Prelim 2.8: Prove that a continuous function $f : \mathbb{R} \to \mathbb{R}$ which sends open sets to open sets must be monotonic.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous which sends open sets to open sets.

By way of contradiction suppose f is not monotonic. Then for some open $U \subset \mathbb{R}$, there exists $x_1, x_2, x_3 \in U$ such that $x_1 < x_2 < x_3$ but $f(x_1) < f(x_2)$ and $f(x_2) > f(x_3)$ (or without loss of generality, $f(x_1) > f(x_2)$ and $f(x_2) < f(x_3)$).

Therefore, construct $S = (x_1, x_3) \subset \mathbb{R}$ which is open in $\mathbb{R}$. Because $f(x_1) < f(x_2)$ and $f(x_2) > f(x_3)$ f achieves a max on S, let this be at $x_0 \in (x_1, x_3)$. Then it follows, for any $B_\epsilon(f(x_0))$, the ball doesn't contain points entirely in $f^{PRE}(U)$, namely $f(x_0) + \epsilon \not\in f^{PRE}(U)$. Therefore, $f^{PRE}(U)$ fails to be open in $\mathbb{R}$ which contradicts f sending open sets to open sets.

2.48: Prove that there is an embedding of the line as a closed subset of the plane, and there is an embedding of the line as a bounded subset of the plane, but there is no embedding of the line as a closed and bounded subset of the plane.

Solution: For embedding of line as closed subset of the plane choose f: $\mathbb{R} \to \mathbb{R}^2$ where $f(x) = (0, x)$. In which case, each x maps to a unique y in $\{0\} \times \mathbb{R}$, also with each y from $\{0\} \times \mathbb{R}$ mapping to a unique $x \in \mathbb{R}$. Also, both f and f^-1 are continuous. Thus, an embedding. Finally, $\mathbb{R}$ itself is closed, so $\{0\} \times \mathbb{R}$ is a closed subset of the plane.

For embedding of line as bounded subset of the plane choose f: $\mathbb{R} \to \mathbb{R}^2$ where $f(x) = (0, tanh(x))$. In which case, each x maps to a unique y in $\{0\} \times (0, 1)$, also with each y from $\{0\} \times (0, 1)$ mapping to a unique $x \in \mathbb{R}$. Also, both f and f^-1 are continuous. Thus, an embedding. Finally, the image is contained in $\{0\} \times (0, 1)$, proving it is a bounded subset of the plane.

By way of contradiction, suppose there is an embedding of the line as a closed and bounded subset of the plane. Therefore, from Heine-Borel, the embedding of the line is compact on the plane. However, because the line embeds into the plane, $\mathbb{R}$ is homeomorphic to $f(\mathbb{R})$, and because $f(\mathbb{R})$ is compact, this would imply that $\mathbb{R}$ is also compact. But $\mathbb{R}$ is not compact, because if we construct a sequence of the $\mathbb{N}$, there is no convergent subsequence, and this is a contradiction.

2.52: Let (A_n) be a nested decreasing sequence of nonempty closed sets in the metric space M.

(a) If M is complete and $\text{diam}A_n \to 0$ as $n \to \infty$, show that $\bigcap A_n$ is exactly one point.

(b) To what assertions do the sets $[n, \infty)$ provide counterexamples?

Solution:

(a) Suppose we construct a sequence (x_n), where $x_n \in A_n$ for all $n \in \mathbb{N}$. Because $\text{diam}A_n \to 0$ as $n \to \infty$, the distance between any two points in A_n becomes arbitrarily small. So then, let $\epsilon > 0$, choose N such that $\text{diam}A_n < \epsilon$, and so $k, m \geq N$ implies $d(x_k, x_m) < \epsilon$ because $\text{diam}A_n \to 0$ as $n \to \infty$, satisfying Cauchy. Moreover, because M is complete, the Cauchy sequence (x_n) converges to a point $x \in M$. Because A_n is closed and nested, x is in each A_n. Moreover, because the $\text{diam}A_n \to 0$ as $n \to \infty$, if there was a y in each A_n then for sufficiently small $\epsilon, d(x, y) > \epsilon$, which implies the intersection is a singleton, and therefore x is the unique point in all A_n.

(b) The intersection of sets $A_n = [n, \infty)$ would be empty, which would provide a counterexample to the assertion: the intersection of nested, closed sets in a complete space is nonempty. A point x cannot be contained in the intersection because the interval $[\lceil x + 1 \rceil, \infty)$ would not contain x, which must be a part of the intersection because $n \to \infty$.