2.4: Write out a proof that the discrete metric on a set M is actually a metric.
Solution:

Suppose we have the discrete metric on a set M.
Let $x, y, z \in M$.
Positive definiteness:
($\Rightarrow$) Suppose d(x, y) = 0. Then from discrete metric definition x = y.
($\Leftarrow$) Suppose x = y. Then d(x, y) = 0 from our definition.
Symmetry: Assume $x \neq y$. Then d(x, y) = 1 and d(y, x) = 1. Thus, d(x, y) = d(y, x).
Triangle inequality: By way of contradiction suppose d(x, z) > d(x, y) + d(y, z). Then it must be so that d(x, z) = 1 and d(x, y) + d(y, z) = 0. This would mean that x = y, y = z, and $x \neq z$. But x = y = z so x = z. This is a contradiction so d(x, z) $\leq$ d(x, y) + d(y, z).
Therefore, the discrete metric on set M is actually a metric.

2.7: Prove that every convergent sequence (p_n) in a metric space M is bounded, i.e., that for some $r > 0$, some $q \in M$, and all $n \in \mathbb{N}$, we have $p_n \in M_r(q)$.
Solution:

Suppose (p_n) is a convergent sequence in metric space M with $p_n \to p$ as $n \to \infty$. Then, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that n > N implies $d(p_n, p) < \epsilon$.

Let $\epsilon = 1$. Then there exists an $N \in \mathbb{N}$ such that n > N implies $d(p_n, p) < \epsilon$. Let S $=\{d(p_n, p) : n \leq N\}$.

Choose r = max S $+\epsilon$, which must hold because S is finite. Choose q = $p \in M$ by definition of convergence. Let $n \in \mathbb{N}$. Then $M_r(q) = \{p_n \in M : d(q, p_n) < r\}$ holds and therefore (p_n) is bounded.

2.12: Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ be a bijection. The sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is a rearrangement of (p_n).

(a) Are limits of a sequence unaffected by rearrangement?

(b) What if f is an injection?

(c) What if f is a surjection?

Solution:

(a) **Converges:** Let (p_n) be a sequence and suppose it converges to limit L. Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$, $d(p_n, L) < \epsilon$. Now suppose $f : \mathbb{N} \to \mathbb{N}$ is a bijection and the sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is a rearrangement of (p_n).

So then, let $\epsilon > 0$. Let S be a set where S = {f(x): $x \leq N, x \in \mathbb{N}$}. Choose $N_1 = maxS$. Let $k > N_1$. Then $d(q_k, L) < \epsilon$ because we know $n = f^{-1}(k) > N$ where $d(p_n, L) < \epsilon$.

Thus, the limit of a convergent sequence is unaffected by rearrangement.

Diverges: Let (p_n) be a sequence and suppose p_n diverges. Then there exists $\epsilon > 0$ for all $N \in \mathbb{N}$ such that there exists $n > N$, $d(p_n, L) \geq \epsilon$.

Choose ϵ from above. Let $N_1 \in \mathbb{N}$. Let S be a set where S = {f$^{-1}(x)$: $x \leq N_1, x \in \mathbb{N}$}. So then, for N = max S, there exists $n > N$, $d(p_n, L) \geq \epsilon$, and therefore there exists $k > N_1$, $d(q_k, L) < \epsilon$.

(b) **Converges:** Generally, the statement would hold because (q_k) in this example would be a rearranged subsequence of (p_n). Combining the proof from part a and the notion that subsequences converge to the same limit, the limit of the subsequence from (p_n), limit of (q_k), and limit of (p_n) would be the same.

Diverges: Let (p_n) be a sequence and suppose (p_n) diverges. Suppose p_n = 1, -1, 1, -1, ... Thus p_n diverges, but suppose $f(x) = 2x$ and $(q_k)_{k\in\mathbb{N}}$ = -1, -1, -1, -1, ... which converges to -1. Thus, when (p_n) diverges, the rearrangement doesn't always model this same behavior.

(c) **Converges:** Let (p_n) be a sequence that converges to 0. Suppose $p_n = \frac{1}{n}$, and let $f(x) = \sqrt{x}$.

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \text{ is a perfect square,} \\ 1, & \text{otherwise.} \end{cases}$$

Thus, f is surjective and (p_k) = 1, 1, 1, 1/2, 1, ... (p_k) doesn't converge to 0, so the rearrangement doesn't always model this same behavior.

Diverges: Let (p_n) be a sequence that diverges. By definition of surjective, every element in (p_n) is an element in (q_k). So then, following the same logic from part a, (q_k) must also diverge.

Prelim 3: Consider $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that for each fixed x_0, $y \mapsto f(x_0, y)$ is continuous and for each fixed y_0, $x \mapsto f(x, y_0)$ is continuous. Find such an f that is not continuous.
Solution:

Suppose

$$f(x,y) = \begin{cases} \frac{xy}{|x|^2+|y|^2}, & x \neq 0 \text{ or } y \neq 0. \end{cases}$$

Then for each fixed x_0,

$$f(x_0, y) = \frac{x_0 y}{|x_0|^2 + |y|^2},$$

which is continuous where $x_0 \neq 0$ or $y \neq 0$, as satisfied by the condition. Similarly, for each fixed y_0,

$$f(x, y_0) = \frac{x y_0}{|x|^2 + |y_0|^2},$$

which is continuous. However, suppose we approach $(0,0)$ as $x \to 0$ and $y \to x$, then

$$f(x,x) = \frac{x^2}{2|x|^2} = \frac{1}{2}$$

approaches $\frac{1}{2}$. Now suppose we approach $(0,0)$ as $y \to 0$ and $x \to -y$, then

$$f(-y,y) = \frac{-y^2}{2|y|^2} = \frac{-1}{2}$$

approaches $\frac{-1}{2}$. Thus, taking different paths along the function results in different limits. Therefore, the function is continuous for a fixed x_0 or for a fixed y_0, but is not continuous for varying x and y.