Andrew

Homework 8:

3.28

Suppose $Z \subset \mathbb{R}$.

1/2) ($\Rightarrow$) Suppose Z is a zero set. Then by definition, for all

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$\varepsilon > 0$ there is a countable covering of Z by open intervals (a_i, b_i) such that $\sum b_i - a_i \leq \varepsilon$. Equivalentally, we can have $[a_i, b_i]$, ~~where δ is tiny such that the sum across~~ all ~~intervals is less than~~ $\varepsilon/2$. ~~So then~~, because adding endpoints won't change the length of the interval, and so $\sum b_i - a_i < \varepsilon$ as desired.

($\Leftarrow$) Suppose for all $\varepsilon > 0$ there is a countable covering of Z by closed intervals $[a_i, b_i]$ with total length $\sum b_i - a_i < \varepsilon$. Following the same approach we can have a countable covering of Z by open intervals (a_i, b_i) without changing the interval length, and thus $\sum b_i - a_i < \varepsilon$.

But they cover less maybe not all pts

2/3) ($\Rightarrow$) Suppose for all $\varepsilon > 0$ there is a countable covering of Z by closed intervals $[a_i, b_i]$ with total length $\sum b_i - a_i < \varepsilon$. Then, choose sets $S_i = [a_i, b_i]$. It follows, $\sum b_i - a_i = \sum \mathrm{diam}(S_i) < \varepsilon$.

($\Leftarrow$) Suppose for all $\varepsilon > 0$, there is a countable covering of Z by sets S_i with total diameter $\sum \mathrm{diam}(S_i) < \varepsilon$. Let $[a_i, b_i]$ represent the endpoints of S_i, and thus $\sum \mathrm{diam}(S_i) = \sum b_i - a_i < \varepsilon$

This doesn't have a well-defined meaning.

1/3) ($\Leftrightarrow$) Because 1-2 are equivalent and 2-3 are equivalent, 1-3 are equivalent.

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3.40 Let $g(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$. Then it follows, $G(x) = \begin{cases} C_1, & x \le 0 \\ x + C_2, & x > 0 \end{cases}$.
Because $g(0) = 0$, it must follow G is continuous at 0, so $\lim_{x \to 0^+} G(x) = \lim_{x \to 0^-} G(x)$, and thus $C_1 = C_2$. However, $\lim_{h \to 0^+} \frac{G(0+h) - G(0)}{h} = 1$ and $\lim_{h \to 0^-} \frac{G(0+h) - G(0)}{h} = 0$, thus G doesn't have a derivative at 0.
Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin(\frac{\pi}{x}), & x > 0 \end{cases}$. Also, suppose $F(x) = \begin{cases} 0 \\ \int \sin(\frac{\pi}{x}), & x > 0 \end{cases}$.
Now consider $\lim_{x \to 0^+} \int_0^x \sin(\frac{\pi}{x}) dx$ Because $\sin(\frac{\pi}{x})$ oscillates more and more near 0 while bounded between -1 and 1, it follows the areas of the curve (above and below) cancel out closer and closer to 0. Therefore $\lim_{x \to 0^+} \int_0^x \sin(\frac{\pi}{x}) dx = 0$, and is thus continuous everywhere, where $F(x)$ is antiderivative

Much more precise estimates needed

3.47 1) Let ϕ be continuous and BWOC suppose there exists point x_0 such that x_0 is contained in the discontinuity set of $\phi \circ f$, and x isn't contained in the discontinuity set of f. However, it must follow $f(x_0)$ is continuous, and because ϕ is continuous, $\phi(f(x_0))$ must be continuous. But, this contradicts the assumption $\phi(f(x_0))$ is discontinuous and contained in the discontinuity set.
2) Suppose $\phi(x) = 0$ and $f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$. Then, $f(x)$'s discontinuity set would be $\{0\}$. However, $\phi(f(0)) = \phi(0) = 0$. Moreover, ϕ is continuous everywhere because it's constant. Thus, the discontinuity set for $\phi(f(x)) = \emptyset$ and $f(x) = \{0\}$, proving non-equality

3) A sufficient condition would be for ϕ to be injective. Suppose ϕ is injective. Then, if for some x_0, $f(x_0)$ is discontinuous, $\phi(f(x_0))$ is also discontinuous from above. Now, if some x_0 is ~~dis~~continuous for $f(x_0)$, where $\phi(f(x_0))$ is injective, it must be mapped to unique $\phi(f(x_0))$ where ϕ is continuous everywhere
4) Not necessarily. Consider $f(x) = \frac{1}{x}$. Thus, the discontinuity set is $\{0\}$. Now consider $\phi(f(x)) = (f(x))^2$. Thus the discontinuity set is also $\{0\}$. However, ϕ isn't injective because $\forall x \neq 0$, $(-f(x))^2 = (f(x))^2$.

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** Original HW was 2.7/3

Homework 10:

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4.4 a) Suppose for all $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta$, $|f_n(x) - f_n(y)| < \varepsilon/3$. Also, $f_n \rightrightarrows f$, so for all $\varepsilon > 0$, there exists N such that for all $n \geq N$ and for all $x \in \mathbb{R}$, $|f_n(x) - f(x)| < \varepsilon/3$, and the same for $y \in \mathbb{R}$. So then, for all $x, y \in \mathbb{R}$ with $|x-y| < \delta$, $|f(x) - f(y)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon$.

b) The same proof above would hold, but instead of having $f_n : \mathbb{R} \to \mathbb{R}$ we can let $f_n : M \to N$ and follow the same logic (uniform continuity follows same rules in different metric spaces

4.7 Consider a sequence of functions f_n in C^0. The graph G_n of f_n is a compact subset of $\mathbb{R}^2$.

a) ($\Rightarrow$) Suppose the sequence (G_n) in $\mathcal{K}(\mathbb{R}^2)$ converges to the graph of a function $f \in C^0$. So then, $G_n \to G$ in $\mathcal{K}(\mathbb{R}^2)$. Because the metric is measuring maximum vertical distance between f_n and f and because $(G_n) \to G$, the maximum distance vertically is $< \varepsilon$, and so using the same ε, N from that convergence, we can say $|f_n(x) - f(x)|$ for all x.

($\Leftarrow$) Suppose (f_n) converges uniformly as $n \to \infty$. Also, consider the sequence (G_n) in $\mathcal{K}(\mathbb{R}^2)$. Then it follows, for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, for all x, $|f_n(x) - f(x)| < \varepsilon$. So then, it directly follows that $\sup |f_n(x) - f(x)| < \varepsilon$ because the largest distance is still $< \varepsilon$. So then $f_n \to f$ and so it follows the corresponding graphs $G_n = \{(f_n(x), x) : x\} \to G = \{(f(x), x) : x$

b) Let $\varepsilon > 0$. Let δ be chosen from uniform convergence above. Then from above, there exists some N such that for all $n \geq N$, for all x, y, $|f_n(x) - f(x)| < \varepsilon/3$ and $|f_n(y) - f(y)| < \varepsilon/3$. Also, because f is uniformly convergent, $|f(x) - f(y)| < \varepsilon/3$ and so $|f_n(x) - f_n(y)| < \varepsilon$ by triangle inequality.

4.9 If $f: \mathbb{R} \to \mathbb{R}$ is continuous and the sequence $f_n(x) = f(nx)$ is equicontinuous, then f must be constant. BWOC suppose the above is true and f isn't constant. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all n, and all x, y, if $|x-y| < \delta$ then $|f(nx) - f(ny)| < \varepsilon$. But, consider arbitrarily large n. Then, let $x - y < \frac{\delta}{n}$. But, $n|f(x) - f(y)| < \varepsilon$ would not hold, so f must be constant.