1 Metric Spaces

Metric space: A function $d : M \times M \to \mathbb{R}$ is a metric if it satisfies:

- **Positive definiteness:** $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- **Symmetry:** $d(x, y) = d(y, x)$.
- **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

Converges: For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(p_n, p) < \epsilon$.
Convergent subsequence: Every subsequence of a convergent sequence in M converges to the same limit as the original sequence.

2 Continuity

Definition: A function $f : M \to N$ is continuous if it preserves sequential convergence: for each sequence (p_n) in M converging to p, the image sequence $(f(p_n))$ converges to $f(p)$.
Theorem: The composite of continuous functions is continuous.
Proposition: The identity map $id : M \to M$ is continuous, as is every constant function $f : M \to N$.
Homeomorphism: A function $f : M \to N$ is a homeomorphism if it is a bijection, continuous, and its inverse $f^{-1} : N \to M$ is also continuous. In this case, M and N are homeomorphic.
Theorem: $f : M \to N$ is continuous if and only if it satisfies the (ϵ, δ)-condition: For each $\epsilon > 0$ and each $p \in M$, there exists $\delta > 0$ such that if $x \in M$ and $d_M(x, p) < \delta$, then $d_N(f(x), f(p)) < \epsilon$.

3 Topology of a Metric Space

Definition: S is closed if it contains all its limit points.
Definition: S is open if for each $p \in S$, there exists $r > 0$ such that $d(p, q) < r \Rightarrow q \in S$.
Theorem: Openness and closedness are dual: the complement of an open set is closed and vice versa.
Theorem: The topology of M (denoted $\mathcal{T}$) satisfies:

- Any union of open sets is open.
- The intersection of finitely many open sets is open.
- $\emptyset$ and M are open.

Corollary: The intersection of any number of closed sets is closed, and the finite union of closed sets is closed.
Theorem: $\lim S$ is a closed set and Mrp is an open set.
Corollary: The interval (a, b) is open in $\mathbb{R}$ and so are $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$. The interval $[a, b]$ is closed.
Corollary: $\lim S$ is the "smallest" closed set that contains S in the sense that if $K \supseteq S$ and K is closed, then $K \supseteq \lim S$.
Theorem: The following are equivalent for continuity of $f : M \to N$:

- The (ϵ, δ)-condition.
- Sequential convergence preservation.
- The preimage of every closed set in N is closed in M.
- The preimage of every open set in N is open in M.

Corollary: A homeomorphism bijects open sets in M to open sets in N.
Inheritance Principle: Every metric subspace N of M inherits its topology from M in the sense that each subset $V \subset N$ which is open in N is actually the intersection $V = N \cap U$ for some $U \subset M$ that is open in M, and vice versa.
Corollary: Every metric subspace of M inherits its closed sets from M.
Corollary: Assume that N is a metric subspace of M and also is a closed subset of M. A set $L \subset N$ is closed in N if and only if it is closed in M. Similarly, if N is a metric subspace of M and also is an open subset of M, then $U \subset N$ is open in N if and only if it is open in M.
Proposition: For metrics:

$$d_E(p, p') = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}, \quad d_{\max}(p, p') = \max\{d_X(x, x'), d_Y(y, y')\}, \quad d_{\text{sum}}(p, p') = d_X(x, x') + d_Y(y, y')$$

where $p = (x, y)$ and $p' = (x', y')$, the following holds:

$$d_{\max} \leq d_E \leq d_{\text{sum}} \leq 2d_{\max}.$$

Convergence in a Product Space: The following are equivalent for a sequence $p_n = (p_{1n}, p_{2n})$ in $M = M_1 \times M_2$: [(a)] (p_n) converges with respect to the metric $d_{\max}$. [(b)] (p_n) converges with respect to the metric d_E. [(c)] (p_n) converges with respect to the metric d_{sum}. [(d)] (p_{1n}) and (p_{2n}) converge in M_1 and M_2 respectively.
If $f : M \to N$ and $g : X \to Y$ are continuous, then so is their Cartesian product $f \times g$, which sends $(p, x) \in M \times X$ to $(f(p), g(x)) \in N \times Y$.
Theorem: d is continuous.
Corollary: The metrics $d_{\max}$, d_E, and d_{sum} are continuous.
Corollary: The absolute value is a continuous mapping $\mathbb{R} \to \mathbb{R}$. In fact, the norm is a continuous mapping from any normed space to $\mathbb{R}$.
Theorem: A metric space M is complete if every Cauchy sequence in M converges to a limit in M.
Theorem: $\mathbb{R}^m$ is complete.
Theorem: Every closed subset of a complete metric space is a complete metric subspace.
Corollary: Every closed subset of Euclidean space is a complete metric space.

4 Compactness

Definition: A subset $A \subset M$ is compact if every sequence (a_n) in A has a subsequence that converges to a limit in A.
Theorem: Every compact set is closed and bounded.
Theorem: The closed interval $[a, b] \subset \mathbb{R}$ is compact.
Theorem: The Cartesian product of m compact sets is compact.
Corollary: Every box $[a_1, b_1] \times \cdots \times [a_m, b_m]$ in $\mathbb{R}^m$ is compact.
Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.
Theorem: Every closed subset of a compact set is compact.
Heine-Borel Theorem: Every closed and bounded subset of $\mathbb{R}^m$ is compact.
Ten Examples of Compact Sets 1. Any finite subset of a metric space, for instance the empty set. 2. Any closed subset of a compact set. 3. The union of finitely many compact sets. 4. The Cartesian product of finitely many compact sets. 5. The intersection of arbitrarily many compact sets. 6. The closed unit ball in $\mathbb{R}^3$. 7. The boundary of a compact set, for instance the unit 2-sphere in $\mathbb{R}^3$. 8. The set $\{x \in \mathbb{R} : \exists n \in \mathbb{N} \text{ and } x = \frac{1}{n}\} \cup \{0\}$. 9. The Hawaiian earring. See page 58. 10. The Cantor set. See Section 8.

Theorem: The intersection of a nested sequence of compact nonempty sets is compact and nonempty.
Corollary: If in addition to being nested, nonempty, and compact, the sets A_n have diameter that tends to 0 as $n \to \infty$, then $A = \bigcap_n A_n$ is a single point.
Theorem: If $f : M \to N$ is continuous and A is compact, then $f(A)$ is compact.
Corollary: A continuous real-valued function on a compact set is bounded and attains its max and min.
Theorem: If M is compact and homeomorphic to N, then N is compact.
Corollary: $[0, 1]$ and $\mathbb{R}$ are not homeomorphic.
Theorem: If M is compact, then a continuous bijection $f : M \to N$ is a homeomorphism.
Theorem: A compact set is absolutely closed and absolutely bounded.
Theorem: Every continuous function on a compact set is uniformly continuous.

5 Connectedness

If M has a proper clopen subset A, then M is disconnected. For there is a separation of M into proper, disjoint clopen subsets. M is connected if it is not disconnected, i.e., it contains no proper clopen subset.
Theorem If M is connected, $f : M \to N$ is continuous, and f is onto, then N is connected. The continuous image of a connected set is connected.
Corollary If M is connected and M is homeomorphic to N, then N is connected. Connectedness is a topological property.
Corollary (Generalized Intermediate Value Theorem) Every continuous real-valued function defined on a connected domain has the intermediate value property.
Theorem $\mathbb{R}$ is connected.
Corollary (Intermediate Value Theorem for $\mathbb{R}$) Every continuous function $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property.
Corollary The following metric spaces are connected: The intervals (a, b), $[a, b]$, the circle, and all capital letters of the Roman alphabet.
Theorem The closure of a connected set is connected. More generally, if $S \subset M$ is connected and $S \subset T \subset \bar{S}$, then T is connected.
Theorem The union of connected sets sharing a common point p is connected.
Theorem Path-connected implies connected.

6 Other Metric Space Concepts

The set S "clusters" at p (and p is a cluster point of S) if each $M_r(p)$ contains infinitely many points of S. The set S "condenses" at p (and p is a condensation point of S) if each $M_r(p)$ contains uncountably many points of S.
Theorem The following are equivalent conditions to S clustering at p. (i) There is a sequence of distinct points in S that converges to p. (ii) Each neighborhood of p contains infinitely many points of S. (iii) Each neighborhood of p contains at least two points of S. (iv) Each neighborhood of p contains at least one point of S other than p.
Proposition $S \cup S' = S$.
Corollary S is closed if and only if $S' \subset S$.
Corollary The least upper bound and greatest lower bound of a nonempty bounded set $S \subset \mathbb{R}$ belong to the closure of S. Thus, if S is closed, then they belong to S.
A metric space M is perfect if $M' = M$, i.e., each $p \in M$ is a cluster point of M.
Theorem Every nonempty, perfect, complete metric space is uncountable.
Corollary $\mathbb{R}$ and $[a, b]$ are uncountable.
Corollary Every nonempty perfect complete metric space is everywhere uncountable in the sense that each r-neighborhood is uncountable.
Theorem The arithmetic operations of $\mathbb{R}$ are continuous.
Lemma For each real number c, the function $\mathrm{Mult}_c : \mathbb{R} \to \mathbb{R}$ that sends x to cx is continuous.
Corollary The sums, differences, products, and quotients, absolute values, maxima, and minima of real-valued continuous functions are continuous (provided the denominator function does not equal zero).
Corollary Polynomials are continuous functions.
A subset S of a metric space M is bounded if for some $p \in M$ and some $r > 0$, $S \subset M_r(p)$.

7 Coverings

Definition A collection U of subsets of M covers $A \subset M$ if A is contained in the union of the sets belonging to U. The collection U is a covering of A. If U and V both cover A and if $V \subset U$ in the sense that each set $V \in V$ belongs also to U, then we say that U reduces to V, and that V is a subcovering of A.
Definition If all the sets in a covering U of A are open, then U is an open covering of A. If every open covering of A reduces to a finite subcovering of A, then we say that A is covering compact.
Theorem For a subset A of a metric space M, the following are equivalent: (a) A is covering compact. (b) A is sequentially compact.
A Lebesgue number for a covering U of A is a positive real number λ such that for each $a \in A$, there is some $U \in U$ with $M_\lambda(a) \subset U$. The choice of this U depends on a, but λ is independent of $a \in A$.
Lebesgue Number Lemma Every open covering of a sequentially compact set has a Lebesgue number $\lambda > 0$.
A set $A \subset M$ is totally bounded if for each $\varepsilon > 0$ there exists a finite covering of A by ε-neighborhoods.
Total Boundedness: A set $A \subset M$ is totally bounded if for each $\epsilon > 0$, there exists a finite covering of A by ϵ-neighborhoods.
Generalized Heine-Borel Theorem 65 A subset of a complete metric space is compact if and only if it is closed and totally bounded.
Corollary A metric space is compact if and only if it is complete and totally bounded.

8 Test questions

1. Continuous function sends open sets to open sets (F, $x^2, \mathbb{R} \to \mathbb{R}, (-1, 1)$ 2. Continuous bijection sends open sets to open sets (?) 3. Continuous bijection is always a homeomorphism (F, $[0, 2\pi) \to S^1$ 4. For continuous function preimage of the open/closed set is open/closed (T) 5. Complement of an open/closed set is closed/open (T) 6. Homeomorphic spaces are topologically equivalent (T) 7. Union of any collection of open/closed sets is open/closed (F, $[1/n, 1], (0, 1) \in \mathbb{R}$ 8. Union of finite number of open/closed sets is open/closed (T) 9. Intersection of any collection of open/closed sets is open/closed (F, $[1/n, 1], (0, 1]$ 10. Intersection of finite number of open/closed sets is open/closed (T) 11. If sequence is Cauchy it converges (F, $(-1/n, 1/n), \{0\}$ 12. If sequence converges it is Cauchy (T) 13. Every compact set is closed and bounded (T) 14. Every closed and bounded set is compact (F, discrete metric) 15. Every closed and bounded set in complete space is compact (F, ...) 16. Continuous function sends compact sets to compact sets (T) 17. Direct product of compact sets is compact (T) 18. Function continuous on a bounded set is uniformly continuous (F, 1/x on (0,1)) 19. Function continuous on a closed set is uniformly continuous (F, $x^2, \mathbb{R}$) 20. Function continuous on a compact set is uniformly continuous (T) 21. Continuous image of a connected set is connected (T)