# Mini-course Machine Learning in Empirical Economic Research

## Lecture 3: Penalized regression and applications in treatment evaluation

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# Setting

Fit a regression curve to model

$$y_i = f(x_i) + \epsilon_i = \beta_0 + x_i'\beta_1 + \epsilon_i$$

- $n$ observations
- $x_i = p_n$-dimensional covariate vector
- $\epsilon_i$ = idiosyncratic error term

# Objective

- prediction (for now)
- training error = measure of *in sample* fit

$$\overline{\text{err}}(\hat{f}) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{f}(x_i)\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - x_i'\hat{\beta}_1\right)^2$$

- EPE (expected prediction error) = measure of fit on a new observation
- assume $\epsilon \perp x$, $\mathbb{E}\epsilon = 0$ and $\mathbb{E}\epsilon^2 = \sigma^2$

# Expected prediction error

- $\mathbb{E}_{\mathcal{T}}$ = expectation operator wrt training sample
- $E_{y,x}$ = integral wrt probability measure of a new $(y, x')$ observation

$$
\begin{aligned}
EPE(\hat{f}) =& \mathbb{E}_{\mathcal{T}} E_{y,x} \left( y - \hat{f}(x) \right)^2 \\
=& \sigma^2 + E_{y,x} \left\{ \left( f(x) - \mathbb{E}_{\mathcal{T}} \hat{f}(x) \right)^2 + \mathbb{E}_{\mathcal{T}} \left( \hat{f}(x) - \mathbb{E}_{\mathcal{T}} \hat{f}(x) \right)^2 \right\} \\
=& \underbrace{\sigma^2}_{\text{irreducible error}} + \underbrace{E_{y,x} \, \text{bias}^2 \left( \hat{f}(x) \right)}_{\text{bias}} + \underbrace{E_{y,x} \, \text{var} \left( \hat{f}(x) \right)}_{\text{variance}}
\end{aligned}
$$

# Gauss-Markov assumptions = OLS is BLUE

OLS is unbiased

$$\text{bias}\left(\hat{f}(x)\right) = \mathbb{E}_{\mathcal{T}}\left(\hat{\beta}^{\text{ols}} - \beta\right) x = 0$$

but has potentially large variance

$$\text{var}\left(\hat{f}(x)\right) \approx \frac{\sigma^2 p_n}{n}$$

- OLS is not well-suited for prediction
  - tries to estimate every component $\beta_j$
  - doesn't trade off noise and predictive power
- if $p_n \gg n$ then OLS is not even computable

# Regression with $p_n \gg n$

- OLS estimator

$$\hat{\beta}^{\text{ols}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$$

- why is this not computable for $p_n \gg n$?
- Idea of ridge regression:

$$\hat{\beta}^{\text{ridge}} = \left(\mathbf{X}'\mathbf{X} + \lambda\,\text{Diag}\left((0, 1, \dots 1)'\right)\right)^{-1}\mathbf{X}'\mathbf{y}$$

- $\lambda =$ regularization parameter

# Ridge regression

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta_0 \in \mathbb{R}, \beta_1 \in \mathbb{R}^{p_n}} \sum_{i=1}^{n} \left(y - \beta_0 - x_i' \beta_1\right)^2 + \lambda \|\beta_1\|_2^2$$

where $\|\beta_1\|_q = \left(\sum_{j=1}^{p_n} |\beta_{1,j}|^q\right)^{1/q}$.

# $L_q$-penalized regression

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta_0 \in \mathbb{R}, \beta_1 \in \mathbb{R}^{p_n}} \underbrace{\sum_{i=1}^{n} \left(y - \beta_0 - x_i' \beta_1\right)^2}_{\text{loss function}} + \underbrace{\lambda \|\beta_1\|_q^q}_{\text{penalty term}}$$

- because of the penalty term "best" in-sample fit is costly
  - reduces *overfitting*
- cost of choosing "large" coefficients $\Rightarrow$ *shrinkage*
- choice of $q$ = choice of $\lambda \mapsto \mathcal{F}_\lambda$

  q = 2 Ridge regression

  q = 1 Lasso regression (Least absolute shrinkage and selection estimator, Tibshirani 1996)

## Intuition of how shrinkage improves prediction

Intercept is not penalized: we will always have (verify!)

$$\hat{\beta}_0^{L_q,\lambda} = \bar{y} - \bar{x}'\hat{\beta}_1^{L_p,\lambda}$$

Predictors of $y$

$$\mathbb{E}[y] \qquad \text{or} \qquad \mathbb{E}[y \mid x]$$

$$\lambda \to \infty \;\; \|\hat{\beta}_1^{L_p,\lambda} - 0\|_q \to 0 \Rightarrow \text{estimate } \mathbb{E}[y]$$

$$\lambda \to 0 \;\; \|\hat{\beta}_1^{L_p,\lambda} - \hat{\beta}_1^{\text{ols}}\|_q \to 0 \Rightarrow \text{estimate } \mathbb{E}[y \mid x]$$

- $L_q$-penalized regression "shrinks" towards the unconditional mean
- "shrink" towards a model that is not complex (=unconditional mean)

## Choice of $\lambda$

- the regularization parameter $\lambda$ is a *tuning parameter*
- chosen by the empirical researcher
- choose $\lambda$ to maximize out-of-sample predictive power (we focus on prediction for now)
  - independent validation sample
  - *k*-fold cross-validation

# Simulation study

- all code is on `https://github.com/adzemski/ML_notes`
- sample size $n = 100$
- number of regressors $p_n = 50$

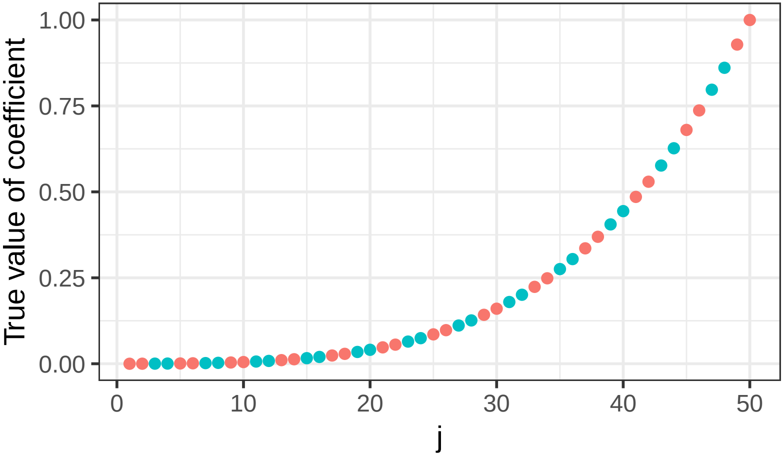


Figure: True values of coefficients

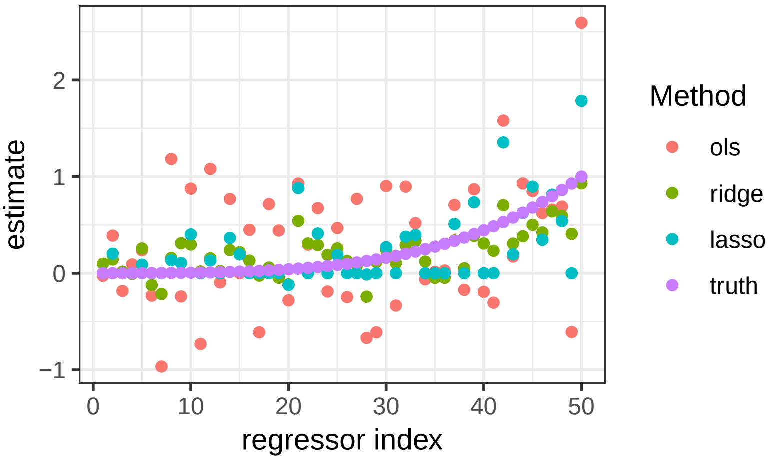


Figure: Estimation results

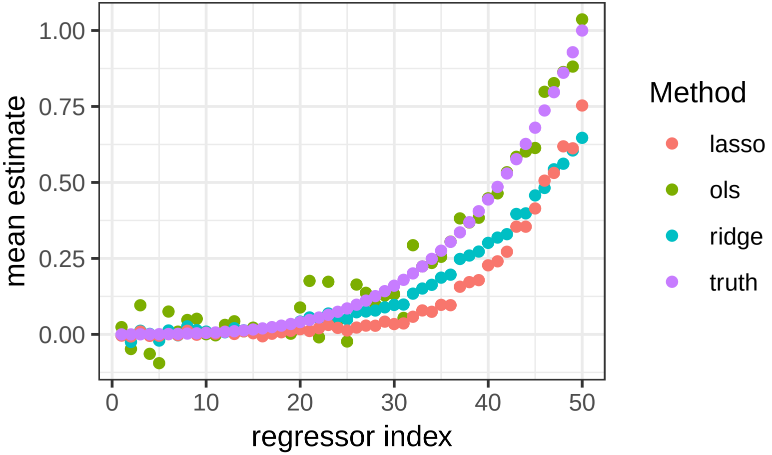


Figure: Expected estimates (average over 200 simulations)

# OLS is terrible for prediction

| | method | mse |
|---|---|---|
| 1 | ols | 27.91 |
| 2 | ridge | 2.89 |
| 3 | lasso | 3.37 |

Table: Mean-squared-error $MSE(f)$

- Not surprising that Ridge performs best (James-Stein estimator, Empirical Bayes theory)

# Variable selection

- an estimator $\hat{\beta}$ selects a variable $x_j$ if $|\hat{\beta}_j| \neq 0$
- variable selection $=$ model selection

# Lasso selects variables

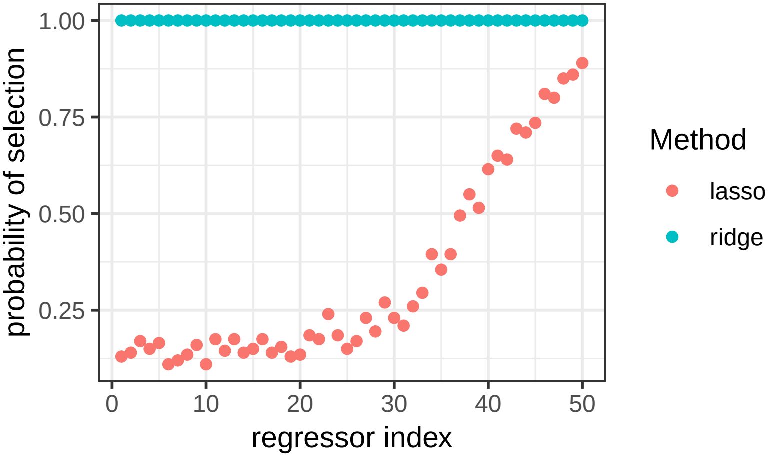


Figure: Probability of including variables (average over 200 simulations)

# Instability of variable selection

- is it a problem for prediction?
- for interpretation?

# Understanding variable selection by the Lasso

for $p_n = 2$ we solve

$$\min_{\beta} \|\mathbf{y} - \beta_0 - \mathbf{x}_1\beta_1 - \mathbf{x}_2\beta_2\|_2^2$$

$$\text{s.t.} \begin{cases} |\beta_1|^2 + |\beta_2|^2 \leq s & \text{if method} = \text{ridge} \\ |\beta_1| + |\beta_2| \leq s & \text{if method} = \text{lasso} \end{cases}$$

- why?

# Contour sets of the loss function

- $\mathbf{X}_{+1}$ = design matrix including intercept $(n \times (1 + p_n))$
- $\hat{\beta}^{\text{ols}}$ = OLS estimator including intercept

contour sets

$$\left\{\beta \in \mathbb{R}^{p_n+1} : \|\mathbf{y} - \mathbf{X}_{+1}\beta\|_2^2 = c\right\}$$

are empty or ellipsoids centered at $\hat{\beta}^{\text{ols}}$ (verify!)

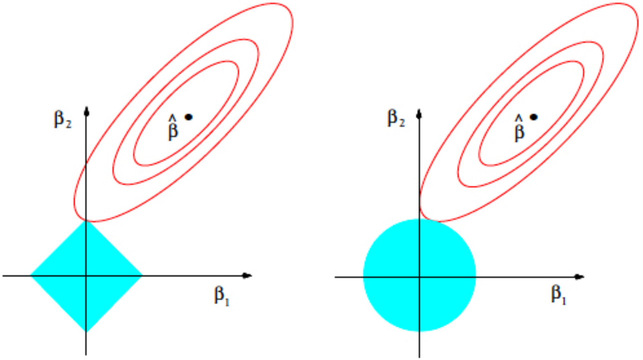


**FIGURE 3.11.** *Estimation picture for the lasso (left) and ridge regression (right). Shown are contours of the error and constraint functions. The solid blue areas are the constraint regions* $|\beta_1| + |\beta_2| \le t$ *and* $\beta_1^2 + \beta_2^2 \le t^2$*, respectively, while the red ellipses are the contours of the least squares error function.*

Figure: Figure 3.11 from Hastie, Tibshirani, and Friedman (2009)

# Series estimation (Newey 1997)

- estimating a smooth regression curve $f : \mathbb{R} \to \mathbb{R}$
- Taylor expansion

$$\begin{aligned} f(x) =& f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + r(x) \\ =& \underbrace{b_0(x) + b_1(x) + b_3(x)}_{\text{approximation by } p_n = 3 \text{ series terms}} + \underbrace{r(x)}_{\text{“small” remainder}} \end{aligned}$$

- justification as non-parameteric technique
  - $p_n \to \infty$ asymptotics
- may make sense to choose orthogonal basis functions
  - e.g. Legendre polynomials, Fourier basis
- generalization
  - non-smooth functions
  - multi-variate functions

# High-dimensional regression vs. non-parametric regression

*"We differ from most of the existing literature that considers $p \ll n$ series terms by allowing $p \gg n$ series terms from which we select $s \ll n$ terms to construct the regression fits. Considering an initial broad set of terms allows for more refined approximations of regression functions relative to the usual approach that uses only a few* **low-order** *[my emphasis] terms."(Belloni, Chernozhukov, and Hansen 2014)*

# A sparse design

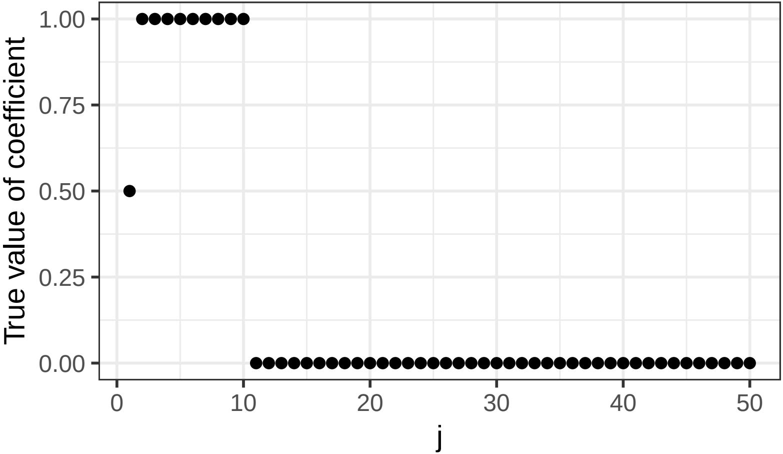


Figure: True values of coefficients

# Lasso detects many of the zero coefficients

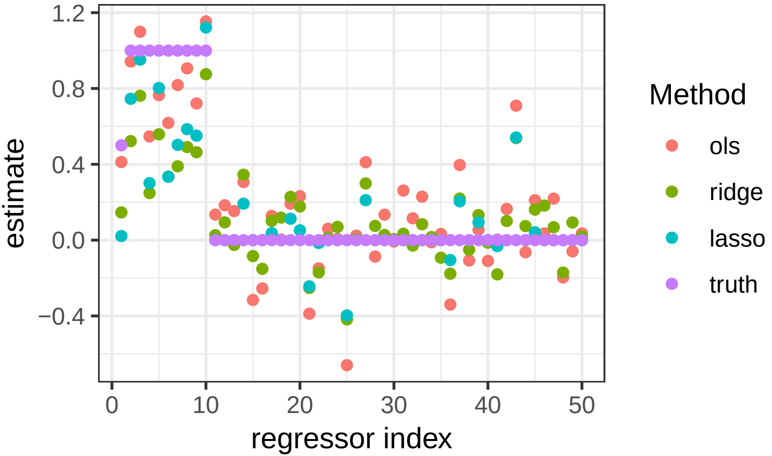


Figure: Estimation results

# Lasso is good at selecting the true model

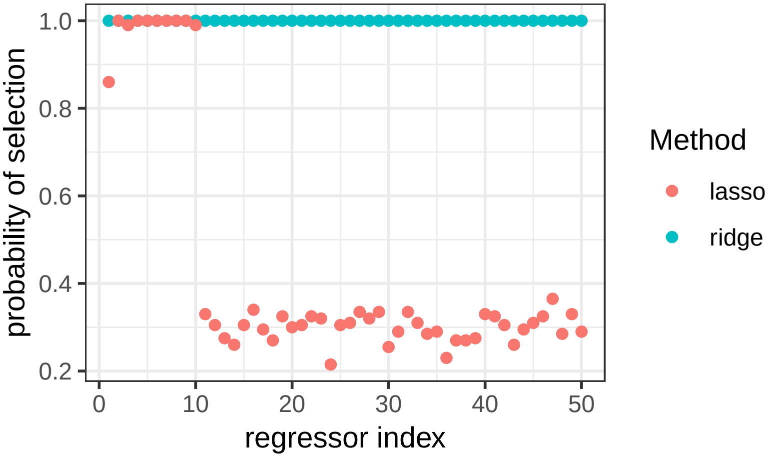


Figure: Probability of including variables (average over 200 simulations)

# But still shrinkage

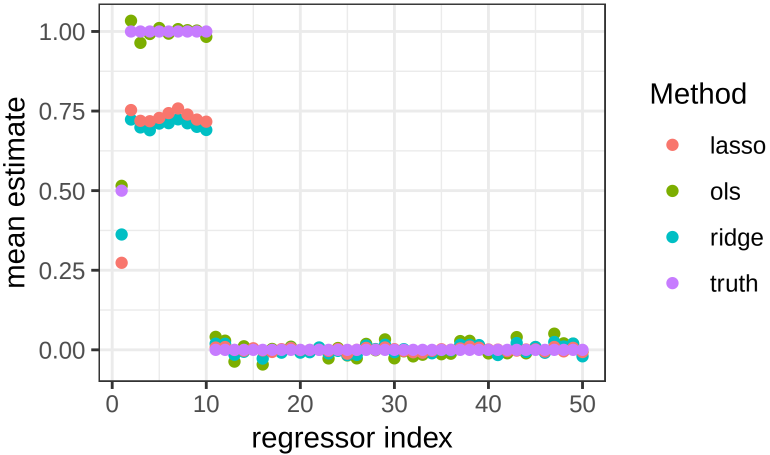


Figure: Expected estimates (average over 200 simulations)

# Post-Lasso

- Run lasso on all possible variables
- Select the variable $j$ with $\hat{\beta}_j^{Lasso} \neq 0$
- Run OLS on the selected variable ($=$ post-selection estimator)

# Always include a variable

- Post-Lasso makes sense if we are interested in the value of the coefficients
- If there is a specific variable of interest we should always select it
- assume $x_1$ = treatment dosage
- Lasso solves

$$\min_{\beta} \sum_{i=1}^{n} \left( y_i - \beta_0 - \beta_1 x_1 - \sum_{j=2}^{p} x_{j,i} \beta_j \right)^2 + \lambda \sum_{j=2}^{p} |\beta|_j$$

(we don't penalize the intercept and the coefficient on $x_1$)

# Penalty matrix

Lasso solves

$$\min_{\beta} \sum_{i=1}^{n} \left(y_i - \beta_0 - \beta' x_i\right)^2 + \lambda \|\Psi\beta\|_1$$

where $\Psi$ is a weight matrix. To exclude the first two coefficients from penalization put

$$\Psi = \mathrm{Diag}\left((0, 0, 1, \ldots, 1)'\right).$$

# Post-Lasso (uncorrelated design)

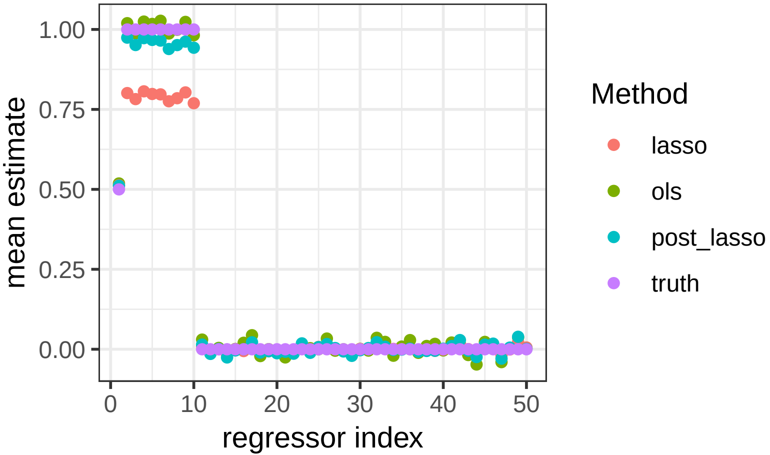


Figure: Expected estimates (average over 200 simulations)

- Why does the post-Lasso *on average* under-estimate the true values?
- Trying to reverse shrinkage has adverse effect on MSE:

| | method | mse |
|---|---|---|
| 1 | ols | 20.63 |
| 2 | lasso | 1.48 |
| 3 | post_lasso | 20.15 |

Table: Mean-squared-error $MSE(\hat{f})$

- Is post-lasso estimator $\hat{\beta}_1^{\text{post}}$ a better estimator than $\hat{\beta}_1^{\text{ols}}$? (homework)

# Introducing correlation

- In uncorrelated design post selection estimator seems to "work"
- now introduce correlation: $\mathrm{cor}(x_1, x_2) = 0.95$, all other variables uncorrelated

# Post-Lasso (correlated design)

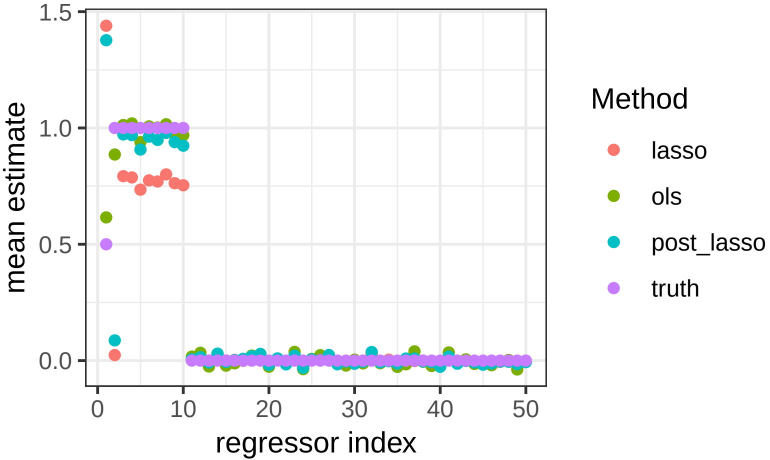


Figure: Expected estimates (average over 200 simulations)

# What happened?

- bias of estimated treatment effect = almost 200% of true effect

# “Double” selection procedure (Belloni, Chernozhukov, and Hansen 2014)

- intuition: detect variables that are highly correlated with $x_1$ and make sure they are always selected
- model: based on standard model for “regression type” treatment evaluation model under unconfoundedness

# Double selection algorithm

outcome equation:

$$y_i = \alpha_{g,0} + \alpha_1 x_{1,i} + \beta'_{g0} x_{-1,i} + r_{g,i} + \zeta_i$$

selection equation:

$$x_{1,i} = \alpha_{m,0} + \beta'_{m0} x_{-1,i} + r_{m,i} + \nu_i$$

1. variables selected from $x_{-1,i}$ by running Lasso on *outcome* equation $= \hat{I}_1$
2. variables selected from $x_{-1,i}$ by running Lasso on *selection* equation $= \hat{I}_2$
3. run post-Lasso on $\hat{I}_1 \cup \hat{I}_2$

# Sparsity assumption

outcome equation:

$$y_i = \alpha_{g,0} + \alpha_1 x_{1,i} + \beta'_{g0} x_{-1,i} + r_{g,i} + \zeta_i$$

selection equation:

$$x_{1,i} = \alpha_{m,0} + \beta'_{m0} x_{-1,i} + r_{m,i} + \nu_i$$

sparsity of linear component:

$$\|\beta_{m0}\|_0 \leq s_n \quad \text{and} \quad \|\beta_{g0}\|_0 \leq s_n,$$

where

$$\|\beta\|_0 = \sum_{j=1}^{p_n} \mathbf{1}(\beta_j \neq 0)$$

# Sparsity assumption

size of remainder (= approximation error):

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}r_{gi}^2\right)^{1/2} \leq \sqrt{\frac{s_n}{n}} \quad \text{and} \quad \left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}r_{mi}^2\right)^{1/2} \leq \sqrt{\frac{s_n}{n}}$$

- $s_n$ has to be small enough

$$\frac{s_n^2 \log^2(n \vee p_n)}{n} = o(1)$$

- Do you expect double selection to fix the “problem” in our simulation above?
- Simulate this yourself (homework).

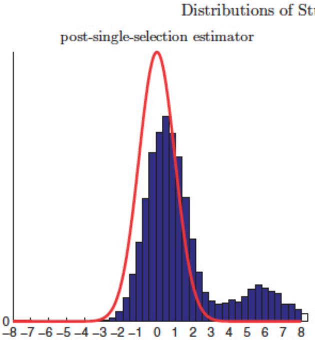


FIGURE 1

The finite-sample distributions (densities) of the standard post-single selection estimator (left panel) and of our proposed post-double selection estimator (right panel). The distributions are given for centered and studentized quantities. The results are based on 10000 replications of Design 1 described in Section 4.2, with $R^2$'s in equation (2.6) and (2.7) set to 0.5.

Figure: Figure 1 from Belloni, Chernozhukov, and Hansen (2014)

# Endogenous treatment

- Belloni, Chernozhukov, Fernández-Val, et al. (2017)
- moment condition model
- treatment effect $\alpha_1$ is defined via moment condition

$$\mathbb{E}_P \psi \left( W, \alpha_1, h_0 \right) = 0$$

- $h_0$ is a functional-valued nuisance parameter that takes values in a space that is approximated well by a "sparse" function space
  - "sparse" = not too complex = entropy is not too large
- the setting in Belloni, Chernozhukov, and Hansen (2014) is a special case (show this!)

- the approach in Belloni, Chernozhukov, and Hansen (2014) requires sparse model for selection equation
- is *not* robust to misspecification of selection equation (propensity score)
- *rebalancing methods* are often robust to misspecification of propensity score (J. M. Robins and Ritov 1997)
- examples of rebalancing methods
    - inverse probability weighting (Cassel, Särndal, and Wretman 1976; J. Robins 1986)
    - inverse probability tilting (Graham, Pinto, and Egel 2012)

# Athey, Imbens, and Wager (2018)

- treatment effect on the treated $\tau$
- combine regression and rebalancing techniques
- $w$ = binary treatment indicator
- $x$ = control variables ($p_n$-vector)
- $y$ = outcome variable
- outcome model

$$y_i = \begin{cases} \beta_c' x_i + \epsilon_i & \text{if } w_i = 0 \\ w_i y_i & \text{if } w_i = 1 \end{cases}$$

- $\beta_c$ is sparse

$$\|\beta_c\|_0 \leq s_n \quad \text{and} \quad \frac{s_n \log(p_n)}{n} = o(1)$$

# A regression approach

- $n_t = \sum_{\{i:w_i=1\}} 1$
- $n_c = \sum_{\{i:w_i=0\}} 1$
- $\bar{x}_t = \sum_{\{i:w_i=1\}} x_i / n_t$
- $\bar{y}_t = \sum_{\{i:w_i=1\}} y_i / n_t$

$$\hat{\tau} = \bar{y}_t - \bar{x}_t' \hat{\beta}_c$$

- Belloni, Chernozhukov, and Hansen (2014) show that $\hat{\beta}_c =$ post-single-selection estimator does not work well (OV bias)

# A balancing approach

balancing estimator

$$\hat{\tau} = \bar{y}_t - \sum_{\{i:w_i=0\}} \hat{\gamma}_i y_i$$

- $\{\hat{\gamma}_i\}_{i:w_i=0}$ weight sequence that re-weighs covariates in control group so that the covariate distribution in control group "looks like" covariate distribution in treatment group

$$\sum_{i:w_i=0} \hat{\gamma}_i = 1$$

# Choice of weights

- $\hat{e}(x_i) =$ estimator of propensity score
- inverse probability weighting

$$\hat{\gamma}_i \propto \frac{\hat{e}(x_i)}{1 - \hat{e}(x_i)}$$

- does not enforce exact balance
- in finite dimensions: does not achieve the semi-parametric efficiency bound (Graham, Pinto, and Egel 2012)

# Intuition for robustness of re-balancing

*"...in a linear model, the bias for estimators based on weighted averaging depends solely on* $\bar{x}_t - \sum_{\{i:w_i=0\}} \hat{\gamma}_i x_i$*. Therefore, getting the propensity model exactly right is less important than accurately matching the moments of* $\bar{x}_t$*. In high dimensions, however, exact re-balancing weights do not in general exist." (Athey, Imbens, and Wager 2018)*

# Approximate residual balancing

$$\hat{\tau} = \bar{y}_t - \bar{x}_t \hat{\beta}_c - \overbrace{\sum_{\{i: w_i = 0\}} \hat{\gamma}_i \qquad \underbrace{\left(y_i - x_i \hat{\beta}_c\right)}_{\text{average over this } = \text{ bias in control group}}}^{\text{approximate OV bias in treatment group}}$$

- $\{\hat{\gamma}_i\}_{i:w_i=0}$ "approximately" balances treatment and control group

# Estimator $\hat{\beta}_c$

- Lasso estimator

$$\hat{\beta}_c = \arg\min_\beta \left\{ \sum_{i:w_i=0} (y_i - x_i'\beta)^2 + \lambda \|\beta\|_1 \right\}$$

- selection is based *only on the outcome equation*
- Lasso estimator, not post-Lasso (why?)
- tuning parameter $\lambda$

# Weight estimation

- $\mathbf{X}_c$ = design matrix in control group ($n \times p_n$)

weight estimation

$$\hat{\gamma} = \arg\min_{\gamma \in \mathbb{R}^{n_c}} \left\{ \zeta \max_{j=1,\ldots,p_n} (\bar{x}_t - \mathbf{X}_c' \gamma)^2 + (1-\zeta)\|\gamma\|_2^2 \right.$$
$$\left. \text{s.t. } \sum_{i:w_i=0} \gamma_i = 1 \text{ and } 0 \leq \gamma_i \leq n_c^{-2/3} \right\}$$

- tuning parameter $\zeta$