1 Differentiation

Basic Rules: f diff'ble $\Rightarrow$ continuous. $(f+g)' = f'+g'$ $(fg)' = f'g+fg'$
$(f/g)' = \dfrac{f'g - fg'}{g^2}$ $(g \neq 0)$
$(g \circ f)'(x) = g'(f(x))f'(x)$ $(c)' = 0$
Polynomial: $\dfrac{d}{dx}(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$
Mean Value Theorem: f cont. on $[a,b]$, diff'ble on $(a,b) \Rightarrow \exists\, \theta \in (a,b)$: $f(b) - f(a) = f'(\theta)(b-a)$
Extrema: If f has a local min/max at θ and is diff'ble, then $f'(\theta) = 0$
Global Lipschitz: $|f'(x)| \leq M \Rightarrow |f(t) - f(x)| \leq M|t-x|$; if $f' \equiv 0 \Rightarrow f$ is constant
Ratio MVT: f, g cont. on $[a,b]$, diff'ble on $(a,b) \Rightarrow \exists\, \theta : \Delta f \cdot g'(\theta) = \Delta g \cdot f'(\theta)$
L'Hôpital's Rule: If $\lim f(x) = \lim g(x) = 0$, $\lim \dfrac{f'}{g'} = L \Rightarrow \lim \dfrac{f}{g} = L$
Derivative Properties: f' has intermediate value property (no jump disc.) $f^{(r)} \Rightarrow f^{(r-1)}$ cont. Smooth $\Rightarrow$ all $f^{(n)}$ exist and cont.
Taylor Approximation: $f(x+h) = P(h) + R(h)$ where $R(h)/h^r \to 0$ as $h \to 0$ $R(h) = \dfrac{f^{(r+1)}(\theta)}{(r+1)!} h^{r+1}$, $\theta \in (x, x+h)$
Inverse Function Thm (1D): If f diff'ble, f' never $0 \Rightarrow f^{-1}$ diff'ble, $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$

2 Riemann Integration

Definition (Partition): A partition P of $[a,b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$.
Mesh of a Partition: $\|P\| = \max_{1 \leq i \leq n}(x_i - x_{i-1})$
Refinement: Q is a refinement of P if $P \subseteq Q$ (i.e., Q adds more points to P).
Partition Pair: Two interlaced sets $P = \{x_0, \ldots, x_n\}$ and $T = \{t_1, \ldots, t_n\}$ with $t_i \in [x_{i-1}, x_i]$.
Effect of Refinement:

- Refining P increases $L(f,P)$ and decreases $U(f,P)$.
- If f is Riemann integrable, then for sufficiently fine partitions, $U(f,P) - L(f,P) < \epsilon$.

Definition (Riemann Sum): Given $f : [a,b] \to \mathbb{R}$, a partition $P = \{x_0, \ldots, x_n\}$ with $a = x_0 < \cdots < x_n = b$, and sample points $T = \{t_1, \ldots, t_n\}$ such that $t_i \in [x_{i-1}, x_i]$, the Riemann sum is:

$$R(f,P,T) = \sum_{i=1}^{n} f(t_i)\Delta x_i, \quad \Delta x_i = x_i - x_{i-1}$$

Definition (Riemann Integrability): f is Riemann integrable if $\exists I$ such that $\forall \epsilon > 0$, $\exists \delta > 0$:

$$\text{mesh}(P) < \delta \Rightarrow |R(f,P,T) - I| < \epsilon$$

Denoted as $\int_a^b f(x)\,dx = I$.
Definition (Darboux Sums):

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i, \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

Definition (Lower and Upper Integrals):

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(f,P), \quad \overline{\int_a^b} f(x)\,dx = \inf_P U(f,P)$$

f is Darboux integrable if the lower and upper integrals are equal.

Theorems and Properties

- **Riemann Integrability Criterion**: f is Riemann integrable iff $\forall \epsilon > 0$, $\exists P$ such that $U(f,P) - L(f,P) < \epsilon$.
- **Refinement Principle**: Refining P increases $L(f,P)$ and decreases $U(f,P)$.
- **Linearity**: If f, g are integrable, so are $f+g$ and cf; $\int(f+g) = \int f + \int g$, $\int cf = c\int f$.
- **Monotonicity**: If $f \leq g$, then $\int_a^b f \leq \int_a^b g$.
- **Absolute Bound**: If $|f(x)| \leq M$ then $|\int f| \leq M(b-a)$.
- **Equivalence Theorem**: Riemann integrability $\Leftrightarrow$ Darboux integrability.
- **Riemann–Lebesgue Theorem**: f is Riemann integrable $\Leftrightarrow$ f is bounded and its discontinuities form a measure zero set.
- **Fundamental Theorem of Calculus**: If f is integrable on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous; if f is continuous at x, then $F'(x) = f(x)$.
- **Antiderivative Theorem**: If $F' = f$ and f is Riemann integrable, then F differs from the indefinite integral by a constant.
- **Integration by Parts**: If f, g differentiable and $f', g' \in R$, then:

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

- **Substitution Rule**: If $g : [c,d] \to [a,b]$ is C^1 with $g' > 0$ and $f \in R$, then:

$$\int_a^b f(y)\,dy = \int_c^d f(g(x))g'(x)\,dx$$

- **Continuity Implies Integrability**: Every continuous function is Riemann integrable.
- **Other corollaries**: If f, g are integrable then:
  - fg is integrable.
  - $|f|$ is integrable.
  - Monotone functions are integrable.
  - $\phi \circ f$ is integrable if ϕ is continuous.
  - f integrable on $[a,c]$ and $[c,b]$ implies integrable on $[a,b]$.

3 Series

Series and Convergence
A series is a formal sum $\sum_{k=0}^{\infty} a_k$ of real numbers. The n-th partial sum is $A_n = \sum_{k=0}^{n} a_k$.
Convergence: The series $\sum a_k$ converges to A if $A_n \to A$ as $n \to \infty$.
Divergence: If the limit does not exist, the series diverges.
Cauchy Criterion for Series: $\sum a_k$ converges $\iff$ for every $\epsilon > 0$, $\exists N$ such that for all $m, n \geq N$:

$$\left|\sum_{k=m}^{n} a_k\right| < \epsilon$$

Necessary Condition: If $\sum a_k$ converges, then $a_k \to 0$.
Geometric Series: $\sum_{k=0}^{\infty} \lambda^k$ converges if $|\lambda| < 1$, and its sum is $1/(1-\lambda)$.
Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges even though $a_k \to 0$.
Convergence Tests:

- **Comparison Test (40)**: If $|a_k| \leq b_k$ for all large k, and $\sum b_k$ converges, then $\sum a_k$ converges.
- **Integral Test (41)**: If f is positive, decreasing, and $f(k) = a_k$, then $\sum a_k$ and $\int f(x)\,dx$ converge or diverge together.
- **Root Test (42)**: Let $\alpha = \limsup_{k \to \infty} \sqrt[k]{|a_k|}$.

$$\alpha < 1 \Rightarrow \text{converges}, \quad \alpha > 1 \Rightarrow \text{diverges}, \quad \alpha = 1 \Rightarrow \text{inconclusive}.$$

- **Ratio Test (43)**: Let $r_k = |a_{k+1}/a_k|$, and define $\rho = \limsup r_k$, $\lambda = \liminf r_k$.

$$\rho < 1 \Rightarrow \text{converges}, \quad \lambda > 1 \Rightarrow \text{diverges}, \quad \text{otherwise inconclusive}.$$

Alternating Series Test: If $a_k \geq 0$, decreasing, and $a_k \to 0$, then $\sum(-1)^{k+1} a_k$ converges.
Radius of Convergence (44): For a power series $\sum c_k x^k$, the radius of convergence is

$$R = \frac{1}{\limsup_{k \to \infty} |c_k|^{1/k}}.$$

Converges absolutely for $|x| < R$, diverges for $|x| > R$.

4 Uniform Convergence and C0

Pointwise and Uniform Convergence
A sequence of functions $f_n : [a,b] \to \mathbb{R}$ converges **pointwise** to a function f if for each $x \in [a,b]$,

$$\lim_{n \to \infty} f_n(x) = f(x).$$

It converges **uniformly** to f if:

$$\forall \varepsilon > 0, \exists N \text{ such that } n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in [a,b].$$

Notation: $f_n \rightrightarrows f$, or $\lim^{\text{unif}}_{n \to \infty} f_n = f$
Key Theorems and Properties

1. If $f_n \rightrightarrows f$ and each f_n is continuous at x_0, then f is continuous at x_0.
2. Convergence in the supremum norm $d(f_n, f) = \sup_{x \in [a,b]} |f_n(x) - f(x)|$ is equivalent to uniform convergence.
3. $C_b([a,b])$, the space of bounded continuous functions on $[a,b]$, is a complete metric space under the sup norm.
4. The subspace $C_0([a,b])$, the set of continuous functions vanishing at the boundary (or some similar condition), is closed in C_b and therefore also complete.
5. If a series $\sum f_k$ converges uniformly, and $\sum |f_k(x)|$ converges for all x, then it converges **absolutely**.
6. **Weierstrass M-test:** If $\|f_k\|_\infty \leq M_k$ and $\sum M_k$ converges, then $\sum f_k$ converges uniformly and absolutely.
7. If $f_n \in R[a,b]$ and $f_n \rightrightarrows f$, then $f \in R[a,b]$, and:

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

8. Indefinite integrals also converge uniformly:

$$\int_a^x f_n(t)\,dt \rightrightarrows \int_a^x f(t)\,dt.$$

9. **Term-by-Term Integration:** If $\sum f_k \rightrightarrows f$ uniformly and each f_k is integrable,
$$\int_a^b \sum_{k=0}^{\infty} f_k(x)dx = \sum_{k=0}^{\infty} \int_a^b f_k(x)dx.$$
10. If $f_n \rightrightarrows f$ and each f_n is differentiable, and if $f_n' \rightrightarrows g$, then f is differentiable and:
$$f'(x) = \lim_{n\to\infty} f_n'(x) = g(x).$$
11. **Term-by-Term Differentiation:** If $\sum f_k \rightrightarrows f$ uniformly and $\sum f_k' \rightrightarrows g$, then:
$$\left(\sum_{k=0}^{\infty} f_k(x)\right)' = \sum_{k=0}^{\infty} f_k'(x).$$

5 Power Series

12. If $r < R$, then a power series converges **uniformly and absolutely** on the interval $[-r, r]$, where R is the radius of convergence.
13. A power series can be **integrated and differentiated term-by-term** within its interval of convergence.
14. **Analytic functions are smooth:** $C^\omega \subset C^\infty$. That is, every analytic function is infinitely differentiable.

6 Compactness and Equicontinuity

A closed and bounded set in $\mathbb{R}^m$ is compact by the **Heine-Borel Theorem**. However, closed and bounded sets in C^0 are rarely compact.
A sequence of functions (f_n) in C^0 is **equicontinuous** if
$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that } |s - t| < \delta \text{ and } n \in \mathbb{N} \Rightarrow |f_n(s) - f_n(t)| < \varepsilon.$$
To distinguish, **pointwise equicontinuity** requires:
$$\forall\varepsilon > 0,\ \forall x \in [a, b]\ \exists\delta > 0 \text{ such that } |x-t| < \delta \text{ and } n \in \mathbb{N} \Rightarrow |f_n(x)-f_n(t)| < \varepsilon.$$
These definitions extend naturally to sets of functions. A set $E \subset C^0$ is equicontinuous if:
$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that } |s - t| < \delta \text{ and } f \in E \Rightarrow |f(s) - f(t)| < \varepsilon.$$
The key is that δ is uniform across all $f \in E$.

15. **Arzelà-Ascoli Theorem:** Every bounded, equicontinuous sequence of functions in $C^0([a, b], \mathbb{R})$ has a uniformly convergent subsequence.
16. **Subsequence Lemma:** If (f_k) is a subsequence of (g_n), then for each k there exists $r \geq k$ such that $f_k = g_r$.
17. **Arzelà-Ascoli Propagation Theorem:** Pointwise convergence of an equicontinuous sequence on a dense subset of the domain implies uniform convergence on the entire domain.
18. **Heine-Borel Theorem in Function Spaces:** A subset $E \subset C^0$ is compact if and only if it is closed, bounded, and equicontinuous.

7 Extras

Critical Point: A point θ is a critical point of f if $f'(\theta) = 0$ or $f'(x)$ does not exist at θ.
Inflection Point: A point θ is an inflection point of f if $f''(\theta) = 0$ and the concavity of f changes around θ.
Cauchy Mean Value Theorem: If f and g are differentiable on (a, b) and continuous on $[a, b]$, then there exists a point $\theta \in (a, b)$ such that:
$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$
Generalized Mean Value Theorem (MVT): If f is differentiable on (a, b) and continuous on $[a, b]$, then there exists $\theta \in (a, b)$ such that:
$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$
Darboux's Theorem: Every function that is Riemann integrable has the Darboux property, meaning the infimum of the upper sums equals the supremum of the lower sums.
Improper Integral: If f is integrable on an unbounded interval or has an infinite discontinuity, then the integral is called improper. It is defined as a limit:
$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$
Absolute Convergence: A series $\sum a_k$ converges absolutely if $\sum |a_k|$ converges.
Conditional Convergence: A series $\sum a_k$ converges conditionally if it converges, but $\sum |a_k|$ does not.
Monotone Convergence Theorem: If a sequence of functions f_n is increasing and converges pointwise to a function f, then:
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$
Fatou's Lemma: If f_n is a sequence of non-negative measurable functions, then:
$$\int_a^b \liminf_{n\to\infty} f_n(x)\,dx \leq \liminf_{n\to\infty} \int_a^b f_n(x)\,dx.$$
Heine's Theorem on Uniform Convergence: If a sequence of continuous functions (f_n) converges uniformly to f on a closed interval $[a, b]$, then f is continuous.
Arzelà-Ascoli Theorem (Uniform Convergence): A family of functions $\{f_n\}$ is uniformly bounded and equicontinuous if and only if it has a uniformly convergent subsequence on a compact set.
Uniform Convergence of Differentiable Functions: If $f_n \to f$ uniformly on $[a, b]$ and each f_n is differentiable, then f is differentiable and $f_n' \to f'$ uniformly on $[a, b]$.
Compactness in Function Spaces: A set of functions is compact in the space of continuous functions if it is bounded, closed, and equicontinuous.
Pointwise Convergence of Power Series: If a power series converges at a point x_0, it converges uniformly on any closed interval within its radius of convergence.

8 Test questions

1. A continuous function is differentiable (F, $f(x) = |x|$) 2. A differentiable function is continuous (T) 3. If a function is differentiable on an interval, its derivative is a continuous function (F, $f(x) = x^2 \sin\left(\frac{1}{x}\right)$) 4. Derivative cannot have jump discontinuities (T) 5. If a function is Riemann integrable it is also Darboux integrable (T) 6. If a function is Darboux integrable it is also Riemann integrable (T) 7. Every Riemann integrable function has finite number of points of discontinuity (f(x) = 1, x is rational, 0 otherwise)) 8. Every Riemann integrable function is bounded (T) 9. Antiderivative is always a continuous function (T) 10. Antiderivative of a continuous function is a differentiable function (T) 11. Every convergent series is absolutely convergent (F, $\sum(-1)^n \frac{1}{n}$) 12. Every absolutely convergent series is convergent (T) 13. Terms of every convergent series tend to zero (T) 14. If terms of a series tend to zero, it is convergent (F, $\sum \frac{1}{n}$) 15. Every convergent sequence of continuous functions converges to a continuous function (F, $f_n(x) = x^n$ on $[0, 1]$) 16. If a sequence of continuous functions converges to a continuous function, the convergence is uniform (F, x^n on $[0, 1]$) 17. Any sequence of bounded and uniformly equicontinuous functions on a compact interval has a uniformly convergent subsequence (T) a. Continuous instead of equicontinuous (F, equicontinuity required) b. Closed instead of compact interval (F, consider something not bounded)