Applied Analysis: 1/21/25

Completeness of $\mathbb{R}$

$x < y$

(i) trichotomy: for any two $x, y \in \mathbb{R}$ either $x<y$ or $y<x$, or $x=z$

(ii) transitivity: if $x<y$ and $y>z$ then $x<z$

$S \subset \mathbb{R}$

Def: Upper bound of $S \subset \mathbb{R}$ is any $U \in \mathbb{R}$ s.t. $x \leq U \ \forall x \in S$

Least upperbound: $w \in \mathbb{R}$ s.t. w is upper bound of S for any upper bound u, $w \leq u$

Ex: $[0,1]$ $\sup[0,1] = \sup[0,1) = 1$

Let $x \in [0,1)$ and let u be an upper bound of $[0,1)$.

We will show $u \geq 1$.

BWOC if $u<1$ then either $u<0$ where $u<x \ \forall x \in [0,1)$ not upper bound or $\frac{1+u}{2} \in [0,1)$ and $u < \frac{1+u}{2}$ not upper bound.

greatest upper (supremum), lowest upper (infimum)

Least upper bound property: Any set $S \subset \mathbb{R}$ with an upper bound has the least upper bound in $\mathbb{R}$.

Ex: $\mathbb{Q}$ does not have least upper bound property

$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$

Triangle inequality

$|x+y| \leq |x| + |y|$

$\forall x, y \in \mathbb{R}$

$|x-z| \leq |x-y| + |y-z|$

$\big||x-y| - |y-z|\big| \leq |x-z|$ (reverse triangular inequality)

$|x-z| - |y-z| \leq |x-y|$

$y \leftrightarrow z$

$|x-y| - |z-y| \leq |x-z|$

$S = \mathbb{N} \cup \{\infty\}$, $\forall x \in \mathbb{N}$ $x < \infty$

- Prove that every subset of S has an upper bound.
- Does every subset of S have a least upper bound?

Let $X \subset S$.

- Then the upper bound is ∞ because $\forall y \in X$, $y \leq \infty$.

- TCS 1) $\infty \in T$
  2) $\infty \notin T$

  T is infinite — T is finite
  ↳ take max

Let T be an infinite set.
BWOC suppose $\exists x \in \mathbb{R}$ s.t. it's a least upper bound.
It follows $\exists y \in T$ s.t. $y > x$.

Suppose that $\infty \in T$ then ∞ is $\sup T$ because $\forall x \in T$, $x \leq \infty$ and $\forall x \neq \infty \Rightarrow x < \infty$. If on the contrary $\infty \notin T$ then there are two alternatives, T is finite or T is infinite. If T is finite then max T exists and is the least upper bound and if T is infinite then ∞ is the limit.
because for any finite x the set $\{y \in S \mid y \leq x\}$ is finite and therefore finite x cannot be an upper bound of an infinite set.

1) <u>Def:</u> A sequence of $\mathbb{R}$ is a function $f: \mathbb{N} \to \mathbb{R}$ $\quad a_n = f(n)$
2) a is called limit of sequence (a_n) if $\forall \varepsilon > 0$ $\quad (a_n)_{n=1}^{\infty}$ vs
   $\exists N$ s.t. $\forall n > N$, $|a_n - a| < \varepsilon$ $\quad \{a_n\}$ vs
3) If sequence has a limit we call it convergent

<u>Prop</u> Every convergent sequence has unique limit.
<u>Proof</u> Assume that a and b are limits of (a_n) then $\forall \varepsilon > 0$, $\exists N_1$ s.t. $\forall n > N_1$, $|a_{n_1} - a| < \varepsilon$ and $\forall \varepsilon > 0$ $\exists N_2$ s.t. $\forall n_2 > N_2$, $|a_{n_2} - a| < \varepsilon$

SW $|a-b| = (a_n - a) + (a_n - b) \leq |a - a_n| + |b - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Let $n = \max(N_1, N_2)$

!!! <u>Next Class: Squeeze theorem</u>
If $a_n \leq b_n$ $\forall n$, $b_n \leq c_n$ $\forall n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A$ and both exist then $\lim_{n\to\infty} b_n$ exists and equals A.

Applied Analysis: 1/23/25

Def: Given sequence (a_n) and increasing function $p: \mathbb{N} \to \mathbb{N}$, sequence $b_n = a_{p(n)} = a_{n_k}$ is called a subsequence.

Prop: If a sequence converges then every subsequence converges to the same limit.

Proof Let (a_n) be a sequence s.t. $\lim_{n\to\infty} a_n = A$ and (a_{n_k}) is a subsequence of (a_n). Then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall k > N$ $|a_k - A| < \varepsilon$ but $n_k \geq k > N$ and therefore $|a_{n_k} - A| < \varepsilon$.

1) Def: Sequence (a_n) is called monotone increasing if $\forall n_1 > n_2$, $a_{n_1} \geq a_{n_2}$.
2) monotone decreasing if $\forall n_1 > n_2$, $a_{n_1} \leq a_{n_2}$.
3) monotone if it is either.

Def: (a_n) is called bounded above if set $\{a_n\}$ is bounded above.

Prop: Increasing sequence bounded above converges.

Proof Let (a_n) be such sequence and $A = \sup\{a_n\}$. We will show that $\lim_{n\to\infty} a_n = A$. Because $A = \sup\{a_n\}$, $A - a_n \geq 0$. ~~BWOC assume $\exists \varepsilon > 0$ s.t. $\forall N$ and $\exists n > N$ s.t. $|A - a_n| > \varepsilon$ and therefore $a_n < A - \varepsilon$.~~ Let $\varepsilon > 0$ be given. ~~Because~~ $A = \sup\{a_n\}$ $\exists n$ s.t. $a_n > A - \varepsilon$ and therefore $\forall n > N$, $a_n \geq a_N > A - \varepsilon$ and $-\varepsilon < 0 \leq A - a_n < \varepsilon$.

Def: For a sequence (a_n) define $s_n = \sup\{a_k \mid k \geq n\}$. If $\lim_{n\to\infty} s_n$ exists it's called $\lim_{n\to\infty} \sup a_n$ (limit supremum)
$d_n = \inf\{a_k \mid k \geq n\}$, $\lim_{n\to\infty} = \lim_{n\to\infty} \inf a_n$ (limit infimum)

Ex $x_n = \begin{cases} \frac{n+1}{n}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$

$\sup\{x_k \mid k \geq n\}$
$= \begin{cases} \frac{n+1}{n}, & n \text{ is odd} \\ \frac{n+2}{n+1}, & n \text{ is even} \end{cases}$

$\inf\{x_k \mid k \geq n\} = 0$

Ex: (x_n) and (y_n) are bounded and $x_n \le y_n$. Show that $\lim_{n\to\infty} \sup x_n \le \lim_{n\to\infty} \sup y_n$ and $\lim_{n\to\infty} \inf x_n \le \lim_{n\to\infty} \inf y_n$

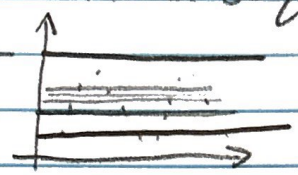
Proof $a_n = \sup\{x_k \mid k \ge n\}$, $b_n = \sup\{y_k \mid k \ge n\}$

$a_n \le b_n$, $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$

Theorem (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence.

Proof



Let (a_n) be a sequence $l \le a_n \le u$. $(l_n), (u_n), k_n : (a_{k_n})$

$l_1 = l$ $\quad \frac{l_1 + u_1}{2}$, $\forall k > 1$ define $l_k = \frac{l_{k-1} + u_{k-1}}{2}$ and $u_k = u_{k-1}$

$u_1 = u$

if there are infinitely many a_n in the interval $\left[\frac{l_{k-1} + u_{k-1}}{2}, u_{k-1}\right]$ and $l_k = l_{k-1}$ and $u_k = \frac{l_{k-1} + u_{k-1}}{2}$ otherwise n_k = smallest n for which $n > n_{k-1}$ and a_{n_k} is on $[l_k, u_k]$

1) $l_k \le a_{n_k} \le u_k$

2) l_k increasing

u_k decreasing

$l_k < u_1$ and $u_k < l_1$

(l_k) and (u_k) converge

3) $|u_k - l_k| = \frac{|u_1 - l_1|}{2^{k-1}}$

$\lim_{k\to\infty} |u_k - l_k| = 0$

$\lim_{k\to\infty} u_k = \lim_{k\to\infty} l_k$

By squeeze, $\lim_{k\to\infty} a_{n_k}$ exists.

Def Sequence (x_n) is called Cauchy if $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n, k > N$ $|x_n - x_k| < \varepsilon$

Theorem: Cauchy sequence equivalent to convergent sequence.

Applied Analysis

1/28/25

(x_n)

$x_n = \frac{(n-1)}{n}(-1)^n \quad l_n = \inf\{x_k \mid k \geq n\} = -1$

Metric Space

Def: Metric space is a set M and a function $d_M : M \times M \to \mathbb{R}$ with properties

1) $d(x,x) = 0$, $d(x,y) > 0$ if $x \neq y$ (positive)
2) $d(x,y) = d(y,x)$ (symmetric)
3) $d(x,y) \leq d(x,z) + d(z,y)$ (Δ inequality)

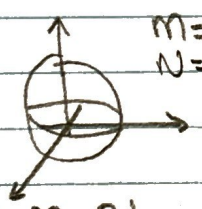
Examples:

1) $M = \mathbb{R}$ $\quad d(x,y) = |x-y|$
2) $M = \mathbb{R}^n$ $\quad d(x,y) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$ (Euclidean metric) $p=2$
3) $M = \mathbb{R}^n$ $\quad d(x,y) = \left(|x_1-y_1|^p + |x_2-y_2|^p + \dots + |x_n-y_n|^p\right)^{1/p}$, $p > 0$ (p-metric)

   $p = 1$ $\quad d(x,y) = |x_1-y_1| + \dots + |x_n-y_n|$

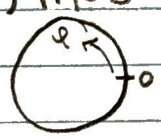
   $p \to \infty$ $\quad d(x,y) = \max(|x_1-y_1|, \dots, |x_n-y_n|)$
4) Any M $\quad d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ (discrete metric)
5) $N \subset M$ $\quad d_N(x,y) = d_M(x,y)$ (induced metric)

   $x, y \in N$



$M = \mathbb{R}^3$
$N = S^2$
$x^2 + y^2 + z^2 = 1$

6) $M = S^1$



$\varphi \in [0, 2\pi]$
φ_1, φ_2

$\min(|\varphi_2 - \varphi_1|, |2\pi - \varphi_2 + \varphi_1|, |2\pi - \varphi_1 + \varphi_2|)$

$|\cos(\varphi_2 - \varphi_1)|$

7) $M = \mathbb{R}$ $\quad d(x,y) = \frac{|x-y|}{1+|x-y|}$

$\frac{|x-x|}{1+|x-x|} = \frac{0}{1} = 0$ ✓

$\frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|}$ ✓

$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z| + |x-z||z-y| + |z-y| + |z-y||x-z|}{1 + |x-z| + |z-y| + |x-z||z-y|}$ $\quad \frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|y-z|}{1+|y-z|}$

$\frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} = 1 + \frac{1}{1+|x-z|} + 1 - \frac{1}{1+|z-y|} = 2 - \frac{1}{1+|x-z|} + \frac{1}{1+|z-y|}$

$d(x,y) = 1 - \frac{1}{1+|x-y|}$ $\quad \frac{1}{1+|x-y|} \geq \frac{1}{1+|x-z|} + \frac{1}{1+|y-z|} - 1$

Sequence: $p: \mathbb{N} \to M$ $\quad p(n)$ (p_n)

Subsequence: $k: \mathbb{N} \to \mathbb{N}$ strictly increasing $\quad p(k(n))$ $\quad p_{k_n}$

Def: $p \in M$ is limit of sequence (p_n) if for every $\varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $d_m(p_n, p) < \varepsilon$

Example: $M = [0,1]$, $d_m(x,y) = |x-y|$, $p_n = \frac{1}{n}$, p_n doesn't converge

$M = S^1$, $d(\varphi_1, \varphi_2) = |\cos(\varphi_1 - \varphi_2)|$, $\varphi_n = 2\pi - \frac{2\pi}{n}$ $\quad \lim_{n\to\infty} \varphi_n = 0$

Theorem: If a sequence in (M,d) has a limit, the limit is unique

Proof: Let $p_n \to p$ then for each $\varepsilon > 0$ $\exists N_1$ s.t. for $n > N_1$, $d(p_n, p) < \varepsilon/2$

If also $p_n \to p'$ " ——————— N_2 ——————, $d(p_n, p') < \varepsilon/2$

Take $N = \max(N_1, N_2)$, then $\forall n > N$, $d(p', p) \le \underbrace{d(p', p_n)}_{<\varepsilon/2} + \underbrace{d(p_n, p)}_{<\varepsilon/2} < \varepsilon$

By ε-principle $d(p', p) = 0$ and by property #1 $p' = p$.

Theorem: Any subsequence of any converging sequence converges to the same limit.

Proof: Let $p_n \to p$ in (M, d) and p_{n_k} be a subsequence.

Then $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall k > N$ $d(p_n, p) < \varepsilon$, but $n_k > k$, therefore $n_k > k > N$ and $d(p_{n_k}, p) < \varepsilon$.

Def: Function $f: N \to M$ between two metric spaces (N, d_N) and (M, d_M) is called continuous if $\forall$ convergent sequence in N, $p_n \to p$ the corresponding sequence $f(p_n) \to f(p)$ in M. (sequential continuity)

Theorem: Composition of continuous functions is continuous

Proof: Let $f: M \to N$ and $g: N \to P$ be continuous. Then for any $(x_n) \subset M$ s.t. $x_n \to x$, $f(x_n) \to f(x)$. But $y_n = f(x_n)$ is a convergent sequence in N therefore because g is continuous, $g(y_n) \to g(f(x))$ and $g(f(x_n)) = (g \circ f)(x_n)$ converges to $(g \circ f)(x)$

Theorem: Function $f: M \to N$ between metric spaces (M, d_M) and (N, d_N) is continuous $\Leftrightarrow$ for every point p in M and every $\varepsilon > 0$ there is $\delta > 0$ s.t. $\underbrace{\forall x \in M,\ d_m(x,p) < \delta}_{B_\delta(p)}$, $d_m(f(x), f(p)) < \varepsilon$

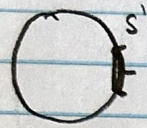
$f(x) \in B_\varepsilon(f(p))$

Applied Analysis: 1/30/25

$M = \mathbb{R}$ $\quad d(x,y) = \frac{|x-y|}{1+|x-y|} = 1 - \frac{1}{1+|x-y|} \leq$

$1 - \frac{1}{1+|x-z|+|z-y|} = \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|} = \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}$

$= d(x,z) + d(z,y)$



a) $\varphi \in [0, 2\pi) \subset \mathbb{R}$ $\quad |\varphi_1 - \varphi_2|$

b) $\min(|\varphi_1 - \varphi_2|, 2\pi - |\varphi_1 - \varphi_2|)$

c) $2\left|\frac{\varphi_1 - \varphi_2}{2}\right|$

discrete $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ $\quad f(p_n) = f(p)$

$\forall \varepsilon\ \exists N, \forall n > N, d(p_n, p) < \varepsilon$

$\exists N, n > N, p_n = p$ (only way to have convergent in discrete)

$f(\varphi): S^1 \to \mathbb{R}$, $\varphi = [0, 2\pi)$, $f(\varphi) = \frac{1}{2\pi - \varphi}$, $\varphi_n \to 2\pi - \frac{1}{n}$, $f(\varphi_n) = \frac{1}{2\pi - (2\pi - \frac{1}{n})} = n$

b, c) $\varphi_n \to 0$, $f(0) = \frac{1}{2\pi}$

a) φ_n diverges

Theorem: The function $f: M \to N$ between spaces (M, d_M) and (N, d_N) is (sequentially) continuous $\Leftrightarrow$ $\forall \varepsilon > 0$ and $p \in M$, $\exists \delta > 0$ s.t. $\forall x \in \underbrace{B_\delta(p)}_{d_M(x,p) < \delta}$, $f(x) \in \underbrace{B_\varepsilon(f(p))}_{d_N(f(x), f(p)) < \varepsilon}$

Let (ε, δ)-condition hold and $\varepsilon > 0$ given. Then, for any sequence $p_n \to p$ in (M, d_M), $\exists K$ s.t. $\forall k \geq K$, $d_M(p_k, p) < \delta$ and for any such p_k, $d_N(f(p_k), f(p) < \varepsilon$ according to (ε, δ)-condition. Therefore, $f(p_k) \to f(p)$

Ex 13

Reminder, Def: Function $f: M \to N$ is continuous if $\forall p_n \to p$, $f(p_n) \to f(p)$

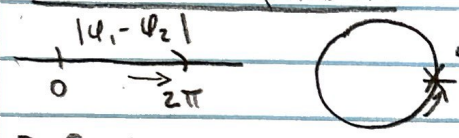
Show that the function $f: M \to N$ is continuous if $\forall p_n \to p$, sequence $f(p_n)$ converges.

Let $p_n \to p$ and sequence $f(p_n)$ converge. $\forall \varepsilon > 0\ \exists N, \forall n > N$, $d(p_n, p) < \varepsilon$. Let $q = \lim_{n\to\infty} f(p_n)$. We have to show $q = f(p)$. We have to show $q = f(p)$. We will do it by showing that for any $\varepsilon > 0$, $d(q, f(p)) < \varepsilon$.

Let $r_n = \begin{cases} p_m, & n = 2m \\ p, & n = 2m+1 \end{cases}$

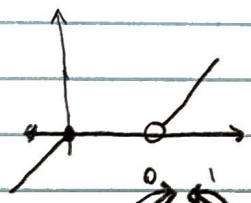
$r_n \to p$, therefore there is limit $f(r_n) \to q$, but $r_{2m+1} = p$, $f(r_{2m+1}) = f(p)$. Because all subsequences of a convergent sequence converge to some limit $q = f(p)$.

<u>Homeomorphism</u>

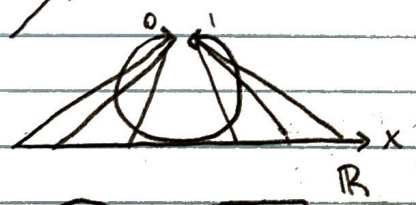


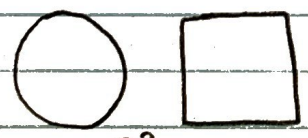
$\min(|\varphi_1 - \varphi_2|, 2\pi - |\varphi_1 - \varphi_2|)$

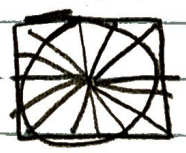
<u>Def:</u> Metric spaces M and N are homeomorphic if there is a continuous bijection $f: M \to N$ for which f^{-1} is also continuous.



$(-\infty, 0]$ $(1, \infty)$







$\mathbb{R}$ $\mathbb{Q}$ not

$\mathbb{Q} \to \mathbb{N}$

$\frac{1}{n} \to 0$

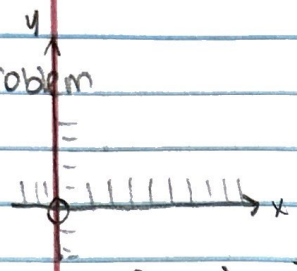
$f(\frac{1}{n})$ Large enough n

$|f(\frac{1}{n}) - m| < \varepsilon = \frac{1}{8}$

Applied Analysis:

2/4/25

* Last problem on HW2



$f(x,y)$, $\frac{1}{y+x^2}$

$P_n \to P$, $f(P_n) \to q$, $q = P$

$$r_n = \begin{cases} P_m, & n = 2m-1 \\ P, & n = 2m \end{cases}$$

$r_n \to P$, $f(r_n) \to s$, $f(r_{2m-1}) = f(P_m) \to q$, $f(r_{2n}) = f(P) \to q$

<u>Def</u>: Subset $S \subset M$ of a metric space has limit (point) $x \in M$ if $\exists$ sequence $(x_n) \subset S$ s.t. $\lim_{n\to\infty} x_n = x$

(ball with radius r)

Point $x \in S \subset M$ is interior point of S if $\exists r > 0$ s.t. $B_r(x) \subset S$

$\{x' \mid d(', x) < r\}$

1) <u>Ex</u>: $S = [0,1)$, $M = \mathbb{R}$, $x_n = 1 - \frac{1}{n} \to \boxed{1}$ → limit point

2) $M = [0,1]$, $S = \{\frac{1}{n}, n \in \mathbb{N}\}$, $x_n = \frac{1}{n} \to \boxed{0}$ * $S \cup \{0\}$ limits

<u>Def</u>: Set $S \subset M$ is closed if it contains all its limits

<u>Ex</u> $[0,1]$

Set $S \subset M$ is open if it consists only of interior points

<u>Ex</u> $[0,2\pi)$ $S = M$, both closed and open

S^1 $[0,1)$

<u>Theorem</u>: Set $S \subset M$ is open if and only if its complement $S^c = M \setminus S$ is closed.

<u>Proof</u> Let $S \subset M$ be open. By contradiction let $(P_n) \subset S^c$ s.t. $P = \lim_{n\to\infty} P_n \notin S^c$. Then $P \in S$. Therefore, there is $r > 0$ s.t. $B_r(P) \subset S$, which means for all x s.t. $d(x,P) < r$, $x \in S$. On the other hand, by definition of limit, $\exists N$ s.t. $d(P_n, P) < r$. Therefore $P_n \in S$. →←

On the other hand, let S^c be closed. If S^c is closed and $\exists x \in S$ s.t. not interior point $\forall \varepsilon > 0$, $\exists y \in S^c$ s.t. $d(x,y) < \varepsilon$. In particular, take $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$ and construct

sequence of $(y_n) \in S^c$ s.t. $d(x, y_n) < \frac{1}{n}$. Because set S^c is closed, $(y_n) \to x \in S^c$. But, $x \in S$. $\to\leftarrow$

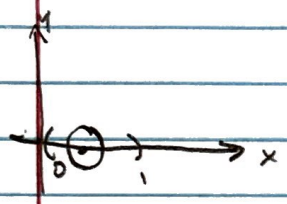
Ex: $(0,1)$ - open $\mathbb{R} \setminus (0,1) = [-\infty, 0] \cup [1, \infty)$

$[0,1]$ - closed $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$ closed open

$(0,1)$ is open in $\mathbb{R}$ but not open in $\mathbb{R}^2$

$x \in (0,1)$

$r = \frac{1}{2}\min\{x, 1-x\}$



For any $r > 0$, $y = (0, \frac{r}{2})$ $d((x,0),(x,\frac{r}{2})) = \frac{r}{2} < r$

$\mathbb{Q} \cap [a,b]$, $\mathbb{Q} \cap (a,b)$

For space M a collection of all open sets is called topology τ

1) Union of any sets $U \in \tau$ is in τ
2) Intersection of finite number of $U \in \tau$ is in τ
3) $M \in \tau$
4) $\emptyset \in \tau$

$x \in$ union

$\exists U \in \tau$ s.t. $x \in U$, $r > 0$, $B_r(x) \in U \Rightarrow B_r(x)$ is in the union

intersect: $x \in U_i \Rightarrow r_i : B_{r_i}(x) \in U_i$, take $r = \min r_i$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

not open

$M = \{a, b\}$

$\tau = \{\emptyset, M\}$

$d(x,y) = \ldots$

$d(a,a) = 0$

$d(b,b) = 0$

$d(b,a) \neq 0$

$\{a\}$ open in (M,d) $r = \frac{1}{2} d(a,b)$

Applied Analysis:

2/6/25

Closed $S \subset M$ if it contains all its limit points
Open $S \subset M$ if all points are interior $B_r(x) \in S$

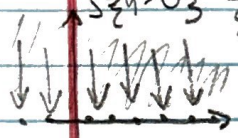
S open $\Leftrightarrow$ S^c closed

<u>Preposition:</u> $B_r(x)$ is open
By def, $y \in B_r(x)$ is the same as $d(x,y) < r$.
There exist s s.t. $d(x,y) < s < r$ $\quad t = r - s$

For any $z \in B_t(y)$ $z \in B_r(x)$
$d(z,x) \leq d(z,y) + d(y,x) < t + s = r$, $B_t(y) \subset B_r(x)$

($\bar{S}$) Closure of set S is the set of all limit points of S.
<u>Proposition:</u> $\bar{S}$ is closed

$S = \{y > 0\}$ $\quad \bar{S} = \{y \geq 0\}$



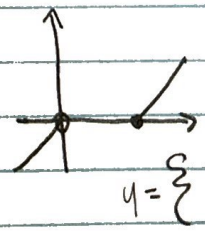
<u>Proof:</u> Let $(p_n) \subset \bar{S}$ and $p_n \to p$; therefore each n there is a sequence $(p_{n,k}) \subset S$ s.t. $\forall \varepsilon > 0$, $\exists L(n) \in \mathbb{N}$ s.t. $d(p_{n,L(n)}, p_n) < \varepsilon/2$. Take any ε and select n and $L(n)$ s.t $d(p_n, p) < \varepsilon/2$. Then, $d(p_{n,L(n)}, p) \leq d(p_{n,L(n)}, p_n) + d(p_n, p) < \varepsilon$
$p_{n,L(n)} \to p$

Set of interior points of S is open

<u>Proposition:</u> If K is closed and $S \subset K$ then $\bar{S} \subset K$.

<u>Def:</u> Open set U ~~which contains~~ with $x \in U$ is called neighborhood of x

$f: M \to N$ $\quad V = f(U)$ is open then V is open
a) is f continuous (No)
(ε, δ) $\forall \varepsilon, \exists \delta$ s.t. $f(B_\delta(p)) \subset B_\varepsilon(f(p))$



b) f is homeomorphic (No)
c) f continuous & bijection (No)
d) $g: \mathbb{R} \to \mathbb{R}$ continuous & surjection, $g(u)$ are open

Applied Analysis:

2/11/25

$f: M \to N$ not continuous, U open in M $f(U)$ is open.

<u>Theorem:</u> Function $f: M \to N$ is continuous if and only if a preimage of every open set in N is open in M.

<u>Proof:</u> Let for every open set U in N, $f^{PRE}(U)$ be open. Let p be a point in $f(M)$ and $\varepsilon > 0$ is given. Then, set $B_\varepsilon(p)$ is open and therefore its preimage is open in M. Also, $f^{PRE}(p) \in p^{PRE}(B_\varepsilon(p))$. Let $x \in f^{PRE}(p)$ and because $f^{PRE}(B_\varepsilon(p))$ is open, $\exists\, \delta$ such that $B_\delta(x) \subset f^{PRE}(B_\varepsilon(p))$. Therefore for each $y \in B_\delta(x)$ $f(y) \in B_\varepsilon(p)$ and ε-δ continuity is satisfied

Conversely, let sequential continuity hold. Then for every sequence $p_n \to p$ in M, sequence $f(p_n) \to f(p)$. Let K be a closed set in N and $f(p_n) \subset K$. Therefore $f(p) \subset K$. (p_n) $f^{PRE}(K)$ and $p \in f^{PRE}(K)$. Conclusion, preimage of every closed set in N is closed in M. Because of open-closed duality, for every open set U in N, $f^{PRE}(U^c)$ is closed and therefore $(f^{PRE}(U^c))^c$ is open.

$f: S^1 \to [0, 2\pi)$ $\qquad f(\varphi) = \varphi$

$d(\varphi_1, \varphi_2) = \min(|\varphi_1 - \varphi_2|, 2\pi - |\varphi_1 - \varphi_2|)$ $\qquad \varphi_n \to 2\pi - \frac{1}{n}$

*In S^1 it converges to 0, on $[0, 2\pi)$ it diverges

Let $U = [0, 1)$. It is open in $[0, 2\pi)$ (for any $0 < x < 1$ choose $\delta = \min(x, 1-x)$ then $B_\delta(x) \subset U$. For $x = 0$ and any $\delta < 1$ $B_\delta(x) \subset U$. In S^1, not open because for $x = 0$ and any δ, $B_\delta(0)$ contains points in $(2\pi - \delta, 2\pi]$

M, d_1, d_2, $\mathbb{R}^n$

<u>Proposition:</u> If there are positive numbers a and b such that $\forall x, y \in M$, $a d_2(x,y) \le d_1(x,y) \le b d_2(x,y)$ then spaces (M, d_1) and (M, d_2) are topologically equivalent

[illegible] are open

$\max(|x_1-y_1|, |x_2-y_2|, \dots, |x_n-y_n|) \le \sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2} \le$

$|x_1-y_1|+\dots|x_n-y_n| \le n \cdot \max(|x_1-y_1|, \dots, |x_n-y_n|)$

<u>Proof:</u> For any $z \in M$ and any r, $B_r(z) \subset U$ according to d_1 then $\forall y \in B_r(z)$ $d_1(y,z) < r$ and therefore $d_2(y,z) \le \frac{1}{a} d_1(y,z) < \frac{r}{a}$ and thus is in $B_{r/a}(z)$ according to d_2.

$d: M \times M \to \mathbb{R}$ Reverse triangle inequality:

$d(p,q)$ $\quad |d(p,q) - d(p,q')| \le d(q,q')$

$d(p,q')$

<u>Completeness:</u> Sequence $(p_n) \subset M$ in a metric space (M,d) is Cauchy if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $m,k \ge N$ $d(p_m, p_k) < \varepsilon$.

<u>Prop:</u> Every convergent sequence is Cauchy.

<u>Proof:</u> $p_n \to p$ $d(p_m, p) < \frac{\varepsilon}{2}$, $d(p_k, p) < \frac{\varepsilon}{2}$, $d(p_m, p_k) \le d(p_m, p) + d(p_k, p) < \varepsilon$.

Q $a_n = \frac{1}{2}(a_n + \frac{2}{a_n}) \to \sqrt{2}$

<u>Def:</u> Space M is complete if every Cauchy sequence in M converges

<u>Ex:</u> $(0,1)$, $x_n = \frac{1}{n}$ is Cauchy but doesn't converge

<u>Homeomorphism:</u> $f: M \to N$ is continuous bijection for which f^{-1} is also continuous.

$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $f(x) = \tan(x)$, $x_n = \frac{\pi}{2} - \frac{1}{n}$, $\tan(\pi/2 - \frac{1}{n})$

<u>Ex 70</u>

(a,b) and $[a,b)$ are not homeomorphic

$f: [a,b) \to (a,b)$

$f(a) \in (a,b)$

$f^{-1}(B_\varepsilon(f(a)))$

Applied Analysis:

2/13/25

rearrangements

$k = f(n) : \mathbb{N} \to \mathbb{N}$

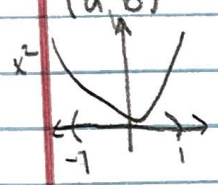
$q_n = P_{f(n)} = P_k$

If $f(n)$ is monotonically strictly increasing, rearrangement is a subsequence

		con	div
a)	Bijection	+	+
b)	Injection	+	−
c)	Surjection	−	+

just choose counterexample for −

(a,b) $[a,b)$



$f : [a,b) \to (a,b)$ $c = f(a)$ $a < c < b$

$(a,c) \cup (c,b) \to (a,b)$

$f^{-1}((a,c)) = U$ open

$f^{-1}((c,b)) = V$ open $U \cup V = (a,b)$ $B_\varepsilon(x)$

$U \cap V = \emptyset$

$\overline{U} = U$

Compactness

B-W: Any sequence in $\mathbb{R}$ which is in $[a,b]$ has a convergent subsequence

Definition: If any sequence in $S \subset M$ has a convergent subsequence, S is compact (sequentially)

Theorem: Every compact set is closed and bounded.

1) Proof: If set S is not closed, there exists sequence $(P_n) \subset S$ s.t. $P_n \to p$ and $p \notin S$. Then, for any subsequence $P_{n_k} \to p \notin S$ which contradicts completeness.

2) If S is not bounded, then for every $q \in S$ and every $r > 0$, there is a point $x \in S$ s.t. $d(q,x) > r$. Choose any $q \in S$ and x_1 st. $d(q,x_1) > 1$. Define $r_1 = 1 + d(q,x_1)$ and choose x_2 s.t. $d(q,x_2) > r_1$. Then for all $n > 1$, define $r_n = 1 + d(q,x_n)$ and choose x_n s.t. $d(q,x_n) > r_n$. For this sequence, $d(x_m, x_n) \geq |d(x_m, q) - d(x_n, q)|$ by reverse triangle inequality. WLOG, let $m > n$, $d(x_m, q) > 1 + d(x_{m-1}, q) > 2 + d(x_{m-2}, q) > \ldots > m - n + d(x_n, q)$ therefore $d(x_m, x_n) > |m - n| \geq 1$. Therefore no subsequence of (x_n) is Cauchy, therefore not Cauchy, therefore doesn't converge.

B-W': Set $[a,b] \subset \mathbb{R}$ is compact.
B-W: Set $[a,b] = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$ is compact.

$S = \mathbb{Q} \cap [0,1]$ is closed and bounded.
$x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right)$, $x_1 = 1$
Doesn't have subsequence which converges in S.

Complete + bounded + closed $\neq$ compact
Ex $\mathbb{N}$ with discrete metric
$x_n = n$, no subsequence with a limit

$M = (x_n) \subset \mathbb{R}$
(y_n)
$d((x_n),(y_n)) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{1/2}$, $\sum x_i^2 < \infty$

$(x)_n = (0, 0, \dots, \underset{n}{1}, 0, 0, 0)$
(Heine-Borel)
Theorem: Every closed and bounded subset of $\mathbb{R}^n$ is compact.
Lemma: Every closed subset of a compact set is compact.
Proof: Let $T \subset S$, S is compact and T is closed. For all $(x_n) \subset T$, $(x_n) \subset S$. Therefore, $\exists x_{n_k} \to x \in S$ because S is compact. But $(x_{n_k}) \subset (x_n) \subset T$ and because T is closed, $x_{n_k} \to x \in T$. Therefore T is compact.

If every closed and bounded subset of M is compact, does it follow that M is complete?

Applied Analysis: 2/18/25

$X = M \times N$ $\quad d_1 = d_M(y_1, y_2) + d_N(z_1, z_2)$

$x = (y, z)$ $\quad d_2 = (d_M^2(y_1, y_2) + d_N(z_1, z_2))^{1/2}$

$d_3 = \max(d_M(y_1, y_2), d_N(z_1, z_2))$

Prop.

If $A \subset M$ and $B \subset M$ are compact, then $A \times B \subset M \times N$ is compact

Proof

Let sequence $(x_n) \subset A \times B$ be given then there are two associated sequences $(a_n) \subset A$ and $(b_n) \subset B$ such that $x_n = (a_n, b_n)$. Because A is compact there is a subsequence a_{n_k} which converges in A. Consider $(b_{n_k}) \subset B$. Because B is compact, this ~~subsequence~~ has a convergent subsequence $b_{n_{k_\ell}} \to b \in B$. Because $a_{n_k} \to a$ and its subsequences converge to a and therefore $a_{n_{k_\ell}} \to a$. Thus $(a_{n_{k_\ell}}, b_{n_{k_\ell}}) \to (a, b) \in A \times B$.

Function $f: M \to \mathbb{R}$. The graph is a subset of $\{M \times \mathbb{R} : d(p, x) \mid p \in M, x \in \mathbb{R}, x = f(p)\}$.

a) If f continuous, then the graph G is closed (in $M \times \mathbb{R}$).

Contradiction: There is a convergent sequence $(p_n, x_n) \to (p, x) \in G$ with $f(p) \neq x$. Then, sequence $(p_n) \to p$ and $x_n = f(p_n) \to x = f(p)$. Contradiction.

b) If f continuous and M is compact, the graph is compact.

Suppose f is continuous and M is compact. We know from A that G is closed. ~~Because M is compact and f continuous~~ ~~Therefore, for any~~ Because M is compact, $(p_n, x_n) \to (p, x) \in G$. ~~Because M is closed~~, there is a subsequence $p_{n_k} \to p \in M$. Because f is continuous, $x_{n_k} = f(p_{n_k}) \to f(p)$ ~~because~~ and according to definition, $(p, f(p)) \in G$.

c) If G is compact, f is continuous

Let $p_n \to p$ be a convergent sequence in M. Because G is compact, there is a subsequence $(p_{n_k}, f(p_{n_k})) \to (p, f(p)) \in G$ (Because G is the graph of $f(p)$)

d) If B is closed, f can be discontinuous

$K_1 \supset K_2 \supset K_3 \supset \dots \supset K_n \supset$

K_i compact and $K_i \neq 0$

$K = \bigcap_{n=1}^{\infty} K_n$ is compact and non-empty

<u>Proof</u>

K_i is non-empty $\Rightarrow \exists p_n \in K_n$ sequence (p_n) has a convergent subsequence in K_1. so $(p_{n_k}) \to p$. Because for every k, $p_{n_k} \in K_\ell$ with $\ell \leq n_k$

<u>Theorem</u>

If $f: M \to N$ is continuous and $A \subset M$ is compact, then $f(A)$ is compact in N.

<u>Proof</u>

Let (q_n) in $f(A)$, for each n there is p_n such that $f(p_n) = q_n$. Because A is compact, there is $p_{n_k} \to p \in A$ and because f is continuous, $q_{n_k} = f(p_{n_k}) \to f(p)$.

Applied Analysis: 2/20/25

HWQ1 $[a,b] \cap \mathbb{Q}$

$N \subset M$ (inherited topology)

- If V is open in N then $\exists$ U open in M s.t. $V = U \cap N$
- For every U open in M, set $V = U \cap N$ is open in N

HWQ2 Point $p \in M$ is limit point of set $S \subset M$ if there is sequence $(p_n) \subset S$ which converges to p.

Prop: p is a limit point of $S \subset M$ if and only if, for any $\varepsilon > 0$ $B_\varepsilon(p) \cap S \neq \emptyset$.

($\Rightarrow$) Let p be a limit point of S. Then, $\exists p_n \to p$ which means that for every $\varepsilon > 0$, $\exists n$ s.t. $d(p, p_n) < \varepsilon$. Therefore, $B_\varepsilon(p) \cap S = p_n$ which isn't empty.

($\Leftarrow$) Let $\varepsilon = 1/n$. Then set $B_{1/n}(p) \cap S$ is non empty. Let $p_n \in B_{1/n}(p) \cap S$ then sequence $p_n \to p$. For any $\varepsilon > 0$, choose $n = \lceil \frac{1}{\varepsilon} \rceil$ then by construction, $\forall k \geq N$ $d(p_k, p) < \varepsilon$

Theorem: If f is continuous and K is compact then $f(K)$ is compact.

1. $K \subset \mathbb{R}$ $f: K \to \mathbb{R}$ then f obtains it's maximum and minimum on K.

Proof: Set $f(K)$ is compact and therefore bounded as subset of $\mathbb{R}$. Therefore, there is $\sup f(K)$. By properties of supremum, there is sequence $y_k \in f(K)$ s.t. $y_k \to \sup f(K)$. Therefore, there is sequence $(x_k) \subset K$. $y_k = f(x_k)$. Because K is compact it has a convergent subsequence $x_{k_\ell} \to x$. Because f is continuous, $y_{k_\ell} \to f(x) = \sup f(K)$. Therefore, $f(x) = \max_{z \in K} f(z)$.

2. If M is compact and N is homeomorphic to M then N is compact.

$(0, 1]$ in $\mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sin x$

3. If f is continuous bijection from M to N and M is compact, then f is homeomorphism.

Proof: f^{-1} is continuous. BWOC let f^{-1} be discontinuous, which means $\exists (q_k) \subset N$ s.t. $q_k \to q \in N$ but $f^{-1}(q_k)$ diverges. ~~Because M is compact~~, Therefore, $\forall p \in M$ $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$ $\exists n \geq N$ with $d(p, p_n) \geq \varepsilon$. Because M is compact $p_{n(N)}$ has a convergent subsequence $p_{n(N_k)} \to p' \in M$, but $f(p_{n(N_k)}) \to q$. Because f is continuous $f(p_{n(N_k)}) \to f(p')$. Therefore, $q = f(p')$ and because f is a bijection, $p' = f^{-1}(q) \in M$ contradicts $d(p', p) \geq \varepsilon$.

Uniform continuity

Def Function $f: M \to N$ is uniformly continuous if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall p, q \in M$ with $d(p,q) < \delta$, $d_N(f(p), f(q)) < \varepsilon$.

$f(x) = 1/x$ on $(0,1]$ is continuous but not uniformly

Theorem: If function f is continuous, it is uniformly continuous on a compact set.

Proof BWOC let f be continuous but not uniformly continuous. Let ε be s.t. $\forall \delta > 0$ there are p, q s.t. $d(p,q) < \delta$ but $d(f(p), f(q)) \geq \varepsilon$. Select a sequence $\delta_n = \frac{1}{n}$ and sequences p_n, q_n s.t. $d(p_n, q_n) < \frac{1}{n}$ but $d(f(p_n), f(q_n)) \geq \varepsilon$. Because p_n and q_n are on a compact set, there is a subsequence $p_{n_k} \to p$ and $q_{n_k} \to q$. Then, $d(p_{n_k}, q_{n_k}) \to 0$ and therefore $d(p,q) = 0$ but $d(f(p), f(q)) \geq \varepsilon$ which contradicts continuity.

Problem 121c: Prove that if K is compact, $M \subset K$, and every $f: M \to \mathbb{R}$ (which is continuous) is uniformly continuous then M is compact

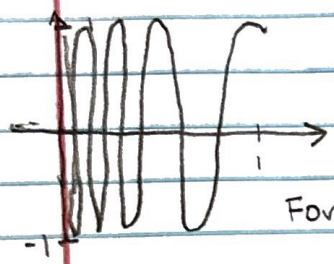
2/25/25

Applied Analysis:

Connected spaces

Def A set $S \subset M$ is disconnected if when viewed as a subspace it has two proper subsets A & B s.t. $A \cup B = S$ and $A \cap B = \emptyset$
$A \sqcup B = S$ and A & B are clopen in S. Otherwise, connected.

1) Ex $S = [a,b) \cup [c,d)$, $b < c$, disconnected
2) $a < b < c$, $S = (a,b) \cup (b,c)$, disconnected
3) $S \subset \mathbb{R}^2$, $(x,y) \in S = \{(x,y) \mid x \in [0,1],\ y = \sin\frac{1}{x},\ x > 0 \text{ and } y \in [-1,1], x = 0\}$



$A = \{(x,y) \mid x = 0 \quad y \in [-1,1]\}$
$B = \{y = \sin\frac{1}{x}\}$

For any $\varepsilon > 0$ and any $y \in [-1,1]$, $x = \arcsin\frac{\varepsilon}{2}$, $y = \frac{\varepsilon}{2}$

Def A set S is called path connected if there is a continuous function $f: [a,b] \to S$ with $f(a) = p$ and $f(b) = q$ for any $p, q \in S$.

Prop If M is connected, and $f: M \to N$ is continuous onto function, then N is connected.

Proof BWOC, let $N = A \cup B$ s.t. A and B are clopen. Consider $f^{PRE}(A)$ and $f^{PRE}(B)$. So $f^{PRE}(A) \sqcup f^{PRE}(B) = M$. Open as a preimage, closed as a complement of open $f^{PRE}(B)$.

Prop Set $[a,b]$ is connected. BWOC let $A \sqcup B = [a,b]$ and A & B are clopen and proper $b \notin A$ ($b \in B$). $A \subset \mathbb{R}$ a bounded subset of $\mathbb{R}$ has an upper limit $\sup A = S$. Let $S \in A$, then $B_\varepsilon(S)$ has points $b \geq x > S$, and therefore S is not interior and A is not open. Alternatively, if $S \notin A$. By properties of $\sup A$ there is a sequence in A converging to S and therefore A is not closed.

Prop Every path connected set is connected.

Proof By contradiction, Let $M = A \sqcup B$. Select $p \in A$ and $q \in B$ then $f^{PRE}(f[a,b] \cap A) \sqcup f^{PRE}(f[a,b] \cap B)$ closed and open, not empty.

Ex: set $S = \{(x,y) \mid r^2 \leq x^2 + y^2 \leq R^2\}$ $r < R$

$S = \{(\rho, \varphi) \mid r \leq \rho \leq R,\ \varphi \in [0, 2\pi)\}$ $\quad \varphi_1 \leq \varphi_2$
$t \in [0,1]$: $f(t) = \rho = \rho_1 + t(\rho_2 - \rho_1)$ $\quad \varphi = \varphi_1 + (\varphi_2 - \varphi_1)t$

$f: M \to \mathbb{R}$

<u>Theorem</u> Continuous function on a connected set exhibits the IV property, If α, β are in $f(S)$, then $\forall\, \alpha < \gamma < \beta$, there exists $x \in S$, $\gamma = f(x)$

<u>Proof</u>: BWOC let $\alpha < \gamma < \beta$ and there is no $x \in S$ s.t. $f(x) = \gamma$. $[\alpha, \beta]$ in $f(S) \cap ([-\infty, \gamma) \cup (\gamma, \infty])$ is disconnected. $A = f(S) \cap (-\infty, \gamma)$, $B = f(S) \cap (\gamma, \infty)$

<u>Prop</u>: If set S is connected, then any T, $S \subset T \subset \bar{S}$ is connected.

<u>Proof</u>: Because $T \subset \bar{S}$ there is a sequence $(x_n) \subset S$ s.t. $x_n \to x$ $\forall x \in T$. BWOC, if $T = A \sqcup B$ is disconnected. $(x_{n_k}) \subset A$, $x \in A$. So $B_\varepsilon(x)$

<u>Theorem</u> If S_α are connected and $p \in S_\alpha$ then $\bigcup_\alpha S_\alpha = S$ is connected.

<u>Ex</u>: If S is connected, is int(S) connected

$\mathbb{R}$ subsets that are path-connected up to homeomorphism,

Applied Analysis: 2/27/25

Covering compactness

1) Def Open covering is a collection of open sets, union of which contains a given set.
2) Subcovering is part of a covering collection which is a covering itself.
3) Finite covering is covering that is a finite set.
2+3) Finite subcovering.

A set $S \subset M$ is (covering) compact if every open cover of S has a finite subcover.

Ex: $\mathbb{R}$ $\mathcal{U} = \{B_1(n) \mid n \in \mathbb{Z}\}$ doesn't have finite subcover
$(0,1)$ $\mathcal{U} = \{(x,1) \mid x > 0\}$ cover with no finite subcover

If for $S \subset M$ every ~~subsequence~~ subsequence has a convergent subsequence then S is comp. alt.

$(\Leftarrow)$ $(x_n) \subset S$. S is covering compact.

Def Les begues number: λ: $\forall x \in S$, $\exists U \in \mathcal{U}$, the ball $B_\lambda(x) \subset U$.
If there is no $\lambda > 0$ then $\lambda = 0$ by definition

Lemma if S is sequentially compact and every covering of S has les begues number, $\lambda > 0$, S is covering compact.

Proof BWOC, $\mathcal{U}$ λ, $a_1 \in S$. There is $U_1 \in \mathcal{U}$, $B_\lambda(a_1) \in U_1$. $K_1 = S \setminus U_1$, $\exists a_2 \in K_1$, $\exists U_2 \in \mathcal{U}$, $B_\lambda(a_2) \in U_2$. $K_2 = K_1 \setminus U_2$... If this process continues to ∞, then (a_n) is a sequence and by supposition, it has a subsequence $a_{n_k} \to a \in S$. Then there exists U s.t. $B_\lambda(a) \subset U$. There exists a_{n_k} for ~~which~~ all $l \geq k$, $d(a_{n_l}, a) < \lambda/2$ but $d(a_{n_k}, a_{n_l}) \leq d(a_{n_k}, a) + d(a_{n_l}, a) < \lambda$ which isn't possible by construction because for $l > k$ $a_{n_l} \notin B_\lambda(a_{n_k})$.

Lemma 2 Every sequentially compact set has a finite ($\lambda > 0$) lesbegues number for every covering.

Proof BWOC. ~~So for every $\lambda_n = \frac{1}{n}$ there~~ There is a covering $\mathcal{U}$ s.t. $\forall \lambda_n = \frac{1}{n}$ there is a point a_n s.t. $B_{\lambda_n}(a_n)$ is not a subset of any U from $\mathcal{U}$. Sequence (a_n) has a conversing subsequence $a_{n_k} \to a$. Let $a \in U_a$ there

Applied Analysis:

3/4/25

Compactness

Set S is compact if for any $(x_n) \subset S$ there is convergent subsequence $x_{n_k} \to k \in S$.

$\Updownarrow$

S is compact if every open covering has a finite subcovering.

Compact $\Rightarrow$ bounded and closed

$\mathbb{R}^m$: bounded and closed $\Rightarrow$ compact (HB)

1) Ex: $(0,1)$ $x_n = \frac{1}{n}$ has no convergent subsequence (Cauchy)

Not compact + not complete

2) Discrete metric

Def: A set $S \subset M$ is <u>totally bounded</u> if for all $\varepsilon > 0$, there is a finite set of open balls, $B_\varepsilon(x_i)$ which is covering of S. $(2^{m \cdot n})$

Theorem: A set S in complete space, which is closed and totally bounded is compact and vice versa. $\Longleftrightarrow$

Proof

Let $(x_n) \subset S$, select $\varepsilon_n = \frac{1}{n}$. For all n, there is a finite collection of open balls, $B_{1/n}(x_i)$ covering S. Therefore, at least one of $B_{1/n}(x_i)$ contains infinite number of elements from $\{x_i\}$. Select such sets x_i and $A_i = B_{1/n}(x_i)$. Let $C_k = \bigcap_{i=1}^{k} A_i$ which is not empty. $x_{n_k} \in C_k$, $d(x_{n_k}, x_{n_\ell}) < \frac{2}{n}$. Therefore x_{n_k} is Cauchy $\to$ converges $\to$ limit in S $\to$ compact.

Applied Analysis:

3/18/25

Ratio MVT (Cauchy) $\frac{\Delta f}{\Delta g} = \frac{f'(x)}{g'(x)}$

Prop - f is continuous on $[a,b]$ and differentiable on (a,b) then there is $c \in (a,b)$ s.t. $(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c)$

Proof - $h(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a))$

$h(a) = 0$ Rolle's theorem to h, so $h'(c) = 0$

$h(b) = 0$

$h'(x) = (f(b)-f(a))g'(x) - (g(b)-g(a))f'(x)$

Theorem (L'Hopital) - Let f, g be continuous and differentiable on (a,b) and $\lim_{x \to a-0} f(x) = \lim_{x \to a-0} g(x) = 0$. If limit $\lim_{x \to a-0} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \to a-0} \frac{f(x)}{g(x)}$ exists and $= L$. $g \neq 0$

Proof - Define $f(a) = 0$ and $g(a) = 0$. Then f and g are continuous on $[a, b-\varepsilon]$. Applying ratio MVT, $\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}$, $c \in (a,x)$. Take limit of both points.

$|x|$ to visualize

$f: (a,b) \to \mathbb{R}$ continuous and differentiable except maybe in $c \in (a,b)$ and $\lim_{x \to c} f'(x) = L$. Then, $f'(c) = L$.

$\frac{f(x)-f(c)}{x-c} = f'(y)$, $y \in (c,x)$ $|f'(y) - L| < \varepsilon$ if $x - c < \delta_+$, $\left|\frac{f(x)-f(c)}{x-c} - L\right| < \varepsilon$

same for δ_-, $\delta = \min(\delta_+, \delta_-)$

$f: (0,1) \to \mathbb{R}$ differentiable and f' is bounded, then there exists $g: [0,1) \to \mathbb{R}$ which is continuous and $f(x) = g(x)$ on $(0,1)$.

~~Theorem~~

f cont on $[a,b]$ and diff on (a,b) then $f'(x)$ has intermediate value property.

$f'(a_1) < f'(b_1)$, then $\forall\, f'(a_1) < L < f'(b_1)$, $\exists\, c \in (a_1, b_1)$, $f'(c) = L$.

Proof - $g(x) = Lx - f(x)$

$g'(a_1) = L - f'(a_1) > 0$

$g'(b_1) = L - f'(b_1) < 0$

Applied Analysis:

3/20/25

$f'(0)$ exists and nowhere else?

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ -x^2, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$f^{(n)}(x) = (f^{(n-1)}(x))'$, $f(x)$ continuous $[a,b]$ $C^0([a,b])$

n-times differentiable

$f_n(x) \in C^n(\mathbb{R})$

$\notin C^{n+1}(\mathbb{R})$ $\quad f_0(x) = |x|$, $f_1(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases} = x|x|$, $f_n(x) = x^n$

Ex II $\quad f: (a,b) \to \mathbb{R}$, $f''(x)$ exists, $\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$

$f'(x)$? $\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$ $\quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = L$

$\forall \varepsilon > 0 \ \exists \delta \ \left| \frac{f(x+h) - f(x)}{h} - L \right| < \varepsilon$ when $|h| < \varepsilon$

$$\left| \frac{f(x+h) - f(x-h)}{2h} \right| = \left| \frac{f(x+h) - f(x)}{2h} + \frac{f(x) - f(x-h)}{2h} - L \right| \leq \left| \frac{f(x+h) - f(x)}{2h} - \frac{L}{2} \right|$$

$$+ \left| \frac{f(x) - f(x-h)}{2h} - \frac{L}{2} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$f(x+h) \approx f(x) + f'(x) \cdot h$, $g(h) = f(x+h) - f(x) - f'(x) \cdot h$

Def: If $f \in C^n((a,b))$, and $x \in (a,b)$ then $P_n(h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k$ is Taylor polynomial. If $f \in C^\infty((a,b))$ $P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(k)}{k!} h^k$ is Taylor series.

Prop: 1. $f \in C^n((a,b))$, $x \in (a,b)$ then $\frac{f(x+h) - P_n(h)}{h^n} \to 0$ as $h \to 0$.

2. $f \in C^{n+1}((a,b))$ then $\exists$ $y = (a,b)$ s.t. $f(x+h) - P_n(h) = \frac{f^{(n+1)}(y)}{(n+1)} h^{n+1}$

Proof: Define $g(h) = f(x+h) - P_n(h)$, $g(0) = 0 = g'(0) = \ldots = g^{(n)}(0)$

Example: $f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$

Applied Analysis:

4/1/25

Riemann Integral: on $[a,b]$ $\quad P=\{x_i\}_{i=0}^{N}$ $\quad a<x_1<x_2\ldots<b$ (Partition)

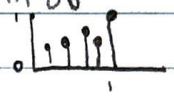
$\tau=\{t_i\}_{i=1}^{N}$ $\quad x_i<t_{i+1}<x_{i+1}$ mesh, $\Delta p=\max(x_{i+1}-x_i)$

Riemann sum $R(f;P,t)=\sum_{i=1}^{N} f(t_i)(x_i-x_{i-1})$. If there is number I s.t. $\forall\varepsilon>0$, $\exists\Delta$ s.t. for all partition pairs with $\Delta p<\Delta$, $|R(f;P,t)-I|<\varepsilon$, then I is Riemann integral.

Ex: 1) All continuous functions are Riemann integrable.

2) Dirichlet $\left(D(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0 & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\right)$ $[0,1]$ not in $\mathcal{R}$

3) $f(x)=\begin{cases}\frac{1}{q} & x=\frac{p}{q} \text{ in lowest terms}\\ 0, & \mathbb{R}\setminus\mathbb{Q}\end{cases}$ on $[0,1]$ is in $\mathcal{R}$



Prop: All $f\in\mathcal{R}$ are bounded

Proof: BWOC let f be unbounded. Take $\varepsilon=1$ and any partition P. f unbounded $\Rightarrow$ there is interval $[x_i,x_{i+1}]$ on which it is unbounded. On that interval, we can select sequence $(t_{i+1})_n\in[x_i,x_{i+1}]$ s.t. $f((t_{i+1})_n)\to\infty$ and therefore $R(f;P,t)\to\infty$

Let $\varepsilon>0$ be given $\quad \Delta,\ K \quad K-1+K-2+\ldots+1+2 \quad \frac{(K-1)K}{2}+2<K^2,\ K^2\Delta$

$R<K^2\Delta+\frac{1}{K}\cdot 1 \quad K=\left\lceil\frac{1}{\Delta^{1/3}}\right\rceil \quad \Delta^{1/3}+\Delta^{1/3}=2\Delta^{1/3}<\varepsilon$

$I=\int_a^b f(x)dx$

1) $[a,b]$ - If $f(x)\le g(x)$ and both integrals exists, $\int_a^b f(x)dx\le\int_a^b g(x)dx$

2) $\int_b^a f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$

$R_1-R_2=\Delta\cdot[f(x_1)-f(x_2)]\le\Delta\cdot 2m$, $|x_1-b|<\Delta$, $|x_2-b|<\Delta$

Prop: All continuous functions are Riemann integrable.

Notice: If f is continuous on $[a,b]$ it is uniformly continuous $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$,

Refinement of partition $P=\{x_i\}$; two partitions P_1 & P_2 $\quad P_1=\{x_i\}$ $\quad P_2=\{y_i\}$

$P=P_1\cup P_2$ common refinement.

$R_1,\Delta_1;R_2,\Delta_2 \quad \Delta x_i=x_{i+1}-x_i,\ R_1=\sum_{i=1}^{N} f(t_i)\Delta x_i,\ R_2=\sum_{k=1}^{M} f(P_k)\Delta y_k$

$\Delta z_{ik} \quad R_1-R_2=\sum_{i,k}[f(t_i)-f(P_k)]\Delta z_{ik}=\sum_{i,k} f(t_i)\Delta z_{ik},\ =\sum_{i,k} f(P_k)\Delta z_{ik}$

$$|R_1-R_2|\le\sum_{i,k}|f(t_i)-f(P_k)|\Delta z_{ik}<\sum\varepsilon_1\Delta z_{ik}=\varepsilon_1(b-a)<\varepsilon/2$$

Applied Analysis

4/3/25

Riemann Integral:

$f: [a,b] \to \mathbb{R}$ bounded, $R(f; P, T)$

$P = \{x_i\}_{i=0}^{N}$ $x_{i+1} > x_i$ partition $x_0 = a$, $x_N = b$

$T = \{t_i\}_{i=1}^{N}$ $x_{i-1} \geq t_{i+1} \geq x_i$

$\Delta P = \max(|x_{i+1} - x_i|) < \Delta$, $|R(f; P, T) - I| < \varepsilon$

$f(t_{i+1}) = \sup_{[x_i, x_{i+1}]}(f(x))$

Upper Darboux sum:

$U(f; P) = \sum_{i=1}^{N} \sup_{[x_{i-1}, x_i]}(f(x))(x_i - x_{i-1})$

Lower Darboux sum:

$L(f; P) = \sum_{i=1}^{N} \inf_{[x_i, x_{i+1}]}(f(x))(x_i - x_{i-1})$ $\quad -m(b-a) \leq L(f; P) \leq R(f; P, T) \leq U(f; P) \leq m(b-a)$

Lower Darboux Integral: $\underline{I} = \sup_P(L(f; P))$

Upper Darboux Integral: $\overline{I} = \inf_P(U(f; P))$

*for Dirichlet, lower is 0, upper is 1

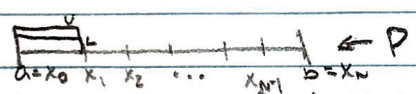
!!! $\boxed{\underline{I} = \overline{I}} \Rightarrow \underline{I} = \overline{I} = I$

* Darboux integrable $\Leftrightarrow$ Riemann Integrable

$\Updownarrow$

$\forall \varepsilon, \exists P$ s.t. $U(f, P) - L(f, P) < \varepsilon$

P is a grid



Find U and L that are same, find partition where such thing exists.

Proof $\underline{I} - L(f, P_L) < \varepsilon/3$ $\quad \overline{I} = \underline{I}$

($\Rightarrow$) $U(f, P_u) - \overline{I} < \varepsilon/3$

$U(f, P_u) - L(f, P_L) < 2\varepsilon/3$

Select $P = P_u \cup P_L$

$U(f, P) - L(f, P) < 2\varepsilon/3$

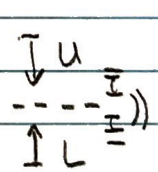
($\Leftarrow$) $|U(f, P_1) - L(f, P_1)| < \varepsilon/3$

$|U(f, P_u) - \overline{I}| < \varepsilon/3$

$|\underline{I} - L(f, P_L)| < \varepsilon/3$

$P = P_1 \cup P_L \cup P_u$

$|\overline{I} - \underline{I}| < |\overline{I} - U(f, P)| + |U(f, P) - L(f, P)| + |\underline{I} - L(f, P)| < \varepsilon$



Applied Analysis:

5/8/25

Riemann Integral:

$f: [a,b] \to \mathbb{R}$, $P = \{x_i\}_{i=0}^{N}$ $x_0 = a < x_1 < \dots < x_n = b$

$\Delta P < \Delta$ $t_i \in [x_{i-1}, x_i]$, $R(f; P, T) = \sum_{i=1}^{N} f(t_i)(x_i - x_{i-1})$ $\Delta P = \max(x_i - x_{i-1})$

$\forall \varepsilon, \exists \Delta, \forall P$ with $\Delta P < \Delta$ $|R(f; P, T) - I| < \varepsilon$, $I = \int_a^b f(x)dx$

Criterion: Exists P s.t. $U(f, P) - L(f, P) < \varepsilon$

$$U(f, P) = \sum_{i=1}^{N} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_{i-1} - x_i)$$

Prop: Riemann integral $\Rightarrow$ bounded

If f on $[a,b]$ is in $\mathcal{R}$ then it has at least one point of continuity.

~~BWOC, $P: x_i = a + \frac{b-a}{N} i$~~

~~$i = 0, \dots, N$ $\Delta P = \frac{b-a}{N} < \Delta$~~

f is in $\mathcal{R}$ & $\exists m > 0$ s.t. $|f(x)| \geq m$. Prove that $\frac{1}{f(x)}$ is in $\mathcal{R}$.

$\forall \varepsilon, \exists P$ s.t. $U(f, P) - L(f, P) < \varepsilon$

$U(\frac{1}{f}, P) - L(\frac{1}{f}, P)$

Oscillations

$f(x)$ on I

$osc(f, I) = \sup_I f(x) - \inf_I f(x)$

$osc(f, x) = \inf[osc(f, I), x \in I]$

f is continuous in x

$osc(f, x) = 0$

f is discontinuous

$\exists \varepsilon \forall \delta \exists y$ $|x - y| < \delta$ & $|f(x) - f(y)| \geq \varepsilon$

$osc(f, x) = \inf \varepsilon$

$$f(x) = \begin{cases} 1/q, x = \frac{p}{q} \text{ lowest terms} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Def. Zero set is a subset of $[a,b]$ which can be covered by countably many open intervals with total length $< \varepsilon$ $\forall \varepsilon$.

1. Finite sets $\{x_i\}$, $(x_i - \frac{\varepsilon}{2N}, x_i + \frac{\varepsilon}{2}$
2. Sets with countably many elements x_i, $(x_i - 2^{i+2}, x_i + 2^{i+2}$

Applied Analysis:

4/10/25

#28 Zero set Z covered by countable open intervals total length $< \varepsilon$.
$\operatorname{diam}\{\frac{1}{n} \mid n \in \mathbb{N}^+\} = 1$

HW 7.3.9 $f'(x) < L < 1$

HW7.3.5 No counterexample.

Riemann Integrable $\Leftrightarrow$ zero set of discontinuity points

FTC: $f:[a,b] \to \mathbb{R}$ in $\mathcal{R}$, for each $x \in [a,b]$, $F(x) = \int_a^x f(t)\,dt$, $F(x)$ exists, continuous, and differentiable in x when $f(x)$ is continuous and then $F'(x) = f(x)$

$$F(x+\delta) = F(x) + \int_x^{x+\delta} f(t)\,dt - M\delta$$

$$|F(x-\delta) - F(x)| < \varepsilon, \quad \delta = \frac{\varepsilon}{m}$$

$$F'(x) = \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} = \lim_{\delta \to 0} \frac{1}{\delta} \int_x^{x+\delta} f(t)\,dt$$

$\forall \varepsilon, \exists \delta$ s.t. $|f(t) - f(y)| < \varepsilon$ when $|x-y| < \delta$.

$$f(x)\cdot\delta - \varepsilon\delta < \int_x^{x+\delta} f(t)\,dt < f(x)\cdot\delta + \varepsilon\delta$$

Ex 39 $f(x) = \begin{cases} 1, & x \in [1,4] \\ 0, & \text{otherwise} \end{cases}$, $g(x) = \begin{cases} 1, & x \in [2,5] \\ 0, & \text{otherwise} \end{cases}$

$$\underbrace{\int_0^3 f(x)g'(x)\,dx}_{=0} = \underbrace{f(3)g(3) - f(0)g(0)}_{1-0=1} - \underbrace{\int_0^3 f'(x)g(x)\,dx}_{=0}$$

$f:[a,b] \to [c,d]$ in $\mathcal{R}$ and $\phi:[c,d] \to \mathbb{R}$ continuous then $\phi \circ f$ is in $\mathcal{R}$ $\phi(f(x))$

Ex 43 a) Let $\psi(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 0 & x = 0 \end{cases}$. $F:[-1,1] \to \mathbb{R}$ in $\mathcal{R}$ $f(\psi(x))$ in $\mathcal{R}$

b) What if $f(x) = \sqrt{x}\sin x$

$$x \xrightarrow{\varphi} y \xrightarrow{f} z$$

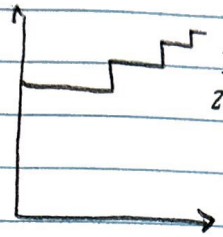
Need: Disc of $f(\varphi(x))$ are zero set. $\varphi(0) = 0$

Have: disc of f are zero set $f(0)$ disc

Zeno's staircase:

$Z:[0,1] \to \mathbb{R}$

$$Z(x) = \begin{cases} \frac{1}{2}, & 0 \le x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \le x \le \frac{3}{4} \\ \frac{7}{8} & \frac{3}{4} \le x < \frac{7}{8} \\ \vdots & \vdots \end{cases}$$



$\frac{2^{n+1}-1}{2^{n+1}}, \quad \frac{2^n-1}{2^n} \le x < \frac{2^{n+1}-1}{2^{n+1}}$

Series

$S = \sum_{n=1}^{\infty} a_n$, Partial sums $S_n = \sum_{k=1}^{n} a_k$

Cauchy criterion

$\forall \varepsilon > 0, \exists N$ s.t. $\forall m \ge N$, $|S_m - S_n| < \varepsilon$

or $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$

1) Geometric series

$a_k = p^k$, $S_n = \frac{p^{n+1}+1}{p-1}$, if $p<1 \to \frac{-1}{p-1} = \frac{1}{1-p}$

$p \ge 1 \to$ diverges

2) Harmonic series

$\sum_{k=1}^{\infty} \frac{1}{k}$ averges $S_n \sim \ln n$ if n and $k = \lfloor \log_2 n \rfloor$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots >$

$S_n > 1 + \frac{1}{2}k$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

Theorem: Alternating series $(a_k \cdot a_{k+1} < 0)$ with $|a_k| > |a_{k+1}|$ converges.

2. sets with countably many ownerns, 2 x is, (N) 2 x 2, ...

Applied Analysis:

4/15/25

$S = \sum_{k=1}^{\infty} a_k$, $S_n = \sum_{k=1}^{n} a_k$, $S = \lim_{n\to\infty} S_n$ $\quad \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$

<u>Cauchy criterion</u>: $\forall \varepsilon, \exists N$ s.t. $\forall m,n \geq N$ $\quad |S_m - S_n| < \varepsilon$

Series is <u>absolutely convergent</u> if $\sum |a_k|$ is convergent

Convergent but not absolutely, <u>conditionally convergent</u>

<u>Ex:</u> 1) $\sum_{k=1}^{\infty} \frac{1}{p^k} = \frac{1}{1-p}$, $|p|<1$

... divergent, $|p| \geq 1$

2) $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, $\approx 2\ln N$; $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$

<u>Prop</u> Series $\sum b_k$ dominates $\sum a_k$ if $|a_k| \leq b_k$. If $\sum b_k$ dominates $\sum a_k$ and $\sum b_k$ converges then $\sum a_k$ converges.

<u>Proof</u> Use cauchy.

<u>Integral test</u>: If $f(x)$ is positive, or non-negative, decreasing function, then some $\sum f(k)$ and $\int_1^{\infty} f(x)dx$ converges or diverges together. $|a_k| \leq f(x)$ when $x \in [k, k+1)$. If $\int_N^{\infty} f(x)dx$ converges, then $\sum a_k$ ~~is~~ converges. $\int f(x)dx$ diverges $\longrightarrow \sum a_k$ diverges

3) $\sum_{k=1}^{\infty} \frac{1}{k^a}$ $\quad a=1$, diverges

$a<1$ $\quad \frac{1}{k^a} > \frac{1}{k}$ diverges

$a>1$, $f(x) = \frac{1}{x^a}$ $\quad \int_1^{\infty} \frac{1}{x^a}dx = \frac{x^{-a+1}}{-a+1}\Big|_1^{B} = \frac{B^{-a+1}-1}{-a+1} \to \frac{1}{a-1}$, converges

$\sqrt[k]{1/p^k} = \frac{1}{p} < 1$

or $\sqrt[k]{|a_k|} < 1$, <u>root test</u>, $\underbrace{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}_{\text{exponential growth}} < 1$ then $\sum a_k$ converges,

> 1 diverges, $|a_k| < \infty$ for large enough k

<u>Ratio test</u>: $\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{p}$

$\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$ converges

$\liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$ diverges

<u>Ex</u>: Suppose $\sum_{k=1}^{\infty} a_k$ absolutely converges, $a_k > 0$. Prove there exists sequence $c_k \to \infty$ s.t. $\sum c_k a_k$ converges

$\sum \frac{1}{k^a}$; $\sum \frac{1}{k^a} \cdot \frac{1}{k^{\frac{1}{2}}}$

Applied Analysis: 4/17/25

$a_n > 0$, $\sum a_n$ convergent

$c_n \to \infty$ s.t. $\sum c_n a_n$ is convergent

$S_n = \sum_{k=1}^{n} a_k$

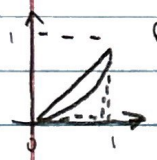
$\sum_{k=0}^{\infty} a_k x^k \quad \alpha = \limsup \sqrt[k]{|a_k|}$ $\quad \alpha < 1$, converges; $\alpha > 1$, diverges

$\limsup \sqrt[k]{|a_k||x|^k} = |x| \limsup \ldots = |x|\alpha$, so $R\alpha = 1 \Rightarrow R = \frac{1}{\alpha}$ $\quad R<1$ conv, $R>1$ div

$\frac{1}{1-x} = \sum x^k$, $R=1$; $\sum a_k \cdot k \cdot x^{k-1} = \frac{1}{x}\sum a_k k x^k$ $\quad \sqrt[k]{|a_k| k} = \sqrt[k]{|a_k|}\underbrace{\sqrt[k]{k}}_{1}$

$f_n(x) \to f(x)$, pointwise convergence (x - fixed, $n \to \infty$)

$\forall \varepsilon, \exists N$ s.t. $\forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon$



$f_n(x) = x^n$ on $[0,1]$

$n \ln 1/x > \ln 1/\varepsilon$, $n > \frac{\ln 1/\varepsilon}{\ln 1/x}$

$f_n(x) \to f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$

Continuous $f_n(x)$: $\forall \varepsilon, \exists \delta$ s.t. $\forall y$ $|x-y| < \delta$, $|f_n(x) - f_n(y)| < \varepsilon$

$\lim_{n\to\infty} \lim_{x\to 1} x^n$; $\lim_{x\to 1^-} \lim_{n\to\infty} = 0$

Def: Uniform convergence - $\forall \varepsilon, \exists N$ s.t. $\forall x \in I$, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$; $f_n \rightrightarrows f$

Ex: $x^n > \varepsilon$, $x > \varepsilon^{1/n}$

Theorem: If $f_n \rightrightarrows f$ and f_n are continuous in x_0, then f is continuous in x_0.

Proof: $\varepsilon/3$-argument

$\forall \varepsilon, \exists \delta$ such that for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$

By uniform convergence, $\exists N$ s.t. $\forall n \geq N$, $|f_n(x_0) - f(x_0)| < \varepsilon/3$ and $|f_n(x) - f(x)| < \varepsilon/3$, & by continuity of f_n there is $\delta > 0$ s.t. $|f_n(x) - f_n(x_0)| < \varepsilon/3$, for $|x - x_0| < \delta$.