PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 16

<u>Boundary Conditions and Uniqueness Theorems</u>

Method of Images

Multipole Expansion

Boundary Conditions and Uniqueness Theorems

With the boundary conditions given in some way, does the problem have no solution, one solution, or more than one solution?

For Laplacian equation

$$\nabla^2 V = 0.$$

First uniqueness theorem: The solution to Laplace's equation in some volume *V* is uniquely determined if V is specified on the boundary surface S.

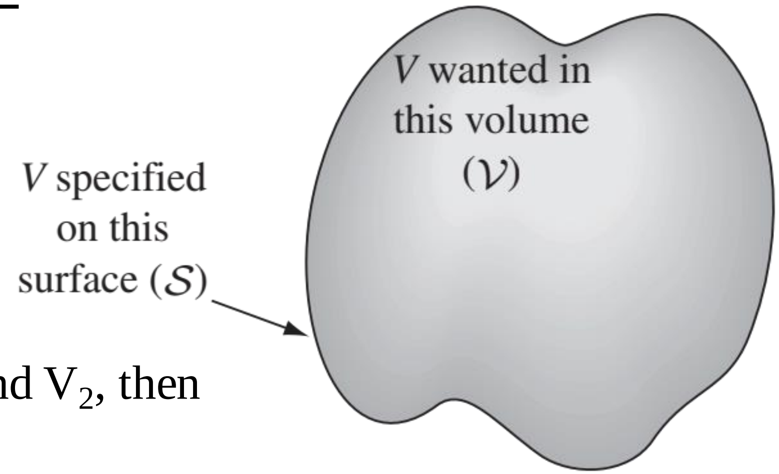
Electric Potential

Boundary Conditions and Uniqueness Theorems

Proof

We have

$$\nabla^2 V = 0.$$



Let the Laplacian equation have two solutions V_1 and V_2, then

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0,$$

both of which assume the specified value on the surface.

We want to prove that they must be equal ($V_1 = V_2$).

Let

$$V_3 \equiv V_1 - V_2.$$

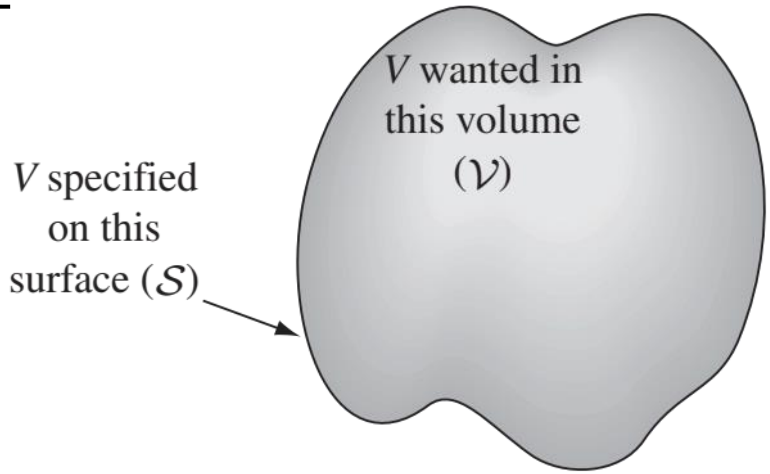
This obeys Laplace's equation,

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

and it takes the value zero on all boundaries (since V_1 and V_2 are equal there).

Boundary Conditions and Uniqueness Theorems

Proof



But Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So the maximum and minimum of V_3 are both zero.

Therefore V_3 must be zero everywhere, and hence

$$V_1 = V_2$$

In case of charges inside γ, we would use the Poisson's equation

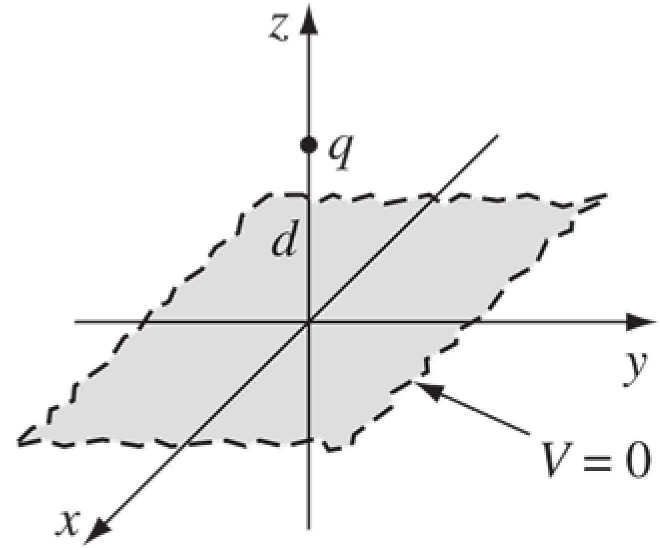
$$\nabla^2 V_1 = -\frac{1}{\epsilon_0}\rho \quad \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho + \frac{1}{\epsilon_0}\rho = 0$$

Again V_3 satisfies Laplace's equation with zero values at boundaries (since $V_1 = V_2$ there) and so V_3 must be zero everywhere.

The Method of images

Suppose a point charge q is held a distance **d** above an infinite grounded conducting plane.



What is the potential in the region above the plane?

It's not just $(1/4\pi\epsilon_0)q/\mathscr{r}$, for q will induce a certain amount of negative charge on the nearby surface of the conductor; the total potential is due in part to q directly, and in part to this induced charge.

But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

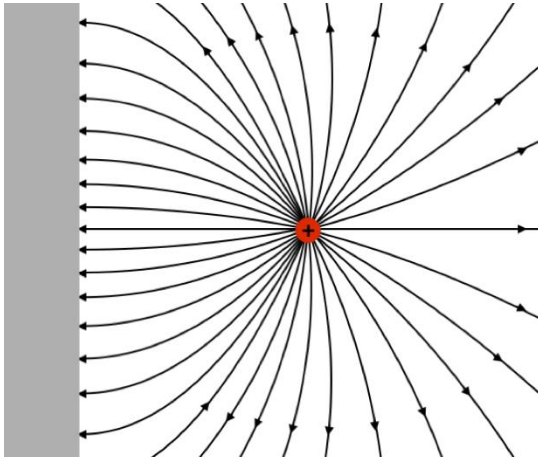
From a mathematical point of view, our problem is to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded), and
2. $V \to 0$ far from the charge (that is, for $x^2 + y^2 + z^2 \gg d^2$).

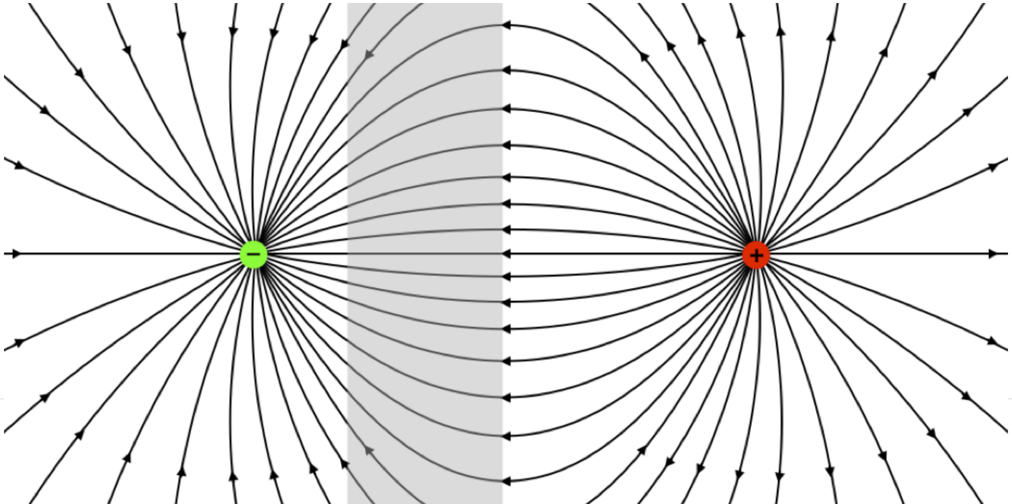
Conductors

The Method of images

Field of a Point Charge Near a Flat Conductor



Field of Point Charge + Flat Conductor = Half a Dipole Field

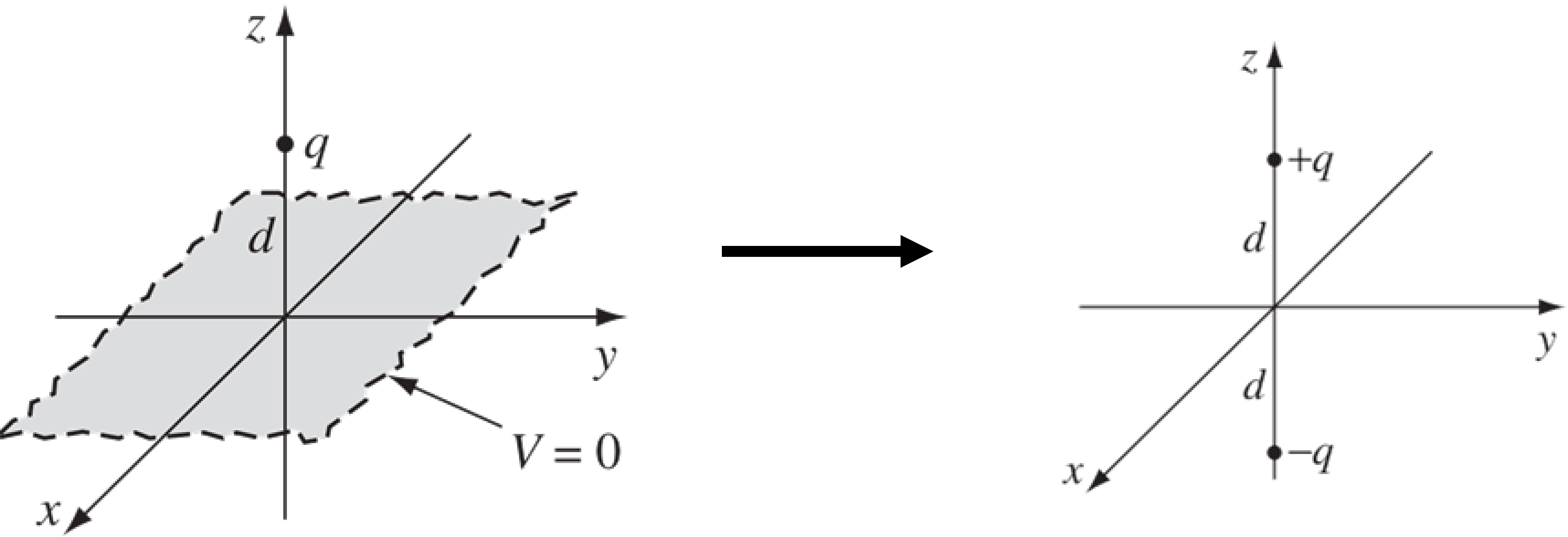


It turns out that this is more than a resemblance of fields. In the region outside the conductor, the field is precisely the same as if the conductor were removed and a second point charge of equal magnitude and opposite sign was symmetrically-placed on the opposite side of where the conducting surface was.

Charge + image charge
(equivalent charge configuration)

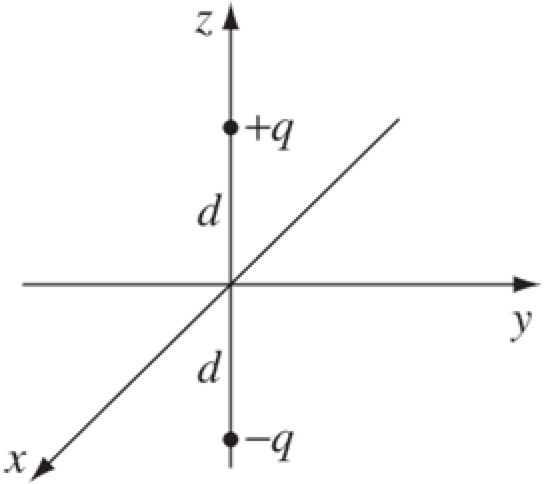
The Method of images

One technique for solving the electric potential of a charge near a conductor is called ***the method of image charges***. This techniques involves “replacing” the conductor with one of more “image charges”, which, together with the total charge, produce an electric potential that satisfies the boundary conditions of the original system. Then, by uniqeness theorem, this potential is the only solution.



This new configuration consists of *two* point charges, **+*q*** at **(0, 0, d)** and **-q** at **(0, 0, -d)**, and *no* conducting plane.

The Method of images



This new configuration consists of *two* point charges, **+*q*** at **(0, 0, d)** and **-q** at **(0, 0, -d)**, and *no* conducting plane.

For this configuration, the potential is:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right].$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

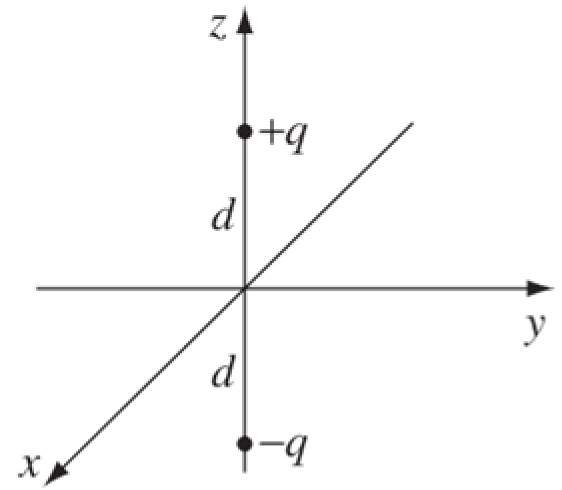
1. $V = 0$ when $z = 0$,
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$,

and the only charge in the region $z > 0$ is the point charge $+q$ at $(0, 0, d)$

Conductors

The Method of images

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right].$$



The electric field at the conductor surface (which is perpendicular to the surface) can then be found from the gradient of the potential:

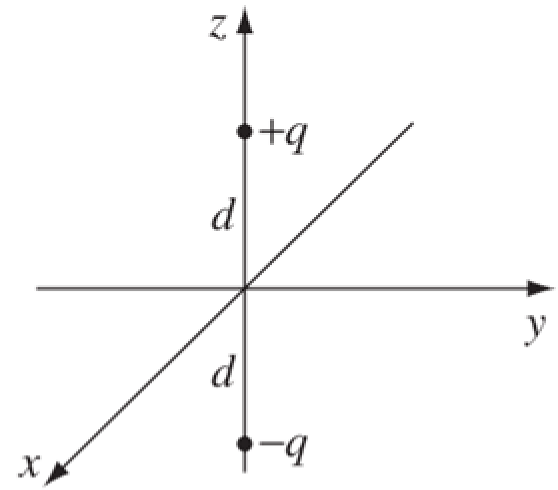
$$E = -\left(\frac{\partial V}{\partial z}\right)_{z=0}$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

$$E = -\frac{qd}{2\pi\varepsilon_0(x^2+y^2+d^2)^{\frac{3}{2}}}$$

The Method of images

Induced Surface Charge on the conductor



$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0}$$

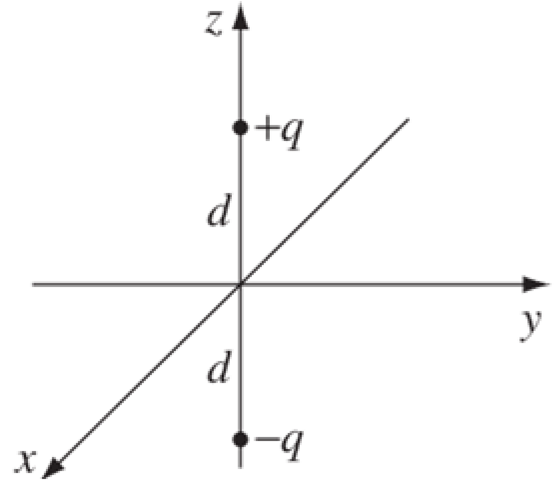
$$\sigma(x, y) = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

Conductors

The Method of images

Induced Surface Charge



The total induced charge

$$Q = \int \sigma \, da$$

This integral, over the xy plane, could be done in Cartesian coordinates, with $da = dx\,dy$, but it's a little easier to use polar coordinates (r, ϕ), with $r^2 = x^2 + y^2$ and $da = r\,dr\,d\phi$. Then

$$\sigma(r) = \frac{-qd}{2\pi(r^2 + d^2)^{3/2}}$$

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2 + d^2)^{3/2}} r\,dr\,d\phi = \left.\frac{qd}{\sqrt{r^2 + d^2}}\right|_0^{\infty} = -q.$$

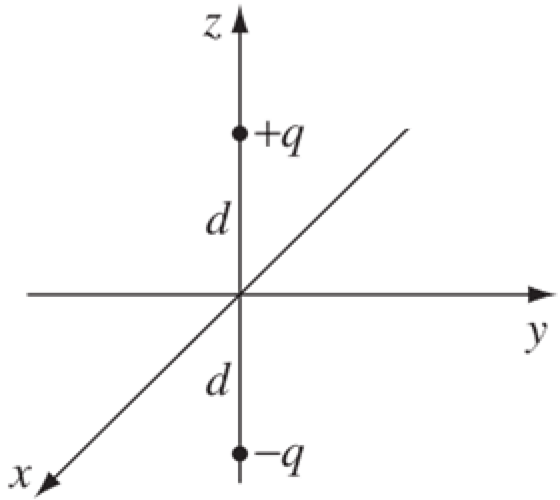
The total charge induced on the plane is $-q$

The Method of images

Force and Energy

The charge q is attracted toward the plane, because of the negative induced charge. Let's calculate the force of attraction. Since the potential in the vicinity of q is the same as in the analog problem (the one with $+q$ and $-q$ but no conductor), so also is the field and, therefore, the force:

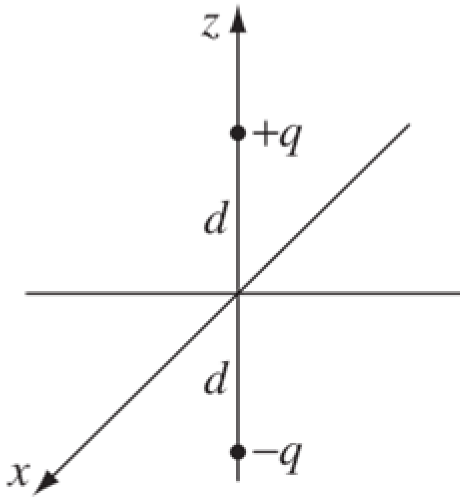
$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}.$$



It is easy to get carried away, and assume that everything is the same in the two problems. **Energy, however, is not the same.**

The Method of images

Force and Energy



With the two point charges and no conductor

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i). \qquad W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}.$$

For a single charge and conducting plane, the energy is determined by calculating the work required to bring q in from infinity.

$$W = \int_{\infty}^{d} \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}\,dz$$

$$= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Bigg|_{\infty}^{d} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}.$$

Energy is half!

As I move q toward the conductor, I do work *only on* q. It is true that induced charge is moving in over the conductor, but this costs me nothing, since the whole conductor is at potential zero. By contrast, if I simultaneously bring in *two* point charges (with no conductor), I do work on *both* of them, and the total is (again) twice as great.

Multipole Expansion

Approximate Potential at a Large Distance

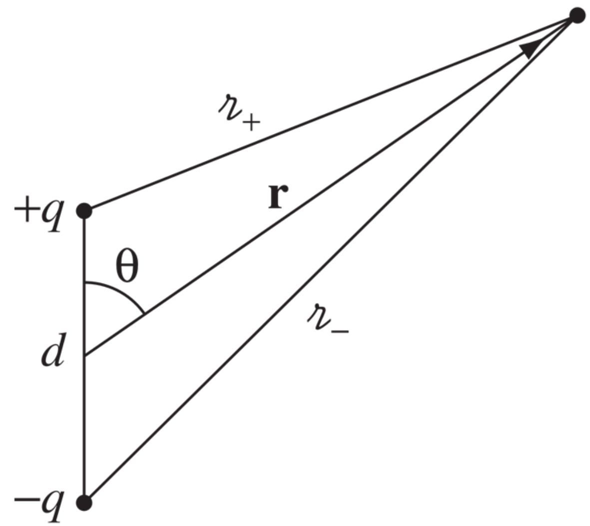
Consider a localized charge distribution, which may comprise continuous distribution of charge, or discrete charges, or both. If we look at this system from very far away (*distance r, which is large compared to the size of the charge distribution*), we won't be able to "resolve" the constituents. It will appear as a point charge with magnitude equal to the total charge (Q) contained in the system. The electric potential can then be approximated by

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{Q}{r}.$$

This decays as $1/r$. If Q is zero, then we get the result as zero, signifying that the potential is very small at that point and no information can be obtained under this crude approximation of replacing the charge system by a point charge. To get information about the potential we will have to invoke a better approximation and go beyond this ***Monopole approximation***.

Potential due to dipoles

A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.



Solution

Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$ Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right),$$

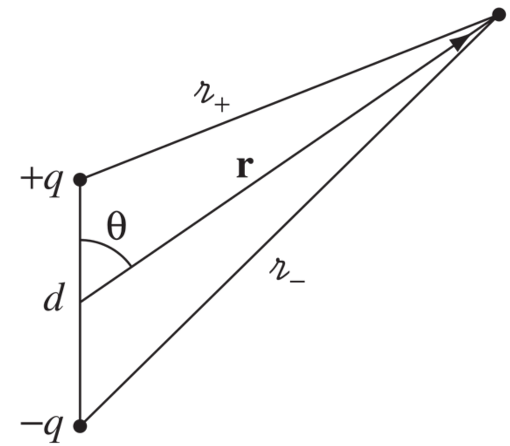
and (from the law of cosines)

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right).$$

Potential due to dipoles

We're interested in the régime $r \gg d$, so the third term is negligible, and the binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).$$



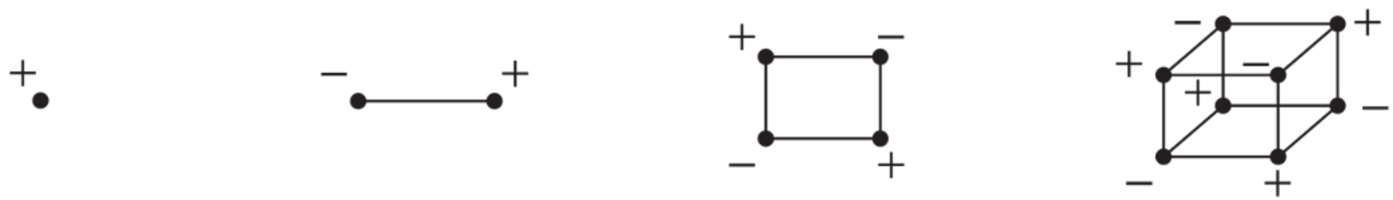
Thus

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta,$$

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$

Potential due to dipoles

The potential of a dipole goes like $1/r^2$ at large r; as we might have anticipated, it falls off more rapidly than the potential of a point charge. If we put together a pair of equal and opposite *dipoles* to make a **quadrupole**, the potential goes like $1/r^3$; for back-to-back *quadrupoles* (an **octopole**), it goes like $1/r^4$; and so on. summarizes this hierarchy; for completeness I have included the electric **monopole** (point charge), whose potential, of course, goes like $1/r$.

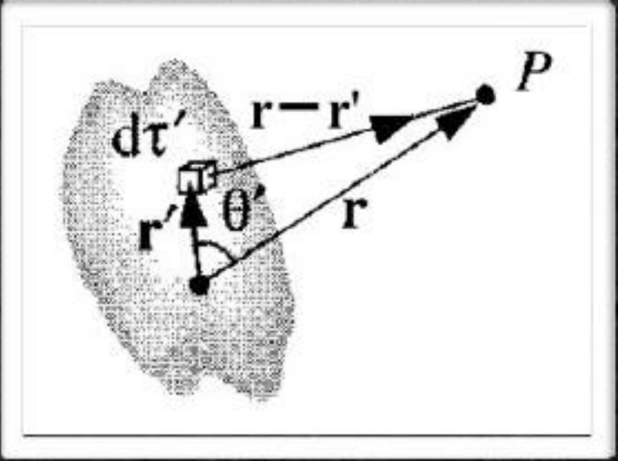


Monopole ($V \sim 1/r$) Dipole ($V \sim 1/r^2$) Quadrupole ($V \sim 1/r^3$) Octopole ($V \sim 1/r^4$)

Multipole Expansion

Consider a charge density $\rho(\mathbf{r}')$ given in some region of space. We are interested in finding the potential at an arbitrary point P lying at position $\mathbf{r}$ outside this charge density, as shown in the figure. The exact potential at this observation point can be written as:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')\, d\tau}{|\mathbf{r} - \mathbf{r}'|}$$



We have,

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= [(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')]^{1/2} \\ &= (r^2 + r'^2 - 2\,\mathbf{r} \cdot \mathbf{r}')^{1/2} \\ &= (r^2 + r'^2 - 2\,r\,r' \cos\theta')^{1/2}. \end{aligned}$$

Multipole Expansion

Therefore,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\frac{1}{\left[1-2\left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\right]^{1/2}}.$$

Now, we are going to use the following remarkable formula (the generating function for Legendre Polynomials):

$$\frac{1}{[1-2uz+u^2]^{1/2}} = \sum_{n=0}^{\infty} u^n P_n(z).$$

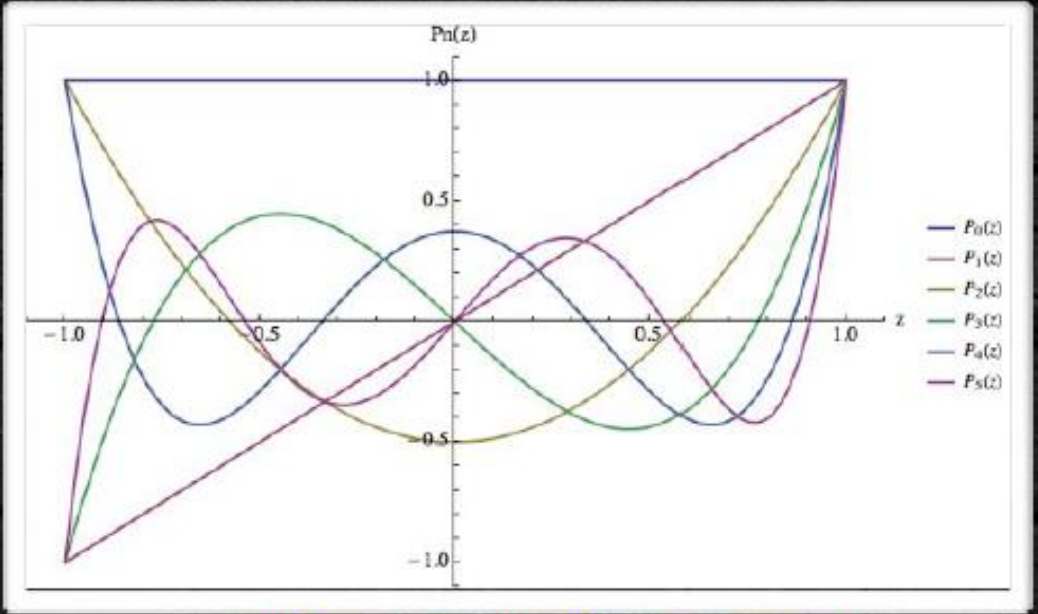
In the above $P_n(z)$ represents the nth degree Legendre polynomial, and $|u|,|z|<1$.

Multipole Expansion

Legendre polynomials are named after the French mathematician Adrien-Marie Legendre.

First few Legendre polynomials:

$$P_0(z) = 1$$
$$P_1(z) = z$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1)$$
$$P_3(z) = \frac{1}{2}(5z^3 - 3z)$$
$$P_4(z) = \frac{1}{8}(35z^4 - 30z^2 + 3)$$
$$P_5(z) = \frac{1}{8}(63z^5 - 70z^3 + 15z)$$



The plots of Legendre polynomials up to $n = 5$.

(The number of zero crossings is equal to the degree of the polynomial.)

Two important relations:

- Rodrigues' formula:

$$P_n(z) = \frac{1}{2^n\, n!}\frac{d^n}{dz^n}(z^2 - 1)^n.$$

Multipole Expansion

Coming back to our problem, we observe that we can identify r'/r as u and $\cos\theta' = z$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\frac{1}{\left[1-2\left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\right]^{1/2}}.$$

$$\frac{1}{[1-2uz+u^2]^{1/2}} = \sum_{n=0}^{\infty} u^n P_n(z).$$

Thereby, giving us the series expansion:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta').$$

Plugging this back in the expression for electric potential, we obtain

Multipole Expansion

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left[\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')\right] d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')\, d\tau'$$

Let's define

$$c_n = \int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')\, d\tau'$$

Then we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{c_n}{r^{n+1}}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{c_0}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^3} + \cdots\right].$$

This gives the **Multipole expansion** for the potential, i.e., $V(\mathbf{r})$ resolved into the contributions from monopole, dipole, quadrupole,... terms.