Sampling and Inference for Beta Neutral-to-the-Left Models of Sparse Networks

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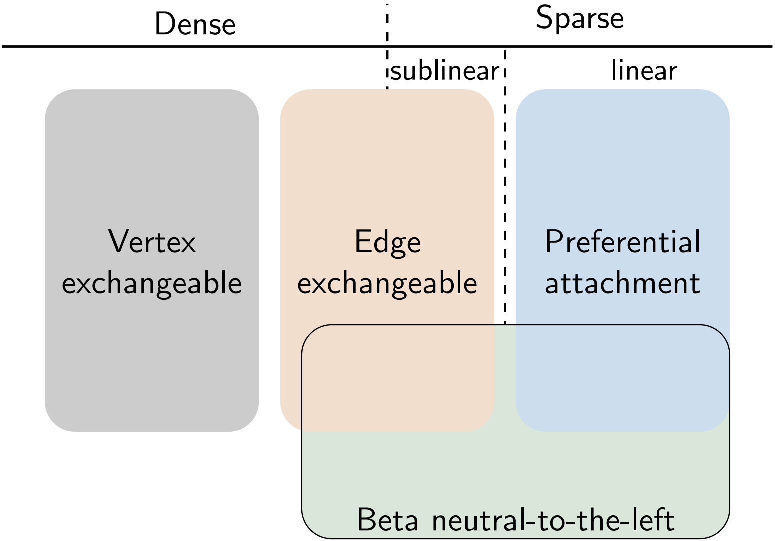
Models for networks



Temporal networks

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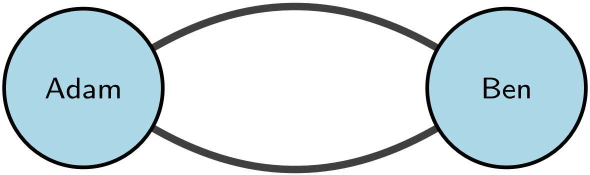

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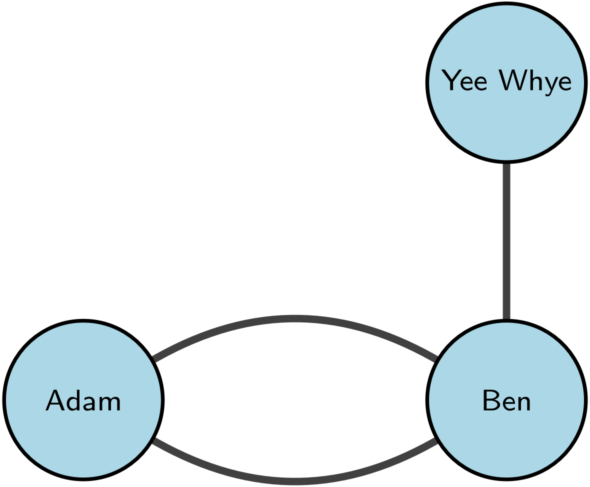

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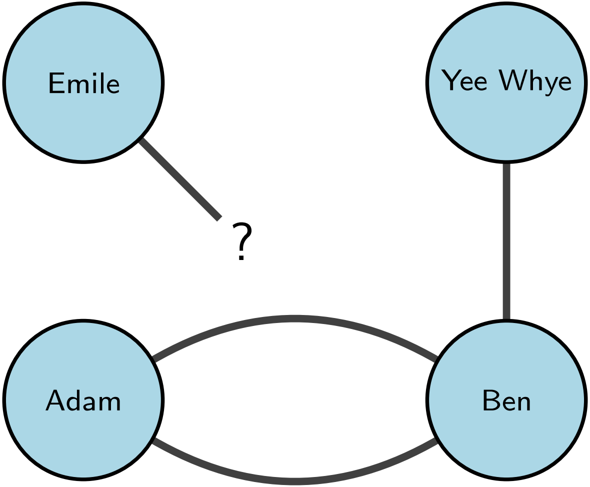

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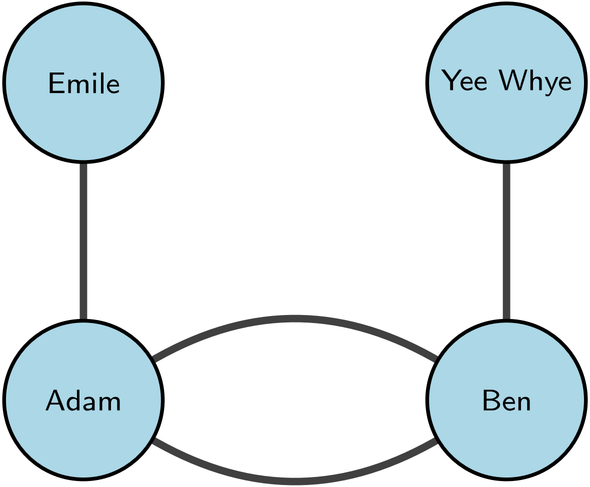
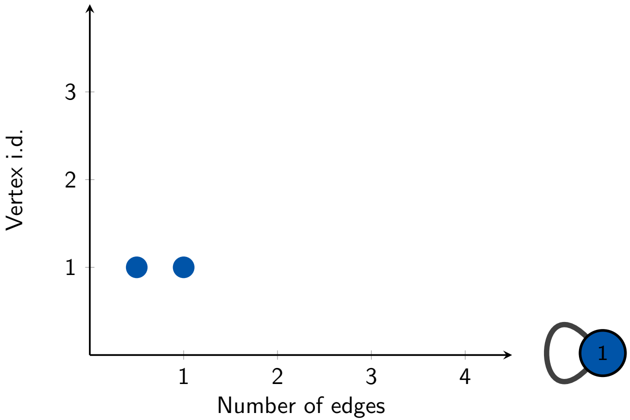


Edges and vertices



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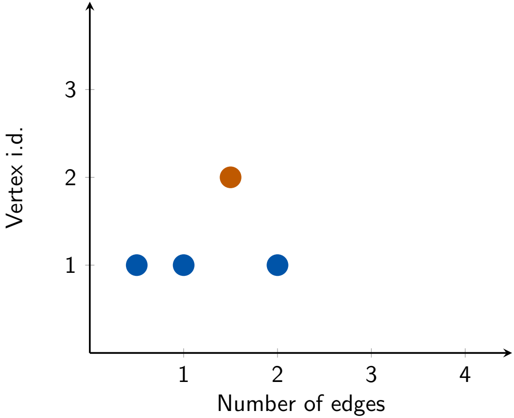

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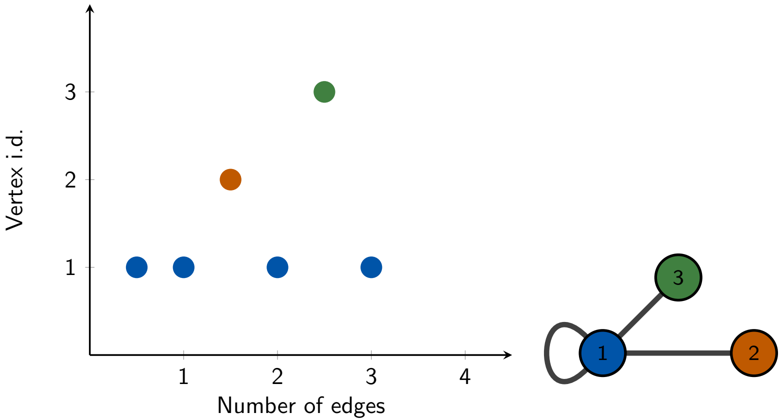

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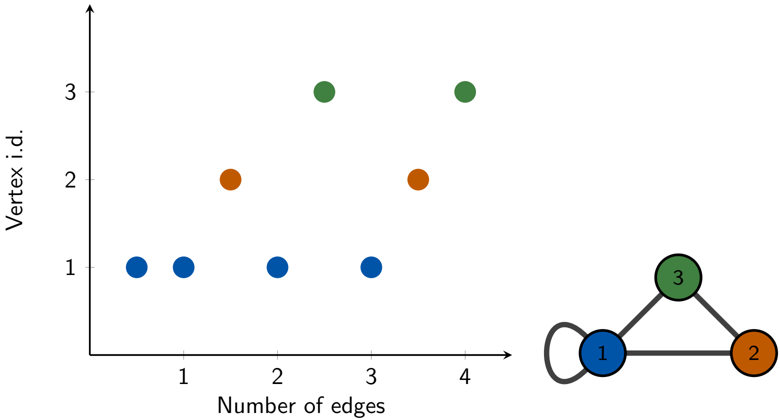
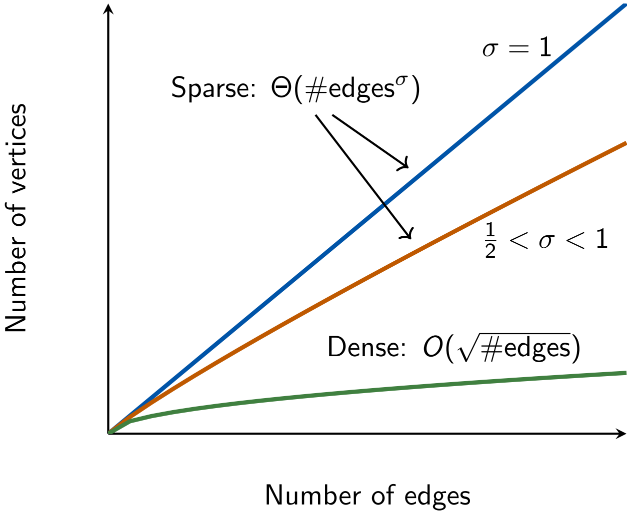
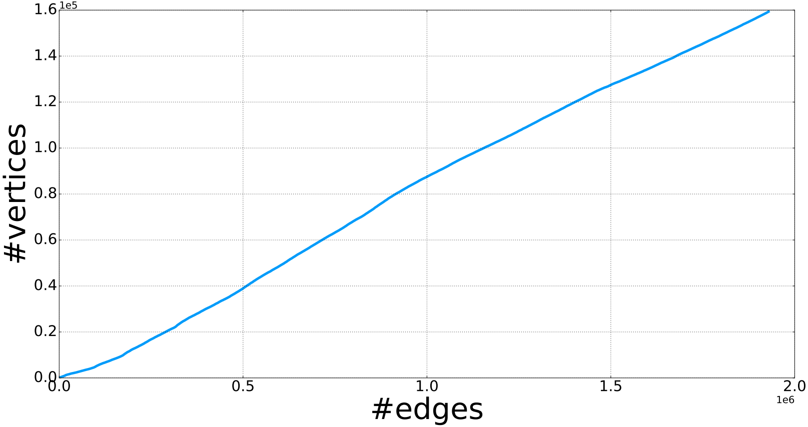


Sparsity

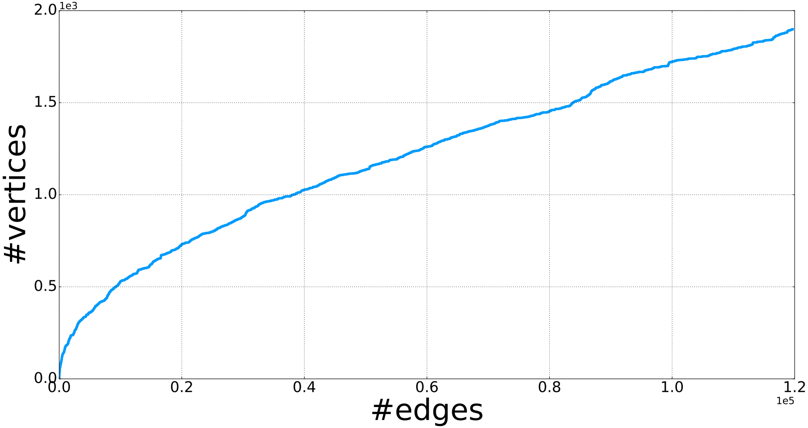


$\sigma = 1$
Sparse: $\Theta(\#\text{edges}^{\sigma})$
$\frac{1}{2} < \sigma < 1$
Dense: $O(\sqrt{\#\text{edges}})$
Number of vertices
Number of edges

Empirical study: Ask Ubuntu [1]



Empirical study: UCI social network [1]



Exchangeable models

A sequence of random variables $X_1, X_2, \ldots$ is *exchangeable* if for a finite permutation σ,

$$X_{\sigma(1)}, X_{\sigma(2)}, \ldots$$

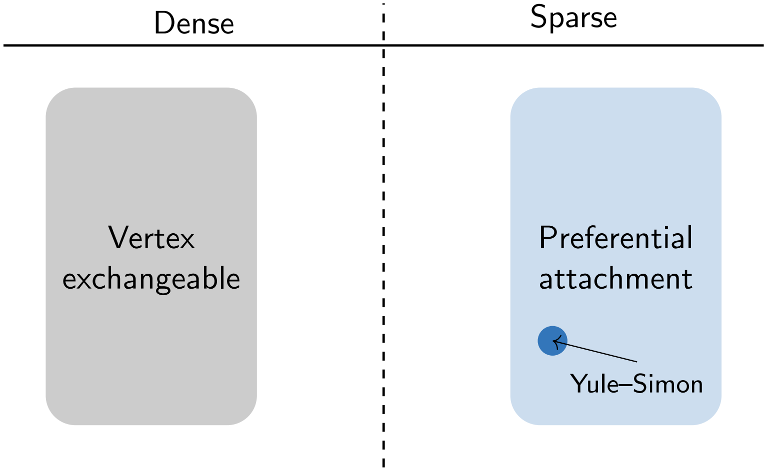
has the same distribution as the original sequence.

Exchangeable models tend to lead to tractable inference.

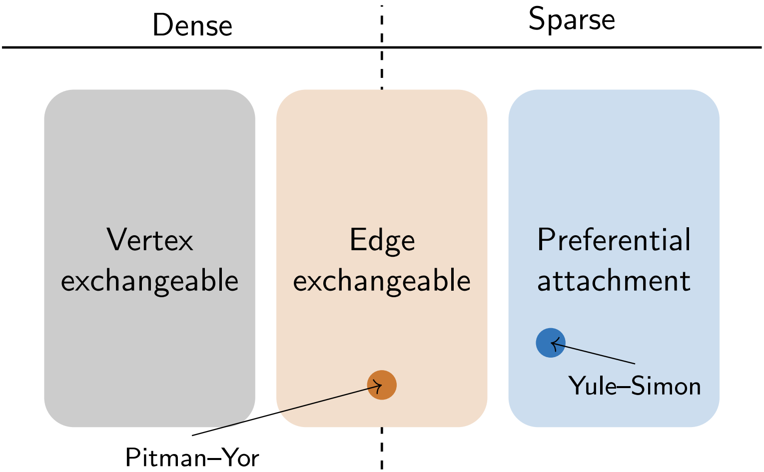
Models

Dense	Sparse
Vertex exchangeable	

Models


Dense
Sparse
Vertex exchangeable
Preferential attachment
Yule–Simon

Models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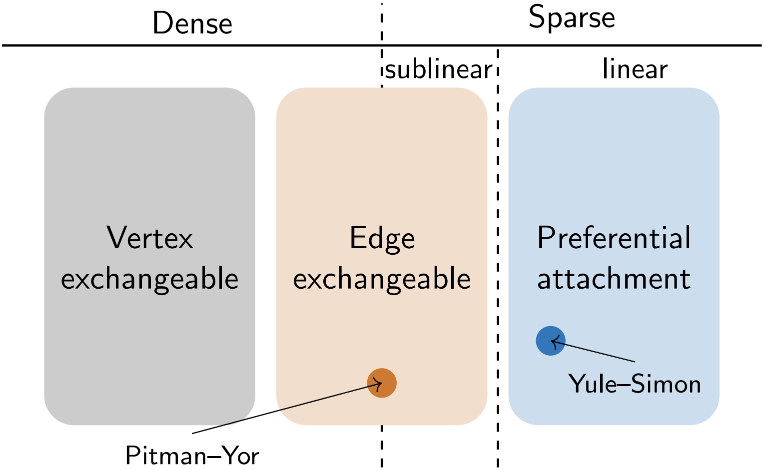
Models


Dense
Sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
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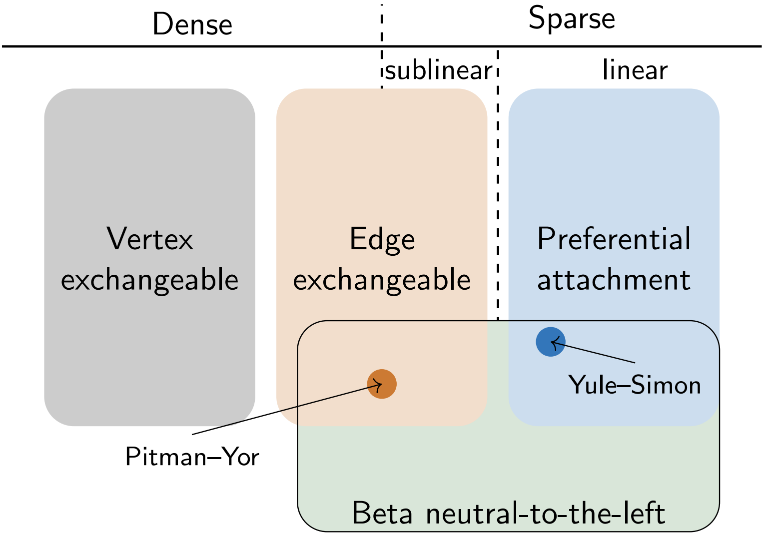
Models


Dense
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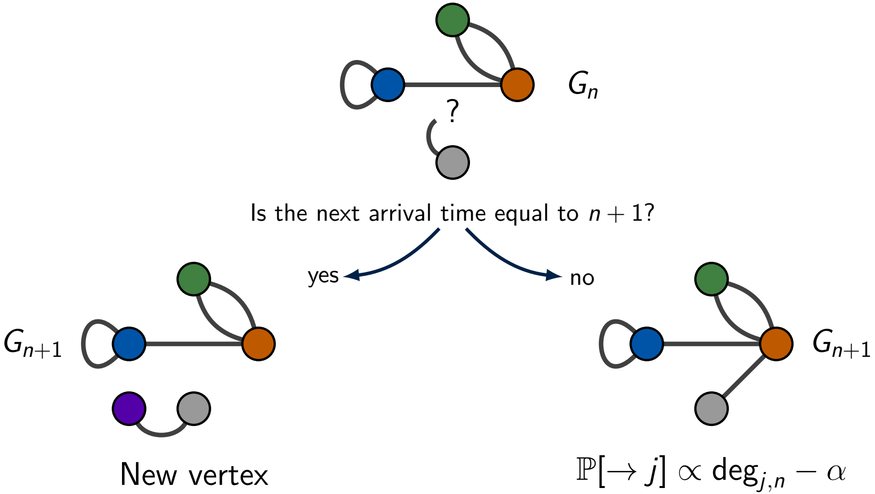
Pitman–Yor
Beta neutral-to-the-left

Beta Neutral-to-the-left Model [2]

1. Generate arrival times $1 = T_1 < T_2 < T_3 < \dots$ in any way.
2. Conditioned on arrival times, generate ends of edges sequentially:



Sampling and inference

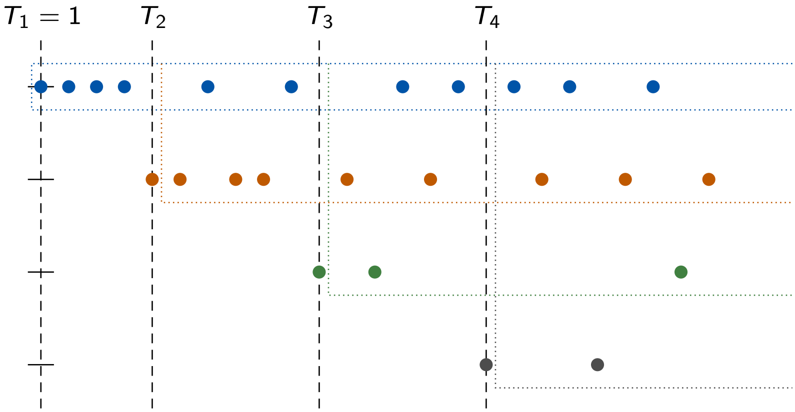
Why a paper on sampling and inference for BNTL models?

Sampling and inference

Why a paper on sampling and inference for BNTL models?

- Inference is notoriously difficult for *non-exchangeable* structures
- Need to identify *exchangeable substructures* for tractable inference

Exchangeable substructure



Neutrality

At each step $n+1$ not corresponding to an arrival time, sample end of edge from discrete distribution

$$P_{n+1} = \left(\frac{d_{1,n} - \alpha}{n - K\alpha}, \ldots, \frac{d_{K,n} - \alpha}{n - K\alpha} \right) .$$

Equivalently (see paper and [2]), can sample from

$$\tilde{P}_K = (\tilde{P}_{1,K}, \ldots, \tilde{P}_{K,K})$$

where $$\frac{\tilde{P}_{j,K}}{\sum_{i=1}^{j} \tilde{P}_{i,K}} = \Psi_j \sim \mathrm{Beta}(1-\alpha, T_j - 1 - (j-1)\alpha) .$$

$\Psi_i \perp\!\!\!\perp \Psi_j$ for all $i \neq j$. These independent increments are called *neutral-to-the-left* and yield a factorized joint probability.

Gibbs structure

The joint probability has **Gibbs structure**

$$P(\text{graph}|T_1, T_2, ...) = \prod_{j \in \text{vertices}} P(\text{choose } j \ d_j - 1 \text{ times out of } T - T_j \text{ trials })$$

- d_j = degree of vertex j
- T = final time
- T_j = arrival time of vertex j

Gibbs sampler

Variable	**Gibbs sampling scheme**
Model variables	Analytic using Gibbs structure
Arrival times	Update each arrival time separately
Arrival order	Initialise in descending degree order use M-H with adjacent swap proposal

Experiments

- Gibbs: parameter recovery
- Gibbs: scalability
- Point estimation with massive graphs

Parameter recovery

- Synthetic data

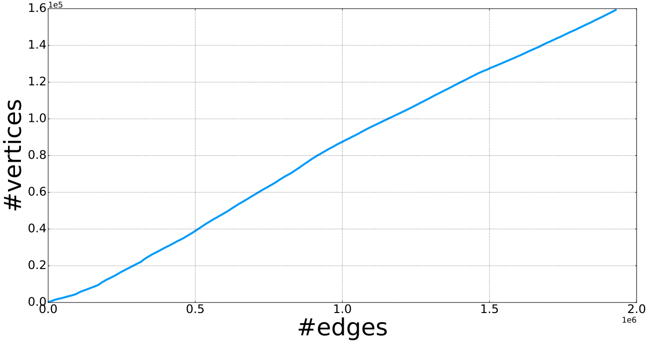
Gen. arrival distn.	Inference model	$\lvert\hat{\alpha} - \alpha^*\rvert$	Pred. log-lik.
$\mathcal{PYP}(1.0, 0.75)$	$(\tau, \mathcal{PYP}(\theta, \tau))$	**0.046** $\pm$ **0.002**	-**2637.0** $\pm$ **0.1**
$\mathcal{PYP}(1.0, 0.75)$	$(\alpha, \text{Geom}(\beta))$	0.049 $\pm$ 0.004	-2660.5 $\pm$ 0.7
Geom(0.25)	$(\tau, \mathcal{PYP}(\theta, \tau))$	0.086 $\pm$ 0.002	-2386.8 $\pm$ 0.1
Geom(0.25)	$(\alpha, \text{Geom}(\beta))$	**0.043** $\pm$ **0.003**	-**2382.6** $\pm$ **0.2**

Scalability

- Runtime linear in #edges
- Most expensive Gibbs update is for arrival times

MLEs for real data

Ask Ubuntu



- Pitman–Yor (edge exchangeable) misspecified
- Non-exchangeable BNTL provide better fit

Conclusion

- BNTL models are *flexible*
- Inference was challenging due to non-exchangeability

Conclusion

- BNTL models are *flexible*
- Inference was challenging due to non-exchangeability
- BNTL models are *tractable* due to exchangeable substructure

Thank you

[1] Jure Leskovec and Andrej Krevl. SNAP Datasets: Stanford large network dataset collection. `http://snap.stanford.edu/data`, June 2014.

[2] Benjamin Bloem-Reddy and Peter Orbanz. Preferential attachment and vertex arrival times. *arXiv preprint arXiv:1710.02159*, 2017.