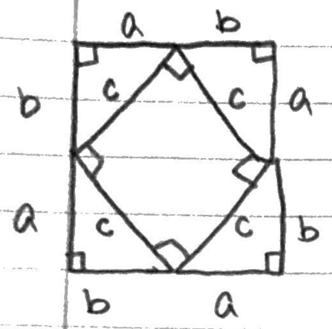
Abstract math (1/16/24)



$$(a+b)^2 = c^2 + 4\left(\frac{ab}{2}\right)$$
$$a^2 + \cancel{2ab} + b^2 = c^2 + \cancel{2ab}$$
$$a^2 + b^2 = c^2$$

Abstract math (1/18/24)

S - collection of elements $\{\}$
$\mathbb{Z}$ - integer
$\mathbb{N}$ - natural numbers $\{1, 2, \ldots\}$
$\mathbb{Q}$ - rational numbers $\{\frac{a}{b} \mid a, b \in \mathbb{Z}$ and $b \neq 0\}$ $\quad : = \mid$
$\mathbb{R}$ - real numbers
$\mathbb{C}$ - complex numbers $\{a + bi \mid a, b \in \mathbb{R}$ and $i^2 = -1\}$

$A \subseteq B$ provided all elements of A are elements of B
$\mathbb{N} \neq \mathbb{Z}$ b/c $\mathbb{Z} \not\subseteq \mathbb{N}$; $\mathbb{Q} \neq \mathbb{R}$ b/c $\mathbb{R} \not\subseteq \mathbb{Q}$
$X = Y :\Longleftrightarrow X \subseteq Y$ and $Y \subseteq X$

$S = \{\{1,2\}, 5, \mathbb{R}, (1,2)\}$ $\quad |S| = 4$
$\{1,2\} \in S$
$\{1,2\} \not\subseteq S$ $\qquad \{\{1,2\}\} \subseteq S$
$|\{1,2\}| = 2$
$|\{\{1,2\}\}| = 1$

$S = \{X \mid X$ a set and $X = X\} \Rightarrow S \in S$

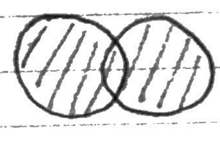
russels paradox: $T = \{X \mid X \notin X\}$ $\qquad T \in T$ iff $T \notin T$

*ZFC $\quad$ * To avoid problems, we want to have a universal set for reference ($X = \{x \in U \mid P(x)\}$ vs $X = \{x \mid P(x)\}$)

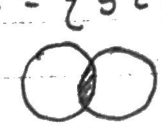
$S = \{x \in \mathbb{C} \mid x^2 = -1\} = \{i, -i\}$

$S = \{x \in \mathbb{R} \mid x^2 = -1\} = \{\ \} = \emptyset$ (empty/null set)

$X \cup Y := \{s \in U \mid s \in X$ or $s \in Y\}$ (X, Y are sets, subsets of U)

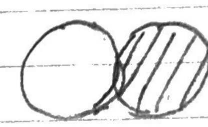


$X \cap Y := \{s \in U \mid s \in X$ and $s \in Y\}$

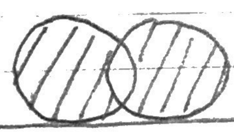


$X \setminus Y$ or $X - Y := \{s \in U \mid s \in X$ and $s \notin Y\}$

X^c or $\overline{X} := \{s \in U \mid s \notin X\}$



$X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$



Abstract math (1/23/24)

$A = \{1, 2, \{4, 7\}, \emptyset, \{\emptyset\}, 5, 7\}$
$B = \{\{1\}, \{2\}, 5, 8, \mathbb{N}\}$

$1 \in A$, $\{1\} \in A$, $\{1\} \subseteq A$
$1 \notin B$, but $\{1\} \in B$, $\{1\} \nsubseteq B$
$4 \notin A$
$\emptyset \in A$, $\emptyset \in B$, $\emptyset \subseteq A$, $\{\emptyset\}$

$X = \{1, 2, 3\} \subseteq \mathbb{N}$
$Y = \{4, 5, 6\} \subseteq \mathbb{N}$

$X \cup Y = \{1, 2, 3, 4, 5, 6\} \subseteq \mathbb{N}$
$X \cap Y = \{\ \} \subseteq \mathbb{N}$
$X^c = \{4, 5, 6, 7, \ldots\} = \mathbb{N} \setminus X$
$X \setminus Y = X$

Interval Notation:

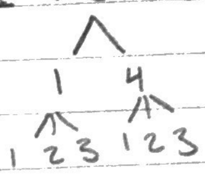
$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$
$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

*Let S be a set, the power set of S, denoted $\mathcal{P}(S)$, is set of all subsets of S.

$S = \{1,2,3\}$ $\quad X = \{1,4\}$

$\mathcal{P}(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$X \times S = \{(x,s) \mid x \in X \text{ and } s \in S\}$



$|X \times S| = |X| \cdot |S|$

$= \{(1,1), (1,2), (1,3), (4,1), (4,2), (4,3)\}$

$\boxed{\text{associative}}$ $A \cup B \cup C = A \cup (B \cup C)$
$A \cap B \cap C = A \cap (B \cap C)$

$\boxed{\text{commutative}}$ $A \cup B = B \cup A$
$A \cap B = B \cap A$

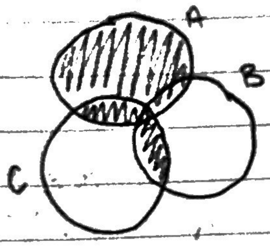
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

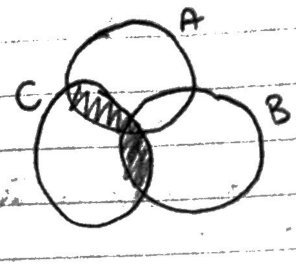
$\boxed{\text{DeMorgan Laws}}$ $(A \cap B)^c = A^c \cup B^c$ $\quad (A \cup B)^c = A^c \cap B^c$

$A^c \subseteq (A \cap B)^c$
$B^c \subseteq (A \cap B)^c$ $\Rightarrow A^c \cup B^c \subseteq (A \cap B)^c$ next argue $(A \cap B)^c \subseteq A^c \cup B^c$

Abstract math (1/25/24)

$A \cup (B \cap C)$ $\quad\quad$ $(A \cup B) \cap C$





<u>Logic</u> - a declarative sentence that's either true or false

*Let A, B be subsets of a universal set U. Then we say A is a subset of B provided for all $x \in U$, $x \in A$ implies $x \in B$

$U = \mathbb{N}$ $\quad$ $x = 1 \in \mathbb{N}$
$A = \{1, 2, 3\}$ $\quad$ $1 \in A, 1 \in B$ so $x \in A = T$, $x \in B = T$
$B = \{1, 2, 3, 4\}$

$x = 4 \in \mathbb{N}$
$4 \notin A, 4 \in B$ $x \in A = F$, $x \in B = T$

$R \equiv S$ - logically equivalent (same truth table)

$P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$ $\quad\quad$ $\sim(P \Rightarrow Q) \equiv P \wedge \sim Q$

Abstract math (2/1/24)

$\forall x \in S, P(x) \Rightarrow Q(x)$

<u>pf</u>: Let $x \in S$ be arbitrary and assume $P(x)$ true >>> logic >>>
show $Q(x)$ must be true

$\square$

ex. Prove that for all integers a, if a^2 divides a then $a \in \{-1,0,1\}$

$\forall a \in \mathbb{Z} \quad a^2 | a \Rightarrow a \in \{-1,0,1\}$

Pf. Let $a \in \mathbb{Z}$ be arbitrary and suppose a^2 divides a. That is, there exists an integer k such that $a = a^2 k$. We want to show $a \in \{-1,0,1\}$.

SW $\quad a = a^2 k \Rightarrow a \in \{-1,0,1\}$

$a^2 k - a = 0$

$a(ak-1) = 0$ then $a = 0$ or $ak = 1 \rightarrow a = k = 1$ or $a = k = -1$

Since $a = a^2 k$, we can write $a(ak-1) = 0$. It follows, by properties of $\mathbb{R}$, that either $a = 0$ or $ak - 1 = 0$, and equivalently $a = 0$ or $ak = 1$. From here, if $a \neq 0$ $a = k = 1$ or $a = k = -1$ since $a, k \in \mathbb{Z}$. So we must have $a = k = 1$ or $a = k = -1$. So we've shown that $a^2 | a \Rightarrow a \in \{-1,0,1\}$ as desired. $\square$

ex. $\forall x \in \mathbb{R}, \; x^2 + 5x < 0 \Rightarrow x < 0$

Pf. Let $x \in \mathbb{R}$ be arbitrary and assume $x^2 + 5x < 0$. We want to show $x < 0$.

S.W. $\quad x^2 + 5x < 0$

$x(x+5) < 0 \Rightarrow (x < 0 \text{ and } x+5 > 0) \text{ or } (x > 0 \text{ and } x+5 < 0)$

$\Downarrow$

$x > 0$ and $x < -5$

insert SW

Pf $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$ contrapositive

Let $x \in \mathbb{R}$ be arbitrary and assume $x \geq 0$. We want to show $x^2 + 5x \geq 0$.

S.W $x \geq 0 \Rightarrow 5x \geq 0$ and $x^2 \geq 0 \Rightarrow x^2 + 5x \geq 0$

Since $x \geq 0$ ~~whenwe~~ know $5x \geq 0$ and $x^2 \geq 0$ so $x^2 + 5x \geq 0$ when $x \geq 0$. Equivalently, $x < 0$ when $x^2 + 5x < 0$ for all $x \in \mathbb{R}$

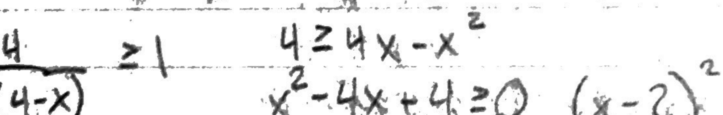
$\square$

ex Prove that every open interval (a,b) on real line contains a rational number and an irrational number.

$\forall a, b \in \mathbb{R}$, where $a < b$, there exists a rational number $q \in (a,b)$ and there exists an irrational number $r \in (a,b)$.

Pf. Let (a,b) be an interval on real line. We need to show there exists $q \in (a,b)$ where $q \in \mathbb{Q}$ and $r \in (a,b)$ where $r \in \mathbb{Q}^c$

SW



$r = ad$ and $0 < d < b - a$, need $r \in$

Abstract math (2/6/24)

$\frac{4}{x(4-x)} \geq 1$ $\quad 4 \geq 4x - x^2$

$x^2 - 4x + 4 \geq 0 \quad (x-2)^2 \geq 0$

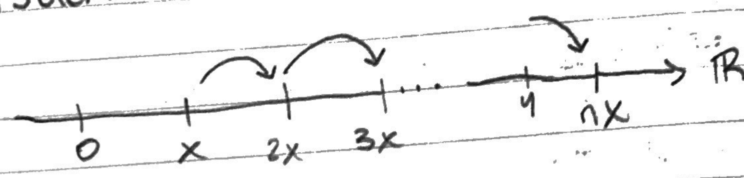
Pf Let $x \in \mathbb{R}$ satisfy $0 < x < 4$. Then $(x-2)^2 \geq 0 \Rightarrow x^2 - 4x + 4 \geq 0 \Rightarrow 4 \geq x(4-x) \Rightarrow \frac{4}{x(4-x)} \geq 1$. The last inequality follows since $0 < x < 4 \Rightarrow$ both $x > 0$ and $x - 4 > 0$ ~~or both $x < 0$ and $x - 4 < 0$~~ so $x(x-4) > 0$.

Every interval (a,b) contains rational and irrational number

<u>Thm</u> Every interval (a,b) contains rational and irrational number

<u>Lemma</u> For any positive real numbers x, y, there exists $n \in \mathbb{N}$ such that $nx > y$.

*archimedian property



For all intervals (a,b), there exists $q \in Q$ such that $q \in (a,b)$ and there exists $r \in Q^c$ such that $r \in (a,b)$.

$\forall (a,b) \subseteq \mathbb{R}, [\exists q \in Q, P(q)] \wedge [\exists r \in Q^c, S(r)]$

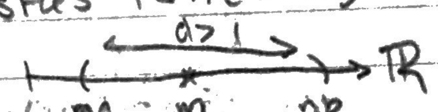
Proof of $\exists q \in Q, q \in (a,b)$: show $\exists m \in \mathbb{Z}, \exists n \in \mathbb{N}, a < \frac{m}{n} < b$

$q = \frac{m}{n} \quad m \in \mathbb{Z}, n \in \mathbb{N}$

$\frac{1}{n} < b - a \qquad 1 < n(b-a)$ *we can find bc archimedian property

pos=y pos=x

Let $n \in \mathbb{N}$ be a natural number that satisfies $1 < n(b-a)$



$na < m < nb \qquad 1 < nb - na$

Then since $nb - na > 1$ there must be $m \in \mathbb{Z}$ s.t. $na < m < nb$

<u>Fact</u> $\sqrt{2} \notin Q$

always false

$P \equiv (\sim P) \Rightarrow (Q \wedge \sim Q) \qquad Q(x) \iff x$ even

$\sim Q(x) \iff x$ odd

True

$$P \equiv (\sim Q) \Rightarrow (Q \wedge \sim Q)$$

$\sim P$ $\quad \sqrt{2} \notin Q \quad \sqrt{2} \in Q \quad \gcd(m,n)=1 \quad \gcd(m,n) \neq 1$

pf BWOC assume $\sqrt{2} \in Q$

$\sqrt{2} \in Q \Rightarrow \sqrt{2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $\gcd(m,n)=1$

$2 = \frac{m^2}{n^2}$ $\quad 2n^2 = m^2 \Rightarrow m^2$ even $\Rightarrow m$ even

$\Rightarrow \exists k \in \mathbb{Z}, m = 2k$

$\Rightarrow 2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2 \Rightarrow n^2$ even $\Rightarrow n$ even

$\Rightarrow \exists j \in \mathbb{Z}, n = 2j$

$\Rightarrow \gcd(m,n) \geq 2 \Rightarrow \gcd(m,n) \neq 1$

Abstract Math (2/8/24)

ex There exists no integers a and b for which $18a + 6b = 1$.

$\forall a, b \in \mathbb{Z}, 18a + 6b \neq 1$

pf BWOC assume there exists $a, b \in \mathbb{Z}$ ~~where~~ such that $18a + 6b = 1$. Then we can write $2(9a + 3b) = 1$, which implies 1 is even; since since $9a + 3b \in \mathbb{Z}$. But 1 is odd and this is a contradiction. Therefore, there exist no integers a, b so that $18a + 6b = 1$.

□

ex For every $x \in [\pi/2, \pi]$ $\sin x - \cos x \geq 1$

BWOC assume there exists $x \in [\pi/2$ such that $\sin x - \cos x < 1$. Then we can write $\sin x - \cos x$

sw $\sin x \geq 0 \quad \cos x \leq 0$

$(\sin x - \cos x)^2 < 1$

$\sin^2 x - 2\sin x \cos x + \cos^2 x < 1$

$(\sin^2 x + \cos^2 x) = 1$

$1 - 2\sin x \cos x < 1$

$\boxed{-2\sin x \cos x < 0 \quad ↯}$

$-2\sin x \cos x \geq 0$

ex Show e is irrational

$$e := \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots$$

lemma: $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 \dots = \frac{a}{1-r}$

pf BWOC assume $e \in \mathbb{Q}$ so that $e = \frac{m}{n}$ where $m, n \in \mathbb{N}$ and $n > 1$.

$$n!\left(\frac{m}{n}\right) = n! \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$(n-1)!\, m = j + \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} \dots$$

$$= j + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots$$

$$= j + \frac{1}{n+1}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} \dots\right)$$

Abstract math (2/13/24)

$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow [P \Rightarrow R]$

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$\wedge$	$P \Rightarrow R$	$\Rightarrow$
T	T	T	T	T	T	T	T
T	F	F	F	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

So assume $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ - true - and show $P \Rightarrow R$

$*$ $P \Rightarrow Q$ is true, $Q \Rightarrow R$ is true

assume P and show R

$P \wedge (P \Rightarrow Q)$ means Q true
$Q \wedge (Q \Rightarrow R)$ means R true

ex $\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{Q}^+, y < x$

*direct pf Let x be an arbitrary positive rational number. Then choose $y = \frac{x}{2}$. We see that $y < x$ and $y \in \mathbb{Q}^+$ as desired.

*contradiction pf BWOC assume $\sim[\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{Q}^+, y < x] \equiv \exists x \in \mathbb{Q}^+, \forall y \in \mathbb{Q}^+, y \geq x$. Let x be such a positive rational number. If $y = \frac{x}{2} < x$ we see there exists a y that is less than x. This is a contradiction, so original claim is there.

ex Show $\sqrt{3} \notin \mathbb{Q}$

pf BWOC assume $\sqrt{3} \in \mathbb{Q}$ so that $\sqrt{3} = \frac{m}{n}$ and $\gcd(m,n)=1$
$\Rightarrow 3n^2 = m^2 \Rightarrow 3 \mid m^2 \Rightarrow 3 \mid m$

ex $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$

direct pf Assume $x \nmid yz \Rightarrow yz = xk + r$, $0 < r < x$

contrapositive pf Assume $\sim[x \nmid y$ and $x \nmid z] \equiv x \mid y$ or $x \mid z$. Either $y = xk$ for $k \in \mathbb{Z}$ or $z = xj$ for $k \in \mathbb{Z}$. WLOG (without loss of generality) suppose $y = xk$, $k \in \mathbb{Z}$. Then $yz = (xk)z = x(kz) = xd$, $d \in \mathbb{Z}$. So $x \mid yz$. Equivalentally
...

Abstract Math (2/20/24)

$[\exists x \in U, P(x)] \wedge [\forall y \in U, \forall z \in U, P(y) \wedge P(z) \Rightarrow y = z]$

$[\exists x \in U, P(x)] \wedge [\forall y \in U, \forall z \in U, y \neq z \Rightarrow \sim P(y) \vee \sim P(z)]$

$\equiv [\exists x \in U, P(x)] \wedge [\forall y \in U, \forall z \in U, y \neq z$

$\sim (\;)$

$\sim(R \Rightarrow S)$
$R \wedge \sim S$

$\sim [\exists x \in U, P(x)] \wedge \sim [\forall y \in U, \forall z \in U, P(y) \wedge P(z) \Rightarrow y = z]$

$\equiv [\forall x \in U, \sim P(x)] \vee [\exists y \in U, \exists z \in U, (P(y) \wedge P(z)) \wedge y \neq z)]$

F　　　　T

$U = \mathbb{Z} \quad P(x): x^2 - 1 = 0$

choose $y = 1, z = -1$

$\Downarrow$

$y^2 - 1 = 1^2 - 1 = 0 \quad P(y)\ T$
$z^2 - 1 = (-1)^2 - 1 = 0 \quad P(z)\ T$

$P(y) \wedge P(z) \wedge y \neq z$

$\forall s \in \mathbb{R}, s \in \emptyset \Rightarrow$ [some statement]

* true no matter the statement b/c first part is F

<u>Claim</u> - Let A be a subset of a universal set U. Then $\emptyset \subseteq A$

$\forall x \in U, x \in \emptyset \Rightarrow x \in A$

<u>pf</u> Let $x \in U$. Then $x \notin \emptyset$ and so implication is true.

Induction: $\forall n \in \mathbb{N}, P(n)$

ex $\forall n \in \mathbb{N}, \sum_{k=1}^{n} k = 1+2+3\ldots+n = \frac{n(n+1)}{2}$

Abstract Math (2/22/24)

Principle of Mathematical Induction (PMI):
Let $P(n)$ be a proposition where n is a natural number. If both:
1) $P(1)$ true;
2) $\forall n \in \mathbb{N}, P(n) \Rightarrow (P(n+1))$,
then $P(n)$ is true for all $n \in \mathbb{N}$

Fact The natural numbers are well ordered; every non-empty subset of $\mathbb{N}$ has a smallest element

Thm $\mathbb{N}$ well ordered $\Leftrightarrow$ PMI

Assumed True

pf $(\Rightarrow)$ BWOC assume $\boxed{\mathbb{N} \text{ well ordered}}$ and PMI fails.

$\sim(R \Rightarrow S) \equiv R \wedge \sim S$

PMI: $\Leftrightarrow \left\{ \begin{matrix} P(1) \text{ true} \\ \forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1) \end{matrix} \right\} \Rightarrow \forall n \in \mathbb{N}, P(n)$

~PMI: $\Leftrightarrow \boxed{\boxed{\left\{ \begin{matrix} P(1) \text{ true} \\ \forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1) \end{matrix} \right\}} \wedge \exists n \in \mathbb{N}, \sim P(n)}$

Assumed True

Let $S = \{n \in \mathbb{N}, \sim P(n) \text{ true}\} = \{n \in \mathbb{N}, P(n) \text{ false}\} \subseteq \mathbb{N}$

$1 \notin S$ $\quad S \neq \emptyset$ by assumption

By Well Ordering Property, S has a smallest element $k \in S$ and $k \neq 1$. $k-1 < k$ and so $k-1 \notin S$ and $k-1 \in \mathbb{N}$. $P(k-1)$ is true and therefore $P(k)$ is true. But $k \in S$, so $P(k)$ is false. So we have $P(k) \wedge \sim P(k)$.

Thm Let $x \in \mathbb{R}$ satisfy $x \geq -1$. Then for all $n \in \mathbb{N}$, we have $(1+x)^n \geq 1+nx$

Bernoulli Inequality $\forall n \in \mathbb{N}, P(n)$ $\qquad P(n) :\Leftrightarrow (1+x)^n \geq 1+nx$

Pf (Induction)

Base case: $P(1) :\Leftrightarrow (1+x)^1 \geq 1+1\cdot x$ $\qquad P(n+1) :\Leftrightarrow (1+x)^{n+1} \geq 1+(n+1)x$

$1+x \geq 1+x$

Inductive step:

Let $n \in \mathbb{N}$ and assume $(1+x)^n \geq 1+nx$.

Show $P(n+1)$

$(1+x)(1+x)^n$ $\qquad$ by induction hypothesis

$\geq (1+x)(1+nx)$

$= \underbrace{1+x+nx}_{1+(1+n)x}+nx^2$

$\geq 1+(n+1)x$

So $(1+x)^{n+1} \geq 1+(n+1)x$ where $(1+x)^n \geq 1+nx$ and inductive step holds.

Then by PMI, $(1+x)^n \geq 1+nx$ for all $n \in \mathbb{N}$ when $x \geq -1$.

Thm Every positive ~~ne~~ integer has a unique prime factorization.

Pf (Induction)

Base case: Let $n=1$. (Holds vacuously)

$n=2 \quad 2=2$

Inductive step: Let $n \in \mathbb{N}$ and assume k has prime factorization whenever $1 \le k \le n$

Strong form

$\left\{ \begin{array}{l} ①\ P(1) \text{ true} \\ ②\ \forall n \in \mathbb{N}\ [P(1) \wedge P(2) \ldots P(n)] \Rightarrow P(n+1) \end{array} \right\} \Rightarrow n \in \mathbb{N},\ P(n)$

Case 1: $n+1 = p$ is prime

Case 2: $n+1 = a \cdot b$ where $1 < a, b < n$

$a = p_1 \ldots p_n$
$b = q_1 \ldots q_s$

$n+1 = p_1 \ldots p_n q_1 \ldots q_s$ has prime factorization. So inductive steps holds, and by SPMI $P(n)$ true for all $n \in \mathbb{N}$.

Abstract Math (2/27/24)

Claim For all integers $k \ge 7$, there exist positive integers $a, b \in \mathbb{Z}$ such that $k = 2a + 3b$

Base case $P(7) = 2a + 3b$ $\quad P(7): \Leftrightarrow \exists a, b \in \mathbb{Z}^+ \quad 7 = 2a + 3b$

$a=2 \quad b=1$

$a=2,\ b=1$
$a=1,\ b=2$
$a=3,\ b=1 \qquad a=2,\ b=2$

Inductive step: $\forall n \in \mathbb{Z}, n \geq 7, \exists a, b \in \mathbb{Z}^+, n = 2a + 3b$

Let $a, b \in \mathbb{Z}^+$ where $P(n)$ is true; that is, $\exists a, b$ where $n = 2a + 3b$.

$n+1 = 2a + 3b$

$n+1 = 2a + 3b + 1$

a or $b > 1$

case 1: $b > 1$

$2(a+1) + 3(b-1) = n+1$

case 2: $a > 1$

$3(b+1) + 2(a-1) = n+1$

$n+1 = (n-1) + 2 =$

ex Abstract Math (2/29/24)

Show that $\forall n \in \mathbb{N}$, $(1+2+3+\ldots+n)^2 = 1^3 + 2^3 + \ldots + n^3$

$P(n): \iff (1+2+\ldots+n)^2 = 1^3 + 2^3 + \ldots + n^3$

Base case: $P(1) \iff 1^2 = 1^3$

Inductive step: $n + n + 1$

$(2n+1)^2$

$(n + (n+1))^2$

$(2n + n)^2$

$(n^2 + 2)^2$

$n^4 + 4n^2 + 4$

$4n^2 + 4n + 1$

$n^3 + 3n^2 + 3n = 4n^2 + 4n$

$4\left(n + \frac{1}{2}\right)^2$

$n^3 = n^2 + n$

$n + r$

$\frac{(n+1)}{3}(n^2 + 2n + 1)$

$4n^2 + 4n + x$

$n^3 + (n+1)^3$

$n^2 + 2n + 1 n^3 + (n+1)^3$

$n^3 + 2n^2 + n$

$+ n^2 + 2n + 1$

$n^3 + 3n^2 + 3n + 1$

$\frac{n^2 + n}{n} = \frac{2n^3}{n}$

$n + 1 = 2n^2$

$2n^3 + 3n^2 + 3n + 1$

$n^3 + 2n^2 + n$

$n^3 +$

$n^2 + 2n + 1$

$$\left(\frac{n(n+1)}{2}\right)^2 \quad \left(\frac{n^2+n}{2}\right)^2 \quad \left(\frac{(n+1)(n+2)}{2}\right)^2 \quad \frac{(n+1)^2(n+2)^2}{4}$$

$$\cancel{\left(\frac{n^2+3n+2}{2}\right)^2}$$

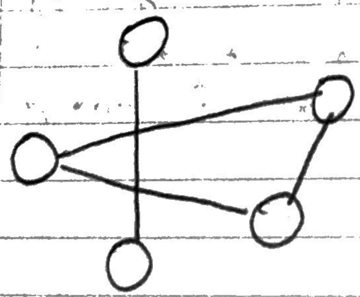
$$= \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4}$$

$$= \frac{(n+1)^2\,(n+2)^2}{4}$$

So inductive step holds.

So by PMI, $\forall n \in \mathbb{N}$, $P(n)$. $\square$



Claim Given any graph with max vertex degree = k, we can color the graph with k+1 colors.

Pf (Induction on # of vertices n)

$\forall n \in \mathbb{N}, \ |G| = n \ \wedge$ max degree $= k \Rightarrow G$ is k+1 colorable

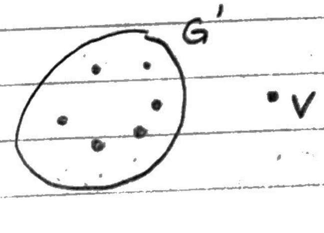
Base case: $P(1) :\Leftrightarrow |G| = 1$, max degree $= 0$; $P(2) :\Leftrightarrow |G| = 2 \wedge$ max degree $= k \Rightarrow G$ is $k+1$ color

1 color

Inductive step: $\forall n \in \mathbb{N}, [P(1) \wedge P(2) \wedge \ldots \wedge P(n)] \Rightarrow P(n+1)$

English - Let n be a natural number and assume for all graphs with no more than n vertices, if max degree $= k$ then G is $k+1$-colorable.

Next, let $|\hat{G}| = n+1$ vertices with max degree $= k$ and show $\hat{G}$ is $k+1$-colorable.



Let G' be $\hat{G} - \{v\}$ where degree$(v) \neq k$.

$|G'| = n$ max degree $= k$

$\Rightarrow$ By I.H. G' is $k+1$ colorable. Then $\deg(v) \leq k$ one of the $k+1$ colors will do so $\hat{G}$ is $k+1$ colorable and inductive step holds and by PMI, $\forall n \in \mathbb{N}, P(n)$. $\square$

Abstract Math (3/5/24)

$f_n := f_{n-1} + f_{n-2}$, $f_1 = f_2 = 1$

$r = \frac{1+\sqrt{5}}{2} > 1$ satisfies $r^2 = r + 1$

Show $\forall n \in \mathbb{N}, \underbrace{f_n \geq r^{n-2}}_{P(n)}$

Base case: $P(1) :\Leftrightarrow f_1 \geq r^{-1}$

$1 \geq r^{-1}$ ✓

Inductive step: Show $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$

Let $n \in \mathbb{N}$ and assume $f_k \geq r^{k-2}$ when $1 \leq k \leq n$.
Show $f_{n+1} \geq r^{n-1}$

$$f_{n+1} := f_n + f_{n-1} \geq r^{n-2} + r^{n-3} = r^{n-3}\underbrace{(r+1)}_{= r^2} = r^{n-1}$$

Def Let A, B be non-empty sets. Define a relation R from A to B as a subset of $A \times B$.

$$R \subseteq A \times B = \{(a,b) \mid a \in A, b \in B\}$$

ex $R \subseteq \mathbb{R} \times \mathbb{R}$
$R = \{(c, \frac{9}{5}c + 32) \mid c \in \mathbb{R}\}$

ex $A = \{1, 2, 3\}$ $B = \{2, 5, 7\}$

$R = \{\}$ *valid $R = \{(1,2), (1,5), (2,7), (3,2)\}$

ex $A = \{0, 1, 2, 3, 4\}$

$R \subseteq A \times A$ where aRb provided $a \leq b$ for all $a, b \in A$

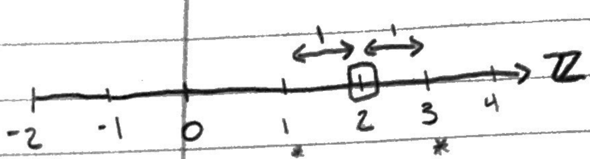
$R = \{(0,0), (0,1), \ldots (0,4), (1,1), \ldots (1,4), \ldots, (4,4)\}$

$|R| = 15$

*If $R \subseteq A \times A$ we say R is a relation on A
1. If aRa $\forall a \in A$, then R is reflexive
2. If $aRb \Rightarrow bRa$ $\forall a, b \in A$, then R is symmetric
3. If $aRb \wedge bRc$ $\forall a, b, c \in A$, then R is transitive

Abstract Math (3/7/24)

ex R a relation on $\mathbb{Z}$ where $\forall a,b \in \mathbb{Z}$, aRb whenever $|a-b| \leq 1$



$|a-b| = \text{dist}(a,b)$

$2R3 :\Leftrightarrow |2-3| \leq 1$

$2 \not R 4 :\Leftrightarrow |2-4| > 1$

① R is reflexive:
Let $a \in \mathbb{Z}$ be arbitrary, then $|a-a| = 0 \leq 1$, so aRa. Since $a \in \mathbb{Z}$ is arbitrary, R is reflexive. □

② R is symmetric:
Let $a,b \in \mathbb{Z}$ be arbitrary and assume aRb; then, $|a-b| \leq 1$ so $|b-a| \leq 1$, and bRa. Since $a,b \in \mathbb{Z}$ are arbitrary, R is symmetric. □

③ R isn't transitive:
$\sim(\forall a,b,c \in \mathbb{Z}, aRb \wedge bRc \Rightarrow aRc) \equiv \exists a,b,c, [aRb \wedge bRc] \wedge a \not R c$
Property not satisfied because of following counterexample.
Choose $a=1, b=2, c=3$. $1R2 \wedge 2R3 \wedge 1 \not R 3$.

Def For $a,b \in \mathbb{Z}$ and $n \in \mathbb{N}$ we say $a \equiv b \pmod{n}$ provided $n \mid b-a$.

Say $R \subseteq \mathbb{Z} \times \mathbb{Z}$ where aRb whenever $a \equiv b \pmod{6}$.

① R is reflexive:
Let $a \in \mathbb{Z}$ be arbitrary. Then $a-a = 0$ and $0 = 6 \cdot 0$ so $6 \mid a-a$ and aRa.

② Let $a,b \in \mathbb{Z}$ and suppose aRb. That is $6 \mid b-a$. So $b-a = 6k$ for some $k \in \mathbb{Z}$. But $a-b = 6 \cdot -k$ and so $6 \mid a-b$ and bRa.

③ Let $a, b, c \in \mathbb{Z}$ and assume aRb and bRc. That is, $b - a = 6k$ and $c - b = 6j$ for integers k and j. Then, $c - a = 6(j+k)$ and aRc.

R is an equivalence relation (satisfies all 3).

$[a] = \{b \in A \mid aRb\}$.

Equivalence of / containing a.

$[7] = \{b \in \mathbb{Z} \mid 7Rb\} = \{6k+1 \mid k \in \mathbb{Z}\}$

$\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3] \cup [4] \cup [5]$

$[a] \neq [b] \Rightarrow [a] \cap [b] = \emptyset$ classes partition $\mathbb{Z}$

Conjecture:
Let R be an equivalence relation on set A, the equivalence classes partition A

ex Let $A = \mathbb{P}(\mathbb{R})$ - set of polynomials with real coefficients

Define R on A so that pRq provided $\deg(p) = \deg(q)$.

$[a] = [b] \Leftrightarrow aRb$

($\Rightarrow$) ~~($\Leftarrow$)~~ $a \in [a] \Rightarrow b \in [b] \Rightarrow aRb$

($\Leftarrow$) $aRb \overset{\text{show}}{\Rightarrow} [a] \subseteq [b] \wedge [b] \subseteq [a]$

Abstract Math (3/26/24)

R an ER on set A provided $R \subseteq A \times A$ satisfying
i) reflexive - $\forall x \in \mathbb{R}$, xRx
ii) symmetric - $\forall a, b \in \mathbb{R}$, $aRb \Rightarrow bRa$
iii) transitive - $\forall x, y, z \in \mathbb{R}$, $xRy \wedge yRz \Rightarrow xRz$

ex R a relation on $\mathbb{R}$ where xRy provided $\lfloor x \rfloor = \lfloor y \rfloor$

$\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$ $\lfloor 1.2 \rfloor = 1$ $\lfloor 7 \rfloor = 7$

$\lfloor x \rfloor = \lfloor x \rfloor$ (reflexive) ✓

$\lfloor x \rfloor = \lfloor y \rfloor \Rightarrow \lfloor y \rfloor = \lfloor x \rfloor$ ✓

$[x] = \{y \in \mathbb{R} \mid xRy\}$ $\bigcup_{k \in \mathbb{Z}} [k, k+1)$

Fact: The set of equivalence classes partition $\mathbb{R}$

$[1.2] = \lfloor 1.2 \rfloor \leq k < \lfloor 1.2 \rfloor + 1 = \{y \in \mathbb{R} \mid \lfloor y \rfloor = 1\}$

$yR1.2 \iff \lfloor y \rfloor = \lfloor 1.2 \rfloor = 1$

ex R a relation on $\mathbb{Z}$ where aRb provided a and b have a common prime factor.

Let A and B be sets. Then a map $f: A \to B$ is a function provided every $a \in A$ is mapped to exactly one element in B.
$A = \text{domain}(f)$
$B = \text{codomain}(f)$
$\text{Range}(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$

ex $f(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$

$\vec{x} \in \mathbb{R}^3$

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$ $\therefore$ $f: \mathbb{R}^3 \to \mathbb{R}^2$

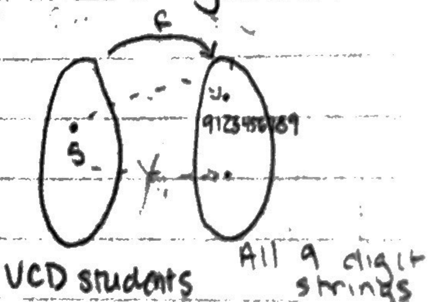
Def A function $f: A \to B$ is relation $f \subseteq A \times B$ where every $a \in A$ appears in exactly one ordered pair $(a, b) \in f$.

$f: A \to B$ and $g: A \to B$, then f and g are equal as functions provided they're equal as sets

$f: \mathbb{R} \to \mathbb{R}$ $g: [-\pi/2, \pi/2] \to \mathbb{R}$
$f(x) = \sin x$ $g(x) = \sin(x)$

$(\pi, 0) \in f$ $(\pi, 0) \notin g$

So $f \neq g$.



$f(s) = $ student #
$f: A \to B$

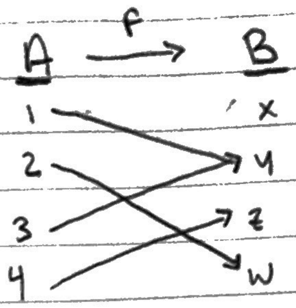
$f: A \to B$ one to one provided $\forall a, a' \in A$, $a \neq a' \Rightarrow f(a) \neq f(a')$.

$f: \mathbb{R} \to \mathbb{R}$ where $f(x) = \sin x$

$f(x) = 1 \implies x = ???$ $g: [-\pi/2, \pi/2] \to \mathbb{R}$ where $g(x) = \sin x$
$g(x) = 1 \Rightarrow x = \pi/2$

Abstract Math (3/28/24)

ex



f not 1-1 :$\iff \exists x_1, x_2 \in A,\ x_1 \neq x_2$ and $f(x_1) = f(x_2)$

choose $x_1 = 1,\ x_2 = 3$

$x_1 \neq x_3 \wedge f(x_1) = y = f(x_2)$

1-1: $x_1, x_2 \in A,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

f not onto :$\iff \exists b \in B,\ \forall a \in A;\ f(a) \neq b$

onto: $\forall b \in B,\ \exists x \in A$ s.t. $f(x) = b$

choose $b = x$. Then if $a \in A$, $f(a) \neq b$.

* 1-1 is the most important when trying to invert

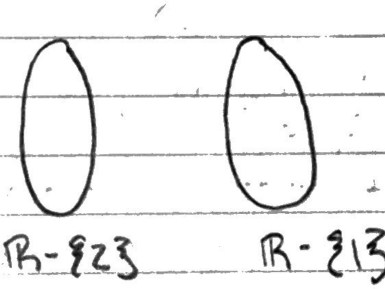
ex $f : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$.

① 1-1 : Let $x_1, x_2 \in Dom(f)$ and suppose $f(x_1) = f(x_2)$, that is $\sqrt{x_1} = \sqrt{x_2} \Rightarrow (\sqrt{x_1})^2 = (\sqrt{x_2^2})^2 \Rightarrow x_1 = x_2$.

② f is not onto since, for example, there is no $x \in Dom(f)$ s.t. $f(x) = -1 \in \mathbb{R}$.

ex $f : \mathbb{R} - \{2\} \longrightarrow \mathbb{R} - \{1\}$

where $f(x) = \frac{x}{x-2}$



Bijective?

1-1:

$f(x_1) = f(x_2) \longrightarrow \frac{x_1}{x_1 - 2} = \frac{x_2}{x_2 - 2} \Rightarrow x_1(x_2 - 2) = x_2(x_1 - 2)$

$\Rightarrow x_1 x_2 - 2x_1 = x_1 x_2 - 2x_2 \Rightarrow x_1 = x_2$

② Let $b \in \mathbb{R} - \{1\}$ be arbitrary.

SW Solve $f(x) = b$.

$\frac{x}{x-2} = b \qquad x = bx - 2b$

$x = \frac{2b}{b-1} \in \mathbb{R} - \{2\}?$

$\Rightarrow 2\left(\frac{b}{b-1}\right) = 2 \Rightarrow b = b-1$ X

$R = \{(1,2), (3,4), (2,2), (8,-1)\} \subseteq \mathbb{Z} \times \mathbb{Z}$

$R^{-1} = \{(2,1), (4,3), (2,2), (-1,8)\}$

$f = \left\{\left(x, \frac{x}{x-2}\right) : x \in \mathbb{R} - \{2\}\right\}$

$f^{-1} = \left\{\left(b, \frac{2b}{b-1}\right) : b \in \mathbb{R} - \{1\}\right\}$

ex $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m,n) = 3m + 5n$

f onto?

cases: $b = 0 \qquad\qquad b = 1$

$(0,0) \longmapsto 0 \qquad (2,-1) \longmapsto 1, \quad (-3,2) \longmapsto 1$

ex $R \subseteq (\mathbb{Z} \times \mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \times \mathbb{Z} \setminus \{0\})$

$(a,b) R (c,d)$ provided $ad = bc$

① Reflexive? $(a,b) R (b,a)$ ✓ b/c $ak = ba$

② Symmetric? $(a,b) R (c,d) \Longrightarrow (c,d) R (a,b)$

$ad = bc \Rightarrow cb = da$

Transitive? $(a,b) R (c,d) \wedge (c,d) R (e,f) \Rightarrow (a,b) R (e,f)$

Abstract Math (4/2/24)

ex $f: \mathcal{P}(\mathbb{Z}_5) \to \mathbb{Z}$ $\quad$ $\mathbb{Z}_5 = \{0,1,2,3,4\}$

$f(X) = |X|$ $\quad$ $aRb :\Leftrightarrow a \equiv b \pmod 5$

$\{[0],[1],[2],[3],[4],[5]\}$

$Dom(f) = \mathcal{P}(\mathbb{Z}_5) =$ all subsets of $\mathbb{Z}_5$

$Codom(f) = \mathbb{Z}$

$[0] = \{5k \mid k \in \mathbb{Z}\}$

$X \to \boxed{f} \to k \in \mathbb{Z}$ $\quad$ $[1] = \{5k+1 \mid k \in \mathbb{Z}\}$

$Range(f) = \{0,1,2,3,4,5\}$

$f \subseteq \mathcal{P}(\mathbb{Z}_5) \times \mathbb{Z}$

$=$

$\{(\emptyset, 0), (\{0\}, 1), (\{1\}, 1), \ldots\}$

ex Let $A \neq \emptyset$ and let $f: A \to A$. Prove $f \circ f = id_A \Rightarrow f$ is bijective

$id_A(x) = x \quad \forall x \in A$ $\quad$ $f^{-1}(f(x)) = x = id_A(x)$

Show f is one to one & onto.

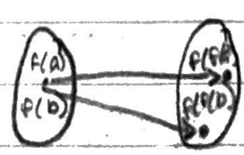
f 1-1 $:\Leftrightarrow \forall a,b \in A \quad a \neq b \Rightarrow f(a) \neq f(b)$

$\forall a,b \in A, \; f(a) = f(b) \Rightarrow a = b$

Pf Let $a, b \in A$ be arbitrary and assume $a \neq b$. Also, assume $f \circ f = id_A$. BWOC assume $a \neq b$ and $f(a) = f(b)$.

$f \circ f(a) = a \neq f \circ f(b) = b$

$f(f(a)) \neq f(f(b))$



f not a function $\rightarrow\leftarrow$

So f is 1-1.

f onto $:\Longleftrightarrow \forall a \in A, \exists x \in A, f(x) = a.$

Pf Let $a \in A$ be arbitrary and choose an $x = f(a) \in A$.
Then $f(x) = f(f(a)) = f \circ f(a) = a$.

So f is a bijection.

ex $f: A \to B$ $\quad g: B \to C$ $\quad g \circ f: A \to C$

$g \circ f$ 1-1 $\Rightarrow f$ 1-1

Pf Assume f not 1-1. Then $\exists x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Then $g \circ f(x_1) = g \circ f(x_2)$.
Then $g \circ f$ isn't 1-1.

$g \circ f$ 1-1 $\Rightarrow g$ 1-1 $\quad f(x) = \sin^{-1}(x)$ $\quad f: [-1,1] \to \mathbb{R}$
$\quad g(x) = \sin(x)$ $\quad g: \mathbb{R} \to \mathbb{R}$

$g \circ f(x) = \sin(\sin^{-1}(x)) = x = id_{[-1,1]}$ is 1-1
but g is not 1-1.

Abstract Math (4/14/24)

ex $f: P(\mathbb{Z}) \to P(\mathbb{Z})$ where $f(X) = X^c$, is f bijective?

$f: A \to A \wedge f \circ f = id_A \Rightarrow f$ is bijective

$f \circ f\ (X^c)^c = X$ $\quad f \circ f = id_{P(\mathbb{Z})}$

Yes, it's bijective

$R \subseteq A \times B \Rightarrow R^{-1} \subseteq B \times A$

$(a,b) \in R \Leftrightarrow (b,a) \in R^{-1}$

$f: A \to B$ is a function, we can always consider the inverse relation $f^{-1}: B \to A$

f^{-1} is a function $\Leftrightarrow$ f is bijective

$X \subseteq A \Rightarrow f(X) = \{ b \in B \mid \exists x \in X, f(x) = b \}$

ex $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$

$X = \{-1, 0, 1, 2\}$ $f(X) = \{0, 1, 4\}$

X is finite $:\Leftrightarrow \exists n \in \mathbb{N}, |X| = n$
X is infinite $:\Leftrightarrow X$ not finite $\equiv \forall n \in \mathbb{N}, |X| \neq n$

Def: we say two sets A and B have same cardinality and we write $|A| = |B|$, provided $\exists$ a function that's bijective $f: A \to B$

ex $A = \{1, 2, 3, 4\}$ $B = \{x, y, z, w\}$
$|A| = |B|$

ex $A = \{1, 2, 3\}$ $B = \{7, 8\}$

f can't be one to one and $|A| \neq |B|$

Abstract Math (4/9/24)

ex $R = \emptyset \subseteq A \times A \Rightarrow \emptyset$ is a relation on A $(A \neq \emptyset)$

reflexive i) $\forall a \in A,\ aRa \equiv \forall a \in A,\ (a,a) \in \emptyset$ X

symmetric ii) $\forall a,b \in A,\ aRb \Rightarrow bRa$ ✓

$\underbrace{(a,b) \in \emptyset}_{F} \Rightarrow (b,a) \in \emptyset$

transitive iii) $\forall a,b,c \in A,\ aRb \land bRc \Rightarrow aRc$ ✓

$\underbrace{(a,b) \in \emptyset \land (b,c) \in \emptyset}_{F} \Rightarrow (a,c) \in \emptyset$

ex R, S E.R. on $A \Rightarrow \underbrace{R \cap S}_{\hat{R}}$ E.R. on A

reflexive i) $\forall a \in A,\ a\hat{R}a \equiv \forall a \in A,\ (a,a) \in \hat{R}$

R is reflexive $\Rightarrow (a,a) \in R$
S is reflexive $\Rightarrow (a,a) \in S$ $\Rightarrow (a,a) \in \hat{R}$

symmetric ii) $\forall a,b \in A,\ a\hat{R}b \Rightarrow b\hat{R}a \equiv \forall a,b \in A,\ \underbrace{(a,b) \in \hat{R}}_{\text{assume}} \Rightarrow (b,a) \in \hat{R}$

R is symmetric $\Rightarrow (a,b) \in R \Rightarrow (b,a) \in R$
S is symmetric $\Rightarrow (a,b) \in S \Rightarrow (b,a) \in S$ $\Rightarrow (b,a) \in \hat{R}$

transitive iii) $\forall a,b,c \in A,\ \underbrace{a\hat{R}b \land b\hat{R}c}_{\text{assume}} \Rightarrow a\hat{R}c$

R is transitive $\Rightarrow (a,b) \in R \land (b,c) \in R$

codomain vs range vs image vs preimage
$\mathbb{Z}_5$ (equivalence classes mod 5?)

Abstract Math (4/16/24)

PMI: If $\left\{\begin{matrix} \text{1) Base case } \checkmark \\ \text{2) Inductive step } \checkmark \end{matrix}\right\} \Rightarrow P(n)$ for all $\mathbb{N}$

R a relation on A:
$R \subseteq A \times A$

$A = \mathbb{Z} \times \mathbb{Z} \setminus \{0\} = \{(a,b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$

$f: A \to Q : (a,b) \mapsto \frac{a}{b}$

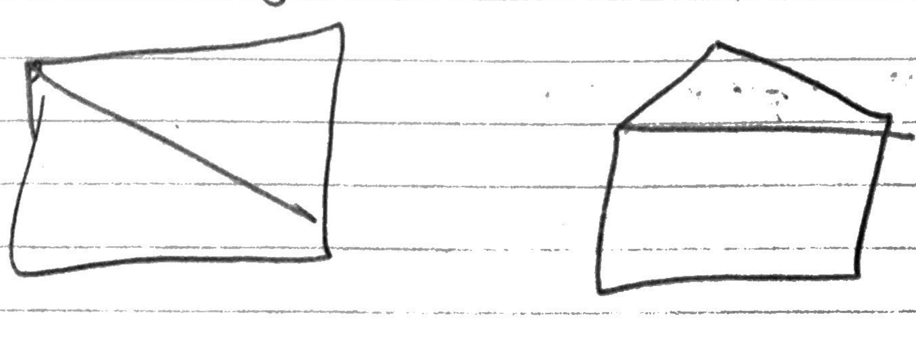
Define R on A s.t. $\forall x, y \in A$, xRy provided $f(x) = f(y)$

$[(a,b)] = \{(c,d) \in A \mid \frac{a}{b} = \frac{c}{d}\}$

Claim: Angle sum of convex n-gon equals $(n-2)\pi$ for $n \geq 3$.

Show $\forall n \in \mathbb{N}$ $P(n) \Rightarrow P(n+1)$
When arguing $P(n+1)$: $\Leftrightarrow$ angle sum of ARBITRARY $n+1$ gon equals $(n+1-2)\pi$.

$P(3)$: $\Leftrightarrow$ angle sum of $\Delta = \pi$



n-gon angle sum $=(n-2)\pi$ by I.H.
3-gon angle sum $=\pi$ by Base case

$\Rightarrow$ n+1 gon angle sum $=(n-2)\pi+\pi=(n-1)\pi$

~~The inductive step holds, thus by PMI~~
So $\forall n\in\mathbb{N}$, $P(n)\Rightarrow P(n+1)$, and by PMI the claim holds.

ex Give proof by contradiction that $\forall a,b\in\mathbb{N}$, $a|b\Rightarrow a\leq b$.

BWOC suppose $\exists a,b$, $a|b \wedge a>b$

$ak=b$ $\quad \cancel{k\in\mathbb{Z}}$, $k\in\mathbb{N}$
$\cancel{a=\frac{b}{k}}$

$a>b \wedge b=ak \Rightarrow a>ak \Rightarrow 1>k$, $k\in\mathbb{N}$

ex $p\in\mathbb{N}\setminus\{1\}$ a prime $:\Leftrightarrow \forall a\in\mathbb{N}, a|p \Rightarrow a=1$ or $a=p$

Fact: $p\in\mathbb{N}$ prime $\Leftrightarrow \forall a,b\in\mathbb{N}$, $p|ab\Rightarrow p|a \vee p|b$

($\Rightarrow$) $p\in\mathbb{N}$ prime $\Rightarrow \forall a,b\in\mathbb{N}$, $p|ab\Rightarrow p|a\vee p|b$

Assume p is prime and assume $p|ab$. $a,b\in\mathbb{N}$ is arbitrary $\neq 1$
If $p|a$ then done. So suppose $p\nmid a$. $\Rightarrow \gcd(a,p)=1$

Hint $\exists x,y\in\mathbb{Z}$, $1=ax+py$

$b=(ab)x+pby \Rightarrow b=p(kx+by)=pj$
$\quad ab = pk$

$\Rightarrow p|b$. So either $p|a$ or $p|b$.

Abstract Math (4/18/24)

Def Let A, B be sets.

① We say $|A| = |B|$ provided we can find a bijection $f: A \to B$

② We say $|A| \leq |B|$ provided there is injection $f: A \to B$

$|\mathbb{N}| = |\mathbb{Z}|$

If $|A| = |\mathbb{N}|$ we say A is countably infinite

If $|A| \leq |\mathbb{N}|$ we'll call it a countable set

$\mathbb{N}$	A
$1 \to$	a_1
$2 \to$	a_2
$3 \to$	a_3
$4 \to$	a_4
$5 \to$	a_5
$\vdots$	$\vdots$

$A = (a_1, a_2, \dots)$

( ) mean sequence - ordering

$\mathbb{N}$	$\mathbb{Z}$
$0 \to$	0
$1 \to$	1
$2 \to$	-1
$3 \to$	2
$4 \to$	-2
$5 \to$	3
$\vdots$	$\vdots$

$\mathbb{N}$ $\quad$ $\mathbb{Q}^+$

So $|\mathbb{Q}^+| = |\mathbb{N}|$

$\mathbb{Q}^+ = \{q_1, q_2, q_3, q_4, \dots\}$

$\mathbb{N}$	$\mathbb{Q}$
1	0
2	q_1
3	$-q_1$
4	q_2
5	$-q_2$
$\vdots$	$\vdots$

$\Rightarrow |\mathbb{Q}| = |\mathbb{N}|$

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 & \dots \\
1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \dots \\
1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \dots \\
1/4 & 2/4 & 3/4 & 4/4 & 5/4 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Fact If A is a set, there is a 1-1 map $f: A \to P(A)$

$\Rightarrow |A| \leq |P(A)|$

$|\mathbb{N}|$ vs $|(0,1)|$

Suppose $|\mathbb{N}| = |(0,1)|$ ← all $\mathbb{R}$

$\Rightarrow$ there's a bijection ; $f: \mathbb{N} \to (0,1)$

$a \in (0,1) \Rightarrow a = 0.d_1 d_2 d_3 d_4 \ldots \quad d_k \in \{0,1\}$

Choose the representation that terminates if there are multiple representations.

(0,1)

$f(1) = a_1 \to 0.d_{11} d_{12} d_{13} \ldots$ $\qquad r = 0.r_1 r_2 r_3 r_4 \ldots$

$f(2) = a_2 \to 0.d_{21} d_{22} d_{23} \ldots$

$f(3) = a_3 \to 0.d_{31} d_{32} d_{33} \ldots$ $\qquad r_k = \begin{cases} 0 : d_{kk} = 1 \\ 1 : d_{kk} = 0 \end{cases}$

$f(4) = a_4 \to 0.d_{41} d_{42} d_{43} \ldots$

$f(5) = a_5 \to 0.d_{51} d_{52} d_{53} \ldots$ $\Rightarrow r$ is not in the list and f is not ~~bijective~~ onto

$\vdots \qquad \vdots$

So $|\mathbb{N}| \neq |(0,1)|$

$g: \mathbb{N} \to (0,1)$

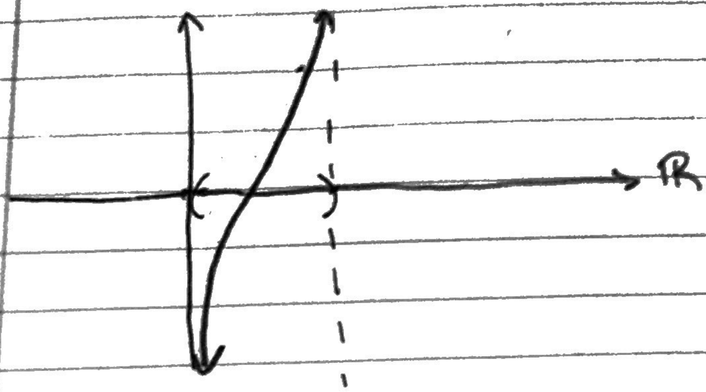
$g(n) = 0.000 \ldots 0100 \ldots$

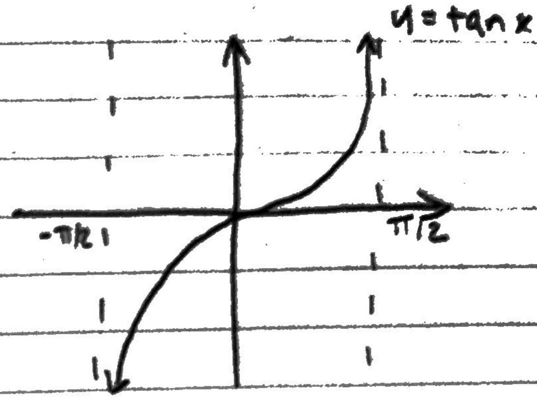
n^{th} spot insert 1

$\Rightarrow g$ is 1-1

$\Rightarrow |\mathbb{N}| < |(0,1)|$

$|(0,1)|$ vs $|\mathbb{R}|$





$f(x) = \tan\left(\frac{4x}{\pi} - \frac{1}{2}\right)$

$f : (0,1) \xrightarrow{\text{bij}} \mathbb{R}$

$\Rightarrow |(0,1)| = |\mathbb{R}|$ $\mathbb{R}$ uncountable

$|\mathbb{R}| = |\mathbb{C}|$

$N < |P(N)| < |P(P(N))| < \ldots$ * Infinity & the mind

$\quad\ \ \| \quad$ infinite

$\ \ |\mathbb{R}|$

Abstract Math (4/25/24)

$Q \cap (0,1)$ countable

$(0,1) \setminus Q$ uncountable

$\Omega = (0,1)$ Choose $x \in (0,1)$ at random

$P(x \in Q \cap (0,1)) = 0$

$P(x \in (0,1) \setminus Q) = 1$

Def: A real-valued sequence is a function
$s : \mathbb{N} \longrightarrow \mathbb{R}$

1) suggest an order
same as {} but with order

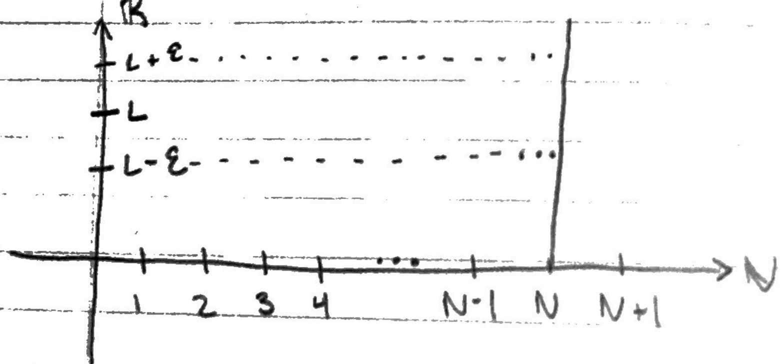
$(s_n) = (s_1, s_2, s_3, \dots)$

write $s(k) = s_k$

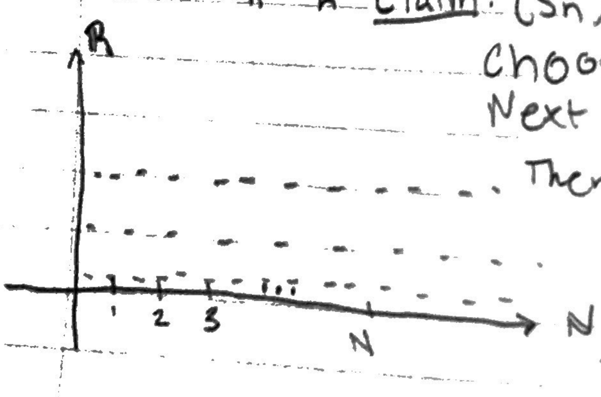
$s_n = \frac{1}{n}$ $(s_n) = (1, \frac{1}{2}, \frac{1}{3} \dots)$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Def: We say a sequence (s_n) converges provided there exists a value $L \in \mathbb{R}$, s.t. for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, s.t. for all $n \in \mathbb{N}$,
$n > N \Longrightarrow |s_n - L| < \varepsilon \Leftrightarrow L - \varepsilon < s_n < L + \varepsilon$



Let $s_n = \frac{1}{n}$ **Claim:** (s_n) converges



Choose $L = 0$ an[illegible] $\varepsilon > 0$ be arbitrary
Next choose $N = \frac{1}{\varepsilon}$.
Then let $n \in \mathbb{N}$ be arbitrary $\wedge$ $N = 10$
$n > N \Rightarrow |s_n - 0| = |\frac{1}{n} - 0|$
assume $n > N$. $= \frac{1}{n} < \frac{1}{10}$
Then $|s_n - L| = |\frac{1}{n} - 0| = \frac{1}{n} < \frac{1}{N} = \varepsilon$.

$s_n \xrightarrow{n} 0$