1.6: Let S be an ordered set and A is a nonempty subset such that $\sup A$ exists. Suppose there is a $B \subset A$ such that whenever $x \in A$, there is a $y \in B$ such that $x \leq y$. Show that $\sup B$ exists and that $\sup B = \sup A$.

Solution:

Let S be an ordered set and A be a nonempty subset such that $\sup A$ exists. Suppose there is a subset $B \subset A$ such that whenever $x \in A$, there exists $y \in B$ such that $x \leq y$. We'll show $\sup B \leq \sup A$ and $\sup B \geq \sup A$, verifying $\sup B = \sup A$

Let $a = \sup A$. Then for all $x \in A, x \leq a$. It follows, for all $y \in B \subset A, y \leq a$. We can then conclude B is bounded above. For now, assume sup B exists. Then it follows $\sup B \leq \sup A$.

Now, let $a = \sup A$. Then for all $x \in A$, there exists a $y \in B$ such that $x \leq y$. Because A is non-empty, we can say B is non-empty. Again, assume $\sup B$ exists. Then it must hold that $\sup B \geq \sup A$.

From above, we know that B is both bounded above and non-empty, thus B has a sup. Moreover, since B has a sup we know that $\sup B \leq \sup A$ and $\sup B \geq \sup A$, and therefore $\sup B = \sup A$.

2.7: Let $(x_n)_{n=1}^{\infty}$ be a sequence.

(a) Show that $\lim_{n\to\infty} x_n = 0$ (that is, the limit exists and is zero) if and only if $\lim_{n\to\infty} |x_n| = 0$.

(b) Find an example of a sequence where $(|x_n|)_{n=1}^{\infty}$ converges but $(x_n)_{n=1}^{\infty}$ diverges.

Solution:

(a) ($\Rightarrow$) Assume that $\lim_{n\to\infty} |x_n| = 0$. That is, for all $\epsilon > 0$, there exists an N_1 such that for all $n > N_1$, we have $||x_n|| < \epsilon$.

Now, we will show that $\lim_{n\to\infty} x_n = 0$. Let $\epsilon > 0$. Let $n > N_1$. Then,

$$|x_n| = ||x_n|| < \epsilon.$$

($\Leftarrow$) Assume that $\lim_{n\to\infty} x_n = 0$. That is, for all $\epsilon > 0$, there exists an N_1 such that for all $n > N_1$, we have $|x_n| < \epsilon$.

Now, we will show that $\lim_{n\to\infty} |x_n| = 0$. Let $\epsilon > 0$. Let $n > N_1$. Then,

$$||x_n|| = |x_n| < \epsilon.$$

(b) Suppose (x_n) = (1, -1, 1, -1, 1, -1, ...).

First we'll show $(|x_n|)_{n=1}^{\infty}$ converges. Choose x = 1. Let $\epsilon > 0$. Choose N = 0. Let $n > N$. Then, $||x_n| - 1| = |1 - 1| = 0 < \epsilon$.

Now we'll show $(x_n)_{n=1}^{\infty}$ diverges. Let $x \in \mathbb{R}$. Choose $\epsilon = 0.5$. Let $n \in \mathbb{N}$.

Case 1: Suppose $x \geq 0$. Choose any $n > N$ such that n is even. Then it follows, $|x_n - x| = |-1 - x| \geq 1 > \epsilon = 0.5$. In which case $(x_n)_{n=1}^{\infty}$ diverges.

Case 2: Suppose $x < 0$. Choose any $n > N$ such that n is odd. Then it follows, $|x_n - x| = |1 - x| \geq 1 > \epsilon = 0.5$. In which case $(x_n)_{n=1}^{\infty}$ diverges.

Therefore, $(|x_n|)_{n=1}^{\infty}$ converges but $(x_n)_{n=1}^{\infty}$ diverges with the given (x_n).

4.5:

(a) Let $x_n = \frac{(-1)^n}{n}$. Find $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$.

(b) Let $x_n = \frac{(n-1)(-1)^n}{n}$. Find $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$.

Solution:

(a) Let $\limsup\limits_{n\to\infty} x_n = \lim_{n\to\infty}(\sup\{x_k \mid k \geq n\})$. Then it follows, $\sup\{x_k \mid k \geq n\}) = \frac{1}{n}$ when n is even and $\sup\{x_k \mid k \geq n\}) = \frac{1}{n+1}$ when n is odd. Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon+1}$. Let $n > N$. Then $|\frac{1}{n} - 0| < \frac{1}{N} < \epsilon$. Therefore, $\limsup\limits_{n\to\infty} x_n = 0$.

Let $\liminf\limits_{n\to\infty} x_n = \lim_{n\to\infty}(\inf\{x_k \mid k \geq n\})$. Then it follows, $\inf\{x_k \mid k \geq n\}) = -\frac{1}{n}$ when n is odd and $\inf\{x_k \mid k \geq n\}) = -\frac{1}{n+1}$ when n is even. Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon+1}$. Let $n > N$. Then $|-\frac{1}{n} - 0| < \frac{1}{N} < \epsilon$. Therefore, $\liminf\limits_{n\to\infty} x_n = 0$.

(b) Let $\limsup\limits_{n\to\infty} x_n = \lim_{n\to\infty}(\sup\{x_k \mid k \geq n\})$. Then it follows, $\sup\{x_k \mid k \geq n\}) = 1$. Clearly, then $\limsup\limits_{n\to\infty} x_n = 1$.

Let $\liminf\limits_{n\to\infty} x_n = \lim_{n\to\infty}(\inf\{x_k \mid k \geq n\})$. Then it follows, $\inf\{x_k \mid k \geq n\}) = -1$. Clearly, then $\liminf\limits_{n\to\infty} x_n = -1$.

4.5: Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences such that $\lim_{n\to\infty} y_n = 0$. Suppose that for all $k \in \mathbb{N}$ and for all $m \geq k$, we have

$$|x_m - x_k| \leq y_k.$$

Show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution:

Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences such that $\lim_{n\to\infty} y_n = 0$. Suppose that for all $k \in \mathbb{N}$ and for all $m \geq k$, we have

$$|x_m - x_k| \leq y_k.$$

We'll now show $(x_n)_{n=1}^{\infty}$ is Cauchy.

Let $\epsilon > 0$. Because $\lim_{n\to\infty} y_n = 0$, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $|y_n| < \epsilon$. Therefore, let $m, k > n$ with $m \geq k$. It follows, $|x_m - xk| \leq y_k \leq |y_n| < \epsilon$, and thus $(x_n)_{n=1}^{\infty}$ is Cauchy.