PHY 102 Introduction to Physics II

Spring Semester 2025

Lecture 28

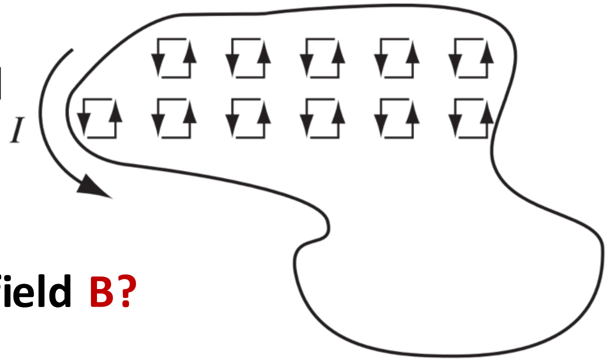
Magnetic Fields in Matter

Torques and Forces on Magnetic Dipoles

Torques on Magnetic Dipoles

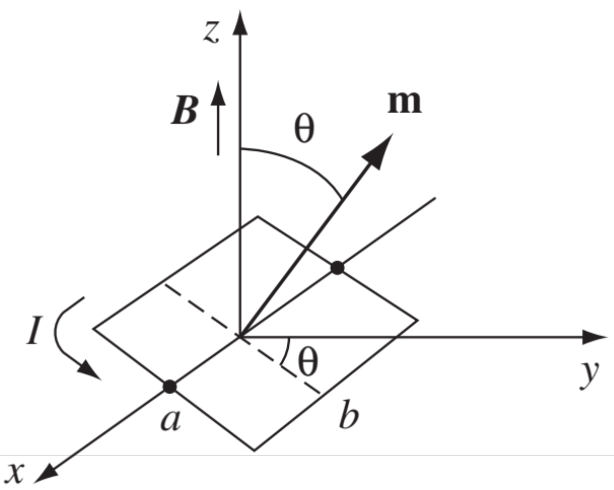
A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field.

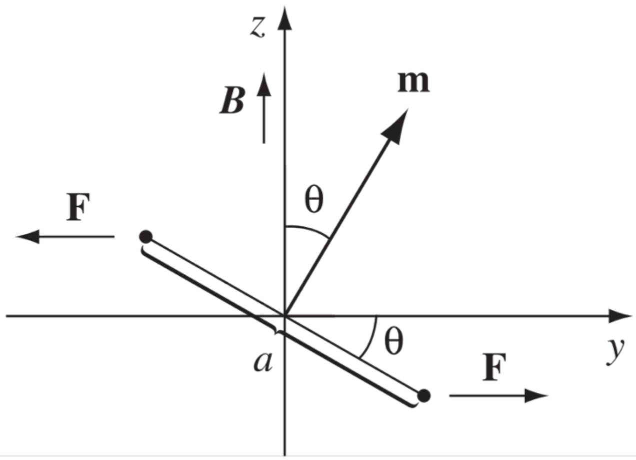
Any current loop could be built up from infinitesimal rectangles, with all the "internal" sides canceling



Calculate the torque on a rectangular current loop in a uniform field B?

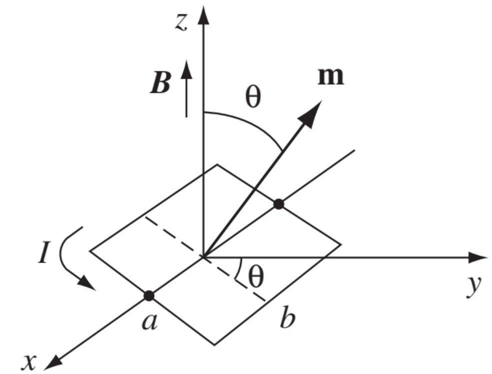
Center the loop at the origin, and tilt it at an angle θ from the z-axis towards the y - axis. Let **B** point in the z - direction.





Torques on Magnetic Dipoles

The forces on the two sloping sides cancel each other and don't rotate.

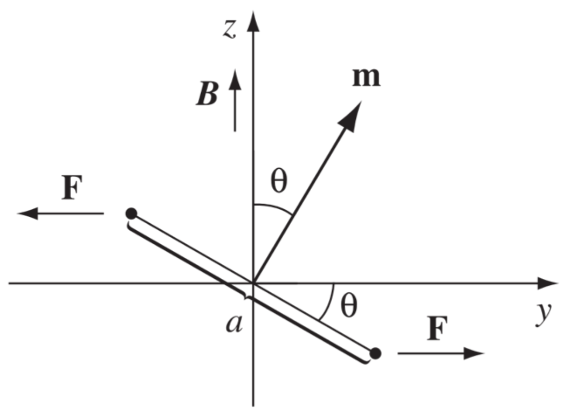


The forces on the "horizontal" sides are likewise equal and opposite (so the net force on the loop is zero), but they do generate a torque:

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}}.$$

The magnitude of the force on each of these segments is

$$F = IbB$$



and therefore

$$\mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}}$$

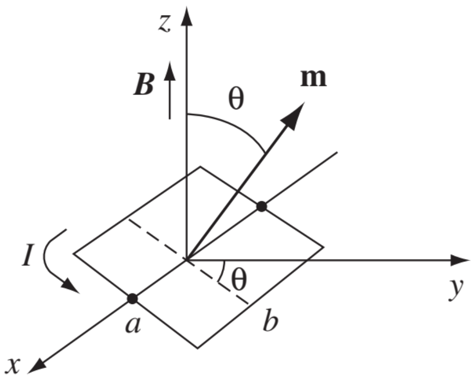
$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B},}$$

$m = Iab$ is the magnetic dipole moment of the loop.

In particular, the torque is in a direction as to line the dipole up parallel to the field. It is this torque that accounts for **paramagnetism**. If it is opposite to **B**, then it accounts for **diamagnetism**.

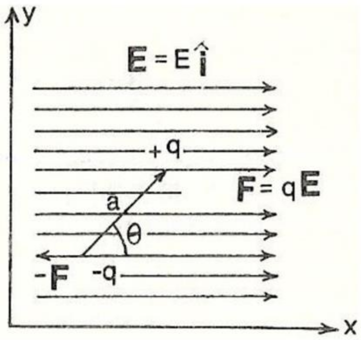
Torques on Magnetic Dipoles

Uniform magnetic field case



$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}}$$

Uniform electric field case



$$\boxed{\boldsymbol{\tau} = \boldsymbol{p}\times\boldsymbol{E} = pE\sin\theta}$$

Every electron constitutes a magnetic dipole (Classically, you may picture it as a charged and spinning tiny sphere). Therefore we may expect paramagnetism to be a universal phenomenon. However, quantum mechanical rules (Pauli's exclusion principle, specifically) dictates that the electrons within a given atom lock together in pairs with opposite signs. Therefore, it effectively neutralizes the torque on the combination. Therefore, paramagnetism normally occurs in atoms or molecules with an odd number of electrons, where the unpaired electron is subjected to a magnetic torque. Because of this alignment the object attains a residual finite magnetic moment, and gets weakly attracted along the direction of external field.

Forces on Magnetic Dipoles

We can calculate the force on the magnetic dipole using the relation

$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B}).$$

- **UNIFORM MAGNETIC FIELD:**

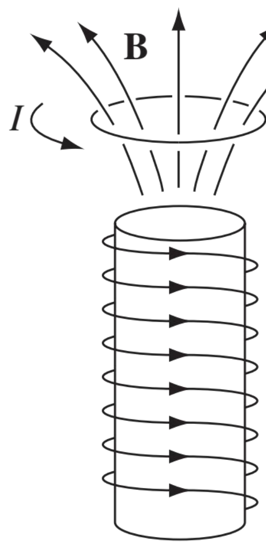
If the field is uniform, then we may proceed as follows:

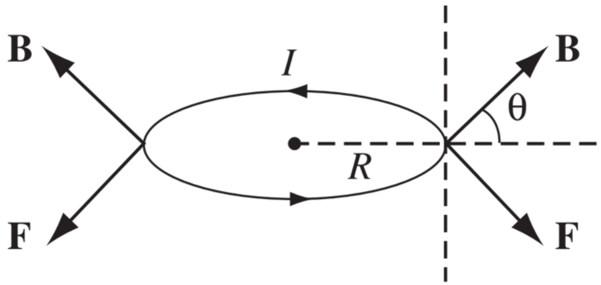
$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B}) = I \left(\oint d\mathbf{l} \right) \times \mathbf{B} = 0.$$

So, in a uniform field the force on the magnetic dipole is zero.

Torques and Forces on Magnetic Dipoles

Non- uniform magnetic field case

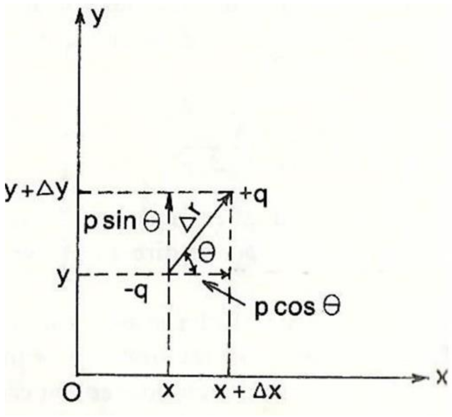




$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})}$$

Proove it as Home Work (HW)

Non- uniform electric field case

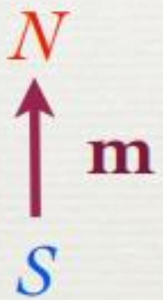


$$\boxed{\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{p}.\boldsymbol{E})}$$

Valid for static scenarios

Similarities between electric and magnetic cases

The similarities between these results for electric and magnetic cases is so striking that, no wonder, early physicists thought that magnetic dipoles consisted of positive and negative "charges" (north and south "poles"), separated by small distance, just like electric dipoles. This is referred to as "Gilbert" model of a magnetic dipole.



"Gilbert" model

"Ampere" model

They even wrote down a "Coulomb's law" for the attraction and repulsion of these poles, and developed the whole of magnetostatics in exact analogy to electrostatics. For many purposes it works nicely -- it gives the correct field of a dipole (at least, away from the origin), and the right torque on a dipole (at least on a *stationary* dipole).

However, physics-wise, this model is flawed. Magnetism is not due to magnetic monopoles, but rather due to moving electric charges. The correct model is referred to as the "Ampere" model.

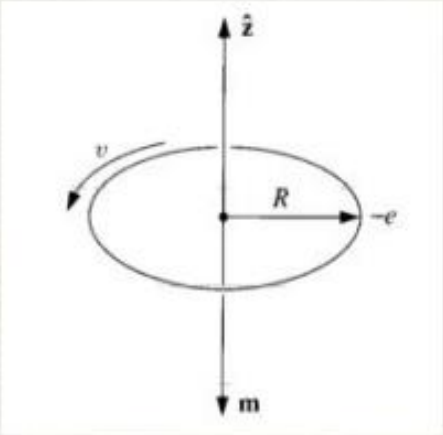
Griffiths suggests to use Gilbert model, may be, to get an intuitive "feel" for a problem, but

Effect of a Magnetic Field on Atomic Orbits

Electrons not only possess spin degree of freedom, but also the orbital degree of freedom (angular momentum) associated with its motion around the nucleus. In the classical picture, let us assume for simplicity that electron revolves in a circular orbit of radius R.

Now, to be precise, the orbital motion does not constitute a steady current. However, the period $T=2\pi R/v$ is so short that we can assume up to a reasonable approximation that the situation is like a steady one with current value:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}.$$



Therefore, the orbital dipole moment ($I\,\pi R^2$) is

$$\mathbf{m} = -\frac{1}{2} evR\,\hat{k}.$$

The negative sign arises because of the negative charge of the electron. This dipole is then subject to a torque ($\mathbf{m}\times\mathbf{B}$) when the atom is placed in a magnetic field. However, it's difficult to tilt the entire orbit than its spin, so the orbital contribution to paramagnetism is small.

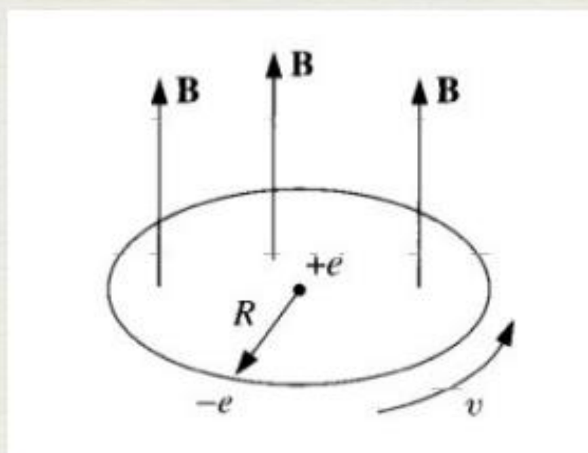
Effect of a Magnetic Field on Atomic Orbits

However, there's a significant effect on the orbital motion. The electron speeds up or slows down, depending on the direction of **B**. Let's see how.

In the absence of any **B**, the centripetal force is provided by electric forces alone, so

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}.$$

When **B** is switched on, there's magnetic Lorentz force also, $-e(\mathbf{v}\times\mathbf{B})$. For simplicity, let's take **B** be perpendicular to the plane of the orbit, as shown:



Effect of a Magnetic Field on Atomic Orbits

Therefore, we have a new speed v-bar, such that

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\overline{v}B = m_e\frac{\overline{v}^2}{R}.$$

Subtract the first equation from this one. We get

$$\begin{aligned} e\overline{v}B &= \frac{m_e}{R}(\overline{v}^2 - v^2) \\ \Rightarrow e\overline{v}B &= \frac{m_e}{R}(\overline{v}+v)(\overline{v}-v) \\ \Rightarrow \Delta v = \overline{v} - v &= \frac{e\overline{v}BR}{m_e(\overline{v}+v)}. \end{aligned}$$

This shows that electron speeds up. Also if change in speed is small, we may write

$$\Delta v = \frac{eBR}{2m_e}.$$

Effect of a Magnetic Field on Atomic Orbits

A change in orbital speed means a change in the dipole moment:

$$\Delta \mathbf{m} = -\frac{1}{2} e(\Delta v) R \,\hat{k} = -\frac{e^2 R^2}{4m_e} \mathbf{B}.$$

We note that change in **m** is opposite to the direction of **B**. Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. However, in the presence of a magnetic field, each atom picks up a little "extra" dipole moment, and these increments are antiparallel to the field giving rise to tiny repulsion. This is the mechanism responsible for **Diamagnetism**.

It is a universal phenomenon, affecting all atoms. However, it is weaker than paramagnetism, and therefore observed mainly in atoms with even number of electrons, where paramagnetism is usually absent.

Magnetization

In the presence of an external magnetic field, matter becomes **magnetized**. If we examine it microscopically, we'll find it to contain many tiny dipoles, with a net alignment along some direction.

We discussed two mechanisms responsible for this **magnetic polarization**:

1. **Paramagnetism:** The dipoles associated with the spins of unpaired electrons experience a torque which tend to align them parallel to the field.

2. **Diamagnetism:** The orbital speed of the electrons is altered to cause change in the orbital dipole moment in a direction opposite to the field.

Whatever the cause, the state of the magnetic polarization is described by the vector quantity $\mathbf{M}$, the **magnetization**, which is defined as

$$\mathbf{M} \equiv \text{magnetic dipole moment per unit volume.}$$

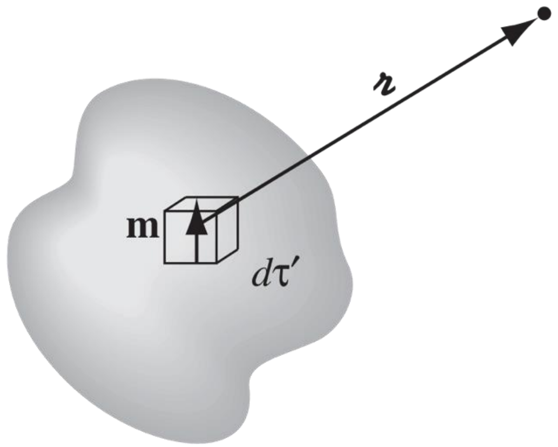
It plays a role analogous to the polarization in electrostatics.

THE FIELD OF A MAGNETIZED OBJECT

Bound Currents

You have a piece of magnetized material- magnetic dipole moment per unit volume $\boldsymbol{M}$ is given

What field does this object produce?



Each volume element element dτ' contains dipole moment $\boldsymbol{M}d\tau'$- potential due to dipole moment 'm'

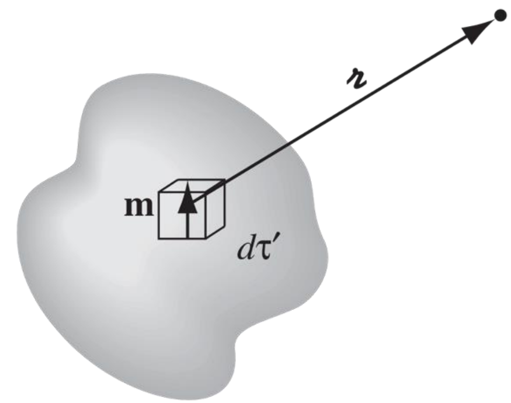
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\eta}}}{\eta^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\eta}}}{\eta^2}d\tau'$$

Magnetization **(M)** is dipole moment per unit volume

Dipole potential of volume element

Bound Currents



For the entire object:

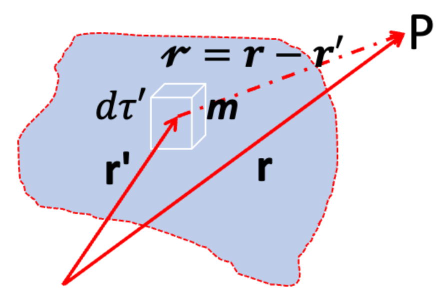
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \left[\mathbf{M}(\mathbf{r}')\times\left(\boldsymbol{\nabla}'\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$[\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}']$$

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(integration must contain primed variables)



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]\,d\tau' - \int \boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'\right\}$$

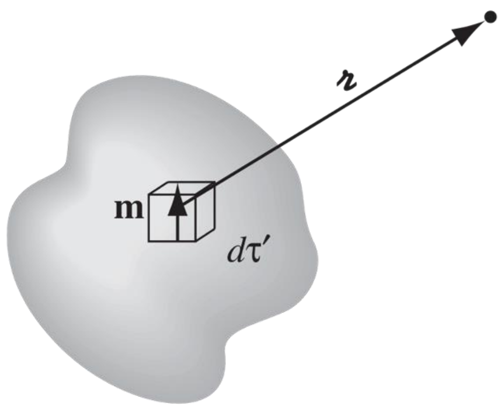
THE FIELD OF A MAGNETIZED OBJECT

Bound Currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}[\nabla' \times \mathbf{M}(\mathbf{r}')]\, d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

Vector identity

$$\int_{v} (\boldsymbol{\nabla} \times \boldsymbol{v})d\tau' = -\int_{S} \boldsymbol{v} \times \boldsymbol{da}$$



Bound Currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}[\nabla' \times \mathbf{M}(\mathbf{r}')]\,d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

The first term looks just like the potential of a volume current,

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M},}$$

the second looks like the potential of a surface current

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}},}$$

where $\hat{\mathbf{n}}$ is the normal unit vector.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'.$$

Bound Currents

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'.}$$

The potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}}$ throughout the material, plus a surface current $\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}}$ on the boundary.

Instead of finding contribution of all infinitesimal dipoles, we first determine the bound currents and then find the field they produce.

Compare with $\sigma_b = \boldsymbol{P}.\hat{\boldsymbol{n}}$ and $\rho_b = -\boldsymbol{\nabla}.\boldsymbol{P}$