**2.60:**

(a) Prove that a continuous function $f : M \to \mathbb{R}$, all of whose values are integers, is constant provided that $M$ is connected.

(b) What if all the values are irrational?

**Solution:**

(a) Assume $f : M \to \mathbb{R}$ is continuous, all of whose values are integers, where $M$ is connected. Then it follows, the intermediate value property must hold. But, if f wasn't constant, there would exist unique $a, b \in \mathbb{Z}$, and by connectedness of $M$, it must take every value between a and b, which would include non-integers which isn't possible given our assumptions. Therefore, f must be constant.

(b) Using the same logic from above, we can use the intermediate value property to show f must be constant. If f wasn't constant, there would exist unique irrational $a, b$, and by connectedness of $M$, it must take every value between a and b, which would include a rational number which isn't possible given our assumptions. Therefore, f must be constant.

**2.88:** Give a direct proof that a closed subset $A$ of a covering compact set $K$ is covering compact.

*Hint:* If $\mathcal{U}$ is an open covering of $A$, adjoin the set $W = M \setminus A$ to $\mathcal{U}$. Is $\mathcal{U} \cup \{W\}$ an open covering of $K$? If so, what does this imply?

**Solution:** Suppose A is a closed subset of K, where K is covering compact. Let $\mathcal{U}$ be an open covering of A. Moreover, let $W = K \setminus A$ which by must be open in K because its complement is A, which is closed in K. Moreover, $\mathcal{U} \cup \{W\}$ is an open covering of K because $\mathcal{U}$ is an open covering of K and W is open in K. Because K is covering compact, every open covering in K reduces to a finite subcovering. So then, this implies there is a finite subcovering for the open covering $\mathcal{U} \cup \{W\}$. So then, if we reconstruct this finite subcovering to only include elements in A, we have constructed a finite subcovering of A for $\mathcal{U}$, proving A is covering compact.

**2.91:** Suppose that $M$ is covering compact and that $f : M \to N$ is continuous. Use the Lebesgue number lemma to prove that $f$ is uniformly continuous.

*Hint:* Consider the covering of $N$ by $\epsilon/2$-neighborhoods $\{N_{\epsilon/2}(q) : q \in N\}$ and its preimage in $M$, $\{f^{-1}(N_{\epsilon/2}(q)) : q \in N\}$.

**Solution:** Suppose that $M$ is covering compact and that $f : M \to N$ is continuous. Consider the covering of $N$ by $\epsilon/2$-neighborhoods $\{N_{\epsilon/2}(q) : q \in N\}$, which forms an open covering of N because each set is an open ball, the collection of which contains every point in N. Moreover, because f is continuous, the preimage of $N$ by $\epsilon/2$-neighborhoods must be open in $M$, and is more so an open covering of $M$ since $M$ is covering compact. This means there exists a finite subcovering of M. By the Lebesgue number lemma, we know that every open covering of our sequentially compact set, M, has a Lebesgue number $\lambda > 0$. Therefore, for any $\epsilon > 0$, there exists $\delta$, namely our Lebesgue number we have guaranteed, such that for all $x, y \in M$ with $d_M(x, y) < \delta$, $d_N(f(x), f(y)) < \frac{\epsilon}{2}$, which we know holds from the Lebesgue number where any $\delta$ ball in M is mapped in some $\epsilon/2$ cover in N, proving uniform continuity.