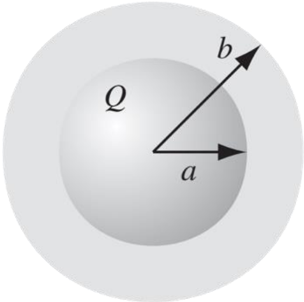
PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 22

Dielectrics

P1 A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity. Find the potential at the center (relative to infinity).



Solution

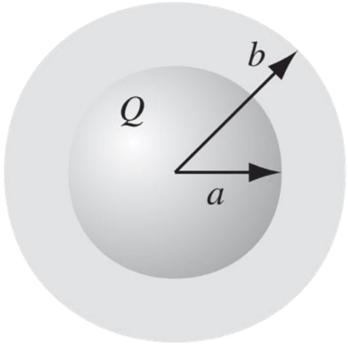
To compute V , we need to know **E**; to find **E**, we might first try to locate the bound charge; we could get the bound charge from **P**, but we can't calculate **P** unless we already know **E**.
So to find **E**, we start from **D**, and is related to free charge.

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

Inside the metal sphere, of course, **E = P = D = 0**.) Once we know **D**, it is a trivial matter to obtain **E.**

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for} \quad a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for} \quad r > b. \end{cases}$$

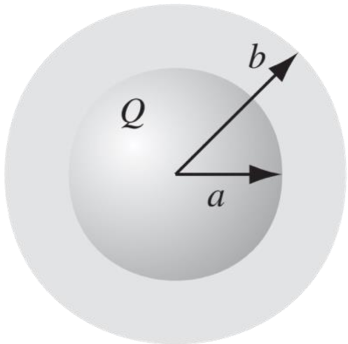


The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{b} \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_{b}^{a} \left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right).$$

As it turns out, it was not necessary for us to compute the polarization or the bound charge explicitly, though this can easily be done:



$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon r^2} \hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$

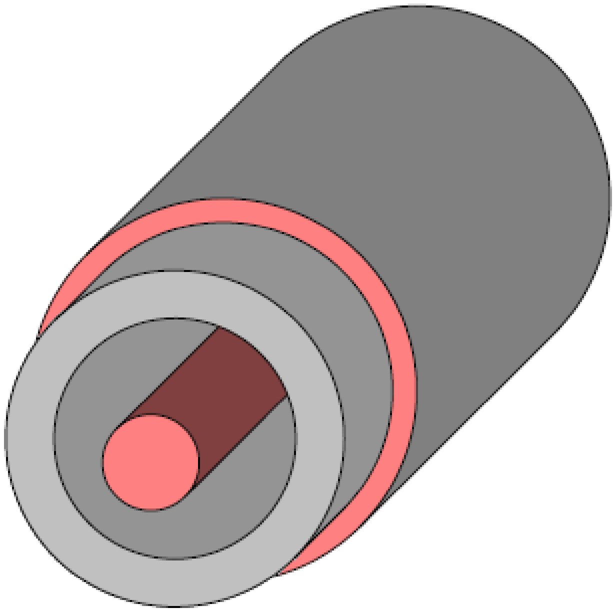
while

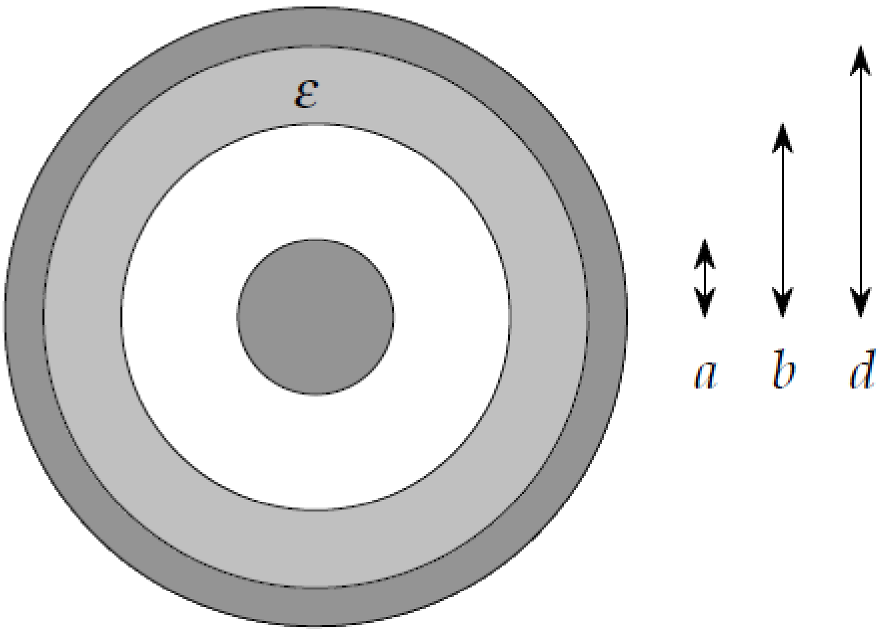
$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi \epsilon b^2}, & \text{at the outer surface,} \\[2ex] \dfrac{-\epsilon_0 \chi_e Q}{4\pi \epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

Notice that the surface bound charge at a is *negative* ($\hat{\mathbf{n}}$ points outward *with respect to the dielectric*, which is $+\hat{\mathbf{r}}$ at b but $-\hat{\mathbf{r}}$ at a). This is natural, since the charge on the metal sphere attracts its opposite in all the dielectric molecules. It is this layer of negative charge that reduces the field, within the dielectric, from $1/4\pi\epsilon_0(Q/r^2)\hat{\mathbf{r}}$ to $1/4\pi\epsilon(Q/r^2)\hat{\mathbf{r}}$. In this respect, a dielectric is rather like an imperfect conductor: on a *conducting* shell the induced surface charge would be such as to cancel the field of Q *completely* in the region $a < r < b$; the dielectric does the best it can, but the cancellation is only partial.

Dielectric filled in Co-axial Cable

Griffiths problem 4.21: A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius d. The space between is partially filled (from b out to d) with material of dielectric constant ε, as shown. Find the capacitance per unit length of this cable.





Assume the charge per unit length on the inner conductor to be λ, and that on the outer to be $-\lambda$. Then construct a Gaussian cylinder at radius s about the center conductor. The fields must be purely radial; there's no flux through the circular ends:

$$\oint \mathbf{D} \cdot d\mathbf{a} = 4\pi Q_f$$

$$D 2\pi s\ell = 4\pi\lambda\ell \quad \Rightarrow \quad \mathbf{D} = \frac{2\lambda}{s}\hat{\mathbf{s}} \quad .$$

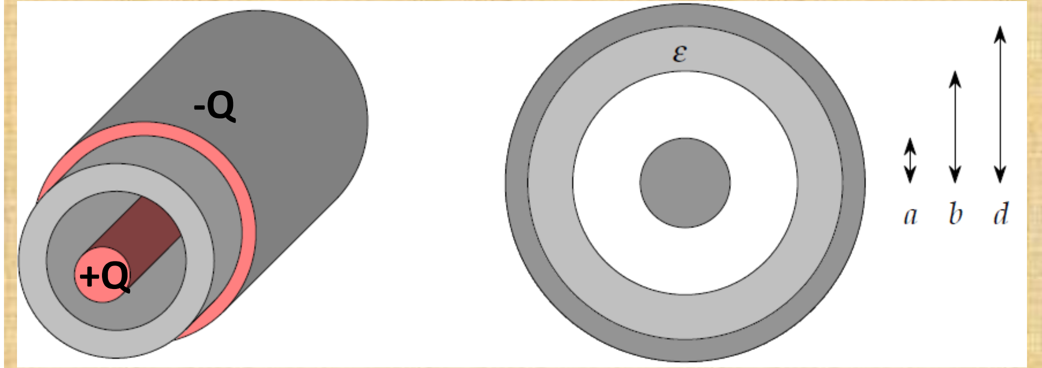
Thus

$$\mathbf{E} = \mathbf{D} = \frac{2\lambda}{s}\hat{\mathbf{s}} \quad (a \le s < b) \quad ,$$

$$= \frac{\mathbf{D}}{\varepsilon} = \frac{2\lambda}{\varepsilon s}\hat{\mathbf{s}} \quad (b \le s < d) \quad .$$

In Gaussian

Dielectrics



Put +Q charge in inner conductor with radius 'a' and –Q charge in conductor with radius 'd'

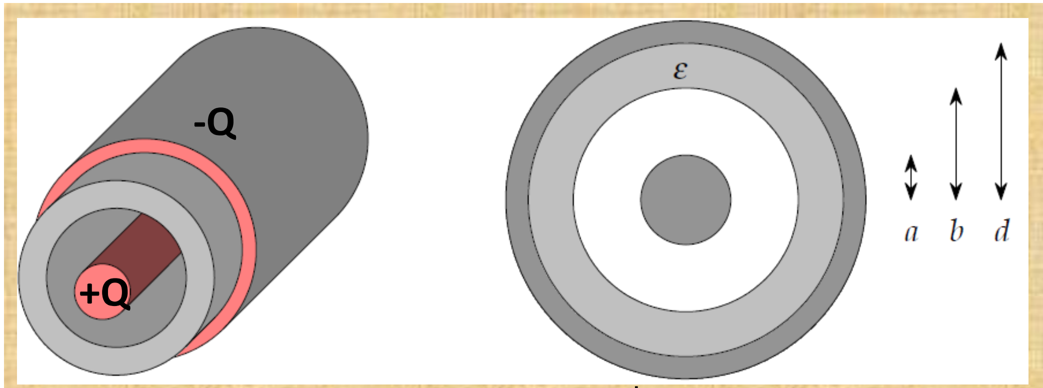
$$\mathbf{D} = \frac{Q}{2\pi s L}\hat{\mathbf{s}}$$

$$E = \frac{D}{\epsilon} \quad \epsilon = \epsilon_0 \epsilon_r$$

Note that $E = 0$ for $s < a$ (metallic conductor)

$\underline{a < s < b}$	$\underline{b < s < d}$
$E = \frac{D}{\epsilon_0 \epsilon_r} = \frac{D}{\epsilon_0}$	$E = \frac{D}{\epsilon}$
($\epsilon_r = 1$, in vaccum)	
$\boldsymbol{E} = \frac{Q}{2\pi \epsilon_0 s L}\hat{\mathbf{s}}$	$\boldsymbol{E} = \frac{Q}{2\pi \epsilon s L}\hat{s}$

Dielectrics



Put +Q charge in inner conductor with radius ‘a’ and –Q charge in conductor with radius ‘d’

$$V(a) - V(d) = V_+ - V_- = -\int_-^+ \boldsymbol{E.dl} = -\int_d^a \boldsymbol{E.dl}$$

$$= -\int_d^a \boldsymbol{E.dl} = \int_a^d \boldsymbol{E.dl} = \int_a^b \boldsymbol{E.dl} + \int_b^d \boldsymbol{E.dl}$$

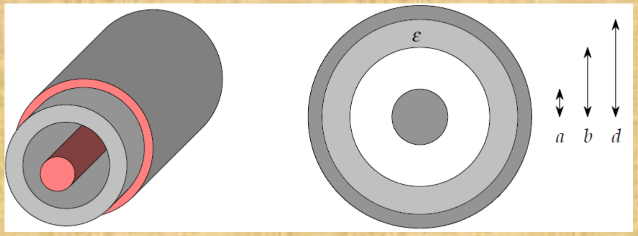
$$V = \int_a^b E.dl + \int_b^d E.dl$$

$$V = \frac{Q}{2\pi\epsilon_0 L}\left[\int_a^b \frac{ds}{s} + \frac{1}{({}^{\epsilon}/_{\epsilon_0})}\int_b^d \frac{ds}{s}\right]$$

$$V = \frac{Q}{2\pi\epsilon_0 L}\left[\ln(\frac{b}{a}) + \frac{1}{(\epsilon_r)}\ln(\frac{d}{b})\right]$$

$$C = \frac{Q}{V} = 2\pi\epsilon_0 \frac{1}{\left\{\ln(\frac{b}{a}) + \frac{1}{(\epsilon_r)}\ln(\frac{d}{b})\right\}}$$

(Capacitance per unit length)



Calculation of bound charges

$$\rho_b = -\nabla \cdot P \qquad \sigma_b = P \cdot \hat{n}$$

Region $\underline{b < s < d}$

$$\boldsymbol{P} = \boldsymbol{D} - \epsilon_0 \boldsymbol{E} = \frac{Q}{2\pi sL} - \epsilon_0 \frac{Q}{2\pi\epsilon sL} = \frac{Q}{2\pi sL}\left[1 - \frac{1}{\epsilon_r}\right] \qquad \boldsymbol{P} = \frac{Q}{2\pi sL}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\hat{\boldsymbol{s}}$$

$$\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\frac{1}{s}\frac{\partial}{\partial s}(sP_s) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{1}{s}\right) \times const = 0$$

$$\sigma_b = (\boldsymbol{P} \cdot \hat{\boldsymbol{n}})_{s=d} = \left[\frac{Q}{2\pi sL}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}\right]_{s=d}$$

$$= \frac{Q}{2\pi dL}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)$$

$$\sigma_b = (P \cdot -\hat{s})_{s=b} = \left[-\frac{Q}{2\pi bL}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}\right]_{s=b}$$

$$= -\frac{Q}{2\pi bL}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right)$$

Energy in Dielectric Systems

Work to charge up a capacitor

$$W = \frac{1}{2} C V^2$$

If the capacitor is filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of the dielectric constant,

$$C = \epsilon_r C_{\mathrm{vac}},$$

The work necessary to charge a dielectric-filled capacitor is increased by the same factor. More (free) charge needs to be pumped, to achieve a given potential, because part of the field is canceled off by the bound charges.

Energy in Dielectric Systems

General formula for the energy stored in any electrostatic system

$$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau$$

The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2} \int \epsilon_r E^2 \, d\tau = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau,$$

in the presence of linear dielectrics.

Proof

Suppose the dielectric material is fixed in position, and we bring in the free charge, a bit at a time

As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will change and with it the bound charge distribution, but we're interested only in the work done on the incremental free charge:

$$\Delta W = \int (\Delta\rho_f) V \, d\tau$$

Since $\nabla \cdot \mathbf{D} = \rho_f$, $\Delta\rho_f = \nabla \cdot (\Delta\mathbf{D})$, where $\Delta\mathbf{D}$ is the resulting change in $\mathbf{D}$, so

$$\Delta W = \int [\nabla \cdot (\Delta\mathbf{D})] V \, d\tau.$$

Energy in Dielectric Systems

Now

$$\nabla \cdot [(\Delta \mathbf{D})V] = [\nabla \cdot (\Delta \mathbf{D})]V + \Delta \mathbf{D} \cdot (\nabla V),$$

and hence (integrating by parts):

$$\Delta W = \int \nabla \cdot [(\Delta \mathbf{D})V]\, d\tau + \int (\Delta \mathbf{D}) \cdot \mathbf{E}\, d\tau.$$

The divergence theorem turns the first term into a surface integral, which vanishes if we integrate over all space. Therefore, the work done is equal to

$$\Delta W = \int (\Delta \mathbf{D}) \cdot \mathbf{E}\, d\tau.$$

The above equation can be applied to any type of material

Energy in Dielectric Systems

Now, if the medium is a linear dielectric, then

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\frac{1}{2}\Delta(\mathbf{D}\cdot\mathbf{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\mathbf{E})\cdot\mathbf{E} = (\Delta\mathbf{D})\cdot\mathbf{E}$$

Thus

$$\Delta W = \int (\Delta\mathbf{D})\cdot\mathbf{E}\,d\tau$$

$$\Delta W = \Delta\left(\frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau\right)$$

The total work done, then, as we build the free charge up from zero to the final configuration, is

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau$$

Energy in Dielectric Systems

General formula for the energy stored in any electrostatic system

$$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau.$$

- Bring in all the charges (free and bound), one by one, with tweezers, and glue each one down in its proper final location and assemble the system.

- This will not include the work involved in stretching and twisting the dielectric molecules

Total energy

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau$$

- With the unpolarized dielectric in place, we bring in the free charges, one by one, allowing the dielectric to respond as it sees fit and assemble the system.

- In this case the “spring” energy is included, albeit indirectly, because the force you must apply to the free charge depends on the disposition of the bound charge; by moving the free charge, the “springs” are stretching automatically.

Optional

Problem

A sphere of radius R is filled with material of dielectric constant ε_r and uniform embedded free charge ρ_f . What is the energy of this configuration?

Solution

From Gauss's law

$$\mathbf{D}(r) = \begin{cases} \dfrac{\rho_f}{3}\mathbf{r} & (r < R), \\ \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

So the electric field is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < R), \\ \dfrac{\rho_f}{3\epsilon_0}\dfrac{R^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

Energy in Dielectric Systems

The purely electrostatic energy is

$$W = \frac{\epsilon_0}{2} \int \epsilon_r E^2 \, d\tau$$

$$W_1 = \frac{\epsilon_0}{2} \left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r} \right)^2 \int_0^R r^2 \, 4\pi r^2 \, dr + \left(\frac{\rho_f}{3\epsilon_0} \right)^2 R^6 \int_R^\infty \frac{1}{r^4} \, 4\pi r^2 \, dr \right]$$
$$= \frac{2\pi}{9\epsilon_0} \rho_f^2 R^5 \left(\frac{1}{5\epsilon_r^2} + 1 \right).$$

Total energy

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau$$

$$W_2 = \frac{1}{2} \left[\left(\frac{\rho_f}{3} \right) \left(\frac{\rho_f}{3\epsilon_0\epsilon_r} \right) \int_0^R r^2 \, 4\pi r^2 \, dr + \left(\frac{\rho_f R^3}{3} \right) \left(\frac{\rho_f R^3}{3\epsilon_0} \right) \int_R^\infty \frac{1}{r^4} \, 4\pi r^2 \, dr \right]$$
$$= \frac{2\pi}{9\epsilon_0} \rho_f^2 R^5 \left(\frac{1}{5\epsilon_r} + 1 \right).$$

$W_1 < W_2$- because W_1 does not include the energy involved in stretching the molecules.

Energy in Dielectric Systems

Let's check that W_2 is the work done on the *free* charge in assembling the system. We start with the (uncharged, unpolarized) dielectric sphere, and bring in the free charge in infinitesimal installments (dq), filling out the sphere layer by layer. When we have reached radius r', the electric field is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < r'), \\[2ex] \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r' < r < R), \\[2ex] \dfrac{\rho_f}{3\epsilon_0}\dfrac{r'^3}{r^2}\hat{\mathbf{r}} & (r > R). \end{cases}$$

The work required to bring the next dq in from infinity to r' is

$$\begin{aligned} dW &= -dq\left[\int_\infty^R \mathbf{E}\cdot d\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot d\mathbf{l}\right] \\ &= -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}\,dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'}\frac{1}{r^2}\,dr\right] \\ &= \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right]dq. \end{aligned}$$

This increases the radius (r'):

$$dq = \rho_f 4\pi r'^2 \, dr',$$

so the *total* work done, in going from $r' = 0$ to $r' = R$, is

$$\begin{aligned} W &= \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1 - \frac{1}{\epsilon_r}\right)\int_0^R r'^5 \, dr' + \frac{1}{\epsilon_r}\int_0^R r'^4 \, dr'\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5 \left(\frac{1}{5\epsilon_r} + 1\right) = W_2. \checkmark \end{aligned}$$

Evidently the energy "stored in the springs" is

$$W_{\text{spring}} = W_2 - W_1 = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5 \left(\epsilon_r - 1\right).$$

Energy in Dielectric Systems

Energy stored in the spring

Picture the dielectric as a collection of tiny proto-dipoles, each consisting of **+q** and **−q** attached to a spring of constant **k** and equilibrium length 0, so in the absence of any field the positive and negative ends coincide.

Let dτ be the volume assigned to each proto-dipole

With the field turned on, the electric force on the free end is balanced by the spring force; the charges are separate by a distance d:

$$qE = kd.$$

In our case

$$\mathbf{E} = \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r}.$$

Energy in Dielectric Systems

The resulting dipole moment is $p = qd$, and the polarization is $P = p/d\tau$, so

$$k = \frac{\rho_f}{3\epsilon_0\epsilon_r d^2} Pr\, d\tau.$$

The energy of this particular spring is

$$dW_{\text{spring}} = \frac{1}{2}kd^2 = \frac{\rho_f}{6\epsilon_0\epsilon_r} Pr\, d\tau,$$

and hence the total is

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r} \int Pr\, d\tau.$$

Now

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \epsilon_0\chi_e \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} = \frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r}\mathbf{r},$$

so

$$W_{\text{spring}} = \frac{\rho_f}{6\epsilon_0\epsilon_r} \frac{(\epsilon_r - 1)\rho_f}{3\epsilon_r} 4\pi \int_0^R r^4\, dr = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5 (\epsilon_r - 1)$$

energy “stored in the springs”