$ and it can be shown easily

$$R^2 = \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} \Rightarrow \frac{d}{dt}R^2 = \frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 0 \Rightarrow \boxed{\vec{\mathbf{r}} \perp \vec{\mathbf{v}}}$$

Useful link: https://www.youtube.com/watch?v=S9_Oe51XkVY

Next Lecture (L 8) : Acceleration
(Reserved slides)

Vectors

Acceleration in plane polar coordinate system

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}})$$

$$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}.$$

$$= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\mathbf{r}}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

radial centripetal tangential Coriolis

Radial acceleration: Due to change of radial speed

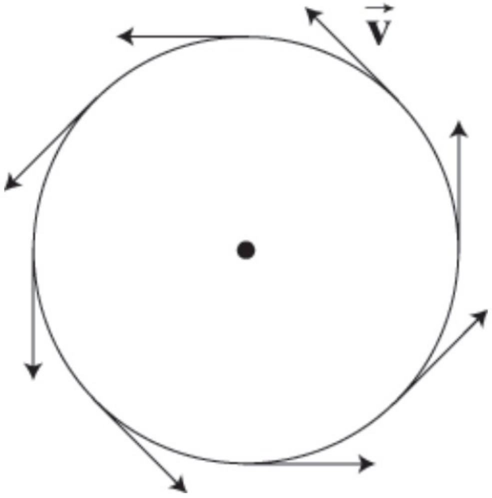
Centripetal acceleration: Due to change of direction of tangential velocity

Tangential acceleration: Due to change of tangential speed

Coriolis acceleration: Due to change of radius and angle, both with time

Vectors (Next day)

Acceleration in polar coordinate system



Uniform circular Motion

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

$$\vec{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

For a circular motion, $r = R$, the radius of the circle.

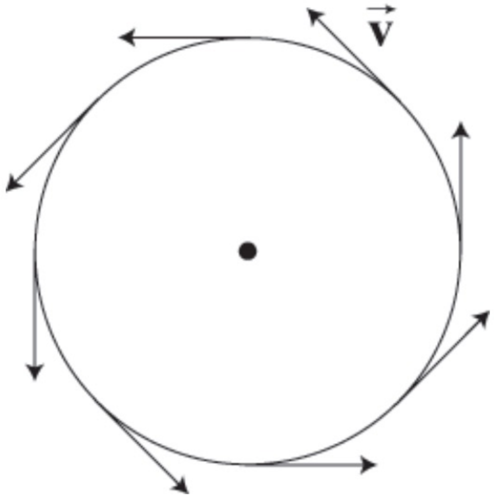
Hence, $\dot{r} = \ddot{r} = 0$

So, $a_\theta = R\ddot{\theta}$ and $a_r = -R\dot{\theta}^2$

Vectors (Next day)

Acceleration in polar coordinate system

Nonuniform circular Motion



For non-uniform circular motion, ω is function of time. Hence, $a_\theta = R\dfrac{d\omega}{dt} = R\alpha$,

where $\alpha = \dfrac{d\omega}{dt}$ is the angular acceleration.

However, the radial acceleration is always $a_r = -R\dot{\theta}^2 = -R\omega^2$

Therefore, an object traveling in a circular orbit with a constant speed is always accelerating towards the center. Though the magnitude of the velocity is a constant, the direction of it is constantly varying. Because the velocity changes direction, the object has a nonzero acceleration.