Sampling and Inference for Beta Neutral-to-the-Left Models of Sparse Networks

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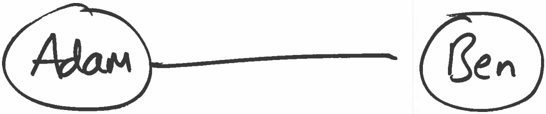
Contents

Temporal networks

Example

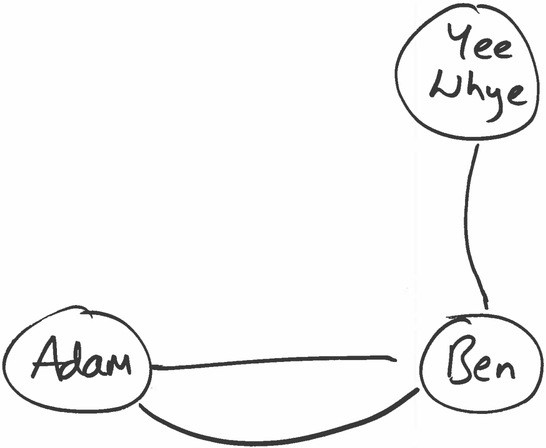
- Messages sent between people over time
- Protein-protein interactions

Temporal networks

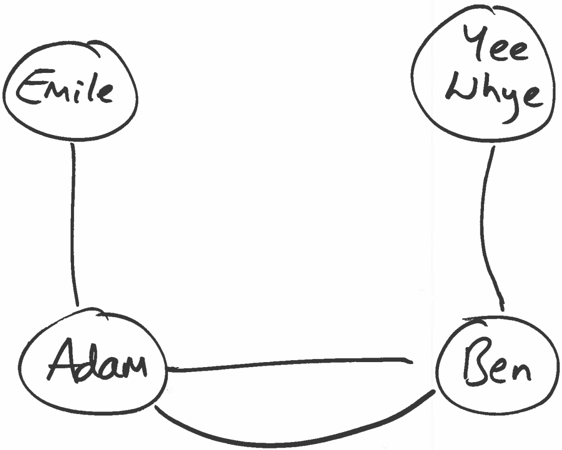


Temporal networks

Temporal networks

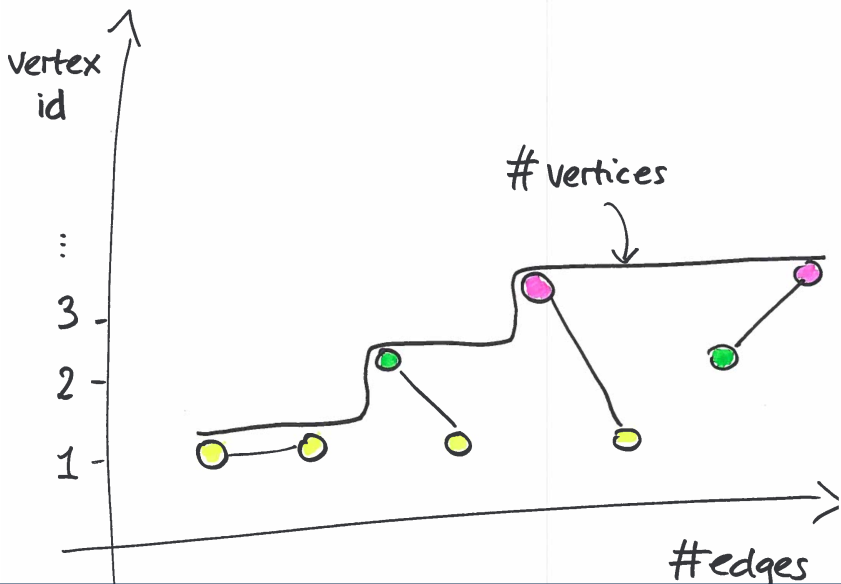


Temporal networks

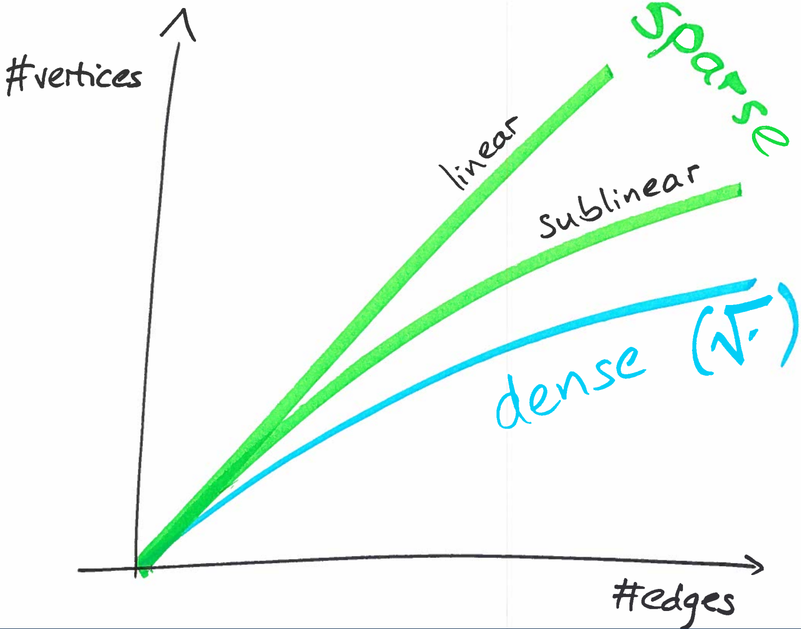


Edges and vertices

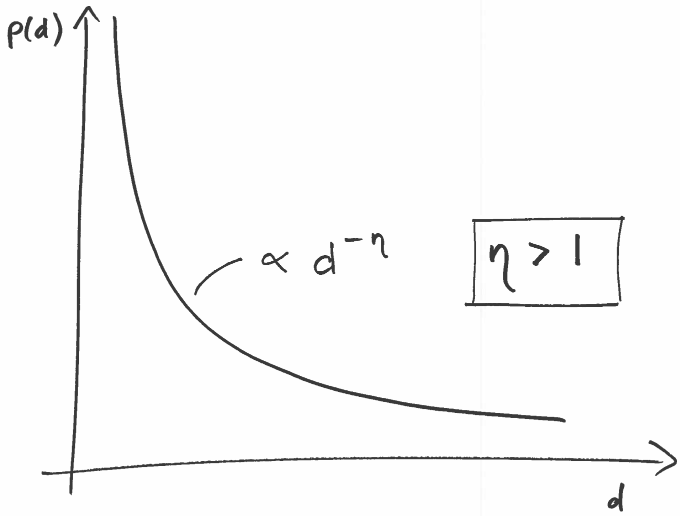
TODO: break picture into 4



Sparsity



Power law degree distribution



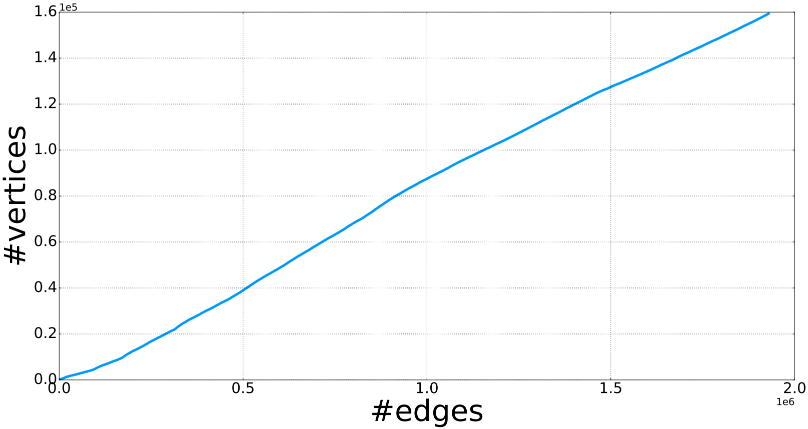
Sparsity and power law

$$
\begin{aligned}
\textbf{Sublinear}\text{ sparsity} &\quad\Longleftrightarrow\quad \eta \in (1,2) \\
\textbf{Linear}\text{ sparsity} &\quad\Longleftrightarrow\quad \eta > 2
\end{aligned}
$$

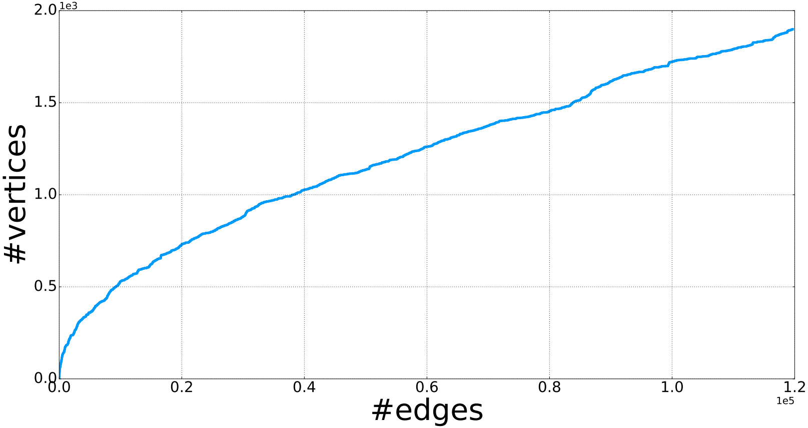
Empirical study

SNAP dataset [1]	# of vertices	# of edges
Ask Ubuntu	159,316	964,437
UCI social network	1,899	20,296
⋮	⋮	⋮

Ask Ubuntu



UCI social network



Models

Vertex exchangeable

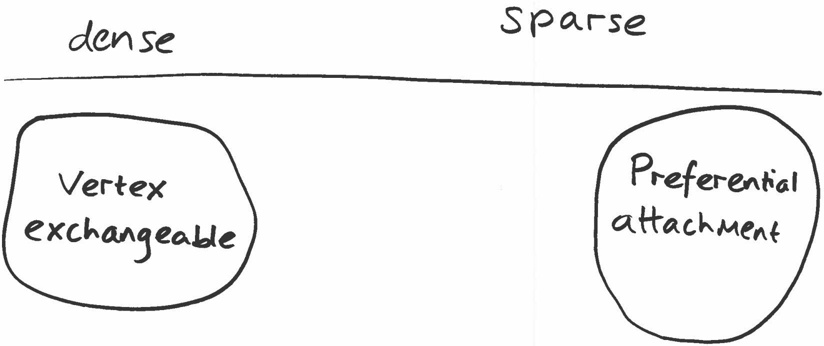
Models

dense

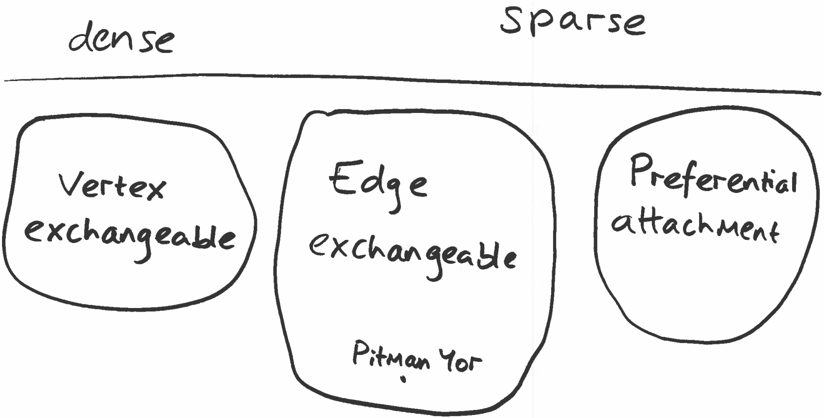
sparse

Vertex exchangeable

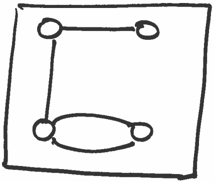
Models


dense
sparse
Vertex exchangeable
Preferential attachment

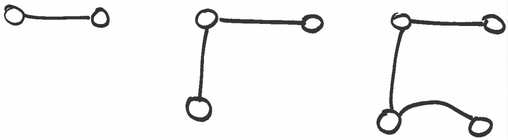
Models

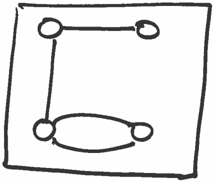

dense
sparse
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

Edge exchangeable models [2], [3]

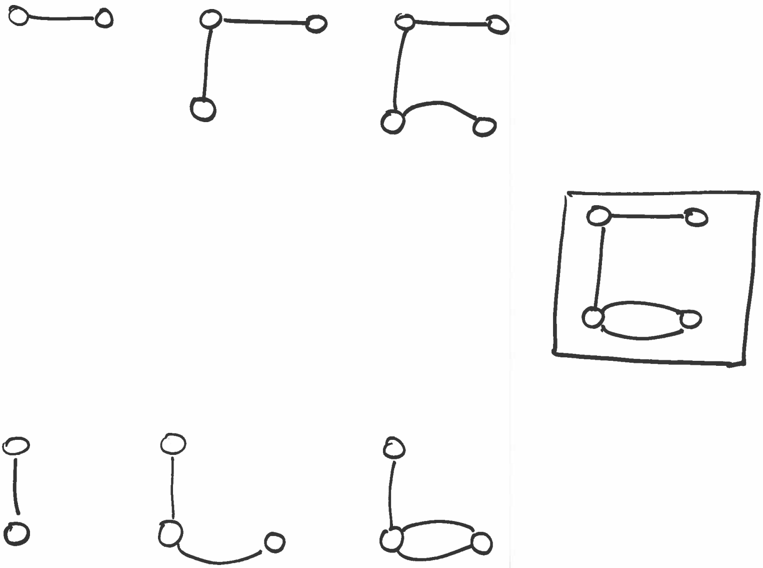


Edge exchangeable models [2], [3]

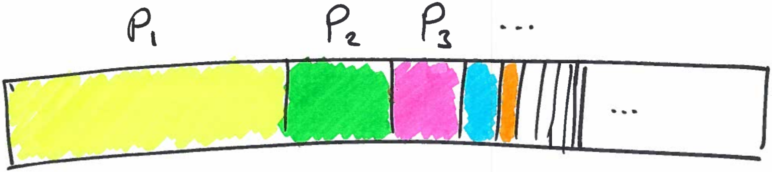




Edge exchangeable models [2], [3]



Paintbox representation

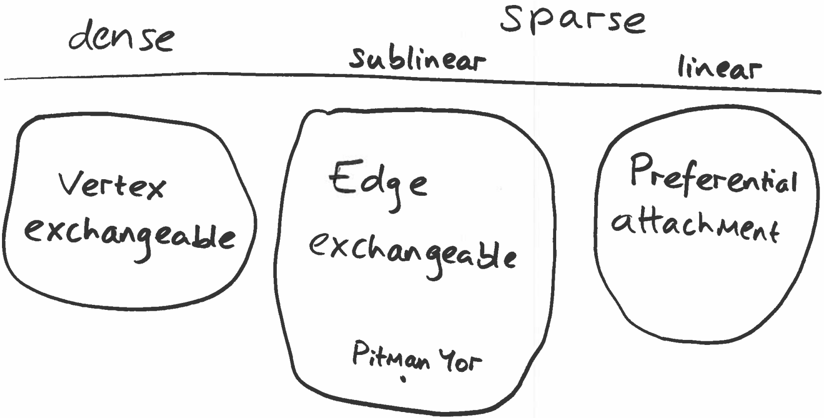


Paintbox representation

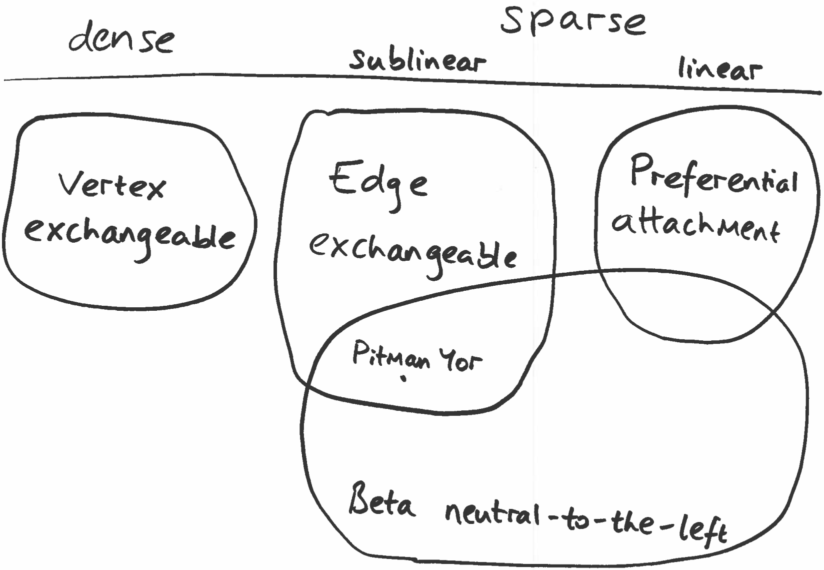
Consequence

- Edge exchangeable models have sublinear sparsity

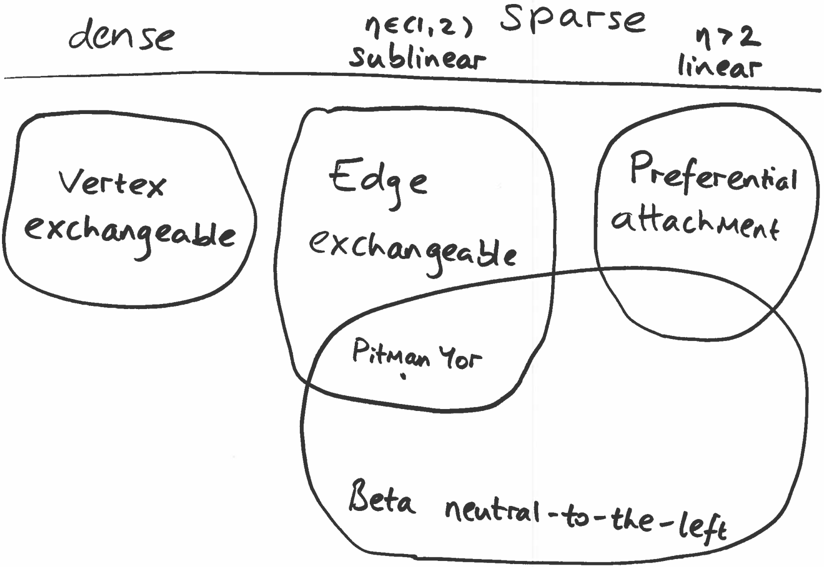
Models


dense
sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

Models

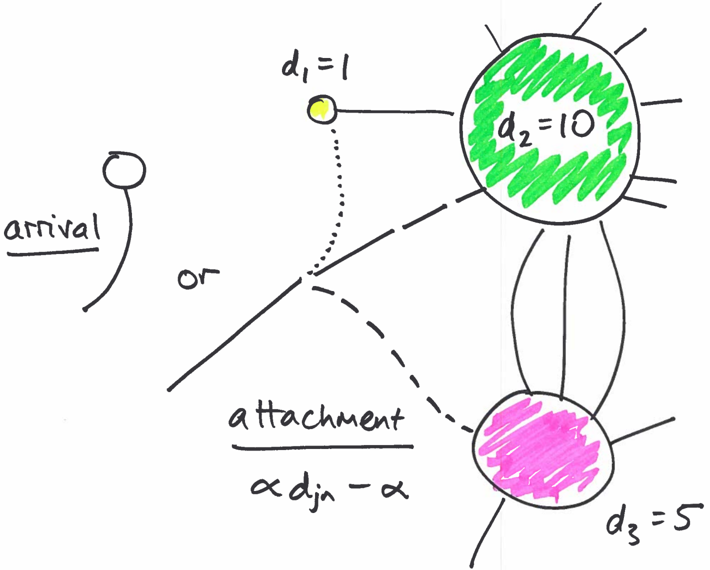

dense
sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
Preferential attachment
Pitman Yor
Beta neutral-to-the-left

Models

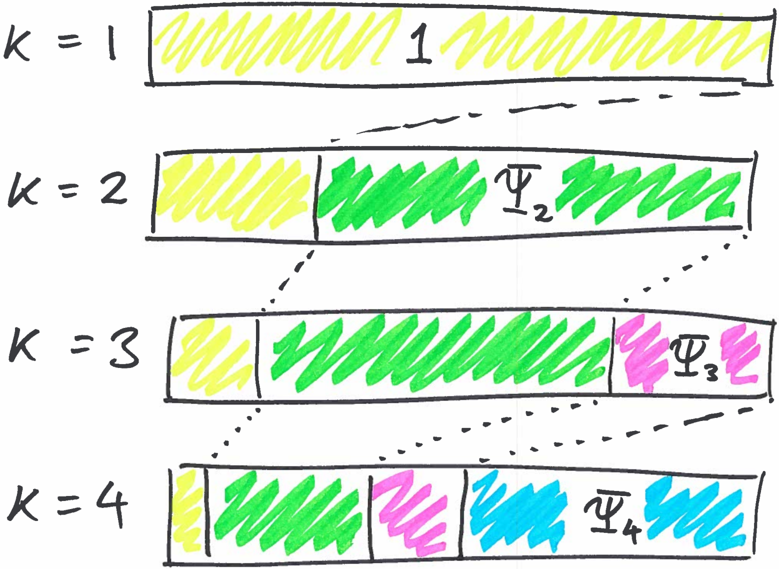


dense
$\eta \in (1,2)$
sublinear
Sparse
$\eta > 2$
linear
Vertex exchangeable
Edge exchangeable
Preferential attachment
Pitman Yor
Beta neutral-to-the-left

Beta Neutral-to-the-left Model [4]



Latent representation



Sampling and inference

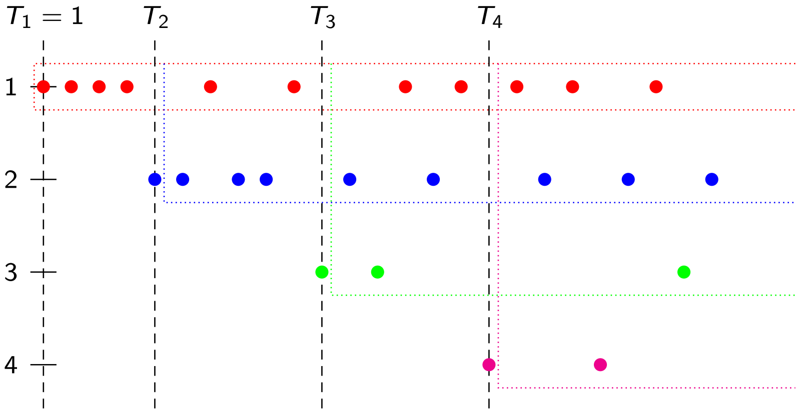
Why a paper on sampling and inference for BNTL models?

Sampling and inference

Why a paper on sampling and inference for BNTL models?

- Inference is notoriously difficult for *non-exchangeable* structures
- Need to identify *exchangeable substructures*

Exchangeable substructure



Gibbs structure

The joint density has **Gibbs structure**

$$p(\text{graph}|\mathbf{T}) = \prod_{j=1}^{K} p(\text{choose } j \ d_j \text{ times out of } n - T_j)$$

- $K = \#\text{vertices}$
- $n = \#\text{edges}$
- $d_j = \text{degree of vertex } j$
- $T_j = \text{arrival time of vertex } j$

Gibbs structure

Explicitly

$$p(\text{graph}|\mathbf{T}) = \frac{\Gamma(d_1 - \alpha)}{\Gamma(n - K\alpha)} \prod_{j=2}^{K} \frac{\Gamma(T_j - j\alpha)\Gamma(d_j - \alpha)}{\Gamma(T_j - 1 - (j-1)\alpha)\Gamma(1-\alpha)}$$

- $K = \#$vertices
- $n = \#$edges
- $d_j =$ degree of vertex j
- $T_j =$ arrival time of vertex j

Available data

Observation	Unobserved variables
Entire history	$\alpha, \phi, \boldsymbol{\Psi}$
Degrees in arrival order	$\alpha, \phi, \boldsymbol{\Psi}, \mathbf{T}$
Snapshot	$\alpha, \phi, \boldsymbol{\Psi}, \mathbf{T}, \sigma$

- $\alpha =$ BTNL parameter $\in (-\infty, 1)$
- $\phi =$ arrival distribution parameters
- $\boldsymbol{\Psi} =$ latent sociabilities
- $\mathbf{T} =$ arrival times
- $\sigma =$ arrival order

Sampling $\boldsymbol{\Psi}$

Beta prior on Ψ_j, plus Gibbs structure, give

$$\Psi_j \mid \text{graph}, \boldsymbol{\Psi}_{\setminus j} \sim \text{Beta}(d_j - \alpha, \bar{d}_{j-1} - (j-1)\alpha) \ ,$$

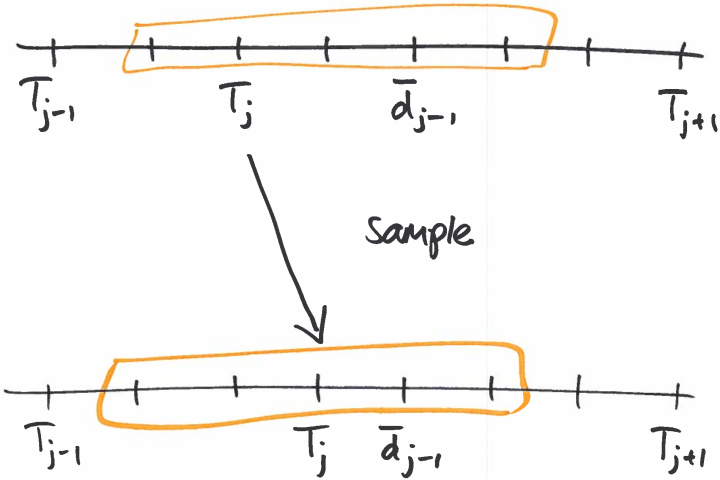
where $\bar{d}_j = \sum_{i=1}^{j} d_i$

Sampling α, ϕ

- One-dimensional unnormalized density for α
- For ϕ depends on arrival distribution family

Sampling **T**

TODO remove $\bar{d}$ from this picture



Sampling σ

- Initialise in descending degree order
- Use Metropolis-Hastings with adjacent swap proposal $\sigma_j \leftrightarrow \sigma_{j+1}$

Point estimation

- Decompose $p_{\alpha,\phi}(\text{graph}) = p_\phi(\mathbf{T})p_\alpha(\text{graph}|\mathbf{T})$
- MLE/MAP estimation for α by optimizing density

Experiments

- I DON'T KNOW
- Synthetic data – parameter recovery
- Scaling in n
- Point estimation with massive graphs

Synthetic data

- Simulate 500 edges from the prior with fixed α
- Arrivals either $\mathcal{PYP}$ or Geom
- Observe final snapshot of the graph only

Gibbs sampler results

Gen. arrival distn.	Inference model	$\lvert\hat{\alpha} - \alpha^*\rvert$	Pred. log-lik.
$\mathcal{PYP}(1.0, 0.75)$	$(\tau, \mathcal{PYP}(\theta, \tau))$	**0.046** $\pm$ **0.002**	-**2637.0** $\pm$ **0.1**
$\mathcal{PYP}(1.0, 0.75)$	$(\alpha, \text{Geom}(\beta))$	0.049 $\pm$ 0.004	-2660.5 $\pm$ 0.7
Geom(0.25)	$(\tau, \mathcal{PYP}(\theta, \tau))$	0.086 $\pm$ 0.002	-2386.8 $\pm$ 0.1
Geom(0.25)	$(\alpha, \text{Geom}(\beta))$	**0.043** $\pm$ **0.003**	-**2382.6** $\pm$ **0.2**

Scalability of Gibbs sampler

- Do we learn from all data?
- How does performance scale?

Scalability of Gibbs sampler

- Do we learn from all data?
- How does performance scale?

	$n = 200$	$n = 20000$
$\lvert\hat{\alpha} - \alpha^*\rvert$	0.12 ± 0.01	0.01 ± 0.00
$\lvert\hat{\beta} - \beta^*\rvert$	0.02 ± 0.00	0.00 ± 0.00
ESS	0.90 ± 0.04	0.75 ± 0.08
Runtime (s)	21 ± 0	2267 ± 2

- Most expensive Gibbs update is for $\mathbf{T}$

MLEs for SNAP datasets

Fitted point estimates

Dataset	Coupled $\mathcal{PYP}(\theta,\alpha)$				Uncoupled $\mathcal{PYP}(\theta,\tau)$		Geom(β)		
	$(\hat{\theta},\hat{\alpha})$	$\hat{\eta}$	Pred. l-l.	$\hat{\alpha}$	$(\hat{\theta},\hat{\tau})$	Pred. l-l.	$\hat{\beta}$	$\hat{\eta}$	Pred. l-l.
Ask Ubuntu	(18080, 0.25)	1.25	-3.707e6	-2.54	(-0.99, 0.99)	-3.678e6	0.083	2.32	**-3.678e6**
UCI social network	(320.4, 4.4e-11)	–	-1.600e5	-4.98	(5.50, 0.52)	**-1.595e6**	0.016	2.10	-1.596e5
EU email	(113.6, 2.5e-14)	–	**-8.06e5**	-1.86	(113.6, 9.2e-10)	**-8.06e5**	0.001	2.00	-8.07e5
Math Overflow	(2575, 0.15)	1.15	-1.685e6	-6.62	(-0.97, 0.997)	-1.670e6	0.025	2.19	**-1.670e6**
Stack Overflow	(297600, 0.11)	1.11	-3.358e8	-8.94	(-1.0, 1.0)	-3.333e8	0.020	2.21	**-3.333e8**
Super User	(20640, 0.24)	1.24	-5.855e6	-4.19	(-0.996, 1.0)	**-5.775e6**	0.067	2.37	-5.775e6
Wikipedia talk pages	(14870, 0.54)	1.54	-3.074e7	-0.25	(-1.0, 1.0)	**-3.066e7**	0.073	2.10	-3.066e7

MLEs for SNAP datasets

$\mathcal{PYP}$ parameter estimates vary coupled and uncoupled

Dataset	Coupled $\mathcal{PYP}(\theta, \alpha)$				Uncoupled $\mathcal{PYP}(\theta, \tau)$		Geom(β)		
	$(\hat{\theta}, \hat{\alpha})$	$\hat{\eta}$	Pred. l-l.	$\hat{\alpha}$	$(\hat{\theta}, \hat{\tau})$	Pred. l-l.	$\hat{\beta}$	$\hat{\eta}$	Pred. l-l.
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MLEs for SNAP datasets

Edge exchangeable models likely misspecified

Dataset	Coupled $\mathcal{PYP}(\theta,\alpha)$				Uncoupled $\mathcal{PYP}(\theta,\tau)$		Geom(β)		
	$(\hat\theta,\hat\alpha)$	$\hat\eta$	Pred. l-l.	$\hat\alpha$	$(\hat\theta,\hat\tau)$	Pred. l-l.	$\hat\beta$	$\hat\eta$	Pred. l-l.
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MLEs for SNAP datasets

Though better than Geom for some datasets

Dataset	Coupled $\mathcal{PYP}(\theta, \alpha)$				Uncoupled $\mathcal{PYP}(\theta, \tau)$		Geom(β)		
	$(\hat{\theta}, \hat{\alpha})$	$\hat{\eta}$	Pred. l-l.	$\hat{\alpha}$	$(\hat{\theta}, \hat{\tau})$	Pred. l-l.	$\hat{\beta}$	$\hat{\eta}$	Pred. l-l.
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Conclusion

- BNTL models are *flexible*
- BNTL models are *tractable*

Future work

- Scalability of inference
  - Metropolis-Hastings to update $\mathbf{T}$ altogether
  - Variational inference for σ

References

[1] Jure Leskovec and Andrej Krevl. SNAP Datasets: Stanford large network dataset collection. `http://snap.stanford.edu/data`, June 2014.

[2] Diana Cai, Trevor Campbell, and Tamara Broderick. Edge-exchangeable graphs and sparsity. In *Advances in Neural Information Processing Systems*, pages 4249–4257, 2016.

[3] Harry Crane and Walter Dempsey. Edge exchangeable models for interaction networks. *Journal of the American Statistical Association*, (just-accepted), 2017.

[4] Benjamin Bloem-Reddy and Peter Orbanz. Preferential attachment and vertex arrival times. *arXiv preprint arXiv:1710.02159*, 2017.