Department of Physics, Shiv Nadar Institution of Eminence

Spring 2025

PHY102: Introduction to Physics-II

Tutorial – 12

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1. If $\vec{B}$ is uniform show that $\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B})$, where $\vec{r}$ is the position vector of the point in question. Show that $\vec{\nabla}.\vec{A} = 0$ and $\vec{\nabla} \times \vec{A} = \vec{B}$.

Solution. Let $\vec{B} = B\hat{k}$ and $\vec{C} = \vec{r} \times \vec{B}$.

$$\therefore \vec{\nabla} \times \vec{C} = \vec{\nabla} \times (\vec{r} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{r} - (\vec{r} \cdot \vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{r})$$

Now

$$(\vec{B} \cdot \vec{\nabla})\vec{r} = B\frac{\partial}{\partial z}(x\hat{i} + y\hat{j} + z\hat{k}) = B\hat{k} = \vec{B}$$

$$(\vec{r} \cdot \vec{\nabla})\vec{B} = 0 \quad \text{as } \vec{B} \text{ is uniform}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\text{and} \quad \vec{\nabla} \cdot \vec{r} = 3$$

Therefore,

$$\vec{\nabla} \times \vec{C} = \vec{B} - 0 + 0 - 3\vec{B} = -2\vec{B}$$

$$\text{or,} \quad \vec{B} = -\tfrac{1}{2}\vec{\nabla} \times \vec{C} = \vec{\nabla} \times \left(-\tfrac{1}{2}\vec{C}\right) = \vec{\nabla} \times \vec{A}$$

$$\therefore \vec{A} = -\tfrac{1}{2}\vec{C} = -\tfrac{1}{2}(\vec{r} \times \vec{B}).$$

$$\vec{\nabla} \cdot \vec{A} = -\tfrac{1}{2}\vec{\nabla} \cdot (\vec{r} \times \vec{B}) = -\tfrac{1}{2}\left[\vec{B} \cdot (\vec{\nabla} \times \vec{r}) - \vec{r} \cdot (\vec{\nabla} \times \vec{B})\right]$$

Now $\vec{\nabla} \times \vec{r} = 0$, $\vec{\nabla} \times \vec{B} = 0$ as $\vec{B}$ is uniform.

Therefore, $\vec{\nabla} \cdot \vec{A} = 0$

$$\vec{\nabla} \times \vec{A} = -\tfrac{1}{2}\vec{\nabla} \times (\vec{r} \times \vec{B})$$

$$= -\tfrac{1}{2}\left[\vec{r}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{r}) + (\vec{B} \cdot \vec{\nabla})\vec{r} - (\vec{r} \cdot \vec{\nabla})\vec{B}\right]$$

Now $\vec{\nabla} \cdot \vec{B} = 0$ and $(\vec{r} \cdot \vec{\nabla})\vec{B} = 0$ because $\vec{B}$ is uniform, $\vec{\nabla} \cdot \vec{r} = 3$ and

$$(\vec{B} \cdot \vec{\nabla})\vec{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = \vec{B}.$$

Thus,

$$\vec{\nabla} \times \vec{A} = -\tfrac{1}{2}\left[-3\vec{B} + \vec{B}\right] = \vec{B}.$$

2. A thin disk of radius a carrying uniform surface charge density σ is rotating with constant angular velocity ω about its axis (z-axis). Suppose there is a uniform magnetic field $\vec{B} = B\hat{\jmath}$. Show that the torque acting on the disk is of magnitude $\frac{1}{4}\pi\sigma\omega B a^4$.

Solution. Consider a ring of radius r and radial thickness dr. The charge on the ring is $\sigma \cdot 2\pi r dr$. It constitutes a current loop of current

$$dI = \frac{\sigma \cdot 2\pi r dr}{T} = \sigma 2\pi r dr \cdot \left(\frac{\omega}{2\pi}\right) = \sigma\omega r dr.$$

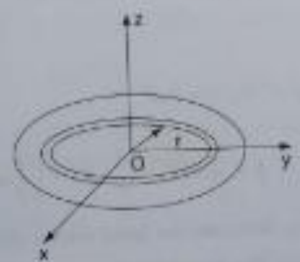


Fig 8.P-20

The magnetic moment of this ring is

$$d\vec{m} = \text{current} \times \text{area of the loop} = dI \cdot \pi r^2 \hat{k} = \pi\sigma\omega r^3 dr \hat{k}.$$

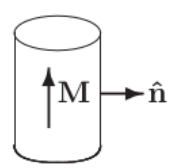
Therefore, torque on the ring is $d\vec{\tau} = d\vec{m} \times \vec{B} = -\hat{i}\pi\sigma\omega B r^3 dr$.

Hence, total torque on the disk is

$$\vec{\tau} = -\hat{i}\pi\sigma\omega B \int_0^a r^3 dr = -\hat{i}\frac{1}{4}\pi\sigma\omega B a^4.$$

3. An infinitely long circular cylinder carries a uniform magnetization **M** parallel to its axis. Find the magnetic field (due to **M**) inside and outside the cylinder.

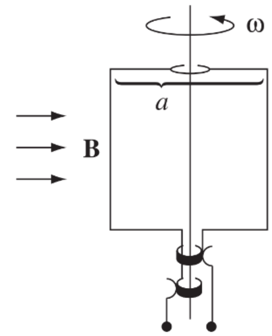
$$\mathbf{J}_b = \nabla \times \mathbf{M} = 0;\ \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}.$$

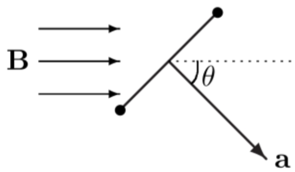


The field is that of a surface current $\mathbf{K}_b = M\hat{\boldsymbol{\phi}}$, but that's just a solenoid, so the field $\boxed{\text{outside is zero,}}$ and inside $B = \mu_0 K_b = \mu_0 M$.

Moreover, it points upward (in the drawing), so $\boxed{\mathbf{B} = \mu_0 \mathbf{M}.}$

4. A square loop (side a) is mounted on a vertical shaft and rotated at angular velocity ω (see figure below). A uniform magnetic field B points to the right. Find the $\mathcal{E}(t)$ for this alternating current generator.





(view from above)

$$\Phi = \mathbf{B}\cdot\mathbf{a} = Ba^2\cos\theta$$
Here $\theta = \omega t$, so
$$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba^2(-\sin\omega t)\omega;$$
$$\boxed{\mathcal{E} = B\omega a^2\sin\omega t.}$$

5. A metal bar of mass m slides frictionless on two parallel conducting rails a distance l apart. A resistor R connected across the rails and a uniform magnetic field B, pointing into the page, fills the entire region.
(a) If the bar moves to the right a speed v, what is the current in the resistor? In what direction does it flow?
(b) What is the magnetic force on the bar? In what direction?
(c) If the bar starts out with a speed v_0 at time $t = 0$, and is the left to slide, what is its speed at a time later time t?
(d) The initial kinetic energy of the bar was ½ mv_0^2. Check the energy delivered to the resistor is exactly ½ mv_0^2.

(a) $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv$; $\mathcal{E} = IR \Rightarrow \boxed{I = \frac{Blv}{R}.}$ (Never mind the minus sign—it just tells you the *direction* of flow: $(\mathbf{v}\times\mathbf{B})$ is *upward*, in the bar, so *downward* through the resistor.)

(b) $F = IlB = \boxed{\frac{B^2l^2v}{R}},$ to the $\boxed{\text{left.}}$

(c) $F = ma = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \Rightarrow \frac{dv}{dt} = -\left(\frac{B^2l^2}{Rm}\right)v \Rightarrow \boxed{v = v_0e^{-\frac{B^2l^2}{mR}t}.}$

(d) The energy goes into heat in the resistor. The power delivered to resistor is I^2R, so
$$\frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R^2}R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t},\ \text{where } \alpha \equiv \frac{B^2l^2}{mR};\quad \frac{dW}{dt} = \alpha mv_0^2e^{-2\alpha t}.$$
The total energy delivered to the resistor is $W = \alpha mv_0^2\int_0^\infty e^{-2\alpha t}dt = \alpha mv_0^2\left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^\infty = \alpha mv_0^2\frac{1}{2\alpha} = \frac{1}{2}mv_0^2.$ ✓