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PHY102: Introduction to Physics-II

Tutorial – 4

1. Evaluate the following integrals:

(a) $\int_{\text{all space}} (r^2 + \mathbf{r}\cdot\mathbf{a} + a^2)\delta^3(\mathbf{r}-\mathbf{a})\,d\tau$, where $\mathbf{a}$ is a fixed vector and a is its magnitude.

(b) $\int_{\mathcal{V}} |\mathbf{r}-\mathbf{b}|^2\delta^3(5\mathbf{r})\,d\tau$, where $\mathcal{V}$ is a cube of side 2, centered on the origin, and $\mathbf{b} = 4\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$.

(c) $\int_{\mathcal{V}}(r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4)\delta^3(\mathbf{r}-\mathbf{c})\,d\tau$, where $\mathcal{V}$ is a sphere of radius 6 about the origin, $\mathbf{c} = 5\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}$, and c is its magnitude.

(d) $\int_{\mathcal{V}} \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})\,d\tau$, where $\mathbf{d} = (1, 2, 3)$, $\mathbf{e} = (3, 2, 1)$, and $\mathcal{V}$ is a sphere of radius 1.5 centered at (2, 2, 2).

Solution:

(a) $a^2 + \mathbf{a}\cdot\mathbf{a} + a^2 = \boxed{3a^2.}$

(b) $\int(\mathbf{r}-\mathbf{b})^2\frac{1}{5^3}\delta^3(\mathbf{r})\,d\tau = \frac{1}{125}b^2 = \frac{1}{125}(4^2+3^2) = \boxed{\frac{1}{5}.}$

(c) $c^2 = 25+9+4 = 38 > 36 = 6^2$, so $\mathbf{c}$ is outside $\mathcal{V}$, so the integral is $\boxed{\text{zero.}}$

(d) $(\mathbf{e} - (2\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}))^2 = (1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + (-1)\,\hat{\mathbf{z}})^2 = 1+1 = 2 < (1.5)^2 = 2.25$, so $\mathbf{e}$ is inside $\mathcal{V}$, and hence the integral is $\mathbf{e}\cdot(\mathbf{d}-\mathbf{e}) = (3,2,1)\cdot(-2,0,2) = -6+0+2 = \boxed{\text{-4.}}$

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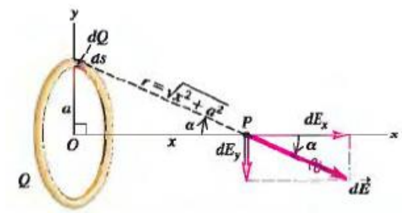
2. A ring shaped conductor with radius *a* carry a total charge *Q* uniformly distributed around it (see Figure below). Find the electric field at a point *P* that lies on the axis of the ring at a distance *x* from its centre. Also, what is the electric field at the centre of the ring? What will happen if *x*>>*a*? Given the infinitesimally small segment (*ds*) on the ring with charge *dQ* acts as a point charge source for electric field.



Sol.

The calculation of $\vec{E}$ is greatly simplified because the field point P is on the symmetry axis of the ring. Consider two segments at the top and bottom of the ring: The contributions $d\vec{E}$ to the field at P from these segments have the same x-component but opposite y-components. Hence the total y-component of field due to this pair of segments is zero. When we add up the contributions from all such pairs of segments, the total field $\vec{E}$ will have only a component along the ring's symmetry axis (the x-axis), with no component perpendicular to that axis (that is, no y-component or z-component). So the field at P is described completely by its x-component E_x.

To calculate E_x, note that the square of the distance r from a ring segment to the point P is $r^2 = x^2 + a^2$. Hence the magnitude of this segment's contribution $d\vec{E}$ to the electric field at P is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}$$

Using $\cos\alpha = x/r = x/(x^2 + a^2)^{1/2}$, the x-component dE_x of this field is

$$dE_x = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2 + a^2}\frac{x}{\sqrt{x^2 + a^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{x\,dQ}{(x^2 + a^2)^{3/2}}$$

To find the *total* x-component E_x of the field at P, we integrate this expression over all segments of the ring:

$$E_x = \int \frac{1}{4\pi\epsilon_0} \frac{x\,dQ}{(x^2 + a^2)^{3/2}}$$

Since x does not vary as we move from point to point around the ring, all the factors on the right side except dQ are constant and can be taken outside the integral. The integral of dQ is just the total charge Q, and we finally get

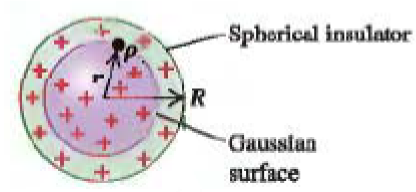
$$\vec{E} = E_x \hat{\imath} = \frac{1}{4\pi\epsilon_0} \frac{Qx}{(x^2 + a^2)^{3/2}} \hat{\imath} \qquad (21.8)$$

EVALUATE: Our result for $\vec{E}$ shows that at the center of the ring $(x = 0)$ the field is zero. We should expect this; charges on opposite sides of the ring would push in opposite directions on a test charge at the center, and the forces would add to zero. When the field point P is much farther from the ring than its size (that is, $x \gg a$), the denominator in Eq. (21.8) becomes approximately equal to x^3, and the expression becomes approximately

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{x^2} \hat{\imath}$$

In other words, when we are so far from the ring that its size a is negligible in comparison to the distance x, its field is the same as that of a point charge. To an observer far from the ring, the ring would appear like a point, and the electric field reflects this.

3. Positive charge Q is distributed uniformly throughout the volume of an insulating sphere with radius R. Find the magnitude of the electric field at a point P a distance r from the centre of the sphere.



EXECUTE: From symmetry the magnitude E of the electric field has the same value at every point on the Gaussian surface, and the direction of $\vec{E}$ is radial at every point on the surface, so $E_\perp = E$. Hence the total electric flux through the Gaussian surface is the product of E and the total area of the surface $A = 4\pi r^2$, that is, $\Phi_E = 4\pi r^2 E$.

The amount of charge enclosed within the Gaussian surface depends on the radius r. Let's first find the field magnitude *inside* the charged sphere of radius R; the magnitude E is evaluated at the radius of the Gaussian surface, so we choose $r < R$. The volume charge density ρ is the charge Q divided by the volume of the entire charged sphere of radius R:

$$\rho = \frac{Q}{4\pi R^3/3}$$

The volume V_{encl} enclosed by the Gaussian surface is $\frac{4}{3}\pi r^3$, so the total charge Q_{encl} enclosed by that surface is

$$Q_{encl} = \rho V_{encl} = \left(\frac{Q}{4\pi R^3/3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

Then Gauss's law,

$$4\pi r^2 E = \frac{Q}{\epsilon_0}\frac{r^3}{R^3} \quad \text{or}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3} \qquad \text{(field inside a uniformly charged sphere)}$$

The field magnitude is proportional to the distance r of the field point from the center of the sphere. At the center $(r = 0)$, $E = 0$.

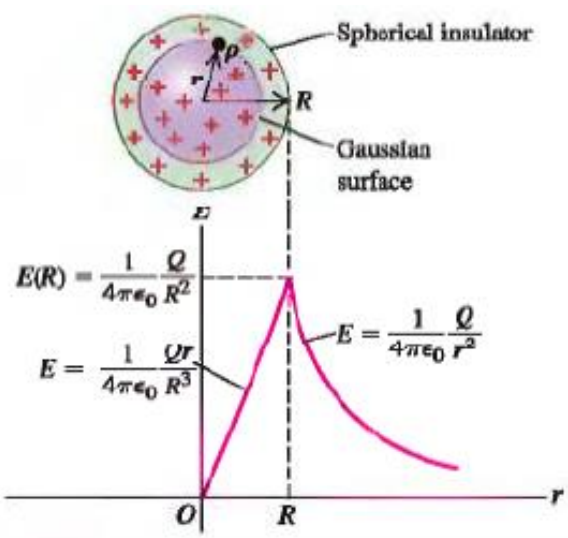
To find the field magnitude *outside* the charged sphere, we use a spherical Gaussian surface of radius $r > R$. This surface encloses the entire charged sphere, so $Q_{encl} = Q$, and Gauss's law gives

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \quad \text{or}$$

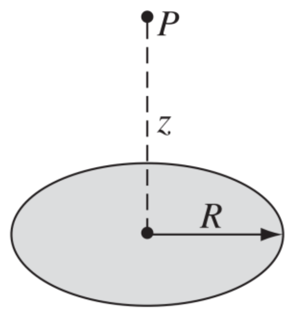
$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \qquad \text{(field outside a uniformly charged sphere)}$$

For *any* spherically symmetric charged body the electric field outside the body is the same as though the entire charge were concentrated at the center.

Figure shows a graph of E as a function of r for this problem. For $r < R$, E is directly proportional to r, and for $r > R$, E varies as $1/r^2$. If the charge is negative instead of positive, $\vec{E}$ is radially *inward* and Q in the expressions for E is interpreted as the magnitude (absolute value) of the charge.



4. Find the electric field a distance z above the center of a flat circular disk of radius R that carries a uniform surface charge σ. What does your formula give in the limit R → ∞? Also check the case z ≫ R.



Solution

Break it into rings of radius r, and thickness dr, and use the E of each ring.

Total charge of a ring is

$$\sigma \cdot 2\pi r \cdot dr = \lambda \cdot 2\pi r,$$

so $\lambda = \sigma dr$ is the "line charge" of each ring.

$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma dr)2\pi r z}{(r^2+z^2)^{3/2}}; \quad E_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R \frac{r}{(r^2+z^2)^{3/2}}dr.$$

$$\boxed{\mathbf{E}_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.}$$

For $R \gg z$ the second term $\to 0$, so $\mathbf{E}_{\text{plane}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma\hat{\mathbf{z}} = \boxed{\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.}$

For $z \gg R$, $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)$, so $[\,] \approx \frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3} = \frac{R^2}{2z^3}$,
and $E = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{2z^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$, where $Q = \pi R^2\sigma$. ✓