**2.39b:** (a) Prove that every convergent sequence is bounded. That is, if $(p_n)$ converges in the metric space $M$, prove that there is some neighborhood $M_r(q)$ containing the set $\{p_n : n \in \mathbb{N}\}$.

(b) Is the same true for a Cauchy sequence in an incomplete metric space?

**Solution:** The same is not true for a Cauchy sequence in an incomplete metric space.

Let $(x_n)$ be a Cauchy sequence in an incomplete metric space. Therefore, it follows that for all $\epsilon > 0$ there exists N such that $k, n \geq N \implies d(x_k, x_n) < \epsilon$. So then, choose $\epsilon = 1$, and therefore there exists N such that $k, n \geq N \implies d(x_k, x_n) < \epsilon$. So then, the sequence past N is bounded by 1. Because the first N-1 terms are finite, the sequence to N is trivially bounded. So then, if the sequence up to N is bounded, and the sequence past N is bounded, $x_n$ must be bounded.

**2.22:** If every closed and bounded subset of a metric space $M$ is compact, does it follow that $M$ is complete? (Proof or counterexample.)

**Solution:** By way of contradiction, suppose M is not complete. Thus, there exists a Cauchy sequence $p_n \to p \notin M$. Because $(p_n)$ is Cauchy, it is also bounded, and therefore there exists closed $S \subset M$ where $(p_n) \in S$. From the question, S is then compact. From compactness, there exists $(p_{n_k}) \to m \in S$. Because $(p_n)$ is Cauchy, $(p_{n_k})$ is Cauchy. So, for all $\epsilon > 0$ there exists $N_1$ such that for all $k, n \geq N_1$ we have $d(p_k, p_n) < \frac{\epsilon}{2}$. Since $(p_{n_k})$ converges to $m \in M$, for all $\epsilon > 0$ there exists $N_2$ such that for all $k \geq N_2$, we have $d(p_{n_k}, m) < \frac{\epsilon}{2}$. Choose N $= max(N_1, N_2)$, then for all $k, n \geq N$, $d(p_n, m) \leq d(p_{n_k}, p_n) + d(p_{n_k}, m) < \epsilon$. Because $p_n \to m, p = m, p \in S$ which is a contradiction.

**2.43:** Assume that the Cartesian product of two nonempty sets $A \subset M$ and $B \subset N$ is compact in $M \times N$. Prove that $A$ and $B$ are compact.

**Solution:** Suppose the Cartesian product of two nonempty sets $A \subset M$ and $B \subset N$ is compact in $M \times N$. Then for all sequences $(x_n, y_n) \subset A \times B$ there exists a subsequence such that $(x_{n_k}, y_{n_k}) \to (x, y) \in A \times B$. Without loss of generality, we'll show A is compact. Suppose that we have sequence $x_n, y_0$ where $y_0$ is fixed. Therefore, there exists $(x_{n_k}, y_0) \to (x, y) \in A \times B$. Moreover, because $y = y_0$, we conclude $x_{n_k} \to x \in A$. The same holds for B, and therefore A and B are each compact.

**2.46:** Assume that $A$ and $B$ are compact, disjoint, nonempty subsets of $M$. Prove that there are $a_0 \in A$ and $b_0 \in B$ such that for all $a \in A$ and $b \in B$ we have

$$d(a_0, b_0) \leq d(a, b).$$

[The points $a_0$, $b_0$ are closest together.]

**Solution:** Suppose that $A$ and $B$ are compact, disjoint, nonempty subsets of $M$. Because A, B are compact, it follows then that $A \times B$ is also compact from Theorem 28. Moreover, since a real-valued function defined on a compact set attains a minimum, we can construct a minimizing sequence $(a_n, b_n)$ in $A \times B$ where $d(a_n, b_n) \to d_0$. It follows from compactness, $(a_n, b_n)$ has a convergent subsequence $(a_{n_k}, b_{n_k})$ where $a_{n_k} \to a_o \in A, b_{n_k} \to b_0 \in B$. Thus, $\lim_{k \to \infty}(a_{n_k}, b_{n_k}) = (a_0, b_0) = d_0$. Therefore, $d(a_0, b_0) \leq d(a, b)$ for all $a \in A, b \in B$.