Tutorial 7

PHY 101

Solutions

Q1. A particle of mass m moves in a one-dimensional potential energy $V(x)=-ax^2+bx^4$, where a and b are positive constants. What will be the angular frequency of small oscillations about the minima of the potential energy?

Sol:

$$U = -ax^2 + bx^4$$

$$\therefore \quad F = -\frac{dU}{dx} = -2ax + 4bx^3$$

At mean position $F = 0$, $2ax = 4bx^3$

$$\text{or} \quad x^2 = \frac{2a}{4b} = \frac{a}{2b} \Rightarrow x = \pm\sqrt{\frac{a}{2b}}$$

$$\therefore \quad x_0 = -\sqrt{\frac{a}{2b}}$$

$$k_{\text{eff}} = \frac{d^2U}{dx^2} \text{ at } x_0 = 4a$$

$$ma' = +4ax \quad \therefore \quad a' = \frac{4a}{m}x$$

Comparing with $a' = \omega^2 x$

$$\omega = \sqrt{\frac{4a}{m}} = 2\sqrt{\frac{a}{m}}$$

Q2. Consider a mass-spring system with m=5kg, μ=7kg/sec, $k=3kg/sec^2$ and a forcing term 2Cos4t N .

(a) Find the steady periodic solution $x_p(t)$and find the amplitude and phase.
(b) Find the position x(t) if x(t=0)=0 m and v=dx/dt at t=0 is 1 m/sec.

Sol:Theory

Periodically forced mass-spring system: $mx''+\mu x'+kx = F_0 \cos \omega t$
or $x'' + dx' + \omega_0^2 x = A \cos \omega t$
where $d = \mu/m$, $\omega_0 = \sqrt{k/m}$, $A = F_0/m$

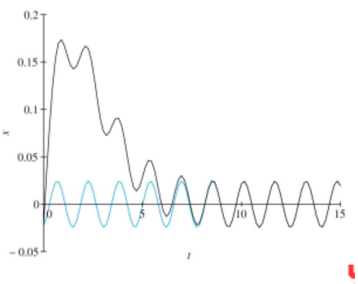
Sinusoidal forcing: $F(t) = A\cos\omega t$

where A is the amplitude and ω is the driving frequency.

General solution:
$x(t) = x_h(t) + x_p(t)$

where

- $x_p(t)$: steady state part (persistent oscillation)
- $x_h(t)$: transient part ($d > 0$) ($x_h(t) \to 0$ for $t \to \infty$)



$$x'' + dx' + \omega_0^2 x = A\cos\omega t \qquad (6)$$

Since $A\cos\omega t = \text{Re}(Ae^{i\omega t})$, any solution $x(t)$ is the real part of a solution $z(t)$ of

$$z'' + dz' + \omega_0^2 z = Ae^{i\omega t} \qquad (7)$$

Solution Strategy:

- Find particular solution of (7)
- Real part $\rightarrow$ particular solution of (6)

Try **complex exponential** for (7):

$$z_p(t) = ae^{i\omega t} \;\Rightarrow\; z_p'' + dz_p' + \omega_0^2 z_p = ((i\omega)^2 + i\omega d + \omega_0^2)ae^{i\omega t} = Ae^{i\omega t}$$

$$\Rightarrow\; [(\omega_0^2 - \omega^2) + i\omega d]a = A$$

$$\Rightarrow\; \frac{a}{A} = \frac{1}{(\omega_0^2 - \omega^2) + i\omega d}$$

Use $\;1/(\alpha + i\beta) = (\alpha - i\beta)/(\alpha^2 + \beta^2)$

$$\Rightarrow\; \frac{a}{A} = \frac{(\omega_0^2 - \omega^2) - i\omega d}{D}$$

where $D = (\omega_0^2 - \omega^2)^2 + \omega^2 d^2$

Amplitude and Phase: Set

$$a/A = Ge^{-i\phi} = G\cos\phi - iG\sin\phi$$

$$\Rightarrow G^2 = \left(\frac{(\omega_0^2 - \omega^2)}{D}\right)^2 + \left(\frac{\omega d}{D}\right)^2 = \frac{(\omega_0^2 - \omega^2)^2 + \omega^2 d^2}{D^2} = \frac{D}{D^2}$$

$\Rightarrow G = 1/\sqrt{D} \equiv G(\omega)$ (*gain*), hence

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 d^2}} \qquad (8)$$

Phase angle:

$$\omega_0^2 - \omega^2 = G\cos\phi,\; \omega d = G\sin\phi$$

where $0 \le \phi < \pi$ (since $\sin\phi \ge 0$)

$$\Rightarrow\; \phi(\omega) = \text{arccot}\left(\frac{\omega_0^2 - \omega^2}{\omega d}\right) \qquad (9)$$

Particular Solution of (6):

$$z_p(t) = ae^{i\omega t} = G(\omega)Ae^{i(\omega t-\phi)} \Rightarrow$$

$$x_p(t) = \text{Re}z_p(t) = GA\cos(\omega t-\phi) \quad (10)$$

General Solution of (6):

$$x(t) = x_h(t) + x_p(t) \quad (11)$$

where $x_h(t) = c_1x_1(t) + c_2x_2(t)$ (12)

and $x_1(t)$, $x_2(t)$ is F.S.S. of

$$x'' + dx' + \omega_0^2 x = 0$$

Steady State and Transient Parts:

- $x_p(t)$: steady state part (persistent oscillation)
- $x_h(t)$: transient part ($d > 0$ $\Rightarrow x_h(t) \to 0$ for $t \to \infty$)

(a) Find the steady periodic solution $x_p(t)$ and determine its amplitude and phase.

Answer: Equation: $5x'' + 7x' + 3x = 2\cos 4t \Rightarrow x'' + 1.4x' + 0.6x = 0.4\cos 4t$

Use complex method: $x_p(t) = \text{Re}z_p(t)$, where z_p is particular solution of

$$z'' + 1.4z' + 0.6z = 0.4e^{4it}$$

Try $z_p = ae^{4it} \Rightarrow (-16 + 5.6i + 0.6)ae^{4it} = 0.4e^{4it}$

$$\Rightarrow a = \frac{0.4}{-15.4 + 5.6i} = \frac{0.4 \times (-15.4 - 5.6i)}{15.4^2 + 5.6^2} = -0.0229 - 0.0083i$$

$$\begin{aligned}
\Rightarrow z_p(t) &= (-0.0229 - 0.0083i)(\cos 4t + i\sin 4t) \\
\Rightarrow x_p(t) &= \text{Re}(z_p(t)) = 0.0083\sin 4t - 0.0229\cos 4t \ \text{(superposition form)}
\end{aligned}$$

To find amplitude and phase compute polar form: $a = A_0e^{-i\phi}$, where

$$\begin{aligned}
A_0 &= \sqrt{0.0229^2 + 0.0083^2} = 0.0244 \\
\phi &= \text{arccot}(-0.0229/0.0083) = 2.7939
\end{aligned}$$

$$\begin{aligned}
\Rightarrow z_p(t) &= A_0e^{i(4t-\phi)} \\
\Rightarrow x_p(t) &= 0.0244\cos(4t - 2.7939) \ \text{(amplitude-phase form)}
\end{aligned}$$

(b) Find the position $x(t)$ if $x(0) = 0,\ x'(0) = 1\,m/s$

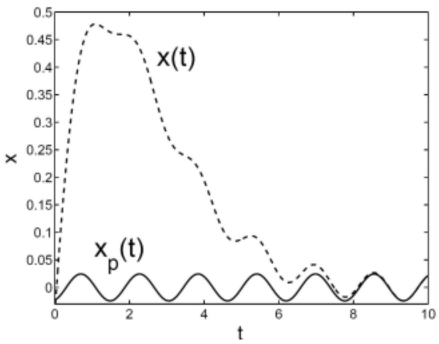
Answer: Find transient part: $x'' + 1.4x' + 0.6x = 0 \Rightarrow p(\lambda) = \lambda^2 + 1.4\lambda + 0.6 = 0$ $\Rightarrow \lambda = -0.7 \pm 0.3317i$

$\Rightarrow x_h(t) = e^{-0.7t}[c_1 \cos(0.3317t) + c_2 \sin(0.3317t)]$ and $x(t) = x_h(t) + x_p(t)$

Match c_1, c_2 to IC: (use superposition form)

$$\left.\begin{array}{rcl} x(0) & = & c_1 - 0.0229 = 0 \Rightarrow c_1 = 0.0229 \\ x'(0) & = & -0.7c_1 + 0.3317c_2 + 4 \times 0.0083 = 1 \Rightarrow c_2 = 2.9630 \end{array}\right\} \Rightarrow$$

$$x(t) = e^{-0.7t}[0.0229\cos(0.3317t) + 2.9630\sin(0.3317t)] + 0.0244\cos(4t - 2.7939)$$



Q3. A damped oscillator consists of a mass 200 gm attached to a spring of constant $100 Nm^{-1}$ and damping constant 5 Nm−1 s. It is driven by a force F = 6 cos wt Newton, where $\omega = 30 s^{-1}$. If displacement in steady state is $x = A \sin(\omega t - \varphi)$ metre, find A and φ. Also calculate the power supplied to the oscillator.

Sol:

Q3

Amplitude of the driving force $F_0 = 6\,N$
Frequency of the driving force $\omega = 30\,s^{-1}$
Mass of damped oscillator $m = 200\,gm = 0.2\,kg$
Spring Constant $k = 100\,N\,m^{-1}$
Damping Constant $R = 5\,N m^{-1} s$

When an external force $F = F_0 \cos \omega t$ acts on a damped oscillator, the amplitude in steady state is given by,

$$A = \frac{F_0}{\omega Z_m} \qquad Z_m = \text{Impedance of the mechanical System}$$

$$Z_m = \left[R^2 + \left(\omega m - \frac{k}{m}\right)^2 \right]^{1/2}$$

and $\tan\phi = \left(\dfrac{\omega m - \frac{k}{m}}{R} \right)$

$$\Rightarrow Z_m = \left[5 \times 5 + \left(30 \times 0.2 - \frac{100}{30}\right)^2 \right]^{1/2}$$

$$= 5.67\ N m^{-1}$$

Q4. The equation of motion is $2 \times 10^{-4}\, d^2x/dt^2 + 4 \times 10^{-2}\, dx/dt + 5x = 0.124 \sin 100t$ where, all quantities are in S.I units. Find (i) Natural frequency of undamped oscillation (ii) Mechanical impedance.

Sol:

Q.4. Given differential equation,

$$2\times10^{-4}\frac{d^2x}{dt^2} + 4\times10^{-2}\frac{dx}{dt} + 5x = 0.124\sin 100t$$

Comparing with general equation

$$m\ddot{x} + R\dot{x} + kx = F_0 \sin \omega t$$

$m = 2\times10^{-4}$ kg, $R = 4\times10^{-2}$ Nsm^{-1}, $k = 5$ Nm^{-1}

$F_0 = 0.124N$

(i) Natural frequency of undamped oscillation.

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$= \frac{1}{2\times3.14}\left[\frac{5}{2\times10^{-4}}\right]^{1/2}$$

$$= 25.16 \text{ Hz}$$

(ii) Mechanical Impedance (Z) $\omega_0 = 2\pi f$

$$\text{Amplitude } A = \frac{F_0}{m\sqrt{\left(\frac{R}{m}\right)^2\omega^2 + (\omega^2-\omega_0^2)^2}} = \frac{F_0}{\omega Z}$$

$$\text{Mechanical Impedance } Z = \frac{m}{\omega}\left(\frac{R^2\omega^2}{m^2} + (\omega^2-\omega_0^2)^2\right)^{1/2}$$

$$\omega^2 - \omega_0^2 = \frac{5}{2\times10^{-4}} - 100^2 = 1.5\times10^4$$

$$\left(\frac{R}{m}\right)^2\omega^2 = \left(\frac{4\times10^{-2}}{2\times10^{-4}}\right)^2 \times (10^2)^2 = 4\times10^8$$

$$\Rightarrow Z = \frac{2\times10^{-4}}{10^{2}}\left(4\times10^{8} + 2.25\times10^{8}\right)$$

$$= 2\times10^{-6}\sqrt{10^{8}(4+2.25)}$$

$$= 2\times10^{-6}\times10^{4}\times\sqrt{6.25}$$

$$= 2\times 2.5\times10^{-2}$$

$$= 0.05\ \mathrm{Nsm^{-1}}$$