Fractional Calculus in Image Processing

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Fractional Calculus

- The fractional-order calculus theory has become a hot research topic in the area of digital image processing in recent years.
- The fractional-order calculus operation is a generalization method of the traditional integral differential.
- The implementation of the fractional-order calculus is more complex than integer-order, but it extends the order of the differential operator and has more freedom degree and flexibility.

Geometric Meaning of fractional derivative and Integration

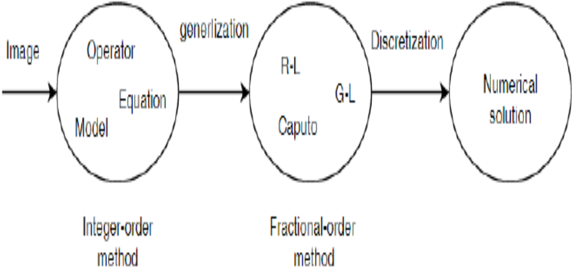
1. Fractional derivative is the generalized slope of its function also called fractional slope and
2. Fractional integral is generalized Euclidean measurement for shaping image also called fractional Euclidean measurement.

Physical Meaning

The physical meaning of fractional differential is the generalized amplitude-and-phase modulation

Here, the amplitude of original signal is changing with its frequency as fractional power exponent, while the phase is generalized Hilbert transform of frequency.

Fractional-order Image Processing Flow



Fractional-order Derivative

The fractional-order derivative possesses non-local property.

$$D^{\alpha}u(x) = \frac{d^{\alpha}}{dx^{\alpha}}u(x) \tag{1}$$

The definition of fractional-order derivative is not unique.
The popular definitions are

- Grüwald-Letnikov
- Riemann-Liouville
- Caputo

Fractional order Derivative Definitions [1]

Grünwald-Letnikov

$$ {}_{a}^{GL}D_{x}^{\alpha}u(x) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\frac{x-a}{h}}(-1)^{k}\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}u\left(x-kh\right) \tag{2}$$

[1] Y. F. Pu et al., "Fractional Differential Mask: A Fractional Differential-Based Approach for Multiscale Texture Enhancement," *IEEE Transactions on Image Processing*, vol. 19, no. 2, pp. 491–511, 2010.

Fractional order Derivative Definitions

Riemann-Liouville

$$ {}_{a}^{RL}D_{x}^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^{m}}{\partial x^{m}} \int_{a}^{x} \frac{u(\zeta)}{(x-\zeta)^{\alpha-m+1}} d\zeta, (m-1) < \alpha < m \quad (3) $$

Caputo

$$ {}_{a}^{C}D_{x}^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} \frac{u^{(m)}(\zeta)}{(x-\zeta)^{\alpha-m+1}} d\zeta, (m-1) < \alpha < m \quad (4) $$

RL fractional differential filter (m=2)

$$
\begin{cases}
C_{-1}^{\alpha} &= \dfrac{1}{\Gamma(3-\alpha)} \\
C_{0}^{\alpha} &= \dfrac{2^{2-\alpha}-3}{\Gamma(3-\alpha)} \\
C_{1}^{\alpha} &= \dfrac{3-3.2^{2-\alpha}-3^{2-\alpha}}{\Gamma(3-\alpha)} \\
&\vdots \\
C_{k}^{\alpha} &= \dfrac{-(k-1)^{2-\alpha}+3k^{2-\alpha}-3(k+1)^{2-\alpha}+(k+2)^{2-\alpha}}{\Gamma(3-\alpha)} \\
&\vdots \\
C_{n-2}^{\alpha} &= \dfrac{-(n-3)^{2-\alpha}+3(n-2)^{2-\alpha}-3(n-1)^{2-\alpha}+(n)^{2-\alpha}}{\Gamma(3-\alpha)} \\
C_{n-1}^{\alpha} &= \dfrac{(2-\alpha)n^{1-\alpha}-2n^{2-\alpha}+3(n-1)^{2-\alpha}-(n-2)^{2-\alpha}}{\Gamma(3-\alpha)} \\
C_{n}^{\alpha} &= \dfrac{(2-3\alpha+\alpha^2)n^{-\alpha}-(2-\alpha)n^{1-\alpha}+n^{2-\alpha}-(n-1)^{2-\alpha}}{\Gamma(3-\alpha)}
\end{cases}
$$

Caputo fractional order differential filter (m=2)

$$
\begin{cases}
C_{-1}^{\alpha} & = \dfrac{1}{\Gamma(3-\alpha)} \\
C_{0}^{\alpha} & = \dfrac{2^{2-\alpha}-3}{\Gamma(3-\alpha)} \\
C_{1}^{\alpha} & = \dfrac{3-3.2^{2-\alpha}-3^{2-\alpha}}{\Gamma(3-\alpha)} \\
 & \vdots \\
C_{k}^{\alpha} & = \dfrac{-(k-1)^{2-\alpha}+3k^{2-\alpha}-3(k+1)^{2-\alpha}+(k+2)^{2-\alpha}}{\Gamma(3-\alpha)} \\
 & \vdots \\
C_{n-2}^{\alpha} & = \dfrac{-(n-3)^{2-\alpha}+3(n-2)^{2-\alpha}-3(n-1)^{2-\alpha}+(n)^{2-\alpha}}{\Gamma(3-\alpha)} \\
C_{n-1}^{\alpha} & = \dfrac{-2n^{2-\alpha}+3(n-1)^{2-\alpha}-(n-2)^{2-\alpha}}{\Gamma(3-\alpha)} \\
C_{n}^{\alpha} & = \dfrac{n^{2-\alpha}-(n-1)^{2-\alpha}}{\Gamma(3-\alpha)}
\end{cases}
\tag{6}
$$

			$\cdots$	0	$\cdots$			
				$\vdots$				
			$\cdots$	0	$\cdots$		$\vdots$	$\vdots$
$\vdots$		$\vdots$	0	$\cdots$	0	$\cdots$	0	0
0	$\cdots$	0	C_{-1}^{α}	C_0^{α}	$\cdots$	C_k^{α}	$\cdots$	C_n^{α}
$\vdots$		$\vdots$	0	$\cdots$	0	$\cdots$	0	0
			$\cdots$	0	$\cdots$		$\vdots$	$\vdots$
				$\vdots$				
			$\cdots$	0	$\cdots$			

(a)

			$\cdots$	0	$\cdots$			
				$\vdots$				
$\vdots$	$\vdots$		$\cdots$	0	$\cdots$			
0	0	$\cdots$	0	$\cdots$	0	$\vdots$		$\vdots$
C_n^{α}	$\cdots$	C_k^{α}	$\cdots$	C_0^{α}	C_{-1}^{α}	0	$\cdots$	0
0	0	$\cdots$	0	$\cdots$	0	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\cdots$	0	$\cdots$			
				$\vdots$				
			$\cdots$	0	$\cdots$			

(b)

			$\cdots$	0	$\cdots$			
				$\vdots$				
			$\cdots$	0	$\cdots$			
$\vdots$		$\vdots$	0	C_{-1}^{α}	0	$\vdots$		$\vdots$
0	$\cdots$	0	$\vdots$	C_0^{α}	$\vdots$	0	$\cdots$	0
$\vdots$		$\vdots$	0	$\vdots$	0	$\vdots$		$\vdots$
			$\vdots$	C_k^{α}	$\vdots$			
		$\cdots$	0	$\vdots$	0	$\cdots$		
		$\cdots$	0	C_n^{α}	0	$\cdots$		

(c)

		$\cdots$	0	C_n^{α}	0	$\cdots$		
		$\cdots$	0	$\vdots$	0	$\cdots$		
			$\vdots$	C_k^{α}	$\vdots$			
$\vdots$		$\vdots$	0	$\vdots$	0	$\vdots$		$\vdots$
0	$\cdots$	0	$\vdots$	C_0^{α}	$\vdots$	0	$\cdots$	0
$\vdots$		$\vdots$	0	C_{-1}^{α}	0	$\vdots$		$\vdots$
			$\cdots$	0	$\cdots$			
				$\vdots$				
			$\cdots$	0	$\cdots$			

(d)

Figure : Fractional differential filters on horizontal and vertical directions (a) F_{y+}^{α}; (b) F_{y-}^{α}; (c) F_{x+}^{α}; (d) F_{x-}^{α}

Discrete Fourier Transform (DFT) (1)

The 2-D DFT of an image $u(x, y)$ of $M \times M$ size is represented as

$$F(u(x,y)) = \widehat{u}(\omega_1, \omega_2) = \frac{1}{M^2} \sum_{x=0}^{M} \sum_{y=0}^{M} u(x,y) e^{\frac{-j2\pi(\omega_1 x + \omega_2 y)}{M}} \tag{7}$$

For 2-D DFT, the shifting property in spatial-domain can be denoted as

$$u(x - x_0, y - y_0) \overset{DFT}{\leftrightarrow} e^{\frac{-j2\pi(\omega_1 x_0 + \omega_2 y_0)}{M}} \widehat{u}(\omega_1, \omega_2) \tag{8}$$

Discrete Fourier Transform (DFT) (2)

The first-order partial difference in x-direction can be denoted as

$$D_x u(x,y) = u(x,y) - u(x-1,y) \tag{9}$$

Thus, the corresponding Fourier transform pair can be represented as

$$D_x u(x,y) \overset{DFT}{\leftrightarrow} \left(1 - e^{\frac{-j2\pi\omega_1}{M}}\right) \widehat{u}(\omega_1,\omega_2) \tag{10}$$

Discrete Fourier Transform (DFT) (3)

Similarly, the second-order partial difference in x-direction can be denoted as

$$D_x^2 u(x, y) = u(x, y) - 2u(x-1, y) + u(x-2, y) \tag{11}$$

Thus, the corresponding Fourier transform pair can be denoted as

$$D_x^2 u(x, y) \overset{DFT}{\leftrightarrow} \left(1 - e^{\frac{-j2\pi\omega_1}{M}}\right)^2 \widehat{u}(\omega_1, \omega_2) \tag{12}$$

Fractional-order derivative using DFT (1)

Hence, the DFT of fractional-order partial difference in x-direction can be represented as

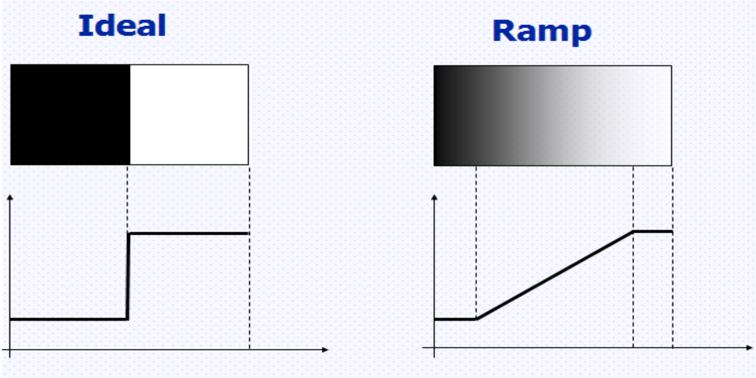
$$D_x^\alpha u(x,y) \overset{DFT}{\leftrightarrow} \left(1 - e^{\frac{-j2\pi\omega_1}{M}}\right)^\alpha \widehat{u}(\omega_1,\omega_2) \tag{13}$$

Similarly, the DFT of fractional-order partial difference in y-direction can be denoted as

$$D_y^\alpha u(x,y) \overset{DFT}{\leftrightarrow} \left(1 - e^{\frac{-j2\pi\omega_2}{M}}\right)^\alpha \widehat{u}(\omega_1,\omega_2) \tag{14}$$

Image Edge detection

Edges are significant local changes of intensity in an image.
Gray level detection:
The ability to measure gray-level transitions in a meaningful way.



Block diagram of filtering operation in image processing

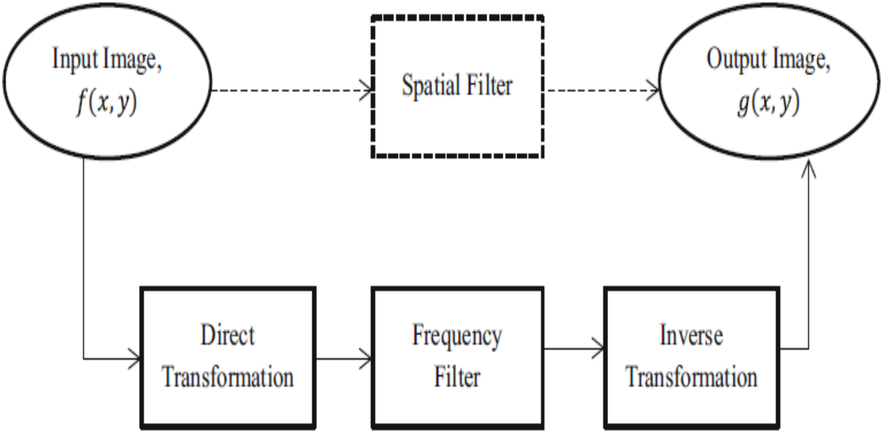


Image Edge detection

Integer-order differentiation operators

1. First order for the gradient
   - Roberts
   - Sobel
   - Prewitt
2. Second order for the Laplacian

Gradient representation

- The gradient is a vector which has magnitude and direction:

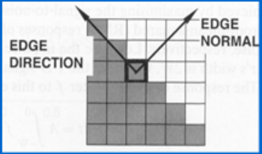
$$magnitude(grad(f)) = \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2}$$

$$direction(grad(f)) = \tan^{-1}\left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\right)$$

or (approximation)

$$\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$$

- **Magnitude:** indicates edge strength.
- **Direction:** indicates edge direction.
  - i.e., perpendicular to edge direction



Block diagram of filtering operation in image processing

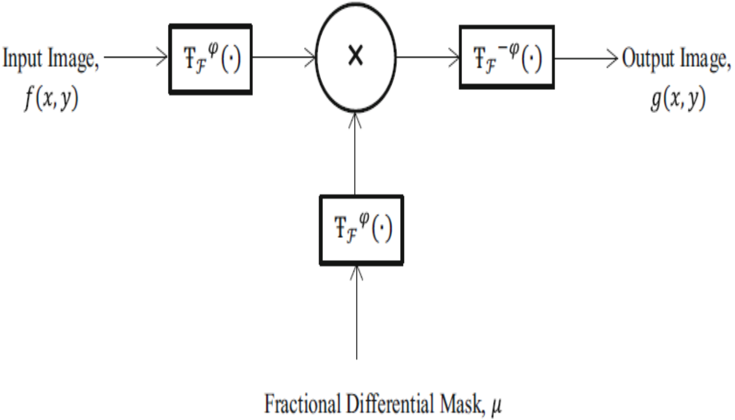


Figure : Block diagram [2]

[2]Sanjay et al., "'Fractional Fourier Transform and Fractional-Order Calculus-Based Image Edge Detection"', Cir. sys. and Sig. Proc., vol. 36, No. 4, pp 1493-1513, 2017

Fractional-order method and other method comparison

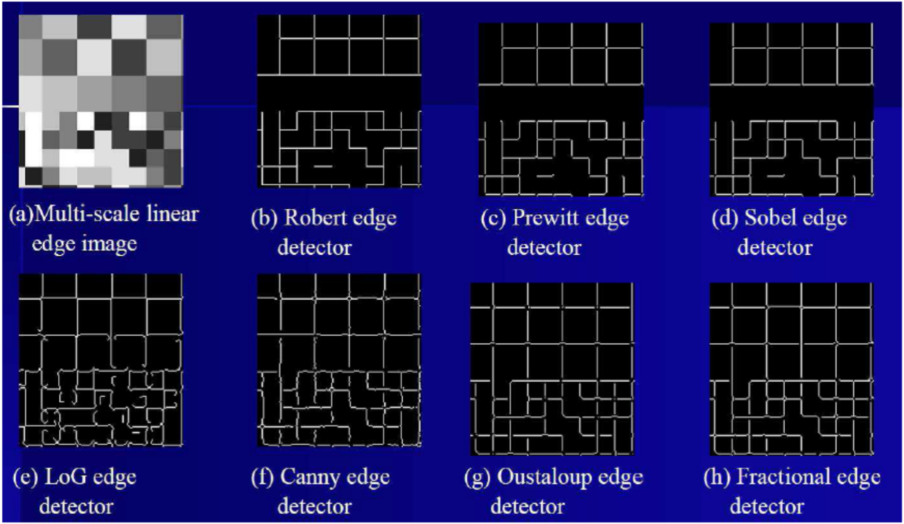


Image Enhancement

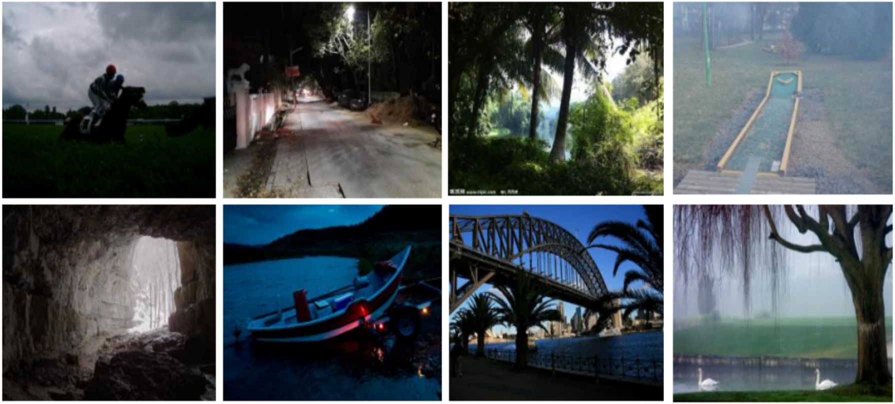
The captured image normally undergoes bad visual effect due to

- The natural condition from the environment, i.e., bad weather, low illumination, or moving objects
- The limitations of the equipment

Image enhancement techniques can be used to achieve visually accepted results or to improve particularly the interested texture in an image.

- Edge or Texture enhancement
- Contrast Enhancement

Examples



(a) (b) (c) (d)

Various conditions that negatively affect image quality, such as (**a**) bad weather; (**b**) dark light; (**c**) one side with darkness and one side with normal contrast; and (**d**) haze or fog.

Classification

Based on different processing spaces, the image enhancement techniques are divided into two categories:

1. Spatial domain methods
   - Point processing methods
   - Local (Neighborhood) processing methods
   - Non-local processing methods
2. Transform domain methods

Example of contrast Enhancement

Figure : a) Dark Image b) Enhanced Image

Example of Edge Enhancement

Figure : a) Textured Image b) Enhanced Image using Sobel

Example of Edge Enhancement

Figure : a) Enhanced Image using G-L Operator

Performance Metrics

- Peak to Signal Noise Ratio
- Entropy
- Average Gradient
- Contrast or Clarity

Available Datasets for Image Enhancement

LIME[3], NUS[4], UAE [5], MEF[6], and VV [7]
CVG-UGR Image database[8]

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[3] Guo et al. "'Low-light image enhancement via illumination map estimation"', IEEE Transactions on Image Procesing, vol. 26, pp. 982-993, 2017

[4] Cheng, D.; Prasad, D.K.; Brown, M.S. Illuminant estimation for color constancy: Why spatial-domain methods work and the role of the color distribution. 2014, 31, 1049-1058

[5] Lynch, S.; Drew, M.; Finlayson, G. Colour constancy from both sides of the shadow edge. In Proceedings of the IEEE International Conference on Computer Vision Workshops, Sydney, NSW, Australia, 2 December 2013; pp. 899-906.

[6] Ma, K.; Zeng, K.; Wang, Z. Perceptual quality assessment for multi-exposure image fusion. IEEE Trans. Image Process. 2015, 24, 3345356.

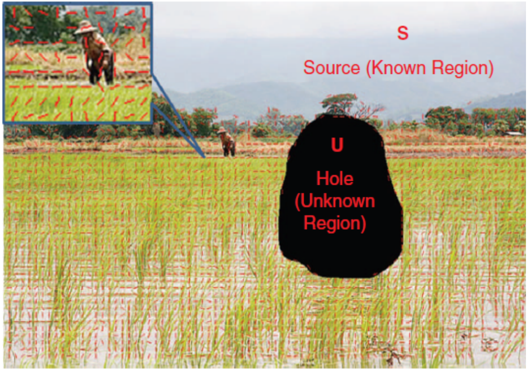
[7] Vonikakis, V.; Andreadis, I.; Gasteratos, A. Fast centreurround contrast modification. IET Image Process. 2008, 2, 194.

[8] http://decsai.ugr.es/cvg/dbimagenes

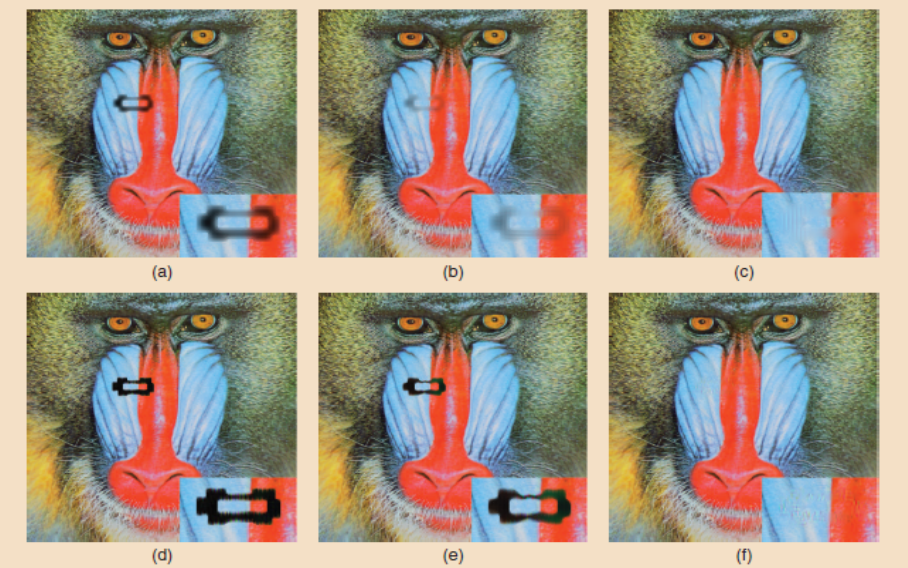
Image Restoration

- Image Denoising
- Imge Deblurring
- Image Inpainting

Example of an inpainting problem



Edge enhancement diffusion



Performance Metrics

- Peak Signal to Noise Ratio
- Structural Similarity Index
- Figure of Merit
- Mutual Information

Competitions

New Trends in Image Restoration and Enhancement Workshop and Challenges on image and video restoration and Enhancement (NTIRE 2019) in Conjuction with CVPR 2019.
(17th June, Long Beach, California)
(Submission Closed on April 14th)

THANK YOU ALL