PHY101: Introduction to Physics I

Monsoon Semester 2023

Lecture 16

Department of Physics, School of Natural Sciences,
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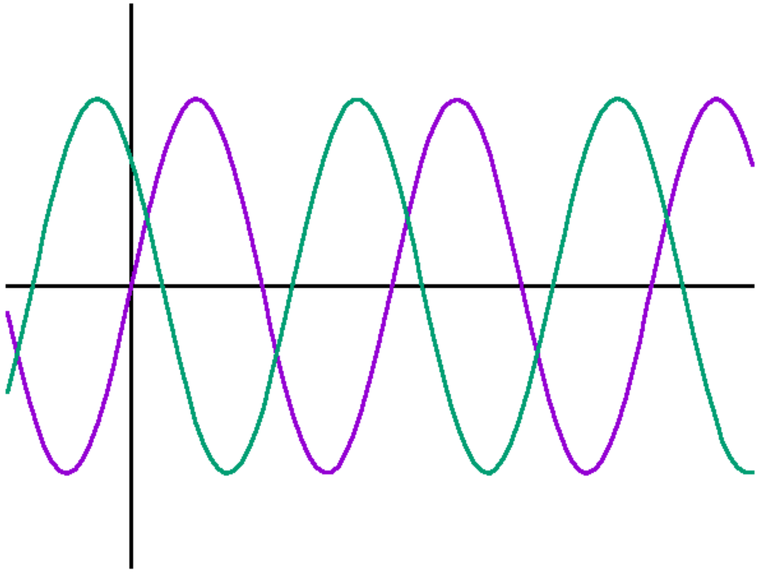
Course instructor:

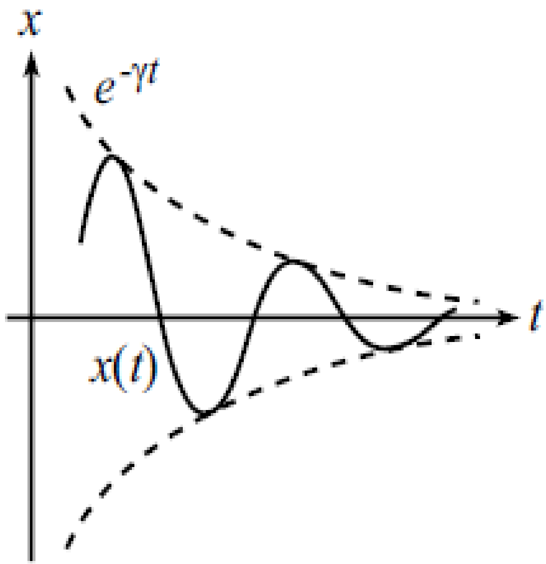
Previous Lecture

Oscillations
Simple harmonic motion



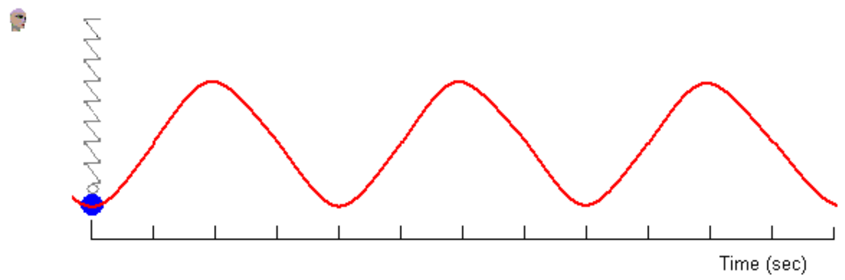
This Lecture

Damped harmonic motion

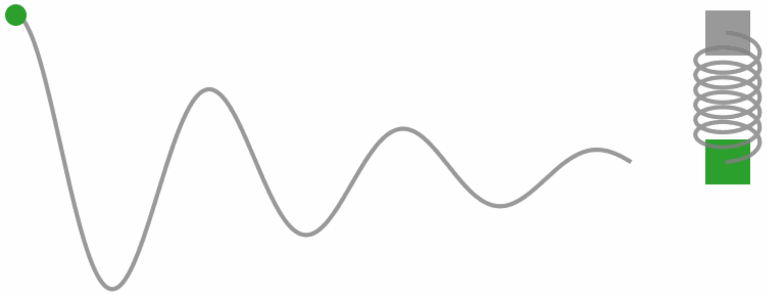


Different types of oscillatory motions (Visual description)

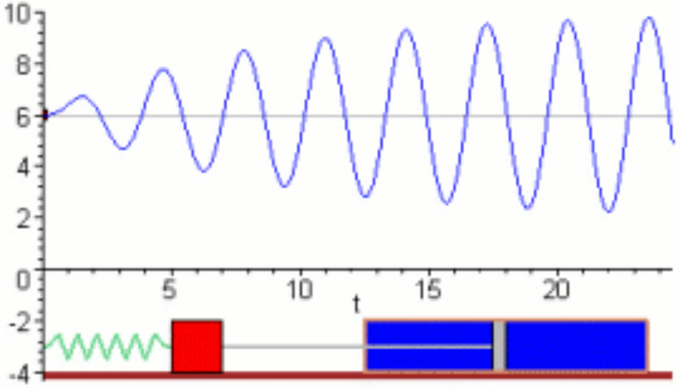
Simple harmonic motion (previous lecture)



Damped harmonic motion (this lecture)



Driven harmonic motion (next lecture)



Examples of Damped Harmonic Oscillations

a) The mass experiences a frictional force as it moves through the air.

b) When the mass oscillates horizontally attached to a string, then there exists frictional forces between mass and surface.

c) There are resistive force acting on the charge in LC circuit, due to wires and internal resistance of the devices.

Solution of linear differential equation

Linear differential equation of order n=2

homogeneous

$$a_2 \frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0$$

or

inhomogeneous

$$a_2 \frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

General solution = Complimentary + Particular solution

For Complementary solution :

1. **Take trial solution : $x=e^{mt}$, m is constant**
2. **m1, m2,……….will be the roots. If all roots are real and distinct, then solution $x=c_1e^{m1t}+c_2e^{m2t}+$……………..**
3. **If some roots are repeated, say m1 repeated k times, then solution will be $(c_1 + c_2t+ \ldots..c_kt^{k-1})e^{m1t}$**
4. **If some roots are complex, (if a+ib then a-ib are roots) solution will be $e^{at}(c_1 \cos(bt) +c_2 \sin(bt)) +$………**

For Particular solution : Trial solution to be assumed depending on the form of f(t)

Damped Free OSCILLATION

Resistive force is proportional to velocity $F_{drag} = -rv$

$$F = m\ddot{x} = -rv - kx$$

$$m\ddot{x} + rv + kx = 0$$

$$\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0}$$

Where, $\beta = \frac{r}{2m}; \omega_0^2 = \frac{k}{m}$

Or sometimes given in the form...

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = 0$$

Where, $\gamma = r/m$ and $\omega_0^2 = \frac{k}{m}$

Solution

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

- The equation is a second order linear homogeneous equation with constant coefficients.

- Solution can be found which has the form: $\boldsymbol{x = Ce^{pt}}$ where C has the dimensions of x, and p has the dimensions of T^{-1}.

$$\dot{x} = pCe^{pt}; \ddot{x} = p^2Ce^{pt}$$

$$m\ddot{x} + r\dot{x} + kx = 0 \Rightarrow Ce^{pt}(mp^2 + rp + k) = 0$$

$$x = Ce^{pt} = 0 \quad \textit{Trivial solution}$$

$$mp^2 + rp + k = 0$$

- Solving the quadratic equations gives us the two roots:

$$p_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}$$

$$p_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

- The general solution takes the form:

$$x = x_1 + x_2 = A_1e^{p_1t} + A_2e^{p_2t}$$

Case I: Overdamped $(\beta^2 > \omega_0^2)$

(Heavy damping)

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

$$x = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

The square root term is +ve: The damping resistance term dominates the stiffness term.

$$x(t) = A_1 \exp(-\beta t + \sqrt{\beta^2 - \omega_0^2}\, t)$$

$$+ A_2 \exp(-\beta t - \sqrt{\beta^2 - \omega_0^2}\, t)$$

- **Non-oscillatory behavior can be observed.**
- **But, the actual displacement will depend upon the boundary conditions**

CaseII: Critical damping $(\beta^2 = \omega_0^2)$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

- The damping resistance term and the stiffness terms are balanced.
- When r reaches a critical value, the system will not oscillate and quickly comes back to equilibrium.

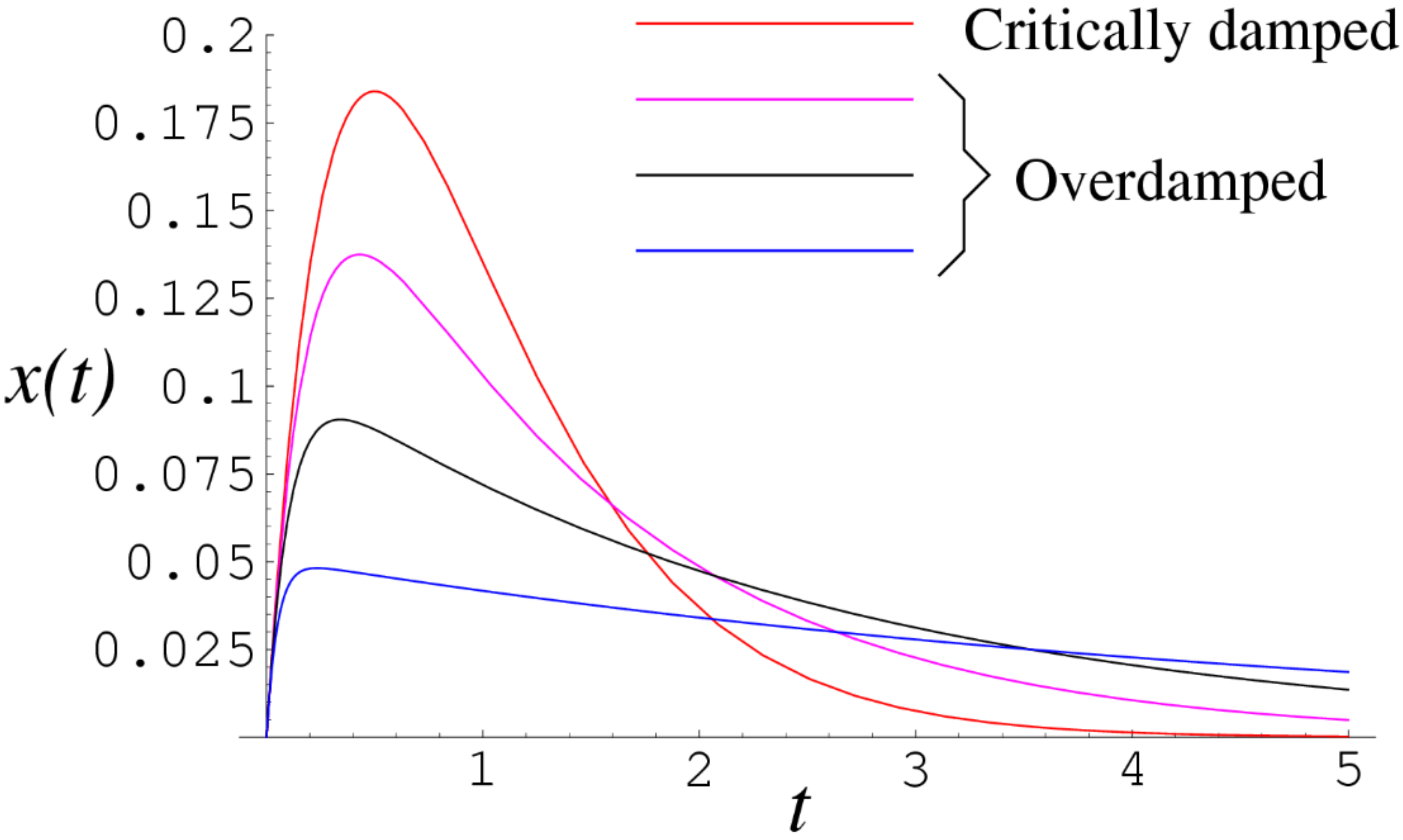
The quadratic equation in p has equal roots, which, in a differential equation solution demands that C must be written as $(A+Bt)$.

$$A \exp(-\beta t).$$

$$Bt \exp(-\beta t)$$

$$\boxed{x(t) = (A + Bt) \exp(-\beta t)}$$

$A=0 \quad B=2$


Critically damped
Overdamped
0.2
0.175
0.15
0.125
x(t)
0.1
0.075
0.05
0.025
1
2
3
4
5
t

Case III: Underdamped $(\beta^2 < \omega_0^2)$

$$\beta = \frac{r}{2m}$$

$$\omega_0^2 = \frac{k}{m}$$

- The square root term is -ve: The stiffness term dominates the damping resistance term.
- The system is lightly damped and gives oscillatory damped simple harmonic motion.

$$x(t) = \exp(-\beta t)[A_1 \exp(-i\sqrt{\omega_0^2 - \beta^2}\ t)$$

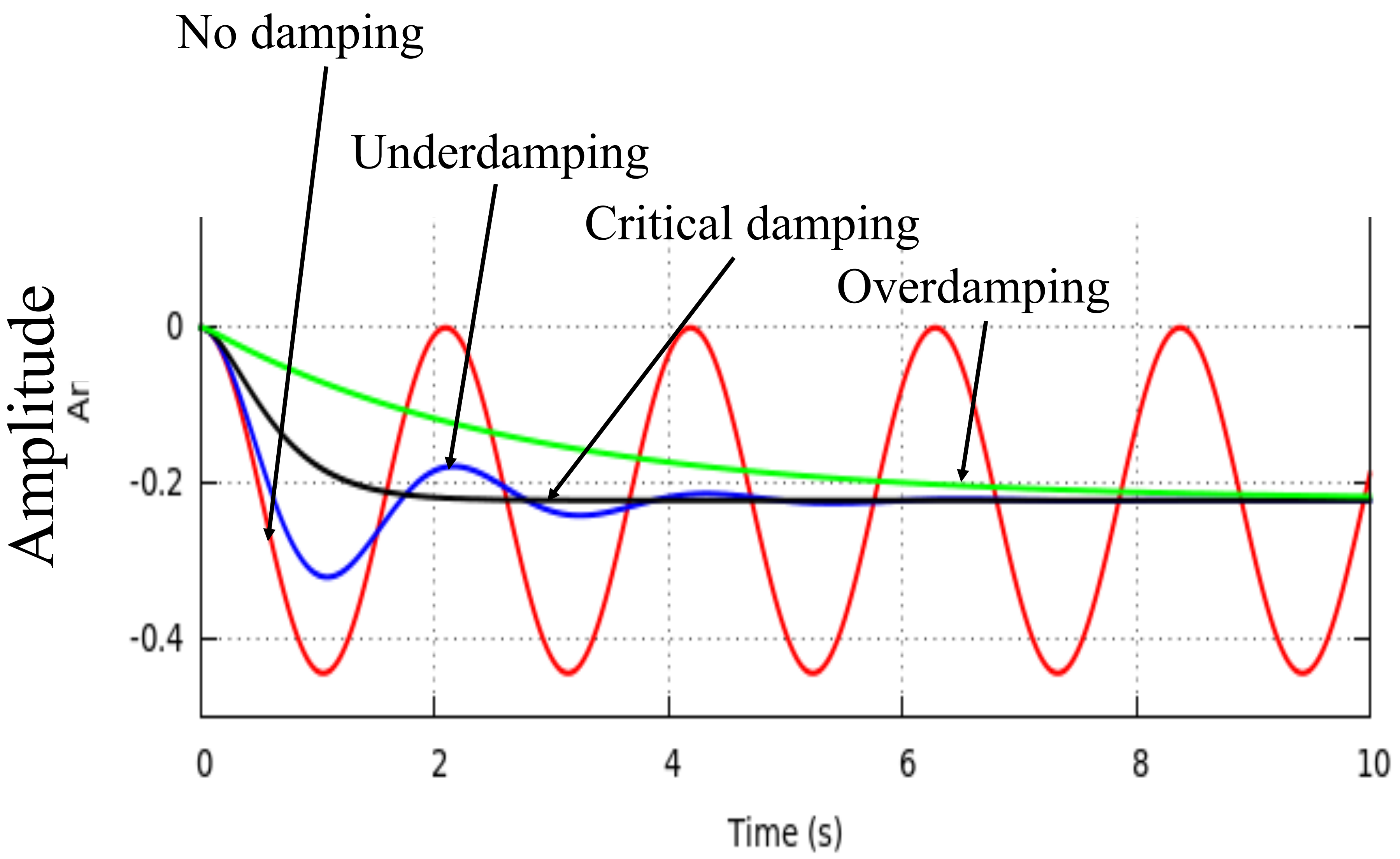
Use $e^{i\theta} = \cos\theta + i\sin\theta$

$$+ A_2 \exp(i\sqrt{\omega_0^2 - \beta^2}\ t)]$$

$$A_1 = A_2 = A/2$$

$$\boxed{x(t) = A\exp(-\beta t)\cos(\omega' t)}$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}} = \sqrt{\omega_0^2 - \beta^2}$$


No damping
Underdamping
Critical damping
Overdamping
Amplitude
0
-0.2
-0.4
0
2
4
6
8
10
Time (s)

Features of underdamped motion

$$x(t) = A\exp(-\beta t)\cos(\omega' t)$$

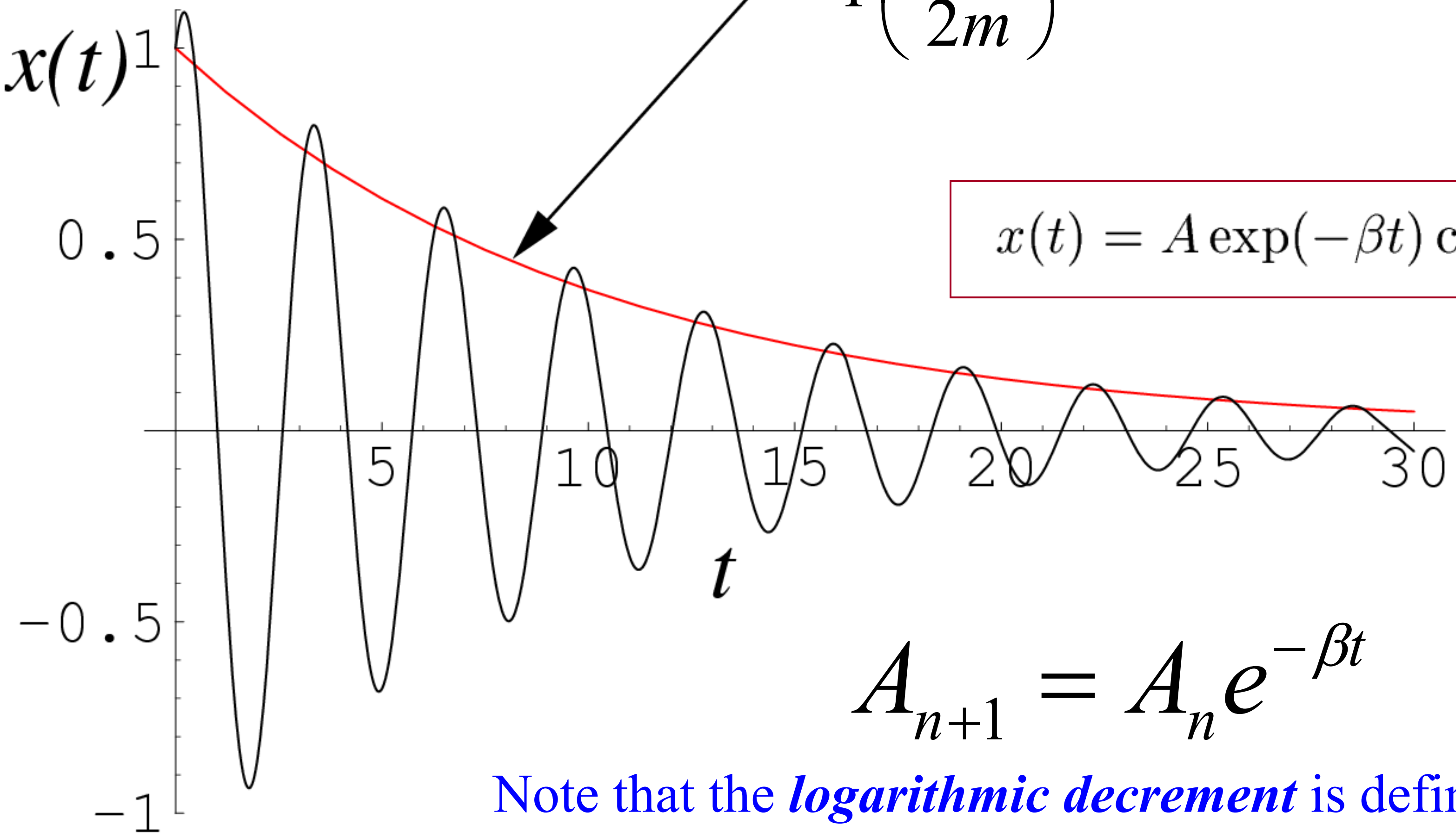
The underdamped motion has two features:

1) *Its frequency is reduced:* $\omega' < \omega_0$ *– which means that the time period is increased and*

2) Its amplitude decays exponentially (as seen in the next graph).

Underdamped oscillations

$$\exp\left(\frac{-rt}{2m}\right) = e^{-\beta t}$$



$$x(t) = A\exp(-\beta t)\cos(\omega' t)$$

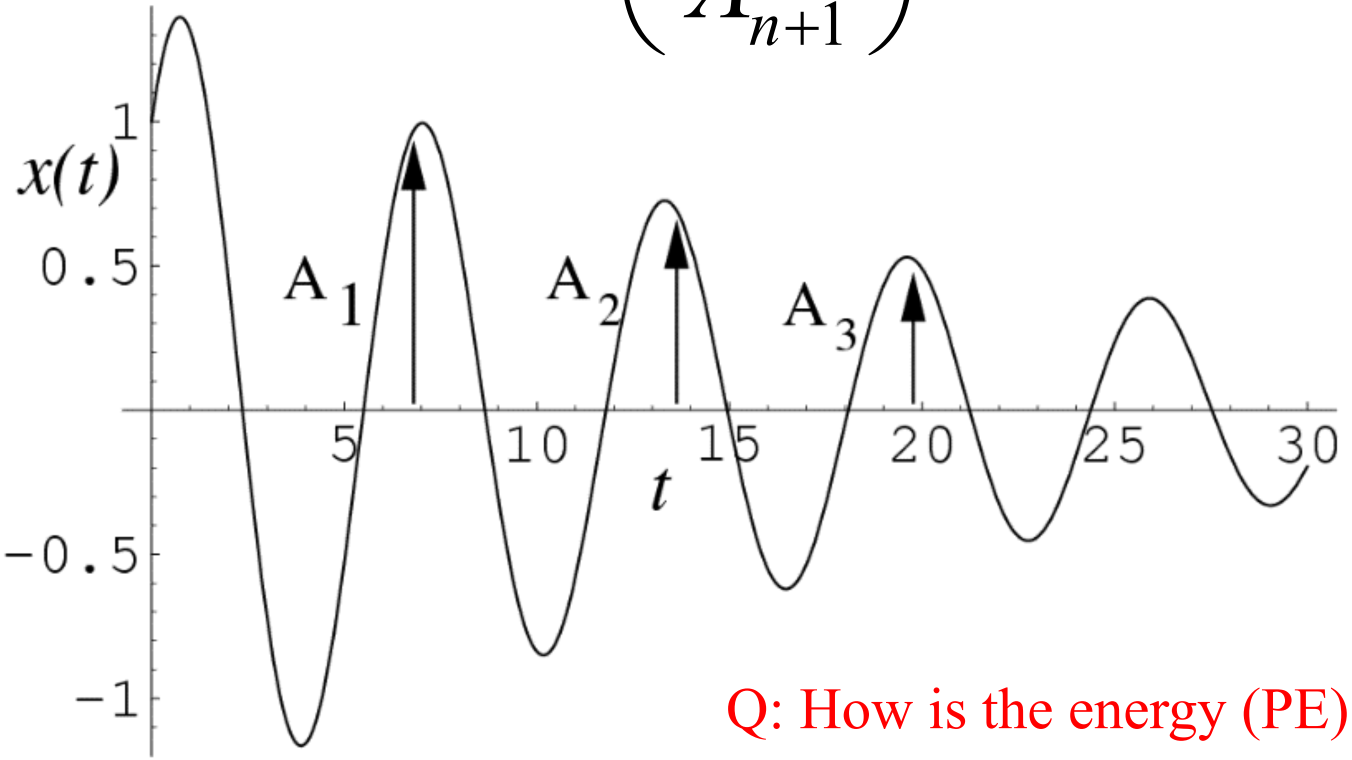
$$A_{n+1} = A_n e^{-\beta t}$$

Note that the ***logarithmic decrement*** is defined as the natural logarithm of the ratio of successive amplitudes:

$$\lambda = \ln\left(\frac{A_n}{A_{n+1}}\right) = \beta t$$

Logarithmic decrement

$$\lambda = \ln\left(\frac{A_n}{A_{n+1}}\right) = \beta t$$



Q: How is the energy (PE) changing??

Relaxation time

Relaxation time is the time taken for the amplitude to decay to *1/e* of its original value.

Note: $1/e = 0.368$

$$A_{n+1} = A_n e^{-\beta t}$$

When t = relaxation time $A_t = A_o e^{-1}$

$$t = \frac{2m}{r}$$

Problem and solution

A damped harmonic oscillator has a frequency of 10 oscillations per second. The amplitude drops to half its value for every 20 oscillations. The time it will take to drop to 1/500 of the original amplitude, is close to:

The time of half amplitude is 2 sec

For damped oscillations,

$A = A_0 e^{-kt}$

So, $A_0/2 = A_0 e^{-2k}$

$k = \ln 2/2$

$A_0/500 = A_0 e^{-kt}$

Thus replacing the value of 'k' we find that,

$t = (\ln 500)/k = (\ln 500)/(\ln 2/2)$

$= (6.21)/(0.69/2) = 18$ secs (approx.)