2.24: For which intervals $[a, b] \subset \mathbb{R}$ is the intersection $[a, b] \cap \mathbb{Q}$ a clopen subset of the metric space $\mathbb{Q}$?

Solution: Intervals $[a, b] \subset \mathbb{R}$ where a,b are irrational satisfy the condition that $[a, b] \cap \mathbb{Q}$ is a clopen subset of the metric space $\mathbb{Q}$.

First suppose $[a, b] \subset \mathbb{R}$ where $a, b \notin \mathbb{Q}$. Then it follows, $[a, b] \cap \mathbb{Q} = (a, b) \cap \mathbb{Q}$. Because $\mathbb{Q}$ inherits its topology from $\mathbb{R}$, and (a, b) is open in $\mathbb{R}$, $[a, b] \cap \mathbb{Q}$ is open in $\mathbb{Q}$. Moreover, $(-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}$, and so the intersection with $\mathbb{Q}$ is open in $\mathbb{Q}$. Thus, $[a, b] \cap \mathbb{Q}$ is also closed in $\mathbb{Q}$.

Now suppose $[a, b] \subset \mathbb{R}$ where $a, b \in \mathbb{Q}$, and thus $[a, b] \cap \mathbb{Q}$ is closed in $\mathbb{Q}$. Also, from above, $(-\infty, a) \cup (b, \infty)$ is open in $\mathbb{Q}$, so this case doesn't give a clopen subset.

Finally suppose $[a, b] \subset \mathbb{R}$ where either a or b is rational, and the other is irrational. Without loss of generality suppose a is rational and b is irrational. Then for any $r > 0$ and $B_r(a)$, the ball contains points outside of $[a, b] \cap \mathbb{Q}$ in $\mathbb{Q}$ and therefore is not open in $\mathbb{Q}$. Moreover, b is a limit point of $[a, b]$ but $b \notin [a, b] \cap \mathbb{Q}$ and therefore is not closed in $\mathbb{Q}$.

2.26: Prove that a set $U \subset M$ is open if and only if none of its points are limits of its complement.

Solution:

($\Rightarrow$) By way of contradiction suppose $U \subset M$ is open and there exists a limit point x which is a limit of its complement. So then, $x \in U$. Moreover, there exists sequence $(x_n) \subset U^c$ where $x_n \to x$. This means for all $r > 0, B_r(x)$ contains a point in U^c by definition of $x_n \to x$. This contradicts U being open because there must be some $r > 0, B_r(x)$ containing points entirely in U. Thus, if $U \subset M$ is open, none of its points are limits of U^c.

($\Leftarrow$) Let $U \subset M$ where none of its points are limits of its complement. We'll show U is open. Let $p \in U$. From above, there exists an $r > 0$ such that $B_r(p) \cap U^c = \emptyset$. So then, if $B_r(p)$ contains point x, $x \in U$.

2.27: If $S, T \subset M$, a metric space, and $S \subset T$, prove that:

(a) $\bar{S} \subset \bar{T}$.

(b) $\text{int}(S) \subset \text{int}(T)$.

Solution:

(a) By way of contradiction, suppose $S \subset T$ and $\bar{S} \not\subset \bar{T}$. Thus, there exists $x \in \bar{S}$ where $x \notin \bar{T}$. Then it follows, there is $(s_n) \subset S$ with $s_n \to x \in \bar{S}, x \notin \bar{T}$. But, $\bar{S} = S \cup$ lim points of S. So, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N, d(x, s_n) < \epsilon$. However, each s_n is also in T because $S \subset T$, and so either x is a limit point of T, or $x \in T$, which means $x \in \bar{T}$. This is a contradiction. Therefore, $\bar{S} \subset \bar{T}$.

(b) By way of contradiction suppose $S \subset T$ and $\text{int}(S) \not\subset \text{int}(T)$. Thus, there exists $x \in int(S)$ where $x \notin int(T)$. Then it follows, for all $\epsilon > 0$, $B_\epsilon(x) \subset S$. So, for all $a \in B_\epsilon(x), a \in S$. So then, $a \in T$ for all a. So then, $B_\epsilon(x) \subset T$, but that means $x \in int(T)$, which is a contradiction. Therefore, $\text{int}(S) \subset \text{int}(T)$.

2.32: Show that every subset of $\mathbb{N}$ is clopen. What does this tell you about every function $f : \mathbb{N} \to M$, where M is a metric space?

Solution: Let N be a subset of $\mathbb{N}$. Suppose N isn't $\mathbb{N}$ or $\varnothing$, both of which are known to be clopen. So, for all $x \in \mathbb{N}$, choose $r = \frac{1}{2}$, and so $B_r(x) = \{\text{x}\}$, which is a singleton set and is contained within N. Therefore, every subset of N is open. Moreover, since every subset of N is open, its complement is also open. Therefore, the complement of N is open, which implies N is closed.

This means f is continuous. Suppose we have a closed set in M. From clopen, we know the pre-image would be closed in $\mathbb{N}$. Now suppose we have an open set in M. From clopen, we know the pre-image would be open in $\mathbb{N}$. Both cases hold.