3.5: Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and for all $x \neq 0$, $f'(x)$ exists. If

$$\lim_{x \to 0} f'(x) = L$$

exists, does it follow that $f'(0)$ exists? Prove or disprove.

Solution:

Prove. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and for all $x \neq 0$, $f'(x)$ exists. By way of contradiction suppose

$$\lim_{x \to 0} f'(x) = L$$

exists and $f'(0)$ does not exist. So there would be a few cases where $f'(0)$ doesn't exist, but we know

$$\lim_{x^+ \to 0} f'(x) = \lim_{x^- \to 0} f'(x)$$

Moreover, the limit approaching x exists and is equal to L. This would imply that there is a discontinuity at $x = 0$. However, this is a contradiction because f is continuous for all x. In any case, the reason for $f'(0)$ not existing would be a contradiction.

3.9:

Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

(a) If there is an $L < 1$ such that for each $x \in \mathbb{R}$, we have $f'(x) < L$, prove that there exists a unique point x such that $f(x) = x$. [That is, x is a fixed point of f.]

(b) Show by example that part (a) fails if $L = 1$.

Solution:

(a) We'll first show that there must exist an x such that $f(x) = x$ and then by contradiction we'll prove x must be unique.

Let $f(x), L < 1$ be given such that for each $x \in \mathbb{R}$, we have $f'(x) < L$. First, let $g(x) = f(x) - x$. Moreover, $g'(x) = f'(x) - 1$, and from our assumption given that $f'(x) < L < 1$ for all x, we know that $g'(x)$ is negative for all x. Moreover, because L is fixed, $g'(x)$ can't approach 0, which means $g(x)$ can't approach a constant value (in which case $f(x) - x$ could stay strictly above/below 0), from which we can conclude $\lim_{x \to -\infty} f(x) - x = \infty$ and $\lim_{x \to \infty} f(x) - x = -\infty$. So then, there exists an $a, b \in \mathbb{R}$ such that $f(a) - a < 0$ and $f(b) - b > 0$. From the intermediate value property, we can conclude that there exists an x such that $f(x) - x = 0$ because f is differentiable.

Now we'll show that the point x for which $f(x) = x$ must be unique. By way of contradiction suppose there exists another point y for which $f(y) = y$. So then, because f is differentiable, if we apply the mean value theorem on the interval (x, y), there must exist c such that $f(x) - f(y) = f'(c)(x - y)$ which is equivalent to $\frac{f(x)-f(y)}{x-y} = f'(c)$ and so $\frac{x-y}{x-y} = f'(c) = 1$. However, this is a contradiction because $f'(c) < 1$ from our assumption. So then, x must be unique.

(b) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + \frac{1}{e^x}$ and $g(x) = f(x) - x = \frac{1}{e^x}$. Thus, $g'(x) = -\frac{1}{e^x}$, satisfying f is differentiable and for each $x \in \mathbb{R}, f'(x) < L = 1$. Moreover, for each $x \in \mathbb{R}, g(x) > 0$, and thus there is in fact no point at all for which $f(x) = x$. This proves the necessity of $f'(x) < L < 1$, a fixed point, compared to $f'(x) < L = 1$.

3.10: Concoct a function $f : \mathbb{R} \to \mathbb{R}$ with a discontinuity of the second kind at $x = 0$ such that f does not have the intermediate value property there. Infer that it is incorrect to assert that functions without jumps are Darboux continuous.

Solution:

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$. Then it follows, as $x \to 0^-$, $f(x) \to -\infty$ and as $x \to 0^+$, $f(x) \to \infty$, and thus neither the right-hand nor left-hand limits exist, affirming the function has a discontinuity of the second kind at x=0. Moreover, if we take a small $\epsilon > 0$, and consider the interval [-ϵ,ϵ], our function takes on values between $-\infty$ and ∞. For the intermediate value property to hold, there must exist an x in this epsilon neighborhood such that $f(x) = 0$ (because $0 \in (-\infty, \infty)$), however such x doesn't exist and thus the intermediate value property doesn't hold at x=0.