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Contents

We referred to several different books in this citation, namely [1][2][3][4].

1 The Classical Action

1.1 Exercise 1: First step

The function f changes in both x_1 as well as in x_2, so when we write the change in the function δf, we have to consider each individual change of the function in terms of x_1 and x_2. To do this, we multiply the gradient of the function $\partial f / \partial x_i$ times x_i to get the change and we do this for each x_1 and x_2. Therefore,

$$\delta f = \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2}. \tag{1}$$

1.2 Exercise 2: Second Step

We consider the change in action. By applying equation (1)

$$\delta S = \delta \left(\int_{t_a}^{t_b} L(x, \dot{x}, t) \mathrm{d}t \right) \tag{2}$$

$$= \int_{t_a}^{t_b} \delta L \mathrm{d}t \tag{3}$$

$$= \int_{t_a}^{t_b} \left(\delta x \frac{\partial L}{\partial x} + \delta \dot{x} \frac{\partial L}{\partial \dot{x}} \right) \mathrm{d}t. \tag{4}$$

We can apply integration by parts to the second term. Using the below formula

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u, \tag{5}$$

we say that $u = \frac{\partial L}{\partial \dot{x}}$ and $v = \dot{x}$. Hence,

$$\delta S = \delta x \frac{\partial L}{\partial \dot{x}} \Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} \delta x \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) \right) \mathrm{d}t. \tag{6}$$

Note that the first term vanishes as $\delta x(t_a) = \delta x(t_b) = 0$. We want action to minimized, so $\delta S = 0$ which means

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0. \tag{7}$$

1.3 Exercise

The lagrangian is $L =$ Kinetic energy - Potential energy. Assuming a point of reference where the potential energy is $U(x)$, we can say the lagrangian consists of only kinetic energy, or

$$L = \frac{1}{2} m \dot{x}^2 - U(x) \tag{8}$$

From equation (8) and equation (7), we have

$$\frac{\partial}{\partial x} \left[\frac{1}{2} m \dot{x}^2 - U(x) \right] - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial}{\partial \dot{x}} \left[\frac{1}{2} m \dot{x}^2 - U(x) \right] \right) = 0. \tag{9}$$

The first term becomes $\partial U / \partial x$ because kinetic energy is only dependant on $\dot{x}$ and potential energy is dependant on x. Similarly, the second term becomes $m\ddot{x}$ for the same reasons. Hence, we finally yield

$$-\frac{\partial U(x)}{\partial x} = m\ddot{x} \tag{10}$$

As the negative gradient of potential energy is force and $\ddot{x} = a$, we yield

$$F = ma. \tag{11}$$

1.4 Exercise

If there is no external force, then we say that the potential energy is 0 as the negative gradient results in force. So, we consider the action of only the kinetic energy component.

$$S = \int_{t_a}^{t_b} \frac{1}{2} m \dot{x}^2 \mathrm{d}t = \frac{1}{2} m \frac{(x_b - x_a)^2}{t_b - t_a}. \tag{12}$$

1.5 Exercise

We consider the change of the hamiltonian with respect to time. If it is equal to zero, then the Hamiltonian does not change and the potential energy does not change on velocity or time explicitly.

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \ddot{x}\frac{\partial L}{\partial \dot{x}} + \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\mathrm{d}L}{\mathrm{d}t}. \tag{13}$$

We know that the change in lagrangian with respect to time is zero, so the third term vanishes. By equation (7), the first two terms vanish since:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \rightarrow \dot{x}\frac{\partial L}{\partial x} - \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) \rightarrow \ddot{x}\frac{\partial L}{\partial \dot{x}} - \dot{x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{14}$$

Hence,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{15}$$

and our statement is proven. If the potential energy does not depend on $\dot{x}$, then $\partial L/\partial \dot{x}$ is only dependent on kinetic energy T. As $T = \frac{1}{2}m\dot{x}^2$, then $\dot{x}(\partial L/\partial \dot{x}) = \dot{x}(m\dot{x}) = m\dot{x}^2$. Hence, the Hamiltonian can be rewritten as

$$H = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x) = \frac{p^2}{2m} + V(x). \tag{16}$$

2 Path-integral in Quantum Mechanics

2.1 Exercise

We know that the action is given as

$$S = \int_{t_a}^{t_b} L(x, \dot{x}, t)\mathrm{d}t = \lim_{\Delta t \to 0} \sum L\Delta t \tag{17}$$

The lagrangian has units of energy as it is given as the difference of kinetic and potential energies. Therefore, the action has units of Energy × time. Planck used $\hbar$ when describing the blackbody spectrum. The energy of a photon given by this is

$$E = \hbar\omega \implies \hbar = \frac{E}{\omega}. \tag{18}$$

Frequency has units of 1/time. Therefore, Planck's constant has units of Energy × time as well.

2.2 Exercise

Let us suppose that the alpha particle is a free particle. From a quick internet search, we found that the typical range of alpha particles in the air is about 4 centimeters, they travel at about $v = 0.05c$, and have a mass of 4 amu (6.64×10^{-27} kg). We will say that the speed is not fast enough to use special relativity (usually we do this when the speeds are about $0.1c$). So, as derived in exercise 1.4, the action is, when assuming (and neglecting units) $t_a = x_a = 0$:

$$S = \frac{1}{2}m\frac{x_b^2}{t_b}. \tag{19}$$

The total time taken follows kinematics, where

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \implies t_b = \frac{x_b}{0.05c}. \tag{20}$$

Hence,

$$\frac{1}{2}mvx_b = \frac{1}{2}(6.64 \times 10^{-27})(0.05 \times 3 \times 10^8)(0.04) = 1.99 \times 10^{-21} \text{ J} \cdot \text{s}. \tag{21}$$

Using Planck's constant $\hbar = 6.62 \times 10^{-34}$ J·s, the ratio $S/\hbar = 3 \times 10^{12}$ which is massive. As $S \gg \hbar$, we can safely say that the alpha particle is within the classical limit.

2.3 Exercise

The fact that normalization is independent of path implies that the transition amplitude is somehow also independent of the path the particle "actually" takes. Thus, in some sense, the particle is taking all the paths simultaneously. In order for this to be a proper probability, it must have magnitude 1 when considering the amplitude between all points, and so a path independent factor must be multiplied dependent only on the endpoints, which is the normalization constant.

2.4 Exercise

We must have A be related to ϵ because it is path independent. As all other variables are integrated over, it can only depend on the time at the endpoints meaning that it can only be a function of the time interval. Furthermore, it must be a function of ϵ and not some other combination like $t_2^2 - t_1^2$ since the interval must be invariant under a time shift $t \to t+c$, as otherwise, calculating an amplitude with the same setup at a later time will yield a different value.

2.5 Exercise

We know that the phase factor is given as $e^{iS/\hbar}$ which means the phase is $\phi = S/\hbar$ since the phase factor is in the form $e^{i\phi}$. The phase difference is then

$$\Delta\phi = \frac{S_2 - S_1}{\hbar}. \tag{22}$$

On path 1, the action is given as $S_1 = \frac{1}{2}mv_1^2 t$ where $v_1 = D/t$. On path 2, the action is given as $S_2 = \frac{1}{2}mv_2^2 t$ where $v_2 = (D+d)/t$. As $d \ll D$, $v = v_1 \approx v_2$ as said in the problem. This then means that, in general, $S = \frac{ms^2}{t}$ where s is an arbitrary distance that is not specified. Keeping only terms in order d/D, we find that

$$S_2 - S_1 = \frac{m}{2t}((D+d)^2 - D^2) \approx \frac{mDd}{t} \approx mvd. \tag{23}$$

Therefore, using $\lambda = h/p$ and $\hbar = h/2\pi$, we can write the phase difference as

$$\Delta\phi = \frac{S_2 - S_1}{\hbar} \approx \frac{2\pi pd}{h} \approx \frac{2\pi d}{\lambda}. \tag{24}$$

2.6   Exercise I

From equation 28 in the packet, we know that

$$\psi(x_b, t_a + \epsilon) = \int_{-\infty}^{\infty} U(x_b, t_a + \epsilon; x_a, t_a)\psi(x_a, t_a)\mathrm{d}x_a \tag{25}$$

We can write the propagator as below. For a small time interval ϵ, we can skip the inner integral.

$$U(x_b, t_a + \epsilon; x_a, t_a) = \frac{1}{A} \exp\left(L\left(\frac{x_b - x_a}{\epsilon}, \frac{x_a + x_b}{2}, t + \frac{\epsilon}{2} \right) \right) \epsilon \tag{26}$$

The lagrangian can be written as the sum of kinetic and potential energy components. Thus,

$$L\left(\frac{x_b - x_a}{\epsilon}, \frac{x_a + x_b}{2}, t_a + \frac{\epsilon}{2} \right) = \frac{1}{2} m \frac{(x_b - x_a)^2}{\epsilon^2} - V\left(x_b + \frac{\eta}{2}, t_a + \frac{\epsilon}{2} \right) \tag{27}$$

$$= \frac{m\eta^2}{2\epsilon^2} - V\left(x_b + \frac{\eta}{2}, t_a + \frac{\epsilon}{2} \right) \tag{28}$$

Now, substituting this propagator into (214) yields

$$\psi(x_b, t + \epsilon) = \frac{1}{A} \int_{-\infty}^{\infty} \exp\left\{ \frac{i\epsilon}{\hbar} \left[\frac{m\eta^2}{2\epsilon^2} - V\left(x_b + \frac{\eta}{2}, t_a + \frac{\epsilon}{2} \right) \right] \right\} \psi(x_b + \eta, t_a)\mathrm{d}\eta \tag{29}$$

$$= \frac{1}{A} \int_{-\infty}^{\infty} \exp\left\{ \frac{im\eta^2}{2\hbar\epsilon} \right\} \exp\left\{ -\frac{i}{\hbar}\epsilon V\left(x_b + \frac{\eta}{2}, t_a + \frac{\epsilon}{2} \right) \right\} \psi(x_b + \eta, t_a)\mathrm{d}\eta \tag{30}$$

2.7   Exercise II

The first integral on the righthand side is just

$$\int_{-\infty}^{\infty} \exp\left\{ \frac{im\eta^2}{2\hbar\epsilon} \right\} \mathrm{d}\eta = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}. \tag{31}$$

Note that this is since,

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}, \tag{32}$$

then we know the integral

$$\int_{-\infty}^{\infty} e^{-ax^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}} \tag{33}$$

by $u-$substitution. Looking at our equation now, we have

$$\psi(x_b, t_a) + \cdots = \frac{1}{A}\left[1 - \frac{i}{\hbar}\epsilon V(x_b, t_a) \right]\left(\psi(x_b, t_a)\sqrt{\frac{2\pi i\hbar\epsilon}{m}} + \ldots \right) \tag{34}$$

Comparing both sides, we see

$$\psi(x_b, t_a) = \psi(x_b, t_a)\frac{1}{A}\sqrt{\frac{2\pi i\hbar\epsilon}{m}} \implies A = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}. \tag{35}$$

3 Free Particle Propagator

3.1 Exercise

For a free particle (zero potential),

$$S_i = \int_{t_{i-1}}^{t_i} dt\, L = \int_{t_{i-1}}^{t_i} dt\, \frac{1}{2} m v^2. \tag{36}$$

Assuming constant velocity between the segments, we have

$$S_i = \int_{t_{i-1}}^{t_i} dt\, \frac{1}{2} m \frac{(x_i - x_{i-1})^2}{\Delta t^2} \tag{37}$$

$$= \Delta t\, \frac{1}{2} m \frac{(x_i - x_{i-1})^2}{\Delta t^2} \tag{38}$$

$$= \frac{m}{2\Delta t} (x_i - x_{i-1})^2, \tag{39}$$

where

$$\Delta t = t_i - t_{i-1} = (t_N - t_0)/N. \tag{40}$$

3.2 Exercise

All the following integrals will be evaluated over $(-\infty, \infty)$, but for the sake of notation, the bounds will be omitted.

$$U(x_N, t_N; x_0, t_0) = C(t) \int \dots \int dx_1 \dots dx_{N-1} \exp\left[\frac{i}{\hbar} S\right] \tag{41}$$

Using our expression for the free particle action for each segment, we get

$$C(t) \int \dots \int dx_1 \dots dx_{N-1} \exp\left[\frac{i}{\hbar} S\right] \tag{42}$$

$$= C(t) \int \dots \int dx_1 \dots dx_{N-1} \exp\left[\frac{i}{\hbar} \sum_{i=1}^{N} S_i\right] \tag{43}$$

$$= C(t) \int \dots \int dx_1 \dots dx_{N-1} \exp\left[\frac{im}{2\hbar\Delta t} \sum_{i=1}^{N} (x_i - x_{i-1})^2\right]. \tag{44}$$

We will now switch variables from the original position variable x_i to a position variable which determines position relative to the classical path $\bar{x}$:

$$y(t) = x(t) - \bar{x}(t). \tag{45}$$

The classical path for a free particle is given by

$$\bar{x}(t) = x_0 + \frac{x_N - x_0}{t_N - t_0}(t - t_0). \tag{46}$$

Since the classical and quantum paths coincide at t_0 and t_N, we have

$$y(t_0) = y(t_N) = 0. \tag{47}$$

In these coordinates,

$$\dot{x} = \dot{\bar{x}} + \dot{y} \tag{48}$$

$$\dot{\bar{x}} = \frac{x_N - x_0}{t_N - t_0}, \tag{49}$$

and so

$$S = \int_{t_0}^{t_N} dt\, \frac{1}{2} m \dot{x}^2 = \int_{t_0}^{t_N} dt\, \frac{1}{2} m (\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2). \tag{50}$$

The first term of this integral is

$$\frac{1}{2} m \frac{(x_N - x_0)^2}{t_N - t_0}. \tag{51}$$

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The second term evaluates to zero due to integration by parts, (47), and the Euler-Lagrange equations

$$\ddot{x} = 0. \tag{52}$$

The last term is equivalent to the action for the free particle in the original coordinates with boundary conditions given by (47). Furthermore it is given in the packet that

$$\int \mathcal{D}[x(t)] = \int \mathcal{D}[y(t)] \tag{53}$$

for these coordinates, and so the total path integral in these coordinates is given by

$$C(t) \int ... \int dx_1...dx_{N-1} \exp\left[\frac{i}{\hbar}S\right] \tag{54}$$

$$= C(t) \exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right] \int ... \int dy_1...dy_{N-1} \exp\left[\frac{i}{\hbar}\int \frac{1}{2}m\dot{y}^2\right]. \tag{55}$$

By using our expression (44) for the action in the path integral with the variable y, we get

$$C(t) \exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right] \int ... \int dy_1...dy_{N-1} \exp\left[\frac{i}{\hbar}\int \frac{1}{2}m\dot{y}^2\right] \tag{56}$$

$$= C(t) \exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right] \int ... \int dy_1...dy_{N-1} \exp\left[\frac{im}{2\hbar\Delta t}\sum_{i=1}^{N}(y_i - y_{i-1})^2\right] \tag{57}$$

$$= C(t) \exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right] \int ... \int dy_1...dy_{N-1} \exp\left[\frac{im}{2\hbar\Delta t}(2y_1^2 + 2y_2^2... + 2y_{N-1}^2 - 2y_1y_2 - 2y_2y_3... - 2y_{N-2}y_{N-1})\right]. \tag{58}$$

To simplify notation, let us define

$$k = \frac{im}{2\hbar\Delta t}. \tag{59}$$

Using the formula (derived from the Gaussian integral by completing the square and doing a u-substitution)

$$\int \exp\left[-\frac{1}{2}ax^2 + Jx\right]dx = \left(\frac{2\pi}{a}\right)^{1/2}e^{J^2/2a}, \tag{60}$$

we can evaluate all the integrals.
Isolating just the terms with y_1, the integral over y_1 becomes

$$\int dy_1 \exp\left[2ky_1^2 - 2ky_1y_2\right]. \tag{61}$$

Letting

$$a_1 = -4k, J_1 = -2ky_2, \tag{62}$$

we get that this integral is

$$\left(-\frac{\pi}{2k}\right)^{1/2}e^{-ky_2^2/2}. \tag{63}$$

The integral over y_2 then becomes

$$\int dy_2 \exp\left[\frac{3}{2}ky_2^2 - 2ky_2y_3\right]. \tag{64}$$

Again, letting

$$a_2 = -3k, J_2 = -2ky_3, \tag{65}$$

we get that this integral is

$$\left(-\frac{2\pi}{3k}\right)^{1/2}e^{-2ky_3^2/3}. \tag{66}$$

From these expressions, we may guess that

$$a_n = -2k\left(\frac{n+1}{n}\right), J_n = -2ky_{n+1} \tag{67}$$

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for $1 \leq n \leq N-1$. The equation for J_n holds because the only factors added to the exponents are of form $J_{n-1}^2/2a_{n-1}$ which are quadratic in y_n and thus cannot impact the linear terms already present in (58). We can verify the equation for a_n with induction. This formula obviously holds for n=1. Noting that

$$-\frac{1}{2}a_{n+1} = 2k + \frac{J_n^2}{2a_n y_{n+1}^2} \rightarrow a_{n+1} = -4k - \frac{4k^2}{-2k\left(\frac{n+1}{n}\right)} = -2k\left(\frac{n+2}{n+1}\right). \tag{68}$$

The first equation was acquired by remembering that $-a_{n+1}/2$ was the coefficient of y_{n+1}^2 in the Gaussian integral and by noting that the original coefficients for all y_n was 2k. We then simply add the factor of $\frac{J_n^2}{2a_n y_{n+1}^2}$, which arises from the calculation of the Gaussian integral for y_n. Since we only care about the coefficient of y_{n+1}^2, we divide this factor out in the equation.

Since in these coordinates $y_N = 0$, we get that J_{N-1}=0. Before this point, the factors of $\exp\left[J^2/2a\right]$ only contribute to the calculation of a and thus do not impact the value of the integration. Thus, the total value of all the integrals over $y_1...y_{N-1}$ will just be

$$\int ... \int dy_1...dy_{N-1} \exp\left[\frac{im}{2\hbar\Delta t}\sum_{i=1}^{N}(y_i - y_{i-1})^2\right] = \prod_{n=1}^{N-1}\left(\frac{2\pi}{a_n}\right)^{1/2}. \tag{69}$$

Inserting the formula for a_n and pulling out the constant factor, we get

$$\prod_{n=1}^{N-1}\left(\frac{2\pi}{a_n}\right)^{1/2} = \left(\left(-\frac{\pi}{k}\right)^{N-1}\prod_{n=1}^{N-1}\frac{n}{n+1}\right)^{1/2}. \tag{70}$$

The product is simply

$$\prod_{n=1}^{N-1}\frac{n}{n+1} = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}...\frac{N-1}{N} = \frac{1}{N}. \tag{71}$$

Putting everything together and inserting the value of k, we get

$$U(x_N, t_N; x_0, t_0) = C(t)\left(\frac{2\pi i\hbar\Delta t}{m}\right)^{\frac{N-1}{2}}\sqrt{\frac{1}{N}}\exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right]. \tag{72}$$

In order to find the normalization factor $C(t)$, we note that

$$\lim_{\Delta t\to 0} U(x_N, t_N; x_0, t_0) = \delta(x_N - x_0), \tag{73}$$

as the particle must approach the initial position as the time interval shortens. Integrating this over x_N, we get

$$\lim_{\Delta t\to 0} C(t)\left(\frac{2\pi i\hbar\Delta t}{m}\right)^{\frac{N}{2}} = 1. \tag{74}$$

The extra factor of $\sqrt{2\pi i\hbar(t_N - t_0)/m} = \sqrt{2\pi i\hbar N\Delta t/m}$ comes from the Gaussian integral over the exponential in the propagator. Furthermore, the exponential due to $\frac{imx_0^2}{2\hbar N\Delta t}$ gets precisely cancelled due to the factor $J^2/2a = -\frac{imx_0^2}{2\hbar N\Delta t}$ which comes from the Gaussian. To have this properly normalized, we must have

$$C(t) = \left(\frac{m}{2\pi\hbar i\Delta t}\right)^{N/2}. \tag{75}$$

We thus get

$$U(x_N, t_N; x_0, t_0) = \left(\frac{m}{2\pi\hbar iN\Delta t}\right)^{1/2}\exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right], \tag{76}$$

where the factors of $C(t)$ have cancelled out the factors from the integration. Noting that $N\Delta t = t_N - t_0$ (the total time interval), we get the final expression for the free particle propagator:

$$U(x_N, t_N; x_0, t_0) = \left(\frac{m}{2\pi\hbar i(t_N - t_0)}\right)^{1/2}\exp\left[\frac{im}{2\hbar}\frac{(x_N - x_0)^2}{t_N - t_0}\right]. \tag{77}$$

4   Schrödinger's Equation from Path Integrals

4.1   Exercise

$$S = \int_{t_0}^{t} \frac{1}{2}mv^2 - V(x)dt' \tag{78}$$

Using the short-time constant-velocity approximation, this becomes

$$S = \delta t * [\frac{1}{2}m\frac{\delta x^2}{\delta t^2} - V(x_0 + \frac{\delta x}{2})] = \frac{m\delta x^2}{2\delta t} - \delta t V(x_0 + \frac{\delta x}{2}), \tag{79}$$

where x_0 is the initial position, $\delta x = x - x_0$ is the change in position, and $\delta t = t - t_0$ is the change in time. Due to the short time scale, the potential is evaluated at the mid-point of the interval, given by $x_0 + \delta x/2$. With this, the propagator is simply

$$U(x_0 + \delta x, t_0 + \delta t; x_0, t_0) = C(t)\exp\left[\frac{im\delta x^2}{2\hbar\delta t} - \frac{iV(x_0 + \frac{\delta x}{2})}{\hbar}\delta t\right]. \tag{80}$$

4.2   Exercise

The first order expansion for $V(x_0 + \frac{\delta x}{2})$ is

$$V(x_0 + \frac{\delta x}{2}) \approx V(x_0) + \frac{\delta x}{2}\frac{dV}{dx}. \tag{81}$$

The propagator can be split into kinetic and potential parts:

$$U(x_0 + \delta x, t_0 + \delta t; x_0, t_0) \tag{82}$$

$$= C(t)\exp\left[\frac{im\delta x^2}{2\hbar\delta t} - \frac{iV(x_0 + \frac{\delta x}{2})}{\hbar}\delta t\right] \tag{83}$$

$$= C(t)\exp\left[\frac{im\delta x^2}{2\hbar\delta t}\right]\exp\left[-\frac{iV(x_0 + \frac{\delta x}{2})}{\hbar}\delta t\right]. \tag{84}$$

In this expression, we will replace $V(x_0 + \frac{\delta x}{2})$ with $V(x_0)$, as the second term in the first order expansion of $V(x_0 + \frac{\delta x}{2})$ will be second order in infinitesimals when multiplied with the factor of δt in the exponential in the propagator. Expanding the potential part, we get

$$U(x_0 + \delta x, t_0 + \delta t; x_0, t_0) = C(t)[1 - \frac{iV(x_0)}{\hbar}\delta t]\exp\left[\frac{im\delta x^2}{2\hbar\delta t}\right]. \tag{85}$$

In view of the integrals cancelling the normalization factor and the results of exercises 2.6.1 and 2.6.2, we will avoid expanding $\exp\left[\frac{im\delta x^2}{2\hbar\delta t}\right]$.

4.3   Exercise

From exercise 2.6.2, we know we can change variables to $\eta = \delta x$, and that if η is small, we can treat the integral as though only the exponential contributes. Using this approximation and inserting our expansions for the propagator and the wavefunctions, we get that

$$\psi(x,t) = C(t)[1 - \frac{iV(x_0)}{\hbar}\delta t]\int d\eta[\psi(x_0,t_0) + \frac{1}{2}\eta^2\frac{\partial^2\psi(x_0,t_0)}{\partial x^2}]\exp\left[\frac{im\eta^2}{2\hbar\delta t}\right]. \tag{86}$$

Evaluating the Gaussian integrals gives

$$\psi(x,t) = C(t)[1 - \frac{iV(x_0)}{\hbar}\delta t]\left\{\psi(x_0,t_0)\left(\frac{2\pi i\hbar\delta t}{m}\right)^{1/2} + \frac{1}{2}\sqrt{2\pi}\left(\frac{i\hbar\delta t}{m}\right)^{3/2}\frac{\partial^2\psi(x_0,t_0)}{\partial x^2}\right\} \tag{87}$$

where the formula

$$\int d\xi\xi^2\exp\left[\frac{im\xi^2}{2\hbar\delta t}\right] = \sqrt{2\pi}\left(\frac{i\hbar\delta t}{m}\right)^{3/2} \tag{88}$$

was used, which can be derived by differentiating (60) with respect to a. To figure out the normalization factor $C(t)$, we take the small δt limit, in which case the terms with factors of higher powers of δt (3/2 vs. 1/2) are significantly smaller than the other terms. In this case,

$$\psi(x_0,t_0) = C(t)\psi(x_0,t_0)\left(\frac{2\pi i\hbar\delta t}{m}\right)^{1/2}. \tag{89}$$

Note that $\psi(x,t)$ was replaced by $\psi(x_0,t_0)$, as when the time interval decreases, the time evolving wavefunction approaches the initial state. With this equality, it is clear that

$$C(t) = \left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2}.$$

(90)

Inserting this equation into (87), expanding, and removing terms second order in δt finally yields

$$\psi(x,t) = \psi(x_0,t_0) - \frac{i}{\hbar}V(x_0)\psi(x_0,t)\delta t + \frac{i\hbar\delta t}{2m}\frac{\partial^2\psi(x_0,t_0)}{\partial x^2}.$$

(91)

This can be rearranged to

$$\frac{\psi(x,t) - \psi(x_0,t_0)}{\delta t} = -\frac{i}{\hbar}V(x_0)\psi(x_0,t) + \frac{i\hbar}{2m}\frac{\partial^2\psi(x_0,t_0)}{\partial x^2}.$$

(92)

Taking the limit $\delta t \to 0$ and multiplying by $i\hbar$, we finally recover the Schrödinger equation:

$$i\hbar\frac{\partial\psi(x,t_0)}{\partial t} = \left(V(x_0) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x_0,t_0).$$

(93)

5 Harmonic Oscillator Propagator

5.1 Exercise

$$\ddot{x}_c = -A\omega^2 \sin(\omega t) - B\omega^2 \cos(\omega t) = -\omega^2 x_c \tag{94}$$

Using the equation

$$f(x) = \int_{s(x)}^{g(x)} h(x,t)dt \tag{95}$$

$$\frac{df}{dx} = \int_{s(x)}^{g(x)} \partial_x h(x,t)dt + h(x,g(x))\frac{dg}{dx} - h(x,s(x))\frac{ds}{dx}, \tag{96}$$

and setting $x = t$, $t = t'$, $h(t,t') = \sin(\omega(t-t'))f(t')$, $g(t) = t$, and $s(t) = t_a$ in the formula, we get

$$\dot{x}_p = -\frac{1}{m}\frac{\cos(\omega(t_a-t))}{\sin(\omega(t_b-t_a))}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' + \frac{1}{m\omega}[\int_{t_a}^{t}\omega\cos(\omega(t-t'))f(t')dt' + \sin(\omega(t-t))f(t)] \tag{97}$$

$$= -\frac{1}{m}\frac{\cos(\omega(t_a-t))}{\sin(\omega(t_b-t_a))}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' + \frac{1}{m}\int_{t_a}^{t}\cos(\omega(t-t'))f(t')dt'. \tag{98}$$

Using the formula again, we get

$$\ddot{x}_p = -\frac{1}{m}\frac{\sin(\omega(t_a-t))}{\sin(\omega(t_b-t_a))}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' - \frac{1}{m}[\int_{t_a}^{t}\omega\sin(\omega(t-t'))f(t')dt' + \cos(\omega(t-t))f(t)] \tag{99}$$

$$= -\frac{1}{m}\frac{\sin(\omega(t_a-t))}{\sin(\omega(t_b-t_a))}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' - \frac{1}{m}\int_{t_a}^{t}\omega\sin(\omega(t-t'))f(t')dt' + \frac{f(t)}{m} \tag{100}$$

$$= -\omega^2 x_p + \frac{f(t)}{m}. \tag{101}$$

At t_a we have

$$x(t_a) = A\sin(\omega t_a) + B\cos(\omega t_a). \tag{102}$$

The factors in x_p are zero at this time because $\sin(\omega(t_a-t))$ will be zero and the bounds of the second integral will match and the integral will vanish. At t_b we have

$$x(t_b) = A\sin(\omega t_b) + B\cos(\omega t_b) + \frac{1}{m\omega}\frac{\sin(\omega(t_a-t_b))}{\sin(\omega(t_b-t_a))}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' + \frac{1}{m\omega}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' \tag{103}$$

$$= A\sin(\omega t_b) + B\cos(\omega t_b) - \frac{1}{m\omega}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' + \frac{1}{m\omega}\int_{t_a}^{t_b}\sin(\omega(t_b-t'))f(t')dt' \tag{104}$$

$$= A\sin(\omega t_b) + B\cos(\omega t_b). \tag{105}$$

Since $x(t_a) = x_a$ and $x(t_b) = x_b$, we can solve for A and B in terms of these variables:

$$x_a\sin(\omega t_b) = A\sin(\omega t_a)\sin(\omega t_b) + B\cos(\omega t_a)\sin(\omega t_b) \tag{106}$$
$$x_b\sin(\omega t_a) = A\sin(\omega t_a)\sin(\omega t_b) + B\sin(\omega t_a)\cos(\omega t_b). \tag{107}$$

Subtracting these equations and factoring out B yields

$$B = \frac{x_a\sin(\omega t_b) - x_b\sin(\omega t_a)}{\cos(\omega t_a)\sin(\omega t_b) - \cos(\omega t_b)\sin(\omega t_a)}. \tag{108}$$

Doing the same for A yields

$$A = \frac{x_a\cos(\omega t_b) - x_b\cos(\omega t_a)}{\cos(\omega t_b)\sin(\omega t_a) - \cos(\omega t_a)\sin(\omega t_b)}. \tag{109}$$

5.2 Exercise

When $f(t)$=0, the action for each segment becomes

$$S_n = \frac{m\omega}{2\sin(\omega T)}[\cos(\omega T)(x_n^2 + x_{n-1}^2) - 2x_n x_{n-1}], \tag{110}$$

since all the integrals with factors of $f(t)$ will vanish. Then the kernel (propagator) is just

$$K = C(T) \int ... \int dx_1...dx_{N-1} \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)} \sum_{n=1}^{N}[\cos(\omega T)(x_n^2 + x_{n-1}^2) - 2x_n x_{n-1}]\right]. \tag{111}$$

We will redefine $x_a = x_0, x_b = x_N$. Again, let us switch to coordinates

$$y(t) = x(t) - \bar{x}(t). \tag{112}$$

Here the classical path is given by

$$\bar{x} = x_c. \tag{113}$$

The Lagrangian in these coordinates becomes

$$L = (\frac{1}{2}m\dot{\bar{x}}^2 - \frac{1}{2}m\omega^2\bar{x}^2) + (m\dot{\bar{x}}\dot{y} - m\omega\bar{x}y) + (\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2). \tag{114}$$

After integrating, the second term is zero due to integration by parts, the boundary conditions for y, and the equation of motion:

$$\int dt(m\dot{\bar{x}}\dot{y} - m\omega\bar{x}y) = m\dot{\bar{x}}y|_{y(t_0)}^{y(t_N)} - m\int dty(\ddot{\bar{x}} + \omega^2\bar{x}) = 0 \tag{115}$$

The action thus becomes

$$S_{cl} + S_n \tag{116}$$

where the action for each segment S_n is now in terms of y instead of x:

$$S_n = \frac{m\omega}{2\sin(\omega T)}[\cos(\omega T)(y_n^2 + y_{n-1}^2) - 2y_n y_{n-1}]. \tag{117}$$

Again, in these coordinates,

$$y_0 = y_N = 0. \tag{118}$$

Bringing the factor of $\exp\left[\frac{i}{\hbar}S_{cl}\right]$ out of the path integral gives

$$K = C(T)\exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}[(x_N^2 + x_0^2)\cos(\omega T) - 2x_0 x_N]\right] \tag{119}$$

$$\times \int ... \int dy_1...dy_N \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)} \sum_{n=1}^{N}[\cos(\omega T)(y_n^2 + y_{n-1}^2) - 2y_n y_{n-1}]\right]. \tag{120}$$

Let us define

$$F(T) = C(T)\int ... \int dy_1...dy_N \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)} \sum_{n=1}^{N}[\cos(\omega T)(y_n^2 + y_{n-1}^2) - 2y_n y_{n-1}]\right]. \tag{121}$$

Then our propagator simply becomes

$$K = F(T)\exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}[(x_N^2 + x_0^2)\cos(\omega T) - 2x_0 x_N]\right]. \tag{122}$$

We know that $C(T)$ is a function of only T from exercise 2.4.1, and since all the integrals in $F(T)$ are evaluated over all other variables $y_1...y_{N-1}$ ($y_0 = y_N = 0$), $F(T)$ is a function of T alone (m and ω are constants).

5.3 Exercise

$$\psi(x, T) = \int dx' K(x, T; x', 0)\psi(x', 0) \tag{123}$$

Inserting

$$\psi(x, 0) = \exp\left[-\frac{m\omega}{2\hbar}(x - a)^2\right] \tag{124}$$

$$K(x, T; x', 0) = F(T)\exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}[(x^2 + x'^2)\cos(\omega T) - 2xx']\right] \tag{125}$$

gives

$$\psi(x, T) = F(T)\int dx' \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}[(x^2 + x'^2)\cos(\omega T) - 2xx'] - \frac{m\omega}{2\hbar}(x' - a)^2\right] \tag{126}$$

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Let us define

$$k = \frac{im\omega}{2\hbar \sin(\omega T)}, l = \frac{m\omega}{2\hbar}. \tag{127}$$

Grouping like terms and bringing constants out of the integral gives

$$\psi(x, T) = F(T)e^{kx^2 \cos(\omega T) - la^2} \int dx' \exp\left[(k\cos(\omega T) - l)x'^2 + (2la - 2kx)x'\right]. \tag{128}$$

This Gaussian can be done by defining

$$b = 2l - 2k\cos(\omega T), J = 2la - 2kx, \tag{129}$$

which gives

$$F(T)\left(\frac{2\pi}{2l - 2k\cos(\omega T)}\right)^{1/2} \exp\left[kx^2 \cos(\omega T) - la^2 + \frac{l^2a^2 + k^2x^2 - 2lakx}{l - k\cos(\omega T)}\right]. \tag{130}$$

Inserting the equations for l, k, and $F(T)$ reduces the first part of this equation to

$$F(T)\left(\frac{2\pi}{2l - 2k\cos(\omega T)}\right)^{1/2} = \left(\frac{1}{\cos(\omega T) + i\sin(\omega T)}\right)^{1/2} = \exp\left[\frac{-i\omega T}{2}\right] \tag{131}$$

by Euler's formula. After inserting the factors of l and k into the fraction and recognizing $\cos(\omega T) + i\sin(\omega T) = e^{i\omega T}$, we get that the exponent is

$$kx^2 \cos(\omega T) - la^2 + \frac{2\hbar i \sin(\omega T)}{m\omega}e^{-i\omega T}(l^2a^2 + k^2x^2 - 2lakx). \tag{132}$$

Inserting the equations for l and k and simplifying gives the exponent as (note that when expanding the coefficient of x^2, the addition with $\cos(\omega T) + i\sin(\omega T) = e^{i\omega T}$ cancelled precisely):

$$-\frac{m\omega}{2\hbar}[x^2 - 2axe^{-i\omega T} + a^2 \cos(\omega T)e^{-i\omega T}]. \tag{133}$$

Combining with our previous result gives the total wavefunction (up to normalization) as

$$\psi(x, T) = C \exp\left\{-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}[x^2 - 2axe^{-i\omega T} + a^2 \cos(\omega T)e^{-i\omega T}]\right\}. \tag{134}$$

The probability distribution is given by

$$|\psi|^2 = \psi^*\psi = |C|^2 \exp\left\{-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}[x^2 - 2axe^{-i\omega T} + a^2 \cos(\omega T)e^{-i\omega T}]\right\} \tag{135}$$

$$\times \exp\left\{\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}[x^2 - 2axe^{i\omega T} + a^2 \cos(\omega T)e^{i\omega T}]\right\} \tag{136}$$

$$= |C|^2 \exp\left\{-\frac{m\omega}{2\hbar}[2x^2 - 2ax(e^{-i\omega T} + e^{i\omega T}) + a^2 \cos(\omega T)(e^{-i\omega T} + e^{i\omega T})]\right\}. \tag{137}$$

Using $e^{-i\omega T} + e^{i\omega T} = 2\cos(\omega T)$, this becomes

$$|C|^2 \exp\left\{-\frac{m\omega}{2\hbar}[2x^2 - 4ax\cos(\omega T) + 2a^2 \cos^2(\omega T)]\right\}. \tag{138}$$

For this to be a properly normalized probability distribution, we must have

$$\int_{-\infty}^{\infty} dx|\psi(x)|^2 = 1. \tag{139}$$

Doing this Gaussian with

$$b = \frac{2m\omega}{\hbar}, J = ba\cos(\omega T) \tag{140}$$

yields

$$|C|^2 \exp\left[-m\omega a^2 \cos^2(\omega T)/\hbar + m\omega a^2 \cos^2(\omega T)/\hbar\right]\left(\frac{\pi\hbar}{m\omega}\right)^{1/2} = |C|^2 \left(\frac{\pi\hbar}{m\omega}\right)^{1/2} = 1, \tag{141}$$

or (up to a phase)

$$C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}. \tag{142}$$

Inserting this normalization into our expression for the wavefunction yields

$$\psi(x,T) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\frac{i\omega T}{2} - \frac{m\omega}{2\hbar}[x^2 - 2axe^{-i\omega T} + a^2\cos(\omega T)e^{-i\omega T}]\right\}. \tag{143}$$

and the probability distribution becomes

$$|\psi|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega}{2\hbar}[2x^2 - 4ax\cos(\omega T) + 2a^2\cos^2(\omega T)]\right\}. \tag{144}$$

5.4 Exercise (New 5.4)

In this case the action for each segment becomes

$$S_n = \frac{m\omega}{2\sin(\omega T)}[\cos(\omega T)(x_{n-1}^2 + x_n^2) - 2x_{n-1}x_n \tag{145}$$

$$+ \frac{2x_{n-1}}{m\omega}\int_{t_{n-1}}^{t_n}\sin(\omega(t_n - t))f(t)dt + \frac{2x_n}{m\omega}\int_{t_{n-1}}^{t_n}\sin(\omega(t - t_{n-1}))f(t)dt$$

$$- \frac{2}{m^2\omega^2}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t}\sin(\omega(t_n - t'))\sin(\omega(t - t_{n-1}))f(t)f(t')dt'dt].$$

As always, we will switch to coordinates

$$y(t) = x(t) - \bar{x}(t). \tag{146}$$

In this case,

$$\bar{x} = x_p + x_c. \tag{147}$$

Furthermore we know the Lagrangian for this system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + f(t)x. \tag{148}$$

In the new coordinates, this is simply

$$L = \left(\frac{1}{2}m\dot{\bar{x}}^2 - \frac{1}{2}m\omega^2\bar{x}^2 + f(t)\bar{x}\right) + (m\dot{\bar{x}}\dot{y} - m\omega\bar{x}y) + \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2 + f(t)y\right). \tag{149}$$

Integrating, the second term becomes,

$$\int dt(m\dot{\bar{x}}\dot{y} - m\omega\bar{x}y) = m\dot{\bar{x}}y|_{y(t_0)}^{y(t_N)} - m\int dty(\ddot{\bar{x}} + \omega^2\bar{x}) = -\int dty f(t) \tag{150}$$

where the last equality is determined by the equations of motions for $\bar{x}$. The precisely cancels the last term in the Lagrangian, and the action for y simply becomes that of the unforced harmonic oscillator:

$$S_{cl} + S_n \tag{151}$$

where S_n defined by (110) is evaluated over y instead of x. Now the propagator for this system is

$$K = C(T)\int \mathcal{D}[x(t)]\exp\left[\frac{i}{\hbar}\sum_{n=1}^{N}S_n\right] = C(T)\exp\left[\frac{i}{\hbar}S_{cl}\right]\int \mathcal{D}[y(t)]\exp\left[\frac{i}{\hbar}(\sum_{n=1}^{N}S_n)\right]. \tag{152}$$

If we define

$$F(T) = C(T)\int \mathcal{D}[y(t)]\exp\left[\frac{i}{\hbar}(\sum_{n=1}^{N}S_n)\right], \tag{153}$$

then,

$$K = F(T)\exp\left[\frac{i}{\hbar}S_{cl}\right]. \tag{154}$$

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As before, we know that $C(T)$ is just a function of T and the path integral integrates over all free variables in the action except T, thus leaving the final $F(T)$ as a function of T alone. To determine $F(T)$, we remember that

$$\lim_{T \to 0} K(x_N, t_N; x_0, t_0) = \delta(x_N - x_0). \tag{155}$$

Integrating over dx_N gives

$$\lim_{T \to 0} F(T) \int dx_N \exp\left[\frac{i}{\hbar} S_{cl}\right] = 1. \tag{156}$$

Looking at the terms in the classical action, we see that as $T \to 0$, or $t_N \to t_0$, all the integrals become significantly smaller than the other terms. Furthermore, the cosine term approaches 1. Thus we can approximate the action as

$$S_{cl} \approx \frac{m\omega}{2\sin(\omega T)}[x_N^2 + x_0^2 - 2x_N x_0] \tag{157}$$

in the small T limit. Doing the Gaussian integration with

$$a = -\frac{im\omega}{\hbar \sin(\omega T)}, J = -\frac{m\omega x_0}{\sin(\omega T)} \tag{158}$$

gives

$$\left(\frac{2\pi i\hbar \sin(\omega T)}{m\omega}\right)^{1/2} F(T) = 1, \tag{159}$$

or

$$F(T) = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega T)}\right)^{1/2}. \tag{160}$$

Inserting this into (154) finally yields the desired propagator for the forced harmonic oscillator:

$$K = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega T)}\right)^{1/2} \exp\left[\frac{i}{\hbar} S_{cl}\right]. \tag{161}$$

5.5   Exercise (Old 5.4)

If $f(t) = f$, the integrals in the action become

$$f \int_{t_0}^{t_N} \sin(\omega(t_N - t))dt = f\frac{1 - \cos(\omega T)}{\omega} \tag{162}$$

$$f \int_{t_0}^{t_N} \sin(\omega(t - t_0))dt = f\frac{1 - \cos(\omega T)}{\omega} \tag{163}$$

$$f^2 \int_{t_0}^{t_N} \int_{t_0}^{t} \sin(\omega(t_N - t)) \sin(\omega(t' - t_0))dt'dt = f^2(-\frac{\omega T \sin(\omega T) + 2\cos(\omega T) - 2}{2\omega^2}) \tag{164}$$

Then

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}[\cos(\omega T)(x_N^2 + x_0^2) - 2x_N x_0 + \frac{2f - 2f\cos(\omega T)}{m\omega^2}[x_0 + x_N] + \frac{2f^2}{m^2\omega^2}\frac{\omega T \sin(\omega T) + 2\cos(\omega T) - 2}{2\omega^2}]. \tag{165}$$

By (161), the propagator is

$$K = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega T)}\right)^{1/2} \exp\left[\frac{im\omega}{2\hbar \sin(\omega T)}\{A + B\}\right] \tag{166}$$

where

$$A = \cos(\omega T)(x_N^2 + x_0^2) - 2x_N x_0 + \frac{2f - 2f\cos(\omega T)}{m\omega^2}[x_0 + x_N] \tag{167}$$

$$B = \frac{2f^2}{m^2\omega^2}\frac{\omega T \sin(\omega T) + 2\cos(\omega T) - 2}{2\omega^2} \tag{168}$$

6 Partition Function

6.1 Exercise

We can think of W as such: we have N particles and we must place n_i particles in the ith box in any order. Here, W will represent the different number of ways one can do that. Hence, W is given as

$$W = \frac{N!}{n_1! n_2! \dots n_i!} \tag{169}$$

Therefore,

$$\ln W = \ln \frac{N!}{n_1! n_2! \dots n_i!} \tag{170}$$

$$= \ln N! - \left(\sum_{i=1 \leq N} \ln n_i! \right) \tag{171}$$

Stirling's formula allows us to approximate factorials as

$$\ln n! \approx n \ln n - n \tag{172}$$

This means

$$\ln W = N \ln N - N - \sum (n_i \ln n_i - n_i) \tag{173}$$

Since $\sum n_i = N$, this means that N and N cancel out and we are left with

$$\ln W = N \ln N - \sum (n_i \ln n_i). \tag{174}$$

6.2 Exercise

We seek to maximize a function $f(n_1, n_2, \dots, n_i)$ with the constraints of

$$\sum n_i = N \tag{175}$$

$$\sum N_i E_i = E \tag{176}$$

$$\frac{\partial f}{\partial n_i} = 0 \tag{177}$$

To do this, we use lagrange multipliers, where we add these constraints to our equation. Hence,

$$f(n_1, \dots, n_i, \alpha, \beta) = \ln W + \alpha \sum_i n_i - \beta \sum_i n_i E_i \tag{178}$$

$$= \sum_i n_i - \sum_i n_i \ln n_i + \alpha \sum_i n_i - \beta \sum_i n_i E_i \tag{179}$$

Using our third constraint, we find that

$$\frac{\partial f}{\partial n_i} = -\ln n_i + \alpha - \beta E_i = 0. \tag{180}$$

Therefore, we can find that

$$n_i = \exp(\alpha - \beta E_i) \tag{181}$$

This equation can be summed over i to show that

$$N = e^\alpha \sum_i e^{-\beta E_i} \implies \alpha = \ln \left(\frac{N}{\sum_i e^{-\beta E_i}} \right). \tag{182}$$

Hence, this implies that

$$n_i = \frac{N \exp(-\beta E_i)}{\sum_j \exp(-\beta E_j)} \tag{183}$$

6.3 Exercise

The average energy can be written as

$$\bar{E} = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \tag{184}$$

The denominator is just the partition function whilst the numerator is

$$-\frac{\mathrm{d}Z}{\mathrm{d}\beta} = \sum_i E_i e^{-\beta E_i}. \tag{185}$$

The average energy is then just

$$\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial \beta}. \tag{186}$$

6.4 Exercise

The variance is the mean squared deviation. Or in other words,

$$(\overline{\Delta E^2}) = \overline{E^2} - \overline{E}^2 \tag{187}$$

We can calculate each part. For the first one, note that

$$\overline{E^2} = \frac{\sum_i E_i^2 \exp(-\beta E_i)}{\sum_i \exp(-\beta E_i)}. \tag{188}$$

From the given equation in the assignment, this implies that

$$\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} \tag{189}$$

We also know that

$$\overline{E}^2 = \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 \tag{190}$$

Combining together gives

$$(\overline{\Delta E^2}) = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 \tag{191}$$

$$= \frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right) + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 - \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 \tag{192}$$

$$= \frac{\partial^2 \ln Z}{\partial \beta^2} \tag{193}$$

6.5 Exercise

The change in energy by a quasi-static change in parameter of $x \to x + \mathrm{d}x$ is

$$\delta E = \frac{\partial E}{\partial x}\delta x. \tag{194}$$

Hence, the macroscopic change in work is simply

$$\delta W = \frac{\sum_i(-\partial E_i/\partial x)\exp(-\beta E_i)}{\sum_i \exp(-\beta E_i)}\delta x. \tag{195}$$

The numerator is simply $-\partial Z/\beta(\partial x)$ and the denominator is just the partition function. Hence,

$$\delta W = \frac{1}{\beta Z}\frac{\partial Z}{\partial x}\delta x = \frac{1}{\beta}\frac{\partial \ln Z}{\partial x}\delta x. \tag{196}$$

6.6 Exercise

If we generalize x to V, then we can use the fact that work done is $\mathrm{d}W = p\mathrm{d}V$ and since

$$\mathrm{d}W = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}\mathrm{d}V, \tag{197}$$

it is apparent that

$$\bar{p} = \frac{1}{Z}\frac{\partial \ln Z}{\partial V} \tag{198}$$

6.7 Exercise (Old)

Generally, we know that for a regular ideal gas, the equation of state is the ideal gas law $pV = nRT$. With this equation, we can relate all intrinsic variables pressure, volume, and temperature together. The above equation can be related to the equation of state because we contain variables of β, V, and p (meaning that β depends on temperature T). This means that if we know either pressure or volume, we can find temperature and vice versa with just the partition function. Most likely, β has units of 1/Energy as the coefficients of exponentials in the partition coefficient ($e = \sum e^{-\beta E_i}$) must be dimensionless. Furthermore, as deduced before, it also must depend on T, so the best value for β should be $1/k_B T$, or the thermodynamic beta as it is called.

6.8 Exercise

Since Z is a function of β and x, we can write by the multivariable chain rule that under a quasi-static change in which parameters x and β change slowly, the change in partition function follows:

$$d\ln Z = \frac{\partial \ln Z}{\partial x}\mathrm{d}x + \frac{\partial \ln Z}{\partial \beta}\mathrm{d}\beta. \tag{199}$$

Substituting equations for work and average energy means

$$d\ln Z = \beta \mathrm{d}W - \overline{E}\mathrm{d}\beta \tag{200}$$

Note that $\mathrm{d}\overline{E} + \mathrm{d}W = \mathrm{d}Q$. So, we can rewrite the above equation as

$$\mathrm{d}\left(\ln Z + \overline{E}\beta\right) = \beta(\mathrm{d}W + \mathrm{d}\overline{E}) = \beta \mathrm{d}Q \tag{201}$$

We know that entropy is given as

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} \tag{202}$$

Hence, this now implies that

$$\beta T \mathrm{d}S = \mathrm{d}\left(\ln Z + \overline{E}\beta\right) \tag{203}$$

$$S = \frac{1}{\beta T}\left(\ln Z + \overline{E}\beta\right) \tag{204}$$

6.9 Exercise

Suppose that two systems have energy states E_i and E_j. As the two systems are interacting weakly, the total energy is $E_t = E_i + E_j$ as nothing is dissipated. So then look at the total partition function.

$$Z = \sum_{i,j}\exp(-\beta E_t) \tag{205}$$

$$= \sum_{i,j}\exp(-\beta(E_i + E_j)) \tag{206}$$

$$= \sum_{i,j}\exp(-\beta E_i)\exp(-\beta E_j) \tag{207}$$

$$= \left(\sum_i \exp(-\beta E_i)\right)\left(\sum_j \exp(-\beta E_j)\right) \tag{208}$$

$$= Z_1 Z_2 \tag{209}$$

So the total partition function will be multiplied to each other

$$Z_t = Z_1 Z_2 \tag{210}$$

By logarithm rules, the natural logarithm is then expressed as

$$\ln Z = \ln Z_1 + \ln Z_2 \tag{211}$$

Therefore, the total average energy follows

$$\overline{E}_t = -\frac{\partial \ln Z_t}{\partial \beta} = -\frac{\partial(\ln Z_1 + \ln Z_2)}{\partial \beta} = -\left(\frac{\partial \ln Z_1}{\partial \beta} + \frac{\partial \ln Z_2}{\partial \beta}\right) = \overline{E_1} + \overline{E_2} \tag{212}$$

which is additive. Similarly, the total entropy follows

$$S_t = \frac{1}{\beta T}(\ln Z_t + \overline{E}_t\beta) = \frac{1}{\beta T}(\ln Z_1 + \overline{E}_1\beta) + \frac{1}{\beta T}(\ln Z_2 + \overline{E}_2\beta) = S_1 + S_2 \tag{213}$$

which is additive as well.

7 Path Integral Calculation of Partition Function

7.1 Exercise

We connect back to equation 46 in the packet where we know that

$$U(x_1, t; x_0, t_0) = \langle x_1 | e^{-i\hat{H}(t-t_0)/\hbar} | x_0 \rangle = \int_{x(t_0)=x}^{x(t)=x_1} \mathcal{D}x \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt\mathcal{L}\right]. \tag{214}$$

Taking $t - t_0 = -i\beta\hbar$ and $x = x_1 = x_0$ gives

$$U(x, -i\beta\hbar; x, 0) = \langle x | e^{-\beta\hat{H}} | x \rangle \tag{215}$$

We also know that $\sum_n |n\rangle\langle n| = 1$, meaning

$$U(x, -i\beta\hbar; x, 0) = \langle x | e^{-\beta\hat{H}} | \sum_j |j\rangle\langle j | x \rangle \tag{216}$$

$$= \sum_j e^{-\beta H} \langle x | j \rangle \langle j | x \rangle \tag{217}$$

$$= \sum_j e^{-\beta H} \langle j | x \rangle \langle x | j \rangle \tag{218}$$

Integrating over x gives us

$$\int dx U(x, -i\beta\hbar; x, 0) = \sum_j e^{-\beta H} \langle j | \int dx |x\rangle\langle x | j \rangle = \sum_j \langle j | e^{-\beta H} | j \rangle = Z. \tag{219}$$

since $\int dx |x\rangle\langle x| = I$.

7.2 Exercise

From our propagator from exercise 5.4, we know that a quantum harmonic oscillator has a propagator of

$$K = F(T) \exp\left[\frac{im\omega}{2\hbar\sin(\omega T)}[(x_N^2 + x_0^2)\cos(\omega T) - 2x_0 x_N]\right]. \tag{220}$$

Substituting $t \to -i\beta\hbar$, we get

$$Z = \int dx U(x, -i\beta\hbar; x, 0) \tag{221}$$

$$= \int dx \left(\frac{m\omega}{2\pi i\hbar\sin(i\beta\hbar\omega)}\right)^{1/2} \exp\left\{\frac{-m\omega}{\hbar i\sin(-i\beta\hbar\omega)}[x^2\cos(-i\beta\hbar\omega) - x^2]\right\} \tag{222}$$

$$= \left(\frac{m\omega}{2\pi i\hbar\sin(i\beta\hbar\omega)}\right)^{1/2} \int dx \exp\left\{\frac{-m\omega x^2}{\hbar\sinh(\beta\hbar\omega)}[\cosh(\beta\hbar\omega) - 1]\right\} \tag{223}$$

$$= \left(\frac{m\omega}{2\pi i\hbar\sin(i\beta\hbar\omega)}\right)^{1/2} \left(\frac{\pi}{\frac{m\omega}{\hbar\sinh(\beta\hbar\omega)}[\cosh(\beta\hbar\omega) - 1]}\right) \tag{224}$$

$$= \frac{1}{2(\cosh(\beta\hbar\omega) - 1)^{1/2}} \tag{225}$$

$$= \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}. \tag{226}$$

7.3 Exercise

From section 4.3, we know that

$$U(x_1, t; x_0, t_0) = \int_{x(t_0)=x_0}^{x(t)=x_1} \mathcal{D}x \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt\mathcal{L}\right] \tag{227}$$

Furthermore, we know from section 1, that the classical action is defined as

$$S = \int_{t_a}^{t_b} L(x, \dot{x}, t) dt. \tag{228}$$

So by substituting the equation into our previous equation, it is easy to show our propagator now becomes

$$U(x_1, -i\tau; x_0) = \int \mathcal{D}x \exp\left[-\frac{1}{\hbar}S_E[x(\tau)]\right] \tag{229}$$

The action in this case can be shown under differentiation rules:

$$t = -i\tau \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = -i\frac{\mathrm{d}x}{\mathrm{d}\tau}. \tag{230}$$

Hence, the Euclidean action goes as

$$\int \mathrm{d}t \left(\frac{m}{2}\dot{x}(t) - V(x(t))\right) \to -i \int \mathrm{d}t \left(\frac{m}{2}\dot{x}(\tau) + V(x(\tau))\right) \tag{231}$$

8 Green's Functions in Quantum Mechanics

8.1 Exercise

Using completeness, we can write the equation in the packet as

$$\int \int \int \int dx_1 dx_2 dx_3 dx_4 \, \langle x_b | \, e^{-i\hat{H}(T-t_2)} \, |x_1\rangle \, \langle x_1 | \, \hat{x} \, |x_3\rangle \, \langle x_3 | \, e^{-i\hat{H}(t_2-t_1)} \, |x_2\rangle \, \langle x_2 | \, \hat{x} \, |x_4\rangle \, \langle x_4 | \, e^{-i\hat{H}(t_1+T)} \, |x_a\rangle . \tag{232}$$

By the properties of position eigenkets and the orthonormality of basis kets,

$$\langle x_i | \, \hat{x} \, |x_j\rangle = x_i \delta(x_i - x_j), \tag{233}$$

we get

$$\int \int \int \int dx_1 dx_2 dx_3 dx_4 x_1 x_2 \delta(x_1 - x_3)\delta(x_2 - x_4) \, \langle x_b | \, e^{-i\hat{H}(T-t_2)} \, |x_1\rangle \, \langle x_3 | \, e^{-i\hat{H}(t_2-t_1)} \, |x_2\rangle \, \langle x_4 | \, e^{-i\hat{H}(t_1+T)} \, |x_a\rangle . \tag{234}$$

Using

$$\langle x_b | \, e^{-i\hat{H}(t_b-t_a)} \, |x_a\rangle = \int_{x_a}^{x_b} \mathcal{D}[x(t)] \exp\left\{ i \int_{t_a}^{t_b} dt \mathcal{L} \right\} \tag{235}$$

and the properties of the Dirac delta function, we get

$$\int \mathcal{D}[x(t)] x_1 x_2 \exp\left\{ i[\int_{t_2}^{T} dt\mathcal{L} + \int_{t_1}^{t_2} dt\mathcal{L} + \int_{-T}^{t_1} dt\mathcal{L}] \right\} = \int \mathcal{D}[x(t)] x_1 x_2 \exp\left\{ i \int_{-T}^{T} \mathcal{L} \right\} \tag{236}$$

and thus

$$\langle x_b | \, e^{-i\hat{H}(T-t_2)} \hat{x} e^{-i\hat{H}(t_2-t_1)} \hat{x} e^{-i\hat{H}(t_1+T)} \, |x_a\rangle = \int \mathcal{D}[x(t)] x_1 x_2 \exp\left\{ i \int_{-T}^{T} \mathcal{L} dt \right\}. \tag{237}$$

Note how we can only add the integrals in (236) if $t_1 < t_2$, as otherwise, the sign of the integral will flip, thus misaligning the bounds of integration. Since the equation (237) can only hold with this time ordering, and since this time ordering is provided by the bounds of integration given by

$$\exp\{-iH(t_2 - t_1)\}, \tag{238}$$

we can associate this term with time ordering:

$$\hat{x} \exp\{-iH(t_2 - t_1)\}\hat{x} \Longleftrightarrow T\{\hat{x}_1 \hat{x}_2\}. \tag{239}$$

This is made more obvious when expanding the Heisenberg kets

$$\hat{x}(t_i) = \hat{x}_i = e^{iHt_i} \hat{x} e^{-iHt_i}. \tag{240}$$

Since time ordering automatically assures the condition $t_1 < t_2$ when

$$T\{\hat{x}(t_1)\hat{x}(t_2)\} = \hat{x}(t_2)\hat{x}(t_1), \tag{241}$$

we have

$$e^{-iH(T-t_2)} \hat{x} e^{iH(t_2-t_1)} \hat{x} e^{-iH(t_1+T)} \tag{242}$$

$$= e^{-iHT} e^{iHt_2} \hat{x} e^{-iHt_2} e^{iHt_1} \hat{x} e^{-iHt_1} e^{-iHT} \tag{243}$$

$$= e^{-iHT} \hat{x}(t_2)\hat{x}(t_1) e^{-iHT} \tag{244}$$

$$= e^{-iHT} T\{\hat{x}(t_1)\hat{x}(t_2)\} e^{-iHT} \tag{245}$$

and thus

$$\langle x_b | \, e^{-iHT} T\{\hat{x}(t_1)\hat{x}(t_2)\} e^{-iHT} \, |x_a\rangle = \int \mathcal{D}[x(t)] x_1 x_2 \exp\left\{ i \int_{-T}^{T} \mathcal{L} dt \right\}. \tag{246}$$

8.2 Exercise

$$\int \mathcal{D}[x(t)]x_1x_2 e^{i\int_{-T}^{T} dtL} = \langle x_b| e^{-iHT}T\{\hat{x}(t_1)\hat{x}(t_2)\}e^{-iHT} |x_a\rangle . \tag{247}$$

We also know that

$$\int \mathcal{D}[x(t)]e^{i\int_{-T}^{T} dtL} = \langle x_b| e^{-2iHT} |x_a\rangle \tag{248}$$

from (235). We can insert completeness of energy eigenstates ($\sum_i |E_1\rangle \langle E_i| = 1$) into $e^{-iHT} |x_a\rangle$, giving

$$\sum_i \langle E_i|x_a\rangle e^{-iHT} |E_i\rangle = \sum_i \langle E_i|x_a\rangle e^{-iE_iT} |E_i\rangle . \tag{249}$$

In the limit $T \to \infty(1 - i\epsilon)$, this sum is dominated by the smallest factor in the exponent, just as with the classical limit of the path integral. This minimum of the exponent is given by the ground state with energy E_0. Then, we have (letting $|E_0\rangle = |\Omega\rangle$)

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iHT} |x_a\rangle = \lim_{T\to\infty(1-i\epsilon)} \sum_i \langle E_i|x_a\rangle e^{-iE_iT} |E_i\rangle = \langle\Omega|x_a\rangle e^{-iE_0*\infty(1-i\epsilon)} |\Omega\rangle . \tag{250}$$

Inserting this (and its conjugate transpose) into our previous expressions, we get

$$\lim_{T\to\infty(1-i\epsilon)} \langle x_b| e^{-iHT}T\{\hat{x}(t_1)\hat{x}(t_2)\}e^{-iHT} |x_a\rangle = \langle x_b|\Omega\rangle\langle\Omega|x_a\rangle e^{-2iE_0*\infty(1-i\epsilon)} \langle\Omega| T\{\hat{x}(t_1)\hat{x}(t_2)\} |\Omega\rangle \tag{251}$$

and

$$\lim_{T\to\infty(1-i\epsilon)} \langle x_b| e^{-2iHT} |x_a\rangle = \langle x_b|\Omega\rangle\langle\Omega|x_a\rangle e^{-2iE_0*\infty(1-i\epsilon)} \langle\Omega|\Omega\rangle . \tag{252}$$

If the vacuum state is normalized

$$\langle\Omega|\Omega\rangle = 1, \tag{253}$$

then

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle x_b| e^{-iHT}T\{\hat{x}(t_1)\hat{x}(t_2)\}e^{-iHT} |x_a\rangle}{\langle x_b| e^{-2iHT} |x_a\rangle} = \langle\Omega| T\{\hat{x}(t_1)\hat{x}(t_2)\} |\Omega\rangle \tag{254}$$

and consequently

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}[x(t)]x_1x_2 e^{i\int_{-T}^{T} dtL}}{\int \mathcal{D}[x(t)]e^{i\int_{-T}^{T} dtL}} = \langle\Omega| T\{\hat{x}(t_1)\hat{x}(t_2)\} |\Omega\rangle = G(t_1, t_2). \tag{255}$$

8.3 Exercise

$$Z[J] = \frac{\int \mathcal{D}[x(t)]e^{i(S+\int dtJ(t)x(t))}}{\int \mathcal{D}[x(t)]e^{iS}}. \tag{256}$$

The denominator is independent of J, so we can consider only the numerator while differentiating, which we denote as $W[J]$. Bringing the functional derivative through the integral, we have

$$\frac{1}{i}\frac{\delta}{\delta J(t_1)}W[J] = \frac{1}{i}\int \mathcal{D}[x(t)]\frac{\delta}{\delta J(t_1)}e^{i(S+\int dtJ(t)x(t))}. \tag{257}$$

Using the properties of the functional derivative given in the handout and the chain rule, this becomes

$$\frac{1}{i}\int \mathcal{D}[x(t)]e^{i(S+\int dtJ(t)x(t))}\frac{\delta(i\int dtJ(t)x(t))}{\delta J(t_1)} = \int \mathcal{D}[x(t)]e^{i(S+\int dtJ(t)x(t))}x(t_1). \tag{258}$$

Since this factor of $x(t_1)$ is independent of J, continued differentiation yields

$$\frac{1}{i}\frac{\delta}{\delta J(t_1)}...\frac{1}{i}\frac{\delta}{\delta J(t_n)}W[J] = \int \mathcal{D}[x(t)]e^{i(S+\int dtJ(t)x(t))}x(t_1)...x(t_n). \tag{259}$$

Setting $J = 0$ eliminates the J term from the exponent and gives

$$\frac{1}{i}\frac{\delta}{\delta J(t_1)}...\frac{1}{i}\frac{\delta}{\delta J(t_n)}W[J]|_{J=0} = \int \mathcal{D}[x(t)]e^{iS}x(t_1)...x(t_n). \tag{260}$$

Reinserting the denominator and using the correlation function's relation to the path integral from the packet, this becomes

$$\frac{1}{i}\frac{\delta}{\delta J(t_1)}...\frac{1}{i}\frac{\delta}{\delta J(t_n)}Z[J]|_{J=0} = \frac{\int \mathcal{D}[x(t)]e^{iS}x(t_1)...x(t_n)}{\int \mathcal{D}[x(t)]e^{iS}} = \langle\Omega| T\{\hat{x}(t_1)\hat{x}(t_2)...\hat{x}(t_n)\} |\Omega\rangle . \tag{261}$$

8.4 Exercise

Under this shift, the Lagrangian becomes

$$L_0' = \frac{1}{2}\dot{x}^2 + \dot{x}\dot{\epsilon} + O(\dot{\epsilon}^2) - \frac{\omega^2}{2}x^2 - \omega^2 x\epsilon + O(\epsilon^2) = L_0 + \dot{x}\dot{\epsilon} - \omega^2 x\epsilon. \tag{262}$$

The action is thus

$$S' = S_0 + \int_{-T}^{T} dt(\dot{x}\dot{\epsilon} - \omega^2 x\epsilon). \tag{263}$$

Expanding the exponential to first order yields

$$\int \mathcal{D}[x(t)]e^{iS'} = \int \mathcal{D}[x(t)]e^{iS_0}[1 + i\int_{-T}^{T} dt(\dot{x}\dot{\epsilon} - \omega^2 x\epsilon)]. \tag{264}$$

Integrating by parts,

$$\int_{-T}^{T} dt\dot{x}\dot{\epsilon} = \dot{x}\epsilon|_{-T}^{T} - \int_{-T}^{T} dt\ddot{x}\epsilon = -\int_{-T}^{T} dt\ddot{x}\epsilon \tag{265}$$

where the first term vanishes due to the boundary conditions of ϵ. Inserting this into the expansion gives

$$\int \mathcal{D}[x(t)]e^{iS'} = \int \mathcal{D}[x(t)]e^{iS_0}[1 + i\int_{-T}^{T} dt(-\ddot{x}\epsilon - \omega^2 x\epsilon)] = \int \mathcal{D}[x(t)]e^{iS_0}[1 - i\int_{-T}^{T} dt\epsilon(\partial_t^2 + \omega^2)x]. \tag{266}$$

Now expanding the total function in (87) in the packet gives

$$\int \mathcal{D}[x(t)]x_1 e^{iS_0} = \int \mathcal{D}[x(t)](x_1 + \epsilon_1)e^{iS_0}[1 - i\int_{-T}^{T} dt\epsilon(\partial_t^2 + \omega^2)x]. \tag{267}$$

Expanding and keeping terms only first order in ϵ gives

$$\int \mathcal{D}[x(t)]x_1 e^{iS_0} = \int \mathcal{D}[x(t)]x_1 e^{iS_0} + \int \mathcal{D}[x(t)]\epsilon_1 e^{iS_0} - i\int \mathcal{D}[x(t)]e^{iS_0}\int_{-T}^{T} dt\epsilon(\partial_t^2 + \omega^2)xx_1. \tag{268}$$

Since $\epsilon_1 = \epsilon(t_1) = \int_{-T}^{T} \epsilon(t)\delta(t - t_1)$, we have

$$\int \mathcal{D}[x(t)]\epsilon_1 e^{iS_0} - i\int \mathcal{D}[x(t)]e^{iS_0}\int_{-T}^{T} dt\epsilon(\partial_t^2 + \omega^2)xx_1 = -i\int \mathcal{D}[x(t)]e^{iS_0}\int_{-T}^{T} dt\epsilon(t)((\partial_t^2 + \omega^2)xx_1 + i\delta(t - t_1)). \tag{269}$$

For the equality in (268) to hold, this term must equal 0:

$$\int \mathcal{D}[x(t)]e^{iS_0}\int_{-T}^{T} dt\epsilon(t)((\partial_t^2 + \omega^2)xx_1 + i\delta(t - t_1)) = 0. \tag{270}$$

Now bringing the path integral past the time integral gives

$$\int_{-T}^{T} dt\epsilon(t)\int \mathcal{D}[x(t)]e^{iS_0}((\partial_t^2 + \omega^2)xx_1 + i\delta(t - t_1)) = 0 \rightarrow \int \mathcal{D}[x(t)]e^{iS_0}((\partial_t^2 + \omega^2)xx_1 + i\delta(t - t_1)) = 0 \tag{271}$$

since $\epsilon(t)$ is an arbitrary function. Again, bringing the path integral past the operator $\partial_t^2 + \omega^2$ gives

$$(\partial_t^2 + \omega^2)\int \mathcal{D}[x(t)]e^{iS_0}xx_1 = -i\delta(t - t_1)\int \mathcal{D}[x(t)]e^{iS_0} \rightarrow (\partial_t^2 + \omega^2)\frac{\int \mathcal{D}[x(t)]e^{iS_0}xx_1}{\int \mathcal{D}[x(t)]e^{iS_0}} = -i\delta(t - t_1). \tag{272}$$

Due to (83) in the packet, we know this is just the Green's function for x and x_1, and so we have the relation

$$(\partial_t^2 + \omega^2)\langle 0| T\{\hat{x}(t)\hat{x}_1(t)\} |0\rangle = -i\delta(t - t_1). \tag{273}$$

8.5 Exercise

Inserting the expression for x' into the equation, we get

$$\int dt \frac{1}{2}\left[x(-\partial^2 - \omega^2)x - i\int dt' G(t,t')J(t')(-\partial^2 - \omega^2)x - i\int dt' J(t')x(-\partial^2 - \omega^2)G(t,t')\right] \tag{274}$$

$$+ \int dt \frac{1}{2}\left[-\int dt' G(t,t')J(t')(-\partial^2 - \omega^2)\int dt'' G(t,t'')J(t'')\right] + \frac{i}{2}\int dt dt' J(t)G(t,t')J(t'). \tag{275}$$

Due to (273), we have

$$(-\partial^2 - \omega^2)G(t,t') = i\delta(t-t'). \tag{276}$$

Inserting this into the second to last term gives

$$-\frac{i}{2}\int dt \int dt' G(t,t')J(t')\delta(t-t'')J(t'') = -\frac{i}{2}\int dt dt' J(t)G(t,t')J(t') \tag{277}$$

which precisely cancels the last term in (274). The first term in (274) can be evaluated using integration by parts, giving

$$\frac{1}{2}\int dt(-x\ddot{x} - \omega^2 x^2) = \frac{1}{2}\left[-x\dot{x}|_{-T}^{T} + \int dt(\dot{x})^2 - \omega^2 x^2\right]. \tag{278}$$

Assuming the x vanishes at the boundary conditions (Note: in the limit of the Green's function, this means that x vanishes at $\pm\infty(1-i\epsilon)$), this is simply

$$\int dt \mathcal{L}_0[x] \tag{279}$$

with $\mathcal{L}_0$ being the harmonic oscillator Lagrangian. The second term in (274) can also be evaluated using integration by parts:

$$\frac{i}{2}\int dt dt' G(t,t')J(t')(\partial^2 + \omega^2)x = \frac{i}{2}\left[\int dt dt' G(t,t')J(t')\dot{x}|_{-T}^{T} - \int dt dt' \partial_t G(t,t')J(t')\dot{x} + \int dt dt' G(t,t')J(t')\omega^2 x\right] \tag{280}$$

$$= \frac{i}{2}\left[\int dt dt' G(t,t')J(t')\dot{x}|_{-T}^{T} - \int dt dt' G(t,t')J(t')x|_{-T}^{T} + \int dt dt' \partial_t^2 G(t,t')J(t')x + \int dt dt' G(t,t')J(t')\omega^2 x\right] \tag{281}$$

$$= \int dt dt' J(t')x(\partial_t^2 + \omega^2)G(t,t') = \frac{1}{2}\int dt dt' J(t')x\delta(t'-t) = \frac{1}{2}\int dt Jx \tag{282}$$

where it was assumed that $\dot{x}$ also vanished at the boundary conditions. Finally, the third term in (274) can be evaluated as

$$-\frac{i}{2}\int dt dt' J(t')x(-\partial^2 - \omega^2)G(t,t') = \frac{1}{2}\int dt dt' J(t')x\delta(t-t') = \frac{1}{2}\int dt Jx. \tag{283}$$

Combining everything, we have

$$\int dt \frac{1}{2}[x'(-\partial^2 - \omega^2)x'] + \frac{i}{2}\int dt dt' J(t)G(t,t')J(t') = \int dt(\mathcal{L}_0[x] + Jx). \tag{284}$$

Using the definition of $Z[J]$, we get

$$Z[J] = \frac{\int \mathcal{D}[x(t)]e^{i(\int dt(\mathcal{L}_0 + Jx)}}{\int \mathcal{D}[x(t)]e^{i\int dt \mathcal{L}_0}} = \exp\left[-\frac{1}{2}\int dt dt' J(t)G(t,t')J(t')\right]\frac{\int \mathcal{D}[x(t)]e^{\frac{i}{2}\int dt x'(-\partial^2 - \omega^2)x'}}{\int \mathcal{D}[x(t)]e^{i\int dt \mathcal{L}_0}}. \tag{285}$$

The exponent of this expression can be integrated by parts to give

$$\int dt \frac{1}{2}[x'(-\partial^2 - \omega^2)x'] = -\frac{1}{2}x'\dot{x}'|_{-T}^{T} + \int dt \frac{1}{2}((\dot{x}')^2 - \omega^2 x'^2) = \int dt \mathcal{L}_0[x'] \tag{286}$$

where it was assumed x' vanished at the boundary conditions as well (meaning the Green's function approaches 0 as $t \to \pm\infty(1-i\epsilon)$). Because $x' = x + \epsilon(t)$, we know a change of variable will not alter the functional integration. Thus,

$$\int \mathcal{D}[x(t)]e^{\frac{i}{2}\int dt x'(-\partial^2 - \omega^2)x'} = \int \mathcal{D}[x'(t)]e^{i\int dt \mathcal{L}_0[x']} = \int \mathcal{D}[x(t)]e^{i\int dt \mathcal{L}_0[x]}. \tag{287}$$

Inserting this into (285) gives

$$Z[J] = \exp\left[-\frac{1}{2}\int dt dt' J(t)G(t,t')J(t')\right]. \tag{288}$$

8.6 Exercise

Since we know

$$Z[J] = \exp\left[-\frac{1}{2}\int dt dt' J(t)G(t,t')J(t')\right], \tag{289}$$

we can take functional derivatives of this to obtain the n-point functions. The first functional derivative is

$$\frac{\delta}{\delta J(t_1)}Z[J] = e^{-\frac{1}{2}\int dt dt' J(t)G(t,t')J(t')}\frac{\delta}{\delta J(t_1)}\left(-\frac{1}{2}\int dt dt' J(t)G(t,t')J(t')\right) \tag{290}$$

$$= -\frac{1}{2}Z[J]\left(\int dt' G(t_1,t')J(t') + \int dt J(t)G(t,t_1)\right). \tag{291}$$

Relabeling $t' \to t$ and noting that $G(t,t_1) = G(t_1,t)$ due to time ordering, this is

$$\frac{\delta}{\delta J(t_1)}Z[J] = -\frac{1}{2}Z[J]\left(2\int dt G(t_1,t)J(t)\right) = -Z[J]\int dt G(t_1,t)J(t). \tag{292}$$

Setting $J = 0$ makes this expression 0, so the 1-point correlation function vanishes. Continuing, the second functional derivative will be

$$\frac{\delta}{\delta J(t_2)}\left(-Z[J]\int dt G(t_1,t)J(t)\right) = -\int dt G(t_1,t)J(t)\frac{\delta}{\delta J(t_2)}Z[J] - Z[J]\frac{\delta}{\delta J(t_2)}\int dt G(t_1,t)J(t). \tag{293}$$

The first term is given by (292) with $t_1 \to t_2$, and the second is

$$-Z[J]G(t_1,t_2). \tag{294}$$

When $J = 0$, the first term vanishes again, and Z[J]=1, leaving

$$\frac{\delta}{\delta J(t_2)}\frac{\delta}{\delta J(t_1)}Z[J]|_{J=0} = -G(t_1,t_2). \tag{295}$$

The third functional derivative will be

$$\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}\frac{\delta}{\delta J(t_3)}Z[J] = \frac{\delta}{\delta J(t_3)}\left[Z[J]\int dt G(t_1,t)J(t)\int dt G(t_2,t)J(t) - Z[J]G(t_1,t_2)\right]. \tag{296}$$

The last term will clearly be 0 when $J(t) = 0$ due to (292). The first term is 0 because when expanding using the product rule, at least one of the integrals will have a factor of $J(t)$, and setting this to 0 will kill this first term. We have thus spotted a pattern in these derivatives: whenever we have an odd number of derivatives, factors of J are multiplied by all terms in the expansions, and setting it to 0 kills all the terms. When there is an even number of derivatives, these factors of J are differentiated due to the product rule, thereby leaving factors of Green's functions and removing factors of J from coefficients of the terms.

8.7 Exercise

Expanding (296) gives

$$\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}\frac{\delta}{\delta J(t_3)}Z[J] = -Z[J]\int dt G(t_1,t)J(t)\int dt G(t_2,t)J(t)\int dt G(t_3,t)J(t) \tag{297}$$

$$+Z[J]G(t_1,t_3)\int dt G(t_2,t)J(t) + Z[J]G(t_2,t_3)\int dt G(t_1,t)J(t) + Z[J]\int dt G(t_3,t)J(t)G(t_1,t_2). \tag{298}$$

Differentiating again with respect to $J(t_4)$ gives

$$\frac{\delta}{\delta J(t_1)}\frac{\delta}{\delta J(t_2)}\frac{\delta}{\delta J(t_3)}\frac{\delta}{\delta J(t_4)}Z[J] = \tag{299}$$

$$= Z[J]\int dt G(t_1,t)J(t)\int dt G(t_2,t)J(t)\int dt G(t_3,t)J(t)\int dt G(t_4,t)J(t) \tag{300}$$

$$-Z[J]\int dt G(t_4,t)J(t)G(t_1,t_3)\int dt G(t_2,t)J(t) + Z[J]G(t_1,t_3)G(t_2,t_4) \tag{301}$$

$$-Z[J]\int dt G(t_4,t)J(t)G(t_2,t_3)\int dt G(t_1,t)J(t) + Z[J]G(t_2,t_3)G(t_1,t_4) \tag{302}$$

$$-Z[J]\int dt G(t_4,t)J(t)G(t_1,t_2)\int dt G(t_3,t)J(t) + Z[J]G(t_1,t_2)G(t_3,t_4). \tag{303}$$

Now $J = 0$ and $Z[J] = 1$, so

$$\frac{\delta}{\delta J(t_1)} \frac{\delta}{\delta J(t_2)} \frac{\delta}{\delta J(t_3)} \frac{\delta}{\delta J(t_4)} Z[J]|_{J=0} = i^4 \langle \Omega | T\{\hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4\} | \Omega \rangle \tag{304}$$

$$= \langle \Omega | T\{\hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4\} | \Omega \rangle = G_{12} G_{34} + G_{13} G_{24} + G_{14} G_{23} \tag{305}$$

where $G_{ij} = G(t_i, t_j)$.

8.8 Exercise

We know that

$$G(t_1, t_2) = \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}[x(t)] x(t_1) x(t_2) e^{iS}}{\int \mathcal{D}[x(t)] e^{iS}}. \tag{306}$$

Furthermore, from section 5.4, we know $(T = t_b - t_a \to T - (-T) = 2T)$.

$$\int \mathcal{D}[x(t)] e^{iS} = K = \left(\frac{m\omega}{2\pi i \sin(2\omega T)}\right)^{1/2} e^{iS_{cl}}. \tag{307}$$

The numerator can be written as

$$\int dx_1 dx_2 x_1 x_2 \int \mathcal{D}[x(t)] e^{iS}. \tag{308}$$

We know when $x_a = x_b = 0$,

$$S_{cl} = \frac{1}{m\omega \sin(2\omega T)} \int_{-T}^{T} \int_{T}^{t} \sin(\omega(T+t)) \sin(\omega(t'-T)) f(t') f(t) dt' dt. \tag{309}$$

In order to calculate the numerator, we can replace S with the discrete approximation S_n given in (145). The numerator then becomes

$$\int dx_1 dx_2 x_1 x_2 \int \mathcal{D}[x(t)] \exp\left[iS_1 + iS_2 + iS_3 + i\sum_{n=4}^{N} S_n\right]. \tag{310}$$

We can now do the same coordinate transform to y_i as in exercise 5.4, and the result is

$$\int dx_1 dx_2 x_1 x_2 \int \mathcal{D}[x(t)] \exp\left[i\sum_{n=1}^{N} S_n\right] = C(T) \exp[iS_{cl}] \int \mathcal{D}[y(t)] (\bar{x}_1 + y_1)(\bar{x}_2 + y_2) \exp\left[i\sum_{n=1}^{N} S_n\right] \tag{311}$$

where S_n is defined by (110) with y instead of x as a variable and with $T \to 2T$. Remembering that one point Green's functions vanish, and hence that the cross terms with single factors of y_i vanish, the integral becomes

$$\exp[iS_{cl}] \left(\bar{x}_1 \bar{x}_2 C(T) \int \mathcal{D}[y(t)] \exp\left[i\sum_{n=1}^{N} S_n\right] + C(T) \int \mathcal{D}[y(t)] y_1 y_2 \exp\left[i\sum_{n=1}^{N} S_n\right]\right). \tag{312}$$

Using the definition of $F(T)$ given in (153) and cancelling the factor coming from the denominator of the Green's function, we get (omitting the limit)

$$G(t_1, t_2) = \frac{\exp[iS_{cl}] \left(\bar{x}_1 \bar{x}_2 F(T) + C(T) \int \mathcal{D}[y(t)] y_1 y_2 \exp\left[i\sum_{n=1}^{N} S_n\right]\right)}{F(T) \exp[iS_{cl}]} = \bar{x}_1 \bar{x}_2 + \frac{\int \mathcal{D}[y(t)] y_1 y_2 \exp\left[i\sum_{n=1}^{N} S_n\right]}{\int \mathcal{D}[y(t)] \exp\left[i\sum_{n=1}^{N} S_n\right]}. \tag{313}$$

The advantage now is that the action is that of the unforced harmonic oscillator. The sum in the exponential can be written as

$$i\sum_{n=1}^{N} S_n = \sum_{n=1}^{N} \frac{im\omega}{2\sin(2\omega T)} [\cos(2\omega T)(y_n^2 + y_{n-1}^2) - 2y_n y_{n-1}] \tag{314}$$

$$= \begin{pmatrix} y_1 & y_2 & \dots & y_{N-1} \end{pmatrix} \begin{pmatrix} 2k\cos(2\omega T) & -2k & 0 & 0 & \dots \\ 0 & 2k\cos(2\omega T) & -2k & 0 & \dots \\ & \dots & & & \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{N-1} \end{pmatrix}. \tag{315}$$

where
$$k = \frac{im\omega}{2\sin(2\omega T)}. \tag{316}$$

In Zee QFT, the formula
$$\langle x_i x_j \rangle = \frac{\int ... \int dx_1 ... dx_N e^{-\frac{1}{2}x \cdot A \cdot x} x_i x_j}{\int ... \int dx_1 ... dx_N e^{-\frac{1}{2}x \cdot A \cdot x}} = A_{ij}^{-1} \tag{317}$$

is derived by differentiating the generalization of (60). Letting
$$A = \begin{pmatrix} -4k\cos(2\omega T) & 4k & 0 & 0 & ... \\ 0 & -4k\cos(2\omega T) & 4k & 0 & ... \\ & ... & & & \end{pmatrix} \tag{318}$$

and
$$a = -4k\cos(2\omega T), b = 4k, \tag{319}$$

we get
$$A_{12}^{-1} = -\frac{b}{a^2}. \tag{320}$$

This was obtained by entering matrices of the form
$$\begin{pmatrix} a & b & 0 \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -(b/a^2) & b^2/a^3 \\ 0 & a^{-1} & -(b/a^2) \\ 0 & 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -(b/a^2) & b^2/a^3 & -(b^3/a^4) \\ 0 & a^{-1} & -(b/a^2) & b^2/a^3 \\ 0 & 0 & a^{-1} & -(b/a^2) \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \tag{321}$$

using Wolfram Alpha. Thus we finally get that
$$G(t_1, t_2) = \lim_{T \to \infty(1-i\epsilon)} (\bar{x}_1 \bar{x}_2 - \frac{4k}{(4k\cos(2\omega T))^2}). \tag{322}$$

Simplifying the last term gives
$$-\frac{1}{4k\cos^2(2\omega T)} = \frac{i\sin(2\omega T)}{2\omega m \cos^2(2\omega T)}. \tag{323}$$

When $x_a = x_b = 0$, we have by (108) and (109) that
$$\bar{x}(t) = -\frac{1}{m\omega} \frac{\sin(\omega(T+t))}{\sin(2\omega T)} \int_{-T}^{T} \sin(\omega(T-t'))f(t')dt' + \frac{1}{m\omega} \int_{-T}^{t} \sin(\omega(t-t'))f(t')dt'. \tag{324}$$

In order for
$$G(t_1, t_2) = \frac{1}{2\omega} e^{-i\omega|t_2-t_1|} \tag{325}$$

to hold, we must have
$$\frac{1}{2\omega} e^{-i\omega|t_2-t_1|} = \lim_{T \to \infty(1-i\epsilon)} \left(\bar{x}_1 \bar{x}_2 - \frac{4k}{(4k\cos(2\omega T))^2} = \cos(\omega|t_2-t_1|) - i\sin(\omega|t_2-t_1|) \right) \tag{326}$$

or
$$\lim_{T \to \infty(1-i\epsilon)} \left(-\frac{1}{m\omega} \frac{\sin(\omega(T+t_1))}{\sin(2\omega T)} \int_{-T}^{T} \sin(\omega(T-t'))f(t')dt' + \frac{1}{m\omega} \int_{-T}^{t_1} \sin(\omega(t_1-t'))f(t')dt' \right) \tag{327}$$

$$\left(-\frac{1}{m\omega} \frac{\sin(\omega(T+t_2))}{\sin(2\omega T)} \int_{-T}^{T} \sin(\omega(T-t'))f(t')dt' + \frac{1}{m\omega} \int_{-T}^{t_2} \sin(\omega(t_2-t'))f(t')dt' \right) \tag{328}$$

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{1}{2\omega} \left[\cos(\omega|t_2-t_1|) - i\sin(\omega|t_2-t_1|) - \frac{i\sin(2\omega T)}{m\cos^2(2\omega T)} \right]. \tag{329}$$

We weren't sure how these expressions would reduce, especially with the factors of $\sim \int f dt$, but our best guess was that the limit somehow killed these terms and reduced the expression to equality.

9 Feynman Diagrams

9.1 Exercise

The first Feynman diagram evaluates to

$$- i\lambda \int_{-T}^{T} dt G(t,t) G(t_1,t) G(t_2,t). \tag{330}$$

The first diagram in the disconnected set is

$$(-i\lambda)^2 \int_{-T}^{T} \int_{-T}^{T} dt' dt G(t_1,t) G(t_2,t) G(t,t') G(t,t') G(t',t_3) G(t',t_4), \tag{331}$$

and the second is

$$- i\lambda \int_{-T}^{T} dt'' G(t'',t'') G(t'',t''). \tag{332}$$

The total disconnected diagram is thus

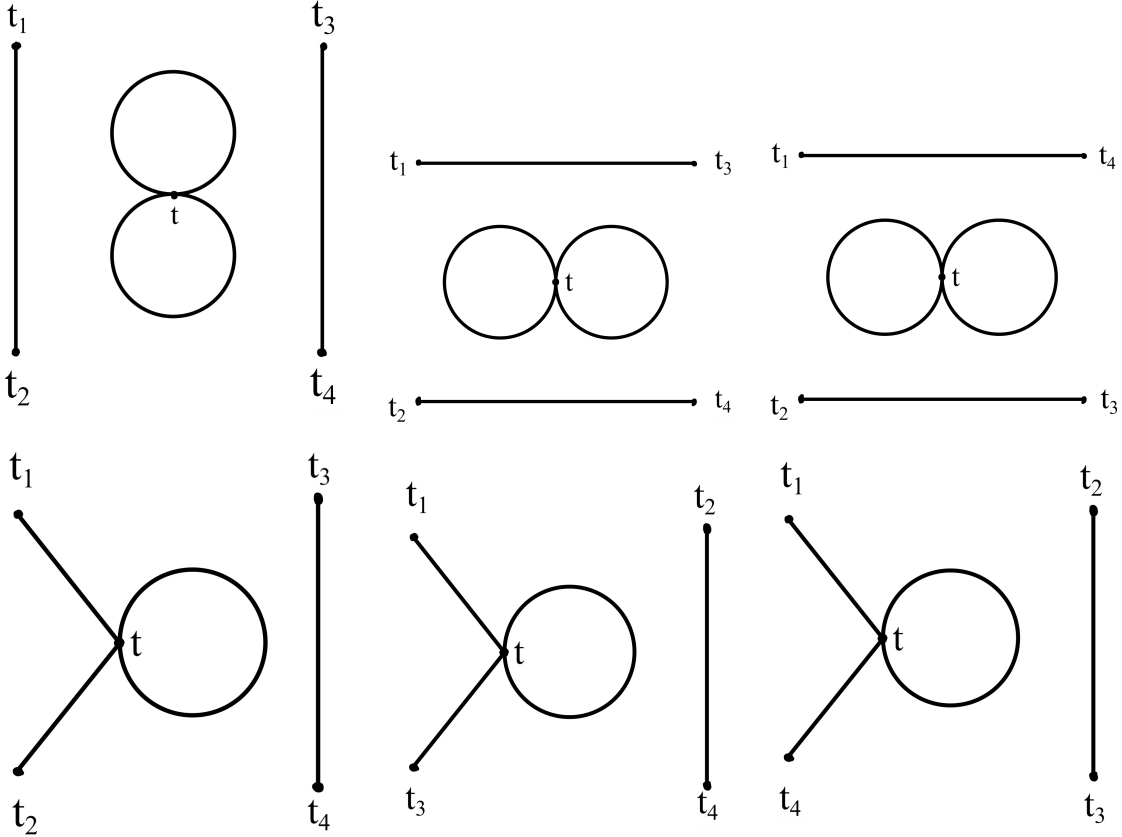$$(-i\lambda)^3 \int_{-T}^{T} \int_{-T}^{T} \int_{-T}^{T} dt'' dt' dt G(t'',t'') G(t'',t'') G(t_1,t) G(t_2,t) G(t,t') G(t,t') G(t',t_3) G(t',t_4). \tag{333}$$
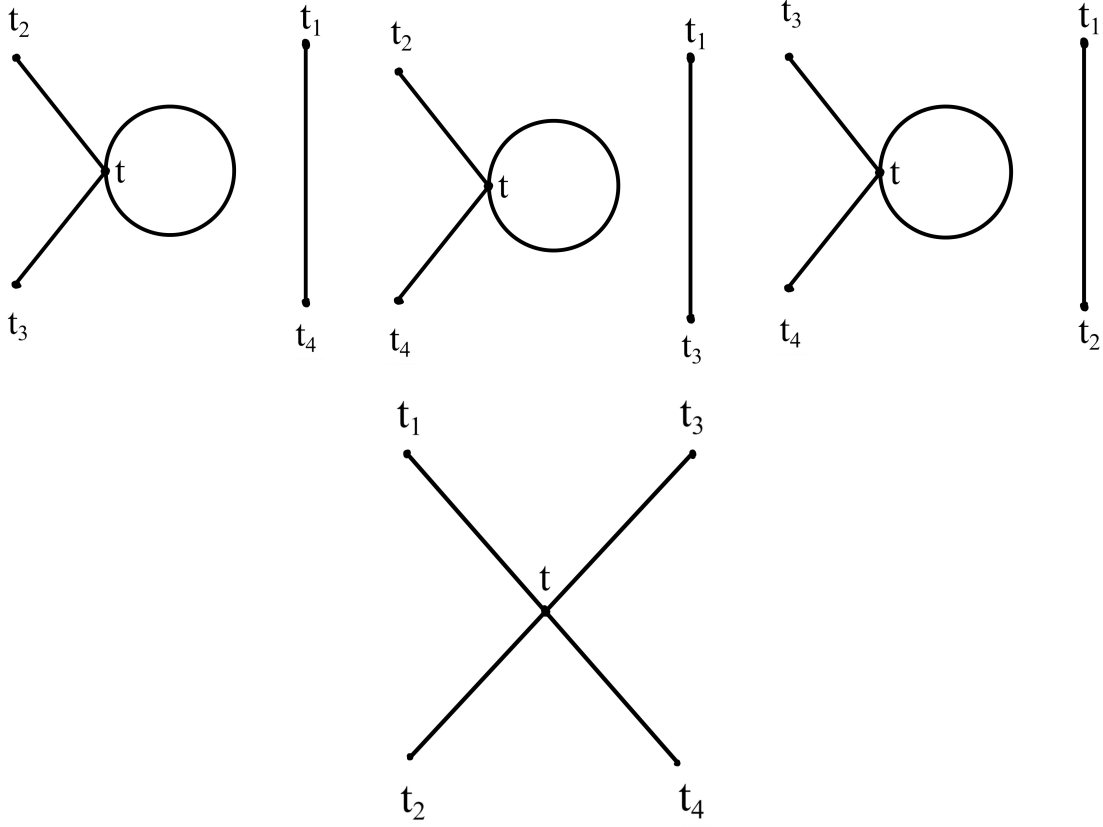
9.2 Exercise

The diagrams for

$$\langle 0 | T\{\hat{x}_I(t_1) \hat{x}_I(t_2) \hat{x}_I(t_3) \hat{x}_I(t_4) \left(-\frac{i\lambda}{4!} \right) \int_{-T}^{T} dt \hat{x}_I^4(t) \} | 0 \rangle \tag{334}$$

are given by all combinations of vertices with 4 external vertices and one internal vertex. These are given by the following:

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The integrals for the diagrams are given by

$$D_1 = -i\lambda \int dt G(t_1, t_2) G(t_3, t_4) G(t, t) G(t, t) \tag{335}$$

$$D_2 = -i\lambda \int dt G(t_1, t_3) G(t_2, t_4) G(t, t) G(t, t) \tag{336}$$

$$D_3 = -i\lambda \int dt G(t_1, t_4) G(t_2, t_3) G(t, t) G(t, t) \tag{337}$$

$$D_4 = -i\lambda \int dt G(t_1, t) G(t_2, t) G(t, t) G(t_3, t_4) \tag{338}$$

$$D_5 = -i\lambda \int dt G(t_1, t) G(t_3, t) G(t, t) G(t_2, t_4) \tag{339}$$

$$D_6 = -i\lambda \int dt G(t_1, t) G(t_4, t) G(t, t) G(t_2, t_3) \tag{340}$$

$$D_7 = -i\lambda \int dt G(t_2, t) G(t_3, t) G(t, t) G(t_1, t_4) \tag{341}$$

$$D_8 = -i\lambda \int dt G(t_2, t) G(t_4, t) G(t, t) G(t_1, t_3) \tag{342}$$

$$D_9 = -i\lambda \int dt G(t_3, t) G(t_4, t) G(t, t) G(t_1, t_2) \tag{343}$$

$$D_{10} = -i\lambda \int dt G(t_1, t) G(t_2, t) G(t, t_3) G(t, t_4). \tag{344}$$

Thus, the total contribution of this term will be

$$\langle 0| T\{\hat{x}_I(t_1)\hat{x}_I(t_2)\hat{x}_I(t_3)\hat{x}_I(t_4)\left(-\frac{i\lambda}{4!}\right)\int_{-T}^{T} dt\hat{x}_I^4(t)\} |0\rangle = s_1(D_1 + D_2 + D_3) + s_2(D_4 + ... + D_9) + s_3 D_{10} \tag{345}$$

where the s_i are the symmetry factors for each diagram. From the packet, we know $s_1 = \frac{3}{4!} = \frac{1}{8}$. The diagrams 4...9 all have 1 loop, so $s_2 = \frac{2}{4!} = \frac{1}{12}$. The 10th diagram has no loops, so s_3 is simply $\frac{1}{4!} = \frac{1}{24}$.

9.3 Exercise

Inserting the factors from 9.2, we get

$$\frac{\langle 0| \, T\{\hat{x}_I(t_1)\hat{x}_I(t_2)\hat{x}_I(t_3)\hat{x}_I(t_4)\} \, |0\rangle + \langle 0| \, T\{\hat{x}_I(t_1)\hat{x}_I(t_2)\hat{x}_I(t_3)\hat{x}_I(t_4) \left(-\frac{i\lambda}{4!}\right) \int_{-T}^{T} dt \hat{x}_I^4(t)\} \, |0\rangle}{1 - \frac{i\lambda}{8} \int dt G(t,t) G(t,t)} \tag{346}$$

$$= \frac{G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23} + s_1(D_1 + D_2 + D_3) + s_2(D_4 + ... + D_9) + s_3 D_{10}}{1 - \frac{i\lambda}{8} \int dt G(t,t) G(t,t)}. \tag{347}$$

Using the expansion

$$\frac{1}{1-x} = 1 + x + x^2... \tag{348}$$

and keeping terms first order in λ, we get (letting $s_1 = \frac{1}{8}$ and plugging in the expressions for $D_1, D_2,$ and D_3):

$$(G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23} + s_1(D_1 + D_2 + D_3) + s_2(D_4 + ... + D_9) + s_3 D_{10}...)(1 + \frac{i\lambda}{8} \int dt G(t,t) G(t,t)...) \tag{349}$$

$$= G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23} \tag{350}$$

$$+ (G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23})(\frac{i\lambda}{8} \int dt G(t,t) G(t,t)) \tag{351}$$

$$- (G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23})(\frac{i\lambda}{8} \int dt G(t,t) G(t,t)) + s_2(D_4 + ... + D_9) + s_3 D_{10}). \tag{352}$$

Noting that the second and third terms cancel, we are left with

$$G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23} + s_2(D_4 + ... + D_9) + s_3 D_{10}. \tag{353}$$

9.4 Exercise

The integral for this diagram is

$$- i\lambda \int_{-T}^{T} dt G(t_1,t) G(t_2,t) G(t,t). \tag{354}$$

Using the Green's function for a harmonic oscillator,

$$G(t_2,t_1) = \frac{1}{2\omega} e^{-i\omega|t_2 - t_1|}, \tag{355}$$

this is

$$- i\lambda \int_{-T}^{T} \frac{dt}{(2\omega)^3} e^{-i\omega(|t_1 - t| + |t_2 - t| + |t - t|)}. \tag{356}$$

Inserting $t_1 = t_2 = 0$, this simplifies to

$$- i\lambda \int_{-T}^{T} \frac{dt}{(2\omega)^3} e^{-2\omega i|t|} = -\frac{i\lambda}{(2\omega^3)} \left[2\int_{0}^{T} dt e^{-2\omega it} \right] \tag{357}$$

because the integrand is even in t. This integral can be evaluated as

$$- \frac{i\lambda}{(2\omega)^3} \frac{2}{-2\omega i} \left[e^{-2\omega iT} - 1 \right] = -\frac{2\lambda}{(2\omega)^4} \left[1 - e^{-2\omega iT} \right]. \tag{358}$$

In the limit $T \to \infty(1 - i\epsilon)$, the second term becomes

$$\lim_{T \to \infty(1-i\epsilon)} e^{-2\omega iT} = e^{-2\omega i\infty} e^{-2\omega\epsilon\infty} = 0 \tag{359}$$

when $\epsilon > 0$, and since the first term always has magnitude 1. Then the diagram evaluates to

$$- \frac{2\lambda}{(2\omega)^4} = -\frac{\lambda}{8\omega^4}. \tag{360}$$

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