EX 0.5.6 - 5660 version

Q. a) Find the Taylor polynomial of degree 4 for $f(x) = \ln(x)$ about the point $x_0 = 1$.

A. $P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{6} + \frac{f^{(4)}(c)(x-c)^4}{24}$

$f(x) = \ln(x) \quad f(1) = 0$

$f'(x) = \frac{1}{x} \quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4} \quad f^{(4)}(1) = -6$

$$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{6} - \frac{6(x-1)^4}{24}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

Q. b) Approximate $f(0.8)$ and $f(1.2)$

A. $P_4(1.2) = (0.2) - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} \approx 0.1823$

$P_4(0.8) = (-0.2) - \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{3} - \frac{(-0.2)^4}{4} \approx -0.2231$

A. c) $f^{(5)}(z) = \frac{24}{x^5}$ $\qquad 0.8 \le z \le 1$

$\qquad 1 \le z \le 1.2$

$R_5(0.8) = \left| \frac{24}{(0.8)^5} \cdot \frac{(-0.2)^5}{120} \right| = 1.9531 * 10^{-4}$ $\qquad |P_4(0.8) - \ln(0.8)| \le 1.9531 \times 10^{-4}$

$R_5(1.2) = \frac{(0.2)^5}{5} = 6.4 * 10^{-5}$ $\qquad |P_4(1.2) - \ln(1.2)| \le 6.4 \times 10^{-5}$

* I'd expect 1.2 to be more accurate due to a smaller error bound.

A. (a.) $|P_4(1.2) - \ln(1.2)| \approx 5.489 * 10^{-5} \leq 6.4 * 10^{-5}$ ✓

$|P_4(0.8) - \ln(0.8)| \approx 7.688 * 10^{-5} \leq 1.9551 * 10^{-4}$ ✓

EX. 1.2.14 - 5660 version

Q. Which of the following fixed point iteration converge to 3? Rank from fastest to slowest:

a) $x \to \frac{9}{x^2} - 2$ b) $x \to \frac{27}{x^3} + 2$ c) $x \to \frac{-18}{x^3} + \frac{11}{3}$

A. (a) cont diff at 3 ✓ $g'(x) = -18x^{-3}$

$g(3) = \frac{9}{3^2} + 2 = 3$ ✓ $g'(3) = \frac{-18}{27} = -2/3$ $|g'(3)| = 2/3 < 1$ ✓

(b) cont diff at 3 ✓ $g'(x) = -81x^{-4}$

$g(3) = \frac{27}{3^3} + 2 = 3$ ✓ $g'(3) = \frac{-81}{81}$ $|g'(3)| = 1 \not< 1$ X

(c) cont diff at 3 ✓ $g'(x) = \frac{54}{x^4}$

$g(3) = \frac{-18}{3^3} + \frac{11}{3} = 3$ ✓ $g'(3) = 54/81$ $|g'(3)| = 2/3 < 1$ ✓

b isn't locally convergent while a and c are. a and c have the same convergence rate indicated by $|g'(3)| = 2/3$. c is simply the derivative from a just with a constant to satisfy $g(x) = x$, which I thought was interesting.

EX. 2.7.2b - 5660 version

Q. Use Taylor expansion for $\left(F: \mathbb{R} \to \mathbb{R},\ \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 5u^2 + v \\ u + e^{-v} \end{pmatrix} \right)$ at $x_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

A. $F(x_0) = F\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 47 \\ 3 + e^{-2} \end{pmatrix}$

$DF\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial u(5u^2+v) & \partial v(5u^2+v) \\ \partial u(u+e^{-v}) & \partial v(u+e^{-v}) \end{pmatrix} = \begin{pmatrix} 10u & 1 \\ 1 & -e^{-v} \end{pmatrix}$

$DF(x_0) = DF\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 30 & 1 \\ 1 & -e^{-2} \end{pmatrix}$

$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 47 \\ 3 + e^{-2} \end{pmatrix} + \begin{pmatrix} 10u & 1 \\ 1 & -e^{-v} \end{pmatrix} \begin{pmatrix} u - 3 \\ v - 2 \end{pmatrix} = \begin{pmatrix} 47 + 10u^2 - 30u + v - 2 \\ 3 + e^{-2} + u - 3 - ve^{-v} + 2e^{-v} \end{pmatrix}$

EX. 3.2.2 · 5660 version

Q. (a) Given the data points $(0,1)$, $(2, e^2)$, $(4, e^4)$ to find degree 2 interpolating polynomial

A.

$$\begin{array}{lll} 0 & 1 & \\ 2 & e^2 & \frac{e^2-1}{2} \\ 4 & e^4 & \frac{e^4-e^2}{2} \quad \frac{\frac{e^4-e^2}{2} - \frac{e^2-1}{2}}{4} \end{array}$$

$$\boxed{P(x) = 1 + \frac{e^2-1}{2} x + \frac{\frac{e^4-e^2}{2} - \frac{e^2-1}{2}}{4} x(x-2)}$$

Q. (b) Approximate e^3

A. $1 + \frac{e^2-1}{2}(3) + \frac{\frac{e^4-e^2}{2} - \frac{e^2-1}{2}}{4}(3)(1) \approx \boxed{25.891}$

(c) $f(x) = e^x$ $\quad$ $f'''(x) = e^x$: maximized at $c = 4$ assuming interval $0 < c < 4$

$f'(x) = e^x$

$f''(x) = e^x$

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\ldots}{n!} f^{(n)}(c) \quad e^4$$

$$|f(x) - P(x)| \le \frac{e^4}{6} |(x-0)(x-2)(x-4)| \;\; \text{at } x=3 \rightarrow |f(x) - P(x)| \le \frac{e^4}{6}$$

$$\boxed{\approx 9.100}$$

(d) actual error: 5.806

actual error < error bound

5.806 < 9.100

Ex. 4.1.8.b - 5660 version

Q. Best line and RMSE from points $(5,7), (9,12), (13,6), (19,4)$

A. $A = \begin{bmatrix} 1 & 5 \\ 1 & 9 \\ 1 & 13 \\ 1 & 19 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 12 \\ 6 \\ 4 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 9 & 13 & 19 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & 9 \\ 1 & 13 \\ 1 & 19 \end{bmatrix} = \begin{bmatrix} 4 & 46 \\ 46 & 636 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 9 & 13 & 19 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \\ 297 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 46 \\ 46 & 636 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 29 \\ 297 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \frac{1}{428} \begin{bmatrix} 636 & -46 \\ -46 & 4 \end{bmatrix} \begin{bmatrix} 29 \\ 297 \end{bmatrix}$$

$$= \frac{1}{428} \begin{bmatrix} 4782 \\ -146 \end{bmatrix} \Rightarrow \boxed{y = \frac{4782}{428} - \frac{146}{428} x}$$

$$r = b - Ax = \begin{bmatrix} 7 \\ 12 \\ 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 5 \\ 1 & 9 \\ 1 & 13 \\ 1 & 19 \end{bmatrix} \frac{1}{428} \begin{bmatrix} 4782 \\ -146 \end{bmatrix} = \begin{bmatrix} -1056/428 \\ 1668/428 \\ 3350/428 \\ -296/428 \end{bmatrix}$$

$$\text{RMSE} = \sqrt{(15{,}207{,}476)/(732{,}736)} \approx \boxed{4.5557}$$