SOLUTIONS SHM

QUESTION 1

We always have

$$\begin{aligned} x(t) &= A\sin(\omega t+\phi_0) \\ dx/dt \equiv \dot{x}(t) &= \omega A\cos(\omega t+\phi_0) \\ d^2x/dt^2 \equiv \ddot{x}(t) &= -\omega^2 A\sin(\omega t+\phi_0) \end{aligned}$$

for any particle in simple harmonic motion. We are told that this one has amplitude $A = 4$ cm $= 0.04$ m, and from the period we can infer the angular frequency $\omega = 2\pi/T = 2\pi/(2\text{ s}) = \pi\text{ s}^{-1}$. Thus, we know that in this case

$$\begin{aligned} x(t) &= 4\,\sin(\pi\,t+\phi_0)\ \ \text{cm} \\ \dot{x}(t) &= 4\pi\,\cos(\pi\,t+\phi_0)\ \ \text{cm s}^{-1} \\ \ddot{x}(t) &= -4\pi^2\,\sin(\pi\,t+\phi_0)\ \ \text{cm s}^{-2} \end{aligned}$$

We now need to find the phase constant, ϕ_0. To do so, we make use of the initial condition, $x = 2$ cm at $t = 0$. This means

$$x(0) \ = \ 4\,\sin(0+\phi_0)\ \text{cm} \ = \ 2\ \text{cm} \quad \Longrightarrow \quad \sin\phi_0 \ = \ 1/2\ .$$

There are *two angles* between 0 and 2π (radians) for which the sine is equal to 1/2: 30°, or $\pi/6$ rad, and 150°, or $5\pi/6$ rad. The phase constant in this problem could be *either* of these, because we are given no more information about any other initial conditions (in particular, we don't know the sign of the initial velocity, which would constrain the sign of $\cos\phi_0$ and thus allow to choose between $\pi/6$ and $5\pi/6$). Simply choosing $\phi_0 = \pi/6$ for definiteness, the position, velocity, and acceleration as functions of time are

$$\begin{aligned} x(t) &= 4\,\sin(\pi\,t+\pi/6)\ \ \text{cm} \\ \dot{x}(t) &= 4\pi\,\cos(\pi\,t+\pi/6)\ \ \text{cm s}^{-1} \\ \ddot{x}(t) &= -4\pi^2\,\sin(\pi\,t+\pi/6)\ \ \text{cm s}^{-2} \end{aligned}$$

QUESTION 2

Again,

$$\begin{aligned} x &= A\sin(\omega t+\phi_0) \\ \dot{x} &= \omega A\cos(\omega t+\phi_0) \\ \ddot{x} &= -\omega^2 A\sin(\omega t+\phi_0) \end{aligned}$$

apply for any simple harmonic oscillation. Thus,

(a) If $x(0) = A$, then $\sin\phi_0 = 1$, which requires $\phi_0 = \pi/2$ rad (or 90°).

(b) If $x(0) = -A$, then $\sin(\phi_0) = -1$, which requires $\phi_0 = -\pi/2$ rad (or $-90°$; or, equivalently, $3\pi/2$ rad or 270°).

(c) If $x(0) = 0$ and $\dot{x}(0) < 0$, then $\sin\phi_0 = 0$ and $\cos\phi_0 < 0$, which requires $\phi_0 = \pi$ rad (or 180°).

(d) If $x(0) = 0$ and $\dot{x}(0) > 0$, then $\sin\phi_0 = 0$ and $\cos\phi_0 > 0$, which requires $\phi_0 = 0$.

(e) If $x(0) = A/2$ and $\dot{x}(0) > 0$, then $\sin\phi_0 = +1/2$ and $\cos\phi_0 > 0$, which requires $\phi_0 = \pi/6$ rad (or 30°).

(f) If $x(0) = A/2$ and $\dot{x}(0) < 0$, then $\sin\phi_0 = +1/2$ and $\cos\phi_0 < 0$, which requires $\phi_0 = 5\pi/6$ rad (or 150°).

QUESTION 3

We always have

$$\begin{aligned} x &= A\sin(\omega t + \phi_0) \\ \dot{x} &= \omega A\cos(\omega t + \phi_0) \\ \ddot{x} &= -\omega^2 A\sin(\omega t + \phi_0) \end{aligned}$$

for any simple harmonic motion. For a block on a spring in particular, we know that the angular frequency is $\sqrt{k/m}$; and we are given $k = 0.4$ N m^{-1} and $m = 25$ g $= 0.025$ kg in this case. So:

(a) $\omega = \sqrt{k/m} = \sqrt{0.4/0.025} = 4$ s^{-1}

(b) $T = 2\pi/\omega = \pi/2$ s, or $\simeq 1.57$ seconds

(c) $f = 1/T = \omega/2\pi = 2/\pi$ s^{-1}, or $\simeq 0.637$ Hz

(d) At $t = 0$, $x = A\sin\phi_0 = 0.10$ m, and $\dot{x} = \omega A\cos\phi_0 = 0.40$ m s^{-1}. From part (a) we know that $\omega = 4$ s^{-1}, and thus $\omega A\cos\phi_0 = 0.40$ implies $A\cos\phi_0 = 0.10$ m. That is,

$$A\ \sin\phi_0 \ = \ A\ \cos\phi_0 \ = \ 0.10 \text{ m} .$$

From this we know that $\sin\phi_0 = \cos\phi_0$ and that both $\sin\phi_0$ and $\cos\phi_0$ are positive. This requires

$$\phi_0 \ = \ \pi/4 \text{ radians} \quad (\text{or } 45°) .$$

But then, $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, so the amplitude is

$$A \;=\; 0.10/\sin\phi_0 \;=\; 0.10/(1/\sqrt{2}) \;=\; 0.1\sqrt{2}\ \text{m} \;\simeq\; 0.1414\ \text{m} \quad (14.14\ \text{cm})\ .$$

(e) We now know all of ω, A, and ϕ_0, so we can write the position and velocity as functions of time:

$$\begin{aligned} x(t) &= 0.1\sqrt{2}\ \sin(4\,t+\pi/4) \quad \text{m} \\ \dot{x}(t) &= 0.4\sqrt{2}\ \cos(4\,t+\pi/4) \quad \text{m s}^{-1} \end{aligned}$$

Evaluating these at $t = \pi/8$ gives

$$x \;=\; 0.1\ \text{m} \;=\; 10\ \text{cm} \quad ; \quad \dot{x} \;=\; -0.4\ \text{m s}^{-1} \;=\; -40\ \text{cm s}^{-1} \qquad \text{at } t = \pi/8\ .$$

(f) The maximum velocity is (in general!) $\dot{x}_{\text{max}} = \omega A$, which in this case is

$$\dot{x}_{\text{max}} \;=\; 0.4\sqrt{2}\ \text{m s}^{-1} \;\simeq\; 0.566\ \text{m s}^{-1}\ .$$

The maximum velocity always occurs at $x = 0$.

(g) The maximum acceleration is (in general!) $\ddot{x}_{\text{max}} = \omega^2 A$, which in this case is

$$\ddot{x}_{\text{max}} \;=\; 1.6\sqrt{2}\ \text{m s}^{-2} \;\simeq\; 2.263\ \text{m s}^{-2}\ .$$

The maximum acceleration is always achieved at $x = -A$, which in this case is

$$x \;=\; -A \;=\; -0.1\sqrt{2}\ \text{m} \;\simeq\; -0.1414\ \text{m} \qquad \text{for } \ddot{x} = \ddot{x}_{\text{max}}$$

QUESTION 4

(a) The angular frequency of any simple pendulum is $\omega = \sqrt{g/L}$, so in this case

$$\omega \;=\; \sqrt{\frac{g}{L}} \;=\; \sqrt{\frac{9.81\ \text{m s}^{-2}}{1\ \text{m}}} \;=\; 3.1321\ \text{s}^{-1}\ .$$

(b) Conservation of momentum (in the horizontal direction in this case) requires

$$[(mv)_{\text{moving thing}} + (mv)_{\text{pendulum bob}}]_{\text{before}} \;=\; [(m_{\text{moving thing}} + m_{\text{pendulum bob}}) \times v]_{\text{after}}$$

$$\begin{aligned} \Longrightarrow \qquad & (100\ \text{g} \times 2\ \text{m s}^{-1}) + (100\ \text{g} \times 0) \;=\; (100\ \text{g} + 100\ \text{g}) \times \dot{s}(0) \\ \Longrightarrow \qquad & \dot{s}(0) \;=\; 1\ \text{m s}^{-1} \\ \Longrightarrow \qquad & \dot{\theta}(0) \;=\; \dot{s}(0)/L \;=\; (+1\ \text{m s}^{-1})/(1\ \text{m}) \;=\; +1\ \text{s}^{-1}\ . \end{aligned}$$

(c) The angular displacement, angular velocity, and angular acceleration of a simple pendulum are given by

$$\begin{aligned}\theta(t) &= \theta_{\max}\,\sin(\omega t+\phi_0)\\ \dot\theta(t) &= \omega\theta_{\max}\,\cos(\omega t+\phi_0)\\ \ddot\theta(t) &= -\omega^2\theta_{\max}\,\sin(\omega t+\phi_0)\end{aligned}$$

where $\theta_{\max}$ is the angular amplitude.

Given the initial conditions $\theta(0)=0$ and $\dot\theta(0)=+1\ \mathrm{s}^{-1}$, we therefore have that $\sin\phi_0=0$ and $\cos\phi_0>0$, which means that (cf. Question 2d above)

$$\phi_0 \;=\; 0\ .$$

Thus, from the value of the initial angular velocity we obtain

$$\dot\theta(0) \;=\; \omega\theta_{\max}\cos\phi_0 \;=\; \omega\theta_{\max}\cos(0) \;=\; \omega\theta_{\max} \;=\; 1\ \mathrm{s}^{-1} \quad\Longrightarrow\quad \theta_{\max} \;=\; \frac{1}{\omega}\ .$$

Numerically,

$$\theta_{\max} \;=\; \frac{1}{\omega} \;=\; \sqrt{\frac{L}{g}} \;=\; \sqrt{\frac{1\ \mathrm{m}}{9.81\ \mathrm{m\,s^{-2}}}} \;=\; 0.3193\ \mathrm{rad} \quad (\text{or } 18.3^\circ)\ .$$

Using $\theta_{\max}$ in radians,

$$\theta_{\max} \;=\; 0.3193 \quad\Longrightarrow\quad \sin\theta_{\max} \;=\; 0.3139 \;=\; 0.98\,\theta_{\max}\ ,$$

so the small-angle approximation is a good one.

Finally, in order to have $\theta=\theta_{\max}$, the time must be such that

$$\sin(\omega\,t+\phi_0) \;=\; 1 \quad\Longrightarrow\quad \omega\,t+\phi_0 \;=\; \pi/2$$

in general. In this particular case, with $\phi_0=0$, this means

$$\omega\,t \;=\; \frac{\pi}{2} \quad\Longrightarrow\quad t \;=\; \frac{\pi}{2\omega} \;=\; \frac{\pi}{2\sqrt{g/L}} \;=\; \frac{\pi}{2}\sqrt{\frac{L}{g}} \simeq\; 0.50\ \text{seconds}\ .$$

The final, numerical value could also be obtained from $t=\pi/(2\omega)$ and the value of ω already calculated in part (a). Either way, notice that the time taken to reach $\theta=\theta_{\max}$, from a starting position of $\theta(0)=0$, is exactly one-quarter of the period ($T=2\pi/\omega=2\pi\sqrt{L/g}$, which is approximately 2 seconds for this example).
