PHY101: Introduction to Physics I

Monsoon Semester 2024

Lecture 25

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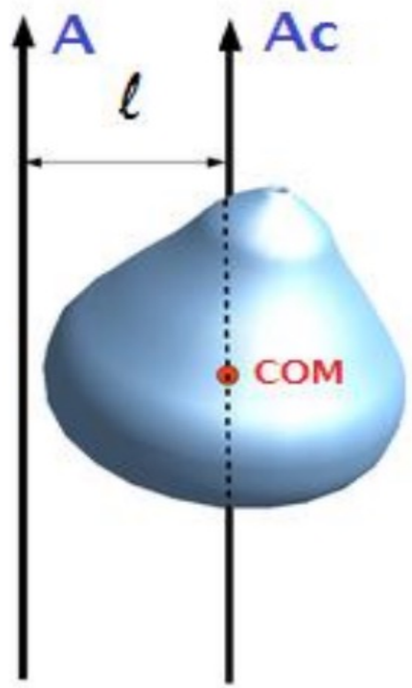
Previous Lecture

Torque
Law of equal area

This Lecture

Moment of Inertia

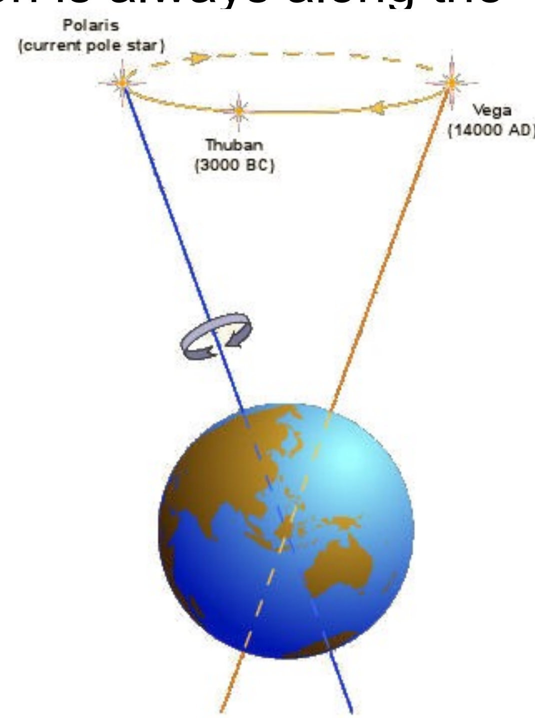
Parallel axis theorem
Perpendicular axis theorem



Rotation about a fixed axis

- The most prominent application of angular momentum is the analysis of motion of rigid bodies.
- The general case of rigid body motion involves free rotation about any free axis, and can be very complicated to deal with.
- For simplicity we restrict ourselves to a special, but important, case: **Rotation about a fixed axis**.
- Fixed axis means: the direction of axis of rotation is always along the same line; the axis itself may translate.

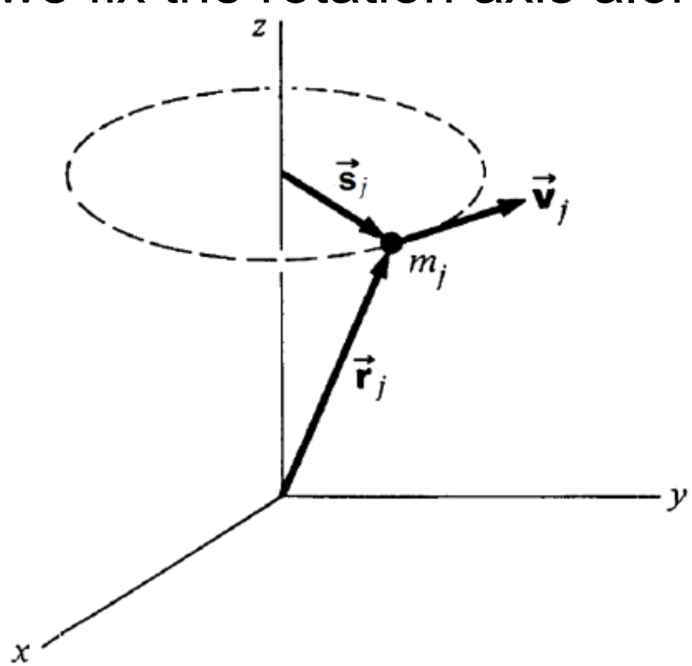
Fixed axes



Precession: not a fixed axis

Rotation about a fixed axis

- For the rotation of a rigid body about a fixed axis, every particle in the body remains at a fixed distance from the axis.
- If the choice of coordinate system is such that the origin lies on the axis of rotation, then for each particle in the (rigid) body $|\vec{r_j}|$ = constant.
- The only way that $\vec{r_j}$ can change while $|\vec{r_j}|$ remains constant is for the velocity to be perpendicular to $\vec{r_j}$.
- If we fix the rotation axis along the z-direction, then



$$|\vec{v}_j| = |\dot{\vec{r}}_j| = \omega\, s_j.$$

Here s_j is the perpendicular distance from the axis of rotation (z-axis in this case) to the particle m_j of the rigid body, and $\vec{s_j}$ is the corresponding vector. ω is the rate of rotation, the angular velocity.

Rotation about a fixed axis

- The *z*-component of angular momentum (or, along the axis of rotation) for the *j*-th particle is:

$$L_{j,z} = (s_j)(m_j v_j) = \left(s_j\right)\left(m_j\ \omega s_j\right) = m_j s_j^2 \omega.$$

- The z-component of the total angular momentum of the entire rigid body is:

$$L_z = \sum_j L_{j,z} = \sum_j m_j s_j^2 \omega$$

Note that since the body is rigid, the angular velocity ω must be same for all the constituent particles.

- The above equation can be written as:

$$L_z = I\omega \quad \text{where} \qquad I = \sum_j m_j s_j^2$$

Rotation about a fixed axis

$$I = \sum_j m_j s_j^2$$

- I is a geometric quantity referred to as the **Moment of Inertia**.
- I depends on the distribution of mass in the body, as well as the location of the axis of rotation.
- For continuously distributed matter we can replace the sum over mass particles by an integral:

$$\sum_j m_j s_j^2 \to \int s^2 \, dm \, .$$

where *dm* is the differential mass element located at a perpendicular distance s from the axis of rotation.

Then,

$$I = \int s^2 \, dm \, .$$

Rotation about a fixed axis

- If ρ is the volumetric mass-density of the object, then

$$dm = \rho\, dV$$

where dV is the volume element located at the distance s from the axis of rotation.

- Since we have chosen the axis of rotation to lie in the z-direction, we get

$$s^2 = x^2 + y^2$$

- Thus,

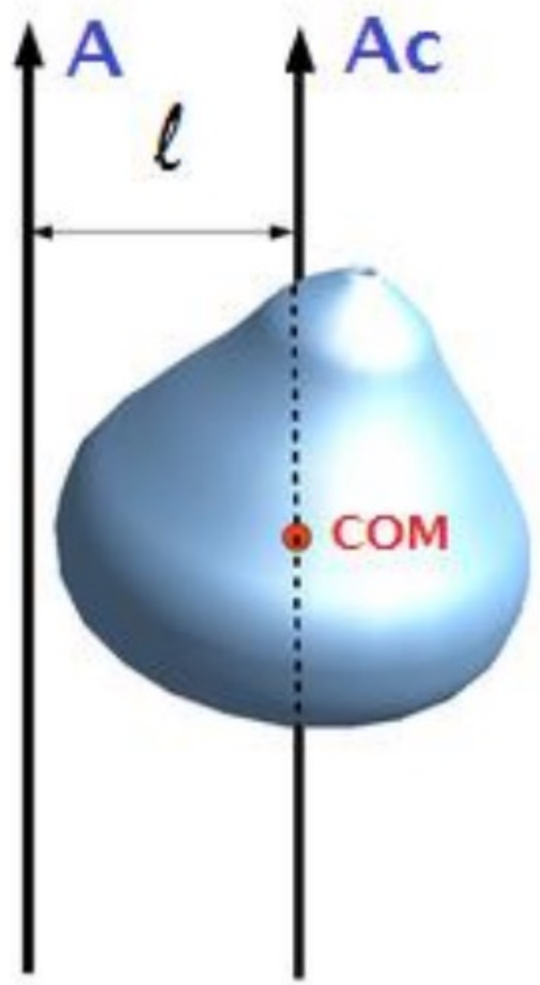
$$I = \int \rho\, s^2 dV = \int \rho (x^2 + y^2) dV\,.$$

Depending on the object under consideration dV may correspond to a length or an area element.

Parallel axis theorem

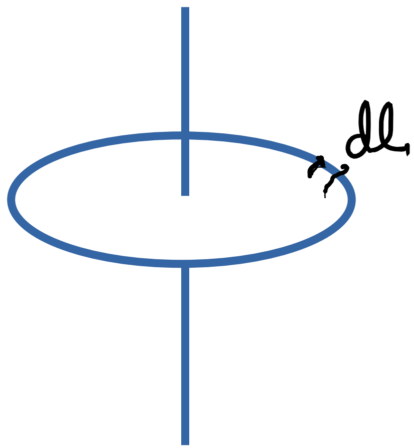
$$I = I_0 + M\, l^2$$

Thus if we know the Moment of Inertia about an axis Ac, passing through the Center of Mass, the Parallel Axis Theorem enables us to easily calculate the Moment of Inertia about any axis A parallel to Ac.



Examples

Find the moment of inertia of a uniform thin ring of mass ***M*** and radius ***R***, around the axis of symmetry of the ring.



Here $I = \int s^2 \, dm \, .$

$$dm = \lambda \, dl$$

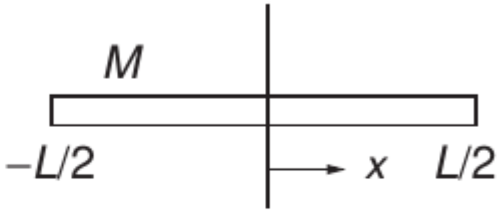
$\lambda =$ mass per unit length $= M/2\pi R$

$s = R$ for all points on the ring

$$I = R^2 \lambda \int dl = R^2 \left(\frac{M}{2\pi R} \right) 2\pi R = M R^2$$

Examples

Find the moment of inertia of a uniform thin stick of mass ***M*** and length ***L***, around a perpendicular axis through its midpoint.

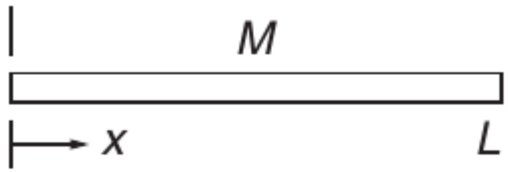


Here $$I = \int s^2 \, dm\,.$$

$$dm = \lambda \, dx$$

$\lambda =$ mass per unit length $= M/L$

$$I = \lambda \int_{-L/2}^{L/2} x^2 dx = \frac{M}{L}\frac{x^3}{3}\Bigg|_{-L/2}^{+L/2} = \frac{1}{12} M L^2$$

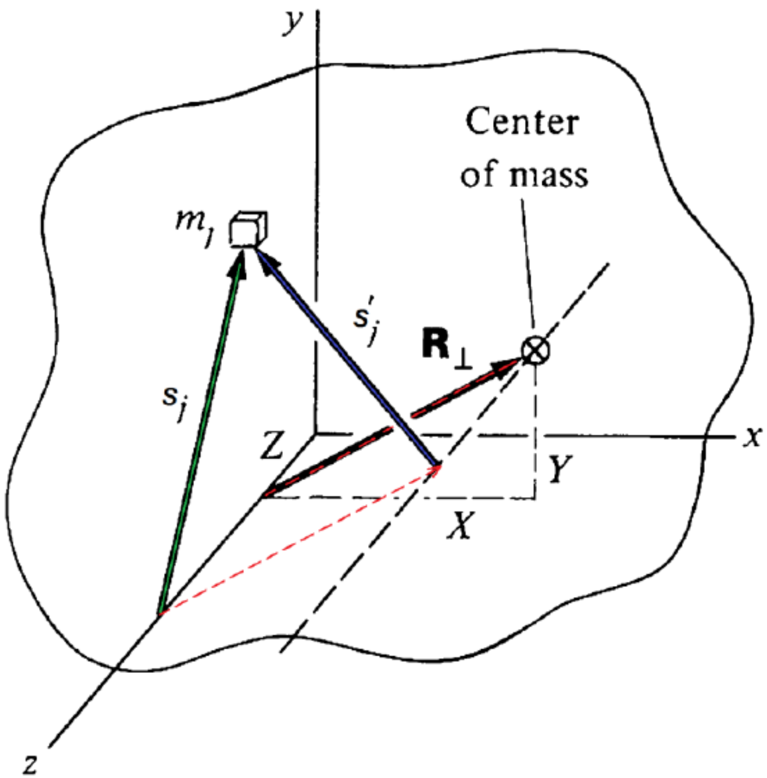


Around a perpendicular axis at its end:

$$I = \lambda \int_0^L x^2 dx = \frac{M}{L}\frac{x^3}{3}\Bigg|_0^L = \frac{1}{3}ML^2$$

Verify the parallel axis theorem for the stick

Proof of parallel axis theorem



Consider a rigid body and let I be its Moment of Inertia about the z-axis.

The vector from the z-axis to the *j*-th particle is

$$\vec{s}_j = x_j\hat{\imath} + y_j\hat{\jmath}$$

and

$$I = \sum_j m_j s_j^2 .$$

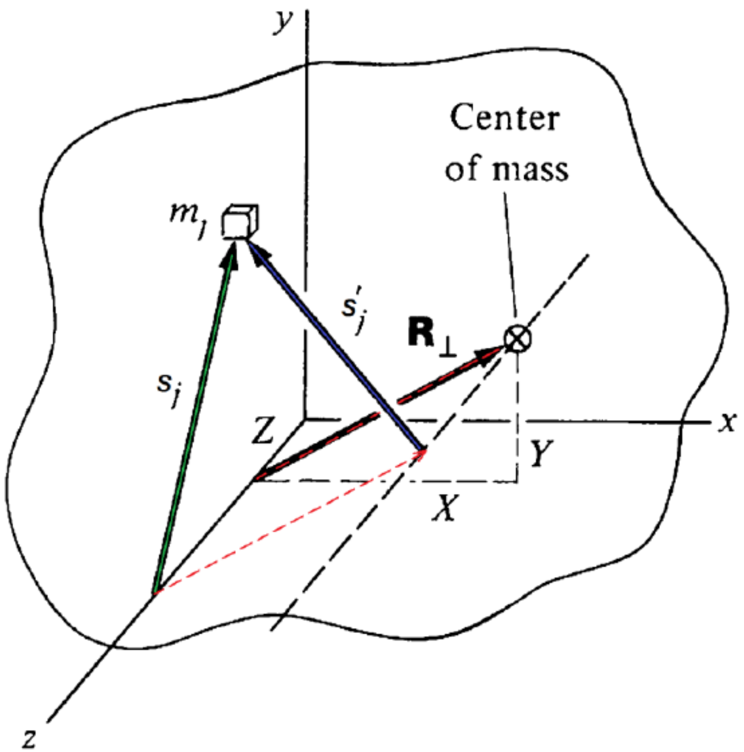
Now let Center of Mass (COM) of the system be situated at $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$.

The perpendicular vector from the z-axis to the COM is $\vec{R}_\perp = X\hat{i} + Y\hat{j}$.

If the vector from the axis through the COM to the *j*-th particle is $S_j^{'}$, then the corresponding Moment of Inertia is

$$I_0 = \sum_j m_j s_j'^2 .$$

Proof of parallel axis theorem



From the figure we see that

$$\vec{s}_j = \vec{s}_j' + \vec{R}_\perp .$$

So,

$$I = \sum_j m_j s_j^2$$

$$= \sum_j m_j \vec{s}_j \cdot \vec{s}_j = \sum_j m_j (\vec{s}_j' + \vec{R}_\perp) \cdot (\vec{s}_j' + \vec{R}_\perp)$$

$$= \sum_j m_j (s_j'^2 + 2\vec{s}_j' \cdot \vec{R}_\perp + R_\perp^2)$$

$$\Longrightarrow I = \sum_j m_j {s_j'}^2 + 2\left(\sum_j m_j s_j'\right) \cdot \vec{R}_\perp + \left(\sum_j m_j\right) \vec{R}_\perp^2$$

$$= I_0 + 2\left(\sum_j m_j (\vec{s}_j - \vec{R}_\perp)\right) \cdot \vec{R}_\perp + M R_\perp^2$$

Proof of parallel axis theorem

$$I = I_0 + 2\left(\sum_j m_j\left(\vec{s}_j - \vec{R}_\perp\right)\right) \cdot \vec{R}_\perp + MR_\perp^2$$

$$= I_0 + 2\left(\sum_j m_j \vec{s}_j - \left(\sum_j m_j\right)\vec{R}_\perp\right) \cdot \vec{R}_\perp + MR_\perp^2$$

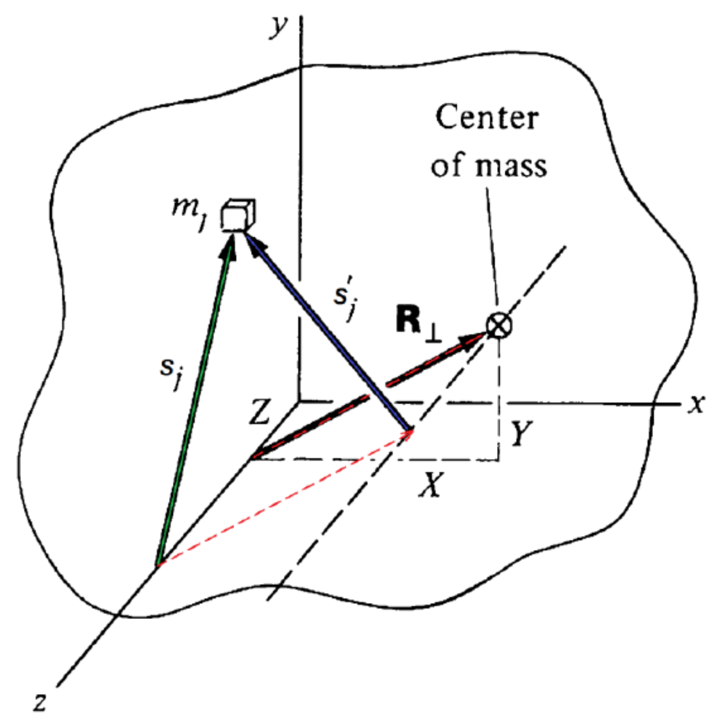
$$= I_0 + 2\left(M\vec{R}_\perp - M\vec{R}_\perp\right) \cdot \vec{R}_\perp + MR_\perp^2$$

$$= I_0 + MR_\perp^2$$

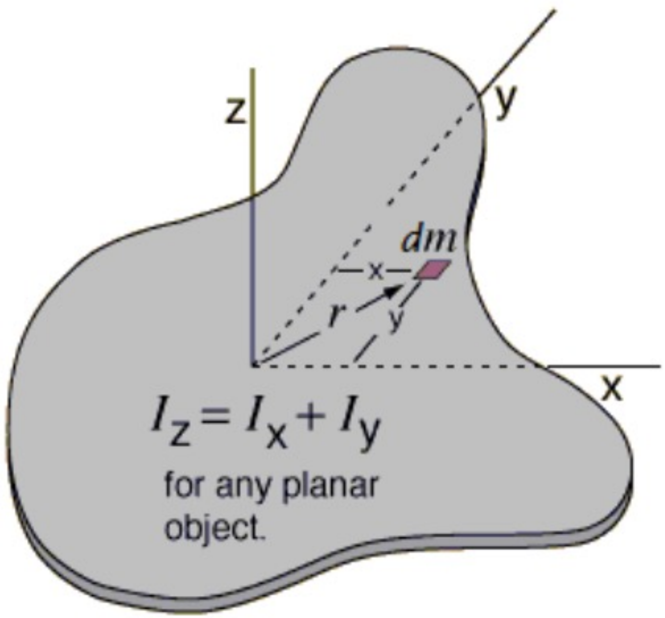
If we write $R_\perp = l$, then

$$I = I_0 + M\, l^2$$

This is the mathematical relation depicting the Parallel Axis Theorem.



Perpendicular axis theorem
(Plane figure theorem)



Consider a planar object lying in the xy plane.

We already saw that the moment of inertia about the z-axis is

$$I_z = \int dm\,(x^2 + y^2).$$

Moment of Inertia about the x-axis is:

$$I_x = \int dm\, y^2.$$

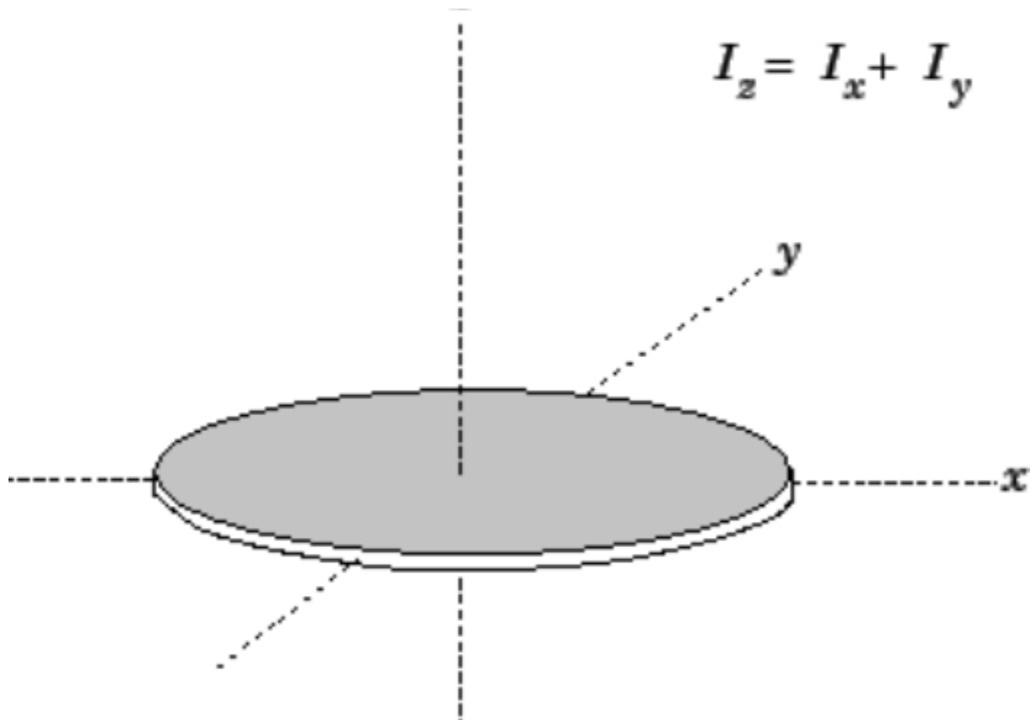
This follows, since there is no length extension of the object along the z-axis, and therefore the total distance of mass element from the x-axis is solely decided by y.

Similarly

$$I_y = \int dm\, x^2.$$

We trivially obtain: $I_z = I_x + I_y.$

If the planar object has a rotational symmetry about the z-axis, then



$$I_z = 2I_x = 2I_y.$$