Sampling and Inference for Beta Neutral-to-the-Left Models of Sparse Networks

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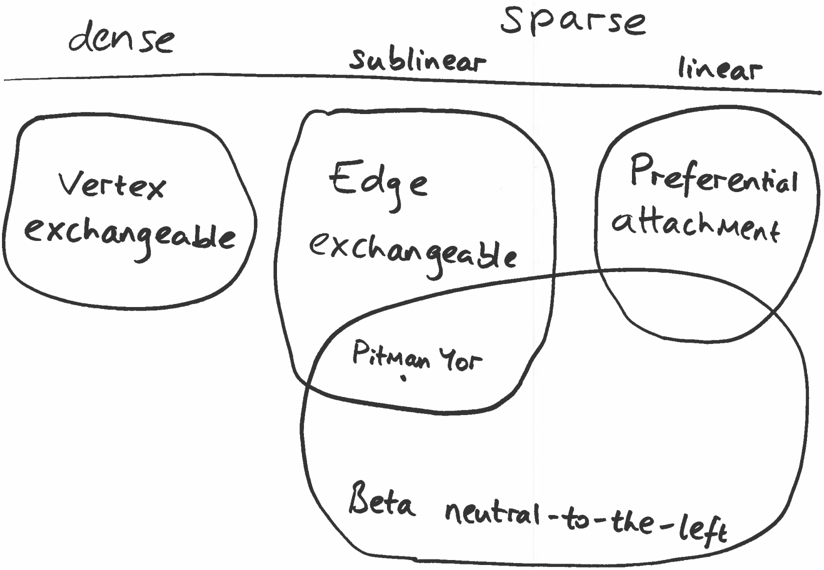
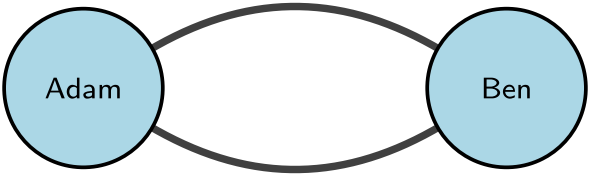
Models for sparse networks



Temporal networks

Temporal networks



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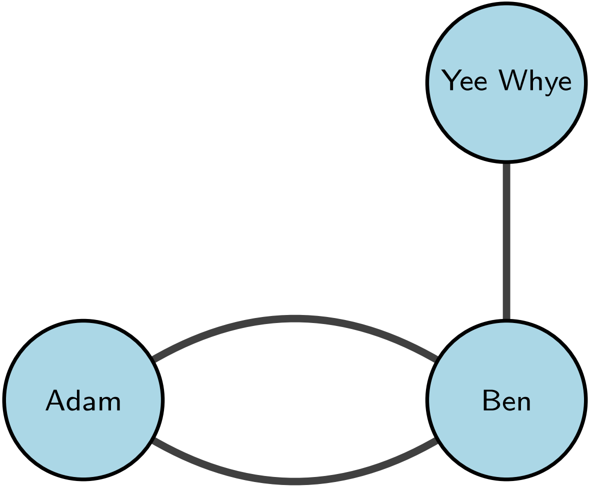
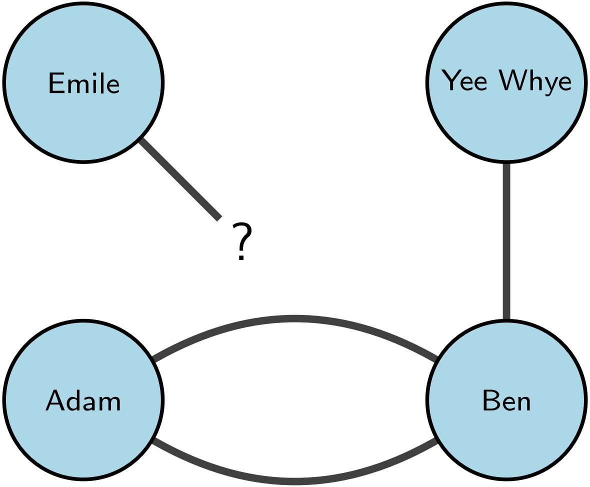
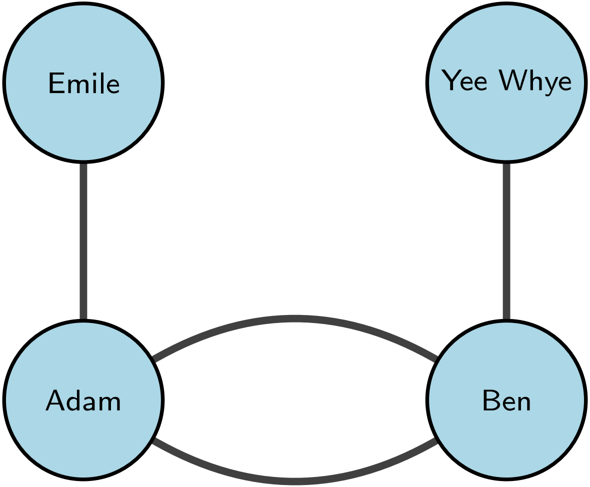


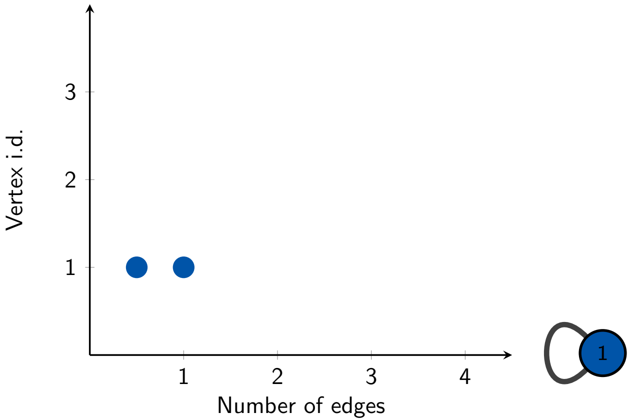
Temporal networks



Temporal networks



Edges and vertices



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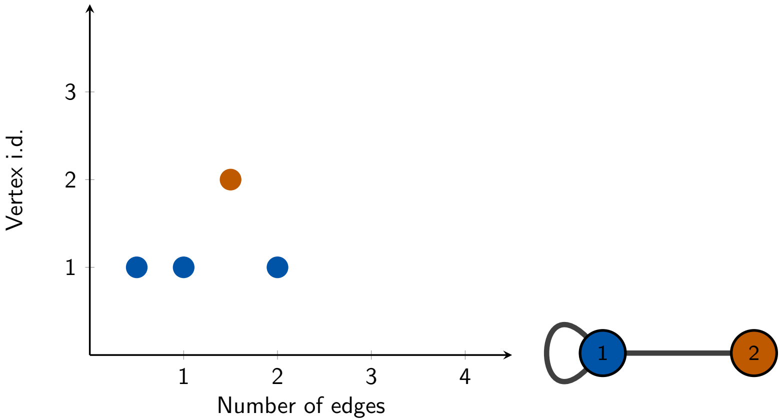
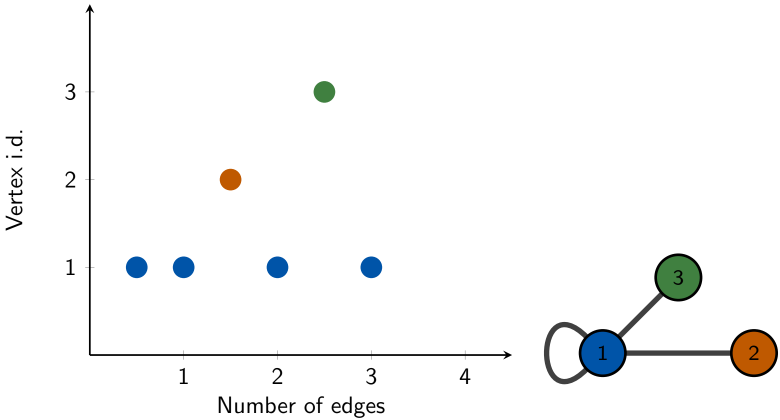
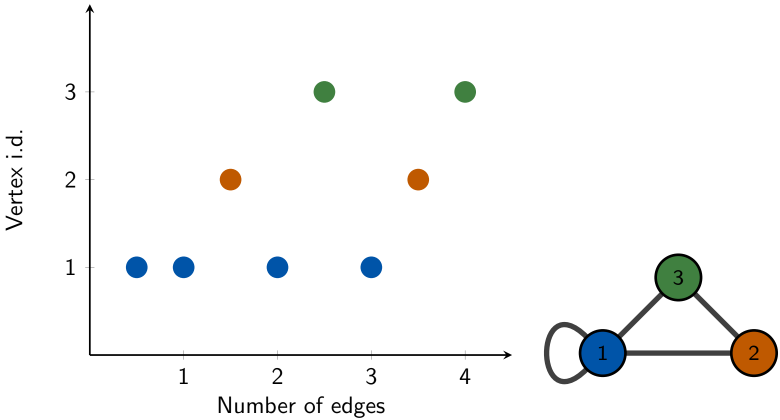


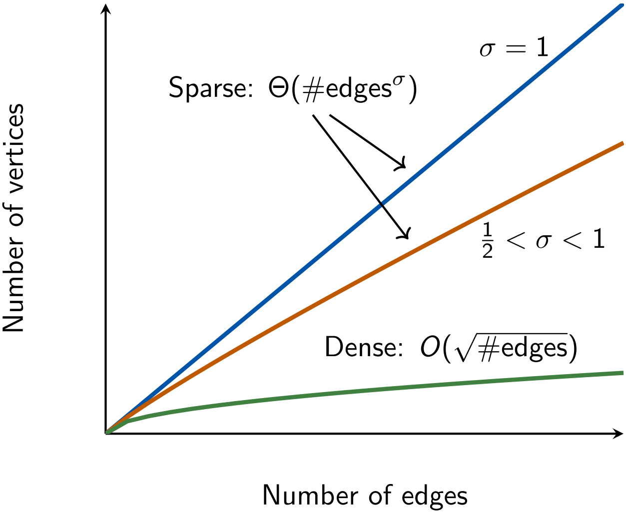
Edges and vertices



Edges and vertices

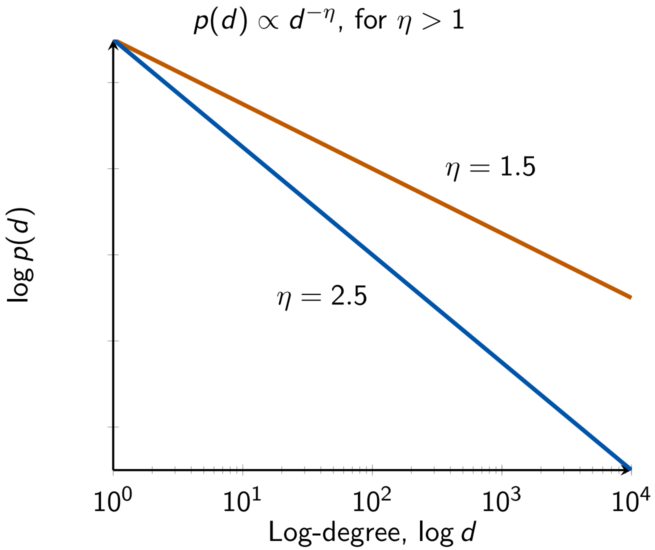


Sparsity



Number of vertices
$\sigma = 1$
Sparse: $\Theta(\#\text{edges}^{\sigma})$
$\frac{1}{2} < \sigma < 1$
Dense: $O(\sqrt{\#\text{edges}})$
Number of edges

Power law degree distribution

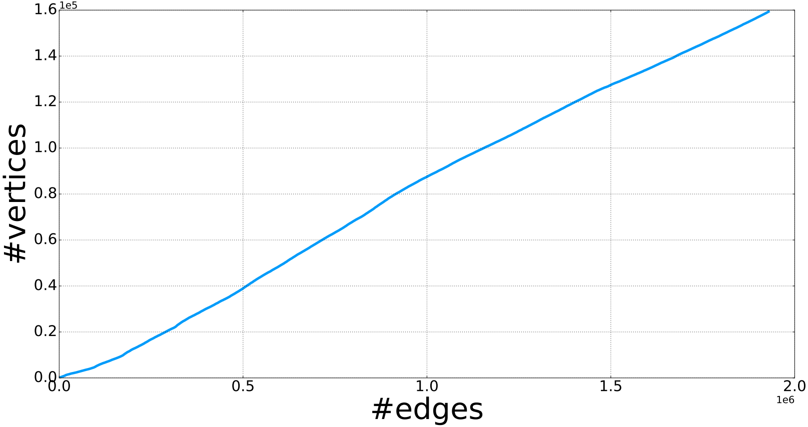


Sparsity and power law

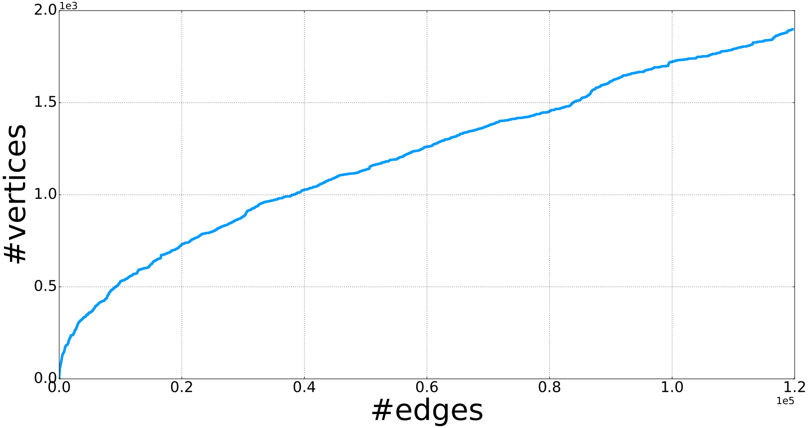
$$\textbf{Sublinear sparsity} \iff \eta \in (1,2)$$

$$\textbf{Linear sparsity} \iff \eta > 2$$

Empirical study: Ask Ubuntu



Empirical study: UCI social network



Exchangeable models

A sequence of random variables $X_1, X_2, \ldots$ is *exchangeable* if for a finite permutation σ,

$$X_{\sigma(1)}, X_{\sigma(2)}, \ldots$$

has the same distribution as the original sequence.

Exchangeable models lead to tractable inference (de Finetti).

Models

Vertex exchangeable

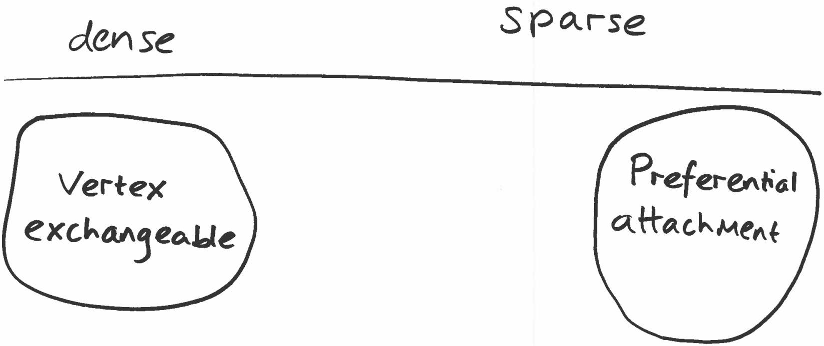
Models

dense

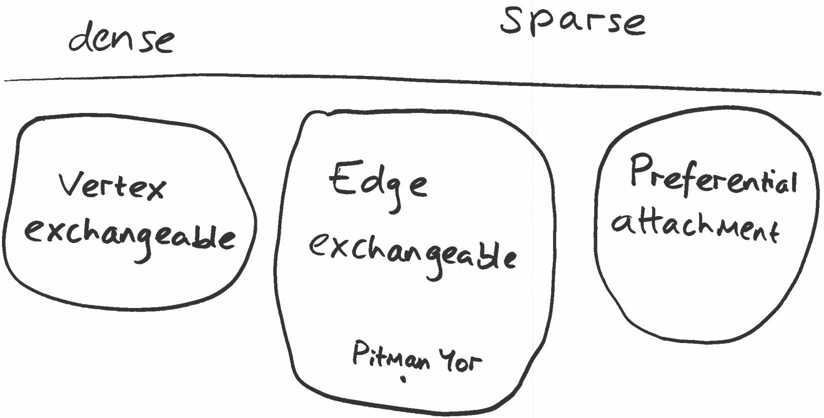
sparse

Vertex exchangeable

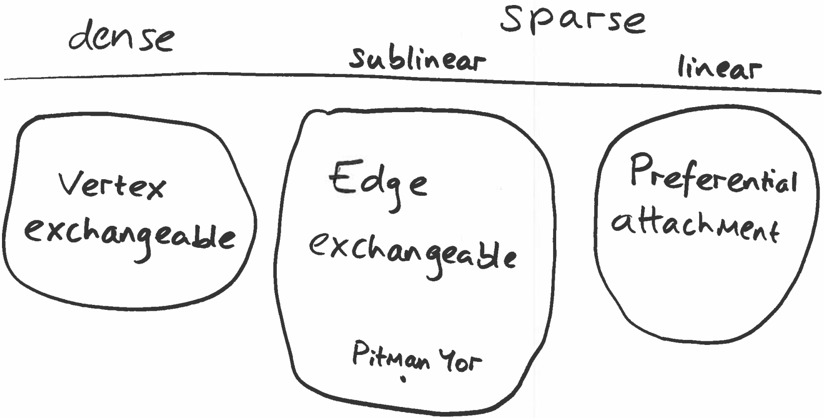
Models


dense
sparse
Vertex exchangeable
Preferential attachment

Models


dense
sparse
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

Models


dense
Sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
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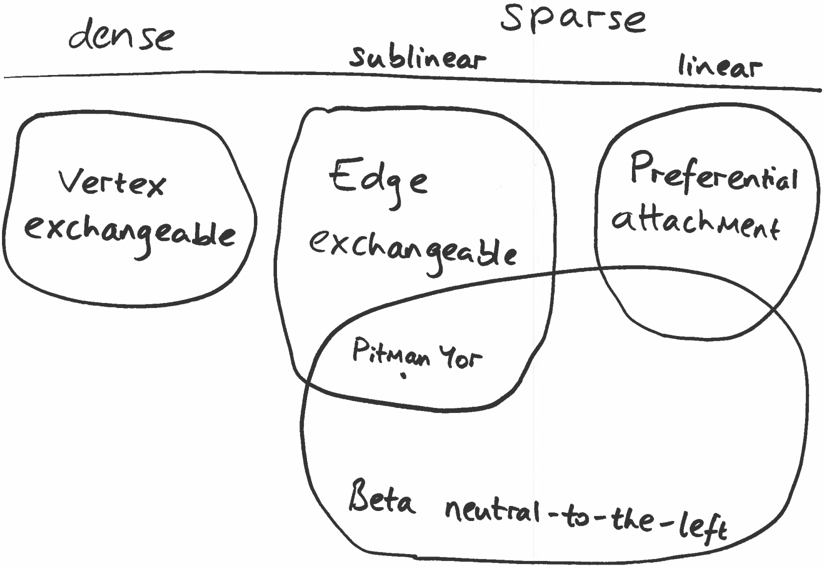
Preferential attachment

Models



dense
sparse
sublinear
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Vertex exchangeable
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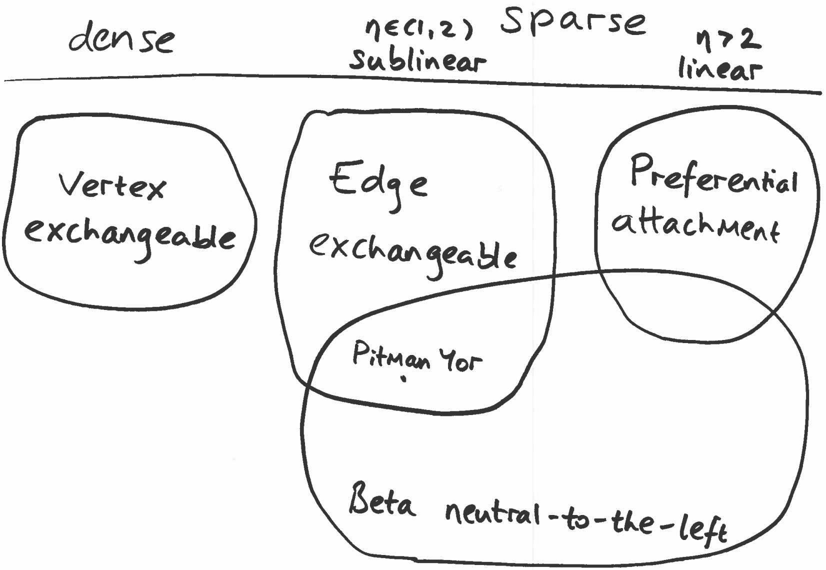
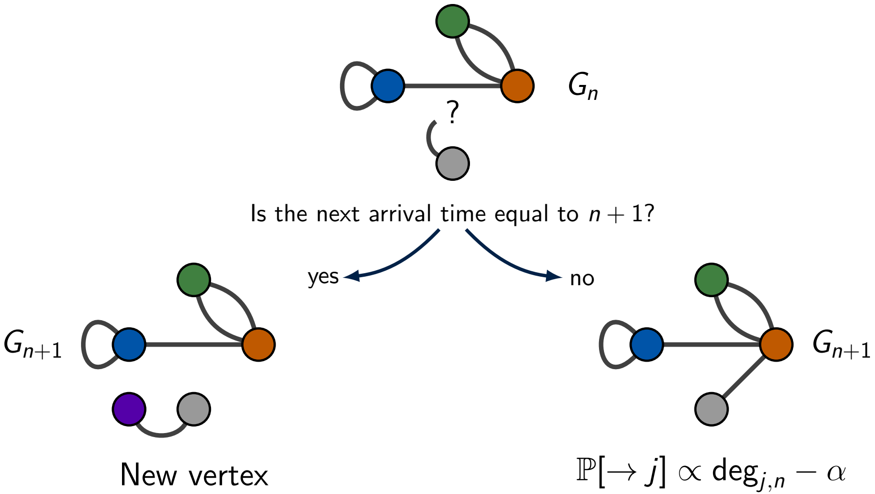
Pitman Yor
Beta neutral-to-the-left

Models


dense
η∈(1,2)
sparse
sublinear
η>2
linear
Vertex exchangeable
Edge exchangeable
Preferential attachment
Pitman Yor
Beta neutral-to-the-left

Beta Neutral-to-the-left Model [4]

1. Generate arrival times $1 = T_1 < T_2 < T_3 < \dots$ in any way.
2. Generate ends of edges sequentially:



Sampling and inference

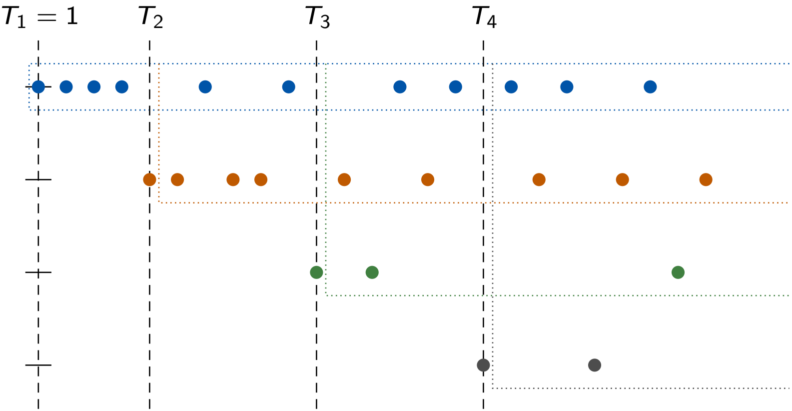
Why a paper on sampling and inference for BNTL models?

Sampling and inference

Why a paper on sampling and inference for BNTL models?

- Inference is notoriously difficult for *non-exchangeable* structures
- Need to identify *exchangeable substructures* for tractable inference

Exchangeable substructure



Gibbs structure

The joint probability has **Gibbs structure**

$$P(\text{graph}|\,T_1, T_2, ...) = \prod_{j \in \text{vertices}} P(\text{choose } j \ d_j - 1 \text{ times out of } T - T_j \text{ trials })$$

- d_j = degree of vertex j
- T = final time
- T_j = arrival time of vertex j

Available data

Observation	Unobserved variables
Entire history	Model variables
Degrees in arrival order	Model variables, arrival times
Snapshot	Model variables, arrival times, arrival order

Gibbs sampler

Variable	Gibbs sampling scheme
Model variables	Analytic using Gibbs structure
Arrival times	Update each arrival time separately
Arrival order	Initialise in descending degree order use M-H with adjacent swap proposal

Experiments

- Gibbs: parameter recovery
- Gibbs: scalability
- Point estimation with massive graphs

Parameter recovery

- Small graph
- Need to learn model variables, arrival times and arrival order

Gen. arrival distn.	Inference model	$\lvert\hat{\alpha} - \alpha^*\rvert$	Pred. log-lik.
$\mathcal{PYP}(1.0, 0.75)$	$(\tau, \mathcal{PYP}(\theta, \tau))$	**0.046** $\pm$ **0.002**	-**2637.0** $\pm$ **0.1**
$\mathcal{PYP}(1.0, 0.75)$	$(\alpha, \text{Geom}(\beta))$	0.049 $\pm$ 0.004	-2660.5 $\pm$ 0.7
Geom(0.25)	$(\tau, \mathcal{PYP}(\theta, \tau))$	0.086 $\pm$ 0.002	-2386.8 $\pm$ 0.1
Geom(0.25)	$(\alpha, \text{Geom}(\beta))$	**0.043** $\pm$ **0.003**	-**2382.6** $\pm$ **0.2**

Scalability

- Runtime linear in #edges
- Most expensive Gibbs update is for arrival times

MLEs for SNAP datasets

- SNAP datasets
- Fit: coupled $\mathcal{PYP}$, uncoupled $\mathcal{PYP}$ and Geom(β) arrivals

MLEs for SNAP datasets

Ask Ubuntu

- Estimates of $\mathcal{PYP}$ parameters vary significantly between coupled and uncoupled
  - $\hat{\theta}, \hat{\alpha} = 18080, 0.25$
  - $\hat{\theta}, \hat{\alpha}, \hat{\tau} = -0.99, -2.54, 0.99$
- Edge exchangeable models misspecified
- Non-exchangeable BNTL provide better fit

Conclusion

- BNTL models are *flexible*
- Inference was challenging due to non-exchangeability

Conclusion

- BNTL models are *flexible*
- Inference was challenging due to non-exchangeability
- BNTL models are *tractable* due to exchangeable substructure

Thank you



[1] Jure Leskovec and Andrej Krevl. SNAP Datasets: Stanford large network dataset collection. `http://snap.stanford.edu/data`, June 2014.

[2] Diana Cai, Trevor Campbell, and Tamara Broderick. Edge-exchangeable graphs and sparsity. In *Advances in Neural Information Processing Systems*, pages 4249–4257, 2016.

[3] Harry Crane and Walter Dempsey. Edge exchangeable models for interaction networks. *Journal of the American Statistical Association*, 2017. In press.

[4] Benjamin Bloem-Reddy and Peter Orbanz. Preferential attachment and vertex arrival times. *arXiv preprint arXiv:1710.02159*, 2017.