PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 6

Operators in Spherical and Cylindrical Coordinate Coordinate System

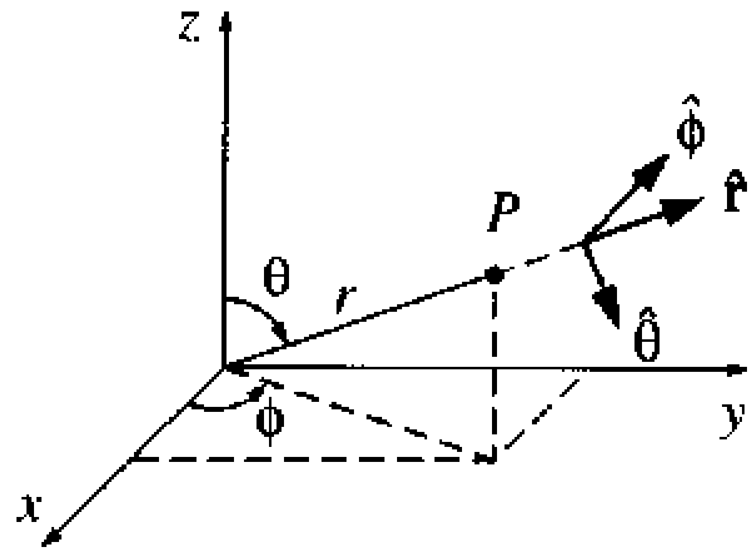
Curvilinear Coordinates :

Spherical Polar Coordinates : (r, θ, ϕ) **polar angle, azimuthal angle,**

$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi,$$

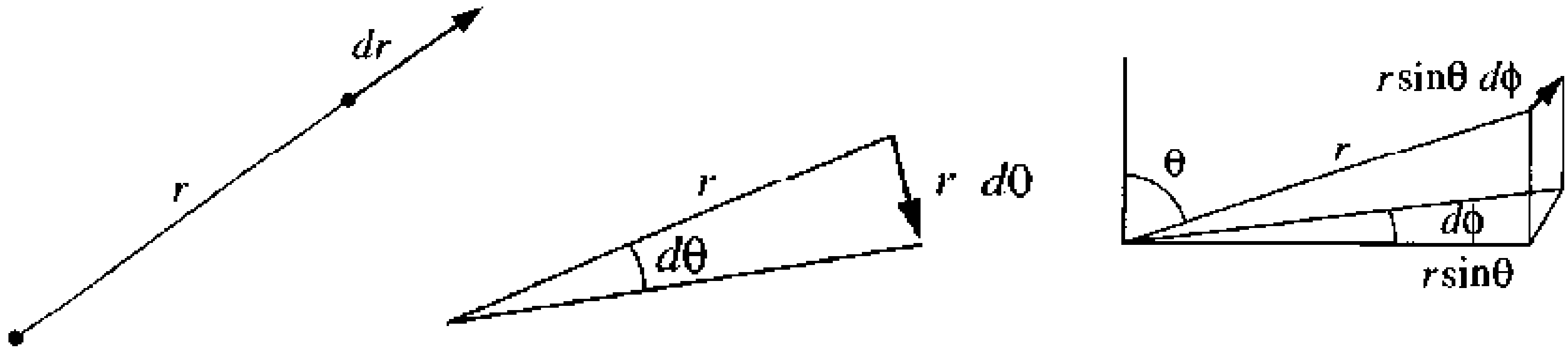
$$z = r \cos\theta.$$



$$\mathbf{A} = A_r \,\hat{\mathbf{r}} + A_\theta \,\hat{\boldsymbol{\theta}} + A_\phi \,\hat{\boldsymbol{\phi}}.$$

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta \cos\phi \,\hat{\mathbf{x}} + \sin\theta \sin\phi \,\hat{\mathbf{y}} + \cos\theta \,\hat{\mathbf{z}}, \\
\hat{\boldsymbol{\theta}} &= \cos\theta \cos\phi \,\hat{\mathbf{x}} + \cos\theta \sin\phi \,\hat{\mathbf{y}} - \sin\theta \,\hat{\mathbf{z}}, \\
\hat{\boldsymbol{\phi}} &= -\sin\phi \,\hat{\mathbf{x}} + \cos\phi \,\hat{\mathbf{y}},
\end{aligned}$$

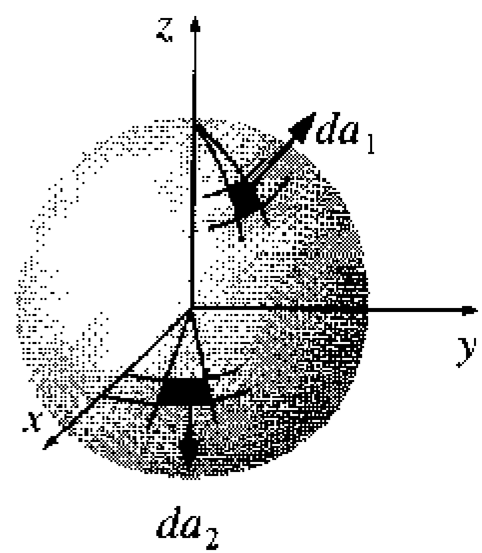
$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}.$$



$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

$$d\mathbf{a}_1 = dl_\theta\,dl_\phi\,\hat{\mathbf{r}} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}.$$

$$d\mathbf{a}_2 = dl_r\,dl_\phi\,\hat{\boldsymbol{\theta}} = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}.$$



Spherical Polar Coordinates

Find the volume of a sphere of radius R.

Solution:

$$\begin{aligned} V &= \int d\tau = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= \left(\int_0^R r^2 \, dr\right)\left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) \\ &= \left(\frac{R^3}{3}\right)(2)(2\pi) = \tfrac{4}{3}\pi R^3. \end{aligned}$$

Differential Operators take very simple form in cartesian coordinate system.
We have already discussed them.
But in terms of curvilinear(spherical/cylindrical), it involves a bit of derivation.
We recommend you, to see the derivation in your prescribed textbook (Griffith).
Here, We give the form of these Differential Operators in curvilinear coordinates directly.

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl:

$$\begin{aligned}\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.\end{aligned}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

Cylindrical coordinates

- Any point in 3-dimensions can be located using (Perpendicular distance from the *z*-axis: s, Azimuthal angle: ϕ, Position on z-axis: z).

- Domain: $0 \leq s < \infty$, $0 \leq \phi < 2\pi$, $-\infty \leq z < \infty$

- Relation between cartesian coordinates (x,y,z) and cylindrical coordinates (s,ϕ,z):

$$x = s\cos\phi$$

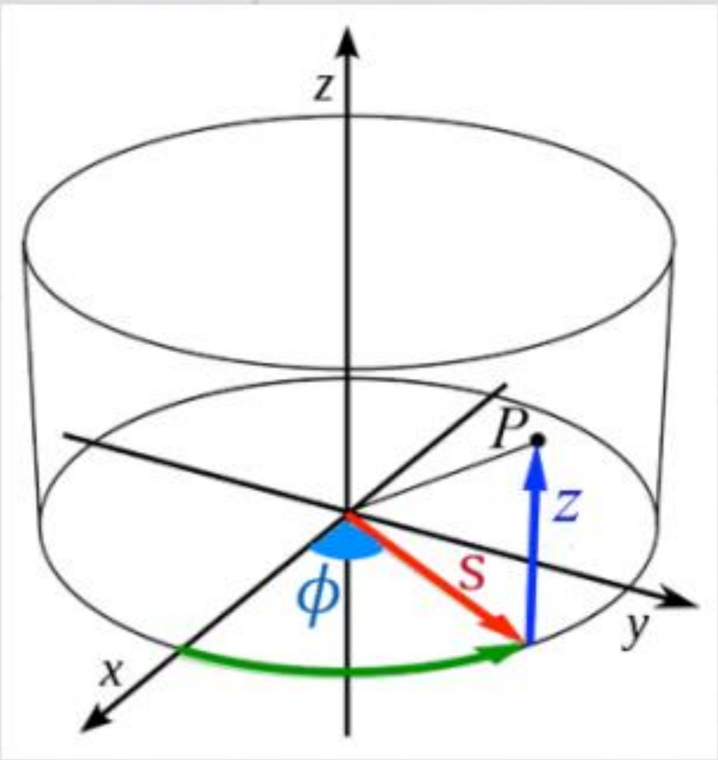
$$y = s\sin\phi$$

$$z = z$$

$$s = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$



Unit vectors for Cylindrical coordinates

- Unit vectors pointing in the direction of increase of s,ϕ,z respectively:

  $\hat{s}, \hat{\phi}, \hat{k}$

- They constitute an orthonormal basis set (just like i, j, k):

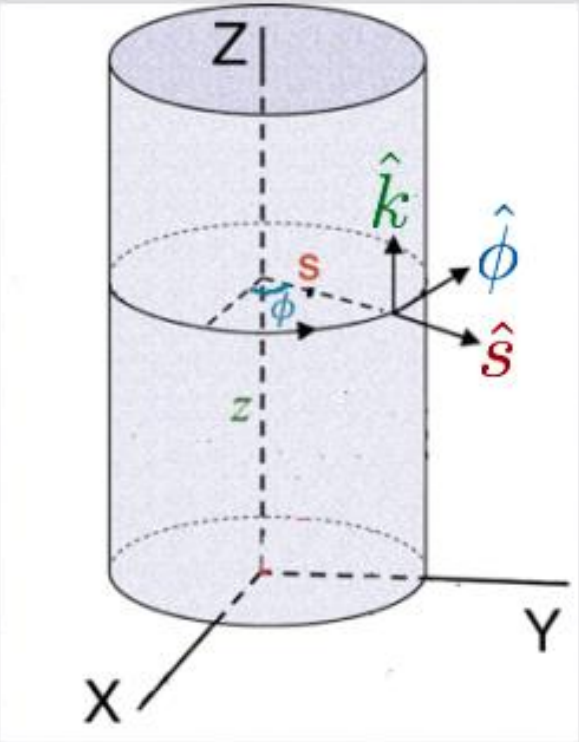
  $$\hat{s}\cdot\hat{s} = \hat{\phi}\cdot\hat{\phi} = \hat{k}\cdot\hat{k} = 1$$
  $$\hat{s}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{k} = \hat{k}\cdot\hat{s} = 0$$

- Any vector **V** can be expressed using these as:

  $$\mathbf{V} = V_s\,\hat{s} + V_\phi\,\hat{\phi} + V_z\,\hat{k}$$

- Expression in terms of i,j,k:

  $$\hat{s} = (\cos\phi)\hat{i} + (\sin\phi)\hat{j}$$
  $$\hat{\phi} = (-\sin\phi)\hat{i} + (\cos\phi)\hat{j}$$
  $$\hat{k} = \hat{k}$$



- The unit vectors $\hat{s}$, $\hat{\phi}$ are not constant--they change direction as we move in space. However , $\hat{k}$ as we know, is fixed.

Infinitesimal displacements

- Infinitesimal displacement in the $\hat{s}$ direction:

$$dl_s = ds$$

- Infinitesimal displacement in the $\hat{\phi}$ direction:

$$dl_\phi = s d\phi$$

- Infinitesimal displacement in the $\hat{k}$ direction:

$$dl_z = dz$$

- General infinitesimal displacement:

$$\begin{aligned} d\mathbf{l} &= dl_s\,\hat{s} + dl_\phi\,\hat{\phi} + dl_z\,\hat{k} \\ &= ds\,\hat{s} + sd\phi\,\hat{\phi} + dz\,\hat{k} \end{aligned}$$

Cylindrical Coordinates

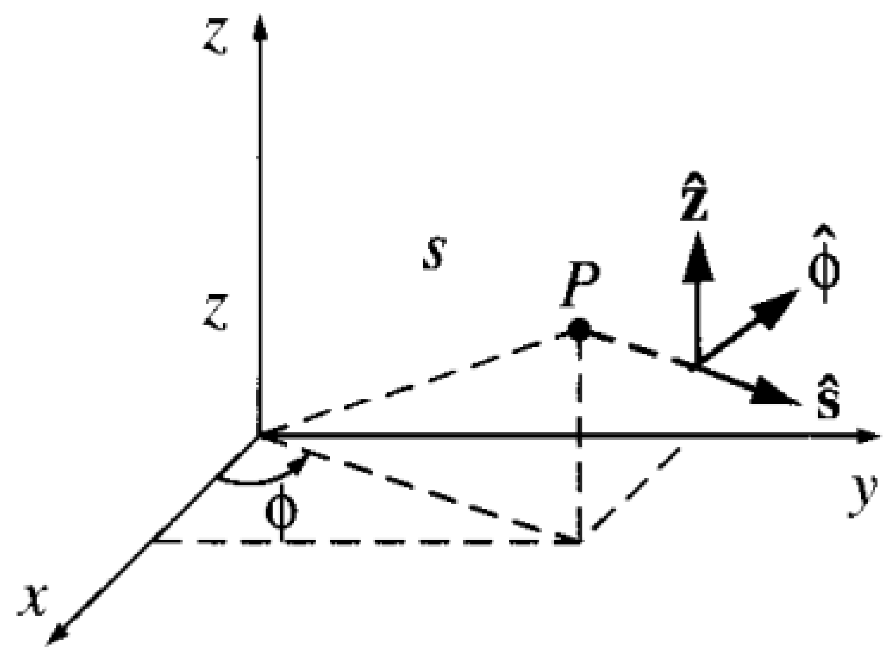
$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}},$$

and the volume element is $d\tau = s\,ds\,d\phi\,dz.$

The range of s is $0 \to \infty$,

ϕ goes from $0 \to 2\pi$,

and z from $-\infty$ to ∞.



Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

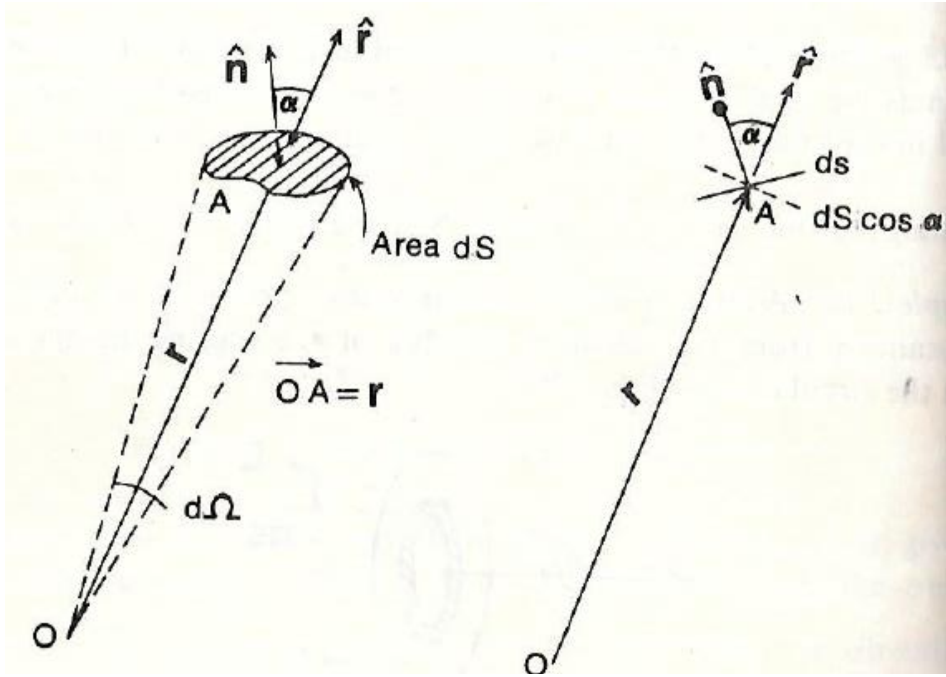
Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

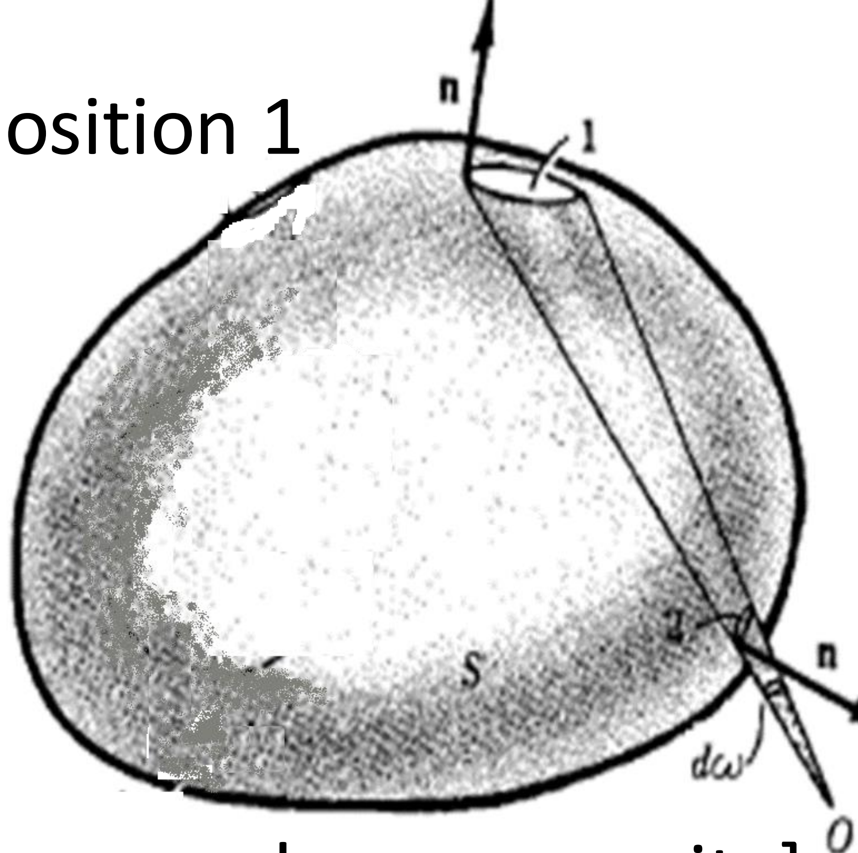
Concepts of Solid Angle



$$d\Omega = \frac{projection\ of\ dS\ perpendicular\ to\ r}{r^2}$$

$$= \frac{dS cos\alpha}{r^2} = \frac{\hat{\boldsymbol{r}}.\hat{\boldsymbol{n}}dS}{r^2}$$

If O is outside S, then at position 1



Solid angle subtended by dS at O

$$d\omega_1 = \frac{\vec{n}.\vec{r}}{r^3} dS$$

At position 2, solid angle subtended by dS at O is

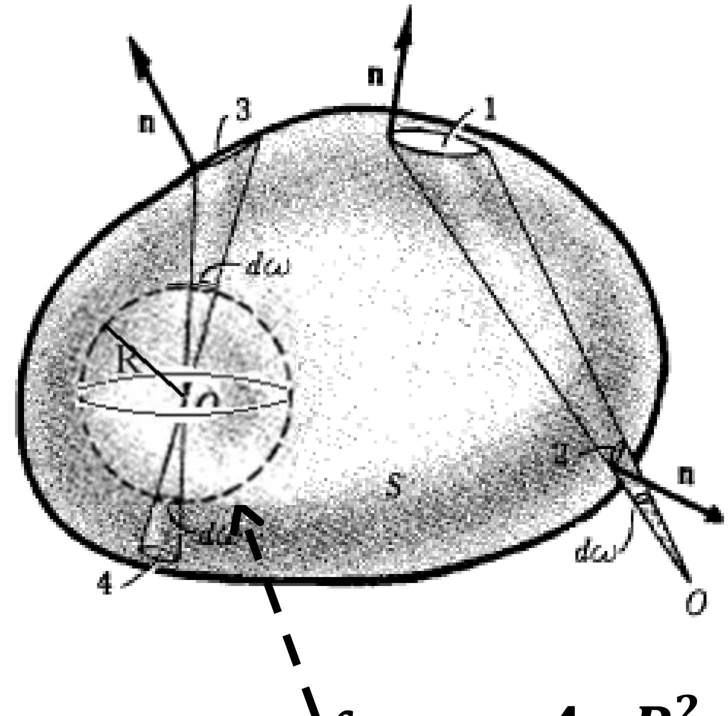
$$d\omega_2 = -d\omega_1 = -\frac{\vec{n}.\vec{r}}{r^3} dS$$

[direction of area normals are opposite]

Integration over these two regions give zero- so, the contribution to solid angle cancels out when O lies outside S. Performing integration over entire surface, we get

$$\iint_S \frac{\vec{\boldsymbol{n}}.\vec{\boldsymbol{r}}}{r^3} dS = 0$$

If O lies inside S



$$\int \boldsymbol{d\omega} = \frac{\boldsymbol{4\pi R^2}}{\boldsymbol{R^2}}$$

At pos. 3, dS contributes to positive solid angle at O

$$d\omega = \frac{\vec{n}.\vec{r}}{r^3} dS$$

At pos. 4, dS also contributes to positive solid angle at O

$$d\omega = \frac{\vec{n}.\vec{r}}{r^3} dS$$

Total contribution for the entire surface is:

$$\iint_S \frac{\vec{n}.\vec{r}}{r^3} dS = 4\pi$$ This equals area of a unit sphere

Examples

Spherical Polar Coordinates

Compute the divergence of the function

$$\mathbf{v} = (r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$$

$$\frac{1}{r^2}\,3r^2\cos\theta + \frac{1}{r\sin\theta}\,r\,2\sin\theta\cos\theta + \frac{1}{r\sin\theta}\,r\sin\theta(-\sin\phi)$$

$$= \quad 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi$$

Consider the example of a vector field $\vec{\boldsymbol{A}} = \hat{\boldsymbol{r}} r^n$

Find out the divergence of the vector field, i.e. $\nabla \cdot \vec{\boldsymbol{A}}$
Using cartesian coordinate system.
(We already did this exercise. But this form of the field is very important.)

$$The\ Answer\ is: \ \nabla \cdot (\hat{\boldsymbol{r}} r^n) = (2+n) r^{n-1}$$

n	3	2	1	0	-1	-2	-3	-4
$\vec{\boldsymbol{A}}$	$\hat{\boldsymbol{r}} r^3$	$\hat{\boldsymbol{r}} r^2$	$\boldsymbol{r}$	$\hat{\boldsymbol{r}}$	$\frac{\hat{\boldsymbol{r}}}{r}$	$\frac{\hat{\boldsymbol{r}}}{r^2}$	$\frac{\hat{\boldsymbol{r}}}{r^3}$	$\frac{\hat{\boldsymbol{r}}}{r^4}$
$\nabla \cdot \vec{\boldsymbol{A}}$	$5r^2$	4r	3	$\frac{2}{r}$	$\frac{1}{r^2}$	0	$-\frac{1}{r^4}$	$-\frac{2}{r^5}$

Consider the example of a vector field $\vec{A} = \hat{\boldsymbol{r}} r^n$,

Find out the divergence of the vector field, i.e. $\nabla \cdot \vec{A}$ **using Spherical coordinates.**

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

$$\vec{A} = \hat{\boldsymbol{r}} r^n \qquad \mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}.$$

For $\vec{A} = \hat{\boldsymbol{r}} r^n, \Rightarrow\ \mathrm{A}_r = r^n, \quad \mathrm{A}_{\boldsymbol{\theta}} = 0, \quad \mathrm{A}_{\boldsymbol{\phi}} = 0$

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) + 0 + 0 \quad \Rightarrow \boldsymbol{\nabla} \cdot (\hat{\boldsymbol{r}} r^n) = (2+n) r^{n-1}$$

For n = -2, $\Rightarrow \boldsymbol{\nabla} \cdot (\hat{\boldsymbol{r}} r^{-2}) = 0$ Is this correct answer for every value of r?

Question: $\nabla \cdot (\hat{\boldsymbol{r}} r^{-2})$= 0 Is this correct answer for every value of r?

$\nabla \cdot (\hat{\boldsymbol{r}} r^{n})$=$(2+n)r^{n-1}$ For n = -2, $\Rightarrow \nabla \cdot (\hat{\boldsymbol{r}} r^{-2})$= 0

Answer: Not quite. At r$=0$, $for\ n=-2$

$$\frac{(2+n)}{r^3} = \frac{0}{0} \Rightarrow undefined$$

$\Rightarrow$We cannot use $\nabla \cdot (\hat{\boldsymbol{r}} r^{n})$=$(2+n)r^{n-1}$ for n = -2 at r=0.

$$\Rightarrow \quad \boldsymbol{\nabla} \cdot (\hat{\boldsymbol{r}} r^{-2}) = 0 \text{ for } r \neq 0$$

Question:

What should be the value of $\boldsymbol{\nabla} \cdot (\hat{\boldsymbol{r}} r^{-2})$at r=0 ?

Cylindrical coordinate system

Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\,s(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial \phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)$$

$$\begin{aligned}
&= \frac{1}{s}2s(2+\sin^2\phi) + \frac{1}{s}s(\cos^2\phi - \sin^2\phi) + 3 \\
&= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3 \\
&= 4 + \sin^2\phi + \cos^2\phi + 3 = \boxed{8.}
\end{aligned}$$

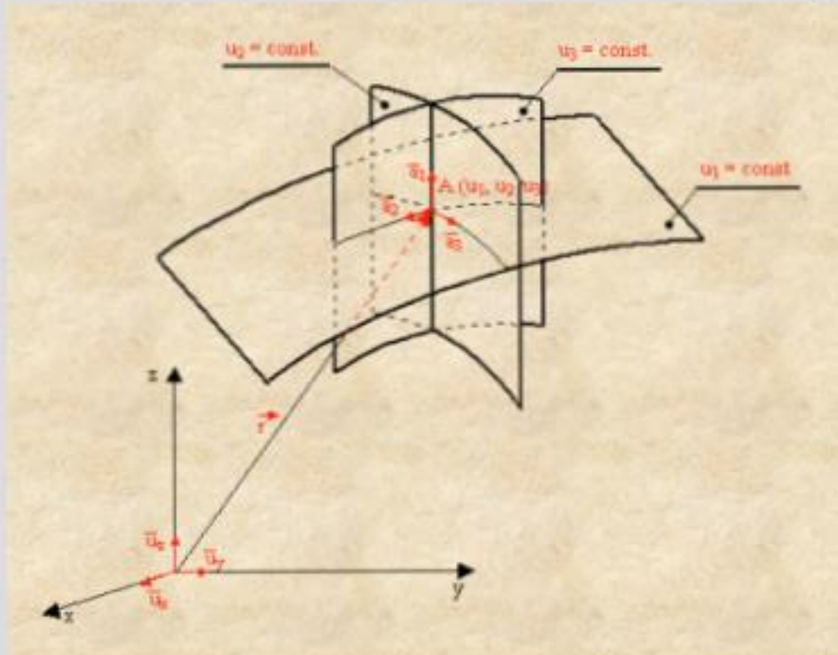
Optional

Rest of the Slides is for revision of spherical and cylindrical coordinates, Which we have already covered in PHY101, But given here for your reference.

Choice of suitable coordinate system: Curvilinear coordinates

- If for a given problem we make a suitable choice of coordinate system, keeping in mind symmetries of the problem, things can simplify a lot.

- Let us consider three independent, unambiguous and smooth functions $f_1(x,y,z)$, $f_2(x,y,z)$, $f_3(x,y,z)$, of cartesian coordinates (x,y,z). We set these functions equal to parameters u_1, u_2, u_3 and consider $u_1=c_1$ (constant), $u_2=c_2$ (constant) and $u_3=c_3$ (constant) surfaces.

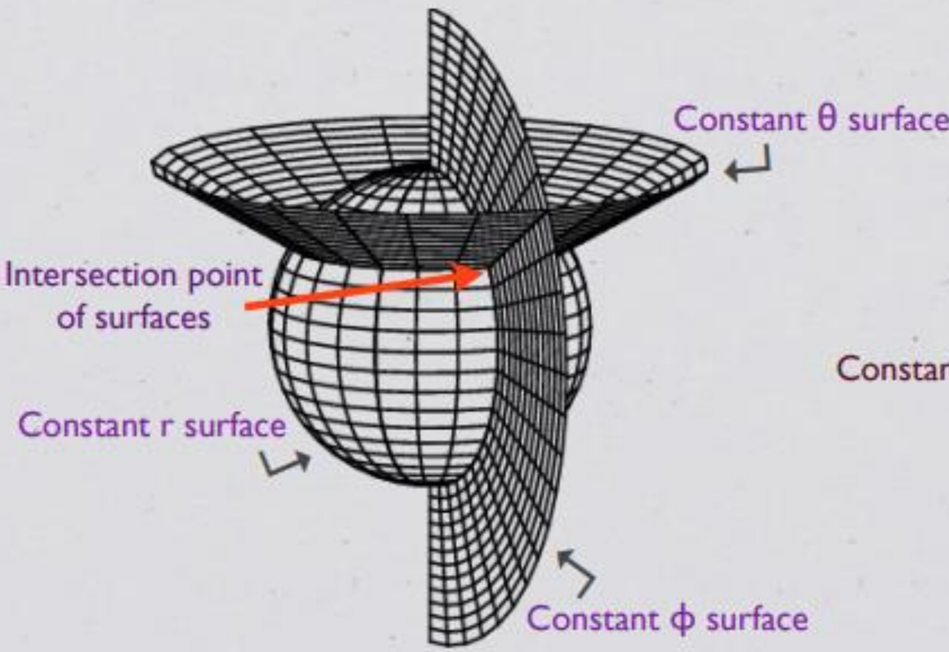


- Common intersection of these surfaces defines one point in the space to which a set of three unique numbers $(u_1, u_2, u_3) = (c_1, c_2, c_3)$ can be assigned. These numbers are called curvilinear coordinates of that point.

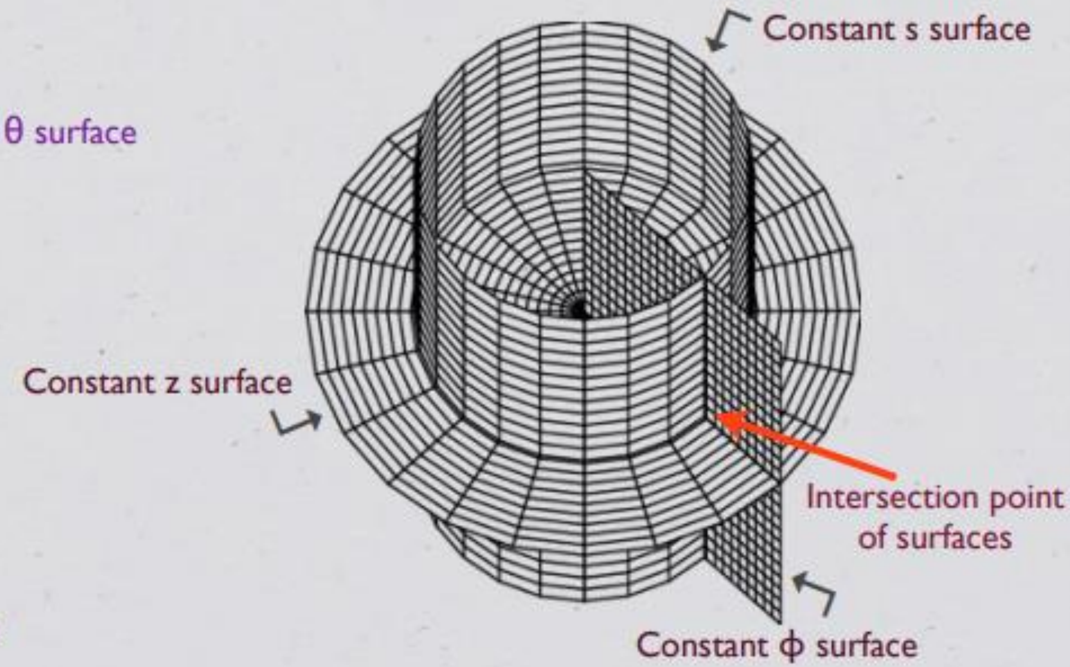
Image Source: http://www.eaeeie.org/theiere/curvilinear/GenCurvilinCoord_files/image003.gif

Choice of suitable coordinate system: Curvilinear coordinates

Spherical coordinates and Cylindrical coordinates are two important examples of curvilinear coordinates. See the slides ahead for relation of these curvilinear coordinates with the cartesian coordinates.



Constant r, θ and ϕ surfaces in Spherical coordinates



Constant s, ϕ and z surfaces in Spherical coordinates

Image Sources: http://www.math.oregonstate.edu/BridgeBook/_media/book/figures/sphgrid.png?cache=cache&w=281&h=322;

Spherical polar coordinates

- Any point in 3-dimensions can be located using: Radial distance from the origin: r, Polar angle: θ, Azimuthal angle: ϕ.
- Domain: $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$
- Relation between cartesian coordinates (x,y,z) and spherical coordinates (r,θ,ϕ):

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

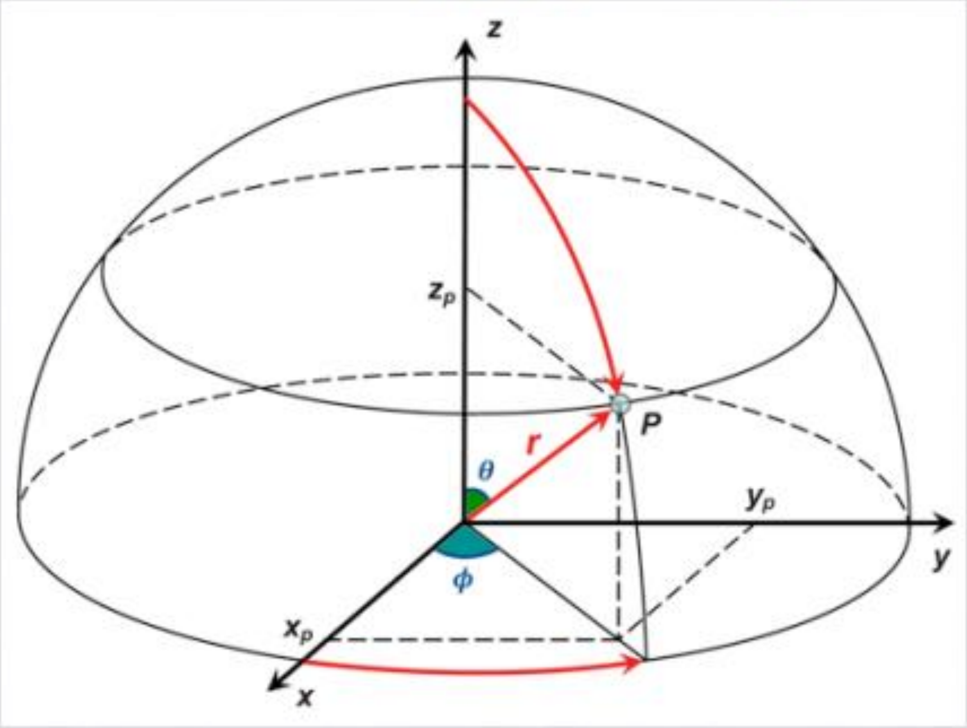
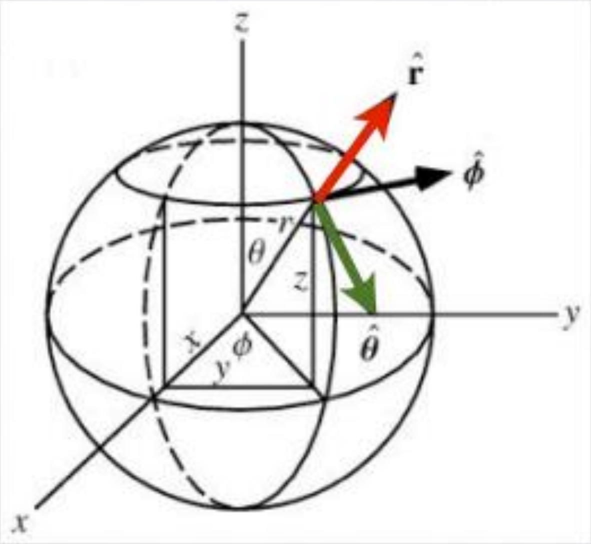


Image Source: http://www.seos-project.eu/modules/laser-rs/images/coordinates-spherical-thumb.png

Unit vectors for Spherical coordinates

- Unit vectors pointing in the direction of increase of r,θ,ϕ respectively:
  $\hat{r}, \hat{\theta}, \hat{\phi}$
- They constitute an orthonormal basis set (just like **i**, **j**, **k**):
  $$\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$$
  $$\hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{r} = 0$$
- Any vector **V** can be expressed using these as:
  $$\mathbf{V} = V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}$$
- Expression in terms of i,j,k:
  $$\hat{r} = (\sin\theta\cos\phi)\hat{i} + (\sin\theta\sin\phi)\hat{j} + (\cos\theta)\hat{k}$$
  $$\hat{\theta} = (\cos\theta\cos\phi)\hat{i} + (\cos\theta\sin\phi)\hat{j} - (\sin\theta)\hat{k}$$
  $$\hat{\phi} = (-\sin\phi)\hat{i} + (\cos\phi)\hat{j}$$

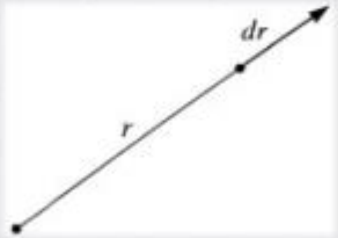


- Unlike i, j, k, the unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ are not constant, rather they change direction as we move in space.

Infinitesimal displacements

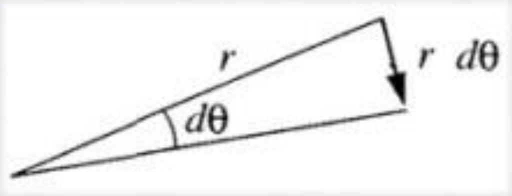
- Infinitesimal displacement in the $\hat{r}$ direction:

$$dl_r = dr$$



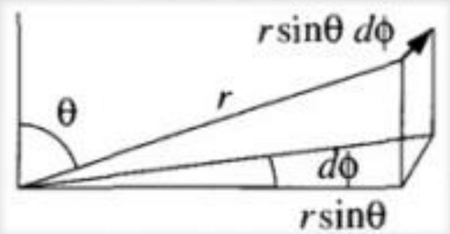
- Infinitesimal displacement in the $\hat{\theta}$ direction:

$$dl_\theta = r d\theta$$



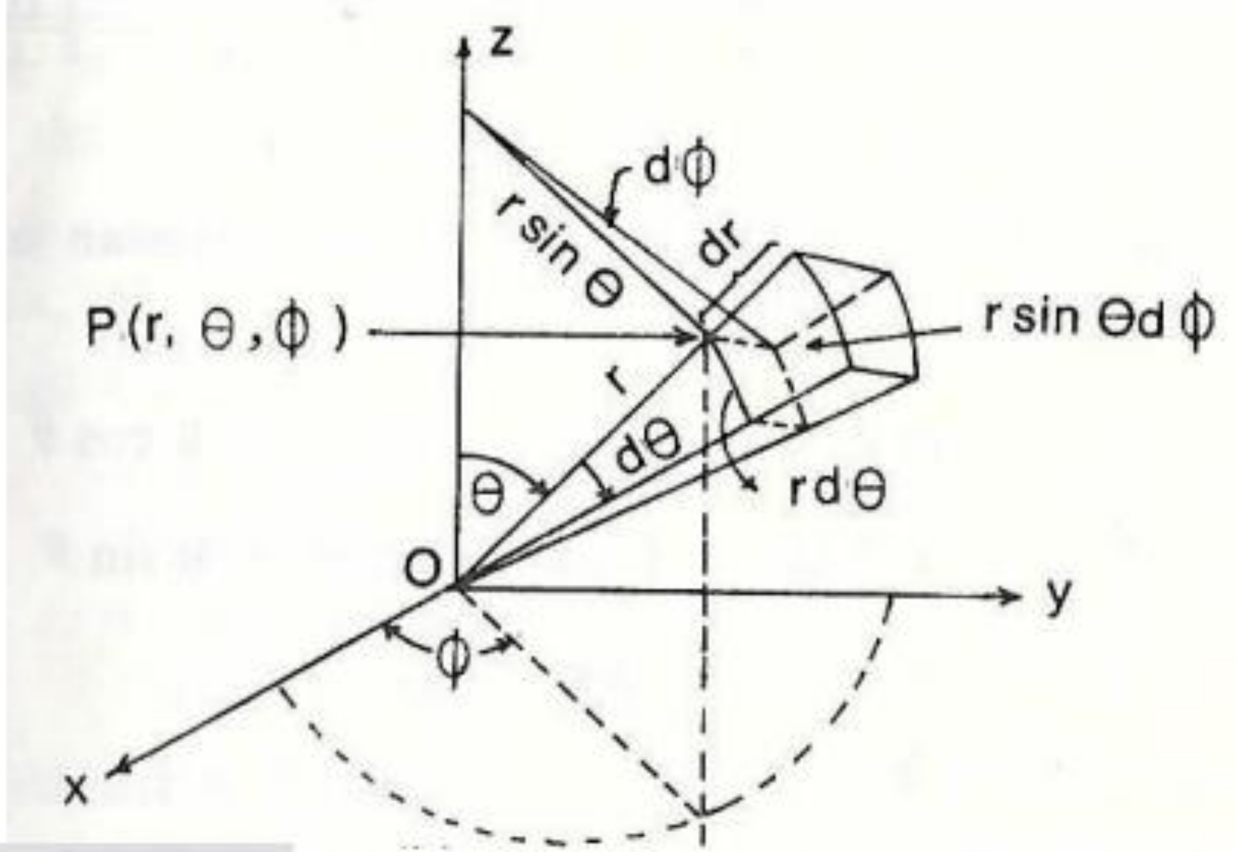
- Infinitesimal displacement in the $\hat{\phi}$ direction:

$$dl_\phi = r \sin\theta \, d\phi$$



- General infinitesimal displacement:

$$\begin{aligned} d\mathbf{l} &= dl_r \, \hat{r} + dl_\theta \, \hat{\theta} + dl_\phi \, \hat{\phi} \\ &= dr \, \hat{r} + r d\theta \, \hat{\theta} + r \sin\theta \, d\phi \, \hat{\phi} \end{aligned}$$



Infinitesimal displacement in the $\hat{r}$ direction:

$$dl_r = dr$$

Infinitesimal displacement in the $\hat{\theta}$ direction:

$$dl_\theta = r d\theta$$

Infinitesimal displacement in the $\hat{\phi}$ direction:

$$dl_\phi = r \sin\theta \, d\phi$$

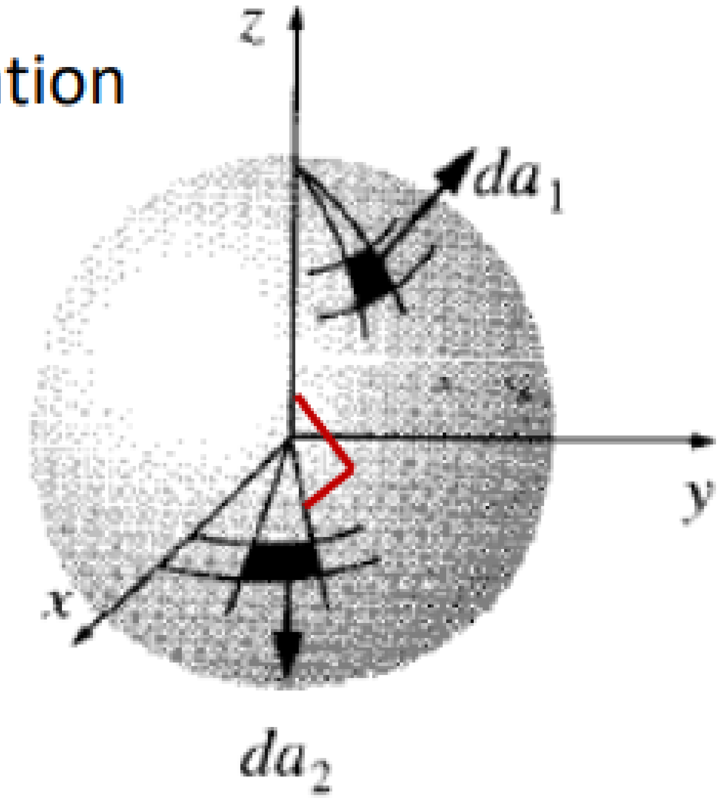
Volume element in spherical polar coordinates

$$dV = dl_r dl_\theta dl_\varphi$$

$$= r^2 sin\theta dr d\theta d\varphi$$

Area element in spherical polar co-ordinates

Area elements depend on the orientation of surfaces- cannot be generalized



If you are integrating over a **surface of a sphere (r = constant)**

$$\boldsymbol{da_1} = dl_\theta dl_\varphi \hat{\boldsymbol{r}} = r^2 sin\theta d\theta d\varphi \hat{\boldsymbol{r}}$$

If surface lies in **x-y plane** ($\boldsymbol{\theta} = \frac{\pi}{2}$) =const)

$$\boldsymbol{da_2} = dl_r dl_\varphi \widehat{\boldsymbol{\theta}} = r dr d\varphi \widehat{\boldsymbol{\theta}}$$

Derivatives

Consider a scalar function T and a vector function $\mathbf{v} = v_r\,\hat{r} + v_\theta\,\hat{\theta} + v_\phi\,\hat{\phi}$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl:

$$\begin{aligned}\nabla \times \mathbf{v} \;=\;& \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} \\ +\;& \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.\end{aligned}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

Cylindrical coordinates

- Any point in 3-dimensions can be located using (Perpendicular distance from the *z*-axis: s, Azimuthal angle: ϕ, Position on z-axis: z).

- Domain: $0 \leq s < \infty$, $0 \leq \phi < 2\pi$, $-\infty \leq z < \infty$

- Relation between cartesian coordinates (x,y,z) and cylindrical coordinates (s,ϕ,z):

$$x = s\cos\phi$$

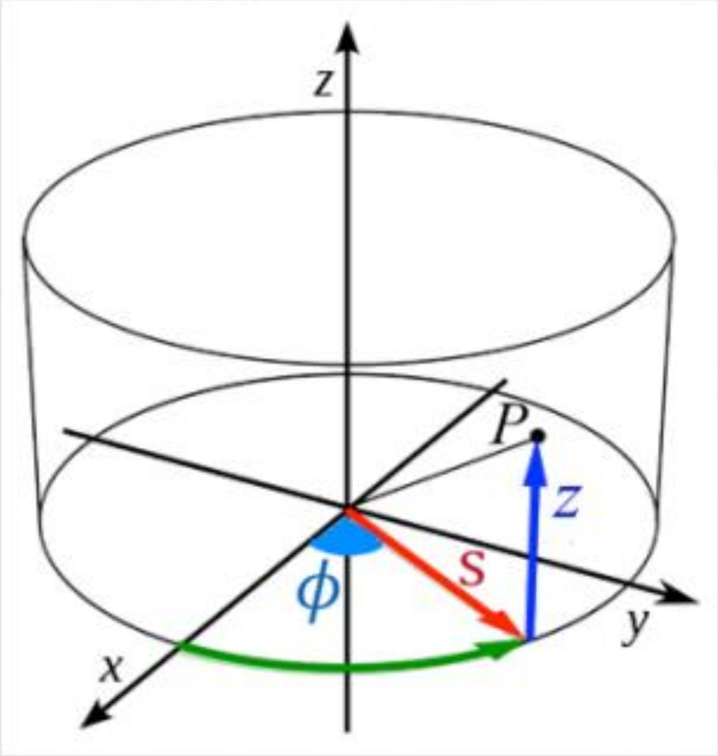
$$y = s\sin\phi$$

$$z = z$$

$$s = \sqrt{x^2 + y^2}$$

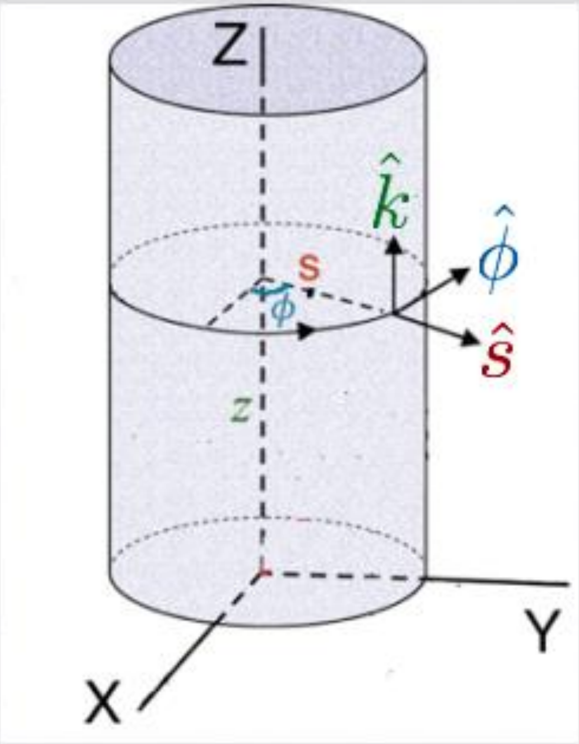
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$



Unit vectors for Cylindrical coordinates

- Unit vectors pointing in the direction of increase of s,ϕ,z respectively:
  $\hat{s}, \hat{\phi}, \hat{k}$
- They constitute an orthonormal basis set (just like i, j, k):
  $$\hat{s}\cdot\hat{s} = \hat{\phi}\cdot\hat{\phi} = \hat{k}\cdot\hat{k} = 1$$
  $$\hat{s}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{k} = \hat{k}\cdot\hat{s} = 0$$
- Any vector **V** can be expressed using these as:
  $$\mathbf{V} = V_s\,\hat{s} + V_\phi\,\hat{\phi} + V_z\,\hat{k}$$
- Expression in terms of i,j,k:
  $$\hat{s} = (\cos\phi)\hat{i} + (\sin\phi)\hat{j}$$
  $$\hat{\phi} = (-\sin\phi)\hat{i} + (\cos\phi)\hat{j}$$
  $$\hat{k} = \hat{k}$$



- The unit vectors $\hat{s}$, $\hat{\phi}$ are not constant--they change direction as we move in space. However , $\hat{k}$ as we know, is fixed.

Infinitesimal displacements

- Infinitesimal displacement in the $\hat{s}$ direction:

$$dl_s = ds$$

- Infinitesimal displacement in the $\hat{\phi}$ direction:

$$dl_\phi = s d\phi$$

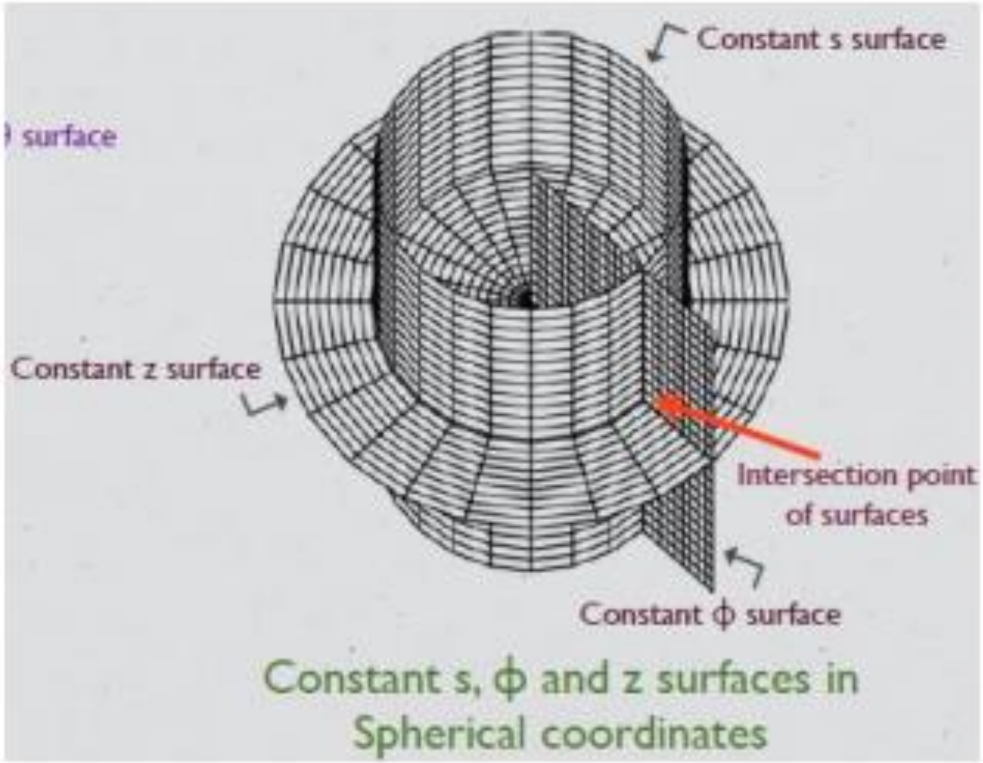
- Infinitesimal displacement in the $\hat{k}$ direction:

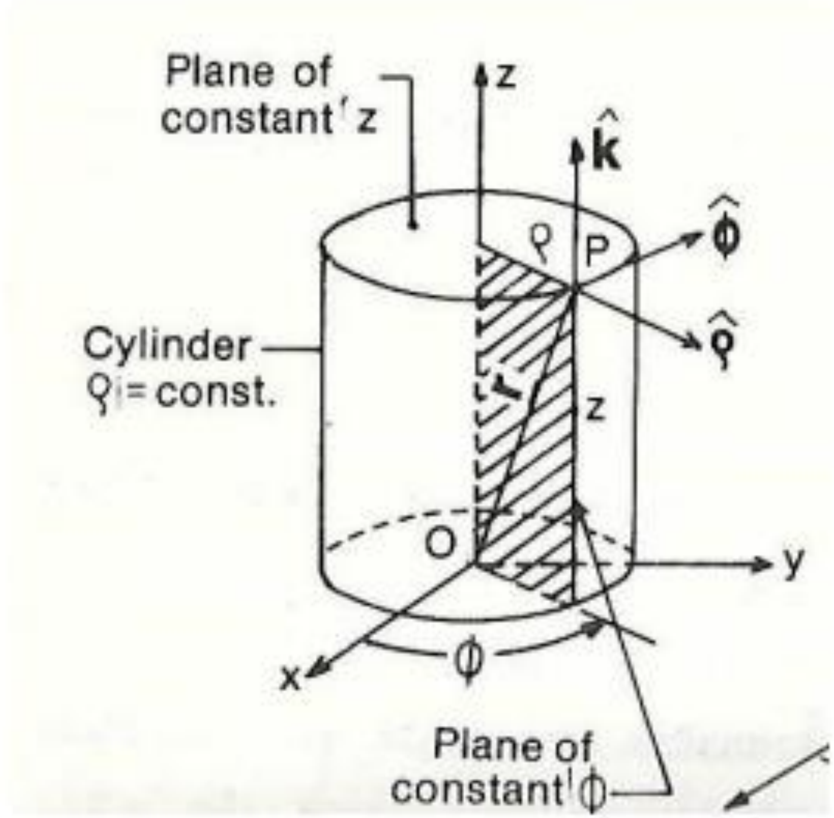
$$dl_z = dz$$

- General infinitesimal displacement:

$$\begin{aligned} d\mathbf{l} &= dl_s\,\hat{s} + dl_\phi\,\hat{\phi} + dl_z\,\hat{k} \\ &= ds\,\hat{s} + s d\phi\,\hat{\phi} + dz\,\hat{k} \end{aligned}$$

The co-ordinates (s, φ, z) or (ρ, φ, z) of a general point P are defined b
the intersection of three surfaces





(i) The surface of a right circular cylinder of radius ρ (or $'s'$ in lef
image) with its axis along the z-axis (surface of constant ρ or s)

(ii) The plane of constant φ

(iii) The plane of constant z.

Infinitesimal displacements

- Infinitesimal displacement in the $\hat{s}$ direction:

$$dl_s = ds$$ or, dρ

- Infinitesimal displacement in the $\hat{\phi}$ direction:

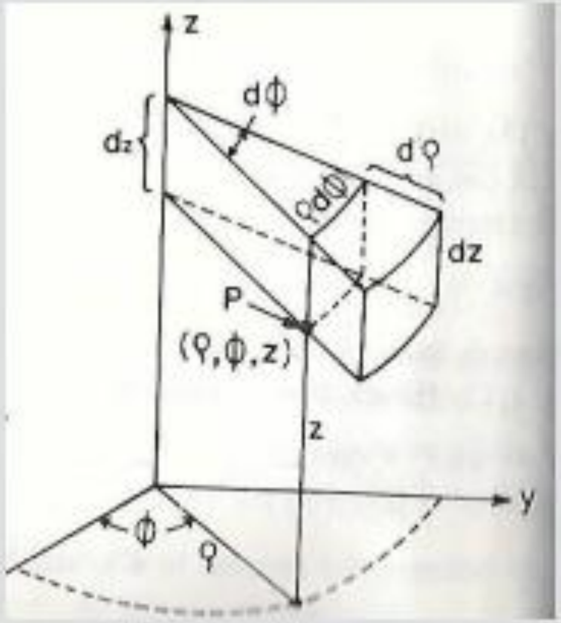
$$dl_\phi = s d\phi$$ or, ρ dϕ

- Infinitesimal displacement in the $\hat{k}$ direction:

$$dl_z = dz$$

- General infinitesimal displacement:

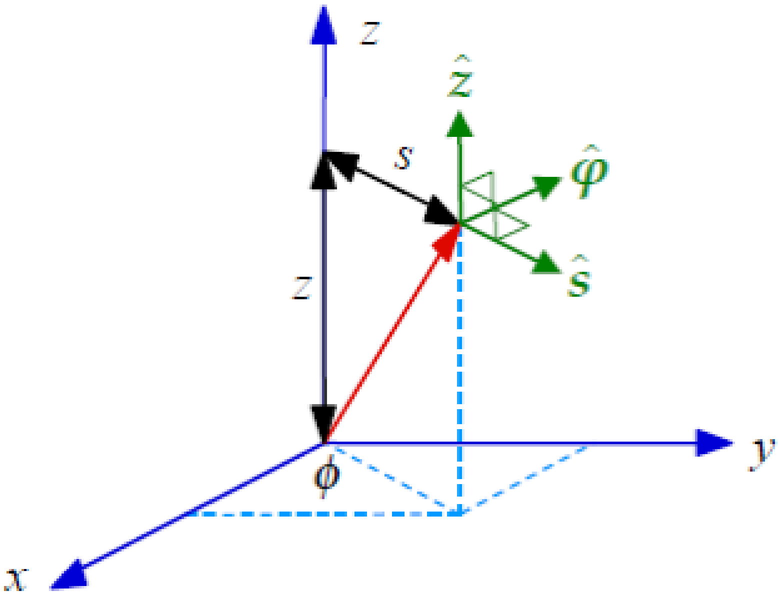
$$d\mathbf{l} = dl_s\,\hat{s} + dl_\phi\,\hat{\phi} + dl_z\,\hat{k}$$
$$= ds\,\hat{s} + s d\phi\,\hat{\phi} + dz\,\hat{k}$$



Volume element in cylindrical coordinates

$$dV = dl_s dl_\varphi dl_z$$
$$= s ds d\varphi dz$$

Area element in cylindrical co-ordinates



$$\boldsymbol{da} = \hat{z} s\, ds\, d\varphi$$

(top of the cylinder)

$$\boldsymbol{da} = \hat{s} s\, dz\, d\varphi$$

(wall of the cylinder)

Derivatives

Consider a scalar function T and a vector function $\mathbf{v} = v_s\,\hat{s} + v_\phi\,\hat{\phi} + v_z\,\hat{k}$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{k}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{k}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$