# PHY 102 Introduction to Physics II

## Spring Semester 2025

# Lecture 33

*Maxwell's Equations in Matter*

# *Maxwell's Equations in Matter*

If we are working with materials that are subject to electric and magnetic polarization there is a more convenient way to write the Maxwell's equation

For inside polarized matter there will be accumulations of "bound" charge and current, over which you exert no direct control.

In the static case, the electric polarization **P** produces a bound charge density

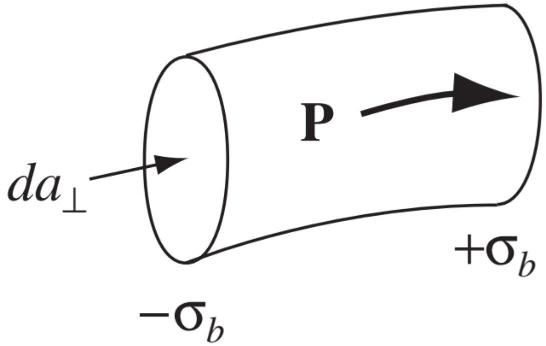
$$\rho_b = -\nabla \cdot \mathbf{P}$$

Likewise, a magnetic polarization (or "magnetization") **M** results in a bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

# Maxwell's Equations in Matter

In the nonstatic case: Any change in the electric polarization involves a flow of (bound) charge (call it $\mathbf{J_p}$), which must be included in the total current.




Suppose in a tiny chunk of polarized material, the polarization introduces a charge density $\sigma_b = \mathbf{P}$ at one end and $-\sigma_b$ at the other.

If $\mathbf{P}$ now increases a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$

The current density

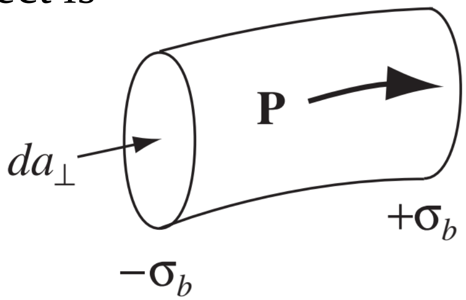
$$\boxed{\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}}$$

**Polarization current**

# *Maxwell's Equations in Matter*

- This polarization current ($\mathbf{J_p}$)has nothing to do with the bound current $\mathbf{J_b}$. The latter is associated with magnetization of the material and involves the spin and orbital motion of electrons
- $\mathbf{J_p}$, is the result of the linear motion of charge when the electric polarization changes.

If **P** points to the right, and is increasing, then each plus charge moves a bit to the right and each minus charge to the left; the cumulative effect is the polarization current $\mathbf{J_p}$




**Checking the consistenty of $\mathbf{J_p}$ with the continuity equation**

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}.$$

$\mathbf{J_p}$ is essential to ensure the conservation of bound charge.

# Maxwell's Equations in Matter

In matter,

the total charge density

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P},$$

the total current density

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}.$$

Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P}),$$

as in the static case, $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}.$

$$\nabla \cdot \mathbf{D} = \rho_f,$$

# Maxwell's Equations in Matter

Ampère's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}.$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t},$$

$$\mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}.$$

# Maxwell's Equations in Matter

In terms of *free* charges and currents, then, Maxwell's equations read

$$\boxed{\begin{array}{ll} \text{(i)} \quad \nabla \cdot \mathbf{D} = \rho_f, & \text{(iii)} \quad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\[2ex] \text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, & \text{(iv)} \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}. \end{array}}$$

**Displacement current**

**Integral Form**

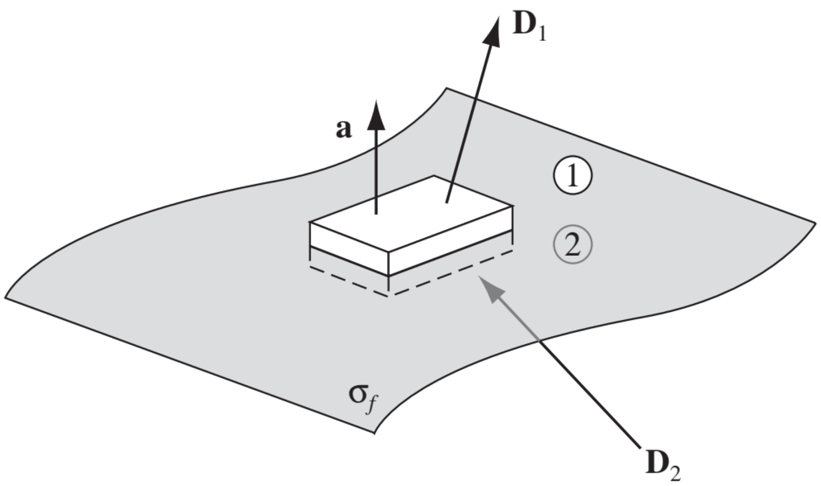
$$\left.\begin{aligned}&\text{(i)} \quad \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ &\text{(ii)} \quad \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0\end{aligned}\right\} \text{ over any closed surface } \mathcal{S}.$$

$$\left.\begin{aligned}&\text{(iii)} \quad \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} \\ &\text{(iv)} \quad \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a}\end{aligned}\right\} \begin{array}{l}\text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}.\end{array}$$

# Boundary Conditions

The positive direction for **a** is from 2 toward 1

(i) $$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$




$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f \, a$$

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.}$$

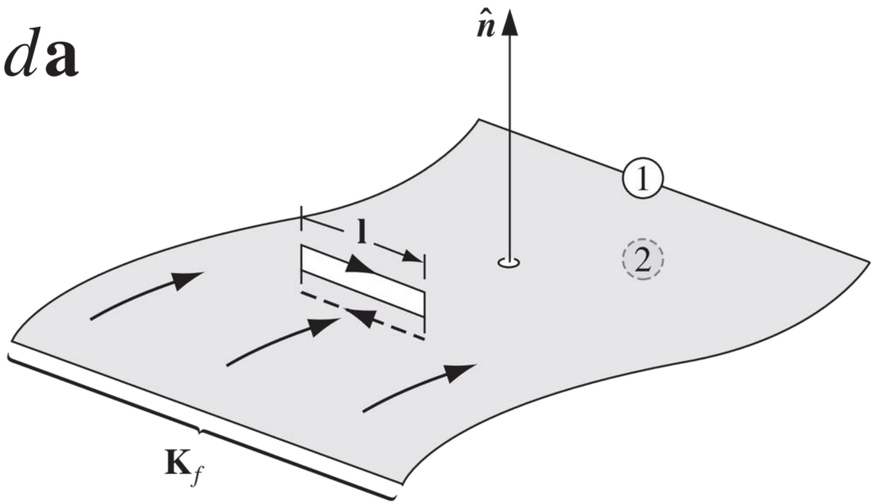
The perpendicular component of D at the interface is discontinuous by an amount of surface charge density $\boldsymbol{\sigma}_{\mathbf{f}}$

(ii) $$\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.}$$

# Boundary Conditions

$$\text{(iii)} \quad \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}$$




$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}.}$$

The components of **E** parallel to the interface are continuous across the boundary

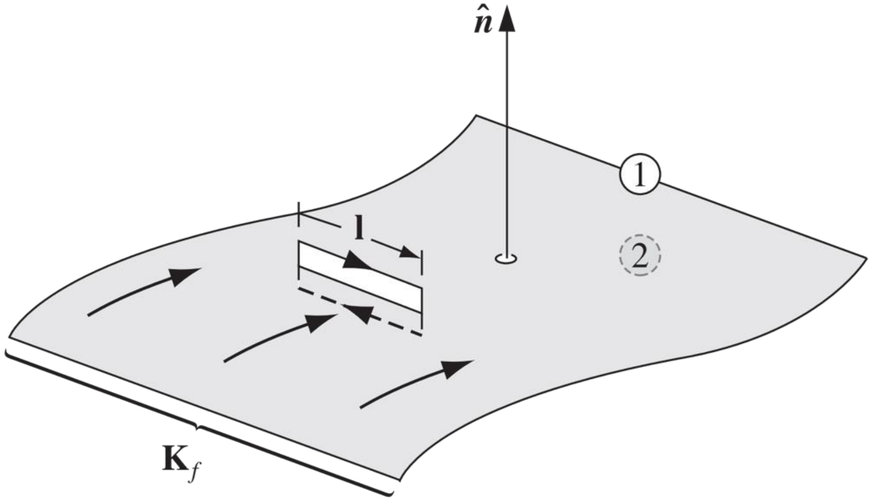
$$\text{(iv)} \quad \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a}$$

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}}$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop.

# Boundary Conditions

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}}$$




$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l},$$

$$\boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.}$$

So the *parallel* components of **H** are discontinuous by an amount proportional to the free surface current density.

# Boundary Conditions

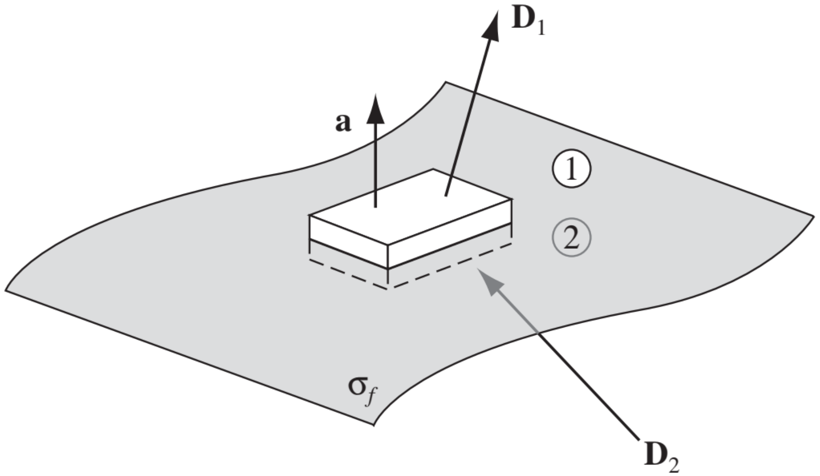
## General boundary conditions for electrodynamics

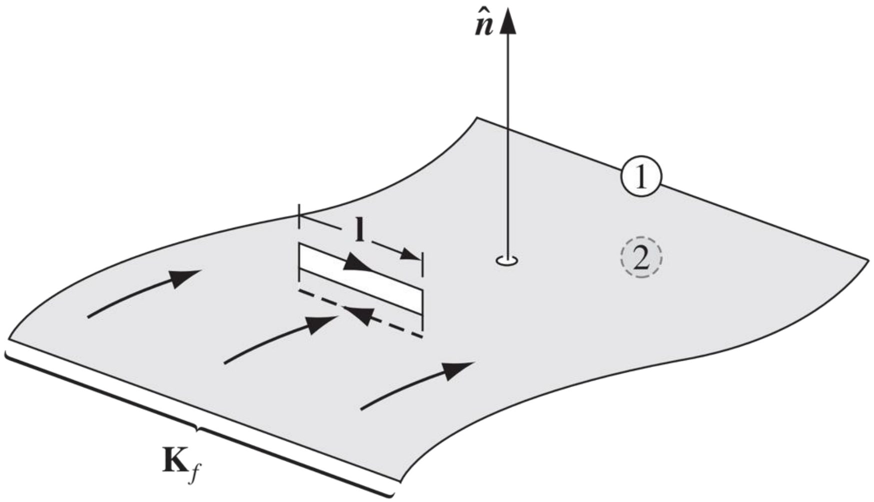
$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.}$$

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.}$$

$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}.}$$

$$\boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.}$$



# Boundary Conditions

In the case of linear media, they can be expressed in terms of **E** and **B** alone:

$$\text{(i)}\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \text{(iii)}\quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0},$$

$$\text{(ii)}\quad B_1^{\perp} - B_2^{\perp} = 0, \qquad \text{(iv)}\quad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

In particular, if there is no free charge or free current at the interface, then

$$\text{(i)}\quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, \qquad \text{(iii)}\quad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0},$$

$$\text{(ii)}\quad B_1^{\perp} - B_2^{\perp} = 0, \qquad \text{(iv)}\quad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}.$$

These equations are the basis for the theory of reflection and refraction.