Spring 2024 **Shiv Nadar Institution of Eminence** 27/02/2024

Department of Physics

PHY102: Introduction to Physics-II, Mid-Sem Examination, Full Marks: 25

Time: 2 hours

Answer all questions.

1. (a) Show that one of the followings is a possible electrostatic field while other is not.
(i) $\vec{E}_1 = 3[(xy)\hat{\imath} + (2yz)\hat{\jmath} + (3xz)\hat{k}]$ (ii) $\vec{E}_2 = 3[(y^2)\hat{\imath} + (2xy + z^2)\hat{\jmath} + (2yz)\hat{\jmath}]$

Solution: Question #1(a)

For electrostatic field $\vec{\nabla} \times \vec{E} = 0$

(i) $$\vec{\nabla} \times \vec{E}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix}$$

$$= \hat{i}(0 - 2y) - \hat{j}(3z - 0) + \hat{k}(0 - x) = -2y\hat{i} - 3z\hat{j} - x\hat{k}$$

$\neq 0 \Rightarrow \vec{E}_1$ is not an electrostatic field

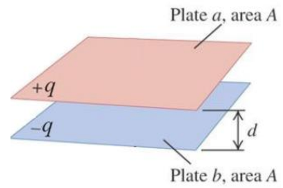
(ii) $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix}$$

$$= \hat{i}(2z - 2z) + \hat{j}(0 - 0) + \hat{k}(2y - 2y)$$

$= 0$

E_2 is an electrostatic field.

(b) Show that the force experienced by a plate carrying the charge $+q$ of an isolated air-filled parallel plate capacitor is $-\frac{q^2}{2\epsilon_0 A}$, where A is the plate area.

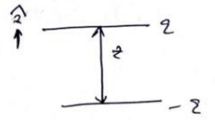


Solution: Question#1(b)

Since the capacitor is isolated, the charge q is constant.

Force experienced by a plate

$F_z = -\frac{\partial V}{\partial z}$



The electrostatic ~~potential~~ energy for a capacitor, $U = \frac{1}{2}CV^2$

$= \frac{1}{2}C \cdot \frac{q^2}{C^2}$ $\quad C = q/V$

$= \frac{q^2}{2C}$

For parallel plate capacitor, $C = \frac{\epsilon_0 A}{z}$

$\therefore F_z = -\frac{\partial}{\partial z}\left(\frac{q^2}{2C}\right) = -\frac{\partial}{\partial z}\left(\frac{q^2}{2}\frac{z}{\epsilon_0 A}\right) = -\frac{1}{2}\frac{q^2}{\epsilon_0 A}$

$\therefore \boxed{\text{Force} = -\frac{q^2}{2\epsilon_0 A}}$

(c) Explain that the bulk and surface of a conductor forms a equipotential region.

[3+3+2 =8]

Solution: Question#1(c)

Since the Electric field inside the conductor is zero.

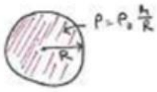
A conductor is an equipotential. For if **a** and **b** are any two points within (or at the surface of) a given conductor, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$, and hence $V(\mathbf{a}) = V(\mathbf{b})$.

2. The volume charge density of a solid sphere of radius R varies as $\rho = \rho_0 \left(\frac{r}{R}\right)$, where ρ_0 is a constant (of appropriate unit) and r is the radial distance measured from the center of the sphere. Using Gauss's law, calculate and plot the electric field at a distance r from the centre of the sphere. [Note: No charge is outside of the sphere]. [3]

Solution: Question#2

The solid sphere has a volume charge density $\rho = \rho_0 \frac{r}{R}$ where r is the radial distance in spherical polar co-ordinates.

<u>Inside sphere:</u> $0 \leq r \leq R$



Consider a Gaussian sphere inside the sphere of radius R. Apply Gauss's law over the Gaussian surface:

$$\oint\!\!\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} \quad \text{— (1)}$$

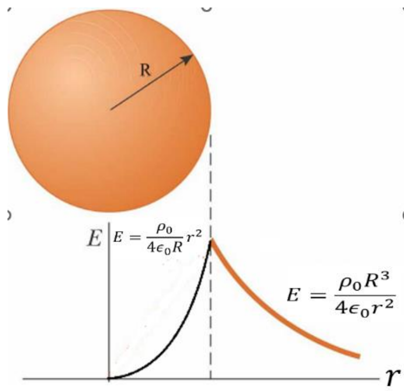
To find Q_{enc} within the sphere of radius 'r', consider volume element $d\tau$ at $\vec{r}'$ from 0

$$E_{in} 4\pi r^2 = \frac{1}{\epsilon_0} \int \rho(r')\, d\tau' \quad [\text{primed variables refers to point inside '}r\text{'}]$$

$$= \frac{1}{\epsilon_0} \int_{r'=0}^{r} \rho_0 \frac{r'}{R} r'^2\, dr' \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \quad [d\tau' = r'^2 dr' \sin\theta\, d\theta\, d\phi]$$

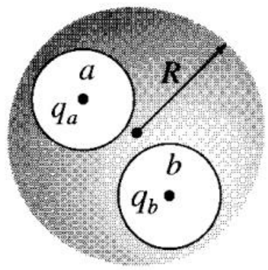
$$= \frac{1}{\epsilon_0} \int_0^{r} \left(\frac{\rho_0 r'}{R}\right) 4\pi r'^2\, dr' = \frac{4\pi\rho_0}{\epsilon_0 R} \int_0^{r} r'^3\, dr' = \frac{4\pi\rho_0}{\epsilon_0 R} \frac{r^4}{4}$$

$$\therefore \boxed{\vec{E}_{in} = \frac{\rho_0 r^2}{4\epsilon_0 R} \hat{r}} \quad [\text{here } \hat{r} \text{ is also the radial direction}] \quad \text{— (2)}$$



+ Similarly do calculation for electric field outside the sphere($r > R$).

3. Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R as shown in figure. At the center of each cavity, a point charge is placed, call these charges q_a and q_b.



   (i) Find the surface charge density σ_a , σ_b , and σ_R at surface of cavities a, and b and the surface of sphere.
   (ii) What is the field outside the conductor?
   (iii) What is the field within each cavity?
   (iv) What is the force on q_a and q_b?
   (v) Which of these answers would change if a third charge q_c, were brought near the conductor? [0.5+1+1+1+0.5=4]

Solution: Question#3

Following are the answers but need to give explanation.

(a) $\boxed{\sigma_a = -\frac{q_a}{4\pi a^2}}$; $\boxed{\sigma_b = -\frac{q_b}{4\pi b^2}}$; $\boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}}$.

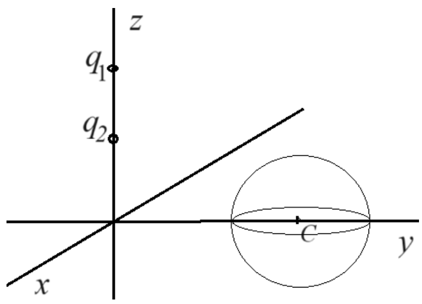
(b) $\boxed{\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}},}$ where $\mathbf{r}$ = vector from center of large sphere.

(c) $\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b,}$ where $\mathbf{r}_a$ ($\mathbf{r}_b$) is the vector from center of cavity a (b).

(d) $\boxed{\text{Zero.}}$

(e) σ_R changes (but not σ_a or σ_b); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on q_a and q_b still zero.

4. Two point charges q_1, and q_2 are placed at locations $(0,0,d_1)$, and $(0,0,d_2)$ as shown in figure. Imagine a spherical region of radius R and centre at $(0,C,0)$, using the properties of electric field and potential give the answers of the following questions.



i. What is the divergence of electric field due to charges q_1, and q_2 at the follwing points $(0,0,d_1)$, $(0,0,d_2)$, $(0,C,R)$, and $(0,C,-R)$?

ii. What is the value of surface integration of the potential V due to point charges q_1, and q_2 over the surface of the sphere shown in figure, $\oint_S V ds$, where S is the surface of the sphere of radius R and centre at $(0,C,0)$?

iii. How much work is required to change the location of charge q_1, at $(0,0,d_1)$ to $(0,0,0)$?

iv. We want to construct an infinitesimal thin charged spherical shell of radius R and centre at $(0,C,0)$ by bringing charge from infinity such that charge is uniformly distributed over the surface with surface charge density σ, If we have only two options,

a) First bring point charges q_1, and q_2 and place it at locations $(0,0,d_1)$, and $(0,0,d_2)$ and then construct the charged spherical shell.

b) First construct this charged spherical shell and then bring point charges q_1, and q_2 and place it at locations $(0,0,d_1)$, and $(0,0,d_2)$ one by one.

Explain: Which option required more work to do?

v. Starting from electric field due to a spherical shell described above at a distance r, $(r > R)$, find the potential due to this spherical shell using the relation between electric field and potential.

vi. Calculate the total work done in constructing the system of charged sphere and point charges q_1, and q_2 for any one of the options a) and b) described above. [1+1+1+1+1+2=7]

Solution: Question#4

(i) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow$at $(0,0,d_1)$, $\frac{q_1}{\epsilon_0}\delta^3(\vec{r}-d_1\hat{k})$, at $(0,0,d_2)$, $\frac{q_2}{\epsilon_0}\delta^3(\vec{r}-d_2\hat{k})$, at $(0,C,\pm R)$, 0

(ii) In the region of sphere $\nabla^2 V = 0$, for Laplace equation using the avarage property of the solution of $\nabla^2 V = 0 \Rightarrow \oint_S \; Vds = \frac{4\pi R^2}{4\pi\epsilon_0}\left(\frac{q_1}{\sqrt{C^2+d_1^2}} + \frac{q_2}{\sqrt{C^2+d_2^2}}\right)$

(iii) Work done = $q_1\Delta V = \frac{q_1 q_2}{4\pi\epsilon_0}\left(\frac{1}{d_2} - \frac{1}{d_1-d_2}\right)$

(iv) Expression of work done in construction of charged system is $\frac{1}{2}\sum_{i=1}^{N} q_i V_i(\boldsymbol{r_i})$ where $V_i(\boldsymbol{r_i})$ is potential due to other charges q_j leaving the q_i. The expression is symmetric in q_i and q_j. Hence work in inconstructing the system of charges does not depend on the sequence of charges introduced in system. This implies that the two case will have same work. Or Work done in constructing the charge configuration total work done is $\frac{1}{2}\epsilon_0 \int E^2 d\tau,$ where $\vec{E}$ is the electric field in the final construction which does not depend on the sequence of charged construction.

(v) $V = -\int_{\infty}^{r}\vec{E}\cdot d\vec{r} = -\int_{\infty}^{r}\frac{4\pi R^2\sigma}{4\pi\epsilon_0 r^2}\hat{r}\cdot dr\hat{r} = \frac{4\pi R^2\sigma}{4\pi\epsilon_0 r}$

(vi) For spherical shell in absence of q_1 and q_2 work in construction is $W_s = \frac{1}{2}\oint_S \; V\sigma ds = \frac{1}{8\pi\epsilon_0 R}(\sigma 4\pi R^2)^2 = \frac{2\pi\sigma^2 R^3}{\epsilon_0}$. or $W_s = \int_0^{\sigma 4\pi R^2}\frac{q dq}{4\pi\epsilon_0 R} = \frac{1}{8\pi\epsilon_0 R}(\sigma 4\pi R^2)^2$

Potential due to spherical shell at distance $r = \frac{4\pi R^2\sigma}{4\pi\epsilon_0 r}$, where $r > R$

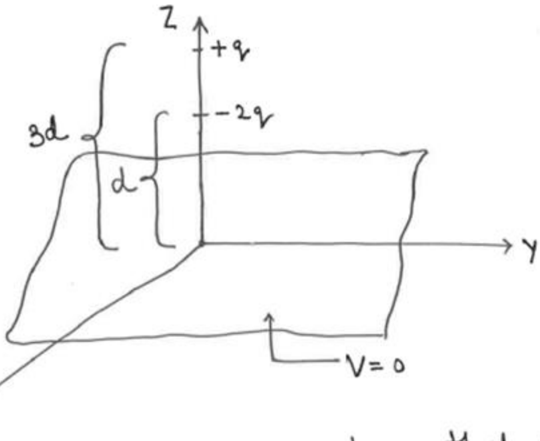
Now for bringing q_2 work done is= $W_{q_2 s} = q_2\Delta V = \frac{4\pi R^2\sigma}{4\pi\epsilon_0}\left(\frac{q_2}{\sqrt{C^2+d_2^2}}\right)$

Now for bringing q_1 work done is= $W_{q_1 s} + W_{q_1 q_2} = q_1\Delta V = \frac{4\pi R^2\sigma}{4\pi\epsilon_0}\left(\frac{q_1}{\sqrt{C^2+d_1^2}}\right) + \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{d_1-d_2}$

Total work is $W_s + W_{q_2 s} + W_{q_1 s} + W_{q_1 q_2}$

5. A point charge $(-2q)$ and another point charge $(+q)$ is placed at a distance 'd' and '$3d$' from the origin above the $x-y$ plane. If the $x-y$ plane has an infinitely long grounded conducting plane, calculate the force on $+q$ charge. [Apply the concept of first uniqueness theorem and results of a point charge in front of a grounded conductor] [3]

Solution: Question#5



0.5 + 0.5

From first uniqueness theorem, we know that a point charge q kept in front of an infinitely large grounded conductor (at a distance d above it) induces an opposite charge (image charge) '−q' at a distance 'd' below the conductor.

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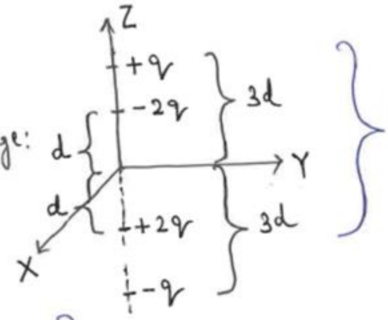
The above problem reduces to formation of induced charges +2q (at distance '−d') and −q (at a distance '−3d')



0.5

Net force on +q charge:

$$\vec{F}_{+q} = \frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} - \frac{q}{(6d)^2}\right]\hat{z}$$

$$= -\frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z}$$

0.5