PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 18

Solutions to Laplace Equations

Special techniques to determine E and V

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$

POISSON's EQUATION

We found that the electric field satisfies the following two equations

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

which along with supplied boundary conditions fully determine $\mathbf{E}$ (*Helmholtz Theorem*). A natural question to ask is what sort of equation(s) does the electric potential satisfy?

The definition of electric field as the negative gradient of potential is equivalent to the first equation above. However if we plug in $\mathbf{E} = -\nabla V$ in the second equation , we obtain

$$\boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0}.$$

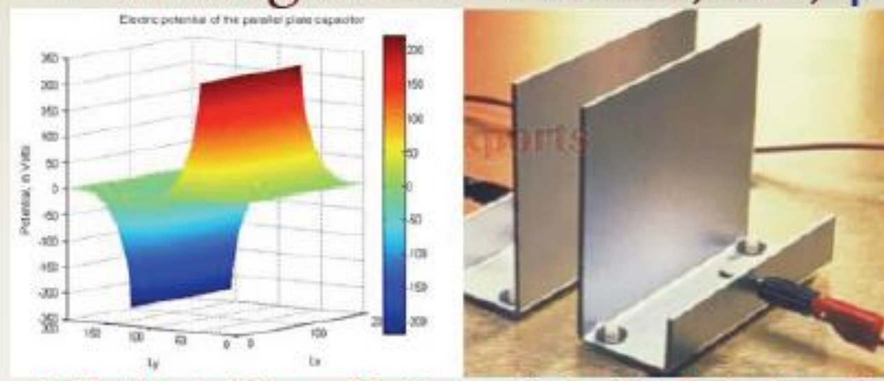
This is referred to as the **Poisson's equation**.

LAPLACE's EQUATION

If there is no charge density in the region of interest, i.e., $\rho=0$ then we obtain

$$\nabla^2 V = 0.$$

This is referred to as the **Laplace's equation**.



2D Electric potential in parallel plates capacitor by solving Laplace's equation

In one dimension it is just $d^2V(x)/dx^2 = 0$, for which the most general solution is $V(x) = Ax + B$ (Equation of a straight line. *This solution should also give you the hint why infinity is not a good choice for reference point in 1D*). The constants (x-*independent*) A and B can be fixed using appropriate boundary conditions. (*Similarly, Poisson's equation may also be analytically solvable in 1D for some special densities* $\rho(x)$.)

For higher dimensions (2 and more) one has to resort to special techniques to solve the Laplace's or Poisson's equation (analytically or numerically).

Image Source: http://www.mathworks.com/matlabcentral/fx_files/37923/1/Laplace2D.jpg

Problem

Find the general solution to Laplace’s equation in spherical co-ordinates for the case when V depends only on ‘r’

Solution:

Laplace’s equation in *spherical* coordinates, for V dependent only on r, reads:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \Rightarrow r^2\frac{dV}{dr} = c \text{ (constant)} \Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \boxed{V = -\frac{c}{r} + k.}$$

Example: potential of a uniformly charged sphere.

Problem

Find the general solution to Laplace's equation for cylindrical co-ordinates assuming V depends only on 's', the radial coordinate.

Solution:

In *cylindrical* coordinates: $\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0 \Rightarrow s\frac{dV}{ds} = c \Rightarrow \frac{dV}{ds} = \frac{c}{s} \quad \Rightarrow \quad \boxed{V = c\ln s + k.}$

Example: potential of a long wire.

Laplace's Equation in 1 dimension

In one dimension the electric potential has dependence only on one variable, say x. Correspondingly, the Laplace's equation reads

$$\frac{d^2V(x)}{dx^2} = 0.$$

Integrating twice we get the general solution for $V(x)$ as

$$V(x) = Ax + B,$$

which is the equation of a straight line. The constants A and B can be determined from the boundary conditions provided.

For example, consider a one dimensional problem where it is given that $V=3$ at $x=-2$ and $V=-2$ at $x=3$. Using these in the general equation we arrive at two coupled linear equations: $3=-2A+B$ and $-2=3A+B$. Solving these we obtain $A=-1$ and $B=1$, thereby giving the solution to the problem as $V(x)=-x+1$ (with proper units). Now the potential at any point can be evaluated, e.g., $V(x=0)=1$ etc.

Averaging property in 1D

$V(x)$ is the average of $V(x+a)$ and $V(x-a)$ for any a:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)].$$

This is easy to prove using the general solution. We have

$$\begin{aligned}\frac{1}{2}[V(x+a) + V(x-a)] &= \frac{1}{2}[A(x+a) + B + A(x-a) + B] \\ &= \frac{1}{2}(2Ax + 2B) = Ax + B = V(x).\end{aligned}$$

Averaging property in 2D and 3D

In 2 dimensions, the value of V at a point (x,y) is the average of those around the point. More precisely, if we draw a circle of any radius R about a point (x,y), then the average value of V calculated over the circle (the circumference) gives the value of the potential at the center, i.e.,

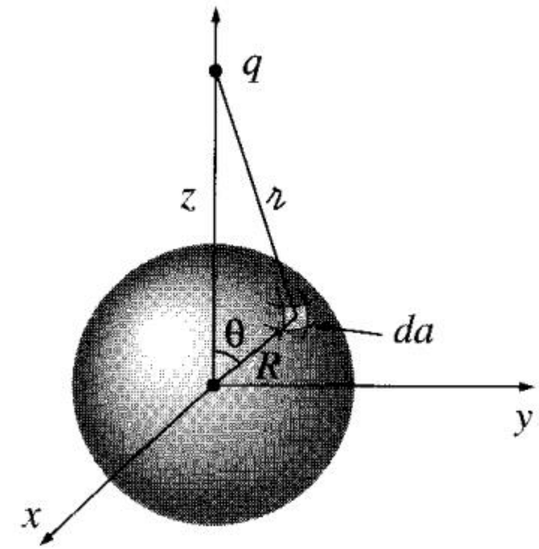
$$V(x,y) = \frac{1}{2\pi R} \oint_{\text{Circle}} V\, dl\,.$$

Similarly, in 3 dimensions the value of V at a point (x,y,z) is the average of the potential at the surface of a sphere centered at (x,y,z)

$$V(x,y,z) = \frac{1}{4\pi R^2} \oint_{\text{Sphere}} V\, dS\,.$$

We will prove this in the next slide considering the case of a point charge place in front of a spherical surface

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da.$$



$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}},$$

where

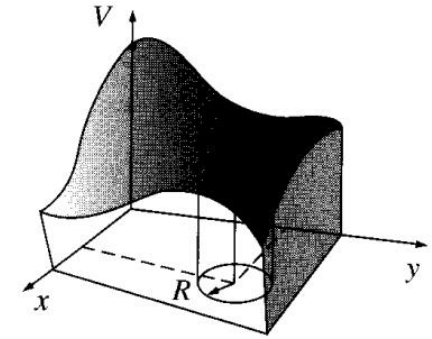
$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta,$$

so

$$\begin{aligned} V_{\text{ave}} &= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \, \sin\theta \, d\theta \, d\phi \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \sqrt{z^2 + R^2 - 2zR\cos\theta} \, \Big|_0^\pi \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} [(z + R) - (z - R)] = \frac{1}{4\pi\epsilon_0} \frac{q}{z}. \end{aligned}$$

z>R

No local extrema



Laplace's equation does not admit any local extrema (maxima or minima) in the region of interest. All extreme values of V occur at the boundaries [points (*zero-dimensional)* in 1 dimension, curves (*1-dimensional*) in 2 dimension, surfaces (*2-dimensional*) in 3 dimensions].

This follows from the averaging property of the solution (***Harmonic functions***) of the Laplace's equation. For example, in 1 dimension, if there were a local maximum (minimum), the potential V at that point would be greater (less) than on either side, and therefore could not be the average. Therefore we conclude that the extrema can occur at boundaries only.

Boundary conditions

As should be evident by now, in electrostatics one is concerned with the static properties (time-independent: charge-densities, electric potentials, electric fields etc.) in some region of interest. This region of interest is taken to be some volume $\mathcal{V}$ enclosed by surface $\mathcal{S}$. The volume can be infinite also, in which case $\mathcal{S}$ extends up to infinity. Moreover, the volume can have "cavities/holes" also, in which case the surface enclosing the volume would be the 'surface enclosing the outer side of the volume *plus* the surface lying on the inside, surrounding the cavities'.

Note that in 1D, a line segment plays the role of a "volume", and two end-points serve as the "surface" enclosing the "volume". In 2D, area plays the role of a "volume", and a curve (say a circle) is the analog of "surface" enclosing the "volume". In 3D we have the usual volume and surface. This notion can be generalized to higher dimensions also.

In problems involving calculation of the electric potential in a given region, to determine solution uniquely we need some boundary conditions. These can be of two types:

(a) **Dirichlet boundary conditions:** The value of the potential is specified at the surface enclosing the volume of interest.

(b) **Neumann boundary conditions:** The value of the normal-derivative of the potential is specified at the surface enclosing the volume of interest.

Laplace’s equation ($\nabla^2 V = 0$) does not itself determine V; in addition suitable boundary conditions must be supplied

1-D solution of Laplace’s equation $$\frac{d^2V}{dx^2} = 0 \qquad V = Ax + B$$

(2 arbitrary constants to be found from 2 boundary conditions)

Either V at each end provided or V at one end and its derivative dV/dx on the other end is provided [(V,V) or (V, dV/dX)]

Insufficient information: when *only* V or dV/dx is provided at *one end*