2019 Eötvös Competition

Translated By: **Ashmit Dutta**

1. An easily moving piston initially divides a thermally insulated horizontal axis cylinder into two parts of equal volume, V_0. In both parts of the cylinder, there exists an ideal gas with pressure p_0. The cylinder's initial temperature is $2T_0$ in left-hand section of the piston and T_0 in the right-hand section of the piston. The piston is moderately conductive and it's heat parameter is characterized by α, i.e in the case of a temperature difference ΔT, a heat flux $\alpha\Delta T$ is flowing through the cylinder per unit time.

   (a) What will be the volume, temperature, and pressure in each section after a long period of time?

   (b) Give as a function of time, the volume of the gases $V_1(t)$ and $V_2(t)$ in each section!

2. Each edge of a cube is made of the same wire which has resistance R. The cube is immersed into a homogeneous induced magnetic field B_0 which is reduced to zero in a time τ. What is the Joule heat generated during the process if the magnetic induction vector forms an acute angle α, β, and γ respectively with the edges of the cube meeting at the vertix? $(\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.)$

3. A horizontal rope is tightened with a force F_0 much greater than its weight. The rope is located in the positive x-axis and one end is at the origin.

   (a) If the end of the rope at the origin is moved towards the positive y-direction perpendicular to the x-axis with a harmonic oscillation of amplitude A and frequency f, transverse waves are generated in the rope which propogate at a speed c (depending on the mass per unit length and tension in the rope). The amplitude of the waves are small, that is, $A \ll c/f$. Give the deflection $y(x,t)$ of the point of the rope with coordinate x at time t!

   (b) What is the average power required to move the end of the rope?

   (c) Now the end of the rope at the origin can move freely in the y direction. It's movement is inhibited by the force $-\gamma v(t)$ which is proportional to the speed $v(t)$ of the end of the rope. On the rope, a sine wave of amplitude A reaches the origin. We find that the wave is partially or possibly completely reflected as a result of which a sine wave of amplitude B moving away from the origin is also formed.

   What is the amplitude of the reflected wave? Enter the B/A ratio! Consider the cases $\gamma \to \infty$ and $\gamma \to 0$ (very strong and very weak attenuation). Is there a damping factor γ at which no wave is reflected from the end of the rope at all?

1. *Solution 1 by Jacob Nie*

(a) "After a long period of time" essentially asks for the equilibrium state of the system. In the equilibrium state, the pressures on both side will be equal (otherwise the piston would move) and the temperatures on both side will be equal (otherwise more heat would be transferred.)

Variables: N_1, N_2 are the number of particles in the left and right sides respectively. (This doesn't change throughout the whole process.) p is the final pressure; V_1, V_2 are the final volumes; and T is the final temperature.

Equations:

$$V_1 + V_2 = 2V_0$$

$$N_1 = \frac{p_0 V_0}{2k_B T_0}, \quad N_2 = \frac{p_0 V_0}{k_B T_0} \quad \text{(Ideal Gas Law)}.$$

The total energy of the gas inside the piston is conserved in the process. The energy of the gas, assuming it is monatomic, is given by $U = \frac{3}{2} N k_B T$. So,

$$\frac{3}{2} N_1 T + \frac{3}{2} N_2 T = \frac{3}{2} N_1 (2T_0) + \frac{3}{2} N_2 T_0.$$

Since $N_2 = 2N_1$, one finds that $\boxed{T = \frac{4}{3} T_0.}$ The ideal gas law applied to each half of the piston at equilibrium gives:

$$pV_1 = \left(\frac{p_0 V_0}{2k_B T_0}\right) k_B \left(\frac{4}{3} T\right) = \frac{2}{3} p_0 V_0$$
$$pV_2 = \left(\frac{p_0 V_0}{k_B T_0}\right) k_B \left(\frac{4}{3} T\right) = \frac{4}{3} p_0 V_0.$$

So, $V_2 = 2V_1 \implies \boxed{V_1 = \frac{2}{3} V_0, V_2 = \frac{4}{3} V_0.}$ And from that, $\boxed{p = p_0.}$

(b) Variables and functions: N_1, N_2 are the number of particles in the left and right sides respecitvely. $T_1(t), T_2(t)$ are the temperatures, $U_1(t), U_2(t)$ are the internal energies, $V_1(t), V_2(t)$ are the volumes, $p_1(t), p_2(t)$ are the pressures.

First, I argue that $p_1(t) = p_2(t)$. This is because we assume the piston moves faster than the heat transfer. Hence, the piston will move quickly enough to counter any pressure differential. (After all, we aren't given any information about how the piston responds to pressure differences, so we must assume this.) Next, I argue that $p_1(t) = p_2(t) =$ constant in the following way: the internal energy of the system remains constant, so $N_1 T_1 + N_2 T_2$ remains constant, so $p_1 V_1 + p_2 V_2$ remains constant. Since $V_1 + V_2$ is constant and $p_1 = p_2$, the pressures must remain constant throughout. Let this pressure be $p \equiv p_1 = p_2$.

Each compartment does $p\frac{dV}{dt}$ work per unit time. By the first law of thermodynamics,

$$\frac{dU_1}{dt} = -\alpha(T_1 - T_2) - p\frac{dV_1}{dt}, \quad \frac{dU_2}{dt} = \alpha(T_1 - T_2) - p\frac{dV_2}{dt}.$$

Note that for each compartment:

$$\begin{aligned} \frac{dU}{dt} &= \frac{3}{2} N k_B \frac{dT}{dt} \\ &= \frac{3}{2} \frac{d}{dt}(pV) \\ &= \frac{3}{2} p \frac{dV}{dt}. \end{aligned}$$

Also using $T = \dfrac{pV}{Nk_B}$ and $N_2 = 2N_1$ we obtain the two coupled equations

$$\begin{aligned}
\frac{dV_1}{dt} &= -\frac{2\alpha}{5N_2k_B}(2V_1 - V_2) \\
\frac{dV_2}{dt} &= \frac{2\alpha}{5N_2k_B}(2V_1 - V_2),
\end{aligned}$$

To de-couple the equations, multiply the first by 2 and the second by -1 and add:

$$\frac{d}{dt}(2V_1 - V_2) = -\frac{6\alpha}{5N_2k_B}(2V_1 - V_2) \implies 2V_1 - V_2 = C\exp\left(-\frac{6\alpha}{5N_2k_B}t\right).$$

At $t = 0$, we know $2V_1 - V_2 = V_0$, so $C = V_0$. Also, $N_2k_B = \dfrac{p_0V_0}{T_0}$. Furthermore, since $V_2 = 2V_0 - V_1$, we find

$$\boxed{V_1 = \frac{V_0}{3}\left[2 + \exp\left(-\frac{6\alpha T_0}{5p_0V_0}t\right)\right], \quad V_2 = \frac{V_0}{3}\left[4 - \exp\left(-\frac{6\alpha T_0}{5p_0V_0}t\right)\right].}$$

This behaves as predicted for $t \to \infty$.

2. The first thing to realize is that the magnetic field B_0 can be decomposed into three parts:

$$B_0 = B_0 \cos\alpha \hat{i} + B_0 \cos\beta \hat{j} + B_0 \cos\gamma \hat{k}.$$

In the same fasion, we will decompose current flowing in the cube:

$$i = i_x \hat{i} + i_y \hat{j} + i_z \hat{k}.$$

Consider Faraday's law which states that the induced voltage V is given as

$$V = -\frac{d\Phi}{dt} \implies 4iR = \frac{Ba^2}{\tau}.$$

There is a factor of 4 because each side of a cube has four resistors. We can then write each component as

$$i_x = \frac{B_0 a^2 \cos\alpha}{4R\tau}, \quad i_y = \frac{B_0 a^2 \cos\beta}{4R\tau}, \quad i_z = \frac{B_0 a^2 \cos\gamma}{4R\tau}.$$

The heat dissipated due to Joule heating is $P = i^2 R$. We need to find the total current I given to be $I = \sum_{k,l} 2(i_k \pm i_l)$. The factor of 2 comes from the fact that there are two identical sides with the same values of current flowing through them which allows us to add them twice. We can then write

$$\begin{aligned}
P &= 2R \left(\sum_{k,l} (i_k \pm i_l) \right)^2 \\
&= 2R((i_x + i_y)^2 + (i_x - i_y)^2 + (i_y + i_z)^2 + (i_y - i_z)^2 + (i_x + i_z)^2 + (i_x - i_z)^2) \\
&= 8R(i_x^2 + i_y^2 + i_z^2) \\
&= 8R \frac{B_0^2 a^4 (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)}{16R^2\tau^2} = \frac{B_0^2 a^4}{2R\tau^2}.
\end{aligned}$$

The heat dissipated Q is then,

$$Q = P\tau = \frac{B_0^2 a^4}{2R\tau}.$$

3. (a) Neglecting attenuation along the rope, the equation of the end of the rope can be described by the equation

$$y(x,t) = A\sin(2\pi ft + \varphi)\cos(2\pi ft + \phi).$$

where φ and ϕ is the phase constant. Since the waves are small, $\cos(\omega t + \phi) \approx 1$ and because the waves propogate at a speed c, we can write that $t = x/c$ and that

$$y(x,t) = A\sin\left[2\pi f\left(t - \frac{x}{c}\right)\right].$$

(b) At a time t, we can write that tangent of the angle the rope makes to the horizontal θ, is given by the slope of the wavefunction. Therefore, we write

$$\tan\theta = \frac{\partial y(x,t)}{\partial x} = -\frac{2\pi Af}{c}\cos\left[2\pi ft - \frac{2\pi f}{c}x\right].$$

We are more specifically worried about the angle at the origin or $x = 0$. Therefore,

$$\tan\theta = -\frac{2\pi Af}{c}\cos(2\pi ft).$$

Since the force that the rope is tightened with is F_0, the vertical force is then

$$F_y = -F_0\tan\theta = F_0\frac{2\pi Af}{c}\cos(2\pi ft).$$

Furthermore, the group velocity of the wave is given to be

$$v_g = \frac{\partial y(x=0,t)}{\partial t} = 2\pi fA\cos(2\pi ft).$$

The power moving the rope is then expressed to be

$$P(t) = F_y v_g = \frac{4\pi^2A^2f^2F_0}{c}\cos^2(2\pi ft).$$

The average power is then

$$\langle P(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\frac{4\pi^2A^2f^2F_0}{c}\cos^2(2\pi ft)dt = \frac{2\pi^2A^2f^2}{c}\cos^2(2\pi t).$$

(c) Consider the end of the rope again. We can write that

$$F_0\sin\theta - \gamma v_y(t) = 0 \implies F_0\frac{\partial y}{\partial x} - \gamma\frac{\partial y}{\partial t} = 0.$$

Now consider the waves attenuating on the rope. We can superimpose the reflected wave and the regular period wavefunction or

$$y(x,t) = A\sin\left[2\pi f\left(t + \frac{x}{c}\right)\right] + B\sin\left[2\pi f\left(t - \frac{x}{c}\right) + \phi\right]$$

where ϕ is the phase constant. We can then write that

$$\frac{\partial y(x,t)}{\partial x} = \frac{2\pi f}{c}A\cos\left[2\pi f\left(t + \frac{x}{c}\right)\right] - \frac{2\pi f}{c}B\cos\left[2\pi f\left(t - \frac{x}{c}\right) + \phi\right].$$

Similarly,

$$\frac{\partial y(x,t)}{\partial t} = 2\pi fA\cos\left[2\pi f\left(t + \frac{x}{c}\right)\right] - 2\pi fB\cos\left[2\pi f\left(t - \frac{x}{c}\right) + \phi\right].$$

Hence, we can go back into our first equation and write

$$\begin{aligned} F_0&\left\{\frac{2\pi f}{c}A\cos\left[2\pi f\left(t + \frac{x}{c}\right)\right] - \frac{2\pi f}{c}B\cos\left[2\pi f\left(t - \frac{x}{c}\right) + \phi\right]\right\}\Bigg|_{x=0} \\ &-\gamma\left\{2\pi fA\cos\left[2\pi f\left(t + \frac{x}{c}\right)\right] - 2\pi fB\cos\left[2\pi f\left(t - \frac{x}{c}\right) + \phi\right]\right\}\Bigg|_{x=0} = 0. \end{aligned}$$

We can use the identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

to simplify our equation to

$$\begin{aligned}F_0 A\cos(2\pi ft) - F_0 B\cos(2\pi ft)\cos\varphi + F_0 B\sin(2\pi ft)\sin\varphi \\ = \gamma c A\cos(2\pi ft) + \gamma c B\cos(2\pi ft)\cos\varphi - \gamma c B\sin(2\pi ft)\sin\varphi.\end{aligned}$$

Using the fact that $\sin\varphi = 0$, we get that

$$B = \frac{F_0 - \gamma c}{F_0 + \gamma c}A.$$

In the limit of $\gamma \to \infty$, we find that $B \approx -A$ which means that the reflected wave is exactly the same as the transmitted one but moves in the opposite direction. In the limit of $\gamma \to 0$, we find that $B = A$ and that the reflected wave is exactly the same as the transmitted one but moves in the same direction.