CSM Midterm Project

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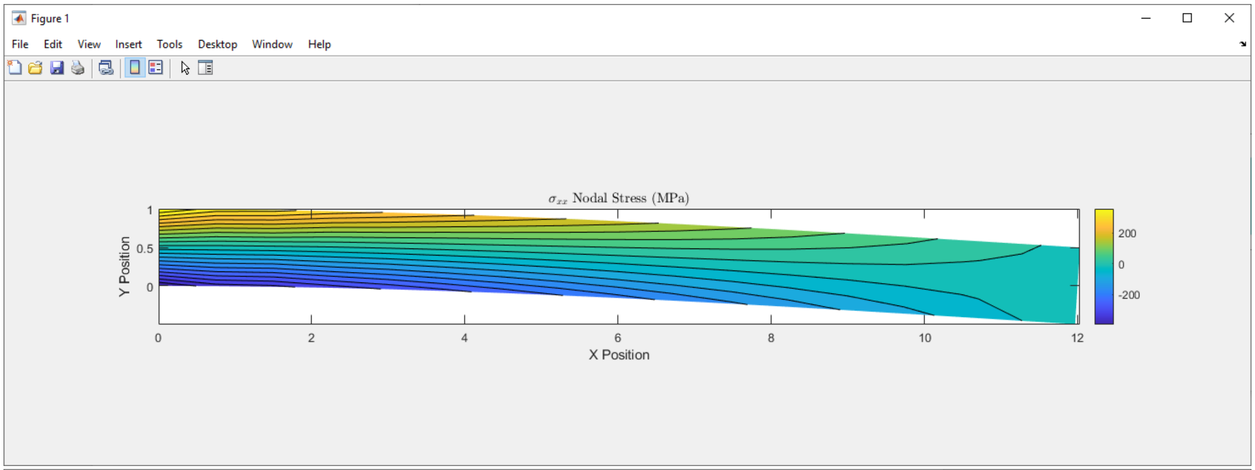
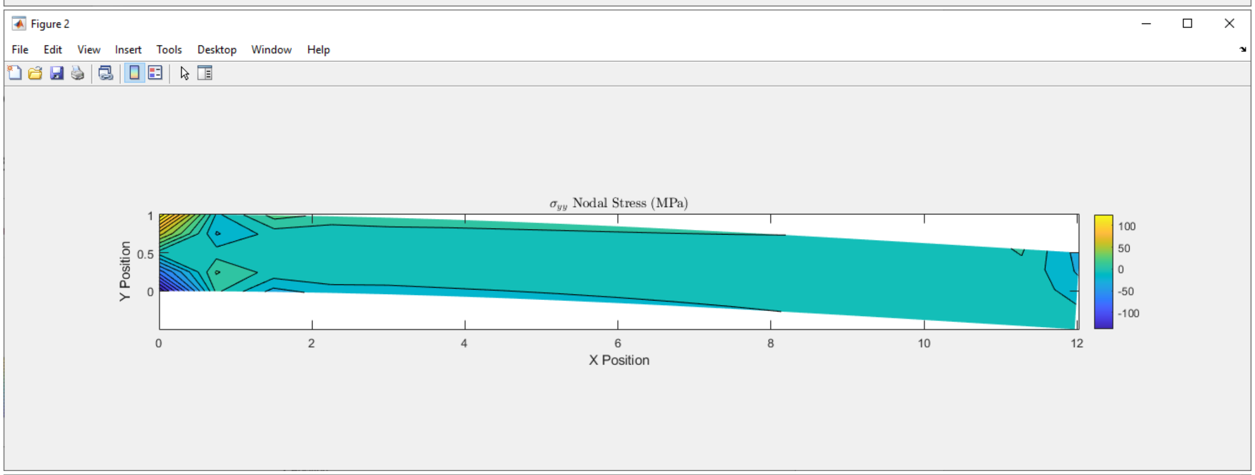
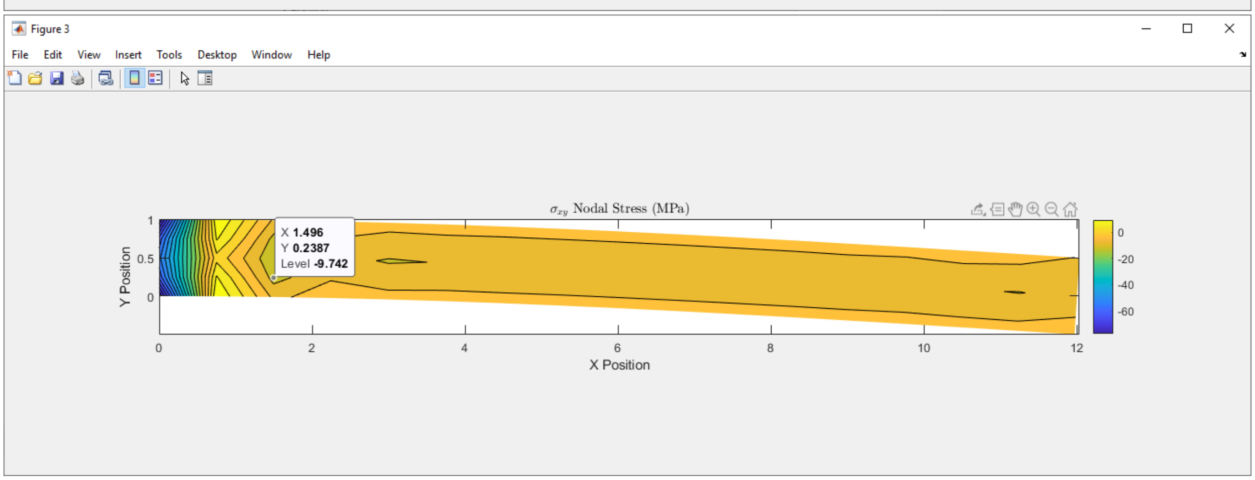
Question 1, Part A

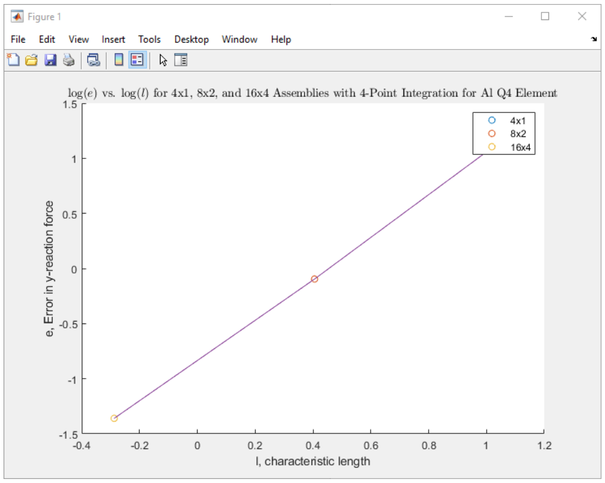
The plots are as follows.

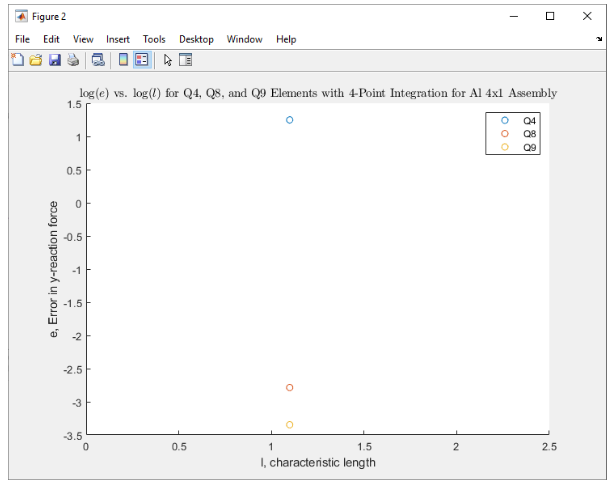




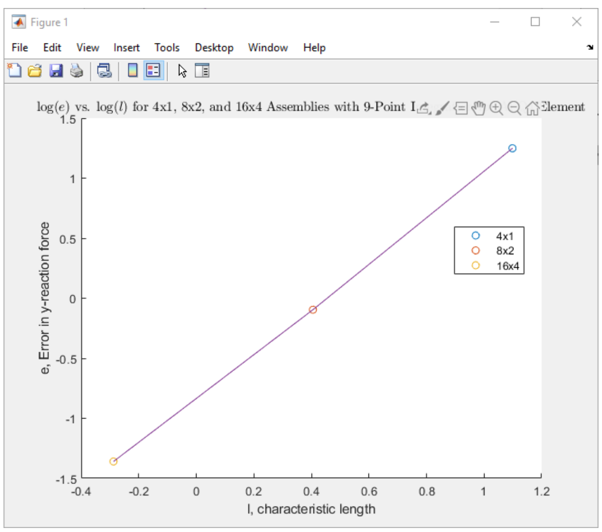


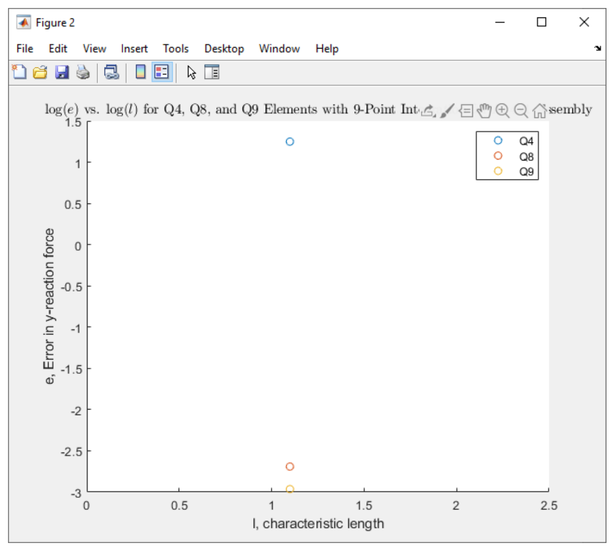
Question 1, Part B





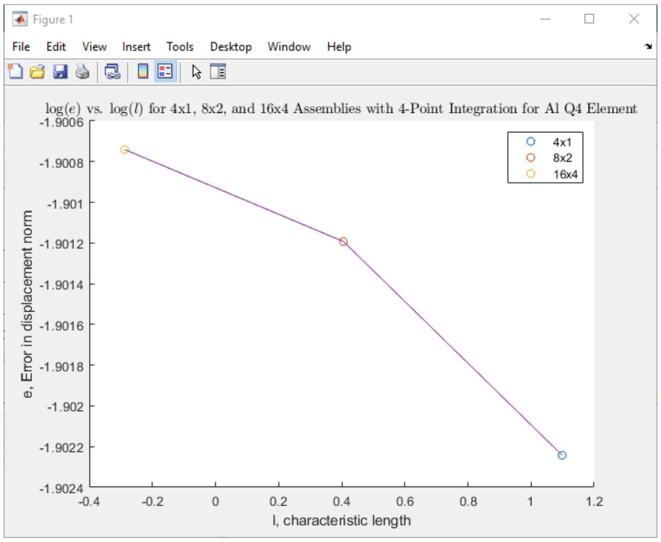
Above is the plot for FEM solution total reaction force against that of the EB beam solution. We obtain highest matching with the EB beam solution with a dense mesh and the Q9 element when using 4 integration points.

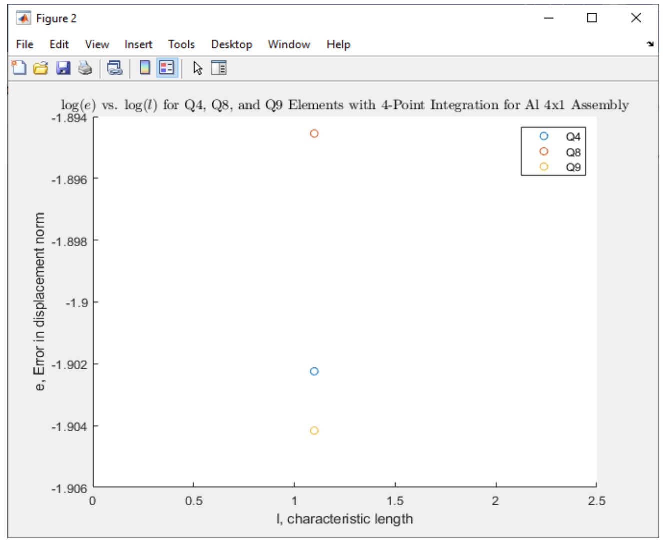




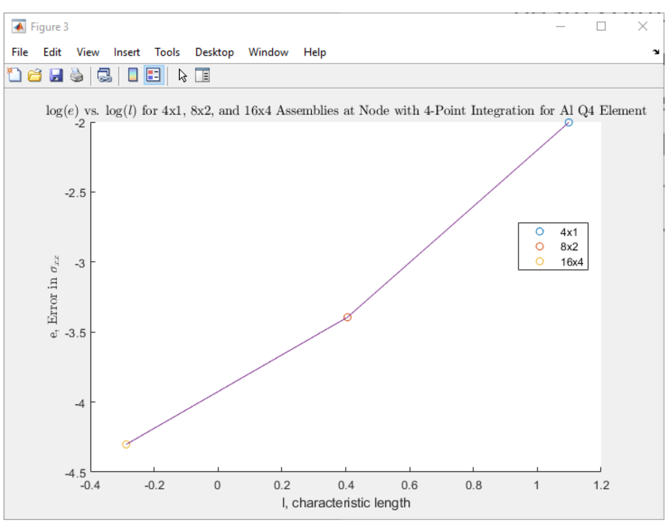
With 9 integration points, we still observe the same trend. However, the error actually increases for the Q8 and Q9 elements. This feels quite unexpected for me because naivety would make me assume that more integration points means more accuracy. This could be because the higher order elements have more variation within the element itself. The more integration points we have, the more we sample that variation, and the more error we would obtain, as it is not as representative of the EB beam solution.

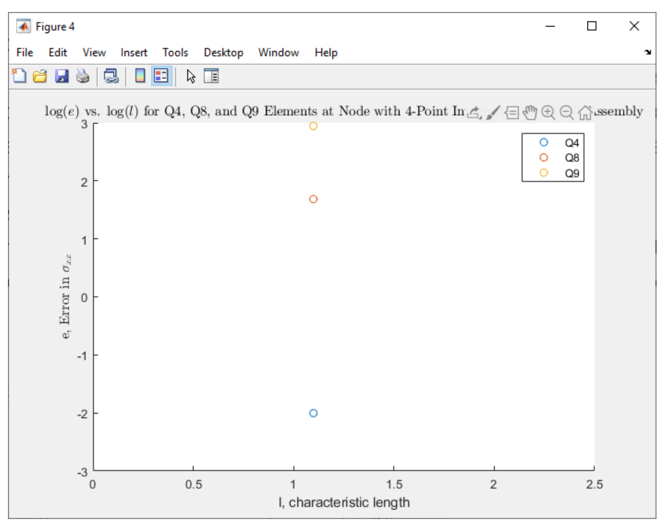
Question 1, Part C



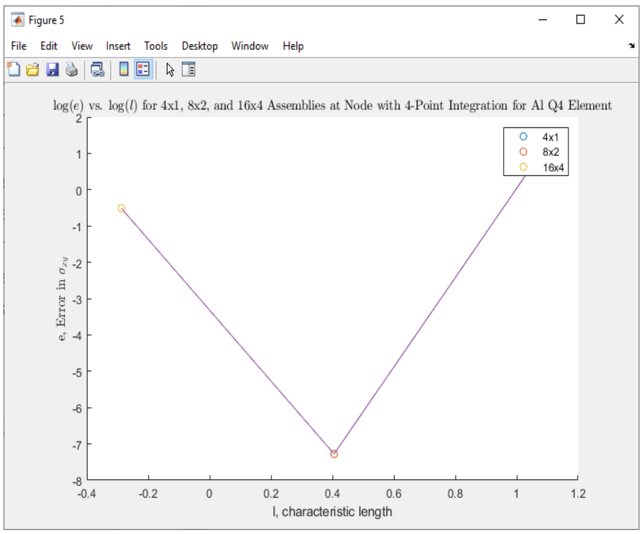


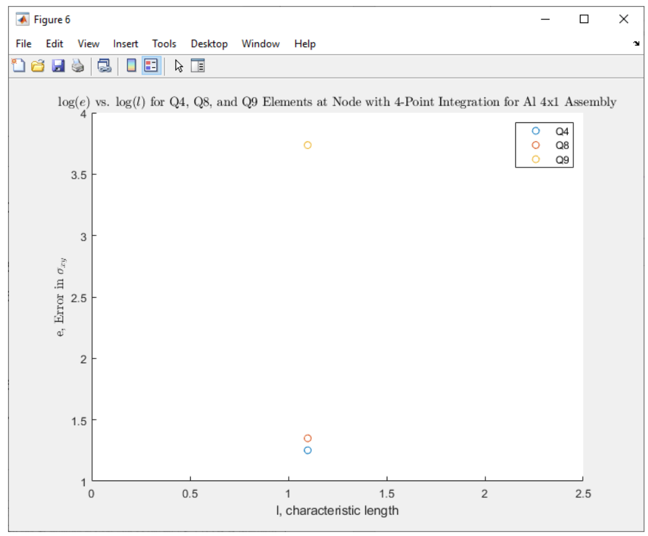
Above are plots for error in displacement norm at a point A of FEM solution relative to EB beam bending solution. As we can see, the error in displacement norm decreases with increasing characteristic length. This is expected, as the EB beam solution cannot displace in x, only in y, whereas the FEM solution does enable for displacements in the x direction. The finer the mesh is, the better the x displacement is handled, making it less representative of the EB beam solution. Interestingly, the Q8 element is the least accurate with respect to the EB beam solution. This might be because Q8 is not complete for quadratic polynomials, and thus it is unable to properly represent the EB beam solution.



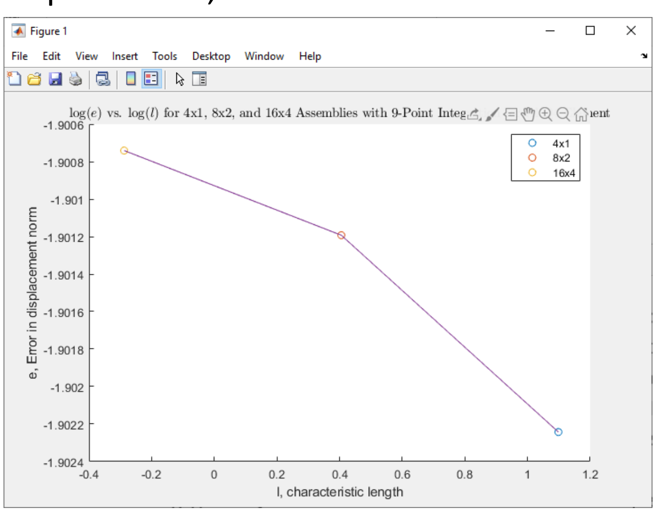


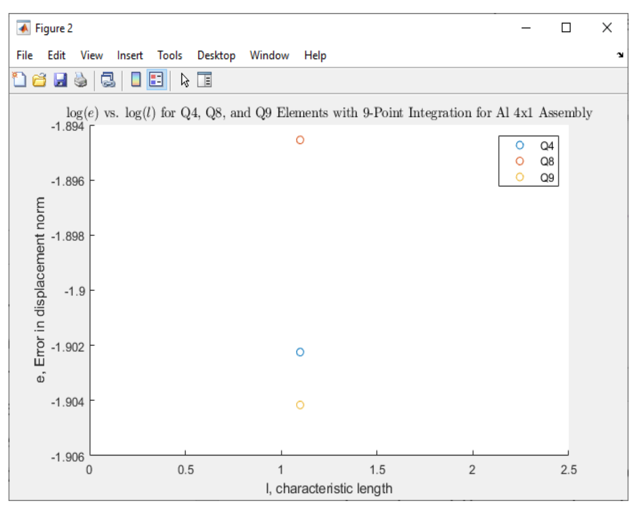
With respect to the bending stress, we obtain a less error for a finer mesh, and greater error for a larger mesh. This is not surprising because the EB bending stress solution has higher order terms in x, compared to y. Furthermore, the bending stress is better represented by the Q4 elements versus the Q8 and Q9. This is expected because the assembly is 4x1, and having additional nodes along the y direction would be less representative of the 1D EB beam solution.



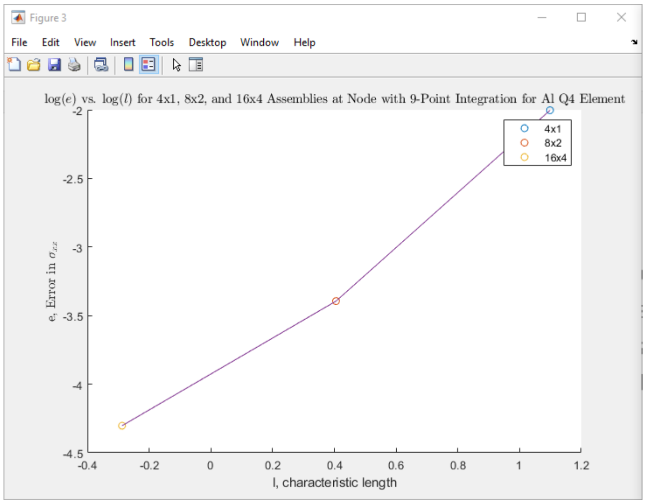


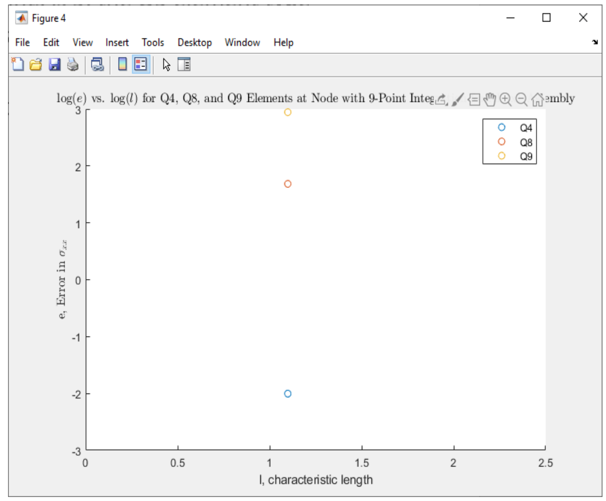
With respect to the shear stress, we obtain a less error for the 8x2 mesh and Q4 elements. I think this is the case because the EB shear stress depends on the third derivative of the moment, making it linear in x. This makes the Q4 element to best match the EB beam solution. I am surprised that the 8x2 mesh performs better than the 16x4 one. This might be because 16x4 could allow for more x-displacement due to the higher number of nodes. As a result, since EB has no x-displacement, then the finer mesh is more inaccurate.

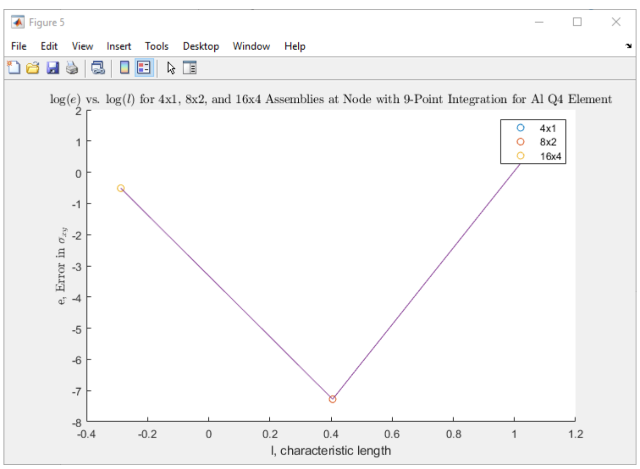


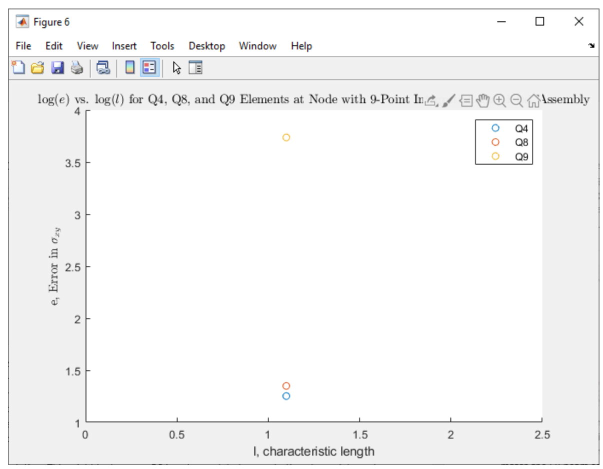


Interestingly, when the number of integration points is 9, the Q9 element type becomes even more accurate than the Q4 for displacement norm. This is because the deflection of the EB beam is of order 4, so having 3 integration points along each axis is able to exactly solve functions of order 4. The number of integration points does not seem to have an effect on the mesh size.



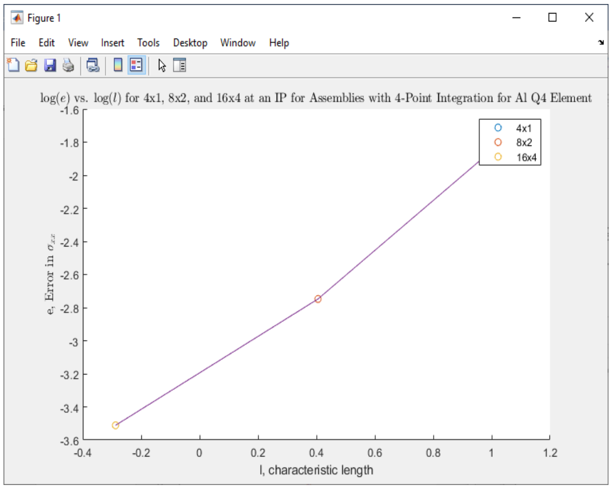


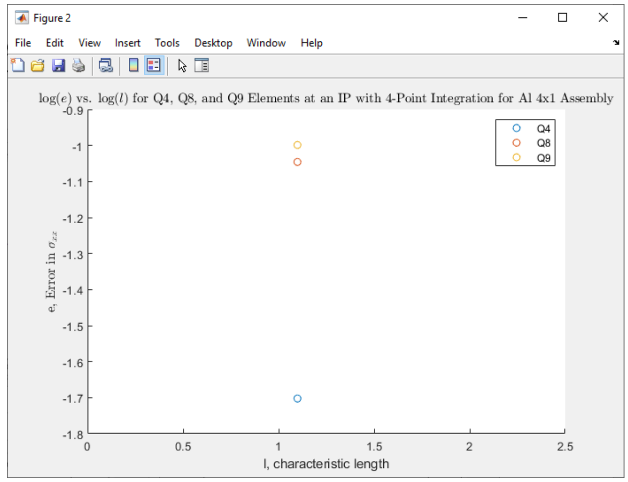


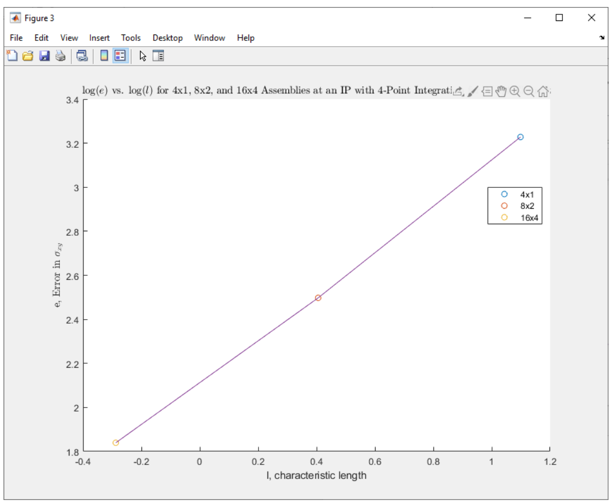


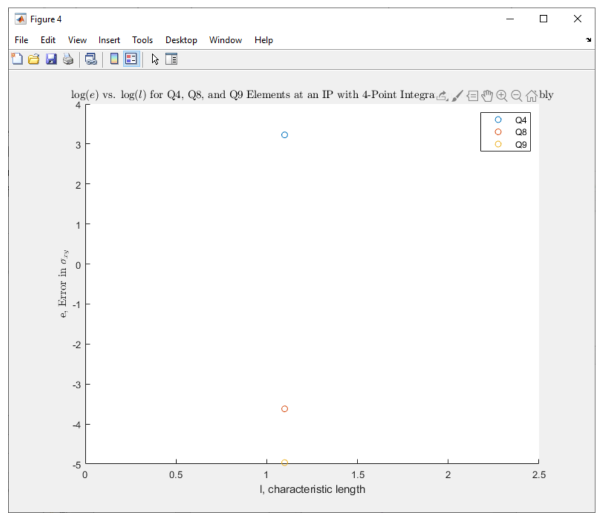
Even when 9 integration points are used, Q4 still remains to be the most accurate with respect to computing both bending and shear stresses.

So far, we've looked at the nodes, but now we shall look at what happens at the integration points. Using 4 point integration,



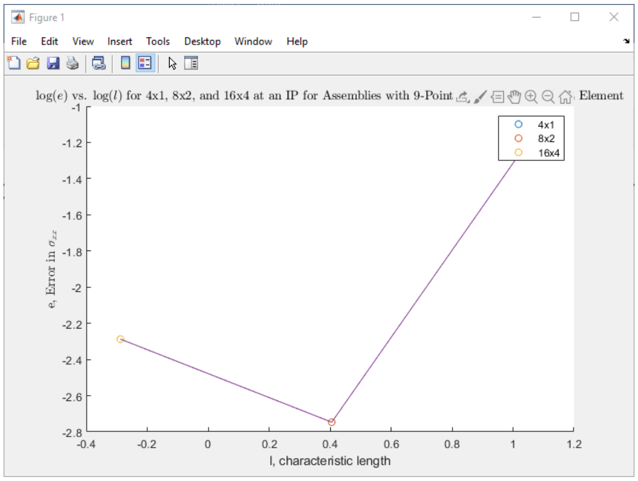


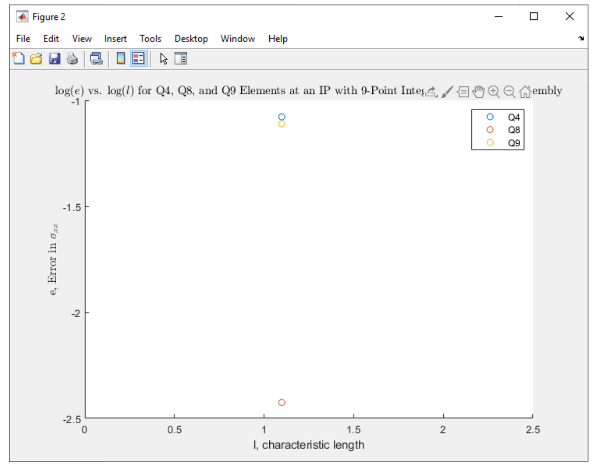


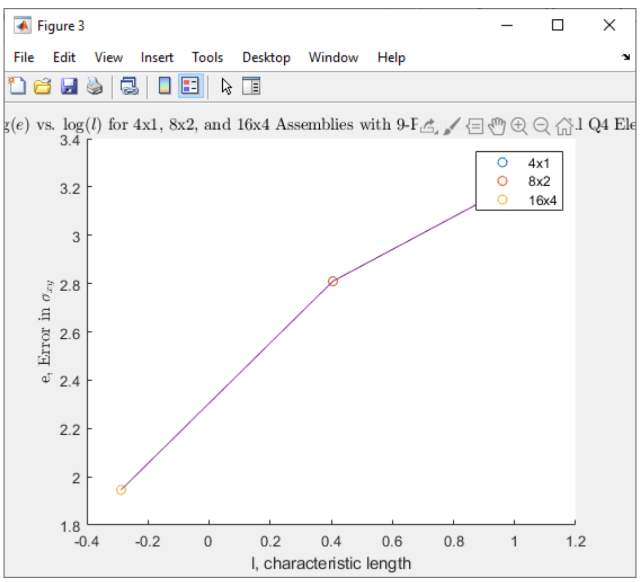


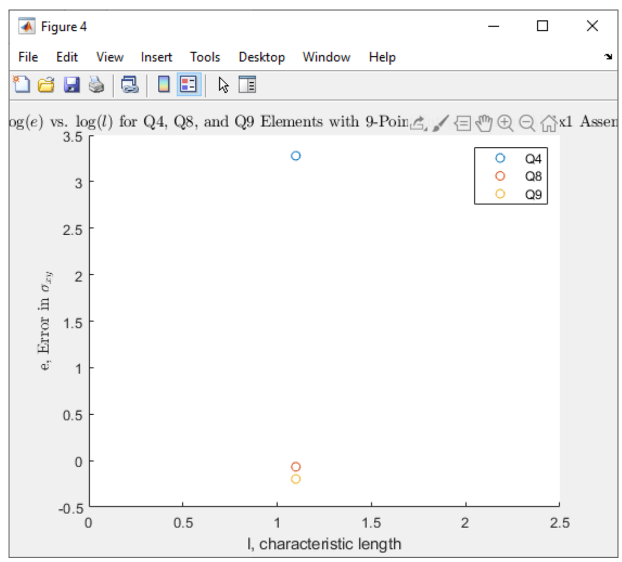
While it's no surprise that the finer mesh gives the more accurate result for the bending stress, it's quite interesting that it does for the shear stress. This is the case because the nodal stresses are approximate. There is no continuity of derivative across elements. It's also quite interesting that the Q9 and Q8 elements are more accurate for the shear stress at the node. This is most likely for the same reason: we are finding stress at a place where it's well defined.

When we change the number of integration points to 9 however, things look a bit different.





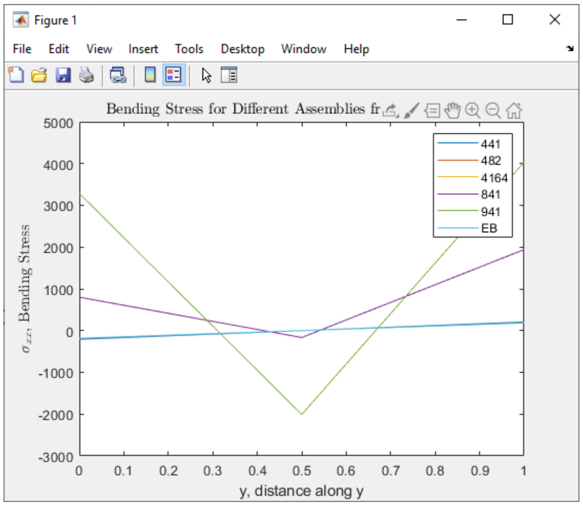


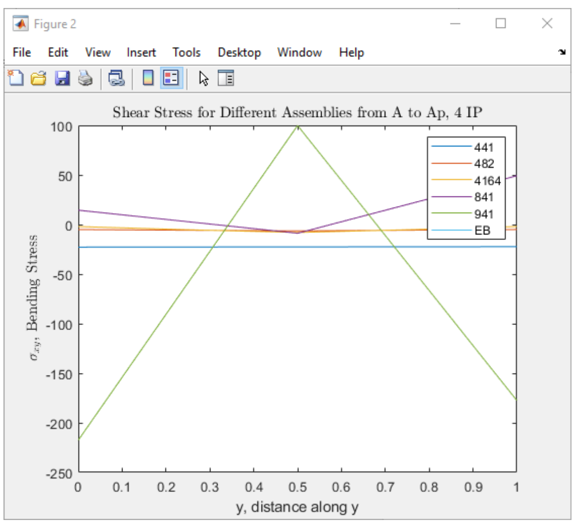


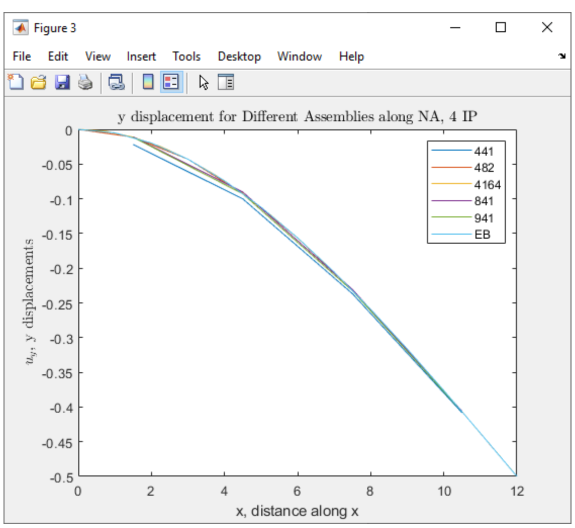
For the bending stress, we get a similar error pattern as we've seen in the case of the nodes. Furthermore, the Q9 element is the most accurate, whereas Q4 is the least accurate. What's even more interesting is the huge difference in error between the Q8 and Q9. This is quite interesting because Q8 is incomplete quadratic, whereas Q9 is a complete quadratic.

Question 1, Part D

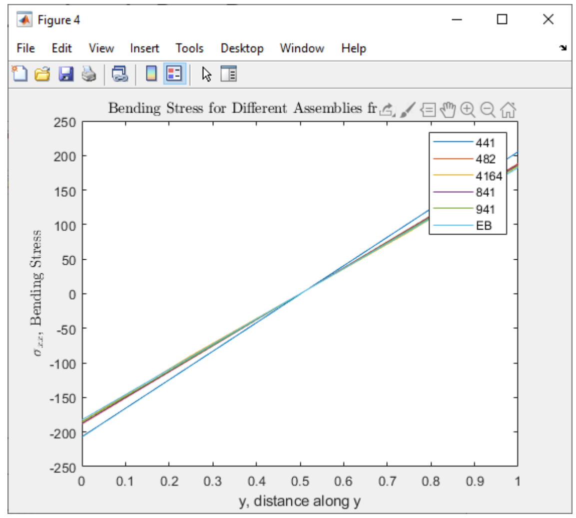
For 4 integration points,

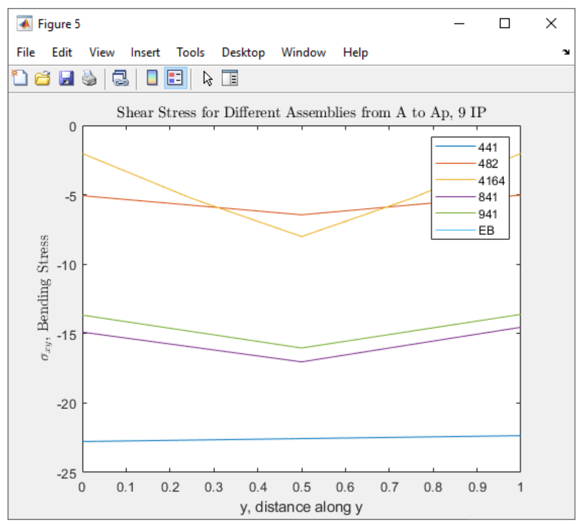


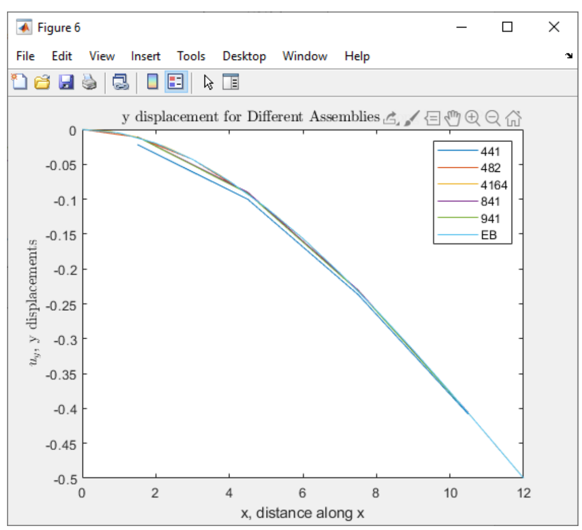




For 9 integration points,







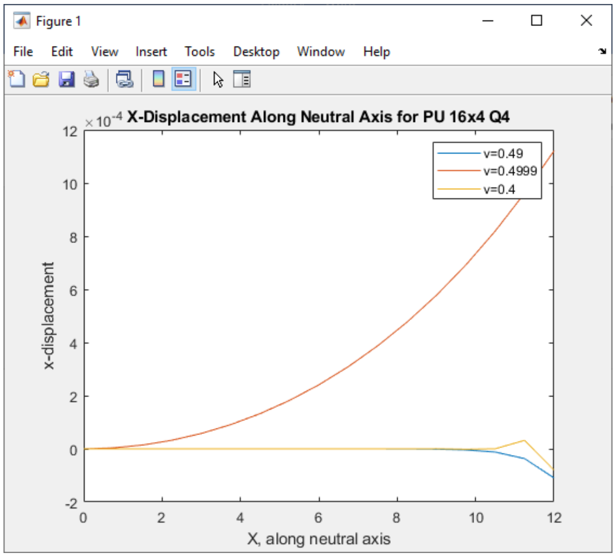
Just as a note to interpret the legend,

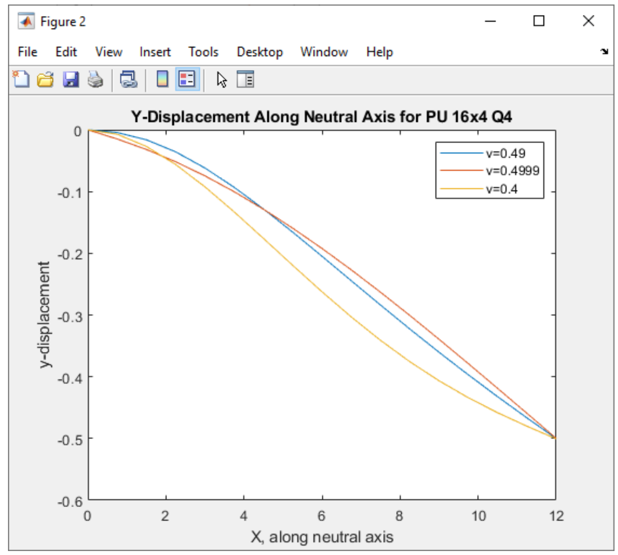
441 -> Q4, 4x1 841 -> Q8, 4x1

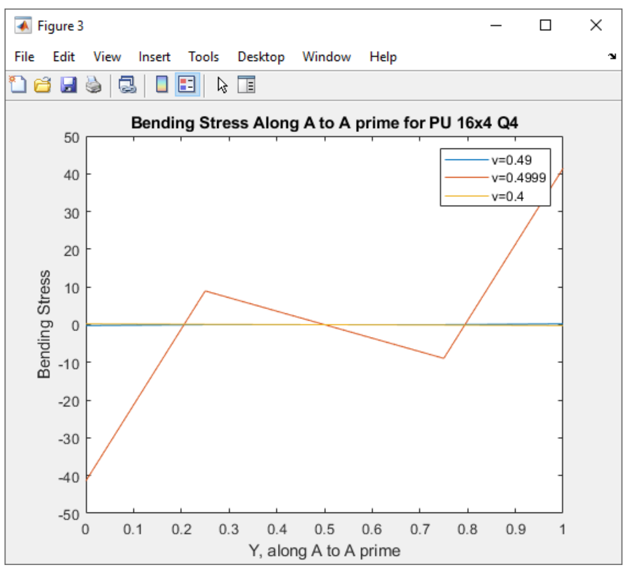
4164 -> Q4, 16x4

As we can see, the number of integration points does not seem to play much of a role in the y displacement of along the neutral axis. All the solutions seem to match quite well with the EB beam solution. This should be no surprise whatsoever. The real discrepancy is in the bending and shear stresses. We note that for both the bending and shear stresses, we obtain higher accuracy to the EB solution when using more integration points. Furthermore, making the mesh more fine makes the solution converge much close to what is seen in the EB beam solution when using a lower number of integration points. This pattern changes, and is arguably more evident in the shear stresses, when using a larger number of integration points. When using a higher number, the higher order elements come closer to the EB beam solution.

Question 2, Part A

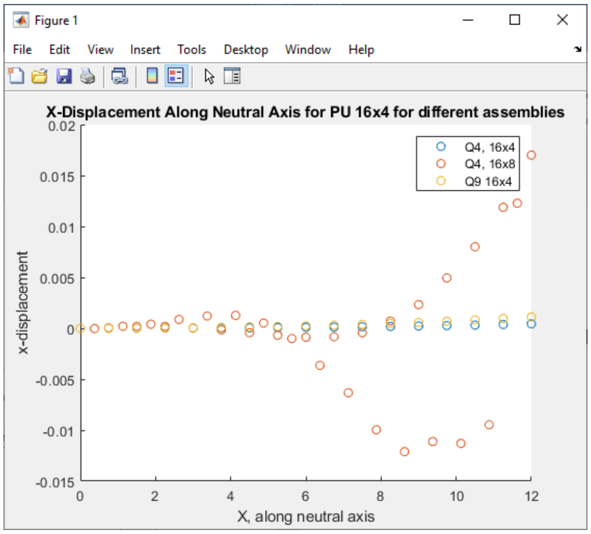


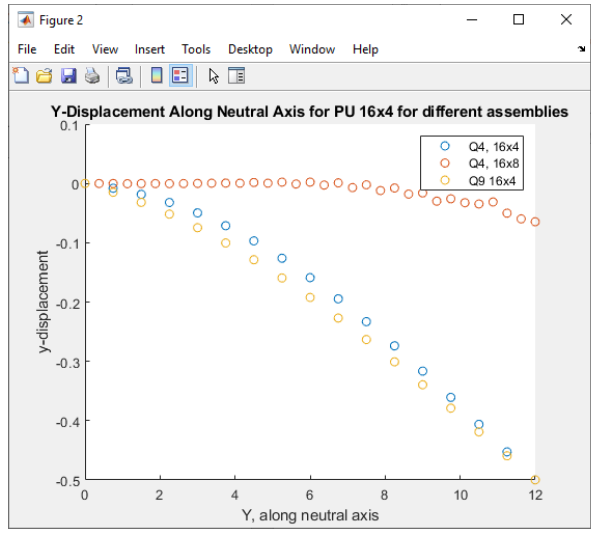


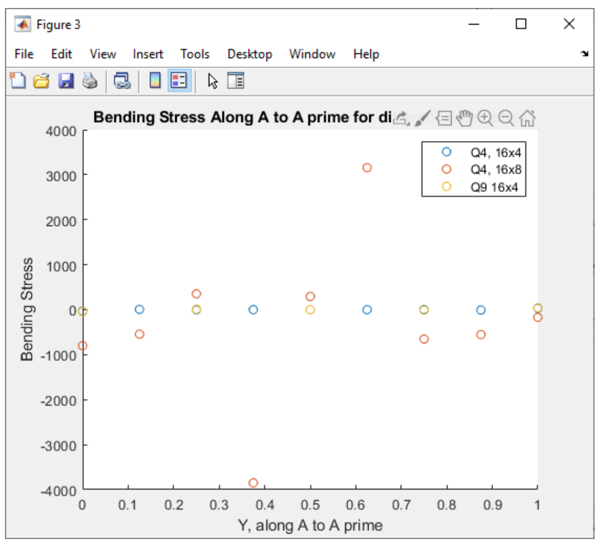


As the Poisson Ratio tends to the limit of 0.5, the bending stress and the x-displacements begin to become extremely large. This is because the D matrix starts to become much larger, making certain terms in the stiffness matrix extremely large. This in turn makes the matrix more and more ill-conditioned. While an inverse exists, it is extremely sensitive. So even the smallest computational inaccuracy will greatly affect the solution. This is in check with what we are observing. It’s not surprising that the y-displacement still remains relatively similar because we enforced displacement conditions on both ends of the cantilever beam.

Question 2, Part B







When the mesh becomes more refined, the solution becomes quite perturbed and very chaotic. This can be seen evidently in the bending stress, x-displacement, and y-displacement plots. This might be because we have large amounts of deformation in the beam, and if that occurs, then it may be harder to capture the displacement precisely in a smaller element. Instead, using a less refined mesh may not give the perfect solution, but a better one. Increasing the order of the element did seem to have that strong of an effect.

EB Beam Solution

$$EI\,\frac{\partial^4 w}{\partial x^4} = q$$

$$w'(0) = 0$$

$$w(0) = 0$$

$$\frac{\partial^4 w}{\partial x^4} = \alpha$$

$$w = Ax^3 + Bx^2 + \cancel{Cx}^{\,0} + \cancel{D}^{\,0} + \frac{\alpha x^4}{24} \qquad w(12) = -.5$$

(1) $w(12) = 1728A + 144B + 864\alpha = -0.5$

(2) $w'' = 6Ax + 2B + \dfrac{\alpha x^2}{2}$

$$M = -EI\,w''(12) = 0 \rightarrow w''(12) = 0$$

$$w''(12) = 72A + 2B + 72\alpha = 0$$

$$\begin{aligned} 72A + 2B &= -72\alpha \\ 1728A + 144B &= -864\alpha - 0.5 \end{aligned}$$

$$\begin{aligned} 864A + 120B &= -0.5 \\ 72A + 2B &= -72\alpha \end{aligned}$$

$$\begin{bmatrix} 864 & 120 \\ 72 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -0.5 \\ -72\alpha \end{bmatrix}$$

$$A = \frac{-1 + 8640\alpha}{-6912} = \frac{1 - 8640\alpha}{6912}$$

$$B = \frac{-62208\alpha + 36}{-6912} = \frac{62208\alpha - 36}{6912}$$

$$A = \frac{1}{6912} - \frac{5\alpha}{4}, \quad B = 9\alpha - \frac{1}{192}$$

Thus, EB beam theory gives

$$w(x) = \left[\frac{1}{6912} - \frac{5\alpha}{4}\right]x^3 + \left[9\alpha - \frac{1}{192}\right]x^2 + \frac{\alpha}{24}x^4$$

To compute stresses, $\sigma_{11} = -\dfrac{My}{I} = -Eyw''$

$$w''(x) = \left[\frac{1}{1152} - \frac{15}{2}\alpha\right]x + \left[18\alpha - \frac{1}{96}\right] + \frac{\alpha}{2}x^2$$

Here, $\alpha = \frac{q}{EI} = \frac{-\rho g H B}{EI}$ ✓

$$V = -EI\frac{\partial^3 w}{\partial x^3}$$

$$V = -EI\alpha x - \left[\frac{1}{1152} - \frac{15}{2}\alpha\right]EI$$

$$\sigma_{12} = -\frac{EI}{HB}\left[\alpha x + \frac{1}{1152} - \frac{15}{2}\alpha\right]$$

To get total reaction force, we can integrate the tractions as follows.

$$\underset{\sim}{t} = \underset{\sim}{\sigma}\,\underset{\sim}{n} = -\underset{\sim}{\sigma}\,\underset{\sim}{e_1} = -\sigma_{11}\underset{\sim}{e_1} - \sigma_{12}\underset{\sim}{e_2}$$

$$f_1\Big|_{x=0} = \int_{-1/2}^{1/2} -\sigma_{11}\,\underset{\sim}{e_1}\,dy\Big|_{x=0}$$

$$= E\int_{-1/2}^{1/2} -\left(18\alpha - \frac{1}{96}\right)\underset{\sim}{e_1}\,y\,dy = 0\,\underset{\sim}{e_1}$$

$$f_2\Big|_{x=0} = \int_{-1/2}^{1/2} -\sigma_{12}\,\underset{\sim}{e_2}\,dy\Big|_{x=0}$$

$$= \frac{EI}{HB}\left[\frac{1}{1152} - \frac{15}{2}\alpha\right]\underset{\sim}{e_2} \approx 5.0635\,\underset{\sim}{e_2}$$

$$= -\sigma_{12}(0)\,\underset{\sim}{e_2}$$