Advanced Problems in Physics Computing
Automatic Matrix Elements Project Description

Resource 1 *Introduction to Elementary Particles* (second edition) by David Griffiths; chapter 7

Resource 2 *Electroweak and Higgs Measurements ...* by Philip Ilten; chapter 2.1

Resource 3 `HelicityBasics.cc` from the Pythia 8 source code

Resource 4 *HELAS: HELicity amplitude ...* by Murayama, Watanabe, and Hagiwara; appendix A

Resource 5 Problem Set 1 and Problem Set 2 solutions

Monte Carlo generators play an in important role in particle physics. They provide a method to connect perturbative quantum field theory with empirically driven phenomenological models, provide a method to efficiently integrate high dimension integrals, and can output particle events similar to what would be seen in colliders. This last point is particularly important, as these events can then be passed through detector simulation, *e.g.* Geant 4. A general purpose Monte Carlo generator typically consists of the following phases:

1. Generation of the hard process. This consists of perturbatively calculating a cross-section following the formalism of Problem Set 2.

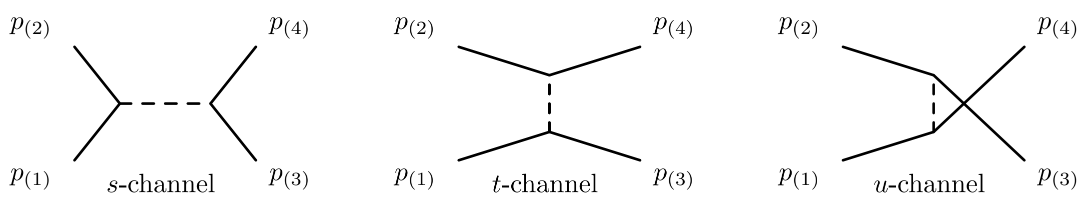
2. Parton showers. The hard process is typically calculated at leading order, or perhaps next-to-leading order, in the expansion of Feynman diagrams representing the process. Parton showers probabilistically apply additional one-to-two splittings, *e.g.* a quark splitting into a quark and a gluon. These parton showers evolve the energy scale of quarks and gluons down to the energy scale at which hadronisation can be performed.

3. Hadronisation. In nature, bare quarks and gluons are not observed, only bound colourless states like mesons and baryons. At this energy scale, typically around 2 GeV, perturbative quantum chromodynamics fails, and so other methods must be used instead to combine quarks and gluons into these states. This process is hadronisation.

4. Particle decays. The bound states formed during hadronisation may be unstable. During this final step, these particles are decayed into stable final state particles. This is typically done through isotropic decays, but can also be done with more sophisticated models.

The focus of this project is Step 1 of the Monte Carlo process, create an automated method to calculate two-to-two cross-sections using perturbative quantum electrodynamics (QED). This project relies heavily on the Dirac spinor formalism of Problem Set 2.

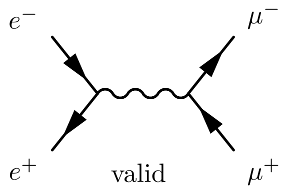
There are three types of diagrams which can contribute to any given two-to-two process, s-channel, t-channel, and u-channel.

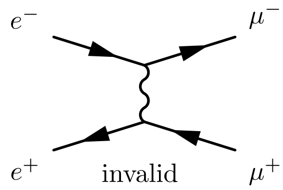


Each of these corresponds to a Mandlestam variable,

$$
\begin{aligned}
s &= (p_{(1)} + p_{(2)})^2 = (p_{(3)} + p_{(4)})^2 \\
t &= (p_{(1)} - p_{(3)})^2 = (p_{(4)} - p_{(2)})^2 \\
u &= (p_{(1)} - p_{(4)})^2 = (p_{(3)} - p_{(2)})^2
\end{aligned}
\tag{1}
$$

where $p_{(i)}$ indicates the momentum four-vector for a given particle. In QED only $\gamma \to f\bar{f}$ vertices are allowed, where f is a fermion and $\bar{f}$ is an anti-fermion. So for $e^+e^- \to \mu^+\mu^-$ the s-channel diagram is allowed, while the t-channel diagram is not allowed, since this would require a $\gamma \to e^+\mu^-$ vertex.





The u-channel diagram only exists for diagrams with identical initial state particles, identical final state particles, or an identical initial state and final state particle. For the process $\gamma\gamma \to e^+e^-$ there is both a t-channel diagram, and a u-channel diagram because the initial state is identical, *e.g.* two photons. The u-channel diagram can be calculated from the t-channel diagram by interchanging the helicity and momentum of two outgoing particles, *e.g.* $p_{(3)} \leftrightarrow p_{(4)}$ and $\lambda_{(3)} \leftrightarrow \lambda_{(4)}$. Anti-symmetrised diagrams, this process of interchanging momenta and helicity, are subtracted rather than added.

The Feynman rules for calculating these diagrams are succinctly outlined on page 243 of Griffiths (second edition) and are summarised here. The elements of a Feynman diagram have the following meanings.

	propagator	incoming line	outgoing line	symbol
spin-1/2	$\frac{i}{\gamma^\mu q_\mu - m}$	$u(q,\lambda)$, $\bar{v}(q,\lambda)$	$\bar{u}(q,\lambda)$, $v(q,\lambda)$	
spin-1 ($m = 0$)	$\frac{-ig_{\mu\nu}}{q^2}$	$\varepsilon_\mu(q,\lambda)$	$\varepsilon^\dagger_\mu(q,\lambda)$	

Identify all the fermion lines in a process. For each of these lines, trace the line backward, against the direction of the arrow. Include the mathematical element for each line and vertex, resulting in something of the form $\bar{u}Xu$, where X is a vertex factor. If the fermion line has an internal propagator, then X becomes XYX where X is still the vertex factors, and Y is the internal propagator for the fermion. The order of these elements is important because they represent vector and matrix multiplication. In this case the multiplication is always of the form (row vector)(matrix)(column vector). Here, ε is the polarisation vector for the photon. The definition of this vector in the Weyl basis can be found in Ilten's thesis, the Pythia 8 source code, or in the *HELAS* paper. The ordering of the external photon lines does not matter, since these are just scalars.

In Problem Set 2, the cross-section for specific helicity configurations was calculated. However, for a process like $e^+e^- \to \mu^+\mu^-$ we usually cannot measure the helicity of the outgoing muons. So instead, we need to calculate the cross-section where the matrix element has been averaged over the final state helicity configurations. Additionally, most experiments work with unpolarised beams so the helicity of the incoming electron and positron also will not be known. In this case the matrix element should be averaged over the initial state helicities, but summed over the final state helicities.

Goal 1 Create a syntax to specify a two-two process, *e.g.* $e^+e^- \to e^+e^-$ could be written as `"e+ e- > e+ e-"` or something similar. All standard model particles except gluon, W bosons, Z bosons, and Higgs bosons should be allowed.

Goal 2 Develop a method to determine all possible diagram contributions to a process. For example the process $e^+e^- \to e^+e^-$ has two contributing diagrams, an s-channel and a t-channel, while $e^+e^- \to \mu^+\mu^-$ only has one contributing diagram. Make sure all relevant QED rules are followed, *e.g.* only $\gamma \to f\bar{f}$ vertices are allowed.

Goal 3 Automatically generate the diagrams of Goal 2 to show the user what is being calculated.

Goal 4 Determine the matrix element for each contributing diagram, following the Feynman rules for QED.

Goal 5 Calculate the cross-section for any given process after specifying the process via the syntax of Goal 1, the initial state momentum four-vectors, and the initial state helicity configuration. The user should be able to specify an unpolarised initial state. Additionally, the total integrated cross-section or the differential cross-section in θ should be calculated. For reference, the cross-section for $e^+e^- \to \gamma\gamma$ with $E_{(e^+)} = E_{(e^-)} = 40$ GeV and the electron-positron system back-to-back is $\approx 1.7 \times 10^{-7}$ mb.