PHY 102 Introduction to Physics II

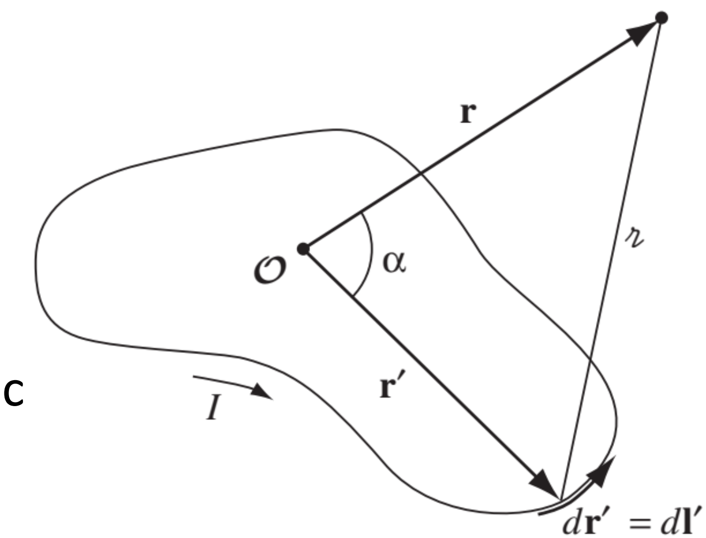
Spring Semester 2025

Lecture 27

$Multipole\ expansion\ of\ vector\ potential$

Multipole expansion of vector potential

We want to find out the *vector potential of localized current distribution*. We would find out an *approximate potential* at far away points due to current flowing in a closed loop



$\frac{1}{\mathscr{r}}$ can be expressed in Legendre Polynomials (same as in multipole expansion of electrostatic potential)

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha),$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\frac{1}{\left[1 - 2\left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\right]^{1/2}}.$$

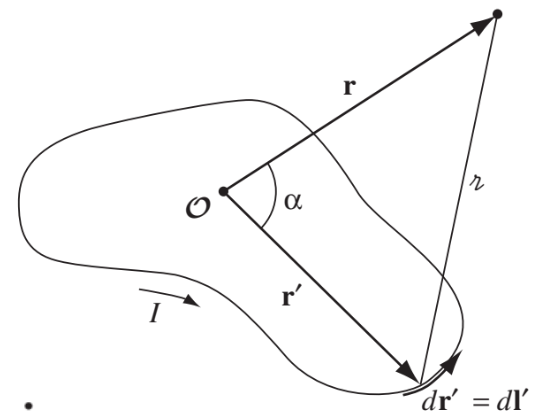
$$\frac{1}{[1 - 2uz + u^2]^{1/2}} = \sum_{n=0}^{\infty} u^n P_n(z).$$

Generating function for Legendre Polynomials

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathscr{r}}\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\mathbf{l}',$$

Multipole expansion of vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\mathbf{l}' + \cdots\right].$$



As in the multipole expansion of V, we call the first term (which goes like $1/r$) the **monopole** term, the second (which goes like $1/r^2$) **dipole**, the third **quadrupole**, and so on.

Multipole expansion of vector potential

First term

$$\frac{\mu_0 I}{4\pi r}\oint \boldsymbol{dl'} \longrightarrow$$

Magnetic monopole term

Zero, because $\oint \boldsymbol{dl'} = \boldsymbol{0}$, total vector displacement around a closed loop

This reflects the fact that there are no magnetic monopoles in nature (an assumption contained in Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, on which the entire theory of vector potential is predicated).

Multipole expansion of vector potential

Second term (Magnetic dipole term)

(Vector integration)

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'.$$

Here $\hat{\boldsymbol{r}}$ is the constant vector

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'.$$

Prove it as Home Work (HW)

Use, Vector identities

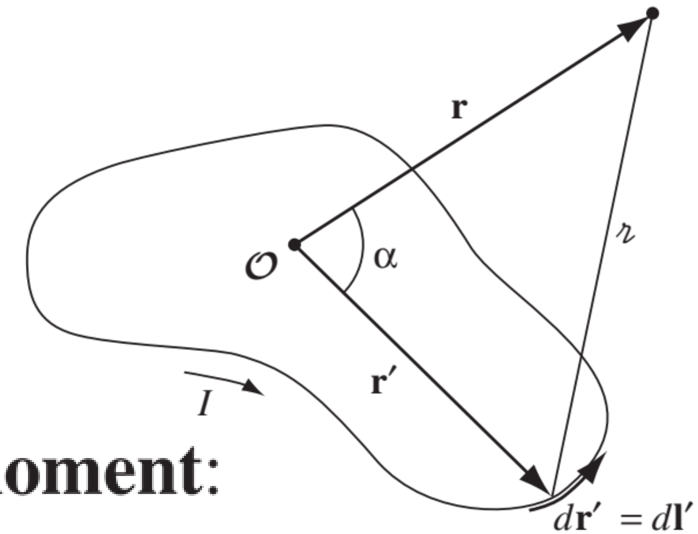
$$\oint T\overrightarrow{\boldsymbol{dl}} = -\int_{S} \vec{\boldsymbol{\nabla}} T \times \overrightarrow{\boldsymbol{da}}$$

Put $\boldsymbol{v} = \boldsymbol{c}T$ in Stoke's theorem, where *'$\boldsymbol{c}$' is a constant vector*

Hint: Consider $T = \hat{\boldsymbol{r}}.\boldsymbol{r}$ and $Proove\ \nabla T = \hat{\boldsymbol{r}}$

Multipole expansion of vector potential

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2},}$$



where $\mathbf{m}$ is the **magnetic dipole moment**:

$$\boxed{\mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}.}$$

a is the "vector area" of the loop.

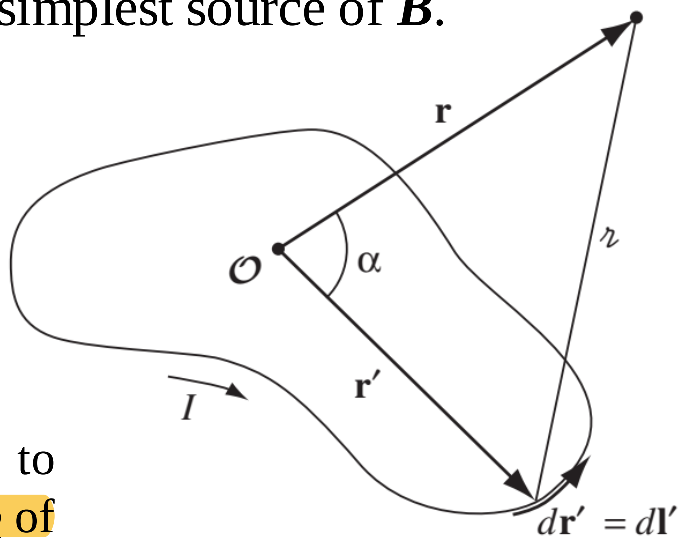
If the loop is flat, a is the ordinary area enclosed, with the direction assigned by the usual right-hand rule (fingers in the direction of the current)

Magnetic Dipole

A *magnetic dipole* is the magnetic equivalent of an electric dipole.
A *magnetic dipole* can be represented as a single-turn current loop of area **A**

An element of *current element* $i\boldsymbol{dl}$ was taken to be the simplest source of $\boldsymbol{B}$.

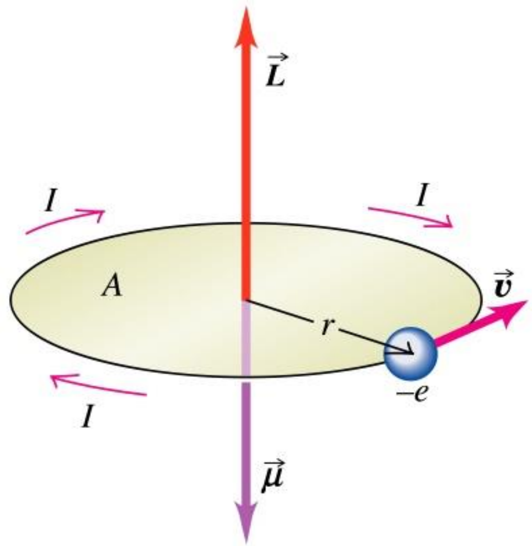
One must be careful in choosing proper current element- it cannot start at one point and terminate at another point. In order to have steady current, currents must flow in closed loops.



The simplest way to make a practical current element is to make a closed loop out of it. It turns out that a *current loop* of area **A** and current i behaves like a *magnetic dipole* of magnitude $\boldsymbol{m} = i\boldsymbol{A}$ and direction given by *right hand rule*

Magnetic Dipole

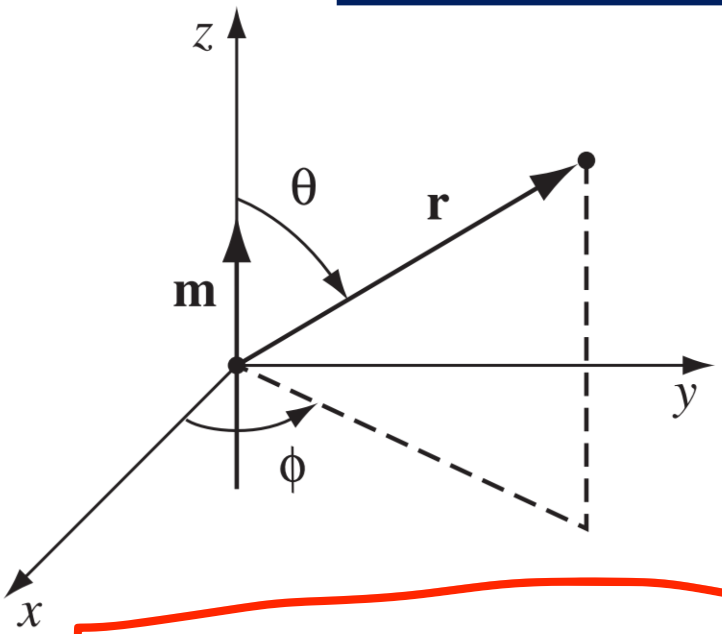
'μ /m' is the magnetic moment



- The magnetic dipole moment is perpendicular to the loop and in the direction of right hand rule.
- **Magnetic dipole is independent of the choice of origin** but electric dipole is dependent (except when total charge is zero!).

The dipole term, $V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2}$, does, in general, change under the origin shift, **except when the total charge Q in the system is zero.** [redacted]

Magnetic field due to a dipole (I)



$$\boldsymbol{A}_{dip}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2}\right)$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m \sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

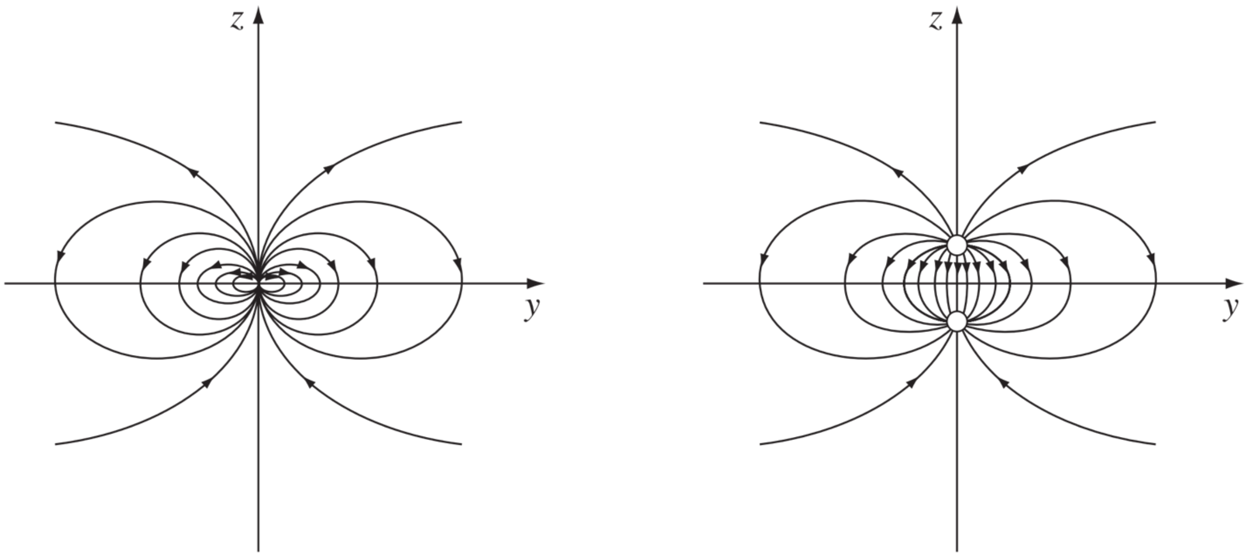
$$\boxed{\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).}$$

In coordinate free form

$$\boxed{\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right].}$$

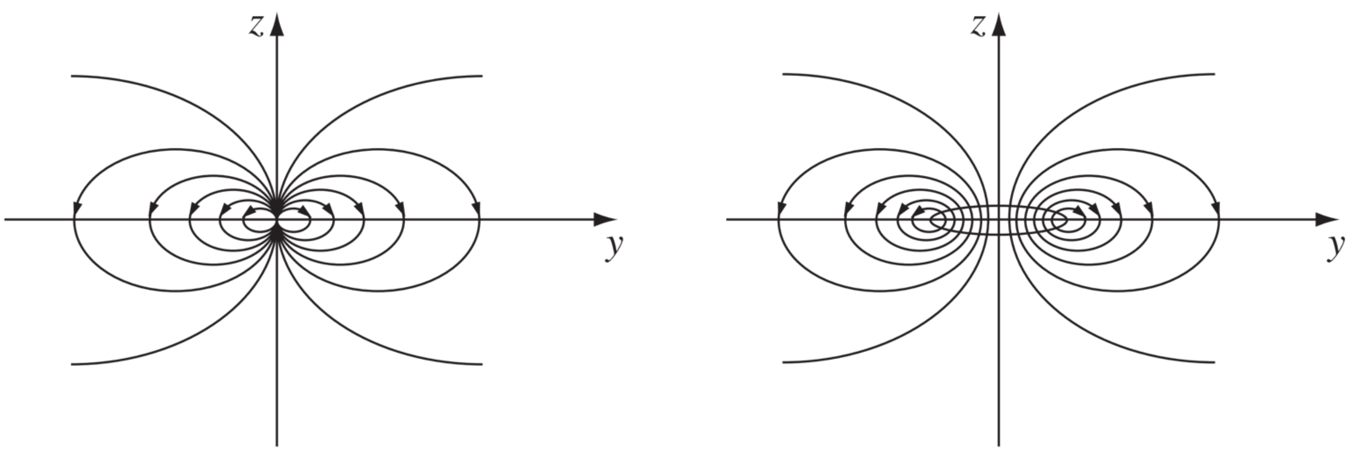
Prove it as Home Work (HW)

Magnetic Dipole



(a) Field of a “pure” dipole (b) Field of a “physical” dipole

→ **Electric Dipole**



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

→ **Magnetic Dipole**