Calculus 3 Review:

2D Vectors

Addition: $u+v = \langle u_1+v_1, u_2+v_2 \rangle$

Scalar Multiplication: $cv = \langle cv_1, cv_2 \rangle$

Magnitude: $|v| = \sqrt{v_1^2 + v_2^2}$

Unit Vector: $\hat{v} = \frac{v}{|v|}$

3D Vectors

→ same principle

Dot product

$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

Commutative: $u \cdot v = v \cdot u$

Distributive: $u \cdot (v+w) = u \cdot v + u \cdot w$

Scalar Multiplication: $c(u \cdot v) = (cu) \cdot v = (cv) \cdot u$

Angle Between Vectors

$\cos\theta = \frac{u \cdot v}{|u||v|}$; if $u \cdot v = 0$ the vectors are orthogonal

Cross Product

The cross product of two vectors results in a new orthogonal vector.

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i(u_2v_3 - u_3v_2) - j(u_1v_3 - u_3v_1) + k(u_1v_2 - u_2v_1)$$

Not Commutative

Distributive

If $u \times v = 0$, the vectors are parallel

Magnitude of Cross Product

$|u \times v| = |u||v|\sin\theta$

Parametric Equations

2D: $x = f(t)$ $y = g(t)$ — $x = 1 + 2t$ — $t = \frac{x-1}{2}$

3D: $x = f(t)$ $y = g(t)$ $z = h(t)$ — $y = 3 + t$ — $y = 3 + \frac{x-1}{2}$

Calculus 3 Review:

Lines in 3D

Ex: Line through (1,2,3) with direction (4,5,6)

$r(t) = r_0 + tv$

$r_0 = \langle x_0, y_0, z_0 \rangle$ $\qquad r(t) = \langle 1,2,3 \rangle + t\langle 4,5,6 \rangle$

$V = \langle a, b, c \rangle$ $\qquad x = 1+4t;\ y = 2+5t;\ z = 3+6t$

Planes in 3D

$a(x-x_0) + b(y-y_0) + c(z-z_0)$ or $ax+by+cz=d$

where $\langle a,b,c \rangle$ is the normal vector

Conic Sections

Circle

$(x-h)^2 + (y-k)^2 = r^2$, centered at (h,k)

Ellipse

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (h,k) is the center, a is semi-major & b is semi-minor

Parabola

Vertical: $(x-h)^2 = 4p(y-k)$ $\quad$ (h,k) is vertex where p is distance from vertex to focus

Horizontal: $(y-k)^2 = 4p(x-h)$

Hyperbola:

Vertical: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Horizontal: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Cylinders:

Circular

$x^2 + y^2 = r^2$

Parabolic

$z = x^2$

Quadric Surfaces:

Ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Calculus 3 Review:

Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of Two Sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Vector Valued Functions

$r(t) = \langle x(t), y(t), z(t) \rangle$

$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$\int r(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle$

Motion in Space

Velocity: $v(t) = r'(t)$

Speed: $\|v(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

Acceleration: $a(t) = r''(t)$

Arc Length

$L = \int_a^b \|r'(t)\|\,dt$ $\quad r(t) = \langle 3t, 4\sin t, 4\cos t \rangle, \; t \in [0, \pi]$

$r'(t) = \langle 3, 4\cos t, -4\sin t \rangle$

$\|r'(t)\| = \sqrt{3^2 + 4\cos t^2 + (-4\cos t)^2} = 5, \quad L = \int_0^\pi 5\,dt = [5t]_0^\pi = 5\pi$

Curvature

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

Normal/Tangent Vectors

Unit Tangent: $T(t) = \frac{r'(t)}{\|r'(t)\|}$

Unit Normal: $N(t) = \frac{T'(t)}{\|T'(t)\|}$

Calculus 3 Review:

Surfaces

$z = f(x,y)$

Level Curves

Curves from slicing a surface at $z = c$.

Limits on Surfaces

$\lim_{(x,y)\to(a,b)} f(x,y) = L$

Continuity on Surfaces

Continuous at (a,b) if:

- $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
- $f(a,b)$ is defined
- $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Partial Derivatives

$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

Tangent Planes

$z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

Linear Approximation

$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

Multivariable Chain Rules

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$z = x^2 + y^2,\ x = t^2\ y = \sin t$

$\frac{dz}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2(t^2)(2t) + 2(\sin t)(\cos t)$

Directional Derivatives

Directional Derivative of $f(x,y)$ at point (x_0, y_0) in the direction of unit vector $u = \langle a, b \rangle$ is:

$D_u f(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b$

or

$D_u f(x_0,y_0) = \nabla f(x_0,y_0) \cdot u$

Calculus 3 Review:

Gradient

$\nabla f(x,y) = \langle f_x, f_y \rangle$

Ex: $f(x,y) = x^2 + y^2$; find directional derivati at $(1,1)$ in direction $u = \langle 3,4 \rangle$

$\sqrt{3^2 + 4^2} = 5$ $\quad u = \langle \frac{3}{5}, \frac{4}{5} \rangle$ $\quad D_u f(1,1) = \nabla f(1,1) \cdot u = \langle 2,2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

$= 14/5$

$f_x = 2x \quad f_y = 2y$

Minima / Maxima / Saddle Points

$z = f(x,y)$

Critical points occur where $f_x(x,y) = 0$ & $f_y(x,y) = 0$

$D = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2$

$D > 0$ and $f_{xx} > 0$: Local minimum

$D > 0$ and $f_{xx} < 0$: Local maximum

$D < 0$: Saddle point

$D = 0$: Inconclusive

Double Integrals over Rectangular Regions

For $f(x,y)$ over rectangular region $R = [a,b] \times [c,d]$:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Ex: $\iint_R (x+y)\,dA$ where $R = [0,1] \times [0,2]$

$$\int_0^1 \int_0^2 (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^1 2x+2\,dx = \left[x^2+2x\right]_0^1 = 3$$

Double Integrals over General Regions

If R is bounded by curves $x = g_1(y)$, $x = g_2(y)$, $y = c$, $y = d$:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

Ex: $\iint_R (x^2+y^2)\,dA$ where R is triangular region with $(0,0), (1,0), (1,1)$:

$0 \le y \le x$; $0 \le x \le 1$

$$\int_0^1 \int_0^x (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^x = \int_0^1 \frac{4x^3}{3} = \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{3}$$

Calculus 3 Review:

Double Integrals in Polar Coordinates

R described in polar coordinates where $x = r\cos\theta$, $y = r\sin\theta$:

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex: $\iint_R (x^2+y^2)\,dA$ where R is bounded by $x^2+y^2=4$

$x^2+y^2=r^2$, so $f(x,y)=r^2$; $R: 0\le r\le 2,\ 0\le\theta\le 2\pi$

$$\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 = \int_0^{2\pi} 4\,d\theta = \left[4\theta\right]_0^{2\pi} = 8\pi$$

Triple Integrals in Rectangular Coordinates

$$\iiint_Q f(x,y,z)\,dV = \int_a^b\int_c^d\int_e^f f(x,y)\,dz\,dy\,dx$$

Ex: $\iiint_Q z\,dV$ where $0\le x\le 1,\ 0\le y\le 2,\ 0\le z\le 3$

$$\int_0^1\int_0^2\int_0^3 z\,dz\,dy\,dx = \int_0^1\int_0^2\left[\frac{z^2}{2}\right]_0^3 dy\,dx = \int_0^1\int_0^2 \frac{9}{2}\,dy\,dx = \int_0^1\left[\frac{9}{2}y\right]_0^2$$

$$= \int_0^1 9\,dx = \left[9x\right]_0^1 = 9$$

Triple Integrals in Cylindrical Coordinates

(r,θ,z) is useful with rotational symmetry around z-axis

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$; $dV = r\,dr\,d\theta\,dz$

Ex: Q is solid cylinder bounded by $x^2+y^2\le 1$ and $0\le z\le 2$

$x^2+y^2\le 1 \to 0\le r\le 1,\ 0\le\theta\le 2\pi$

$0\le z\le 2$

$$\int_0^{2\pi}\int_0^1\int_0^2 z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left[\frac{z^2 r}{2}\right]_0^2 = \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = \int_0^{2\pi}\left[r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} 1\,d\theta = \left[\theta\right]_0^{2\pi} = 2\pi$$

Calculus 3 Review

Triple Integrals in Spherical Coordinates

(ρ, θ, ϕ) are ideal for spherical regions

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

ρ: radial distance from origin

ϕ: angle from positive z-axis

θ: angle from positive x-axis in xy plane

Ex: Evaluate $\iiint_Q \rho^2 \, dV$ where Q is sphere or radius $= 2$

$0 \le \rho \le 2, \ 0 \le \phi \le \pi, \ 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^5 \sin\phi}{5}\right]_0^2 = \int_0^{2\pi}\int_0^{\pi} \frac{32\sin\phi}{5}$$

$$= \int_0^{2\pi}\left[-\frac{32\cos\phi}{5}\right]_0^{\pi} = \int_0^{2\pi} \frac{64}{5} \, d\theta = \left[\frac{64}{5}\theta\right]_0^{2\pi} = \frac{128\pi}{5}$$

Vector Fields

2D $F(x,y) = P(x,y)i + Q(x,y)j$; 3D $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y$

Line Integrals

A line integral of a vector field F along curve C parameterized by $r(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$ is:

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt$$

Ex: Evaluate $\int_C F \cdot dr$ for $F(x,y) = \langle x, y \rangle$ along C parameterized by $r(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$

$r'(t) = \langle 1, 2t \rangle \qquad F(r(t)) = \langle t, t^2 \rangle$

$F(r(t)) \cdot r'(t) = t \cdot 1 + t^2 \cdot 2t = 2t^3 + t$

$$\int_0^1 2t^3 + t \, dt = \left[\frac{t^4}{2} + \frac{t^2}{2}\right]_0^1 = 1$$

Conservative Vector Fields

A vector field is conservative if there exists a scalar potential function $f(x,y,z)$:

$$F = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

1. Line integral of F over C depends only on endpoints.
2. Line integral around a closed curve is 0: $\oint_C F \cdot dr = 0$

Calculus 3 Review:

Fundamental Theorem for Line Integrals

If F is a conservative vector field with potential function, f, then the line integral along C from B to A is:

$$\int_C F \cdot dr = f(B) - f(A)$$

Ex: Verify if $F(x,y) = \langle 2x, 2y \rangle$ is conservative and evaluate $\int_C F \cdot dr$ where C is a curve from A(1,1) to B(2,3)

potential: $f(x,y)$ $\quad \frac{\partial}{\partial x} = 2x \quad \frac{\partial}{\partial y} = 2y$

$f(x,y) = x^2 + g(y)$

$f(x,y) = x^2 + y^2$ $\quad \int_C F \cdot dr = f(2,3) - f(1,1) = 13 - 2 = 11$

Green's Theorem

Relates a simple closed curve C to a double integral:

$$\int_C F \cdot dr = \int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Ex: $F(x,y) = \langle -y, x \rangle$ around curve C, unit circle $x^2 + y^2 = 1$, CCW.

$\frac{\partial Q}{\partial x} = 1 \quad \frac{\partial P}{\partial y} = -1 \quad \int_C F \cdot dr = \iint_R (1 - (-1))\,dA = 2\pi$

Divergence

Divergence measures the net rate of flow out of a point.

Divergence of vector field $F(x,y,z) = Pi + Qj + Rk$ is:

$$\text{div } F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Curl

Curl measures the rotation of the field around a point

$$\text{curl } F = \nabla \times F = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

Divergence Theorem

$$\iint_S F \cdot n\,dS = \iiint_V (\nabla \cdot F)\,dV$$

S (closed surface enclosing V), $\nabla \cdot F$ (divergence) n (normal vector on S), left (surface integral, flux) right (volume integral, divergence through S over V)

Calculus 3 Review:

Ex: Evaluate flux of $F = \langle x, y, z \rangle$ across surface of unit sphere $x^2 + y^2 + z^2 = 1$

$$\operatorname{div} F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

$$\iiint_V (\nabla \cdot F)\, dV = \iiint_V 3\, dV = 3 * \frac{4}{3}\pi = 4\pi$$

Surface Integrals

surface integral computes flux of vector field F through surface S given parametrically by $r(u,v)$

$$\iint_S F \cdot n\, dS = \iint_D F(r(u,v)) \cdot (r_u \times r_v)\, dA$$

$r_u = \frac{\partial r}{\partial u}$

$r_v = \frac{\partial r}{\partial v}$

or

$z = f(x,y)$, flux can be expressed as:

$$\iint_S F \cdot n\, dS = \iint_D \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R\right) dA,$$

where $F = \langle P, Q, R \rangle$

Ex: Find the flux of $F = \langle 0, 0, z^2 \rangle$ through surface $z = 1 - x^2 - y^2$

$$n = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}} \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle$$

$\frac{\partial z}{\partial x} = -2x$, $\frac{\partial z}{\partial y} = -2y$ $\quad \iint_S F \cdot n\, dS = \iint_D (0 + 0 + z^2)\, dA = \iint_D (1 - x^2 - y^2)^2\, dA$

$$\iint_D (1 - r^2)^2\, r\, dr\, d\theta, \quad r \in [0,1],\ \theta \in [0, 2\pi]$$

$$\int_0^{2\pi}\int_0^1 (1-r^2)^2\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r - 2r^3 + r^5\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{2} + \frac{r^6}{6}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\, d\theta = \left[\frac{\theta}{6}\right]_0^{2\pi} = \frac{\pi}{3}$$