Sampling and Inference for Beta Neutral-to-the-Left Models of Sparse Networks

Benjamin Bloem-Reddy, Adam Foster, Emile Mathieu, Yee Whye Teh

Department of Statistics, University of Oxford

Contents

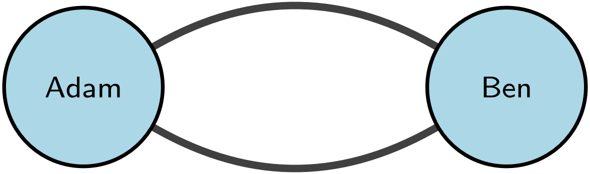
Temporal networks

Examples

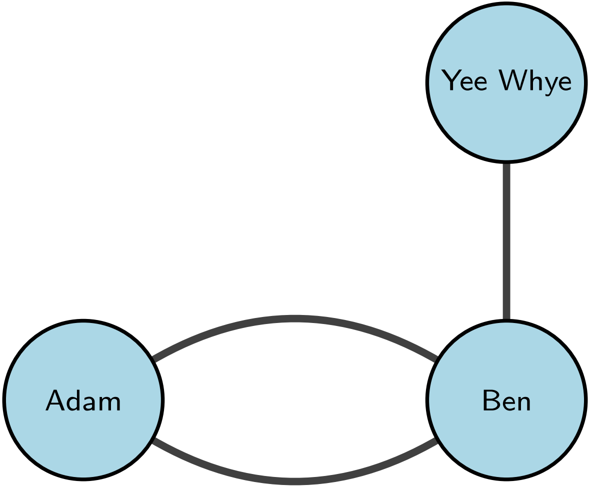
- Messages sent between people over time
- Protein-protein interactions

Temporal networks

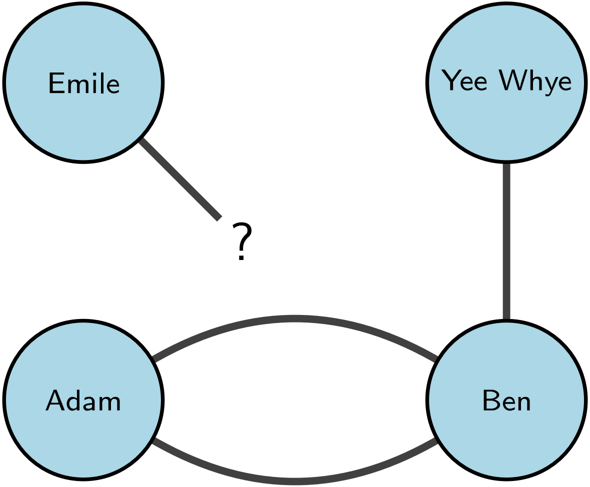
Temporal networks



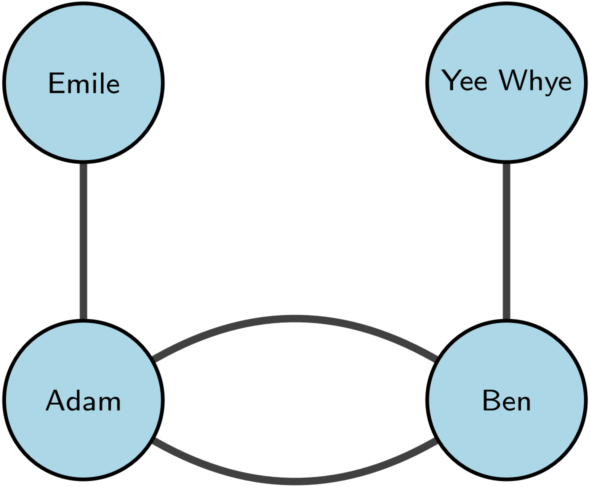
Temporal networks



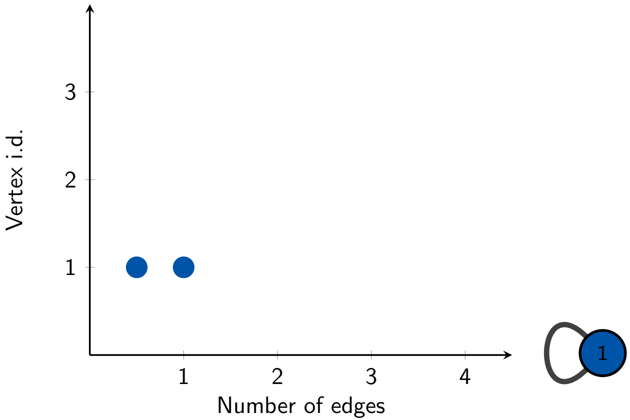
Temporal networks



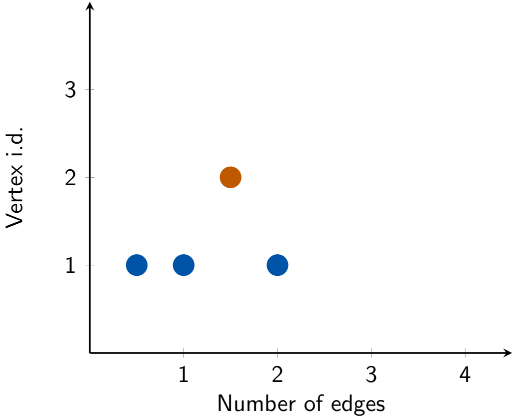
Temporal networks



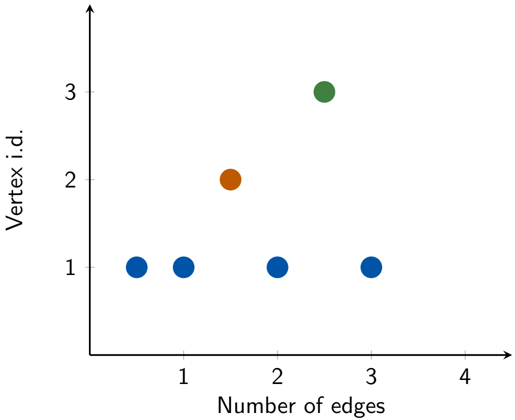
Edges and vertices

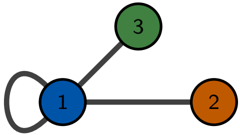


Edges and vertices

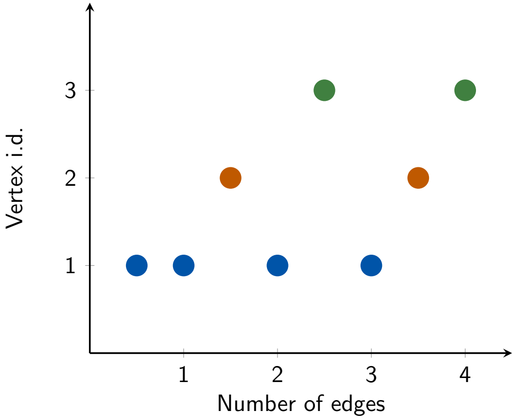


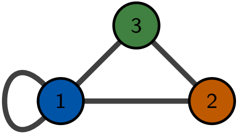
Edges and vertices



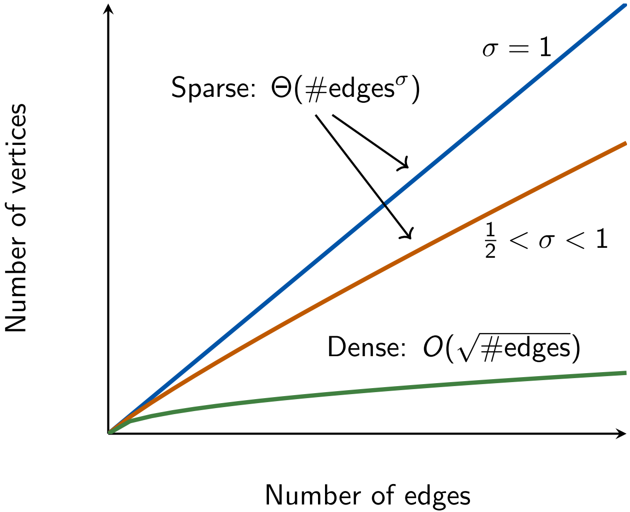


Edges and vertices



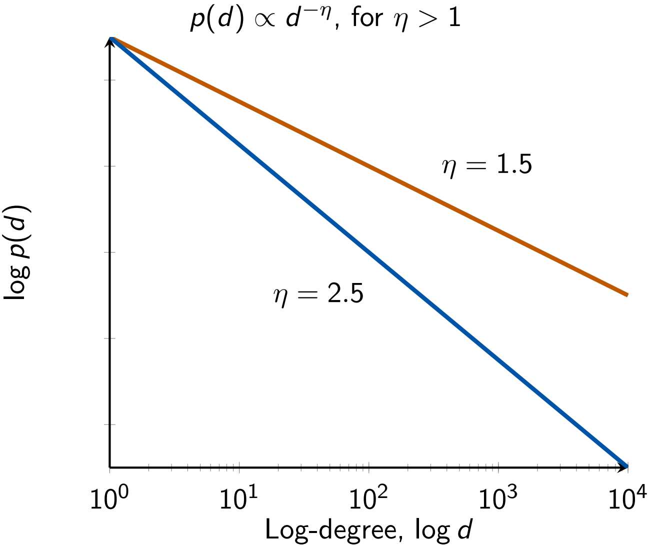


Sparsity



$\sigma = 1$
Sparse: $\Theta(\#\text{edges}^{\sigma})$
$\frac{1}{2} < \sigma < 1$
Dense: $O(\sqrt{\#\text{edges}})$
Number of vertices
Number of edges

Power law degree distribution



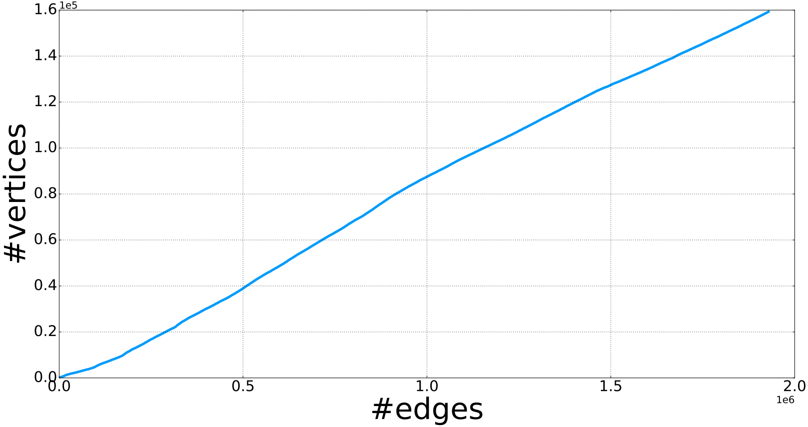
Sparsity and power law

$$\textbf{Sublinear}\text{ sparsity} \iff \eta \in (1,2)$$
$$\textbf{Linear}\text{ sparsity} \iff \eta > 2$$

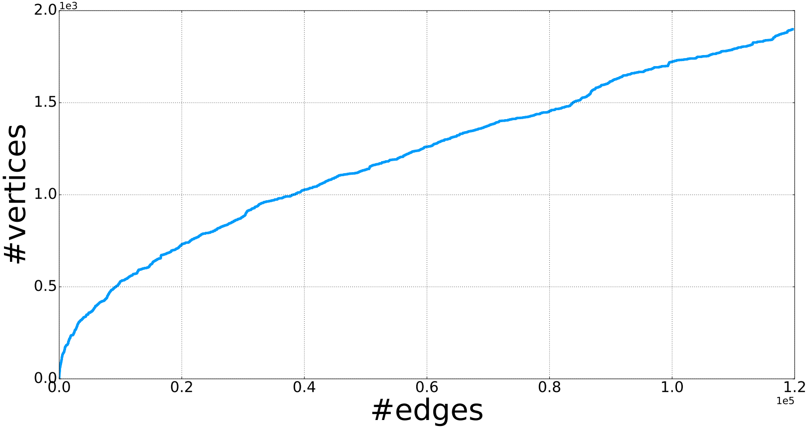
Empirical study

SNAP dataset [1]	# of vertices	# of edges
Ask Ubuntu	159,316	964,437
UCI social network	1,899	20,296
⋮	⋮	⋮

Ask Ubuntu



UCI social network



Models

Vertex exchangeable

Models

dense | sparse

Vertex exchangeable

Models

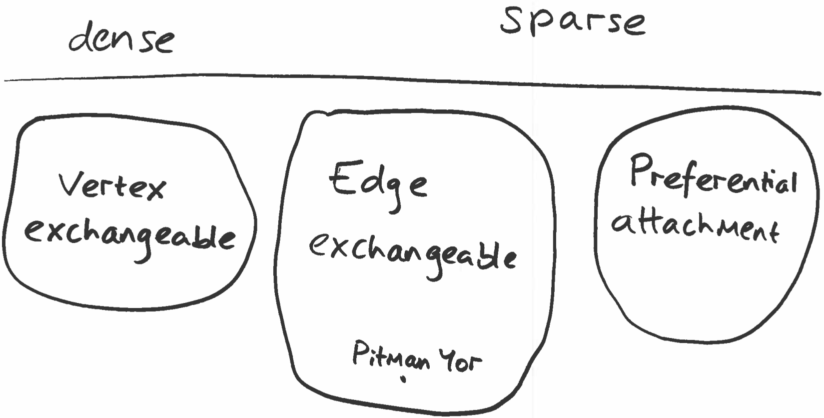
dense

sparse

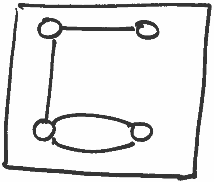
Vertex exchangeable

Preferential attachment

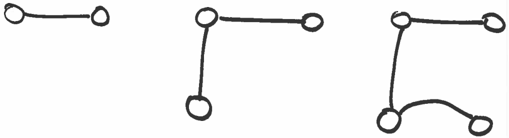
Models

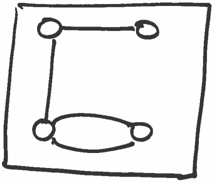

dense
sparse
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

Edge exchangeable models [2], [3]

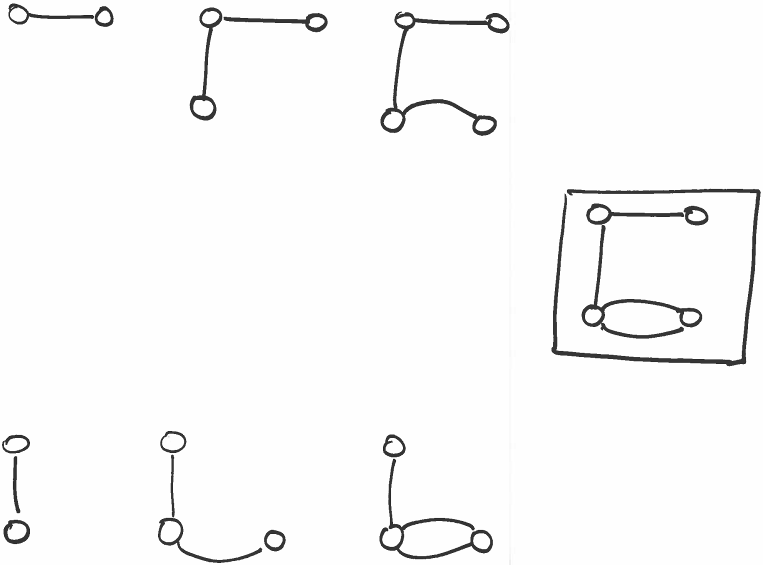


Edge exchangeable models [2], [3]

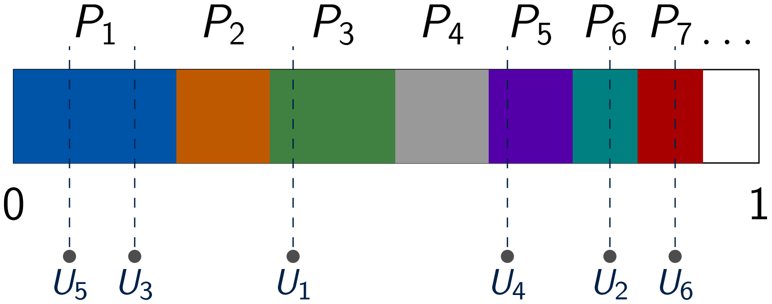




Edge exchangeable models [2], [3]

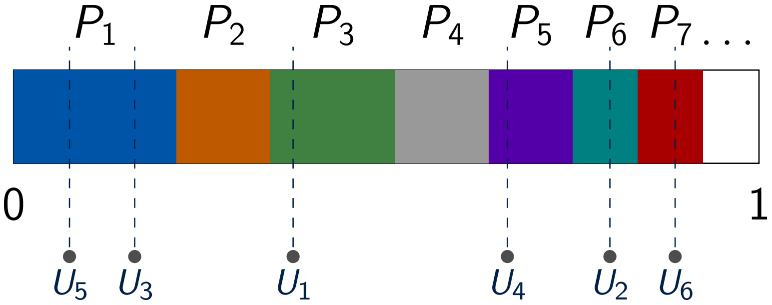


Factorizable ("rank one") paintbox representation

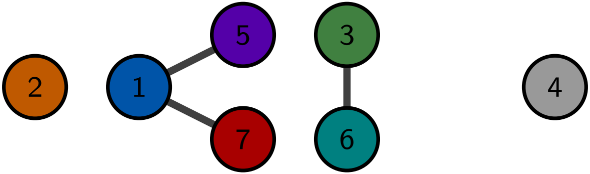


Sequence of edges: $((3,6),(1,5),(1,7))$

Factorizable ("rank one") paintbox representation



Sequence of edges: $((3,6),(1,5),(1,7))$

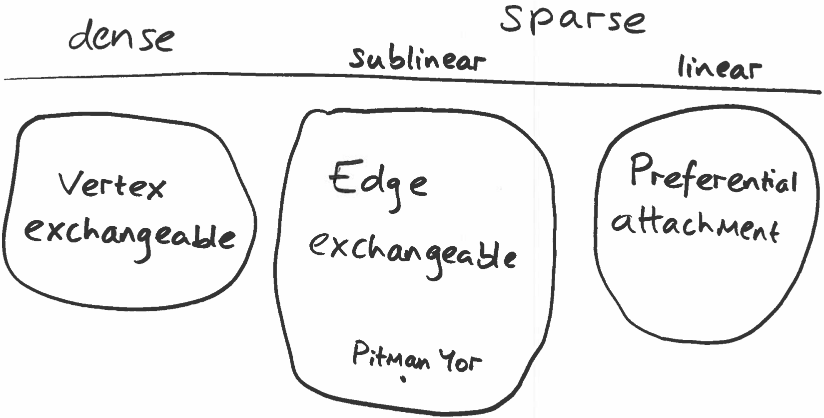


Paintbox representation

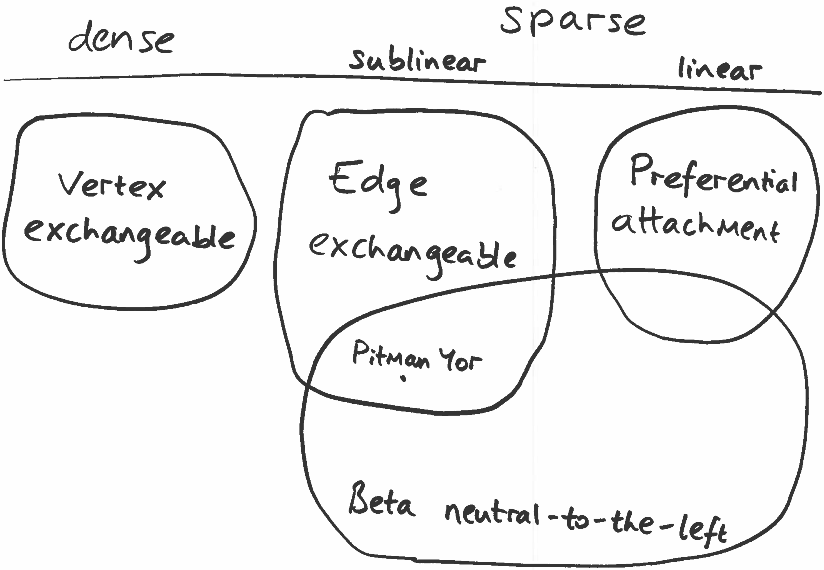
Consequences for edge exchangeable models:

- Rate of vertex arrival gets slower and slower
  $\Rightarrow$ sublinear sparsity: $\#\text{vertices} = o(\#\text{edges})$
- Edges pile up
  $\Rightarrow$ linear scaling of degrees: $d_{j,n} = \Theta(n)$

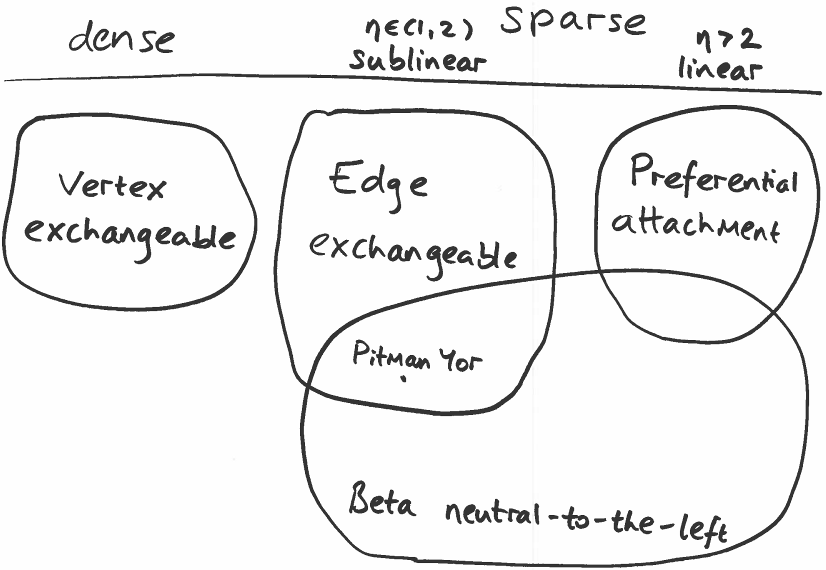
Models


dense
sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

Models

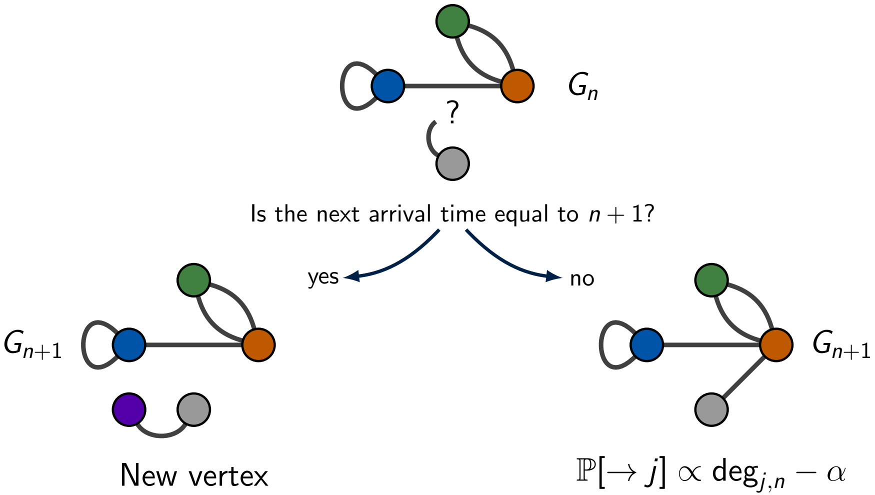

dense
sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
Preferential attachment
Pitman Yor
Beta neutral-to-the-left

Models

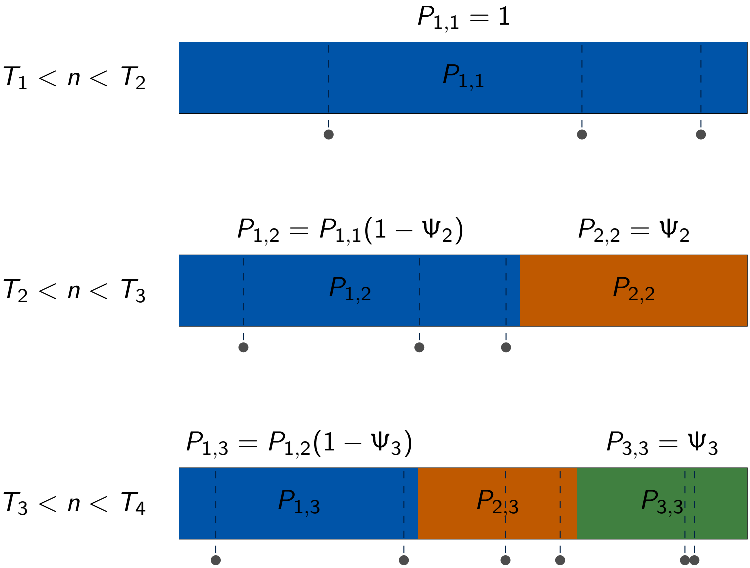

dense
η∈(1,2)
sublinear
sparse
η>2
linear
Vertex exchangeable
Edge exchangeable
Preferential attachment
Pitman Yor
Beta neutral-to-the-left

Beta Neutral-to-the-left Model [4]

1. Generate arrival times $1 = T_1 < T_2 < T_3 < \ldots$ in any way.
2. Generate ends of edges sequentially:



Sequence of paintboxes representation



Sampling and inference

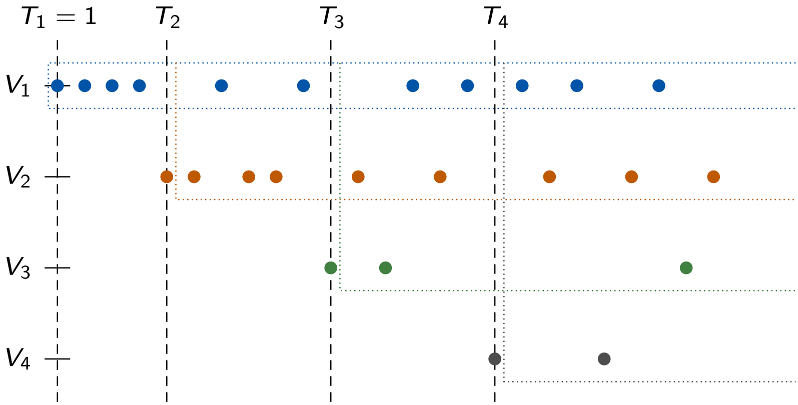
Why a paper on sampling and inference for BNTL models?

Sampling and inference

Why a paper on sampling and inference for BNTL models?

- Inference is notoriously difficult for *non-exchangeable* structures
- Need to identify *exchangeable substructures*

Exchangeable substructure



Gibbs structure

The joint probability has **Gibbs structure** due to left-neutrality

$$P(\text{graph}|\mathbf{T}) = \prod_{j=1}^{K} P(\text{choose } j \ d_j - 1 \text{ times out of } n - T_j \text{ trials })$$

- $K = \#$vertices
- $n = \#$edges
- $d_j =$ degree of vertex j
- $T_j =$ arrival time of vertex j

Gibbs structure

Explicitly,

$$p(\text{graph}|\mathbf{T}) = \frac{\Gamma(d_1 - \alpha)}{\Gamma(n - K\alpha)} \prod_{j=2}^{K} \frac{\Gamma(T_j - j\alpha)\Gamma(d_j - \alpha)}{\Gamma(T_j - 1 - (j-1)\alpha)\Gamma(1-\alpha)}$$

- $K = \#$vertices
- $n = \#$edges
- $d_j =$ degree of vertex j
- $T_j =$ arrival time of vertex j

Available data

Observation	**Unobserved variables**
Entire history	$\alpha, \phi, \mathbf{\Psi}$
Degrees in arrival order	$\alpha, \phi, \mathbf{\Psi}, \mathbf{T}$
Snapshot	$\alpha, \phi, \mathbf{\Psi}, \mathbf{T}, \sigma$

- $\alpha =$ BTNL parameter $\in (-\infty, 1)$
- $\phi =$ arrival distribution parameters
- $\mathbf{\Psi} =$ latent sociabilities
- $\mathbf{T} =$ arrival times
- $\sigma =$ arrival order

Gibbs sampler

Variable	**Gibbs sampling scheme**
α	MCMC, e.g. slice sampling
ϕ	Depends on arrival dist. family Λ_ϕ
$\boldsymbol{\Psi}$	$\Psi_j \mid \text{graph}, \boldsymbol{\Psi}_{\setminus j} \sim \text{Beta}(d_j - \alpha, \bar{d}_{j-1} - (j-1)\alpha)$ where $\bar{d}_j = \sum_{i=1}^{j} d_j$ can marginalise out $\boldsymbol{\Psi}$
$\mathbf{T}$	Simple update for T_j, can't move past neighbours
σ	Initialise in descending degree order use M-H with adjacent swap proposal $\sigma_j \leftrightarrow \sigma_{j+1}$ fast to compute due to Gibbs structure

Point estimation

If entire history observed,**maximum a posterior** (or **maximum likelihood**) estimates for α, ϕ computable

Experiments

- Gibbs: parameter recovery
- Gibbs: scalability
- Point estimation with massive graphs

Parameter recovery

- Simulate 500 edges with fixed α
- Arrivals either $\mathcal{PYP}$ or Geom
- Observe final snapshot of the graph

Gen. arrival distn.	Inference model	$\lvert\hat{\alpha} - \alpha^*\rvert$	Pred. log-lik.
$\mathcal{PYP}(1.0, 0.75)$	$(\tau, \mathcal{PYP}(\theta, \tau))$	**0.046** $\pm$ **0.002**	-**2637.0** $\pm$ **0.1**
$\mathcal{PYP}(1.0, 0.75)$	$(\alpha, \text{Geom}(\beta))$	0.049 $\pm$ 0.004	-2660.5 $\pm$ 0.7
Geom(0.25)	$(\tau, \mathcal{PYP}(\theta, \tau))$	0.086 $\pm$ 0.002	-2386.8 $\pm$ 0.1
Geom(0.25)	$(\alpha, \text{Geom}(\beta))$	**0.043** $\pm$ **0.003**	-**2382.6** $\pm$ **0.2**

Scalability

- Simulate with fixed α and Geom(β) arrivals

	100 edges	10000 edges
$\lvert\hat{\alpha} - \alpha^*\rvert$	0.12 ± 0.01	0.01 ± 0.00
$\lvert\hat{\beta} - \beta^*\rvert$	0.02 ± 0.00	0.00 ± 0.00
Effective Sample Size	0.90 ± 0.04	0.75 ± 0.08
Runtime (s)	21 ± 0	2267 ± 2

- Runtime linear in #edges
- Most expensive Gibbs update is for $\mathbf{T}$

MLEs for SNAP datasets

- SNAP datasets
- Fit point estimates for α, ϕ
- Fit: coupled $\mathcal{PYP}$, uncoupled $\mathcal{PYP}$ and Geom(β) arrivals

MLEs for SNAP datasets

Ask Ubuntu

- Estimates of $\mathcal{PYP}$ parameters vary significantly between coupled and uncoupled
  - $\hat{\theta}, \hat{\alpha} = 18080, 0.25$
  - $\hat{\theta}, \hat{\alpha}, \hat{\tau} = -0.99, -2.54, 0.99$
- Edge exchangeable models misspecified ($\eta > 2$)
- Using Geom estimates η well

Conclusion

- BNTL models are *flexible*
- BNTL models are *tractable*

References

[1] Jure Leskovec and Andrej Krevl. SNAP Datasets: Stanford large network dataset collection. `http://snap.stanford.edu/data`, June 2014.

[2] Diana Cai, Trevor Campbell, and Tamara Broderick. Edge-exchangeable graphs and sparsity. In *Advances in Neural Information Processing Systems*, pages 4249–4257, 2016.

[3] Harry Crane and Walter Dempsey. Edge exchangeable models for interaction networks. *Journal of the American Statistical Association*, 2017. In press.

[4] Benjamin Bloem-Reddy and Peter Orbanz. Preferential attachment and vertex arrival times. *arXiv preprint arXiv:1710.02159*, 2017.