PHY 102 Introduction to Physics II

Spring Semester 2025

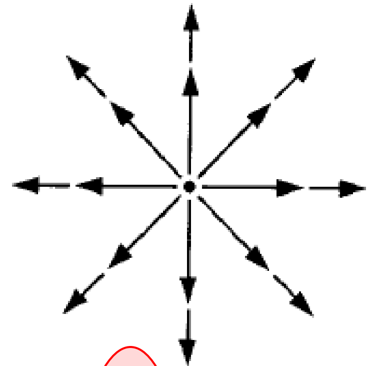
Lecture 7

Dirac Delta function

Divergence of a vector function

$$\vec{A} = \hat{\boldsymbol{r}} r^n$$

$\vec{A}$ is *a divergent* vector field for all values of '*n*'



An example of divergent field

$$\vec{A} = \frac{\hat{\boldsymbol{r}}}{r^2}$$

n	3	2	1	0	-1	-2	-3	-4
$\vec{A}$	$\hat{\boldsymbol{r}} r^3$	$\hat{\boldsymbol{r}} r^2$	$\boldsymbol{r}$	$\hat{\boldsymbol{r}}$	$\frac{\hat{\boldsymbol{r}}}{r}$	$\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}$	$\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^3}$	$\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^4}$
$\nabla \cdot \vec{A}$	$5r^2$	4r	3	$\frac{2}{r}$	$\frac{1}{r^2}$	0	$-\frac{1}{r^4}$	$-\frac{2}{r^5}$

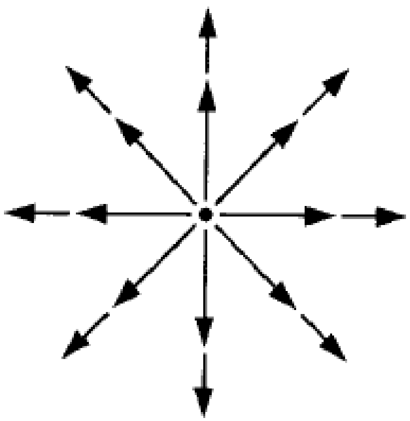
$\nabla \cdot \vec{A}$ increasing with r

$\nabla \cdot \vec{A}$ decreasing with r

Divergence of a vector function

Divergence of $\hat{\mathbf{r}}/r^2$

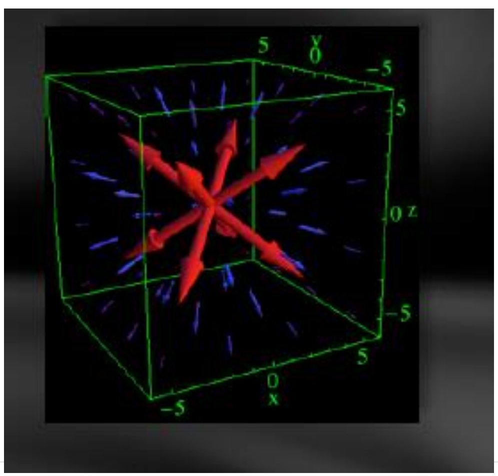
2D



$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$$

Clearly $\mathbf{v}$ is directed radially outwards. We would expect that its divergence is nonzero. In fact we expect it to be large. However, we find that

3D



$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

This is surprising!

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

Divergence of a vector function

Divergence of $\hat{\mathbf{r}}/r^2$

Furthermore, Gauss's divergence theorem asserts that

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S}$$

Consider a volume $\mathcal{V}$ enclosed by a spherical surface $\mathcal{S}$ whose center lies at the origin and has radius R(>0). With our result $\boldsymbol{\nabla}\cdot\mathbf{v}=0$ of the previous slide we obtain LHS=0. However for the RHS, on the surface S,

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{R^2} \qquad d\mathbf{S} = \hat{\mathbf{r}} R^2 \sin\theta\, d\theta\, d\phi$$

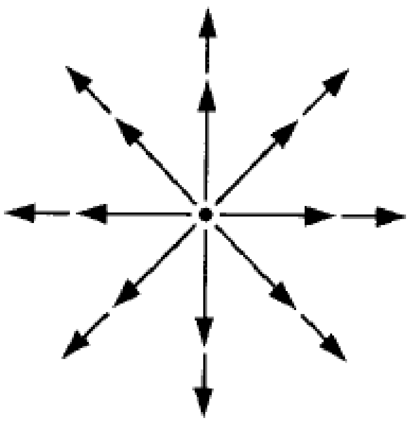
which gives

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \left(\frac{\hat{\mathbf{r}}}{R^2}\right) \cdot (\hat{\mathbf{r}} R^2 \sin\theta) = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi = 4\pi.$$

Thus we have a contradiction! (or do we have?)

Divergence of $\hat{\mathbf{r}}/r^2$

2D



The answer is that $\nabla \cdot \vec{\boldsymbol{v}} = 0$ everywhere except at the origin, but at the origin our calculation is no good, since $r = 0$, and the expression for $\vec{\boldsymbol{v}}$ blows up.

Evidently the entire contribution must be coming from the point **r = 0**!

From Gauss's theorem, the surface integral is independent of R and is given **4π**, which should be equal to

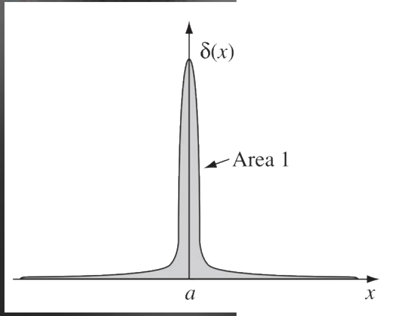
$$\int_v (\boldsymbol{\nabla}.\boldsymbol{v})d\boldsymbol{\tau} = \mathbf{4}\pi$$

eg:- The density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its integral is finite—namely, the mass of the particle.

"Dirac delta function"

The one-dimensional Dirac delta function

- Strictly speaking Dirac delta "function" is a distribution or a generalized function.



- It has the following peculiar properties:

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

- Thus it can be viewed as an infinitely high, infinitesimally narrow spike, with area 1.

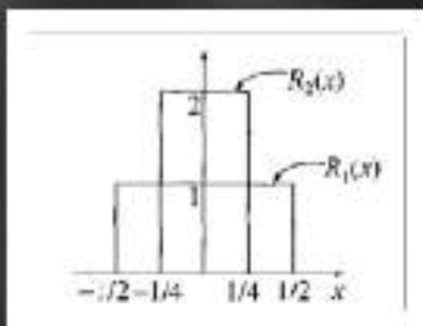
- We may shift the spike at the origin (x=0) to some other point, say $x = a$. In this case we have

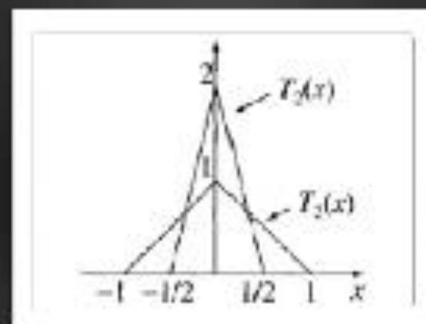
$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a, \\ \infty, & \text{if } x = a, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x-a)\, dx = 1.$$

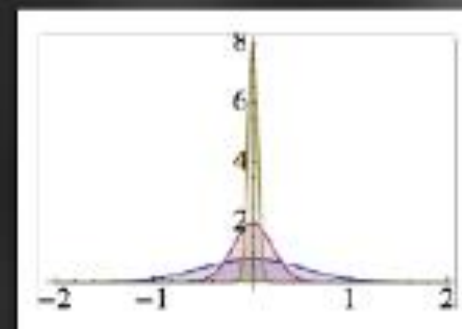
Dirac delta function

The one-dimensional Dirac delta function

- It can be considered as the limit of sequence of functions, such as rectangles $R_n(x)$ of height n and width $1/n$, or isosceles triangles $T_n(x)$, of height n and base $2/n$, with $n \to \infty$. Note that the area remains fixed at 1.



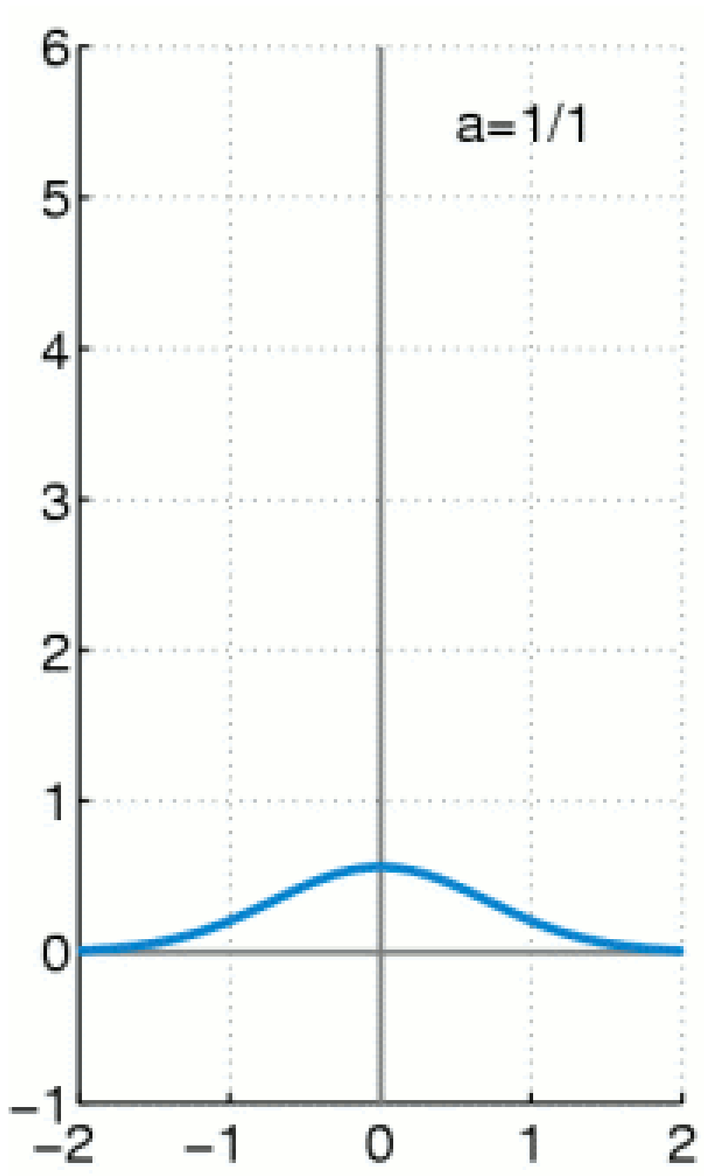




- There are many other ways to realize the Dirac delta function. For example, it can be thought of as the limiting case of a Gaussian curve whose width (variance) approaches 0.

$$\delta(x-a) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

Dirac delta function



Dirac delta function

The one-dimensional Dirac delta function

Important properties:

$$\int_{\Delta x} \delta(x-a)\,dx = \begin{cases} 1 & \text{if } a \text{ lies within } \Delta x, \\ 0 & \text{otherwise.} \end{cases}$$

Sifting Property

$$\int_{\Delta x} f(x)\delta(x-a)\,dx = \begin{cases} f(a) & \text{if } a \text{ lies within } \Delta x, \\ 0 & \text{otherwise.} \end{cases}$$

In the above relations Δx represents the domain of integration (line).

The one-dimensional Dirac delta function

Sifting Property

Evaluate the integral

$$\int_0^3 x^3 \delta(x-2)\,dx.$$

Solution

The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

Dirac delta function

The one-dimensional Dirac delta function

Some more properties:

$$\delta(f(x)) = \sum_i \frac{1}{\left| \left[\frac{df}{dx} \right]_{x=x_i} \right|} \delta(x - x_i)$$

Here $f\,(x)$ is assumed to have only simple zeros, located at $x{=}x_{i.}$ As a special case of the above relation we have

$$\delta(kx) = \frac{1}{|k|}\delta(x), \text{ for any nonzero constant } k.$$

In particular, for $k = -1$

$$\delta(-x) = \delta(x).$$

The above relations are intended to be used under an integral sign.

The two-dimensional Dirac delta function

In two dimensions we have, for $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j}$,

$$\delta^2(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')$$

with the property

$$\int_{\Delta S} \delta^2(\mathbf{r} - \mathbf{r}')\, d^2r = \begin{cases} 1 & \text{if } \mathbf{r}' \text{ lies within } \Delta S, \\ 0 & \text{otherwise.} \end{cases}$$

Here d^2r represents the area element $dx\,dy$, and the integral is over an area ΔS. Note that above property follows from the behavior of one-dimensional delta functions constituting the two-dimensional delta function.

Also, if $f(\mathbf{r})$ be a function of $\mathbf{r}$, then

$$\int_{\Delta S} f(\mathbf{r})\delta^2(\mathbf{r} - \mathbf{r}')\, d^2r = \begin{cases} f(\mathbf{r}') & \text{if } \mathbf{r}' \text{ lies within } \Delta S, \\ 0 & \text{otherwise.} \end{cases}$$

The three-dimensional Dirac delta function

In three dimensions we have, for $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$,

$$\delta^3(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$$

with the property

$$\int_{\Delta V} \delta^3(\mathbf{r}-\mathbf{r}')\,d^3r = \begin{cases} 1 & \text{if } \mathbf{r}' \text{ lies within } \Delta V, \\ 0 & \text{otherwise.} \end{cases}$$

Here d^3r represents the volume element $dx\ dy\ dz$, and the integral is over a volume ΔV. Also, if $f(\mathbf{r})$ be a function of $\mathbf{r}$, then

$$\int_{\Delta V} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{r}')\,d^3r = \begin{cases} f(\mathbf{r}') & \text{if } \mathbf{r}' \text{ lies within } \Delta V, \\ 0 & \text{otherwise.} \end{cases}$$

These results can be readily generalized to other coordinate systems (e.g., spherical) by incorporating corresponding Jacobian of transformation, and also to higher dimensions.

Divergence of $\hat{\mathbf{r}}/r^2$

The problem is in the calculation of divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

The cancellation in r^2/r^2 giving 1 is allowed only when $r\neq 0$. Thus $\nabla\cdot\mathbf{v}=0$ is true only for $r\neq 0$. Actually, there's a source located at $r=0$. In terms of 3-dimensional Dirac-delta function we have,

$$\nabla \cdot \mathbf{v} = 4\pi\delta^3(r)$$

which gives $$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{v})\, d\tau = \int_{\mathcal{V}} 4\pi\delta^3(r)\, d\tau = 4\pi.$$

Therefore, things are consistent with Gauss's theorem!

The three-dimensional Dirac delta function

Properties

1. $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

2. $\int_{-\infty}^{\infty} \delta^3(\vec{r})\, d\,v = 1$

3. $\int_v f(\vec{r})\, \delta^3(\vec{r} - \vec{r}') = f(\vec{r}')$

4. $\overline{\nabla}.\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$

Gauss Divergence Theorem Never failed just it need a function which can define at a point.