Logic of compound statements

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form & equivalence

A. Argument - sequence of statements used to demonstrate the truth of an assertion

1. Conclusion - assertion at the end of sequence

2. Premise - preceding statements

B. Logical analysis helps analyze an argument's form so you can determine whether the truth of the conclusion follows the truth of the premises

1. → logic sometimes defined as the science of necessary inference (science of reasoning)

If $\overbrace{\text{the bell rings}}^{p}$ or $\overbrace{\text{the flag drops}}^{q}$, then $\overbrace{\text{the race is over}}^{r}$

$\overbrace{\text{If the race is not over}}^{\text{not } r}$, then $\overbrace{\text{the bell hasn't rung}}^{\text{not } p}$ and $\overbrace{\text{the flag hasn't dropped}}^{\text{not } q}$

2. Common form of arguments is: If p or q, then r

a. Or: If not r, then not p and not q

b. Example: If logic is easy or I study hard, then I will get an A in this course

statements

A. Statement - (proposition) sentence that is true or false but not both

1. Example: "Two plus two equals four" - true

2. Example: "Two plus two equals six" - false

3. Examples: $x^2 + 2 = 11$ depends on value of x

compound statements

A. Compound Statements - uses symbols to build more complicated logical expressions out of simpler ones

1. NOT($\sim$) AND($\wedge$) OR($\vee$)

a. Given a statement p, you can write NOT p, $\sim p$, or $\neg p$ in some languages

b. Given a statement q, $p \wedge q$ = p AND q, $p \vee q$ = p OR q

c. Order of Operations - $\sim$ is first

i. $\sim p \wedge q = (\sim p) \wedge q$, (NOT p) AND q

ii. Override through parentheses

t r u t h v a l u e s

A. Truth values - statement is either true or false

1. E.g. Negation of p (statement variable) is "not p" ($\sim p$), and has opposite truth value from p

a. If p is true, $\sim p$ is false; if p is false, $\sim p$ is true

2. The conjunction of two statements p and q as p AND q ($p \wedge q$) is only true if both are true

a. If either p or q or both are false, $p \wedge q$ is false

b. Specialization - rule of inference where something is true in particular

i. Conjunction $p \wedge q$, then one can infer that any operand in conjunction is true in particular

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

*neither-nor is $\sim p \wedge \sim q$

3. The disjunction of statement variables p and q is "p OR q", $p \vee q$, is true when either p is true or q is true, or both p and q are true; it is false only when both p and q are false

a. Generalization - rule of inference, e.g. if p is true, then you can infer $p \vee q$ is true

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. Domain - set of all possible inputs (p, q)

1. Don't worry much about it, bc inputs are true or false

c o m p l i c a t e d e x p r e s s i o n s

A. Statement Forms - expression made up of statement variables (p, q, r) and logical connectives ($\sim, \wedge, \vee$)

B. Truth Table - displays truth values for all possible truth values for statement variables (p, q, r)

C. Exclusive OR (XOR) - p OR q means "p or q but not both" = $(p \vee q) \wedge \sim(p \wedge q)$" p or q and not both p and q

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

D. $(p \wedge q) \vee \sim r$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

l o g i c a l e q u i v a l e n c e

A. Logical Equivalence - 2 statement forms are either both true or both false, same truth values,

1. Logical equivalence of statement forms P and Q : $P \equiv Q$
2. Test Logical Equivalence

a. Make truth table w/ values for P and Q

b. Check combination of truth values to see if truth values for P and Q match

i. If each row of truth value for P and Q match, P and Q are logically equivalent

i. If a row in P is different from a value in Q, P and Q are not logically equivalent

B. Double negative property - $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

C. Showing Nonequivalence

1. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent

p	q	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F
T	F	T $\neq$	F
F	T	T $\neq$	F
F	F	T	T

2. You can substitute concrete statements for p and q to prove nonequivalence

a. Let p be $0<1$ and q $1<0$

$\Rightarrow \sim(p \wedge q)$ is "It is not the case that both $0<1$ and $1<0$ are true

$\Rightarrow \sim p \vee \sim q$ is "It is not the case that $0<1$ is true and it is not the case that $1<0$ is true

D. De Morgan's Laws - The negation of an AND statement is logically equivalent to the or statement where each component is negated; the negation of an OR statement is logically equivalent to the negation of an AND statement where each component is negated

1. $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent, same for $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

2. Example:

I don't have a cat or dog

= I don't have a cat and I don't have a dog

The bus was late and Tom's watch was slow

The bus was not late and Tom's watch wasn't slow

= The bus was not late nor was Tom's watch slow

3. Inequalities and Morgan's Laws

a. Negation of $-1 < x \leq 4$: x is greater than -1 and less than or equal to 4

It's not the case that x is less than or equal to -1 and greater than 4

= x is not less than or equal to -1 nor is it greater than 4

4. When applying De Morgan's Laws, there must be complete statements on either side of AND or OR

t a u t o l o g i e s

A. Tautology - statement form that is always true regardless of the truth values the substituted statements may have

B. Contradiction - statement form that is always false regardless of the truth values the substituted statements may have

C. Truth and falsity of the above statements is due to logical structure, not the meanings of the statements

1. $p \vee \sim p$ is a tautology, $p \wedge \sim p$ is a contradiction

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

all true = tautology

all False = contradiction

B. Logical Equivalence in Tautology and Contradictions

1. If t is a tautology and c is a contradiction $p \wedge t \equiv p$ and $p \wedge c \equiv c$, $p \vee t \equiv t$ and $p \vee c \equiv p$

p	t	c	$p \wedge t$	$p \wedge c$	$p \vee t$	$p \vee c$
T	T	F	T	F	T	T
F	T	F	F	F	T	F

s i m p l i f y i n g s t a t e m e n t s

A. Verify logical equivalence of $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

1. Use De Morgan's Laws on the 1st statement

$\sim(\sim p \wedge q) = \sim(\sim p) \vee \sim q$

$\Rightarrow (\sim(\sim p) \vee \sim q) \wedge (p \vee q) \equiv p$

2. Use Double Negative Law

$\sim(\sim p) = p$

$\Rightarrow (p \vee \sim q) \wedge (p \vee q) \equiv p$

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:* ?	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

3. Use distributive laws

$(p \vee \sim q) \wedge (p \vee q) = p \vee (\sim q \wedge q)$

$\Rightarrow p \vee (\sim q \wedge q) \equiv p$

* $3 \cdot (2+5) = 3 \cdot 2 + 3 \cdot 5$

$p \vee (q \wedge r) = p \vee q \wedge p \vee r$

4. Use contradiction Law

$\sim q \wedge q = c$

$\Rightarrow p \vee c \equiv p$

$p \equiv p$

B. Theorems 2.1.1 can be used to proved logical equivalence, but cannot be used to prove nonequivalence

e x e r c i s e s

① If p, then q

p, therefore q

b. p = "all algebraic expressions can be written in prefix notation

q = "$(a+2b)(a^2-b)$ can be written in prefix notation

If all expressions can be written in prefix notation, then $(a+2b)(a^2-b)$ can be written in prefix notation

③ $p \vee q$

$\sim p \Rightarrow q$

Therefor logic is confusing

⑤ a. A statement is a sentence that is exclusively true or false. The statement is true because $\sqrt{1024} = 32$ which is an integer

⑥ a. $s \wedge i$

b. $\sim s \vee \sim i$

⑧ a. $(h \wedge w) \wedge \sim s$

d. $(\sim w \vee \sim s) \wedge h$

⑨ a. $p \vee q$

⑩ a. $p \wedge q \wedge r$

c. $p \wedge (\sim q \vee \sim r)$

⑪ Inclusive OR

Both conditions are satisfied if a team wins games 1, 3, and 4

⑫ a. $\sim p \wedge q$

p	q	$\sim p \wedge q$
T	T	F
T	F	F
F	T	T
F	F	F

⑭ $p \wedge (q \wedge r)$

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
F	T	T	T	F
F	F	T	F	F
F	T	F	F	F
T	F	F	F	F
F	F	F	F	F

⑯ ? $p \vee (p \wedge q) \equiv p$

p	q	$p \vee (p \wedge q)$
T	T	T
T	F	T
F	T	F
F	F	F

Logically equivalent bc they both have the same truth values

(21) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

p	q	r	$(p\wedge q)\wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
F	T	T	F	F
F	F	T	F	F
F	T	F	F	F
T	F	F	F	F
F	F	F	F	F

logically equivalent because both truth values are the same

(23) $(p \wedge q) \vee r \equiv p \wedge (q \vee r)$

p	q	r	$(p \wedge q) \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
F	T	T	T	F

Not logically equivalent because the truth values for both statements aren't all the same

(27) It's not the case that the connector is loose nor is the machine unplugged

= The connector isn't loose and the machine isn't unplugged

(32) $x > -2$ and $x < 7$

$x \leq -2$ or $x \geq 7$

(36) $x < 1$ and $x \geq -3$

$x \geq 1$ or $x < -3$

(25) Hal isn't a math major nor is his sister a CS major

(34) $x < 2$ or $x > 5$

$x \geq 2$ and $x \leq 5$

$2 \leq x \leq 5$

(38) (num_orders ≤ 100 or num_instock > 500) and num_instock ≥ 200

form of $(p \wedge q) \vee r$

$\sim((p \wedge q) \vee r) \equiv \sim(p \wedge q) \wedge \sim r \equiv (\sim p \wedge \sim q) \wedge \sim r$

p	~p	~p ∨ p
T	F	T
F	T	T

(40) $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$

$\equiv (p \wedge q) \vee ((\sim p \vee p) \wedge (\sim p \vee \sim q))$

$\equiv (p \wedge q) \vee (t \wedge (\sim p \vee \sim q))$

$\equiv (p \wedge q) \vee (t \wedge (\sim (p \wedge q))$

p	q	t	$p \wedge q$	$(p \wedge q) \vee (t \wedge (\sim (p \wedge q))$
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	T	F	T

tautology

(41) $(p \wedge \sim q) \wedge (\sim p \vee q)$

p	q	$(p \wedge \sim q)$	$(\sim p \vee q)$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	F	T	F	F
F	T	F	T	F
T	T	F	T	F
F	F	F	T	F

contradiction

(44)

$x > 2$ and $x \leq 0$

none

46 a. $p \oplus p$ where $\oplus$ = XOR

p	$p \oplus p$ $\equiv c$
T	F all False ⇒
F	F contradiction

$p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$

$\Rightarrow p \oplus p \equiv (p \vee p) \wedge \sim(p \wedge p)$

$\equiv p \wedge \sim p \equiv c$

c. ? $(p \oplus q) \wedge r \equiv p \oplus (q \oplus r)$

p	q	r	$p \oplus q$	$(p \oplus q) \wedge r$	p	q	r	$q \oplus r$	$p \oplus (q \oplus r)$
T	T	T	F	F ✗	T	T	T	F	T
T	T	F	F	F	T	T	F	T	F
T	F	T	T	T	T	F	T	T	F
F	T	T	T	T	F	T	T	F	F
F	F	T	F	F	F	F	T	T	T
F	T	F	T	F	F	T	F	T	T
T	F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	F	F	F	F

not logically equivalent, truth table is not exactly the same

47 "Yeah, yeah" is considered a double negative because it is said sarcastically

48 $(p \wedge \sim q) \vee (p \wedge q)$

$\equiv p \wedge (\sim q \vee q)$ a) distribution

$\equiv p \wedge t$ b) negation law

$\equiv p$ d) identity law

q	$\sim q$	$\sim q \vee q$	$\sim q \wedge q$	p	$p \wedge t$
T	F	T	F	T	T
F	T	T	F	F	F
		t	c		

50 $(p \wedge \sim q) \vee p$

$\equiv p$ by absorption law

? why use commutative law

? how does absorption

53 $\sim[(\sim p \wedge q) \vee (\sim p \wedge \sim q)] \vee (p \wedge q)$

$\equiv (\sim(\sim p \wedge q)) \wedge (\sim(\sim p \wedge \sim q)) \vee (p \wedge q)$ De Morgan's Law

$\equiv (p \vee \sim q) \wedge (p \vee q) \vee (p \wedge q)$ De Morgan's Law

$\equiv [p \vee (\sim q \wedge q)] \vee (p \wedge q)$ Distribution Law

$\equiv (p \vee c) \vee (p \wedge q)$ Negation Law

$\equiv p \vee (p \wedge q)$ Identity Law

$\equiv p$ Absorption Law

* Use distribution law to be faster