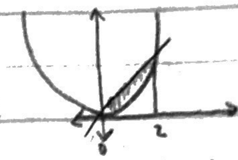
Calculus 2 Review:

Region Between Curves

$A = \int_a^b (f(x) - g(x))\,dx$

$y = x^2,\ y = 2x\ ;\ x = 0\ ;\ x = 2$

$A = \int_0^2 (2x - x^2)\,dx$



$= \left[x^2 - \frac{x^3}{3}\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

Disk/Washer Method

Disk

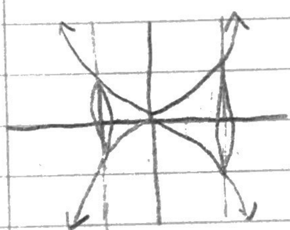
$V = \pi \int_a^b [R(x)]^2\,dx$

Washer

$V = \pi \int_a^b \left([R_{outer}(x)]^2 - [R_{inner}(x)]^2\right)dx$

Washer

Region bounded by $y = x^2$ and $y = 4$ about the x-axis



$x^2 = 4$

$x = \pm 2$

$V = \pi \int_{-2}^{2} \left[4^2 - (x^2)^2\right]dx$

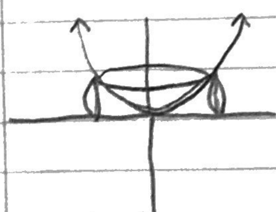
$= \pi \int_{-2}^{2} (16 - x^4)\,dx$

$= \pi \left[16x - \frac{x^5}{5}\right]_{-2}^{2}$

$= \pi \left(\left[32 - \frac{32}{5}\right] - \left[-32 + \frac{32}{5}\right]\right)$

Disk

Region bounded by $y = x^2$ and $x = 1$ about the y-axis



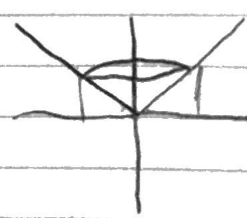
$x = \sqrt{y}$

$V = \pi \int_0^1 (\sqrt{y})^2\,dy = \pi \int_0^1 y\,dy$

$= \pi \left[\frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}$

Volume by Shells

$V = 2\pi \int_a^b (\text{radius}) * (\text{height})\,dx$



Rotate $y = x$ about y-axis from $x = 0$ $x = 1$

$V = 2\pi \int_0^1 (x)(x)\,dx = 2\pi \int_0^1 x^2\,dx = 2\pi \left[\frac{x^3}{3}\right]_0^1 = \frac{2\pi}{3}$

Calculus 2 Review:

Length of Curves

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Length of $y = \frac{x^3}{3}$ from $x=0$ to $x=1$

$$L = \int_0^1 \sqrt{1+(x^2)^2}\, dx = \int_0^1 \sqrt{1+x^4}\, dx$$

Surface Area

$$SA = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Revolve $y=x^2$ about x-axis from $x=0$ to $x=1$

$$SA = 2\pi \int_0^1 x^2\sqrt{1+(2x)^2}\, dx$$

Physics Applications

Work: $W = \int_a^b F(x)\, dx$

Center of mass: $\bar{x} = \frac{1}{A}\int_a^b x \cdot \rho(x)\, dx$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Ex: $\int xe^x$ $\quad u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= x \cdot e^x - \int e^x dx = x\cdot e^x - e^x + C$$

Trig Identities

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\tan^2\theta = 1 - \frac{2\tan\theta}{\tan 2\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta = 1 + \tan^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Trig Integrals

$$\int \sin^2(x)\, dx = \int \frac{1-\cos(2x)}{2}\, dx = \frac{1}{2}\int (1-\cos(2x))\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$$

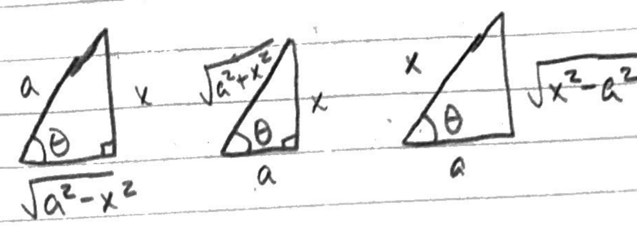
Calculus 2 Review:

Trig Sub

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$

$\sqrt{a^2+x^2}$ $\quad x = a\tan\theta$

$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$



$$\int \frac{1}{\sqrt{4-x^2}}\,dx \qquad x = 2\sin\theta,\ dx = 2\cos\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{4-(2\sin\theta)^2}}\, 2\cos\theta\, d\theta = \int \frac{2\cos\theta}{2\cos\theta}\, d\theta = \int d\theta = \theta + C$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C$$

Partial Fractions

$$\frac{px+q}{(x-a)(x-b)} \rightarrow \frac{A}{x-a} + \frac{B}{x-b} \qquad \frac{px+q}{(x-a)^2} \rightarrow \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} \rightarrow \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \qquad \frac{px^2+qx+r}{(x-a)(x^2+bx+c)} \rightarrow \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

Ex: $\displaystyle\int \frac{1}{x^2-x} = \int \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$ $\qquad 1 = A(x-1) + Bx$

$\qquad 1 = (A+B)x - A$

$\qquad A + B = 0$

$\qquad -A = 1$

$\qquad A = -1,\ B = 1$

$$\int\left(-\frac{1}{x} + \frac{1}{x-1}\right)dx = -\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx$$

$$= \ln|x-1| - \ln|x| + C$$

Numerical Integration

Trapezoidal $\displaystyle\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$

$\displaystyle\int_0^1 e^x\,dx$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$ $\qquad \displaystyle\int_0^1 e^x\,dx \approx \left(\frac{1-0}{2(4)}\right)\left(e^0 + 2\left(e^{0.25} + e^{0.5} + e^{0.75}\right) + e^1\right)$

Simpson's $\displaystyle\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left(f(x_0) + 4\sum_{\text{odd}} f(x_i) + 2\sum_{\text{even}} f(x_i) + f(x_n)\right)$

Calculus 2 Review:

Improper Integrals:

Infinite $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx \qquad \int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$\lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1}\right) = 0 + 1 = 1$

Discontinuous

$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx \qquad \int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}$

$\lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0^+} \left(2\sqrt{1} - 2\sqrt{a}\right) = 2(1-0) = 2$

Sequences

ordered list of numbers : $\{a_n\}$ $\qquad \lim_{n\to\infty} \frac{1}{n} = 0$

Series

sum of an infinite sequence $\qquad \sum_{n=1}^{\infty} a_n$

Geometric Series

$\sum_{n=0}^{\infty} ar^n$ converges if $|r|<1$ and its sum is: $S = \frac{a}{1-r}$

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, the series $\sum a_n$ diverges

Integral Test

If $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ both converge or diverge.

$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1$ integral converges so series converges

Calculus 2 Review:

Comparison Test

If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges
If $0 \le b_n \le a_n$ and $\sum b_n$ diverges, then $\sum b_n$ diverges

Alternating Series Test (Leibniz's Test)

$\sum (-1)^{n-1} a_n$ converges if a_n is decreasing $a_{n+1} \le a_n$ & $\lim_{n\to\infty} a_n = 0$

Ratio Test

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

- $L < 1$ the series converges absolutely
- $L > 1$ the series diverges
- $L = 1$ the test is inconclusive

Root Test

$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

Power Series

$\sum_{n=0}^{\infty} c_n (x-a)^n$; converges if $|x-a| < R$ where R is the radius of convergence

Taylor Series (any center a)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

Maclaurin Series (e^x, $\sin x$, $\ln(x)$) (centered at $a=0$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Parametric Equations

$x = f(t) \quad y = g(t)$

$x = t^2, \; y = t^3 \qquad t = \pm\sqrt{x}$

$y = (\pm\sqrt{x})^3 = \pm x^{3/2}$