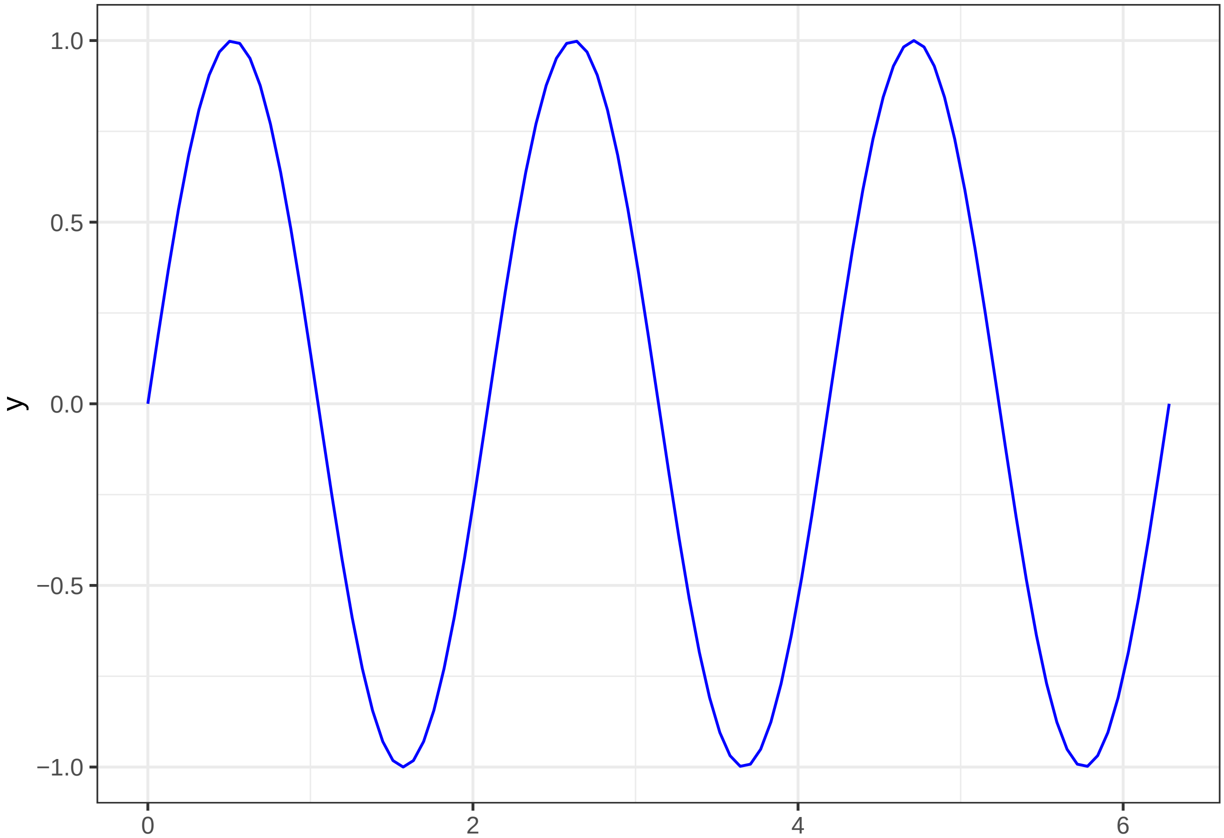
# Unit 2 Lecture 2: Bias-variance tradeoff

September 21, 2021

In this activity, we will explore the bias-variance tradeoff in the context of natural spline fits.

As our true function $f$, let us use the following sine curve:

```
f = function(x)(sin(3*x))
ggplot() +
  stat_function(fun = f, colour = "blue") +
  xlim(0,2*pi) + theme_bw()
```



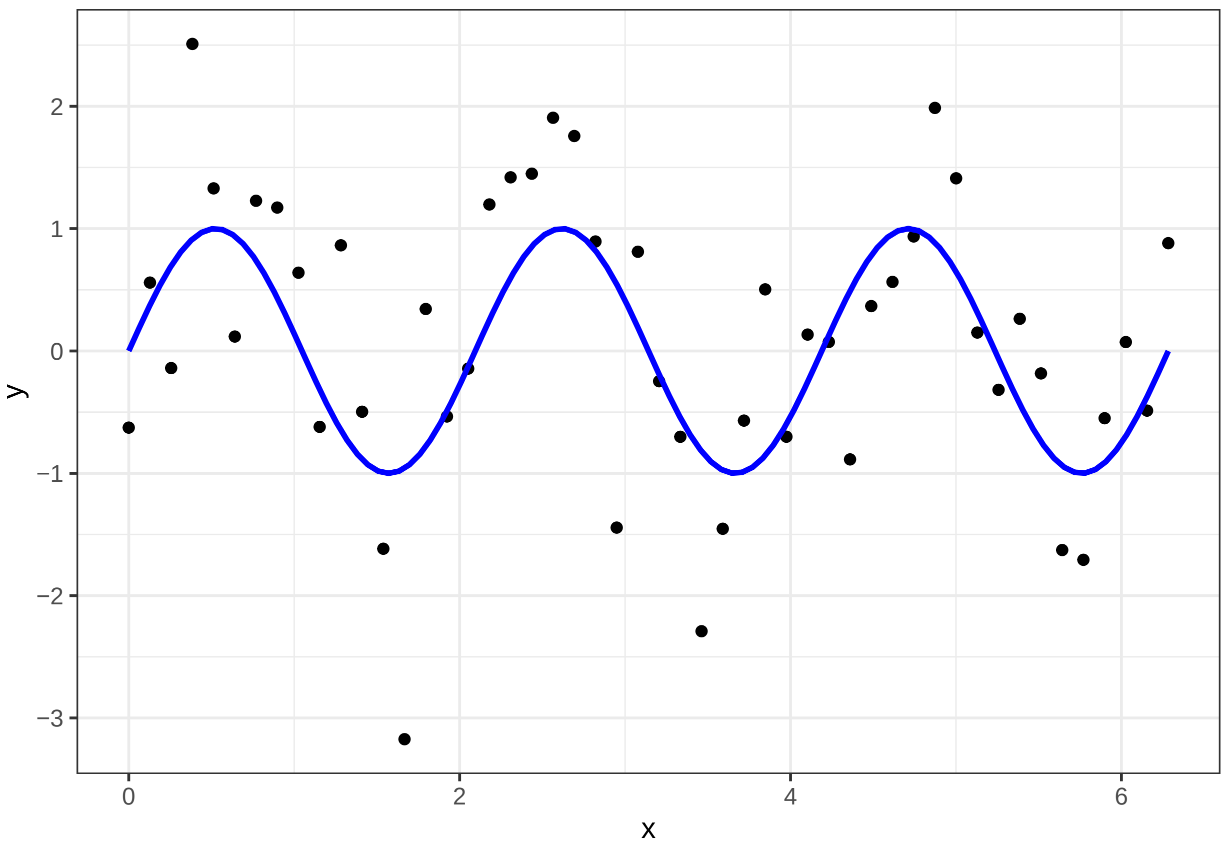

## Training and testing

Let us start with the function `f`. Let us create a training set

```
set.seed(1)
n = 50
sigma = 1
train_data = tibble(x = seq(0, 2*pi, length.out = n),
                    y = f(x) + rnorm(n, sd = sigma))
train_data
```

```
## # A tibble: 50 x 2
##        x      y
##    <dbl>  <dbl>
##  1 0     -0.626
##  2 0.128  0.559
##  3 0.256 -0.140
##  4 0.385  2.51
##  5 0.513  1.33
##  6 0.641  0.118
##  7 0.769  1.23
##  8 0.898  1.17
##  9 1.03   0.640
## 10 1.15  -0.620
## # ... with 40 more rows
```

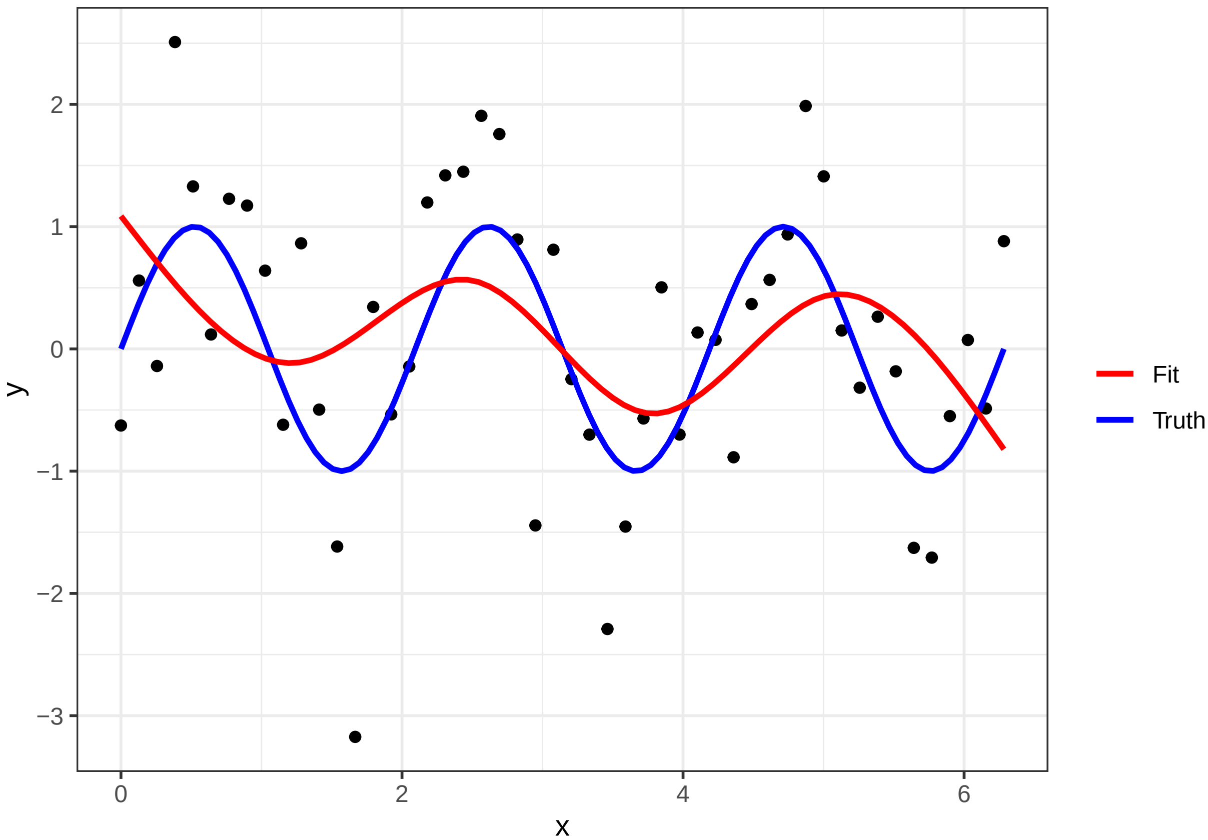
```
train_data %>%
  ggplot(aes(x = x, y = y)) +
  geom_point() +
  stat_function(fun = f, colour = "blue", size = 1) +
  theme_bw()
```




Let us train a natural spline fit with 5 degrees of freedom on this data:

```
spline_fit = lm(y ~ splines::ns(x, df = 5), data = train_data)
train_data %>%
  ggplot(aes(x = x, y = y)) +
  geom_point() +
  stat_function(fun = f, aes(colour = "Truth"), size = 1) +
```

```
  geom_smooth(method = "lm",
              formula = "y ~ splines::ns(x, df = 5)",
              aes(colour = "Fit"), se = FALSE) +
  scale_colour_manual(values = c("red", "blue")) +
  theme_bw() + theme(legend.title = element_blank())
```




Next, we compute the training and test error for this fit:

```
### training error
y_hat_train = predict(spline_fit, newdata = train_data)
head(y_hat_train)
```

```
##         1         2         3         4         5         6
## 1.0863047 0.8955427 0.7090961 0.5312804 0.3664109 0.2188031
```

```
train_data %>%
    mutate(y_hat_train = y_hat_train) %>%
    summarise(training_error = mean((y_hat_train-y)^2))
```

```
## # A tibble: 1 x 1
##   training_error
##            <dbl>
## 1           1.16
```

```
### test error

# create a large test set
N = 50000
test_data = tibble(x = seq(0, 2*pi, length.out = N),
```

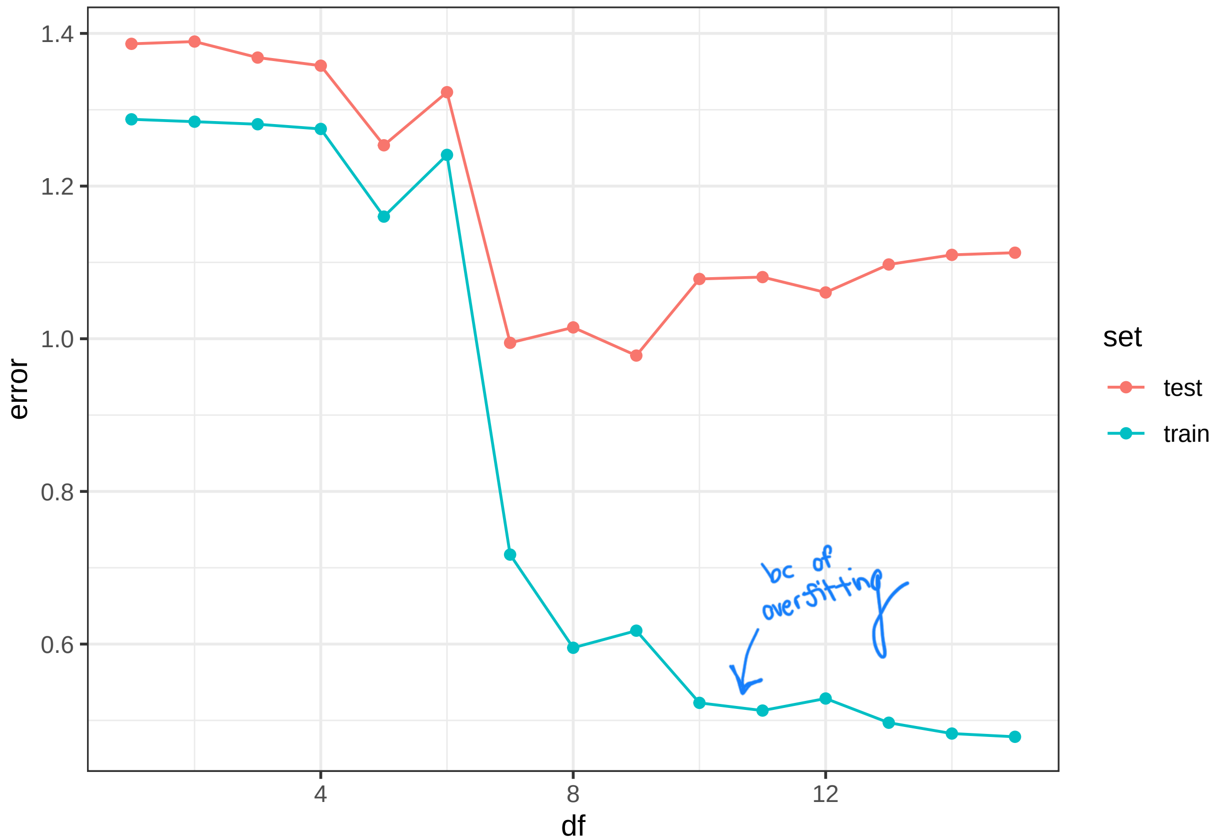
```r
                  y = f(x) + rnorm(n, sd = sigma))

# compute test error
y_hat_test = predict(spline_fit, newdata = test_data)
test_data %>%
    cbind(y_hat_test) %>%
    summarise(test_error = mean((y_hat_test-y)^2))
```

```
##   test_error
## 1   1.253463
```

## Varying the degrees of freedom

Next let's do the same thing, but for `df` varying between 1 and 15.

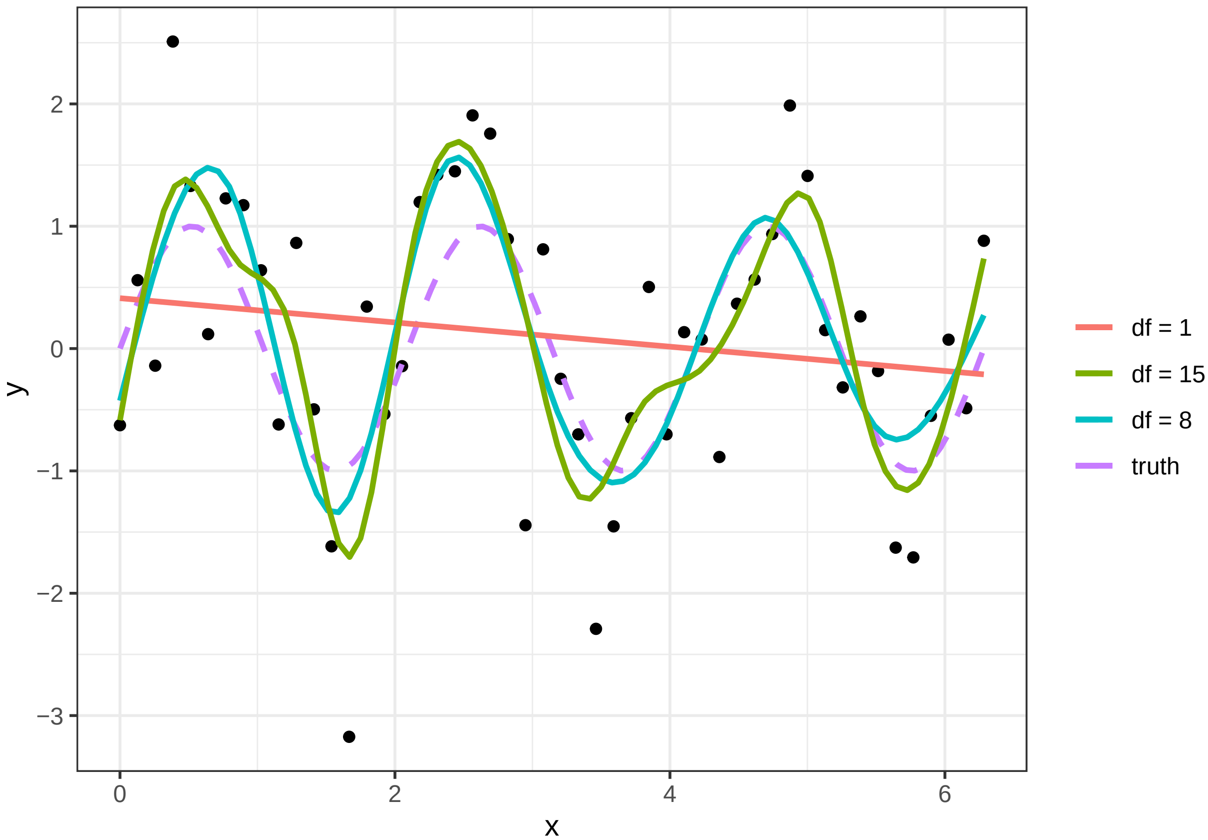
```r
max_df = 15
error_test = numeric(max_df)
error_train = numeric(max_df)
for(df in 1:max_df){
  formula = sprintf("y ~ splines::ns(x, df = %d)", df)
  spline_fit = lm(formula = formula, data = train_data)
  y_hat_train = predict(spline_fit, newdata = train_data)
  y_hat_test = predict(spline_fit, newdata = test_data)
  error_train[df] = train_data %>%
    cbind(y_hat_train) %>%
    summarise(mean((y_hat_train-y)^2)) %>%
    pull()
  error_test[df] = test_data %>%
    cbind(y_hat_test) %>%
    summarise(mean((y_hat_test-y)^2)) %>%
    pull()
}

tibble(df = 1:max_df, error_train, error_test) %>%
  pivot_longer(cols = -df, names_to = "set",
               names_prefix = "error_",values_to = "error") %>%
  ggplot(aes(x = df, y = error, color = set)) +
  geom_point() + geom_line() + theme_bw()
```




What is the best choice for `df`?

Let's visualize a few of these fits to see if these train and test errors make sense:

```
train_data %>%
  ggplot(aes(x = x, y = y)) +
  geom_point() +
  stat_function(fun = f, aes(colour = "truth"),
                size = 1, linetype = "dashed") +
  geom_smooth(method = "lm",
              formula = "y ~ splines::ns(x, df = 1)",
              aes(colour = "df = 1"), se = FALSE) +
  geom_smooth(method = "lm",
              formula = "y ~ splines::ns(x, df = 8)",
              aes(colour = "df = 8"), se = FALSE) +
  geom_smooth(method = "lm",
              formula = "y ~ splines::ns(x, df = 15)",
              aes(colour = "df = 15"), se = FALSE) +
  theme_bw() + theme(legend.title = element_blank())
```




Does the best choice of `df` from before make sense?

## Fitting splines to many random training sets

Recall that the expected test error, bias, and variance are quantities averaged over the randomness in the training data. Therefore, let us repeatedly generate the training data to compute them in the above example.

```
resamples = 100
train_data_resamples = tibble(x = rep(seq(0, 2*pi, length.out = n), resamples),
                              y = f(x) + rnorm(n*resamples, sd = sigma),
                              resample = rep(1:resamples, each = n))
```

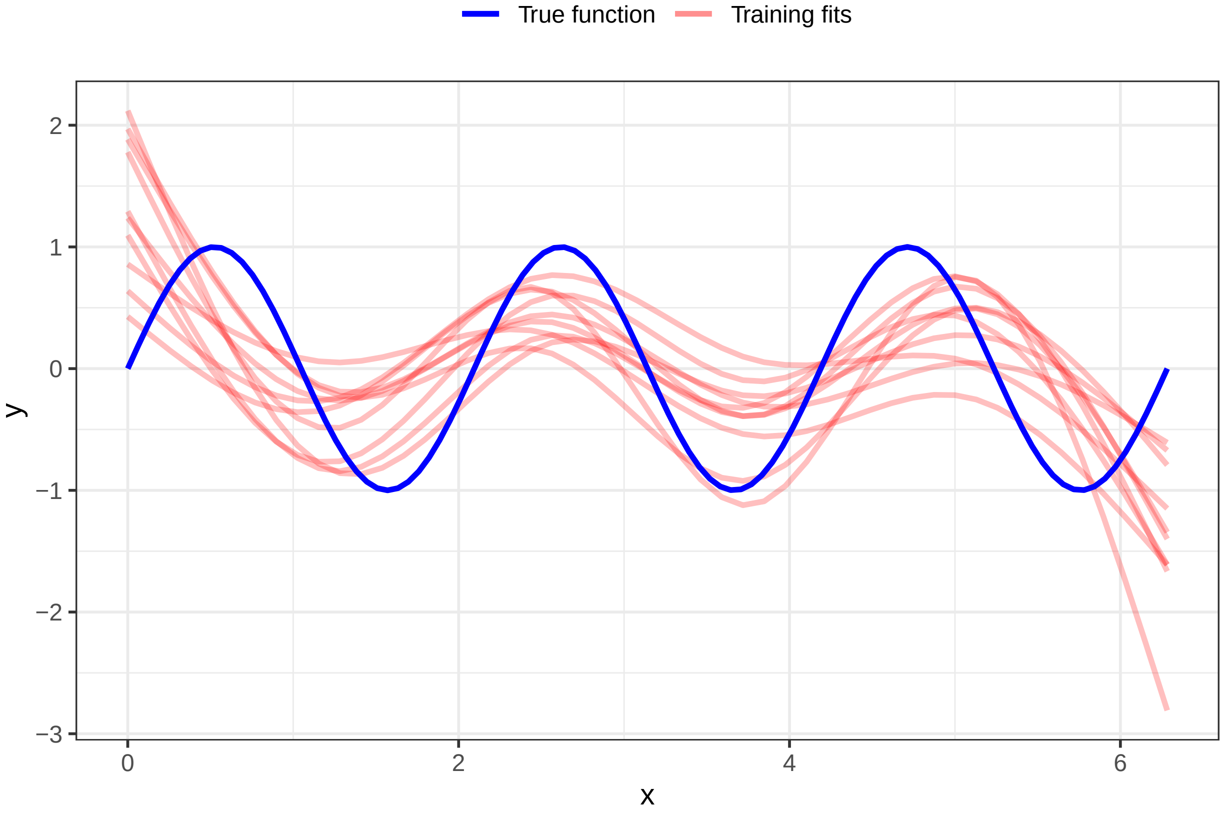
Next, let us fit the natural spline model with 5 degrees of freedom to each resampled dataset:

```
# function that fits a degree 5 natural spline
spline_model_5_df = function(data) {
  lm(y ~ splines::ns(x, df = 5), data = data)
}

# fit the natural spline to each resampled dataset
training_results = train_data_resamples %>%
  group_by(resample) %>%                                  # group by resample
  nest() %>%
  mutate(model = map(data, spline_model_5_df)) %>%        # fit each model
  mutate(fitted = map2(data, model, add_predictions)) %>%  # add predictions
  select(resample, fitted) %>%
  unnest(fitted) %>%
  ungroup()
```

```
training_results
```

```
## # A tibble: 5,000 x 4
##    resample     x      y   pred
##       <int> <dbl>  <dbl>  <dbl>
##  1        1 0     -0.620 0.858
##  2        1 0.128  0.417 0.736
##  3        1 0.256 -0.215 0.616
##  4        1 0.385  1.07  0.502
##  5        1 0.513  0.345 0.395
##  6        1 0.641  2.71  0.298
##  7        1 0.769  1.46  0.214
##  8        1 0.898  1.34  0.144
##  9        1 1.03   0.448 0.0921
## 10        1 1.15   1.37  0.0600
## # ... with 4,990 more rows
```

Let's plot the first ten of these fits:




# Bias, variance, and expected test error

## Separately for each data point

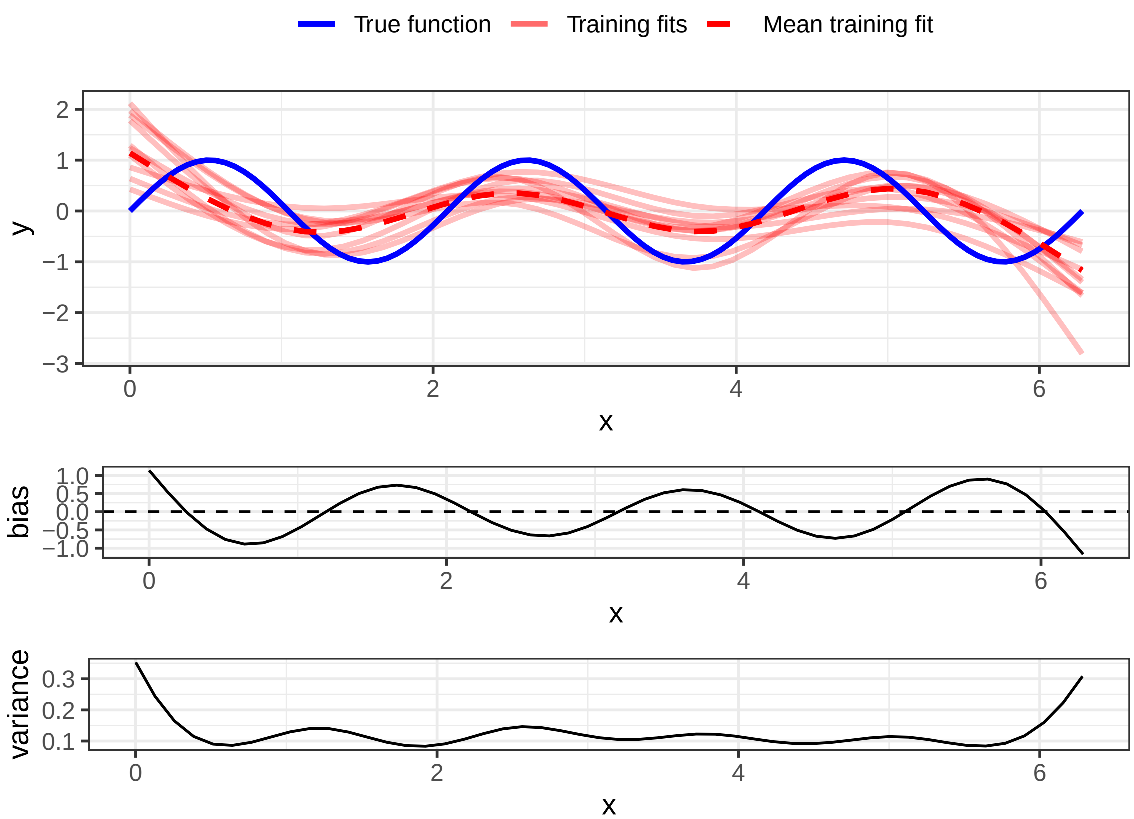
Let's compute the mean prediction, bias, and variance for each value of `x` by averaging over the resamples:

```
training_results_summary = training_results %>%
  mutate(true_fit = f(x)) %>%
  group_by(x) %>%
```

```
  summarise(mean_pred = mean(pred),
            bias = mean(pred - true_fit),
            variance = var(pred))
training_results_summary
```

```
## # A tibble: 50 x 4
##        x mean_pred    bias variance
##    <dbl>     <dbl>   <dbl>    <dbl>
##  1 0         1.14    1.14     0.353
##  2 0.128     0.902   0.527    0.244
##  3 0.256     0.669  -0.0265   0.165
##  4 0.385     0.446  -0.468    0.115
##  5 0.513     0.238  -0.761    0.0902
##  6 0.641     0.0502 -0.888    0.0861
##  7 0.769    -0.113  -0.853    0.0960
##  8 0.898    -0.247  -0.681    0.113
##  9 1.03     -0.345  -0.410    0.130
## 10 1.15     -0.405  -0.0895   0.140
## # ... with 40 more rows
```

Let us plot these and see what we get:




How do we interpret the bias in terms of the first plot above? When is it above zero and when is it below zero?

How do we interpret the variance in terms of the first plot above? Why is it larger at the edges of the data?

### Overall bias, variance, and ETE

To get the overall squared bias and variance, we average across data points:

```
bias_variance = training_results_summary %>%
  summarize(sq_bias = mean(bias^2),
            variance = mean(variance),
            irreducible_error = sigma^2)
bias_variance
```

```
## # A tibble: 1 x 3
##   sq_bias variance irreducible_error
##     <dbl>    <dbl>             <dbl>
## 1   0.328    0.126                 1
```

Based on this information, how do we compute expected test error?

```
bias_variance %>% mutate(expected_test_error = ???)
```

### Sanity check 1: does the variance match the formula?

What is the formula for overall variance of the fit for a linear regression model?

```
# formula for mean variance:
variance_formula = ???
```

Does this match the variance obtained above?

### Sanity check 2: does ETE match the definition?

Let us calculate the ETE from its definition by generating test points:

```
training_results %>%
  mutate(y_test = f(x) + rnorm(n*resamples, sd = sigma)) %>%
  group_by(resample) %>%
  summarise(test_error = mean((pred - y_test)^2)) %>%
  summarise(expected_test_error = mean(test_error))
```

```
## # A tibble: 1 x 1
##   expected_test_error
##                 <dbl>
## 1                1.46
```

Does this quantity match the ETE computed above?

## Varying the degrees of freedom

Next let's do the same thing, but for `df` varying between 1 and 15. We wrap everything in a function for convenience:

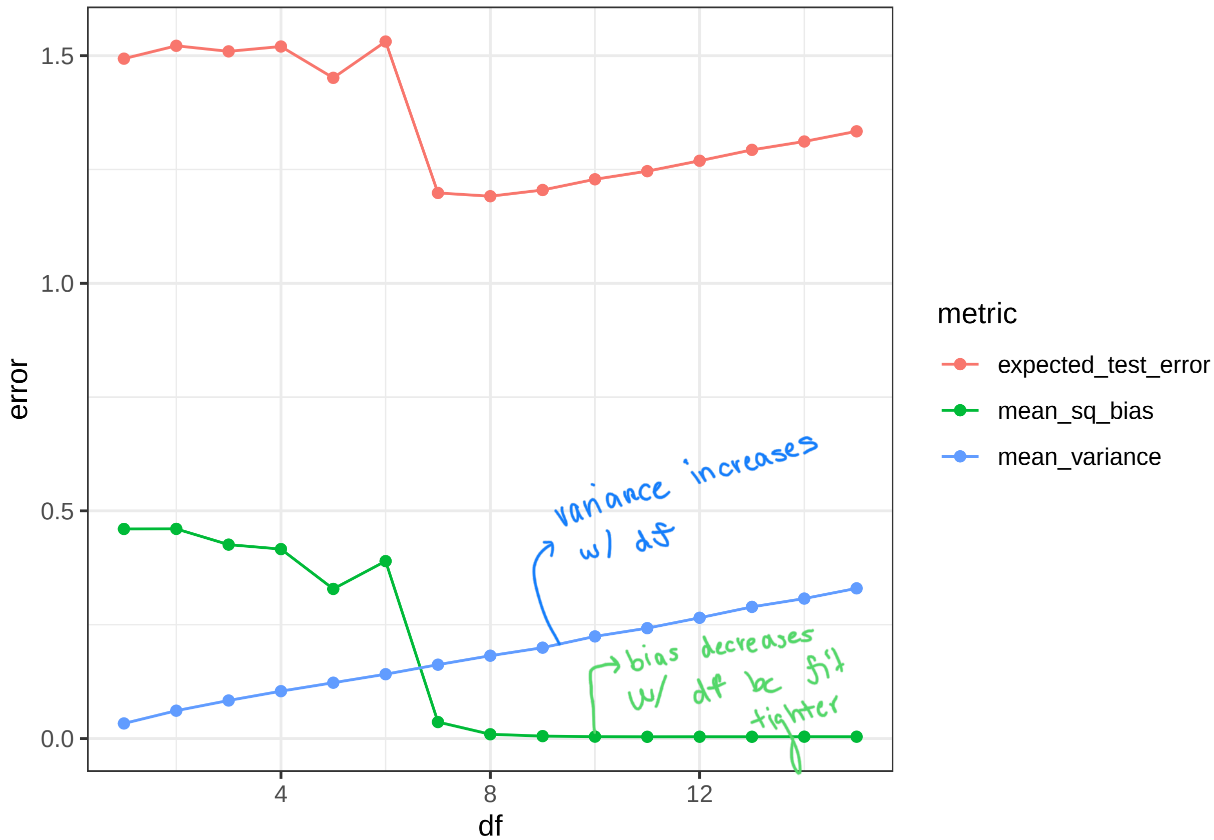
```
bias_variance_tradeoff = function(f, sigma, n, resamples){
  # set seed for reproducibility
  set.seed(1)

  # generate training data
  train_data_resamples = tibble(x = rep(seq(0, 2*pi, length.out = n), resamples),
                                y = f(x) + rnorm(n*resamples, sd = sigma),
                                resample = rep(1:resamples, each = n))
```

```r
  # function that fits a degree `df` natural spline to `data`
  spline_model = function(data, df) {
    lm(y ~ splines::ns(x, df = df), data = data)
  }

  # fit natural spline of each df to each resampled dataset
  training_results = train_data_resamples %>%
    crossing(df = 1:15) %>%
    group_by(resample, df) %>%
    nest() %>%
    mutate(model = map2(data, df, spline_model)) %>%
    mutate(fitted = map2(data, model, add_predictions)) %>%
    select(resample, df, fitted) %>%
    unnest(fitted) %>%
    ungroup()

  # compute bias, variance, and ETE
  training_results_summary = training_results %>%
    mutate(true_fit = f(x)) %>%
    group_by(df, x) %>%
    summarise(bias = mean(pred - true_fit),
              variance = var(pred)) %>%
    summarise(mean_sq_bias = mean(bias^2),
              mean_variance = mean(variance)) %>%
    mutate(expected_test_error = mean_sq_bias + mean_variance + sigma^2)

  # plot the bias, variance, and ETE
  p = training_results_summary %>%
    pivot_longer(-df, names_to = "metric", values_to = "error") %>%
    ggplot(aes(x = df, y = error, colour = metric)) +
    geom_line() + geom_point() + theme_bw()
  plot(p)
}
```

Let's try out this function on the example from before:

```r
f = function(x)(sin(3*x))
sigma = 1
n = 50
resamples = 100
bias_variance_tradeoff(f, sigma, n, resamples)
```

```
## `summarise()` has grouped output by 'df'. You can override using the `.groups` argument.
```




What trends do we observe in this plot? What appears to be the best degrees of freedom? Why does the variance curve appear linear? What does it mean that the bias is zero from `df` = 8 onwards?

## Varying the noise level

What happens if we increase the noise standard deviation? What happens if we decrease it?

```
# try increasing noise level
```

```
# try decreasing noise level
```

## Varying the sample size

What happens if we decrease the sample size? What happens if we increase it?

```
# try decreasing sample size
```

```
# try increasing sample size
```

## Varying the complexity of the underlying function

What happens if we decrease the complexity of the underlying function `f`? What happens if we increase it?

```
# try decreasing complexity of f
```

```
# try increasing complexity of f
```