EX. 5.1.2

a) $f'(x) = e^x$

$f'(0) = e^0 = \boxed{1}$

b) $f'(x) = \boxed{\dfrac{f(x+h) - f(x-h)}{2h}}$

d) $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} \boxed{- \dfrac{h^2}{6} f'''(c)}$ where $x-h<c<x+h$

e) $f'''(x) = e^x$ $\qquad x-h < c < x+h$

$\dfrac{h^2}{6} e^{x-h} > \left|\dfrac{h^2}{6} f'''(c)\right| > \dfrac{h^2}{6} e^{x+h}$

$\boxed{\left(-\dfrac{h^2}{6} e^{x-h}, -\dfrac{h^2}{6} e^{x+h}\right)}$

g) $\boxed{\dfrac{h^2}{6} f'''(x) \le \dfrac{h^2}{6} e^{x+h}}$

EX. 5.1.4

a) $f'(x) = \cos x$

$f'\left(\frac{\pi}{3}\right) = \cos \pi/3 = \boxed{-\frac{1}{3}}$

b) $f'(x) = \boxed{\dfrac{f(x+h) - f(x-h)}{2h}}$

d) $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} \boxed{- \dfrac{h^2}{6} f'''(c) \text{ where } x-h<c<x+h}$

e) $f'''(x) = -\cos x$ $\qquad x-h < c < x+h$

$-\dfrac{h^2}{6} \cos(x-h) < \left|\dfrac{h^2}{6} \cos(c)\right| < -\dfrac{h^2}{6} \cos(x+h)$

$\boxed{\left(\dfrac{h^2}{6} \cos(x+h), \dfrac{h^2}{6} \cos(x-h)\right)}$

g) $\boxed{\dfrac{h^2}{6} f'''(x) \le \dfrac{h^2}{6} \cos(x-h)}$

$e^{-\frac{x^2}{2}}$ $\frac{-2x}{2}$ $-\frac{e^{-\frac{x^2}{2}}}{2}$

EX. 5.1.8

Taylor's Theorem:

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cancel{\frac{h^3}{6}f'''(c_1)}$ $O(h^3)$

$f(x+2h) = f(x) + (2h)f'(x) + \frac{(2h)^2}{2}f''(x) + \cancel{\frac{(2h)^3}{6}f'''(c)}$ $O(h^3)$

$f'(x) = \frac{4(f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cancel{\frac{h^3}{6}f'''(c)}) - 1(f(x) - (2h)f'(x) + \frac{(2h)^2}{2}f''(x) + \cancel{\ldots}}{2h}$ $O(h^3)$

$f'(x) = \frac{3f(x) + 2hf'(x) + O(h^3) - 3f(x)}{2h} = \boxed{f'(x) + O(h^3)}$ ✓

EX. 5.1.10

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x)$

$f(x-2h) = f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4}{3}h^3f'''(x) + \frac{2}{3}h^4f^{(4)}(x)$

$\frac{4f(x+h) - 3f(x) - f(x-2h)}{6h} = \frac{6hf'(x) - \frac{6}{3}h^3f'''(x) + \ldots}{6h}$

$\frac{1}{3}h^2f'''(x)$

* Second order with error of $O(h^2)$

5.1.18

$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(c)$

$f'(x)_{correct} - f'(x)_{machine} = f'(x) - \frac{f(x+h) - f(x)}{h} + \frac{\varepsilon_2 - \varepsilon_1}{h}$

$\left|\frac{\varepsilon_2 - \varepsilon_1}{h}\right| \le \frac{2\varepsilon_{machine}}{h}$ $\boxed{E(h) \equiv \frac{h}{2}f''(c) + \frac{2\varepsilon_{machine}}{h}}$

$E'(h) = -\frac{2\varepsilon_{machine}}{h^2} + \frac{M}{2}$

$\boxed{h = \left(\frac{4\varepsilon_{machine}}{M}\right)^{1/2}}$

$\frac{81}{24} - \frac{12}{3} + \frac{1}{2} + \frac{1}{8}$

$-1.5 + \frac{24}{3} - \frac{27}{6}$

$\frac{21}{6} - \frac{9}{6} = \frac{12}{6}$

EX. 5.1.20

$$\begin{aligned}
1.\quad & f(x-3h) = f(x) - 3hf'(x) + \frac{9h^2}{2}f''(x) - \frac{27h^3}{6}f'''(x) + \frac{81}{24}h^4 f^{(4)}(x) - O(h^5) \\
-6.\quad & f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3}h^3 f'''(x) + \frac{2h^4}{3} f^{(4)}(x) + O(h^5) \\
12.\quad & f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5) \\
3.\quad & f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5)
\end{aligned}$$

$-10f(x)$

$$f'''(x) = \frac{0f(x) + 0f'(x) + 0f''(x) + 2h^3 f'''(x) + 0f^{(4)}(x) + O(h^5)}{2h^3}$$

$$\boxed{= f'''(x) + O(h^5)}$$

EX. 5.2.4.b

$$\int_0^1 \frac{1}{1+x^2}\,dx \qquad \tan^{-1}(x)\Big|_0^1 \qquad \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

one panel $m=1$

$$\frac{1}{6}\left(f(0) + 4f(1/2) + f(1)\right) = 0.7833$$

$\boxed{\text{error} = 2.06 * 10^{-3}}$

two panels $m=2$

$$\frac{1}{6}\left(f(0) + 2f(1/4) + 4(1/2) + 2(3/4) + f(1)\right) = 0.7854$$

$\text{error} = 6.01 * 10^{-6}$

four panels $m=4$

$$\frac{1}{6}\big(f(0) + 4f(1/8) + 2f(1/4) + 4f(3/8) + 2f(1/2) + 4f(5/8) + 2f(3/4) + 4f(7/8) + f(1)\big) = 0.7854$$

$\text{error} = 3.78 * 10^{-8}$

EX. 5.2.6.b

$$\int_0^1 \frac{e^x - 1}{x}\,dx \approx 1.3179$$

* Solved in colab

$1 \cdot f(1/2) \approx 1.2974$, error $\approx 2.0 * 10^{-2}$

$1/2 \cdot (f(1/4) + f(3/4)) \approx 1.3127$, error $= 5.2 * 10^{-3}$

$1/4 \cdot (f(1/8) + f(3/8) + f(5/8) + f(7/8)) \approx 1.3166$, error $\approx 1.3 * 10^{-3}$

EX. 5.3.2.b

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

$r_{11} = 1/2\, h_1 (f(0) + f(1))$
$r_{21} = 1/2\, r_{11} + h_2 (f(1/2))$
$r_{31} = 1/2\, r_{21} + h_3 (f(1/4) + f(3/4))$
$r_{22} = (4 * r_{21} - r_{11})/3$
$r_{32} = (4 * r_{31} - r_{21})/3$
$r_{33} = (16 * r_{32} - r_{22})/15$

$$\boxed{\begin{array}{l} r_{33} \approx 0.7855 \\ \text{error} \approx 1.31 * 10^{-4} \end{array}}$$