Sampling and Inference for Beta Neutral-to-the-Left Models of Sparse Networks

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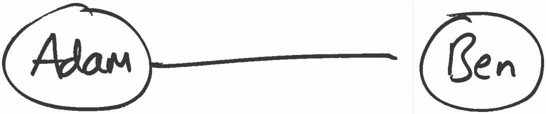
Temporal networks

Example

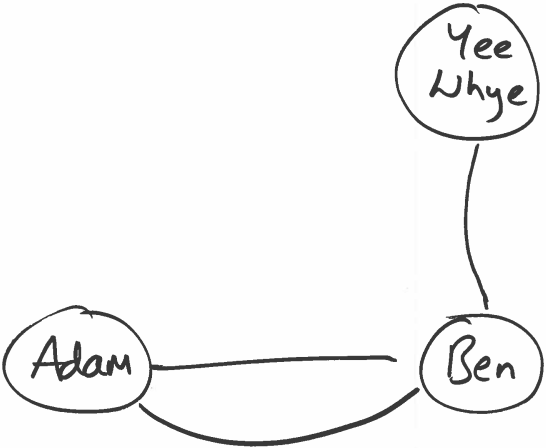
- Messages sent between people over time

Temporal networks

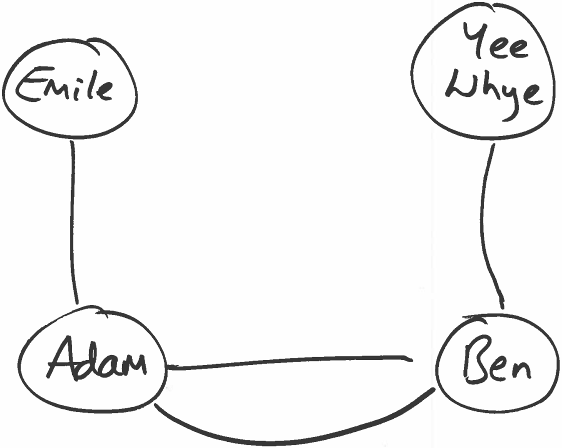


Temporal networks

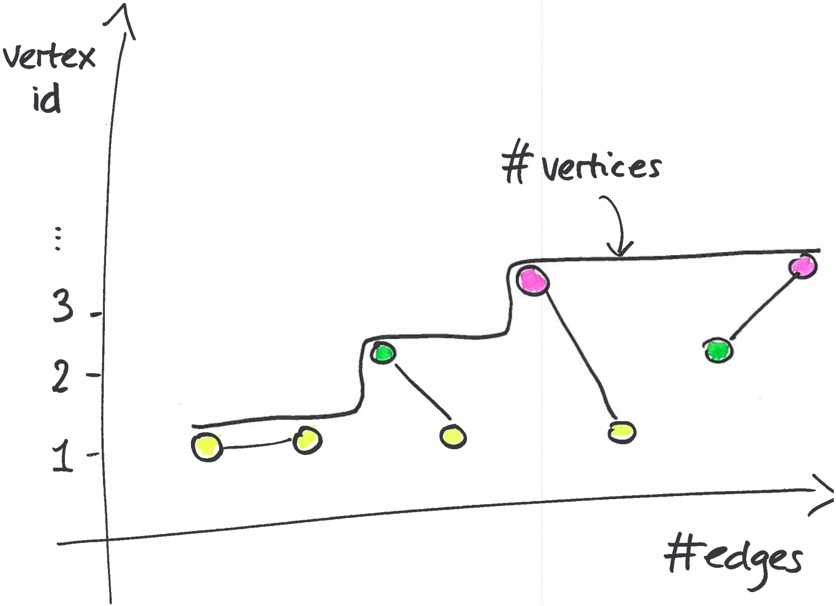
Temporal networks



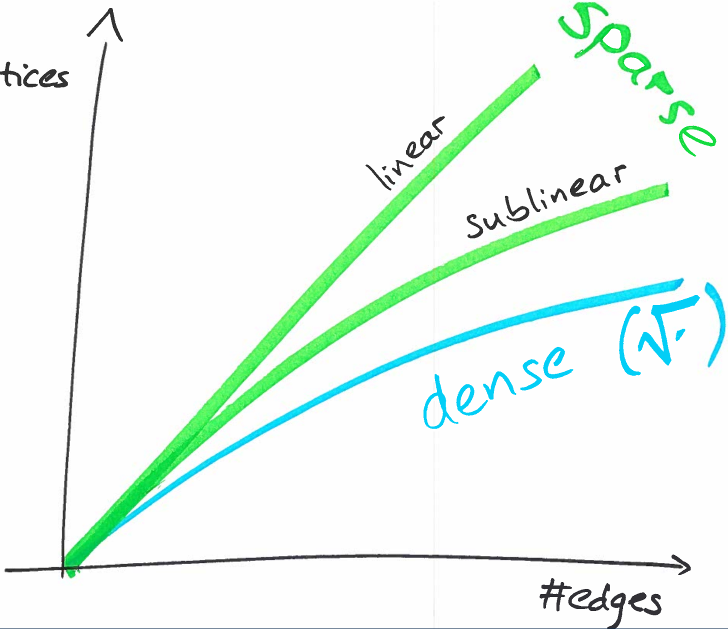
Temporal networks



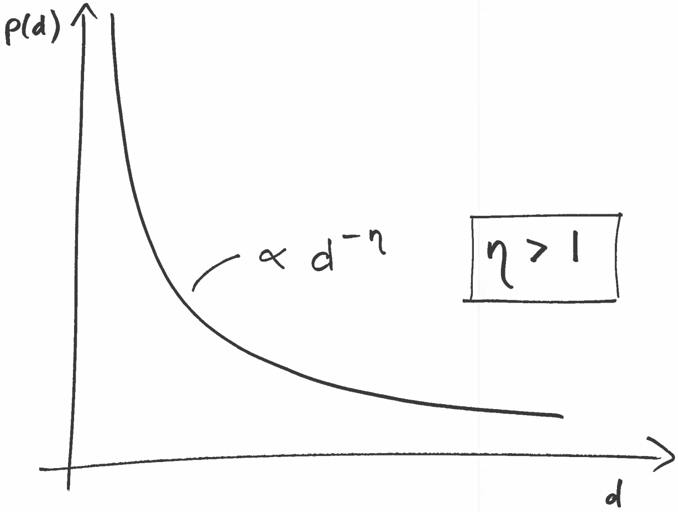
Edges and vertices



Sparsity



Power law degree distribution



Sparity and power law

$$\textbf{Sublinear sparsity} \quad \Longleftrightarrow \quad \eta \in (1,2)$$

$$\textbf{Linear sparsity} \quad \Longleftrightarrow \quad \eta > 2$$

Models

Vertex exchangeable

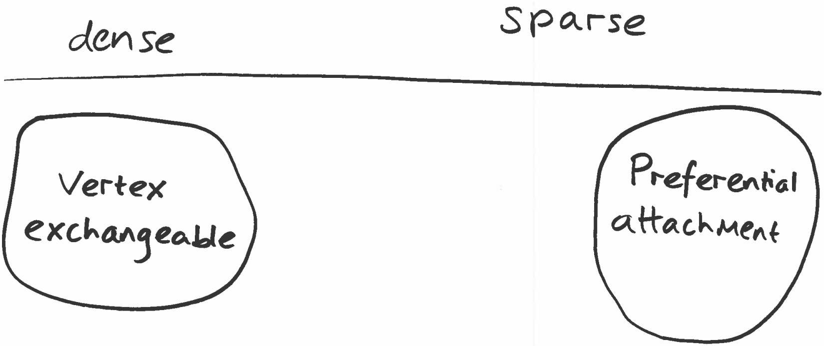
Models

dense

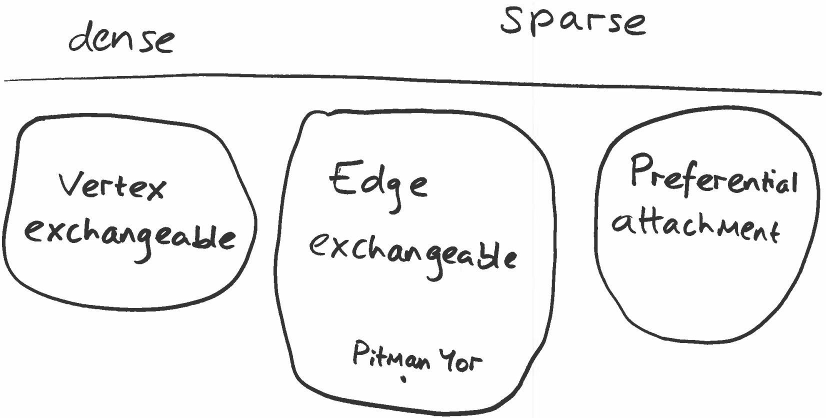
sparse

Vertex exchangeable

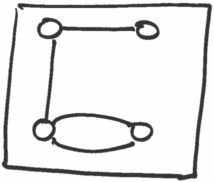
Models


dense
sparse
Vertex exchangeable
Preferential attachment

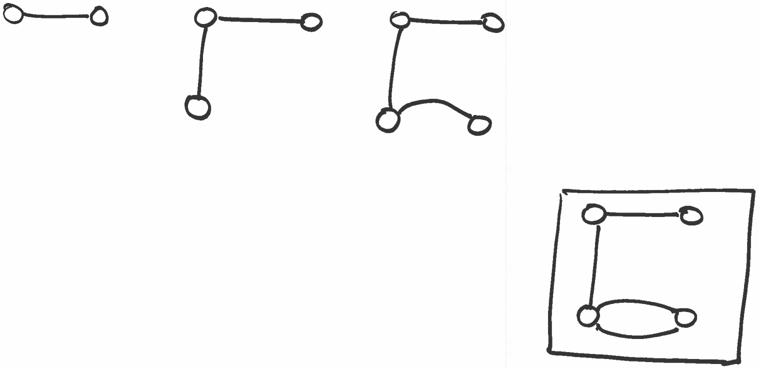
Models


dense
sparse
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

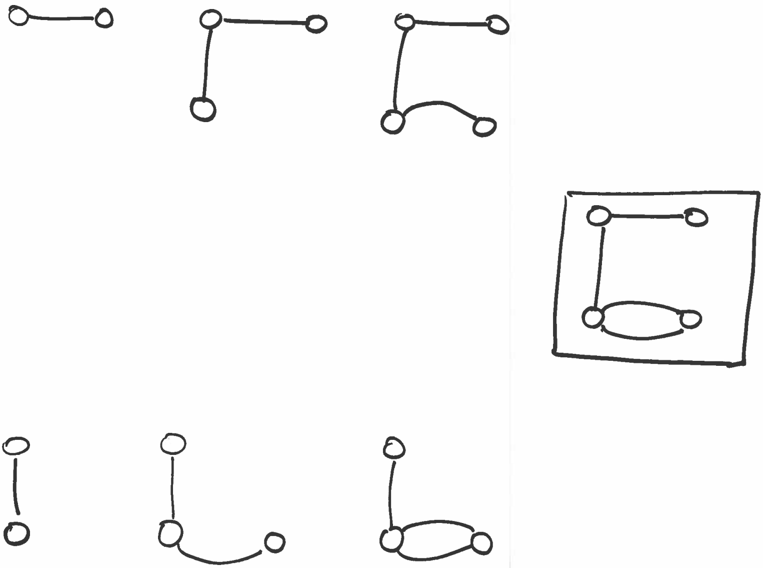
Edge exchangeable models [9], [8]



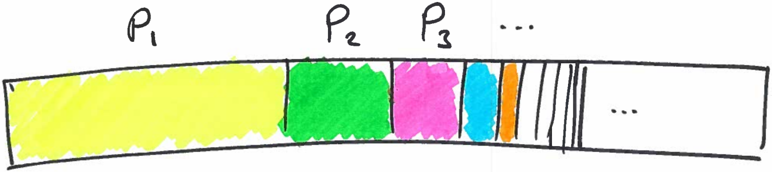
Edge exchangeable models [9], [8]



Edge exchangeable models [9], [8]



Paintbox representation

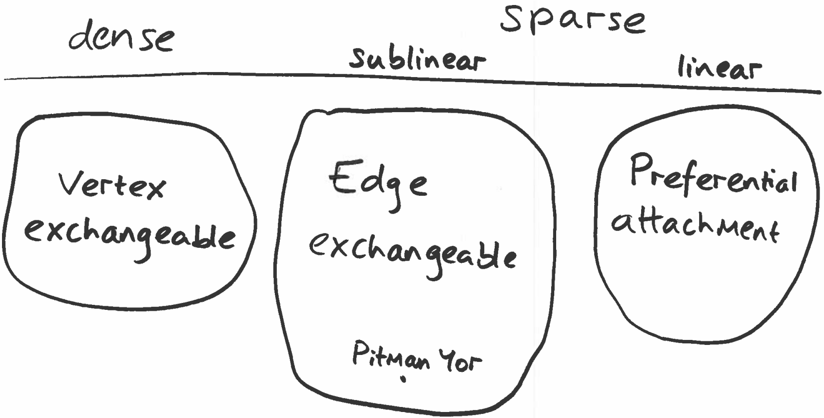


Paintbox representation

Consequence

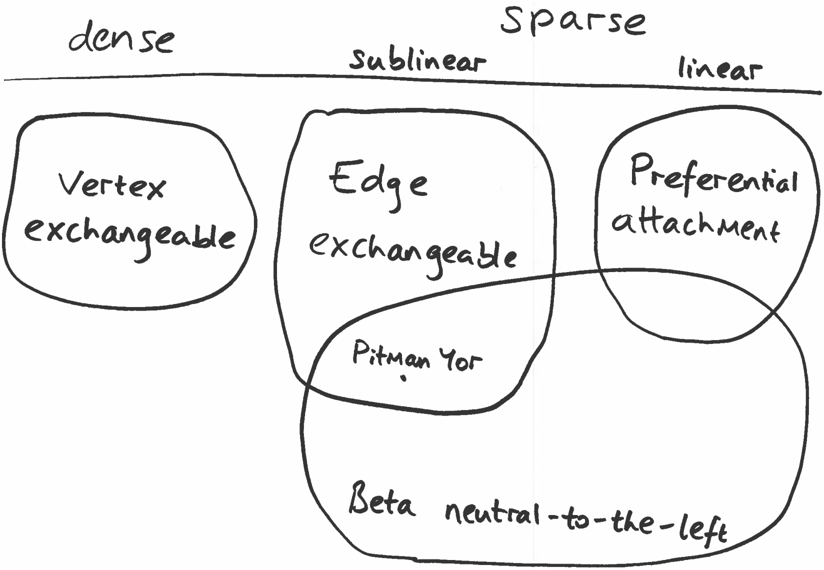
- Edge exchangeable models have sublinear sparsity

Models



dense
sparse
sublinear
linear
Vertex exchangeable
Edge exchangeable
Pitman Yor
Preferential attachment

Models



dense
sparse
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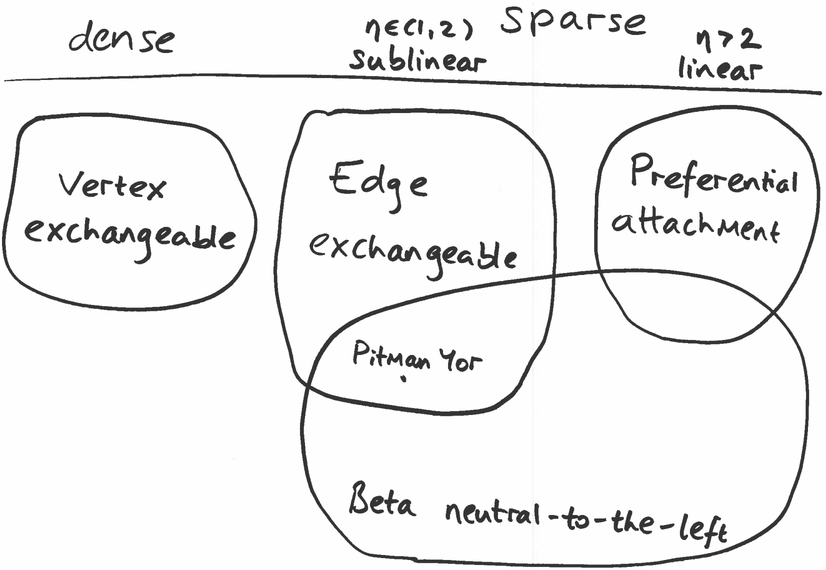
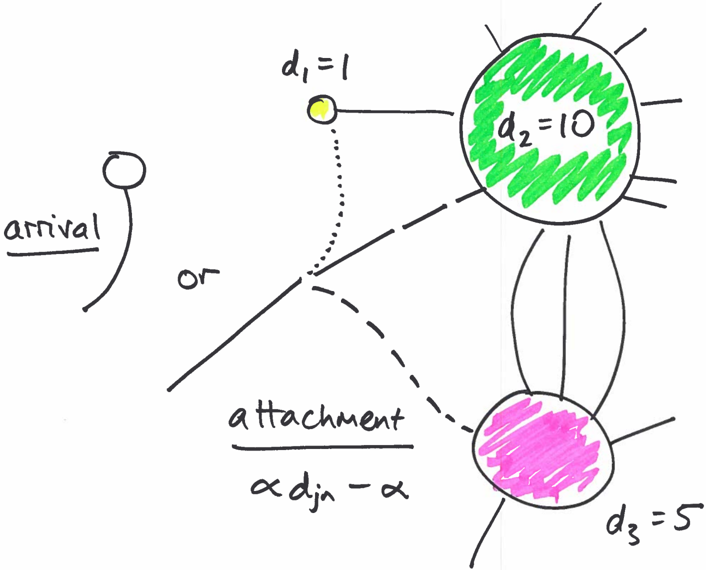
Pitman Yor
Beta
neutral-to-the-left

Models

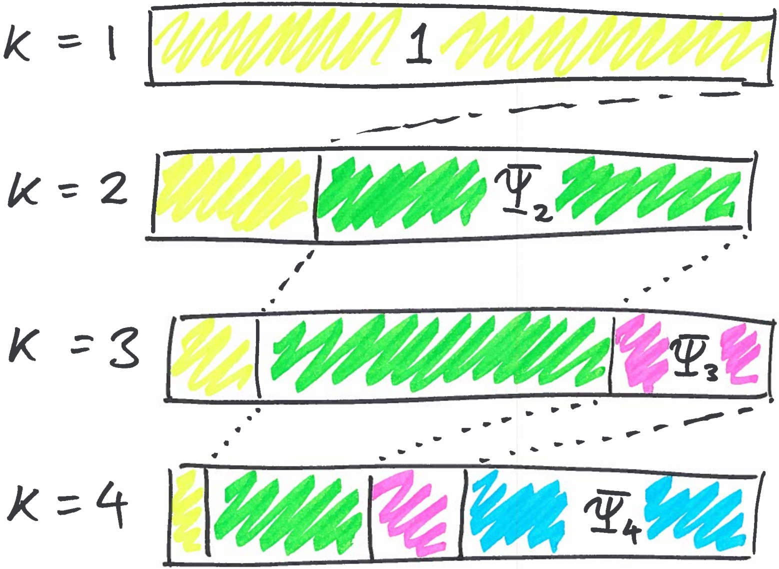


dense
$\eta \in (1,2)$
sublinear
sparse
$\eta > 2$
linear
Vertex exchangeable
Edge exchangeable
Preferential attachment
Pitman Yor
Beta neutral-to-the-left

Beta Neutral-to-the-left Model [10]



Latent representation



Available data

Observation	**Unobserved variables**
Entire history	$\alpha, \phi, \mathbf{\Psi}$
Degrees in arrival order	$\alpha, \phi, \mathbf{\Psi}, \mathbf{T}$
Snapshot	$\alpha, \phi, \mathbf{\Psi}, \mathbf{T}, \sigma$

- α = BTNL parameter $\in (-\infty, 1)$
- ϕ = arrival distribution parameters
- $\mathbf{\Psi}$ = latent sociabilities
- $\mathbf{T}$ = arrival times
- σ = arrival order

Gibbs structure

The joint density has **Gibbs structure**

$$p(\text{graph}|\mathbf{T}) = \frac{\Gamma(d_1 - \alpha)}{\Gamma(n - K\alpha)} \prod_{j=2}^{K} \frac{\Gamma(T_j - j\alpha)\Gamma(d_j - \alpha)}{\Gamma(T_j - 1 - (j-1)\alpha)\Gamma(1-\alpha)}$$

- T_j = arrival time of vertex j
- d_j = degree of vertex j
- K = #vertices
- n = #edges

Sampling $\boldsymbol{\Psi}$

Beta prior on Ψ_j, plus Gibbs structure, give

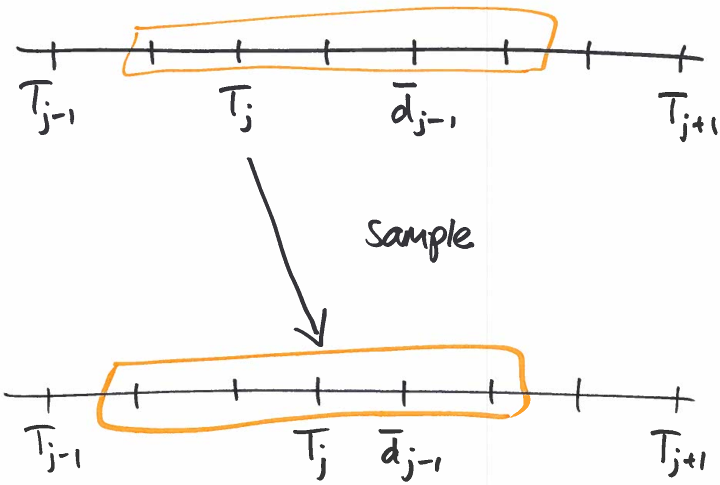
$$\Psi_j \mid \text{graph}, \boldsymbol{\Psi}_{\setminus j} \sim \text{Beta}(d_j - \alpha, \bar{d}_{j-1} - (j-1)\alpha) \ ,$$

where $\bar{d}_j = \sum_{i=1}^{j} d_i$

Sampling α, ϕs

- One-dimensional unnormalized density for α
- For ϕ depends on arrival distribution family

Sampling **T**



Sampling σ

- Initialise in descending degree order
- Use Metropolis-Hastings with adjacent swap proposal $\sigma_j \leftrightarrow \sigma_{j+1}$

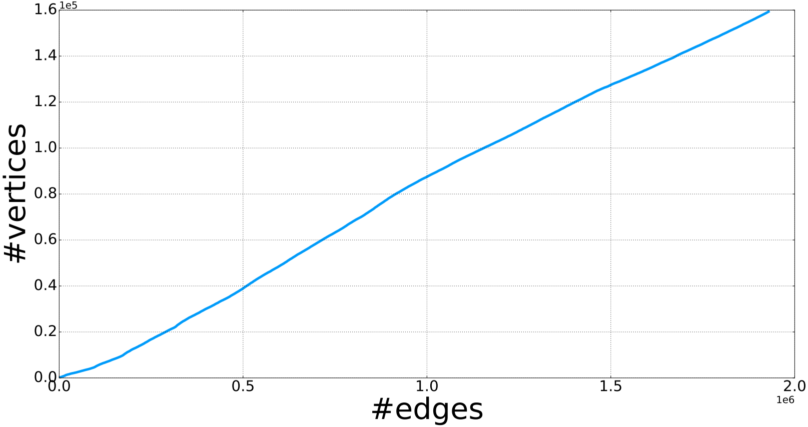
Point estimation

- Decompose $p_{\alpha,\phi}(\text{graph}) = p_\phi(\mathbf{T})p_\alpha(\text{graph}|\mathbf{T})$
- MLE/MAP estimation for α by optimizing unnormalized density

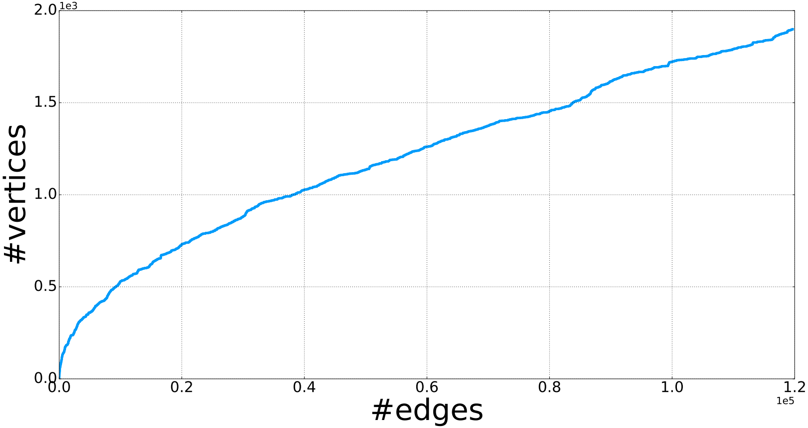
Empirical study

SNAP dataset [2]	# of vertices	# of edges
Ask Ubuntu	159,316	964,437
UCI social network	1,899	20,296
⋮	⋮	⋮

Ask Ubuntu



UCI social network



Experiments

- Synthetic data – parameter recovery
- Scaling in n
- Point estimation with massive graphs

Synthetic data

- Simulate 500 edges from the prior with fixed α
- Arrivals either $\mathcal{PYP}$ or Geom
- Observe final snapshot of the graph only

Gibbs sampler results

Gen. arrival distn.	Inference model	$\lvert\hat{\alpha} - \alpha^*\rvert$	Pred. log-lik.
$\mathcal{PYP}(1.0, 0.75)$	$(\tau, \mathcal{PYP}(\theta, \tau))$	$\mathbf{0.046} \pm \mathbf{0.002}$	$-\mathbf{2637.0} \pm \mathbf{0.1}$
$\mathcal{PYP}(1.0, 0.75)$	$(\alpha, \text{Geom}(\beta))$	0.049 ± 0.004	-2660.5 ± 0.7
Geom(0.25)	$(\tau, \mathcal{PYP}(\theta, \tau))$	0.086 ± 0.002	-2386.8 ± 0.1
Geom(0.25)	$(\alpha, \text{Geom}(\beta))$	$\mathbf{0.043} \pm \mathbf{0.003}$	$-\mathbf{2382.6} \pm \mathbf{0.2}$

Scalability of Gibbs sampler

- Do we learn from all data?
- How does performance scale?

Scalability of Gibbs sampler

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- How does performance scale?

	$n = 200$	$n = 20000$
$\lvert\hat{\alpha} - \alpha^*\rvert$	0.12 ± 0.01	0.01 ± 0.00
$\lvert\hat{\beta} - \beta^*\rvert$	0.02 ± 0.00	0.00 ± 0.00
ESS	0.90 ± 0.04	0.75 ± 0.08
Runtime (s)	21 ± 0	2267 ± 2

- Most expensive Gibbs update is for $\mathbf{T}$

MLEs for SNAP datasets

Fitted point estimates

Dataset	Coupled $\mathcal{PYP}(\theta, \alpha)$				Uncoupled $\mathcal{PYP}(\theta, \tau)$		Geom(β)		
	$(\hat{\theta}, \hat{\alpha})$	$\hat{\eta}$	Pred. l-l.	$\hat{\alpha}$	$(\hat{\theta}, \hat{\tau})$	Pred. l-l.	$\hat{\beta}$	$\hat{\eta}$	Pred. l-l.
Ask Ubuntu	(18080, 0.25)	1.25	-3.707e6	-2.54	(-0.99, 0.99)	-3.678e6	0.083	2.32	**-3.678e6**
UCI social network	(320.4, 4.4e-11)	–	-1.600e5	-4.98	(5.50, 0.52)	**-1.595e6**	0.016	2.10	-1.596e5
EU email	(113.6, 2.5e-14)	–	**-8.06e5**	-1.86	(113.6, 9.2e-10)	**-8.06e5**	0.001	2.00	-8.07e5
Math Overflow	(2575, 0.15)	1.15	-1.685e6	-6.62	(-0.97, 0.997)	-1.670e6	0.025	2.19	**-1.670e6**
Stack Overflow	(297600, 0.11)	1.11	-3.358e8	-8.94	(-1.0, 1.0)	-3.333e8	0.020	2.21	**-3.333e8**
Super User	(20640, 0.24)	1.24	-5.855e6	-4.19	(-0.996, 1.0)	**-5.775e6**	0.067	2.37	-5.775e6
Wikipedia talk pages	(14870, 0.54)	1.54	-3.074e7	-0.25	(-1.0, 1.0)	**-3.066e7**	0.073	2.10	-3.066e7

MLEs for SNAP datasets

$\mathcal{PYP}$ parameter estimates vary coupled and uncoupled

Dataset	Coupled $\mathcal{PYP}(\theta, \alpha)$				Uncoupled $\mathcal{PYP}(\theta, \tau)$		Geom(β)		
	$(\hat{\theta}, \hat{\alpha})$	$\hat{\eta}$	Pred. l-l.	$\hat{\alpha}$	$(\hat{\theta}, \hat{\tau})$	Pred. l-l.	$\hat{\beta}$	$\hat{\eta}$	Pred. l-l.
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MLEs for SNAP datasets

Edge exchangeable models likely misspecified

Dataset	Coupled $\mathcal{PYP}(\theta, \alpha)$				Uncoupled $\mathcal{PYP}(\theta, \tau)$		Geom(β)		
	$(\hat\theta, \hat\alpha)$	$\hat\eta$	Pred. l-l.	$\hat\alpha$	$(\hat\theta, \hat\tau)$	Pred. l-l.	$\hat\beta$	$\hat\eta$	Pred. l-l.
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MLEs for SNAP datasets

Though better than Geom for some datasets

Dataset	Coupled $\mathcal{PYP}(\theta, \alpha)$				Uncoupled $\mathcal{PYP}(\theta, \tau)$		Geom(β)		
	$(\hat{\theta}, \hat{\alpha})$	$\hat{\eta}$	Pred. l-l.	$\hat{\alpha}$	$(\hat{\theta}, \hat{\tau})$	Pred. l-l.	$\hat{\beta}$	$\hat{\eta}$	Pred. l-l.
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Conclusion

- BNTL models are *flexible*
- BNTL models are *tractable*

Future work

- Scalability of inference
  - Metropolis-Hastings to update $\mathbf{T}$ altogether
  - Variational inference for $\mathbf{T}$ [11]
- Recency-weighted preferential attachment

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