Automatic matrix element generation

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Abstract

In this Paper, we present a code for an automated method to calculate two-to-two cross-sections using perturbative quantum electrodynamics. For a given process the contributing Feynman diagrams will be generated and the scattering cross-section for different helicity configuration can be calculated. Using monte carlo integration one can obtain the total cross-section. The code generates the diagrams right, but calculates the wrong values for the scattering cross-section.

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1 Introduction

In particle physics, Monte Carlo generators are very important, since they provide a method to integrate high dimension integrals and can output particle events similar to what is observed in colliders. In general, a Monte Carlo generator consists of four steps [1].
The first step is the perturbative calculation of a cross-section. The second step is the Parton shower, which probabilistically applies an additional one to two splittings, e.g. a quark splitting into a quark and a gluon. Parton showers evolve the energy scale from a few hundred GeV down to a few GeV, where hadronization can be performed. The third step is Hadronization, where the quarks and gluons are combined to colourless hadrons. The last step is Particle decays. Here the in the hadronization formed unstable particles decay into final state stable particles.
This project is about the first step in the Monte Carlo generation. The goal is to create an automated method to calculate two-to-two cross-sections using perturbative quantum electrodynamics. We consider only two-to-two scattering and build the minimum diagrams necessary for this process to work and then calculate the cross-section from it.
In chapter 2 we present the necessary physical theory and the formulas implemented in the code. Chapter 3 shows the structure of the code and how to use it. In chapter 4 we present the results of this project and in chapter 5 we discuss the results, the advantage, and disadvantage of the code structure, and possible work that could be undertaken in the future.

2 Theory

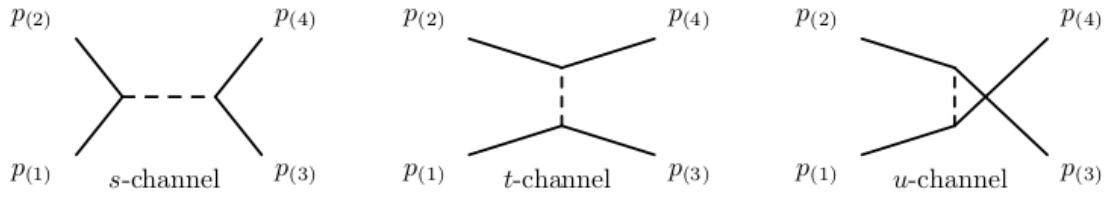


Figure 1: *Feynman diagrams for the s-, t- and u- channel [2].*

We distinguish between three diagrams, which can contribute to a two-to-two process s-,t- and u-channel, where each of them corresponds to a Mandelstam variable

$$s = (p_{(1)} + p_{(2)})^2 = (p_{(3)} + p_{(4)})^2 \tag{1}$$
$$t = (p_{(1)} - p_{(3)})^2 = (p_{(4)} - p_{(2)})^2 \tag{2}$$
$$u = (p_{(1)} - p_{(4)})^2 = (p_{(3)} - p_{(2)})^2 \,, \tag{3}$$

where p_1, p_2 are the four momenta of the incoming particles and p_3, p_4 of the outgoing particles. These variables are as the matrix amplitude Lorentz invariant. From the QED Lagrangian [3]

$$\mathcal{L}_{QED} = \sum_n \bar{\psi}_n (i\gamma^\mu \partial_\mu - m_n)\psi_n - \sum_n q_n - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}_n \gamma^\mu A_\mu \psi_n, \tag{4}$$

we can obtain the Feynman rules for the above diagrams. In this project, we will use the rules from [4]. For spin-1/2 particles, the external lines dor particles are given by the spinors $u(p, \lambda)$ and for antiparticles by the ant- spinor $\nu(p, \lambda)$, with the helicity $\lambda = \pm 1$, which is the spin of the particle projected along the direction of the particle momentum and the momentum four-vector p. The spinors in the Weyl basis are given by [5]

$$u(p,\lambda) = \begin{pmatrix} \kappa(p,\lambda)\sqrt{p^0 - \lambda q} \\ \kappa(p,\lambda)\sqrt{p^0 + \lambda q} \end{pmatrix} \quad \nu(p,\lambda) = \begin{pmatrix} -\lambda\kappa(p,-\lambda)\sqrt{p^0 + \lambda q} \\ \lambda\kappa(p,-\lambda)\sqrt{p^0 - \lambda q} \end{pmatrix} \,, \tag{5}$$

where $q = \sqrt{p^1 p^+ p^2 p^2 + p^3 p^3}$ and κ is given by,

$$\kappa = \begin{cases} \xi \begin{pmatrix} q + p^3 \\ ip^2 + p^1 \end{pmatrix} & \text{for} \lambda = +1, \quad \begin{pmatrix} o \\ 1 \end{pmatrix} \quad \text{as } p^3 \to -q \,, p^2 = 0 \,, p^1 \to +0 \\ \xi \begin{pmatrix} ip^2 - p^1 \\ q + p^3 \end{pmatrix} & \text{for} \lambda = -1, \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{as } p^3 \to -q \,, p^2 = 0 \,, p^1 \to +0 \end{cases} \tag{6}$$

with the normalization $\xi = \frac{1}{\sqrt{2(q^2 + qp^3)}}$. While the external lines for spin-1 particles are given by the polarization vectors [6]

$$\epsilon(p,\lambda) = \begin{cases} \lambda \begin{pmatrix} 0 \\ \frac{p_x p_z}{|\vec{p}|p_T} - \frac{ip_y}{p_T} \\ \frac{p_y pz}{|\vec{p}|p_T} + \frac{ip_x}{p_T} \\ \frac{-p_T}{|p|} \end{pmatrix} \text{for } \lambda = \pm 1 \\ \lambda \begin{pmatrix} 0 \\ p_z \\ 0 \\ 0 \end{pmatrix} \text{as } p_T \to 0 \end{cases} , \tag{7}$$

where $p_T = \sqrt{p_x^2 + p_y^2}$ is the transverse momentum. In experiments, the cross-section is measured, which is the process probability, per unit time, over the particle flux of the initial state. It is given in units of area, typically given in barns, where $1b = 10^{-28}m^2$.

In the center of mass (CM) frame the differential scattering cross-section is given by [4]

$$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S}{(p^0_{(1)} + p^0_{(2)})^2} \sqrt{\frac{\sum_{i=1}^3 (p^i_{(3)})^2}{\sum_{i=1}^3 (p^i_{(1)})^2}} (\mathcal{M} * \mathcal{M}) sin(\theta) d\theta d\phi \, , \tag{8}$$

where $S = \frac{1}{2}$ if the outgoing particles are identical and $S = 1$ otherwise. In the CM frame where both incoming particles are aligned with the z-axis, we can calculate the momenta for the outgoing particles by

$$p_{(3)} = (p^0_{(1)}, qsin(\theta)cos(\phi), qsin(\theta)sin(\phi), qcos(\theta)) \tag{9}$$

$$p_{(4)} = (p^0_{(1)}, -qsin(\theta)cos(\phi), -qsin(\theta)sin(\phi), -qcos(\theta)) \, , \tag{10}$$

in terms of θ and ϕ, where $q = \sqrt{(p^0_1)^2 - \frac{1}{2}(m_3^2 + m_4^2)}$. In the lab frame, we get for differential scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{p_3^2 S |\mathcal{M}|^2}{m_2 |p_1| (|p_3|(E_1 + m_2 c^2) - |p_1| E_3 cos(\theta))} \, . \tag{11}$$

In both of the above equations for the differential cross section M is the matrix element, which we can get from the Feynman diagrams. For the process $e^+e^- \to \mu^+\mu^-$, where only the s-channel diagram is valid, we get for the matrix element:

$$\mathcal{M} = \frac{-4\pi\alpha}{|p_{(1)} + p_{(2)}|^2} [\bar{u}_{(3)} \gamma^\mu \nu_{(4)}][\bar{\nu}_{(2)} \gamma_\mu u_{(1)}] \, . \tag{12}$$

If we consider a process we quarks are involved like for $e^+e^- \to q\bar{q}$, we have to replace the charge e with the quark charge $Q|e|$, and count each quark three times, one for each color [3]. For u, c, and t quarks we have $Q = 2/3$, while for d,s, and b quarks we have $Q = -1/3$. Since the cross sections are proportional to the square of the charge of the final-state particle, we obtain the cross-section by inserting a factor Q^2 and sum over the colors, which gives us a factor of $3Q^2$ in the above formulas. For the total cross-section we need to perform the integration over ϕ and θ. For the numerical integration

we will use the monte carlo integration. If we consider a two dimensional integral over a function $f(x, y)$, we can approximate it by averaging samples of the function f at uniform random points within the interval. With this method, the integral for a two-variable function is [1],

$$\int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} f(x, y)dxdy \approx < f > (x_{max} - x_{min})(y_{max} - y_{min}) \, , \tag{13}$$

where $< f >$ is the average value for $f(x, y)$ when randomly sampled with uniform x and y.

3 Code structur

The code is written in Python version 3.6. To draw the Feynman diagrams the external Package "feynman", which is publicly accessible [7]. This code consists of three python files ps1.py, ps2.py, and ameg.py, and one XML file Particle.xml. The ps1.py file contains the Vector and Matrix class [8], while the ps2.py file consists of the classes which provide us the particle data, the Dirac matrices, the spinors, and polarization vectors, and the Integrator class, which carries out the monte carlo integration to get the total cross-section. The classes and methods contained in ps1.py and ps2.py are used in the ameg.py file, which is the code for this project. The code consists of two classes: The Process class and the Scattering class. We only consider a QED two-two process. All standard model particles except for gluon, W/Z bosons, and Higgs bosons are allowed.

3.1 Process Class

The Process class aims to draw the valid Feynman diagrams for a given process. The user specifies the process by a string consisting of 5 space-separated elements, where the first two elements are the incoming particles and the last two the outgoing. The incoming and outgoing particles are separated by the symbol ->, e.g. "e+ e- -> mu+ mu-".
Given the process, the initializer first checks if the given process is valid, e.g if the given particles are standard model particles, the flavor or the charge conservation is violated, etc. If the process is invalid, an error message will be returned to the user.

The internal functions "i_channel_exist()", where i stand for one of the channels, checks if the channel "i" exists for the given proccess. The internal function "i_channel()" returns the diagram for the "i" channel. Using these two functions the method "diagram()" draws the possible diagrams for the given process.

3.2 Scattering Class

The aim of the Scattering class is to calculate the differential cross-section. Using the Integrator method from ps2.py we get the total cross-section. It takes as input the Process class (with the initialized process), the momenta pp1, and pp2 of the incoming particles and the helicity configuration h. The momenta, and the helicity configuration have to be given in the same order as the particles in the process.
The user is able to initialize the helicities of the incoming particles by setting

$$h = [i, j, None, None] \,, \tag{14}$$

where "i" and "j" are the helicities of the incoming particles. The user initializes the class by writing "Scattering(Process, pp1, pp2, h)". The internal method "mc()" calculates the matrix elements for the existing diagrams. The method "dxs(theta, phi)" method calculates the differential cross-section for a given theta and phi.Given the helicity structure of 14 the algorithm for calculating the differential cross-section, will sum the matrix elements over the possible helicity configurations of the outgoing particles.
In the case, where we have an unpolarized beam, the helicities of the incoming particles will not be known. The user can identify this case by setting "i" and "j" in 14 to "None". The algorithm will then

average the matrix elements over the initial state helicities and sum over the final state helicities. It is also possible to calculate the differential cross-section for a fixed helicity configuration for the outgoing particles. In this instance, the user has to change the ”None” values in 14 into the desired values for the helicities of the outgoing particles.

The momenta and the helicities have to be specified in the lab frame since the code is doing the calculations in this frame.
It is possible to tell dxs() in which frame it should calculate the differential cross-section. the default argument is set to frame =”lab”. By setting it to frame =”CM” it is doing the calculation in the CM frame. With this feature, one can check if the calculation is done right, since if we sum over all helicity configurations the scattering cross-section in the lab and CM frame should be the same.

4 Results

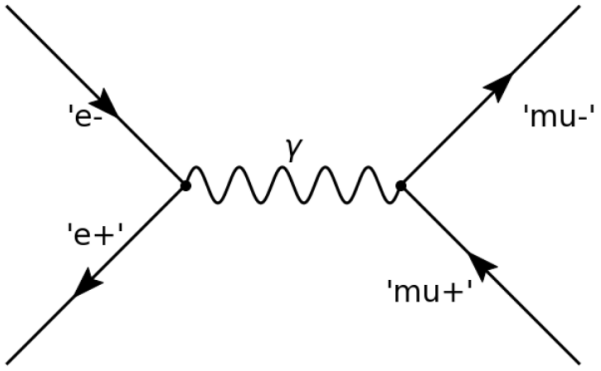
In this section, we will present the results of the Process and the scattering class

4.1 Process class

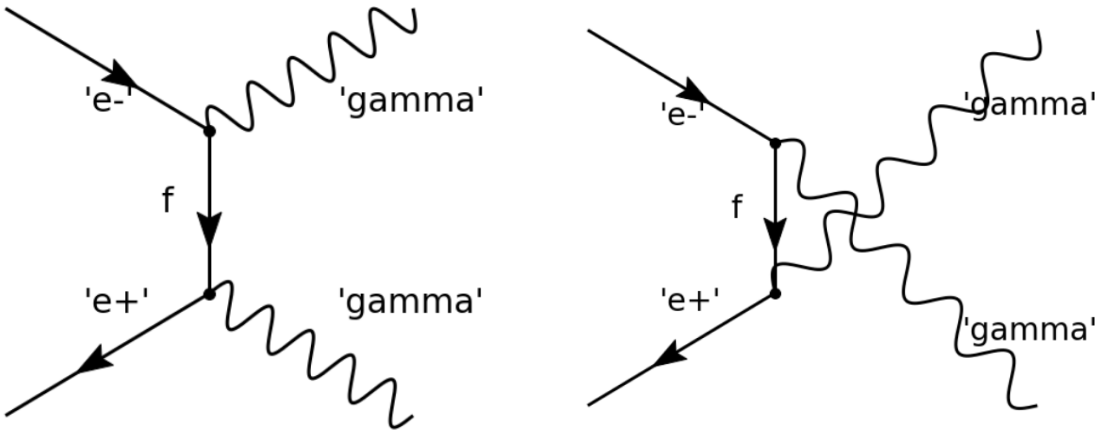
The task of the Process class is to generate the valid tree-level diagrams for a given process. We will look at the process 'e+ e- $\rightarrow \mu^+\mu^-$' and pair annihilation 'e+ e- $\rightarrow \gamma\gamma$'. By calling

$$a = Process(”e+\ e-\ ->\ mu+\ mu-”) \tag{15}$$
$$a.diagram() \ , \tag{16}$$

the code generates the following diagram,



which is the only valid diagram for this process. For this process, the t-channel diagram is not allowed, since it would require a $\gamma \rightarrow e^+\mu^-$. The u-channel diagram does not exist, since it only exists for diagrams with identical initial state particles, identical final state particles, or an identical initial state and final state particle. For the process 'e+ e- $\rightarrow \gamma\gamma$' the following diagrams are generated.



The s diagram does not exist, since it would require $f \rightarrow \bar{f}f$ and $f \rightarrow \gamma\gamma$. The diagram() method generates also for other processes the right diagrams.

4.2 Scattering class

For the validation of the Scattering class, we will calculate the scattering cross-section for processes where we have the literature value. From problem set 2 we have the scattering corss-section for the process $e^+e^- \to \mu^+\mu^-$ for all possible helicity, where you can find in the left table the literature values [ps2] and in the right, the ones from the Scattering class.

Table 1: *Literature values*

λ_{e^-}	λ_{e^+}	λ_{μ^-}	λ_{μ^+}	σ[mb]
-1	-1	-1	-1	1.6×10^{-26}
-1	-1	-1	1	2.8×10^{-20}
-1	-1	1	-1	2.8×10^{-20}
-1	-1	1	1	1.6×10^{-26}
-1	1	-1	-1	1.2×10^{-15}
-1	1	-1	1	2.2×10^{-09}
-1	1	1	-1	2.2×10^{-09}
-1	1	1	1	1.2×10^{-15}
1	-1	-1	-1	1.2×10^{-15}
1	-1	-1	1	2.2×10^{-09}
1	-1	1	-1	2.2×10^{-09}
1	-1	1	1	1.2×10^{-15}
1	1	-1	-1	1.6×10^{-26}
1	1	-1	1	2.8×10^{-20}
1	1	1	-1	2.8×10^{-20}
1	1	1	1	1.6×10^{-26}

Table 2: *Values from the scattering class*

λ_{e^-}	λ_{e^+}	λ_{μ^-}	λ_{μ^+}	σ[mb]
-1	-1	-1	-1	1.6×10^{-28}
-1	-1	-1	1	2.8×10^{-26}
-1	-1	1	-1	2.8×10^{-26}
-1	-1	1	1	1.6×10^{-28}
-1	1	-1	-1	1.2×10^{-117}
-1	1	-1	1	2.2×10^{-15}
-1	1	1	-1	2.2×10^{-15}
-1	1	1	1	1.2×10^{-17}
1	-1	-1	-1	1.2×10^{-17}
1	-1	-1	1	2.2×10^{-15}
1	-1	1	-1	2.2×10^{-15}
1	-1	1	1	1.2×10^{-17}
1	1	-1	-1	1.6×10^{-28}
1	1	-1	1	2.7×10^{-26}
1	1	1	-1	2.7×10^{-26}
1	1	1	1	1.6×10^{-28}

The cross-section for $e^+e^- \to \gamma\gamma$ with $E_{e^+} = E_{e^-} = 40$ GeV and the electron-positron system back to back is $\sigma \approx 1.7 \times 10^{-7}$ mb [ps2]. The value we get by the code is $\sigma_{CM} \approx 2.16 \times 10^{-12}$ mb in the CM frame and $\sigma_{lab} \approx 6.82 \times 10^{-36}$ mb in the lab frame. A third way how to check if the code is giving us the right values is by calculating the scattering cross-section averaged over the initial states helicities and summed over the final states helicities in the CM-frame and lab-frame. In both frames, we should get the same value. As we could see for the process $e^+e^- \to \gamma\gamma$ we do not get the same value for both frames. For the process $e^+e^- \to \mu^+\mu^-$ we get $\sigma_{CM} \approx 8.70 \times 10^{-9}$ and $\sigma_{lab} \approx 3.61 \times 10^{-31}$.
As we can see our code does not give us the right values for all the checks we made.

5 Conclusion

In this paper a code for generating the Feynman diagrams and calculating the scattering cross-section was presented. The Process class, which checks if the given process is valid and then generates the diagrams, performed very code in the tests. It recognizes valid and invalid processes right and generates the right diagrams for a given process. In chapter 4 we showed the results for the processes $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$.
The Scattering class on the other hand performed very poorly. The values for the scattering cross-sections do not agree with the literature values. Also, the cross-sections averaged over the initial state and summed over the final state helicities in the lab- and CM-frame does not agree.
For this project, we chose a very simple way to build the code. Only QED rules were used, which means that we only have one vertex $\gamma \to f\bar{f}$. Due to the simplicity of QED one can just test if each of the diagrams is possible, which make the code very simple. The disadvantage of this approach is that it makes it difficult to scale for different things like extending the code for QCD or electroweak rules.
The smarter design decision would be to make a fully flexible automatic code. This can be achieved by taking the QED vertex and then build all possible diagrams, which can get very complicated. The advantage of it is that one can than build an arbitrary large matrix element you want to calculate.
Work that can be done in the future is to expand this project by including electroweak and QCD vertices.

References

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