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$\forall x, y \in \mathbb{F}: |x-y| \leq ||x|-|y||$ (reverse triangle inequality)

Let $x, y \in \mathbb{F}$.

Assume $|x-y| < ||x|-|y||$

Then $|x-y| = ||x|-|y|| + m$ where $m \in \mathbb{R}$. So

$|x|-|y| < |x-y| < |x|-|y|$

$m \in \mathbb{R}$

$|x|-|x+m| < |x-x+m| < |x|-|x+m|$

best upper bound
= lowest upper bound
= sup A

① upper bound $\forall x \in A \quad u \geq x$
② no lower upper bound
$\forall w < u \quad \exists x \in A: x > w$

highest lower bound
inf A

$\sqrt{2} = \sup\{x \in \mathbb{Q} : x^2 \leq 2$

① Let $x \in \mathbb{Q}$ with $x^2 \leq 2$

$|x| \cdot |x| = x^2 \leq 2 = \sqrt{2} \cdot \sqrt{2}$

So $|x| < \sqrt{2}$

② Let $w \in \mathbb{R}$, $w < \sqrt{2}$

Theorem:
The rational ~~num~~ dense in the real numbers.

$\forall x, y \in \mathbb{R}: x < y, \Rightarrow \exists r \in \mathbb{Q}: x < r < y$

Let $x, y \in \mathbb{R}$ with $y > x$

Then $y - x > 0$. Then $\exists n \in \mathbb{N}$ s.t. $(y-x) \cdot n > 1$

$\hookrightarrow \frac{1}{n} < y - x$

$\exists p \in \mathbb{Z} : \frac{p}{n} < x \leq \frac{p+1}{n}$

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- $x = \sup A$

① x is an upper bound of A
$\forall a \in A: x \geq a$

② x is lowest upper bound
$\forall \varepsilon > 0, \exists a \in A: a > x - \varepsilon$

<u>Completeness axiom</u>

Every bonded set has a supremum.

$\mathbb{Q}$ is an ordered field but not complete

$\mathbb{Z}$ is an ordered ring and complete (finite set)

*Dedekind cut

sup
inf

$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ sup: 1
inf: 0

$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$ sup: 1
inf: 0

$\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ sup: 1
inf: 1/2

$\left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ sup: $\frac{1}{2}$
inf: -1

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4},$

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<u>sequence</u>: function $\mathbb{N} \to \mathbb{R}$

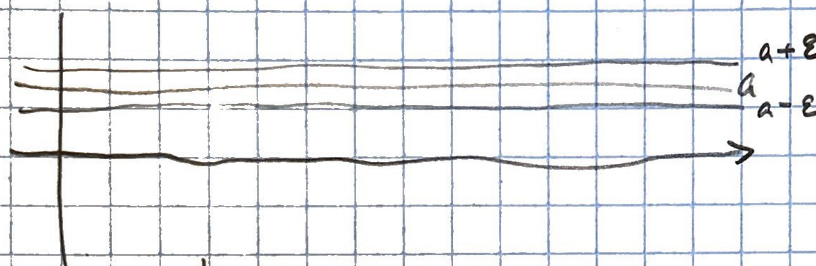
<u>properties</u> $\forall n \; a_{n+1} - a_n \geq 0$ increasing

$\phantom{\forall n \; a_{n+1} - a_n} > 0$ strictly increasing

monotone - increasing & decreasing

bounded: $\exists u \in \mathbb{R}, L \in \mathbb{R} \; \forall n \; L < s_n < U$

$\exists a, \forall \varepsilon > 0, \exists N \; \forall n > N. \; |a_n - a| < \varepsilon$



$\lim_{n \to \infty} \frac{1}{n} = 0$

<u>Proof:</u>

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon}$

Let $n > N$.

Then $-\varepsilon < \frac{1}{n} - 0 < \frac{1}{N} = \varepsilon$

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$\frac{1}{n} = \varepsilon \quad \frac{1}{\varepsilon} = n \quad \frac{1}{n} - 0 < \varepsilon \quad \frac{1}{\varepsilon} = N$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon}$. Let $n > N$

Then. $-\varepsilon < \frac{1}{n} < \frac{1}{N} = \varepsilon \not< \varepsilon$

$\lim \frac{1}{n+1} = 0$

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon+1}$

Let $n > N$

Then $-\varepsilon < \frac{1}{n} < \frac{1}{N} = \varepsilon$

Theorem

If $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} s_{n+1} = L$

Pf

$\lim_{n\to\infty} s_n = L$

$\Leftrightarrow \forall \varepsilon > 0 \ \exists N \ \forall n > N \quad L-\varepsilon < s_n < \varepsilon \ (*)$

So let $\varepsilon > 0$. Choose N according to $(*)$

Let $n > N$.

Then $n+1 > N$. So $L - \varepsilon < s_{n+1} < L + \varepsilon$

$\lim \frac{1}{5n} = 0$

Let $\varepsilon > 0$

Choose $N = \frac{1}{5\varepsilon}$

Let $n > N$

Then $-\varepsilon < \frac{1}{5n} < \frac{1}{5N} = \varepsilon$

$\lim 5 + \frac{1}{n} = 5$

Let $\varepsilon > 0$

Choose $N = \frac{1}{\varepsilon}$

Let $n > N$

$5 - \varepsilon < 5 < 5 + \frac{1}{n} < 5 + \frac{1}{N} = 5 + \varepsilon$

$\lim \frac{2n+3}{n-5} = 2$

Let $\varepsilon > 0$.

Choose $N = \frac{13}{\varepsilon} + 5$

Let $n > N$.

Then, $2 - \varepsilon < 2 = \frac{2n}{n} < \frac{2n+3}{n-5} = \frac{2(n-5)}{n-5} + \frac{13}{n-5} = 2 + \frac{13}{n-5}$

$n \geq 5$

$< 2 + \frac{13}{N-5} = 2 + \varepsilon$

$N - 5 = \frac{13}{\varepsilon}$

$N = \frac{13}{\varepsilon} + 5$

$$\lim \frac{\sin n}{n}$$

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon}$

Let $n > N$.

Then, $-\varepsilon = \frac{-1}{N} \leq \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} < \frac{1}{N} = \varepsilon$

$$\lim \frac{5n^4 + 3n}{10n^4 - 3n^3 + 3n + 7} = \frac{1}{2}$$

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon}$

Let $n > N$.

Then, $\frac{1}{2} - \varepsilon < \frac{1}{2} = \frac{5n^4}{10n^4} < \frac{5n^4 + 3n}{10n^4 - 3n^3 + 3n + 7} < \frac{1}{2} + \varepsilon$

$n \geq 2$

>0 for $n \geq 2$

$$\frac{5n^4 - \frac{3}{2}n^3 + \frac{3}{2}n + \frac{7}{2}}{10n^4 - 3n^3 + 3n + 7} + \frac{\frac{3}{2}n^3 + \frac{3}{2}n - \frac{7}{2}}{10n^4 - 3n^3 + 3n + 7} > 0$$

$$-\varepsilon < 0 < \frac{\frac{3}{2}n^3 + \frac{3}{2}n - \frac{7}{2}}{10n^4 - 3n^3 + 3n + 7} < \frac{3n^3}{3n^4} = \frac{1}{n} < \frac{1}{N} = \varepsilon$$

conv $\exists L \; \forall \varepsilon > 0 \; \exists N \; \forall n > N \quad |s_n - L| < \varepsilon$

nonconv $\forall L \; \exists \varepsilon > 0 \; \forall N \; \exists n > N \quad |s_n - L| \geq \varepsilon$

$\exists L \quad \forall \varepsilon > 0 \quad \exists N \quad \forall n > N \quad |s_n - L| < \varepsilon$

$\forall L \quad \exists \varepsilon > 0 \quad \forall N \quad \exists n > N \quad |s_n - L| \geq \varepsilon$

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$\lim 1-\frac{1}{\sqrt{n}} = 1$

$\frac{1}{\sqrt{n}} = \varepsilon$

$\sqrt{N}$

$\frac{1}{\varepsilon^2} = N$

Let $\varepsilon > 0$.

Choose $N = \frac{1}{\varepsilon^2}$

Let $n > N$.

$1-\varepsilon < \qquad < 1+\varepsilon$

$-\varepsilon < 0 < -\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\varepsilon}}$

$\lim (-1)^n$ DNE

$\in \quad \in \quad \varepsilon \quad \in \quad \varepsilon \quad \in \quad \varepsilon \quad \in$

Let $L \in \mathbb{R}$
Choose $\varepsilon = \frac{1}{2}$
Let $N \in \mathbb{R}$
Choose $n > N$, $\begin{cases} n \text{ even if } L < 0 \\ n \text{ odd if } L \geq 0 \end{cases}$

$|(-1)^n - L| \geq 1 > \varepsilon$

$S_n = n$

$\lim S_n = \infty$

$\forall M \; \exists N \; \forall n > N \; S_n > M$

Let $L \in \mathbb{R}$

Choose $\varepsilon = \frac{1}{2}$

Let $N \in \mathbb{R}$

Choose $n = \lceil \max(N, L, 0) \rceil + 1$

<u>Thm</u>

If $\lim s_n = a$ and $\lim s_n = b$, then $a = b$.

proof by contradiction

Assume $s_n \to a$ and $s_n \to b$ where $a \neq b$.

Choose $\varepsilon = \frac{|a-b|}{3} > 0$. Then, since $s_n \to a$, $\exists N_a$ s.t. $\forall n > N_a$: $|a - s_n| < 3$. ______ $s_n \to b$ ~~$\exists N_b$~~ s.t. N_b: $|b - s_n|$
$< \varepsilon$.

Choose $n \geq \max\{N_a, N_b\}$. Then $3\varepsilon = |a-b| \leq |a - s_n| + |s_n - b| < 2\varepsilon$

$3\varepsilon \not< 2\varepsilon$

proof 2

Assume $s_n \to a$, $s_n \to b$, WTS $s_n = b$.

Let $\varepsilon > 0$. Then $\exists N_a, N_b$ s.t. ______

Choose $n > \max\{N_a, N_b\}$

Then $|a-b| \leq |a - s_n| + |b - s_n| < 2\varepsilon$.

Therefore $|a-b| = 0$ and $a = b$ ∎

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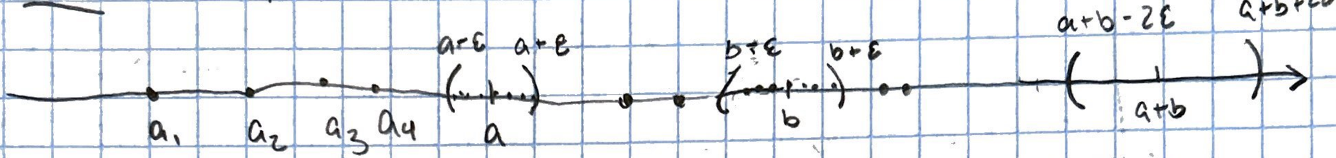
$a_n \to a \qquad b_n \to b \qquad c_n = a_n + b_n$

ex $\quad a_n = \frac{1}{n} \to 0 \quad b_n = 1 - \frac{1}{n^2} \to 1 \quad c_n = 1 + \frac{1}{n} - \frac{1}{n^2} \to 1$

Thm $\lim c_n = a + b$



Fact: $\lim a_n = a \iff \forall a$: only finitely many a_n outside $(a-\varepsilon, a+\varepsilon)$

Want: $\forall \varepsilon > 0 \ \exists N \ \forall n \in N \quad a+b-\varepsilon < c_n < a+b+\varepsilon$

$a - \frac{\varepsilon}{2} \qquad a + \frac{\varepsilon}{2} \qquad b - \frac{\varepsilon}{2} \qquad b + \frac{\varepsilon}{2} \rightarrow a+b-\varepsilon \qquad a+b+\varepsilon$

Let $\varepsilon > 0$.

Choose N_a s.t. $\forall n > N_a \quad a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$ since $a_n \to a$
Choose N_b s.t. $\forall n > N_b \quad b - \frac{\varepsilon}{2} < b_n < b + \frac{\varepsilon}{2}$ since $b_n \to b$

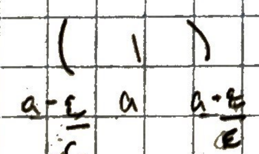
Choose $N = \max\{N_a, N_b\}$

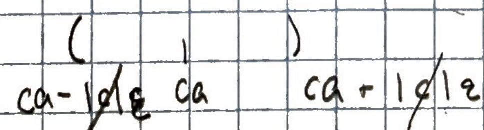
Let $n > N$.

Then $a - \frac{\varepsilon}{2} + b - \frac{\varepsilon}{2} < a_n + b_n < a + \frac{\varepsilon}{2} + b + \frac{\varepsilon}{2}$

$$a+b-\varepsilon < a_n + b_n < a+b+\varepsilon$$

$c \cdot a_n \to c \cdot a$





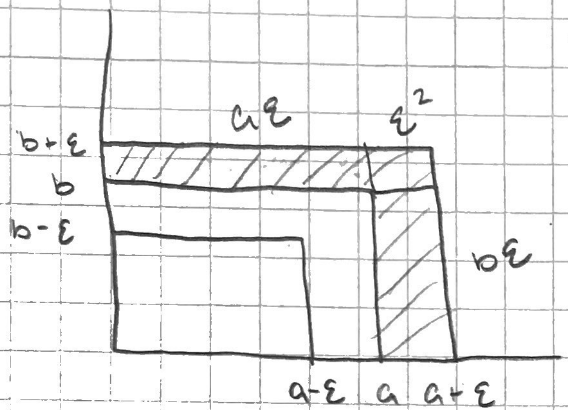
~~Proof #2~~

If $c=0$, $c \cdot a_n = 0$ so $ca_n \to 0$. Assume $c \neq 0$.

Let $\varepsilon > 0$.

Choose N s.t. $\forall n \geq N$ $\quad a - \frac{\varepsilon}{|c|} < a_n < a + \frac{\varepsilon}{|c|}$

Then $ac - \varepsilon < a_n c < ac + \varepsilon$



$$(a-\varepsilon)(b-\varepsilon) < a_n b_n < (a+\varepsilon)(b+\varepsilon)$$
$$= ab + (a+b)\varepsilon + \varepsilon^2$$

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$$
$$= |a_n \underbrace{(b_n - b)}_{\to 0} + \underbrace{\underbrace{(a_n - a)}_{\to 0} b}_{\to 0}|$$

$$\underbrace{|a_n| < |a| + \varepsilon}_{0}$$

$a_n \neq 0$, $a_n \to a \neq 0$

$\Rightarrow \frac{1}{a_n} \to \frac{1}{a}$

Let $\varepsilon > 0$.

Choose N s.t. $\forall n \geq N$

$a_n - \varepsilon < a < a_n + \varepsilon$

$\frac{1}{a-\varepsilon} < \frac{1}{a_n} < \frac{1}{a+\varepsilon}$

Let $\varepsilon > 0$.

Choose N s.t. $\forall n \geq N$ $\quad \frac{1}{a_n} - \varepsilon < a < \frac{1}{a_n} + \varepsilon$

Then

1. $a > 0$
2. $a - \varepsilon > 0$

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Limit Laws

If $a_n \to a^{\in\mathbb{R}}$, $b_n \to b^{\in\mathbb{R}}$, $c \in \mathbb{R}$

1. $a_n + b_n \to a+b$
2. $c \cdot a_n \to c \cdot a$
3. $a_n \cdot b_n \to a \cdot b$
4. $\frac{1}{a_n} \to \frac{1}{a}$

Q. If $a_n + b_n \to L$, do (a_n) and (b_n) converge

Squeeze Theorem

<u>want</u> $\forall \varepsilon > 0 \ \exists N \ \forall n > N \quad a-\varepsilon < b_n < a+\varepsilon$

<u>have</u> $\forall \varepsilon > 0 \ \exists N_a \ \forall n > N_a \quad a-\varepsilon < a_n < a+\varepsilon$

$\forall \varepsilon > 0 \ \exists N_c \ \forall n > N_c \quad a-\varepsilon < c_n < a+\varepsilon$

$\exists \tilde{N} \ \forall n \geq \tilde{N} \quad a_n \leq b_n \leq c_n$

Let $\varepsilon > 0$.

Choose $N = \max\{N_a, N_c, \tilde{N}\}$

Let $n > N$.

Then $\underset{①}{a-\varepsilon < a_n} \underset{③}{\leq} b_n \underset{③}{\leq} c_n \underset{②}{<} a+\varepsilon$

a_n decreasing

example $\quad b_n = \frac{\log n}{n}$

a_n monotone bounded

$\Rightarrow a_n$ convergent

$\exists L \ \forall \varepsilon > 0 \ \exists N \ \forall n \geq N \quad L-\varepsilon < a_n < L+\varepsilon$

Choose $L = \sup\{a_n\}$

Let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ s.t. $a_N > L-\varepsilon$

Let $n > N$.

Then $L-\varepsilon < a_N < a_n \leq L < L+\varepsilon$

Choose $L = \inf\{a_n\}$

Let $\varepsilon > 0$.

Choose $N \in \mathbb{N}$ s.t. $a_N < L+\varepsilon$

Let $n > N$.

Then $L-\varepsilon < L \leq a_n \leq a_N < L+\varepsilon$

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Subsequences

$$\{(k, s_k)\} \supseteq \{(n_k, s_{n_k})\}$$

$$\updownarrow$$

$$\{(k, s_{n_k})\}$$

- ordered
- infinitely long

$$\begin{array}{ccccc} s_2 & s_4 & s_6 & s_8 & s_{10} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ s_{n_1} & s_{n_2} & s_{n_3} & s_{n_4} & s_{n_5} \end{array}$$

$n_k = 2k$

Index sequence

$n_k : \mathbb{N} \to \mathbb{N}$

$k \to n_k$

strictly increasing

$(1,-1), (2,1), (3,-1), \ldots$

$-1, \underline{1}, -1, \underline{1}, \ldots \underline{\ }$

$(2,1), (4,1), (6,1), \ldots \longleftrightarrow (1,1), (2,1), (3,1), \ldots$

Theorem

$a_n \xrightarrow[n\to\infty]{} L \iff$ all subsequences have $a_{n_k} \xrightarrow[k\to\infty]{} L$

$\forall \varepsilon > 0\ \exists N\ \forall n > N\ \ L-\varepsilon < a_n < L+\varepsilon$

$\forall \varepsilon > 0\ \exists K\ \forall k > K\ \ L-\varepsilon < a_{n_k} < L+\varepsilon$

Let $\varepsilon > 0$.

Choose $K = N$ from $*$

Let $k > K$.

Then $n_k \geq k > K = N$, so $L-\varepsilon < a_{n_k} < L+\varepsilon$

Subsequential Limit

a limit of a sequence

example The set of subsequential

① limits of $\left(\frac{1}{n}\right)$ is $\{0\}$

② $\left((-1)^n\right)$ $\{-1, 1\}$

③ $(\sin n)$ $[-1, 1]$

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$S_n \to a \iff$ every subsequence $S_{n_k} \to a$

allowing for $a \in \{\pm\infty\}$ $\iff$ set of subs limits $= \{a\}$

Fact

For any countable set there exists a sequence s_n with set of ss limits $\supseteq S$.

$S \subseteq \mathbb{R} \cup \{\pm\infty\}$

1st	1	2	3	4	5	6	7
2nd	1	2	3	4	5	6	7
3rd	1	2	3	4	5	6	7

Ex

(S_n)

$\exists (S_{n_k^q}) \to q \quad \forall q \in \mathbb{Q} \cap [0,1]$

$S_{n_k^{1/2}} \to \frac{1}{2}$

$S_{n_k^{1/3}} \to \frac{1}{3}$

Claim for some $r \in [0,1]$ $(r \in \mathbb{R} \setminus \mathbb{Q})$

$\exists S_{n_i^r} \to r$

① Choose n_1^r so that $|S_{n_1^r} - r| < 1$

② Choose $n_2^r > n_1^r$ s.t. $|S_{n_2^r} - r| < \frac{1}{2}$

⋮

Choose $n_i^r > n_{i-1}^r$ s.t. $|S_{n_i^r} - S_{n_{i-1}^r}| < \frac{1}{i}$

Theorem (Bolzano - Weierstraß)

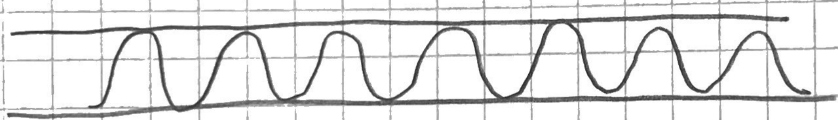
Every bounded sequence has a convergent subsequence.

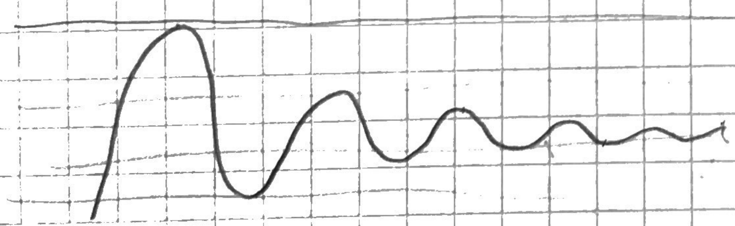
<u>Lemma</u>

Every sequence has a monotone subsequence

<u>remark</u>

then done w/ BW, since monotone subsequence of a bounded sequence is bounded, and thus convergent.





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__Last time__

$\underline{\varepsilon / \delta}$ $\forall x \, \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall y \; |y-x| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

may depend on x & ε

$\underline{\text{seq}}$ $\forall x \; \forall x_n \to x \quad f(x_n) \to f(x)$

$f(x) = e^x$

Let $\varepsilon > 0$.
Choose $N =$
Let $n > P$

Then, $|e^{x_n} - e^x| = |e^x (e^{x_n - x} - 1)|$

$x_n \to x$
want $e^{x_n} \to e^x$
$\iff e^x e^{x_n - x} \to e^x$
$e^{x_n - x} \to 1$

$$1 + (x_n - x) \le e^{x_n - x} \le 1 + (x_n - x) + (x_n - x)^2 \quad \text{for } (x_n - x) < 1$$

0 0 0

1 1 1

squeeze

$f(x) = e^x$
Let $x \in \mathbb{R}$
Let $\varepsilon > 0$
Choose $\delta = \max\{1, \frac{\varepsilon}{2e^x}\}$
Let $y \in \mathbb{R}$ with $|y - x| < \delta$

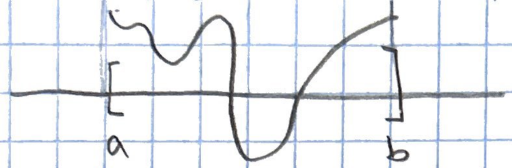
$1 + t \le e^t \le 1 + t + t^2$

$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots$

$$|e^y - e^x| = |e^x(e^{y-x} - 1)| \le e^x(\delta + \delta^2) \le e^x \cdot 2\delta \le \varepsilon$$

$$1 - \delta \le 1 + y - x \le e^{y-x} \le 1 + y - x + (y-x)^2 < 1 + \delta + \delta^2$$

$f: [a,b] \to \mathbb{R}$ continuous



<u>Thm</u>

f is bounded

<u>Pf</u> Assume f is not bounded.

$\forall n \; \exists x_n \in [a,b] : |f(x_n)| > n$

B-W $\exists x_{n_k}$ convergent to some $x \in [a,b]$

Since f is continuous, $f(x_{n_k}) \to f(x)$ but $|f(x_{n_k})| > n_k$

bounded but unbounded

$\rightarrow\leftarrow$

<u>Extreme Value Theorem</u>

$f: [a,b] \to \mathbb{R}$ continuous

$\Rightarrow \exists \underline{x}, \overline{x} \in [a,b]$ s.t. $\forall x \in [a,b]: f(\underline{x}) \leq f(x) \leq f(\overline{x})$

[$f(x)$ takes on both minimum and maximum)

<u>pf:</u> (maximum)

Define $U = \sup\limits_{x \in [a,b]} f(x)$ (exist since $f(x)$ bounded)

$y_n \to U$ where each $y_n = f(x_n)$ for some $x_n \in [a,b]$

B-W $\to \exists x_{n_k} \to x$, then $f(n_k) \to f(x)$

$\overset{\|}{y_{n_k}}$ (cont)

$\downarrow$

U

$x_n = (-1)^n \left(1 - \frac{1}{n}\right)$

$f(x_n) = \left(1 - \frac{1}{n}\right)^2 \to 1$

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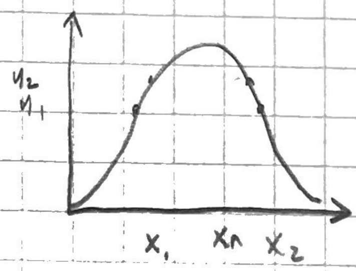
Last time

EVT

$f: [a,b] \to \mathbb{R}$ continuous

closed

$\Rightarrow \exists \underline{x}, \bar{x} \in [a,b]$ s.t. $\forall x \in [a,b]$ $f(\underline{x}) \leq f(x) \leq f(\bar{x})$

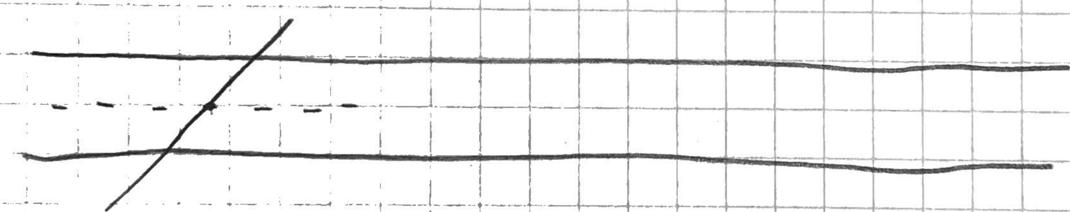


$Y = \{f(x) : x \in [a,b]\}$

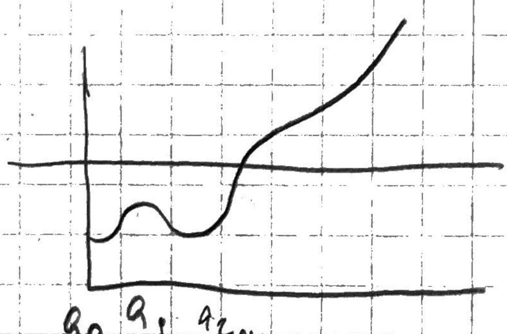
$\sup Y$ $\exists (y_n) \subseteq Y$ with $\lim y_n = \sup Y$

$\exists (x_n) \subseteq [a,b]$ with $f(x_n) = y_n$

x_n



Let $f: [a,b] \to \mathbb{R}$ continuous
with $f(a) < 0 < f(b)$



$a_n \to A$ $\quad$ a_n is increasing and bounded by b_0

$b_n \to B$ $\quad$ b_n is decreasing ______ a_0

$|b_n - a_n| = |b_0 - a_0| \cdot \frac{1}{2^n} \to 0$

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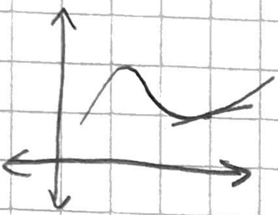
Derivatives

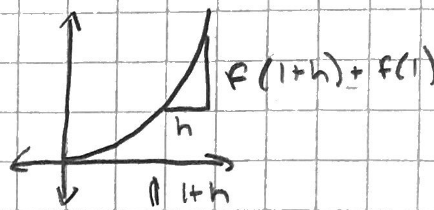


instantaneous speed

speed = $\frac{\text{distance}}{\text{time}}$

example

$f(t) = t^2$



$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h}$$

$$= \frac{1 + 2h + h^2 - 1}{h}$$

$$\lim_{h \to 0} 2 + h = 2$$

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \frac{t^2 + 2t\cancel{h} + h^{\cancel{2}} - t^2}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{2t + h}{1} = 2t$$

Definition

$f: (a,b) \to \mathbb{R}$, $x \in (a,b)$

Then, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Continuous

$\lim_{y \to x} f(y) = f(x)$ $\qquad \lim_{h \to 0} f(x+h) = f(x)$

$\lim_{h \to 0} (f(x+h) - f(x)) = 0$

Theorem

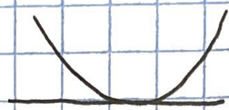
f differentiable in x $\Rightarrow$ f continuous in x

Fact not $\Leftarrow$

$f(x)=|x|$ for $x=0$

$$\lim_{h\to 0}\frac{|h|-0}{h}$$

$$f(x)=\begin{cases} x & x\in\mathbb{Q} \\ 0 & x\in\mathbb{Q}\end{cases}\qquad \frac{f(h)-f(0)}{h}=\begin{cases}\frac{h}{h}=1, & h\in\mathbb{Q}\\ \frac{0}{h}=0, & h\notin\mathbb{Q}\end{cases}$$



$$f(x)=\begin{cases} x^2 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}\end{cases}$$

$$\frac{f(h)}{h}=\frac{h^2}{h}=h\to 0$$

differentiable ($\exists$ continuous)

$(f+g)'(x)=f'(x)+g'(x)$ <u>always</u>: $f'(x)$ and $g'(x)$ exist

$$\lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \quad\Big|\quad \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$(f\cdot g)'(x)=\lim_{n\to\infty}\frac{f(x+h_n)\,g(x+h_n)-f(x)\,g(x)}{h_n}$$

$$f'(x)g(x)+f(x)g'(x)=\lim_{n\to\infty}\frac{f(x+h_n)g(x)-f(x)g(x)+f(x)g(x+h_n)-f(x)g(x)}{h_n}$$

$$\lim_{n\to\infty}\frac{f(x+h_n)-f(x)}{h_n}\cdot \cancel{g(x)}\,g(x+h_n)+\frac{g(x+h_n)-g(x)}{h_n}\cdot f(x)$$

$$=\lim_{h\to 0}\underbrace{\frac{f(x+h_n)g(x+h_n)-f(x)g(x)}{h_n}}_{\to(f\cdot g)'(x)}+\underbrace{\frac{f(x+h_n)-f(x)}{h_n}}_{\to f'(x)}\underbrace{(g(x)-g(x+h_n))}_{\to 0}$$

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$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Let I be an open interval.

$f, g : I \to \mathbb{R}$ be differentiable at $c \in I$

<u>Chain Rule:</u>

$f(g)'(c) = f'(g(c)) \cdot g'(c)$

<u>pf</u>

$$d(y) = \begin{cases} \frac{f(y) - f(g(c))}{y - g(c)} & y \neq g(c) \\ f'(g(c)) & y = g(c) \end{cases}$$

$$\frac{f(g(x)) - f(g(c))}{x - c} = d(g(x)) \cdot \frac{g(x) - g(c)}{x - c}$$

<u>Case 1:</u> $g(x) \neq g(c)$

$$\frac{f(g(x)) - f(g(c))}{x - c} = \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}$$

<u>Case 2:</u> $g(x) = g(c)$

$$\frac{f(g(x)) - f(g(c))}{x - c} = f'(g(c)) \cdot \frac{g(x) - g(c)}{x - c} \nearrow 0$$

$$= 0$$

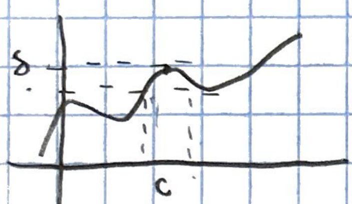
$$(f \circ g)(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c}$$

$$= \lim_{x \to c} d(g(x)) \frac{g(x) - g(c)}{x - c}$$

$$= f'(g(x)) \cdot g'(c)$$

Def

Let $f: A \to \mathbb{R}$. f has a local maximum at $c \in A$
if $\exists \delta > 0$ s.t. $\forall x \in A$ when $|x-c| < \delta$ then $f(x) \leq f(c)$



Let I be an open interval and suppose $f: I \to \mathbb{R}$ is differentiatable at $c \in I$. If f has a local max Imin at c, $f'(c) = 0$

Pf

Assume f is diff at c

Thus $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$

Assume f has a local max. Then, $\exists \delta > 0$ where $x \in I$ and $|x-c| < \delta$ ~~a~~ $f(x) \leq f(c)$ $(f(x)-f(c) \leq 0)$

$$\frac{f(x)-f(c)}{x-c} = \begin{cases} \leq 0 & x > c \\ \geq 0 & x < c \end{cases}$$

Pick two sequences, one approaching c from the right and the other from the left. They need to be within δ of c and converge to c. (a_n, b_n)

$c-\delta < a_n < c \quad \forall n \quad a_n \to c$

$\frac{f(a_n)-f(c)}{f-c} \geq 0 \ \forall n \quad f'(c) \geq 0$

$c < b_n < c+\delta \quad \forall n \quad b_n \to c$

$\frac{f(b_n)-f(c)}{f-c} \leq 0 \ \forall n \quad f'(c) \leq 0$

f is diff $\Rightarrow$ f is cont $\Rightarrow$ IVP

<u>Darboux's Thm</u>

Suppose $f: [a,b] \to \mathbb{R}$ is diff
If $f'(a) < \alpha < f'(b)$ $\exists c \in (a,b)$
$f'(c) = \alpha$

<u>Rolle's Thm</u>

$f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$
and diff on (a,b). If $f(a) = f(b)$ then
$\exists c \in (a,b)$ where $f'(c) = 0$.

<u>Two cases</u>
<u>Case 1</u>: $\max = c_1 \in [a,b]$ $\quad f(x) \leq f(a)\ \forall x$
$\min = c_2 \in [a,b]$ $\quad f(x) \geq f(a)\ \forall x$

WLOG suppose $c_1 = a$ and $c_2 = b$. Assume $c_1 = c_2$
thus $\min = \max \Rightarrow c_1 = c_2$.
$\exists \delta > 0$ where $x \in I$ and $|x - c| < \delta$, $f(x) = f(c)$
$f(x) - f(c) = 0$

<u>Case 2</u>: $\max = c_1 \in (a,b)$
$\min = c_2 \in (a,b)$

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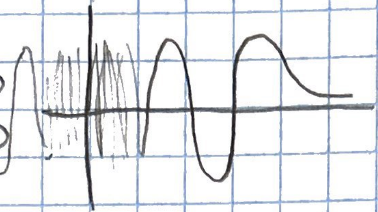
Last time
- Chain rule

$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



$f'(x) = \cos\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2}$

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \overset{?}{=}$

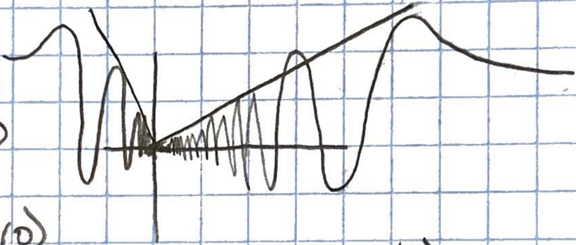
$\forall x_n \to 0: \lim_{n \to \infty} \sin\left(\frac{1}{x_n}\right) \overset{?}{=}$

$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}, \quad \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Final

$x_n = \frac{1}{2\pi n}, \quad \sin\left(\frac{1}{x_n}\right) = \sin(2\pi n) = 0$

$g(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



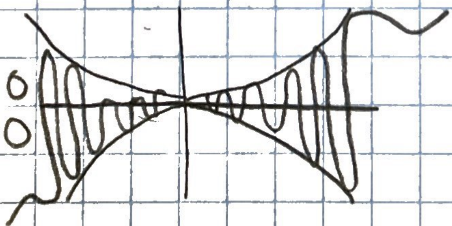
$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$

$g'(x) = x\cos\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2} + \sin\left(\frac{1}{x}\right)$

$\lim_{h \to 0} \frac{h\sin\left(\frac{1}{h}\right)}{h}$

$\lim_{h \to 0} \sin\frac{1}{h}$ DNE

$h(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



$c'(x) = -\cos\frac{1}{x} + 2x\sin\left(\frac{1}{x}\right)$

$f'(a) > 0$

$f'(b) < 0$

$\lim_{h \to 0} \frac{c(h) - c(0)}{h} = \frac{c(h)}{h} = h\sin\frac{1}{h} = 0$

$a < \xi < b$

$f'(c) = 0$

$f'(a) > \alpha, \; f'(b) < \alpha \Rightarrow f'(\xi) = 0$

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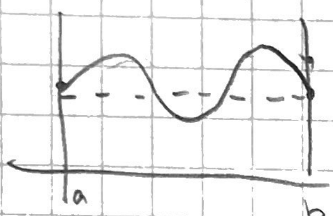
Rolle's Thm



$a<b$,

$f:[a,b] \to \mathbb{R}$, differentiable, $f(a)=f(b)$

$\Rightarrow \exists x \in (a,b): f'(x)=0$

Proof

Idea Show there is a min or max in (a,b)

Case 1: $\{\underline{x}, \bar{x}\} \subseteq \{a,b\} \Rightarrow f$ is constant

$\Rightarrow f'(x)=0 \ \forall x \in (a,b)$

Case 2: $\underline{x} \notin \{a,b\} \Rightarrow f'(\underline{x})=0$

Case 3: $\bar{x} \notin \{a,b\} \Rightarrow f'(\bar{x})=0$

Mean Value Theorem

$a<b$

$f:[a,b] \to \mathbb{R}$ differentiable

$\Rightarrow \exists x \in (a,b): f'(x) = \frac{f(b)-f(a)}{b-a}$

Consider $g(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$

Then $g(a)=f(a)-\frac{f(b)-f(a)}{b-a}a$

$= \frac{(b-a)f(a)-af(b)+af(a)}{b-a}$

$= \frac{bf(a)-af(b)}{b-a}$

$g(b)= \dots = \frac{bf(a)-af(b)}{b-a}$

g differentiable, $g(a)=g(b)$

$\Rightarrow \exists x$ with $g'(x)=0$

$0=g'(x)=f'(x)-\frac{f(b)-f(a)}{b-a} \Rightarrow \square$

$f'(x) = 0 \ \forall x$

$\Rightarrow f(x) = c$

Pf Suppose not. $f(a) < f(b)$ (WLOG)

$\underset{MVT}{\Rightarrow} \exists x \in (a,b)$ with $f'(x) = \frac{f(b)-f(a)}{b-a} \neq 0$

$f'(x) > 0$ everywhere

$\Rightarrow f$ strictly increasing

Pf otherwise, $\exists a < b$ with $f(a) \geq f(b)$

$\underset{MVT}{\Rightarrow} \exists x \in (a,b)$ with $f'(x) = \frac{f(b)-f(a)}{b-a} \leq 0$

$$\frac{f(x)}{g(x)} = \frac{f'(0)(x)}{g'(0)(x)} + \frac{f''(\xi)\frac{x^2}{2}}{g''(\lambda)\frac{x^2}{2}}$$

$$\frac{f'(0)}{g'(0)} + \frac{f''(\xi)\frac{x}{2}}{g''(\lambda)\frac{x}{2}}$$

$f'(x_1) = 4$ MVT

$f'(x_2) = 1$ MVT

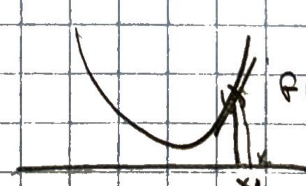
$f'(x_3) = -2$ MVT

$f'(x_4) = 0$ Darboux, IVT + Rolle

Assume $f(0) = g(0)$, f, g differentiable, and $\lim \frac{f'(x)}{g'(x)} = L$

Then, $\lim_{x \to 0} \frac{f(0)}{g(0)} = L$

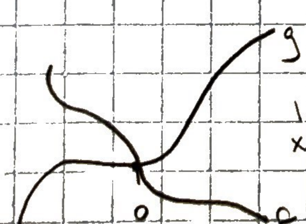
NOT: $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$



$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

$f(x) = f(x_0) + f'(x_0)(x - x_0)$

$+ f''(\xi)\frac{(x-x_0)^2}{2}$



$$\lim_{x \to 0} \frac{f(x)}{g(x)} \approx \frac{f'(0)\cancel{(x-x_0)}}{g'(0)\cancel{(x-x_0)}}$$

Sequence

a_n $\boxed{\text{converges}}$ to $a \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists N$ s.t. $|a_n - a| < \varepsilon\ \forall n > N$

Let $\varepsilon > 0$.
Choose $N =$
Let $n > N$.
Then, ...

a_n $\boxed{\text{diverges}}$ if $\exists \varepsilon > 0\ \forall N\ \exists n > N$ s.t. $|a_n - a| \geq \varepsilon$

$\boxed{\text{Convergent} \Rightarrow \text{bounded}}$

tail is infinite
everything before is finite
finite can't be unbounded

$\boxed{\text{Squeeze}}$

$a_n \leq x_n \leq b_n\ \forall n$, $a_n \to L$ and $b_n \to L \Rightarrow x_n \to L$

$\exists N_1$
$\exists N_2$
choose $\max\{N_1, N_2\}$

$\boxed{\text{monotone inc}}$ if $a_n \leq a_{n+1}$
$\boxed{\text{dec}}$ if $a_n \geq a_{n+1}$

a_n is monotone; a_n convergerges $\Leftrightarrow$ a_n is bounded

$a_n \to a \Leftrightarrow \forall a_{n_k} \to a$

Different subsequential limits $\Rightarrow$ diverges

Monotone sequence and convergent subsequence $\Rightarrow a_n$ converges

$\boxed{\text{BW}}$

Lemma: Every sequence has monotone subsequence

* Every bounded sequence has a convergent subsequence

A sequence is $\boxed{\text{Cauchy}}$ if $\forall \varepsilon > 0\ \exists N$ s.t. $|a_m - a_n| < \varepsilon\ \forall m, n > N$

Cauchy $\Rightarrow$ bounded

Converges $\Leftrightarrow$ Cauchy

Continuity

Let $f: A \to \mathbb{R}$ and c be a limit point of A.

$\lim_{x \to c} f(x) = L$ if $\forall \varepsilon > 0\ \exists \delta > 0$

where $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Continuous at $c \in A$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in A$ where $|x - c| < \delta$ we have $|f(x) - f(c)| < \varepsilon$

$\boxed{\text{EVT}}$ A is closed and bounded $[\ ]$ $\exists \underline{x}, \overline{x}$

Thus $\exists a_n$ which converges to sup

and $\exists b_n$ which converges to inf

$\boxed{\text{IVT}}$ If f is continuous on $[a,b]$ and $f(a) \leq d \leq f(b)$, then $\exists c \in (a,b)$ where $f(c) = d$

$\boxed{\text{Uniformly continuous}}$

f is uniformly continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in A$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Differentiation

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Differentiable $\Rightarrow$ continuous ($|x|$ shows it's not both ways)

Local max at $c \in A$ if $\exists \delta > 0$ s.t. $\forall x \in A$ with $|x - c| < \delta$, $f(x) \leq f(c)$

$\boxed{\text{MVT}}$

Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$