EX. 1.1.2

a) $x^5 + x = 1$

The function $f: \mathbb{R} \to \mathbb{R}$, $x \to x^5 + x - 1$ is continuous.
$f(0) = -1 < 0$ and $f(1) = 1 > 0$ so IVT holds and
$\forall x_* \varepsilon (0,1) : f(x_*) = 0$

b) $\sin(x) = 6x + 5 \equiv \sin(x) - 6x + 5$

The function $f: \mathbb{R} \to \mathbb{R}$, $x \to \sin(x) - 6x + 5$ is
continuous. $f(0) = 5 > 0$ and $f(1) = \sin(1) - 1 < 0$
so IVT holds and $\forall x_* \varepsilon (0,1): f(x_*) = 0$

c) $\ln(x) + x^2 = 3$

The function $f: \mathbb{R} \to \mathbb{R}$, $x \to \ln(x) + x^2 - 3$ is continuous
where $x > 0$. $f(1) = -2 < 0$ and $f(2) = \ln(2) + 1 > 0$
so IVT holds and $\forall x_* \varepsilon (1,2) : f(x_*) = 0$

EX. 1.1.4 (bisection method)

a) $x^5 + x = 1$

Step 0: $a_0 = 0$ $\quad$ $b_0 = 1$
$f(a) = -1$ $\quad$ $f(b_0) = 1$
$x_0 = \frac{1}{2}$ $\quad$ $f(x_0) = -0.4688$

Step 1: $a_1 = \frac{1}{2}$ $\quad$ $b_1 = 1$
$f(a_1) = -0.4688$ $\quad$ $f(b_1) = 1$
$x_1 = 3/4$ $\quad$ $f(x_1) = -0.0127$

Step 2: $a_2 = 3/4$ $\quad$ $b_2 = 1$
$f(a_2) = -0.0127$ $\quad$ $f(b_2) = 1$
$x_2 = 7/8$

The approximate solution $x_2 = 7/8$ is within $1/8$ of the true root

b) $\sin(x) = 6x + 5$

Step 0: $a_0 = 0$ $b_0 = 1$

$f(a_0) = 5$ $f(b_0) = \sin(1) - 1$

$x_0 = \frac{1}{2}$ $f(x_0) = \sin(\frac{1}{2}) - 8$

Step 1: $a_1 = \frac{1}{2}$ $b_1 = 1$

$f(a_1) = \sin(\frac{1}{2}) - 8$ $f(b_1) = \sin(1) - 1$

$x_1 = 3/4$ $f(x_1) = \sin(3/4) - \frac{18}{4} - 5$

Step 2: $a_2 = 3/4$ $b_2 = 1$

$f(a_2) = \sin(3/4) - \frac{18}{4} - 5$ $f(b_2) = \sin(1) - 1$

$x_2 = 7/8$

The approximate solution $x_2 = 7/8$ is within $1/8$ of the true root

c) $\ln(x) + x^2 = 3$

Step 0: $a_0 = 1$ $b_0 = 2$

$f(a_0) = -2$ $f(b_0) = \ln(2) + 1$

$x_0 = \frac{3}{2}$ $f(x_0) = -0.3445$

Step 1: $a_1 = \frac{3}{2}$ $b_1 = 2$

$f(a_1) = -0.3445$ $f(b_1) = \ln(2) + 1$

$x_1 = 7/4$ $f(x_1) = 0.6221$

Step 2: $a_2 = \frac{3}{2}$ $b_2 = 7/4$

$f(a_2) = -0.3445$ $f(b_2) = 0.6221$

$x_2 = 13/8$

The approximate solution $x_2 = 13/8$ is within $1/8$ of the true root

Followed example 1.1.7

EX.1.1.8

a) $\text{true-err}(k) = |c_k - x_*| \Rightarrow \text{true-err}(k) \le \text{err-bound}(k)$

$\Rightarrow \text{true-err}(k) \le \text{err-bound}(k) \Rightarrow \text{err-bound}(k) \le tol$

$\Rightarrow \text{true-err}(k) \le tol$

* bisection method $\Rightarrow |c_0 - x_*| \le \frac{b-a}{2} \Rightarrow \text{err-bound}(0) = \frac{b-a}{2}$

$\Rightarrow \text{true-err}(0) = |c_0 - x_*| \le \text{err-bound}(0)$

$\Rightarrow \text{err-bound}(k) = (b-a)2^{-(k+1)} \Rightarrow \boxed{(b-a)2^{-(k+1)} \le tol}$

b) $(b-a)2^{-(k+1)} \le tol \equiv \frac{b-a}{tol} \le 2^{k+1} \equiv \log_2(b-a) - \log_2(tol) \le$

$\boxed{\equiv k \ge \log_2(b-a) - \log_2(tol) - 1 \Rightarrow k = \log_2(b-a) - \log_2(tol) -}$

c) numerals$(k) = k+2$

d) 15

e) 42

] calculated in colab notebook

CP.1.2.8 $g(x) = \frac{2x-1}{x^2} \equiv g(x) = (2x-1)(x^{-2})$

$g(x) = \frac{2x-1}{x^2}$, $r=1$ a)

a) The function $\left(g: (0,\infty) \to \mathbb{R},\ x \to \frac{2x-1}{x^2}\right)$ is cont diff in $(0,\infty)$

b) $g(1) = \frac{2(1)-1}{1} = 1$, so $g(r) = r$

c) $g'(x) = (-2x^{-3})(2x-1) + (2)(x^{-2}) = -4x^{-2} + 2x^{-3} + 2x^{-2}$, $g'(1) = 0 < 1$, so $|g'(r)| < 1$

* Fixed Point Iteration is locally convergent to fixed point $r=1$

$g(x) = \cos x + \pi + 1$, $r = \pi$ b)

a) The function $(g: \mathbb{R} \to \mathbb{R},\ x \to \cos x + \pi + 1)$ is cont. diff. in $\mathbb{R}$

b) $g(\pi) = -1 + \pi + 1 = \pi$, so $g(r) = r$

c) $g'(x) = -\sin x$, $g'(\pi) = 0 < 1$, so $|g'(r)| < 1$

*Fixed Point Iteration is locally convergent to fixed point $r=\pi$

$g(x) = e^{2x} - 1$, $r = 0$ c)

a) The function $(g: \mathbb{R} \to \mathbb{R},\ x \to e^{2x} - 1)$ is cont. diff. in $\mathbb{R}$.

b) $g(0) = e^0 - 1 = 0$, so $g(r) = r$

c) $g'(x) = 2e^{2x}$, $g'(0) = 2e^0 = 2 \ge 1$, so $|g'(r)| \ge 1$

* The theorem isn't satisfied so we can't make a conclusion

EX. 1.2.14

a) $x \to \frac{1}{2}x + \frac{1}{x}$, cont diff from $(0,\infty)$

$\sqrt{2}$ $\left(g(\sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}}\right) \Rightarrow 2 = \frac{2}{2} + \frac{\sqrt{2}}{\sqrt{2}}$ ✓, $g(r) = r$

$g'(x) = \frac{1}{2} - x^{-2}$, $g'(\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = \boxed{0} < 1$, so $|g'(r)| < 1$

$\boxed{\text{* Locally convergent to } \sqrt{2}}$

b) $x \to \frac{2}{3}x + \frac{2}{3x}$, cont diff from $(0,\infty)$

$g(\sqrt{2}) = \frac{2\sqrt{2}}{3} + \frac{2}{3\sqrt{2}} = \frac{2(2)}{3\sqrt{2}} + \frac{2}{3\sqrt{2}} = \frac{6}{3\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}}$, $g(r)=r$

$\frac{2}{3}x^{-1}$

$g'(x) = \frac{2}{3} - \frac{2}{3}x^{-2}$, $g'(\sqrt{2}) = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3} < 1$, so $|g'(r)| < 1$

$\boxed{\text{* Locally convergent to } \sqrt{2}}$

$\frac{1}{2}x^{-1}$

c) $x \to \frac{3}{4}x + \frac{1}{2x}$, cont diff from $(0,\infty)$

$g(\sqrt{2}) = \frac{3\sqrt{2}}{4} + \frac{1}{2\sqrt{2}} = \frac{3\sqrt{2}}{4} + \frac{\sqrt{2}}{4} = \frac{4\sqrt{2}}{4}$, $g(r) = r$

$g'(x) = \frac{3}{4} - \frac{1}{2}x^{-2}$, $g'(\sqrt{2}) = \frac{3}{4} - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2} < 1$, so $|g'(r)| < 1$

$\boxed{\text{* Locally convergent to } \sqrt{2}}$

All 3 converge to fixed point $\sqrt{2}$ the fastest will be a then b, then c. a will converge with superlinear convergence b and c will converge with linear convergence but $g_2'(\sqrt{2}) < g_3'(\sqrt{2})$ so b will converge faster than c.

EX.1.3.8

$f(x) = x^n - ax^{n-1}$, $g(x) = x^n$, $f_\varepsilon(x) = (1+\varepsilon)x^n - ax^{n-1}$

a) $x^n - ax^{n-1} = 0 \Rightarrow x^{n-1}(x-a)$ $\quad r = 0, a$

$\Delta r \approx -\dfrac{\varepsilon g(r)}{f'(r)}$ $\quad f'(x) = nx^{n-1} - a(n-1)x^{n-2}$

$f'(a) = na^{n-1} - a(n-1)a^{n-2} = na^{n-1} - (n-1)a^{n-1} = a^{n-1}$

$g(a) = a^n$

$\Delta r \approx -\dfrac{\varepsilon a^n}{a^{n-1}} = \boxed{-\varepsilon a}$ $\quad r + \Delta r = a - \varepsilon a = \boxed{a(1-\varepsilon)}$

b) $f_\varepsilon(x) = (1+\varepsilon)x^n - ax^{n-1}$

$(1+\varepsilon)x^n - ax^{n-1} \Rightarrow x^{n-1}((1+\varepsilon)x - a)) = 0$

$1+\varepsilon x = a \Rightarrow \boxed{x \approx \dfrac{a}{1+\varepsilon}}$

$a(1-\varepsilon)$ vs $\dfrac{a}{1+\varepsilon}$ aren't identical but for a small ε we can expect near identical answers

CP.1.3.2

$\sin(x^3) - x^3$

a) $\sin(x^3) - x^3 = \left(x^3 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{15} \dots\right) - x^3 = -\frac{1}{3!}x^9 + \frac{1}{5!}x^{15}$

lowest degree

*multiplicity is $\boxed{\text{nine}}$ as expanding by hand 9 times would result in a term that isn't 0

EX.1.4.2

a) $x^3 + x^2 - 1 = f(x) = 0$, $f'(x) = 3x^2 + 2x$ $\quad x_0 = 1$

$x_1 = 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$

$x_2 = \frac{4}{5} - \frac{0.152}{3.52} = \boxed{0.7568}$

b) $x^2 + \frac{1}{x+1} - 3x = f(x) = 0$, $f'(x) = 2x - \frac{1}{(x+1)^2} - 3$ $\quad x_0 = 1$

$x_1 = 1 - \dfrac{-\frac{3}{2}}{-\frac{5}{4}} = \boxed{-0.2}$

$x_2 = -0.2 - \dfrac{1.89}{-4.9625} = \boxed{0.1809}$

c) $f(x) = 5x - 10 = 0$, $f'(x) = 5$, $x_0 = 1$

$x_1 = 1 - \frac{-5}{5} = \boxed{2}$

$x_2 = 2 - \frac{0}{5} = \boxed{2}$

EX. 1.4.4

a) $f(x) = 32x^3 - 32x^2 - 6x + 9 = 0$, $r = -1/2$, $r = 3/4$

$f(-1/2) = 0$ $f(3/4) = 0$

$f'(x) = 96x^2 - 64x - 6 = 0$

$f'(-1/2) = 50$ $f'(3/4) = 0$

$f''(x) = 192x - 64 = 0$

$f''(-1/2) = -160$ $f''(3/4) = 80$

$r = -1/2$: According to theorem 1.11, $f(-1/2) = 0$ and $f'(-1/2) = 50 \Rightarrow$ convergence to $r = -1/2$ is quadratic where $e_{i+1} \approx e_i^2 \left|\frac{-160}{2(50)}\right| = 1.6 e_i^2$

$r = 3/4$: According to theorem 1.12, $f(3/4) = 0$ and $f'(3/4) = 0$, $f''(3/4) = -160 \Rightarrow$ convergence to $r = 3/4$ is linear where $e_{i+1} \approx \frac{1}{2} e_i$

b) $f(x) = x^3 - x^2 - 5x - 3 = 0$, $f'(x) = 3x^2 - 2x - 5 = 0$, $f''(x) = 6x - 2$

$r = -1$, $r = 3$

$f(-1) = 0$, $f'(-1) = 0$, $f''(-1) = -8$

$f(3) = 0$, $f'(3) = 16$, $f''(3) = 16$

$r = -1$: According to theorem 1.12, $f(-1) = f'(-1) = 0$ and $f''(-1) = -8 \Rightarrow$ convergence to $r = -1$ is linear where $e_{i+1} \approx \frac{1}{2} e_i$

$r = 3$: According to theorem 1.11, $f(3) = 0$ and $f'(3) = 16 \Rightarrow$ convergence to $r = 3$ is quadratic where $e_{i+1} \approx e_i^2 \left|\frac{16}{2(16)}\right| = e_i^2/2$

$f(x) = 94\cos^3 x - 24\cos x + 177\sin^2 x - 108\sin^4 x - 72\cos^3 x \sin^2 x - 65$

$f'(x) = -282\cos^2 x \sin x + 24\sin x + 177\sin x\cos x - 108\sin^3 x\cos x$