Solutions to Jaan Kalda's Problems in Kinematics

With detailed diagrams and walkthroughs

Edition 2.0.0

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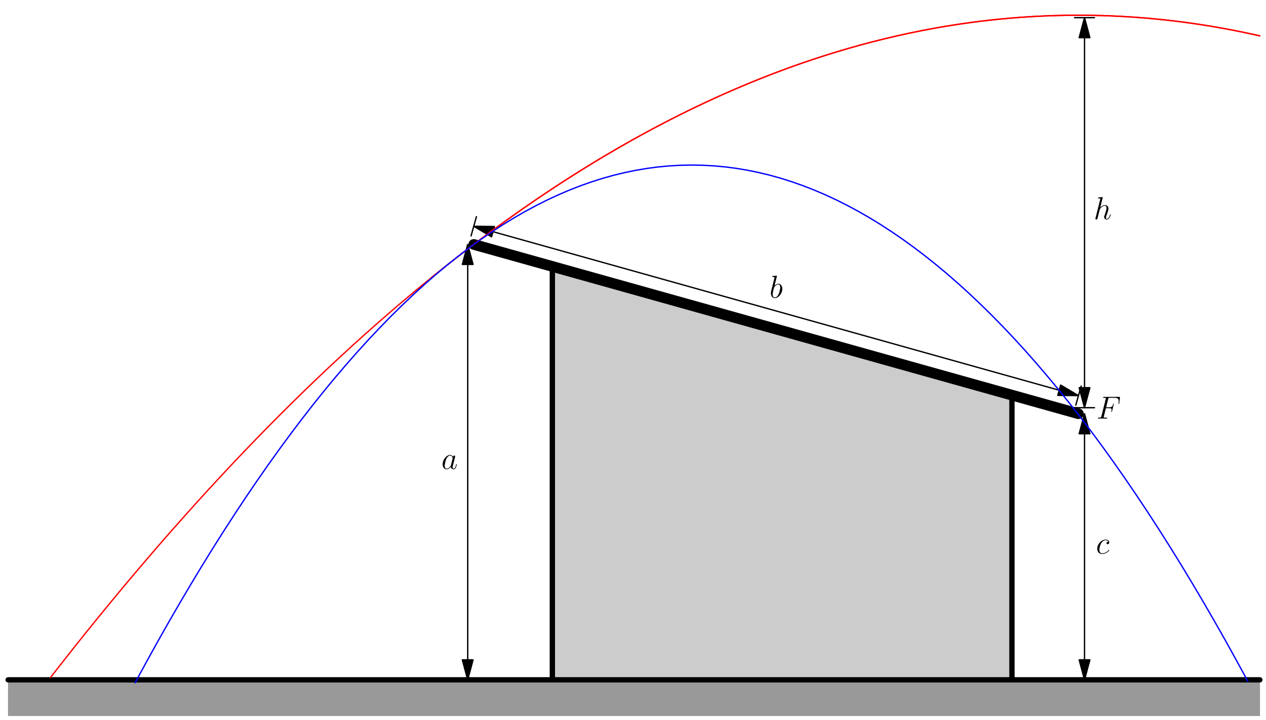

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Preface

Jaan Kalda's handouts are beloved by physics students both in for a quick challenge, to students preparing for international Olympiads. As of writing, the current kinematics handout (ver 2.0) has 66 unique problems and 45 main 'ideas'.

This solutions manual came as a pilot project from the online community at artofproblemsolving.com. Although there were detailed hints provided, full solutions have never been written. The majority of the solutions seen here were written on a private forum given to those who wanted to participate in making solutions. In an amazing show of an online collaboration, students from around the world came together to discuss ideas and methods and created what we see today.

This project would not have been possible without the countless contributions from members of the community. Online usernames were used for those who did not wish to be named:

Rakshit, Evan Kim, Ameya Deshmukh, Alan Abraham, dbs27, Heramb Podar, Anant Lunia, Sumgato, Viraj Jayam

Structure of The Solutions Manual

Each chapter in this solutions manual will be directed towards a section given in Kalda's kinematics handout. There are six major chapters: velocities, accelerations/displacements, optimal trajectories, rigid bodies/hinges/ropes, miscellaneous topics, and revision problems. If you are stuck on a problem, cannot make progress even with the hint, and come here for reference, look at only the start of the solution, then try again. Looking at the entire solution wastes the problem for you and ruins an opportunity for yourself to improve.

Contact Us

Despite editing, there is almost zero probability that there are *no* mistakes inside this book. If there are any mistakes, you want to add a remark, have a unique solution, or know the source of a specific problem, then please contact us at hello@physoly.tech. The most current and updated version of this document can be found on our website physoly.tech.

Please feel free to contact us at the same email if you are confused on a solution. Chances are that many others will have the same question as you.

1 Solutions to Velocities Problems

This section will consist of the solutions to problems from problem 1-9 of the handout. In this section we will be analyzing the usage of *reference frames*. Reference frames are defined as point or perspective of an action or motion from different objects or people. For example, two different reference frames of a car moving on a round could be one of an observer standing right by the road who sees the car to be moving on the road or the person inside the car who does not see the car to be moving at all. Reference frames are extremely important in physics as they allow us to solve seemingly complex problems which become greatly simplified with the use of reference frames.

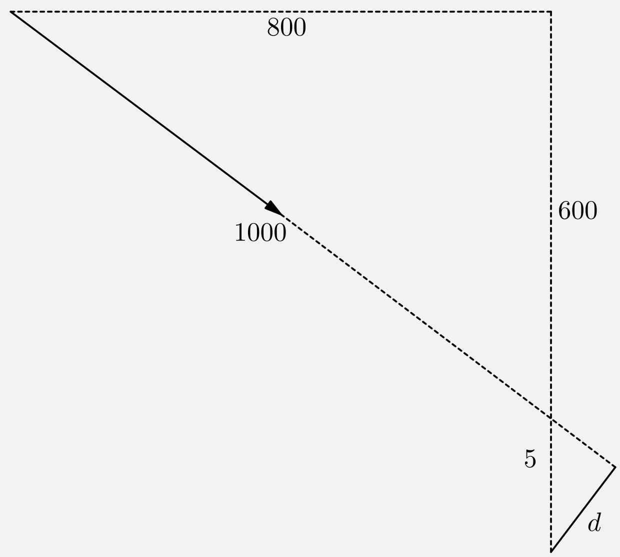
pr 1. In a time $2t$, the barge moved a distance of 6 km in the ground frame, so this implies that the speed of the water is:

$$v_{\text{water}} = \frac{\Delta d}{2t} = \frac{6\text{ km}}{1.5\text{ hours}} = \boxed{4\text{ km/h}}$$

In the water's reference frame, the barge is stationary and the boat travels at a constant speed $v_{\text{boat rel water}}$ relative to the water, where:

$$v_{\text{boat rel water}} = \frac{\Delta d}{2t} = \frac{30 - 6\text{ km}}{1.5\text{ hours}} = \boxed{16\text{ km/h}}$$

pr 2. Moving into the frame of the red plane, we see the blue plane with a diagonally directed velocity.



The closest approach would be when the faster plane's path makes a perpendicular line with the slower plane. This turns out into a geometry problem where we have two similar right triangles. We can break up the velocity of the blue plane into components (since the displacement is in the same direction as velocity, this is also the components of its displacement). The top triangle is a $3-4-5$ right triangle so the bottom right triangle must also be a $3-4-5$ right triangle.

Now all we need to know now is to determine how far away the blue plane is when it is directly overhead the red plane. The time it takes to reach this point is:

$$t = \frac{20\text{ km}}{800\text{ km/h}} = 0.025\text{ h}$$

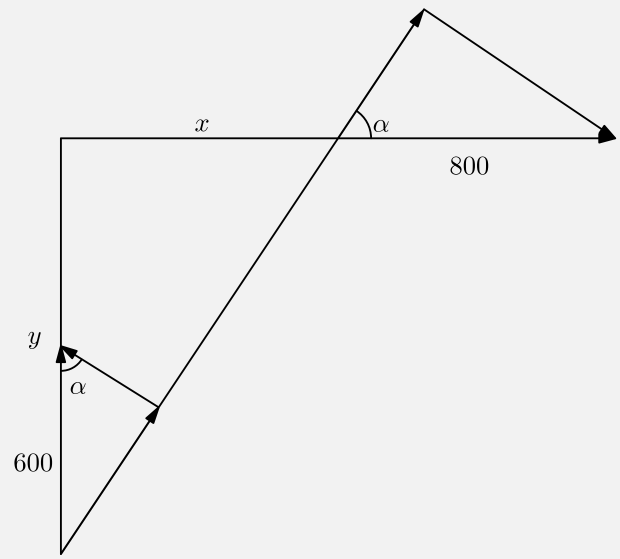
and the vertical distance it travels during this time is:

$$\Delta y = (600\text{ km/h})(0.025\text{ h}) = 15\text{ km/h}$$

meaning the vertical separation is 5 km/h. Therefore:

$$d = \boxed{4 \text{ km}}$$

Solution 2: Let us work in the lab frame this time, but break the velocities of the two planes into a direction perpendicular and towards the other plane. We only need to worry about this radial component. Originally, the two planes will be nearing each other but will eventually get farther apart. The point at which this happens is when the radial component of their velocities are directed in the same direction and have the same magnitude. If we measured their radial acceleration at this point, it would be zero.



Thus, we must have:

$$600 \sin\alpha = 800 \cos\alpha \implies \tan\alpha = \frac{4}{3}$$

Due to similar triangles, we must also have:

$$\frac{4}{3} = \frac{y}{x}$$

Let $t = 0$ be when the fast plane is directly above the slow plane. The vertical separation at this point is 5 km. Therefore, we have $x = 800t$ and $y = 5 - 600t$ We want:

$$\frac{5 - 600t}{800t} = \frac{4}{3}$$

Solving for t and plugging it into x and y can give you the separation, which turns out to be $\boxed{4 \text{ km}}$.

Solution 3: We can use generalized coordinates. The distance between the two planes is:

$$\vec{d} = (20 - 800t)\hat{x} + (-20 + 600t)\hat{y} = \vec{s} + \vec{v}t$$

where $\vec{s} \equiv 20\hat{x} - 20\hat{y}$ and $\vec{v} \equiv -800\hat{x} + 600\hat{y}$, which represents the relative velocity. As with before, we want the relative velocity to be perpendicular to the displacement $\vec{d}$. One way of doing it is maximizing the dot product:

$$|\vec{d} \times \vec{v}| = |\vec{s} \times \vec{v} + \vec{v} \times \vec{v}t| = |\vec{s} \times \vec{v}|$$

At the maximum, this cross product has to be equal to $|\vec{d}||\vec{v}|$. Therefore, all we need is to evaluate:

$$|\vec{d}| = \frac{|\vec{s} \times \vec{v}|}{|\vec{v}|} = \boxed{4 \text{ km}}$$

Solution 4: Here's a standard calculus method. The distance between the two planes after a time t is:

$$d^2 = (20 - 800t)^2 + (20 - 600t)^2$$

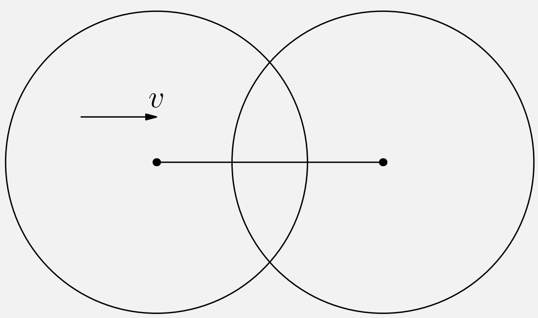
d is maximized when d^2 is maximized or when:

$$\begin{aligned}\frac{d}{dt}\left((20-800t)^2 + (20-600t)^2\right) &= 2(20-800t)(-800) + 2(20-600t)(-600)\\ 0 &= 4(20-800t) + 3(20-600t)\\ 0 &= 80 - 3200t + 60 - 1800t\\ t &= 7/250\end{aligned}$$

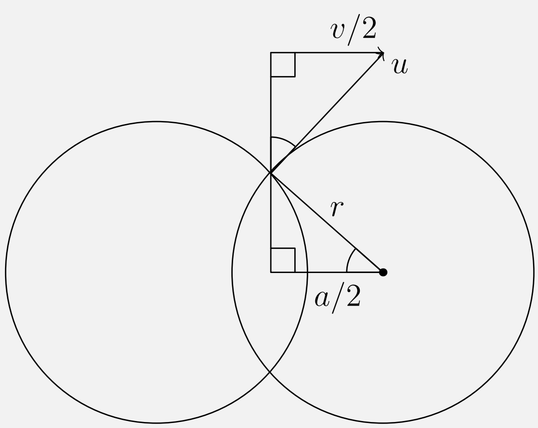
Plugging in $t = 7/250$ into the distance formula gives:

$$d = \boxed{4\text{ km}}$$

pr 3. Consider the following diagram:



Moving into the reference frame moving leftward at velocity $v/2$ gives us the following diagram (where u is the velocity of intersection):



Using SAS similarity we find that

$$\begin{aligned}\frac{u}{v/2} &= \frac{r}{\sqrt{r^2 - (a/2)^2}}\\ u &= \frac{v}{2r\sqrt{r^2-(a/2)^2}} = \boxed{\frac{v}{2\sqrt{1-(a/2r)^2}}}\end{aligned}$$

Solution 2: We start similarly to Solution 1 and move into a reference frame moving left at $v/2$.

Let $y = \sqrt{r^2 - a^2/4}$ be the height of of the intersection above the centers of the hoop. We then see that

$$\frac{dy}{dt} = \frac{dy}{da} \cdot \frac{da}{dt} = \frac{1}{\sqrt{(2r/a)^2 - 1}} \cdot \frac{v}{2}$$

Therefore,

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{\frac{v^2}{4} + \frac{v^2}{4} \cdot \frac{1}{(2r^2/a) - 1}}$$

$$u = \boxed{\frac{v}{2\sqrt{1 - (a/2r)^2}}}$$

pr 4. We notice that the graph is quadratic so we can fit it to the equation

$$\begin{aligned} \alpha &= \frac{\pi}{180}\left(-\frac{60}{49}(t-7)^2 + 60\right) \\ &= -\frac{\pi}{147}(t-7)^2 + \pi/3 \end{aligned}$$

where α is in radians and t is in minutes.

Since we know that the upward ascending velocity is constant, it is

$$\begin{aligned} v_y &= L\alpha'(0) = 1000\left(\frac{14\pi}{147}\right) \\ &= 299\,\mathrm{m/min} = \boxed{4.99\,\mathrm{m/s}} \end{aligned}$$

The height is simply

$$h = v_y t = \boxed{2000\,\mathrm{m}}$$

At $t = 7\,\mathrm{min}$, the change in elevation angle is momentarily 0, which means that the velocity vector also points at 60 degrees.

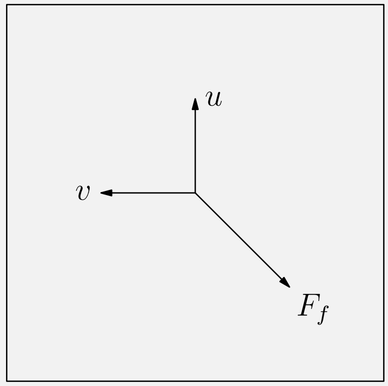
Thus we can get

$$v_x = v_y \tan(30^\circ) \approx \boxed{2.8\,\mathrm{m/s}}$$

You don't need the equation of the curve to perform calculations, but even without it, the answer can appear a bit off. e.g. the initial slope you get could be:

$$4^\circ/0.2 \text{ min} = 0.0698 \text{ rad}/12 \text{ sec} = 0.00582 \text{ sec}^{-1}$$

pr 5.

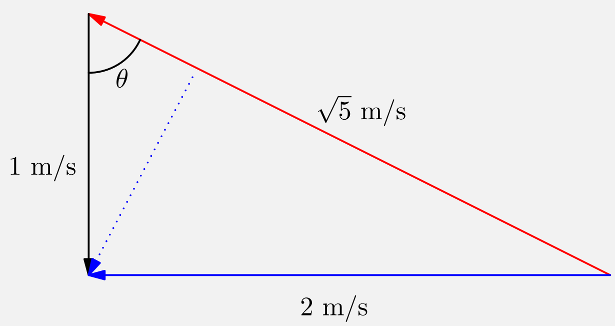


In the board's frame of reference, there is only a horizontal force (the friction force), which has a constant direction that is anti-parallel to the velocity. Thus, the chalk moves in a $\boxed{\text{straight line}}$.

pr 6. Let us assume that the block is originally pushed leftwards in the frame of the ground and the conveyor is travelling upwards.

In the frame of the conveyor belt, the block is moving with a speed of $\sqrt{5}$ m/s. This is represented by the red vector. Due to friction opposing the motion, the direction of motion relative to the belt will be constant. The magnitude will steadily decrease to zero.

To move back to the frame of the ground, we can add back the velocity of the conveyor belt, as shown below.

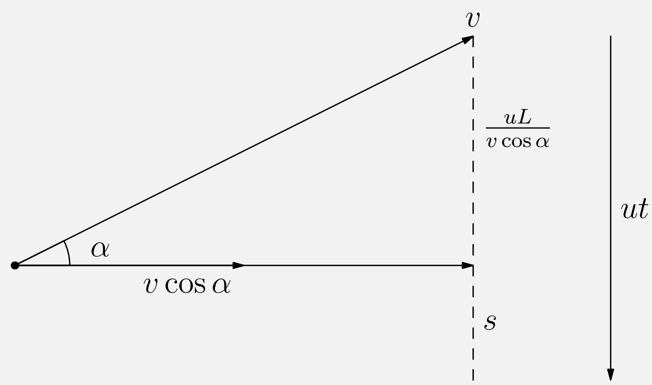


The blue vector shows the velocity of the block relative to the ground. Initially, it is 2 m/s but as friction reduces the magnitude of the red vector (which represents the velocity relative to conveyor belt), the blue vector will decrease to a minimum. This minimum occurs when it forms a right angled triangle (represented by the dotted lines).

Therefore, the minimum velocity of the block relative to the ground is

$$v = 1\sin\theta = \boxed{\frac{2}{\sqrt{5}}}$$

pr 7.



Denote the dashed line by the wall which is a distance L away from the point source.

We express the lateral displacement of the ball as the sum of two components: lateral displacement in the air's frame of reference, and the lateral displacement of the moving frame.

In the air's frame the displacement is given by

$$ut_{\text{air}} = \frac{uL}{v\cos\alpha}$$

and the lateral displacement in the moving frame is given by s. This gives us

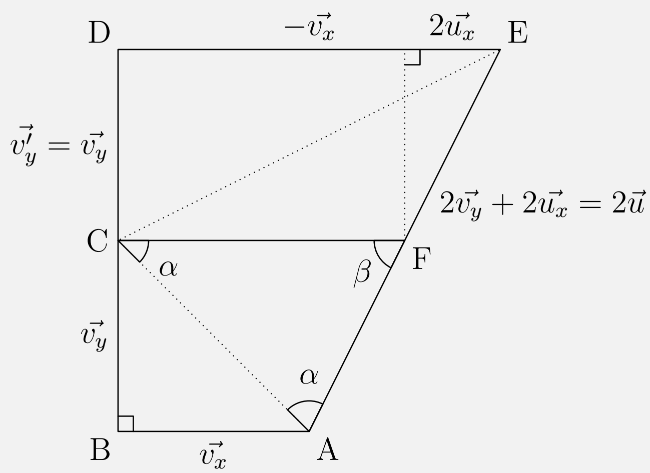
$$ut = \frac{uL}{v\cos\alpha} + s \implies t = \boxed{\frac{s}{u} + \frac{L}{v\cos\alpha}}$$

pr 8. Draw a right trapezoid as follows:

We decompose $\vec{v}$ into parallel and perpendicular components, $\vec{v} = \vec{v_x} + \vec{v_y}$; let us mark points A, B and C so that $AB = \vec{v_x}$ and $BC = \vec{v_y}$ (then, $AC = \vec{v}$).

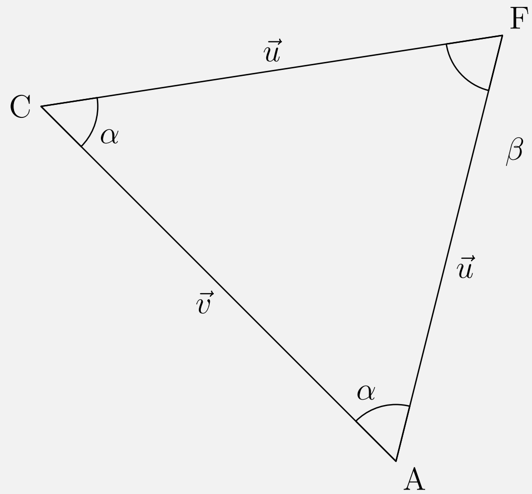
Next we mark points D, E and F so that $CD = \vec{v_y'} = \vec{v_y}$, $DE = -\vec{v_x}$, and $EF = 2\vec{u_x}$; then, $CF = \vec{v_y'} - \vec{v_x} + 2\vec{u_x} \equiv \vec{v'}$ and $AF = 2\vec{v_y} + 2\vec{u_x} \equiv 2\vec{u}$.

Due to the problem conditions, $\angle ACF = 90°$.



We now can see that ΔACF is an isoceles triangle containing the lengths provided in the figure below.

Let us also mark point G as the centre of AF; then, FC is both the median of the right trapezoid $ABDF$ (and hence, parallel to AB and the $x-$axis), and the median of the triangle ACF.

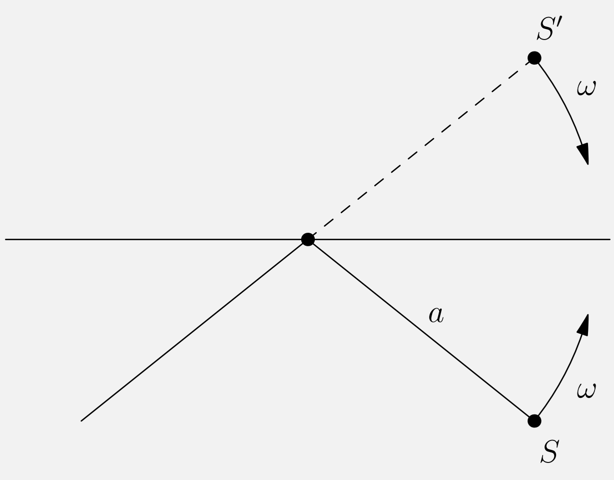


By splitting ΔACF into it's median, we find,

$$u\cos\alpha = \frac{v}{2} \implies u = \boxed{\frac{v}{2\cos\alpha}}.$$

For this to also happen, we see that $\beta = \boxed{180 - 2\alpha}$ because ΔACF is an isoceles triangle.

pr 9. We move into the reference frame that is rotating clockwise at ω about the center of the mirror (i.e. the mirror is stationary).



In this frame of reference, the image S' has angular velocity ω clockwise about the center of the mirror.

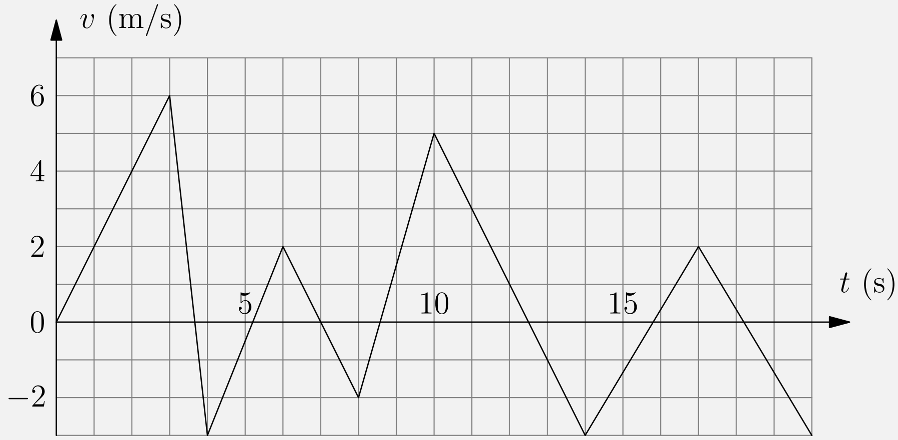
Moving back into the reference frame where S is stationary, we see that S' is moving with angular velocity 2ω about the center of the mirror, so the image has speed

$$v = \boxed{2\omega a}$$

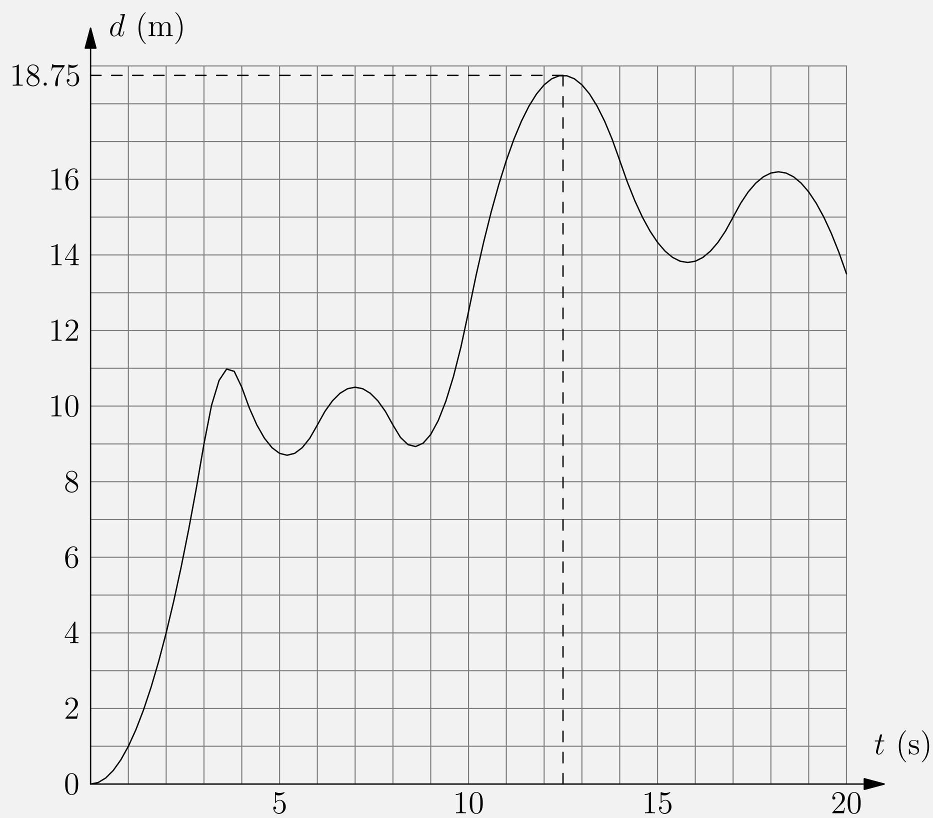
2 Solutions to Accelerations/Displacements Problems

This chapter will contain problems 10-16 of the handout. In this section, we will be analyzing the usage of non-constant velocity and graphs. When an object does not have a constant velocity, there is an acceleration present meaning that the object is either speeding up or slowing down. The implementation of acceleration and graphs allows us to figure out many problems. In this section, we will also be looking at the implementation of cutting into tiny pieces and integrating to derive a result.

pr 10. Consider the following graph:



Since the particle starts from the origin, the distance graph is simply the area under the velocity graph:



We need to find the maximum displacement, so our answer is $\boxed{18.75\text{ m}}$

pr 11. Let us divide the displacement into tiny pieces, $s = \sum \Delta s$ where $\Delta s = v\Delta t$.

If the function $v(t)$ were known, the last formula would have been completed our task, because $\sum v(t)\Delta t$ is the sum of rectangles making up the area under the $v - t$-graph.

However, the acceleration is given to us as a function of v, hence we need to substitute Δt with Δv.

While trying to do that, we can introduce the acceleration (which is given to as a function of v):

$$\Delta t = \Delta v \cdot \frac{\Delta t}{\Delta v} = \frac{\Delta v}{\Delta v/\Delta t} = \frac{\Delta v}{a}.$$

In other words

$$s = \sum \frac{v}{a}\Delta v = \int \frac{v}{a(v)} dv.$$

This tells us that the answer is equal to the area under a graph which depicts $\frac{v}{a(v)}$ as a function of v.

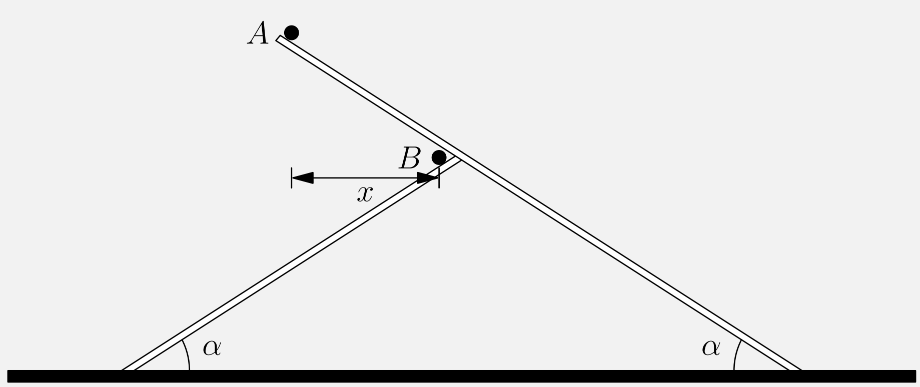
Applying a quartic least-squares fit to some of the discernible data points, we can see that the curve $a(v)$ is well approximated by the function $0.00617211v^4 - 0.0301639v^3 + 0.0581573v^2 + 0.0546369v + 0.000715828$.

Taking the integral, we can see that

$$\begin{aligned}\int_0^4 \frac{v}{a(v)} dv &\approx \int_0^4 \frac{v}{0.00617211v^4 - 0.0301639v^3 + 0.0581573v^2 + 0.0546369v + 0.000715828} dv \\ &\approx \boxed{39 \text{ m}}\end{aligned}$$

Don't worry if your answer isn't exactly the same as ours, as this result may be difficult to determine by hand with the graph provided. A rough approximation (within reasonable limits) would suffice.

pr 12.



In the reference frame of ball A, ball B accelerates to the left with

$$a_B = 2g \sin\alpha \cos\alpha$$

We can find that the initial length $|AB|$ is

$$\frac{g\left(t_1^2 - t_2^2\right)\sin\alpha}{2}$$

Therefore,

$$x = \frac{g\left(t_1^2 - t_2^2\right)\sin\alpha\cos\alpha}{2}$$

Since there is no relative acceleration in the y-direction, we need

$$\frac{a_B t^2}{2} = x$$

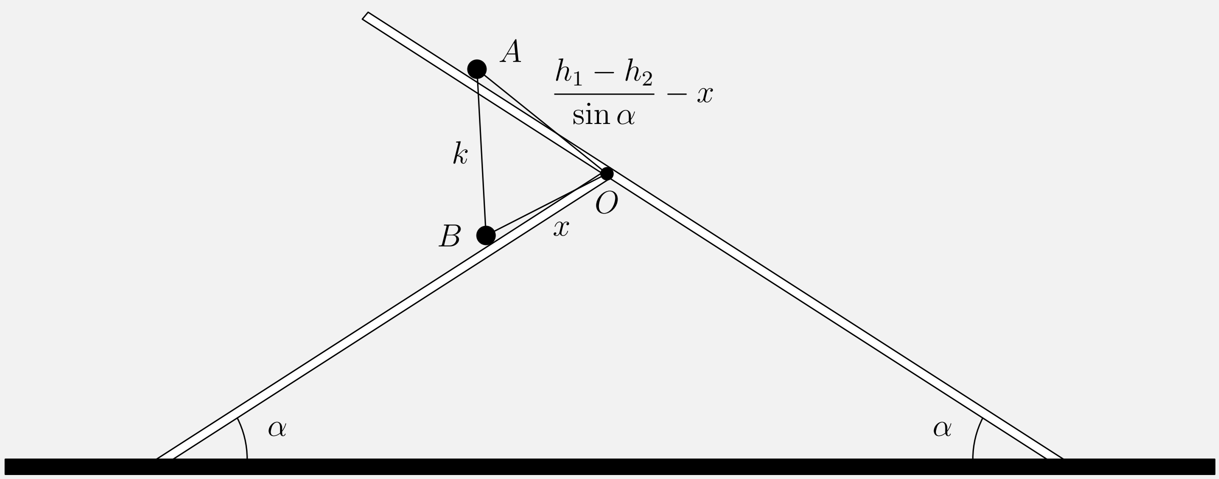
$$g t^2 \sin\alpha\cos\alpha = \frac{g\left(t_1^2 - t_2^2\right)\sin\alpha\cos\alpha}{2}$$

$$t = \boxed{\sqrt{\frac{t_1^2 - t_2^2}{2}}}$$

Solution 2: Each ball will accelerate with the same acceleration down their platform, meaning that they will travel the same distance in the same timeframe.

Let x be the distance traveled by the individual balls and k be the distance between the two balls. Let the height of the ball at point A be h_1 and the height of the ball at point B be h_2.

If you draw a diagram you will find that there is a triangle formed by the position of the two balls and the intersection of the planks. The lengths of the triangle are $x, x - \frac{h_1 - h_2}{\sin\alpha}, k$.



By the Law of Cosines, we have

$$k^2 = x^2 + \left(\frac{h_1 - h_2}{\sin\alpha} - x\right)^2 - 2x\left(\frac{h_1 - h_2}{\sin\alpha} - x\right)\cos(2\alpha)$$

Let $\beta = \frac{h_1 - h_2}{\sin\alpha}$ for simplicity.

Simplifying the expression, we get that

$$k(x) = \sqrt{2x^2(1 + \cos(2\alpha)) - 2x\beta(1 + \cos(2\alpha)) + \beta^2}$$

After taking the derivative of the quadratic and setting it equal to zero, we get that

$$x_m = -\frac{B}{2A} = \frac{\beta}{2}$$

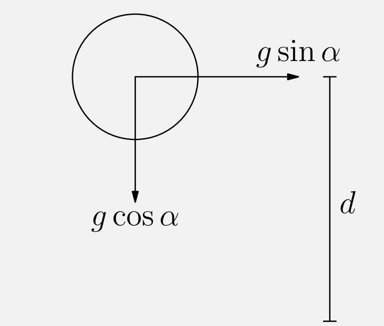
Using acceleration along the ramp we can also find that

$$\frac{h_1}{\sin\alpha} = \frac{gt_1^2 \sin\alpha}{2} \qquad \frac{h_2}{\sin\alpha} = \frac{gt_2^2 \sin\alpha}{2}$$
$$x_m = \frac{gt_m^2 \sin\alpha}{2} = \frac{h_1 - h_2}{2\sin\alpha}$$

Plugging in everything we find that

$$\begin{aligned} t_m &= \sqrt{\frac{h_1 - h_2}{g\sin^2\alpha}} \\ &= \sqrt{\frac{\frac{gt_1^2 \sin^2\alpha}{2} - \frac{gt_2^2 \sin^2\alpha}{2}}{g\sin^2\alpha}} \\ &= \boxed{\sqrt{\frac{t_1^2 - t_2^2}{2}}} \end{aligned}$$

pr 13.



Let t be the time it takes the ball to hit the ramp. Therefore, we find that

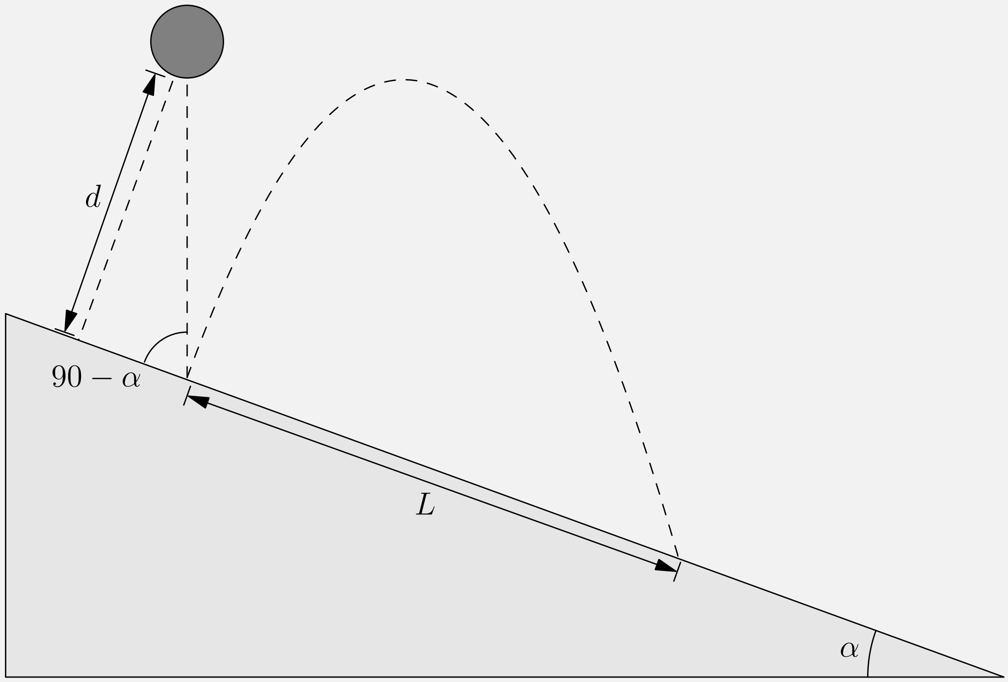
$$d = \frac{1}{2}at^2 \implies d = \frac{1}{2}g\cos\alpha t^2 \implies t = \sqrt{\frac{2d}{g\cos\alpha}}.$$

Now, we note that the total time $T = 3t$ because the ball travels a distance d to collide with the ramp, bounces up a distance d to the vertex of its parabolic trajectory, and then falls back down for the final distance d.

This means that the distance between both bouncing points s is found by

$$s = \frac{1}{2}g\sin\alpha(T^2 - t^2)$$
$$s = \frac{1}{2}g\sin\alpha(8t^2) = 4g\sin\alpha\left(\frac{2d}{g\cos\alpha}\right)$$
$$s = \boxed{8d\tan\alpha}$$

Solution 2: Consider the following diagram:



We rotate the plane by α counterclockwise such that gravity now has acceleration $g\cos\alpha$ in the y-direction and $g\sin\alpha$ in the x-direction.

When the ball hits the plane, it strikes with velocity $v_0 = \sqrt{\dfrac{2gd}{\sin(90-\alpha)}}$ at an angle $90-\alpha$ to the inclined plane.

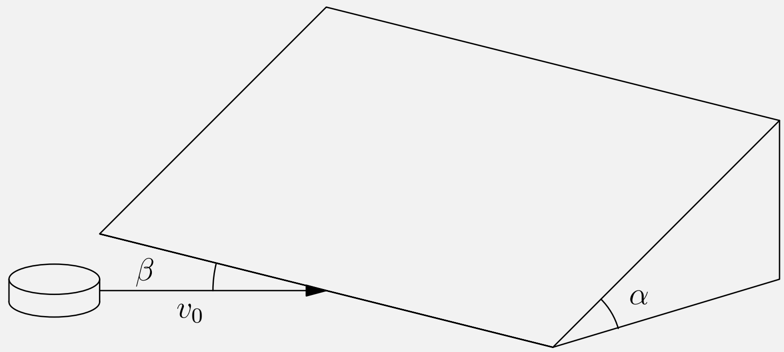
Then, we use modified projectile motion to get that

$$t = \frac{2v_0\sin(90-\alpha)}{g\cos(\alpha)} = \frac{2v_0}{g}$$

We also have that

$$\begin{aligned} L &= v_0\cos(90-\alpha)\,t + \frac{1}{2}g\sin(\alpha)\,t^2 \\ &= \frac{2v_0^2\cos(90-\alpha)}{g} + \frac{4v_0^2 g\sin(\alpha)}{2g^2} \\ &= \frac{2v_0^2\sin(\alpha)}{g} + \frac{2v_0^2\sin(\alpha)}{g} \\ &= \boxed{8gd\tan\alpha} \end{aligned}$$

pr 14.



When on the plane, the puck experiences no change in its x-velocity, which is

$$v_0 \cos\beta = 5 \text{ m/s}$$

However, it experiences an acceleration parallel to the plane with magnitude

$$a = g \sin\alpha$$

We note from the trajectory given that the puck drops 2.5 m below the apex of its trajectory while undergoing a horizontal displacement of $x = 5$ m.
The time it takes to complete this motion is

$$t = \frac{x}{v_0 \cos\beta} = 1 \text{ s}$$

Therefore, we have that

$$\frac{gt^2 \sin\alpha}{2} = 2.5$$
$$\sin\alpha = \frac{5}{gt^2}$$
$$\alpha \approx \boxed{30^\circ}$$

pr 15. Due to symmetry the turtles meet at the centroid of the triangle formed, and form an equilateral triangle at any instant. The velocity of the first turtle with respect to the other is obviously

$$0.1 \cos(60^\circ) \text{ m/s}$$

Thus, the relative velocity of separation is

$$v(1 + \cos(60^\circ)) = \frac{3v}{2}$$

Since this is constant, the time taken for the turtles to meet is

$$t = \frac{d}{\frac{3v}{2}} = \frac{2d}{3v} = \boxed{6.7 \text{ s}}$$

Solution 2: The path length of any turtle in the motion is simply

$$ds = dr\sqrt{1 + \left(\frac{rd\theta}{dr}^2\right)}$$

Using polar coordinates, one can deduce that

$$\begin{aligned}\frac{dr}{dt} &= -10\sin(60^\circ) \\ v\frac{d\theta}{dt} &= \frac{10}{2} = 5\end{aligned}$$

Hence we have

$$\frac{d\theta}{dr} = -\frac{1}{r\sqrt{3}} \Rightarrow ds = \frac{2dr}{\sqrt{3}}$$

Since $t = \frac{ds}{10}$, we find the total time by integrating this expression:

$$\begin{aligned}\int_0^T dt &= -\frac{2}{\sqrt{3}v}\int_{\frac{1}{\sqrt{3}}}^0 dr \\ T &= \frac{2d}{3v} = \boxed{6.7 \text{ s}}\end{aligned}$$

pr 16. Note that

$$dk = \frac{ds}{L(t)} = \frac{vdt}{L + ut}.$$

Integrating both sides gives us

$$\int_0^1 dk = \int_0^T \frac{vdt}{L + ut}$$

Using a s substitution $s = L + ut \Rightarrow dx = udt$ and rearranging the integral gives us

$$\begin{aligned}1 = \frac{v}{u}\int_0^T \frac{dx}{s} \implies \frac{u}{v} &= \ln(ut + L)\,|_0^T \\ \frac{u}{v} &= \ln\left(\frac{uT + L}{L}\right)\end{aligned}$$

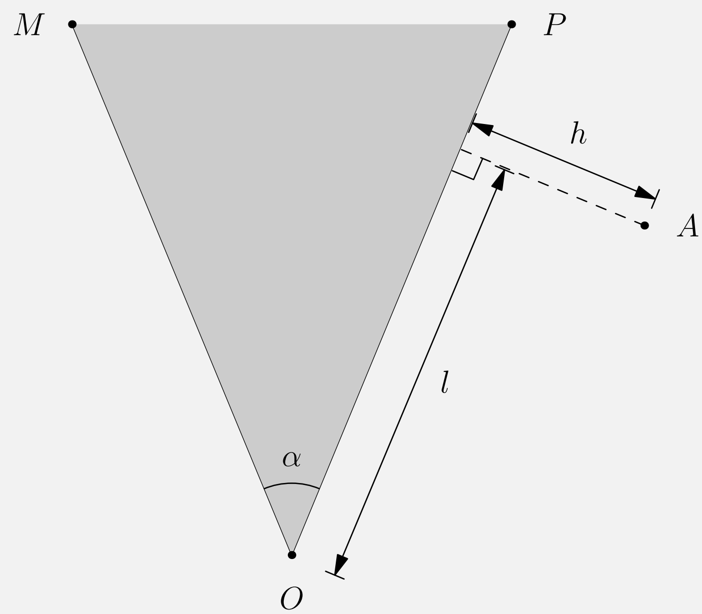
Substituting the values given in the problem tells us

$$\begin{aligned}100 = \ln\left(\frac{uT + L}{L}\right) \implies e^{100} &= \frac{uT}{L} + 1 \\ T &= \boxed{e^{100} - 1}\end{aligned}$$

3 Solutions to Optimal Trajectories Problems

This section will contain problems 17-22 of the handout. In this section, we will be analyzing trajectories. You should have a good understanding of conics (parabolas in general) and projectile motion before getting into this section. There are a lot of proofs and sometimes more they have more math than what you would call physics. However, don't be demotivated, there are a lot of great problems in this section and some of these problems have ties to other areas of physics.

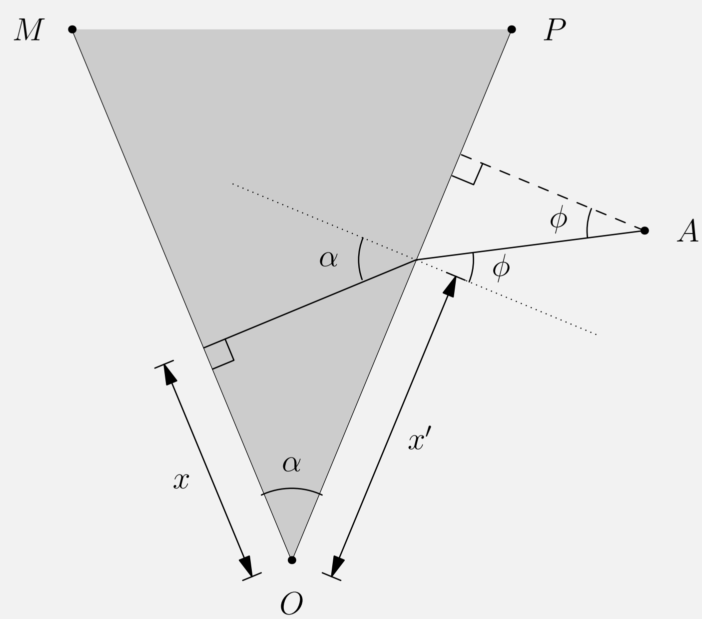
pr 17.



One extreme case that we must consider first is directly travelling along the path AO, which gives

$$t = \frac{\sqrt{l^2 + h^2}}{v}$$

We'll deal with this later, but we first use fact 5, as shown in the following diagram:



We use ϕ as defined above to make calculations easier and we get that

$$\frac{\sin\phi}{\sin\alpha} = \frac{v}{u}$$
$$\phi = \arcsin\left(\frac{v\sin\alpha}{u}\right)$$

Since

$$x' = l - h\tan\phi$$
$$x = (l - h\tan\phi)\cos\alpha$$

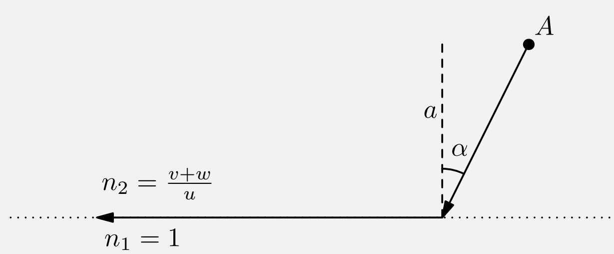
We also have that

$$\begin{aligned} t &= \frac{h}{v\cos\phi} + \frac{x\tan\alpha}{u} \\ &= \frac{h}{v\cos\phi} + \frac{(l-h\tan\phi)\sin\alpha}{u} \\ &= \frac{h}{v\cos\phi} - \frac{h\sin\phi\sin\alpha}{u\cos\phi} + \frac{l\sin\alpha}{u} \\ &= \frac{h}{v\cos\phi} - \frac{h\sin^2\phi}{v\cos\phi} + \frac{l\sin\alpha}{u} \\ &= \frac{h\cos\phi}{v} + \frac{l\sin\alpha}{u} \end{aligned}$$

However, we must also note that, when $\phi \geq \arctan\left(\frac{l}{h}\right)$, the boy never actually reaches side OP. Therefore, our answer is

$$\boxed{\begin{array}{ll} x = (l-h\tan\phi)\cos\alpha,\ t = \dfrac{h\cos\phi}{v} + \dfrac{l\sin\alpha}{u} & \text{if } \phi < \arctan\left(\dfrac{l}{h}\right) \\ x = 0,\ t = \dfrac{\sqrt{l^2+h^2}}{v} & \text{otherwise} \end{array}}$$

pr 18. **Solution 1 (Fermat's Principle):**



Let us move into the reference frame of the river such that everywhere in the water, the boy is travelling at a constant speed u. This might seem troublesome at first because his destination would be moving, but that doesn't trouble us at all. As it will soon be made clear, the angle α the boy makes with the shore-line will be independent of how far away the target is.

Consider a light beam that starts off from A and ends up travelling with a speed of $v+w$ parallel to the shoreline. Since it will take the fastest path, the boy will need to mimic this behavior. Snell's Law gives:

$$n_1 \sin(90^\circ) = n_2 \sin\alpha \implies \sin\alpha = \frac{u}{v+w} \implies \alpha = \boxed{\arcsin\left(\frac{u}{v+w}\right)}$$

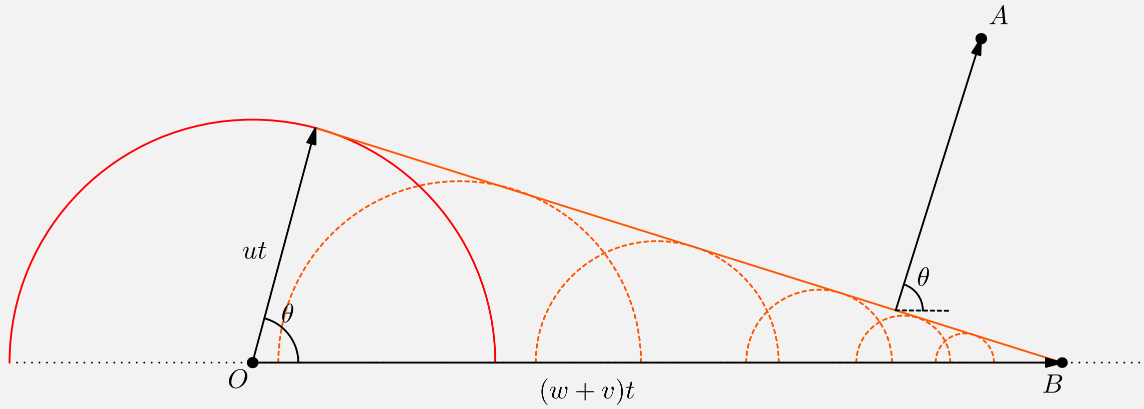
The total time the boy spends swimming is:

$$t = \frac{d}{u} = \frac{a}{u\cos\alpha}$$

Relative to the water, the boy swims a horizontal distance $a\tan\alpha$. The water during this time flows a distance $wt = \frac{wa}{u\cos\alpha}$ in the opposite direction. Therefore, the horizontal distance x is:

$$x = \boxed{a\left(\frac{w}{u\cos\alpha} - \tan\alpha\right)}$$

Solution 2 (Huygens Principle):



Consider the same setup as before by moving into the frame of the river. This time however, the path is reversed. The boy starts running from point O along the shore and eventually starts swimming to location A. Imagine the boy emitting Chernenko radiation as he moves as shown in the diagram. The wave speed is u while the speed of the boy is $w+v > u$. Physically, the outlines of all the circles represent the superposition of all the points in which the boy can be at after a time t.

Due to Huygen's Principle, we can see that this forms a wavefront that is moving towards A at a speed of u. We can let this wavefront evolve until a part of it eventually reaches the point A. The path that this part of the wave takes will represent the optimal path of the boy, that is, perpendicular to the wavefront. We can determine the angle θ by considering two extreme paths the boy can take.

First, the boy can start swimming immediately and reach a distance ut after a time t. During this period, the boy can also run a distance $(w+v)t$. The angle θ is thus given by:

$$\cos\theta = \frac{u}{w+v}$$

and thus the angle $\alpha = 90^\circ - \theta$ normal to the shore is

$$\alpha = \boxed{\arcsin\left(\frac{u}{w+v}\right)}$$

This is the same angle found in the first solution and as a result we can copy the exact steps to determine x.

pr 19. We split up v into it's vertical and horizontal components. From here we can see that each parameter x, y and z as a function of t is

$$z = v_0 t \sin\alpha - \frac{1}{2}gt^2 \tag{1}$$
$$x = v_0 \cos\alpha t \tag{2}$$
$$t = \frac{x}{v_0 \cos\alpha} \tag{3}$$

From here, we substitute equation 3 into equation 1 to yield

$$z = v_0 \left(\frac{x}{v_0\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{v_0\cos\alpha}\right)^2.$$

Simplifying with trigonometry yields

$$z = x\tan\alpha - \frac{gx^2}{2v_0^2}\sec^2\alpha$$
$$z = x\tan\alpha - \frac{gx^2}{2v_0^2}(\tan^2\alpha + 1)$$
$$0 = \frac{gx^2}{2v_0^2}\tan^2\alpha - x\tan\alpha + \frac{gx^2}{2v_0^2}$$

Here, we find a quadratic. For the region of space $\mathcal{R}$ to exist, the discriminant of this quadratic must be greater than zero. This tells us

$$x^2 - 4\left(\frac{gx^2}{2v_0^2}\right)\left(z + \frac{gx^2}{2v_0^2}\right) \geq 0$$
$$x^2 - \frac{2gx^2}{v_0^2}z + \frac{g^2x^4}{v_0^4} \geq 0$$
$$\frac{2gx^2}{v_0^2}z \leq x^2 - \frac{g^2}{x^4}v_0^4$$

Simplifying this final expression gives us the answer. The region of space of $\mathcal{R}$ is

$$\boxed{z \leq \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}}$$

pr 20. Note that for a parabola with equation $x^2 = 4p(z-k)$, the focus is located at $(0, k+p)$

In problem 19, we have the equation

$$z \leq \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

and with some manipulation we obtain

$$x^2 = \frac{v_0^4}{g^2} - \frac{2v_0^2}{g}z.$$

Factoring the equation gives

$$x^2 = -\frac{2v_0^2}{g}\left(z - \frac{v_0^2}{2g}\right)$$

$$x^2 = 4\left(-\frac{v_0^2}{2g}\right)\left(z - \frac{v_0^2}{2g}\right)$$

Therefore, the focus of the parabola is at

$$\left(0, \frac{v_0^2}{2g} - \frac{v_0^2}{2g}\right) = (0, 0).$$

In problem 19, we assumed that the cannon was located at $(0, 0)$, and so we are done.

pr 21. First, we make the following claim:

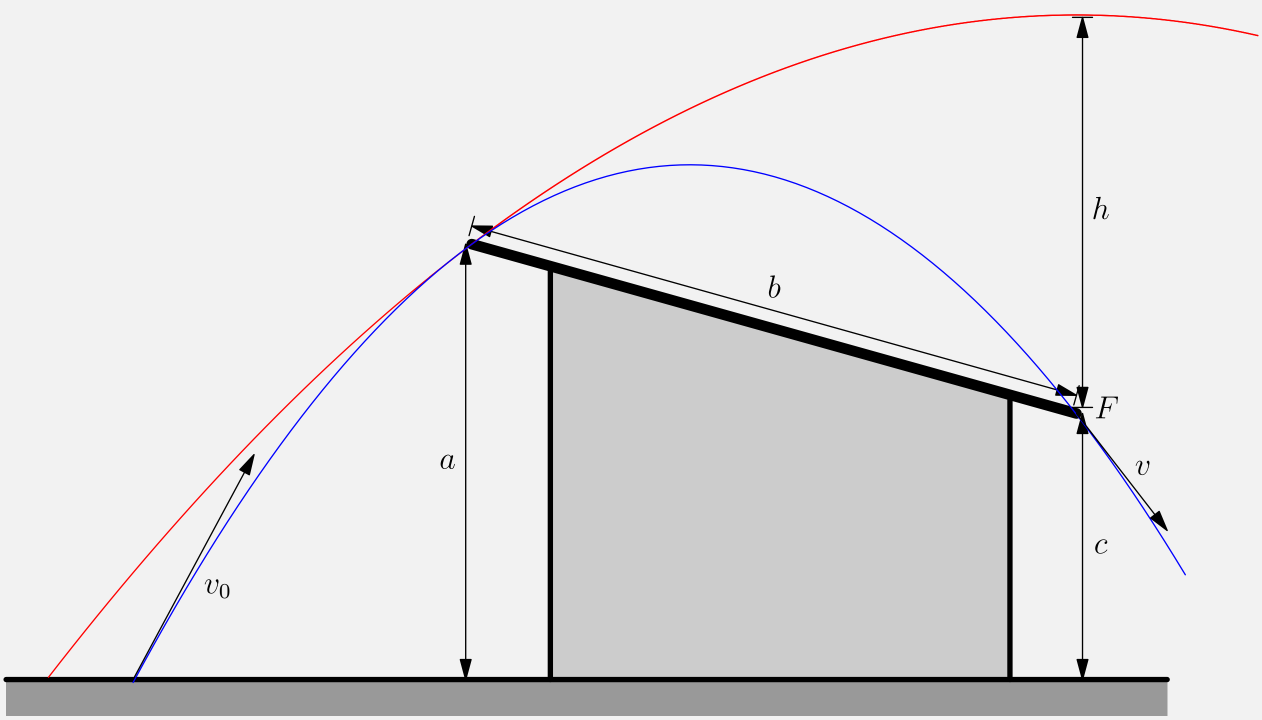
Claim: The optimal-velocity trajectory must contain both endpoints of the roof along its path.

Proof: Assume for the sake of contradiction that the optimal-velocity trajectory hits neither one of the two endpoints of the roof. Then, we can clearly see that reducing the velocity by an small amount would still result in the stone clearing the roof.

Now assume that the optimal-velocity trajectory hits only one of the two endpoints of the roof. In both cases, the thrower can displace themself horizonally by an small amount, resulting the stone hitting neither one of the two endpoints.

Thus, the optimal-velocity trajectory must contain both endpoints of the roof.

By idea 28, we can set the rightmost point of the roof (point F) to be the focus of the region $\mathcal{R}$ of all possible trajectories. Optimally, this parabola should pass through the left end of the roof.



By fact 9, we have that

$$h = \frac{a + b - c}{2}$$

We know that if the projectile is thrown straight up, it hits the top of the red parabola, so

$$\frac{1}{2}v^2 = gh$$
$$v = \sqrt{g(a+b-c)}$$

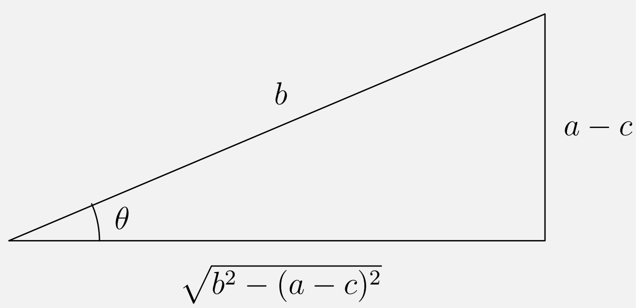
By idea 32, we have that

$$\frac{1}{2}v^2 + gh = \text{constant}$$
$$\frac{1}{2}v_0^2 = \frac{1}{2}v^2 + gc$$
$$v_0 = v_{\text{min}} = \boxed{\sqrt{g\left(a+b+c\right)}}$$

Solution 2: We begin this solution by also proving that the optimal-velocity trajectory must pass through the two endpoints of the roof.

Then, we set of coordinates of F to be $(0,0)$, so the coordinates of the left end of the roof are $\left(\sqrt{b^2-(a-c)^2}, a-c\right)$, where we have taken the absolute value of the x-coordinate to make calculations easier.

Let $\theta = \arctan\left(\frac{a-c}{\sqrt{b^2-(a-c)^2}}\right)$, let the initial launch angle (to the horizontal) be α, and let the initial velocity of the stone be v_0.



The equation for the slope of the roof is given by

$$y = x\tan\theta$$

Along the slope of the roof, we have that

$$x = v_0 t\cos\alpha$$
$$y = v_0\sin\alpha t - \frac{gt^2}{2}$$
$$v_0\sin\alpha t - \frac{gt^2}{2} = v_0 t\cos\alpha\tan\theta$$
$$t = \frac{2v}{g}\left(\sin\alpha - \cos\alpha\tan\theta\right)$$

It suffices to maximize the horizontal distance travelled, which is

$$x = \frac{2v^2\cos\alpha}{g}\left(\sin\alpha - \cos\alpha\tan\theta\right)$$

Taking the derivative with respect to α, we get that

$$\begin{aligned}
\frac{dx}{d\alpha} &= \frac{2v^2}{g}\left(2\tan\theta\sin\alpha\cos\alpha - \sin^2\alpha + \cos^2\alpha\right) \\
&= \frac{2v^2}{g\cos\theta}\cos\left(\theta - 2\alpha\right) \\
\implies \alpha &= \frac{\pi}{4} + \frac{\theta}{2}
\end{aligned}$$

From this, we can find that

$$\begin{aligned}
\sin\alpha &= \sqrt{\frac{a+b-c}{2b}} \\
\cos\alpha &= \sqrt{\frac{-a+b+c}{2b}} \\
\cos\theta &= \frac{\sqrt{b^2-(a-c)^2}}{b} \\
\tan\theta &= \frac{a-c}{\sqrt{b^2-(a-c)^2}}
\end{aligned}$$

Then, we must have that

$$\begin{aligned}
vt\cos\alpha &= b\cos\theta \\
\frac{2v^2\cos\alpha\left(\sin\alpha - \cos\alpha\tan\theta\right)}{g} &= b\cos\theta \\
v &= \sqrt{\frac{gb\cos\theta}{2\cos\alpha(\sin\alpha - \cos\alpha\tan\theta)}}
\end{aligned}$$

Plugging everything in, this simplifies (quite miraculously) to

$$v = \sqrt{g(a+b-c)}$$

Applying conservation of energy to find the velocity at the ground, we see that

$$\boxed{v_0 = \sqrt{g(a+b+c)}}$$

pr 22. Assume you are at $(0,0)$, and let the target be at (x_0, y_0), $x_0 > 0$.

The trajectory of the projectile is defined by

$$y = x\tan(\theta) - \frac{gx^2}{2v^2\cos^2(\theta)}$$

the slope of the projectile (as a function of x) is thus

$$\frac{dy}{dx} = \tan(\theta) - \frac{gx}{v^2\cos^2(\theta)}.$$

It suffices to prove that

$$\tan(\theta)\left(\tan(\theta) - \frac{gx_0}{v^2\cos^2(\theta)}\right) = -1$$

since $v^2\cos^2(\theta) = \dfrac{gx_0^2}{2(x_0\tan(\theta) - y_0)}$, it suffices to prove that

$$\tan(\theta)\left(\tan(\theta) - \frac{2(x_0\tan(\theta) - y_0)}{x_0}\right) = -1$$

we know that

$$v = \sqrt{\frac{gx_0^2}{2\cos^2(\theta)(x_0\tan(\theta) - y_0)}}.$$

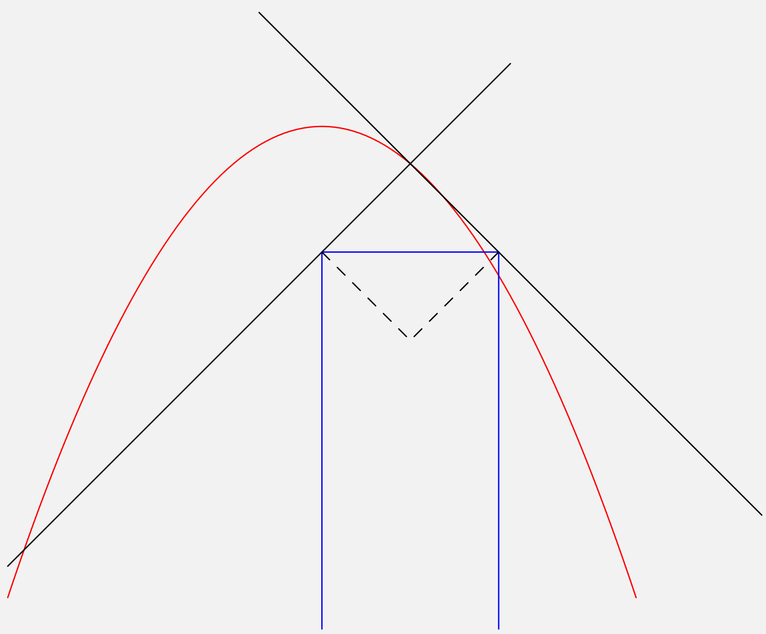
Minimizing v means maximizing

$$\cos^2(\theta)(x_0\tan(\theta) - y_0) = x_0\sin(\theta)\cos(\theta) - y_0\cos^2(\theta)$$

therefore, taking the derivative, we get that we need $-\cot(2\theta) = \dfrac{y_0}{x_0}$.
Plugging this in, we can verify that

$$\tan(\theta)\left(\tan(\theta) - 2\tan(\theta) + \frac{2y_0}{x_0}\right) = \tan(\theta)(-\cot(\theta)) = -1$$

Solution 2:



Due to idea 28, together with facts 6, 7, and 9, a vertical ray directed at the target is reflected by the projectile's trajectory to the focus, i.e. to the cannon.

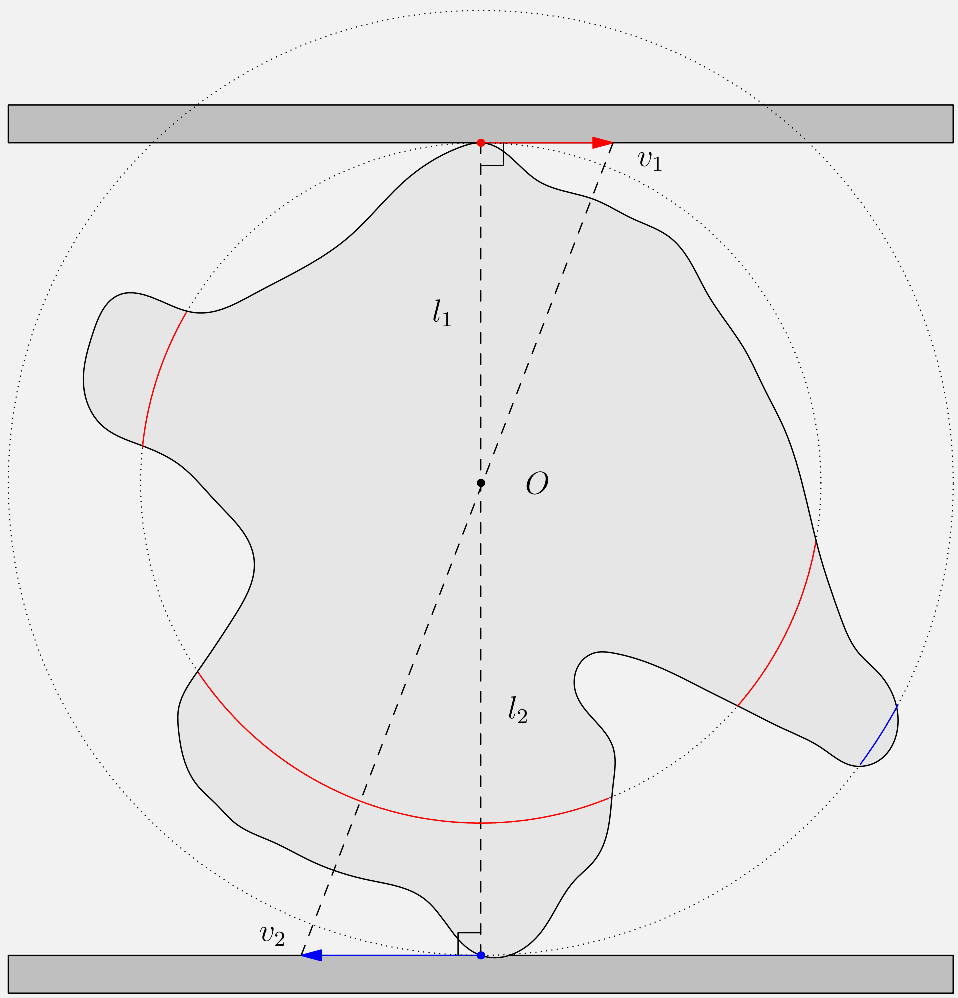
When making use of idea 26, we see that this projectile's trajectory is also optimal for shooting the cannon's position from the location of the target; hence, the projectile's trajectory reflects a vertical ray directed to the cannon towards the target.

If we combine these two observations we see that a vertical ray directed to the cannon is rotated after two reflections from the trajectory by 180°, which means that the reflecting surfaces must have been perpendicular to each other.

4 Solutions to Rigid Bodies/Hinges/Ropes Questions

This section will contain problem 23-27 of the handout. Some of the hardest problems in kinematics come from rigid bodies/hinges/ropes. These types of problems generally want the solver to calculate a certain aspect of motion of a singular system or multiple systems of objects interacting with each other. When solving these types of problems. it is best to analyze how the motion will happen before getting right into the equations.

pr 23.



We know by idea 33 that the instantaneous axis of rotation O of the object exists.

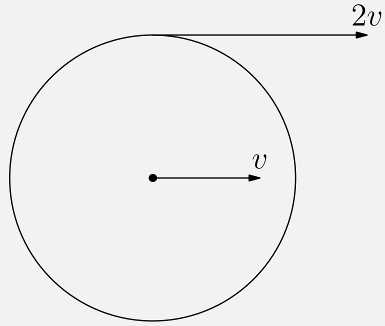
Let l_1 and l_2 be the distance from O to the top and bottom boards, respectively.

In fact, we have that

$$\frac{l_1}{l_2} = \frac{|v_1|}{|v_2|}$$

By the properties of the instantaneous axis of rotation, we know that all points with speed $|v_1|$ lie on a circle centered at O with radius l_1, and all points with speed $|v_2|$ lie on a circle centered at O with radius l_2.

pr 24.



As the wheel is rolling, we have that $\omega = \frac{v}{R}$.

The speed of the highest point in the lab frame is

$$v + \omega R = 2v$$

Therefore, we find that the centripetal force at the highest point is

$$a_c = \frac{(2v)^2}{r}$$

The speed of highest point in frame of wheel's centre is $\omega R = v$.

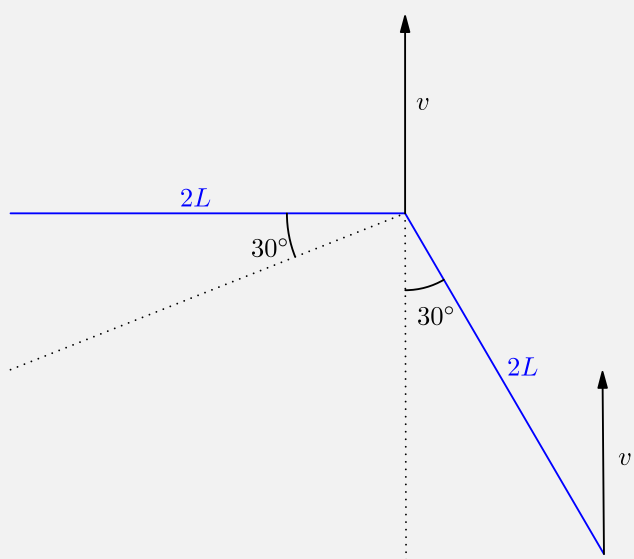
Therefore, the centripetal force in the wheels center is

$$a_c = \omega^2 R = \frac{v^2}{R}$$

As both frames are inertial frames,

$$\frac{v^2}{R} = \frac{4v^2}{r} \implies \boxed{r = 4R}$$

pr 25. **a)**



First we can find that the horizontal projection of the acceleration is $\frac{v_0^2}{2l}$.

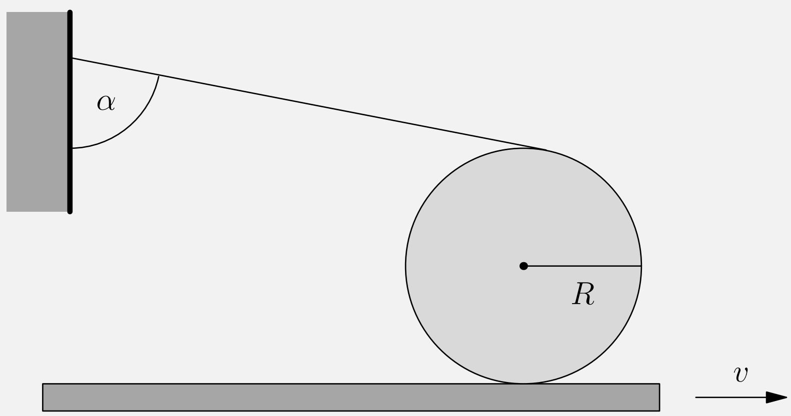
Then, since the velocity of the joint and the end are equal, there can be no centripetal acceleration, so the direction of the total acceleration must be perpendicular to the right rod, thus

$$\frac{v_0^2}{2l} = a\cos 30^\circ \implies a = \boxed{\frac{v_0^2}{\sqrt{3}l}}$$

b) If we take the frame of reference moving upward at v_0, it is essentially the same setup and thus

$$a = \boxed{\frac{v_0^2}{\sqrt{3}l}}$$

pr 26.



From constraints on the thread we can determine that

$$\omega R = v_{\mathrm{CM}} \sin\alpha$$

For a no-slipping condition, we have

$$v_{\mathrm{CM}} = v_0 - \omega R \Rightarrow v_{\mathrm{CM}} = \boxed{\frac{v_0}{1+\sin\alpha}}$$

Solution 2: We move into the frame moving left with velocity v

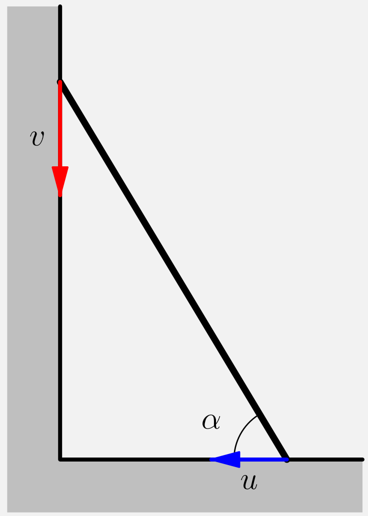
$$\sin\alpha = \frac{v_0 dt}{(v-v_0)dt} = \frac{v_0}{v-v_0}$$

$$v_0 = \frac{v\sin\alpha}{1+\sin\alpha}$$

Moving back into the reference frame of the wall, we get

$$v_0 = v - \frac{v\sin\alpha}{1+\sin\alpha} = \boxed{\frac{v}{1+\sin\alpha}}$$

pr 27.



Since the length of the rod is constant, we can consider the equation

$$x^2 + y^2 = L^2$$

Differentiating with respect to time, this gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$xu + yv = 0$$
$$u = -v\frac{y}{x} = \boxed{-v\tan\alpha}$$

Now, we find the acceleration

$$a = \frac{dv}{dt} = \frac{d(u\tan\alpha)}{dt}$$

Since u is constant, we have

$$a = u\frac{d\tan\alpha}{dt} = u\sec^2\alpha\frac{d\alpha}{dt}$$

The angular velocity $\frac{d\alpha}{dt}$ is simply

$$\frac{d\alpha}{dt} = \frac{u\cos\alpha + v\tan\alpha\sin\alpha}{L} = \frac{u}{L\cos\alpha}$$

This means that

$$a = u\sec^2\alpha\frac{d\alpha}{dt} = \boxed{\frac{u^2}{L\cos^3\alpha}}$$

5 Solutions to Miscellaneous Problems

This section will contain problem 28-34 of the handout. These problems are ones that are too unique or different to be placed under one singular category. Therefore, all of these problems got their own category by themselves, the Miscellaneous Problems Category. These problems are generally pretty interesting, but beware! You may have to take out a compass and straightedge for some of these problems!

pr 28. Consider a reference frame moving with speed u opposite to the direction of the cars.

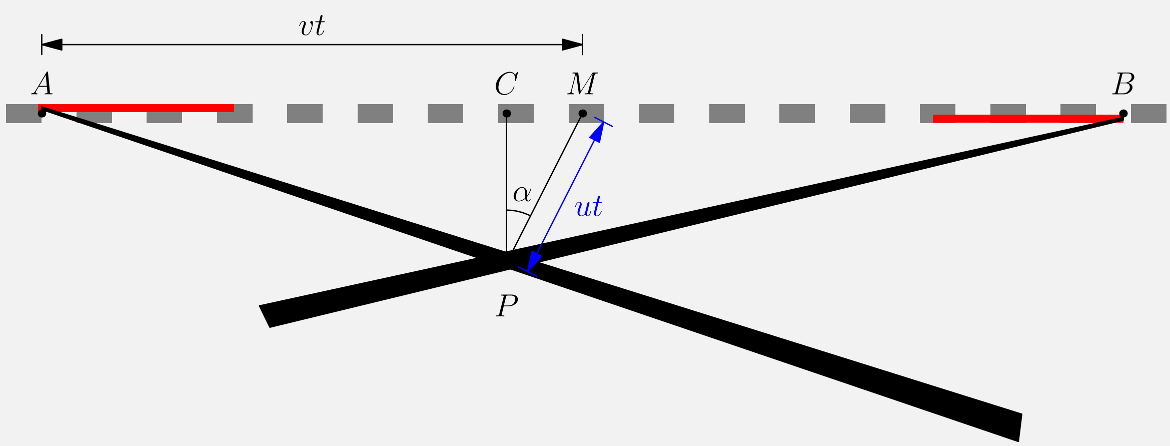
Cars at the end will be moving with speed $v+u$, and the distance between them will be $v\tau$.

Cars at the front will be moving at speed u (because they are stopped) and have a distance l between them.

Now we take ratios of speed to length and equate them to get

$$\frac{v\tau}{v+u} = \frac{l}{u}$$
$$v\tau u = vl + ul$$
$$v\tau u - ul = vl \implies u(v\tau - l) = vl$$
$$u = \boxed{\frac{v}{v\tau/l - 1} \approx 3.4 \text{ m/s}}$$

pr 29.



Consider the diagram above, where A is the engine of the left train, B is the engine of the right train, M is the midpoint of AB, P is the intersection of the smoke trails, and C is that intersection projected onto line AB.

Since both trains travel at $v = 50$ km/h, M is where the two trains met. In the time it took the trains to travel from M to A and B, the wind caused the smoke to drift from point M to point P.

Let us set an arbitrary scale of 1 cm = 100 km.

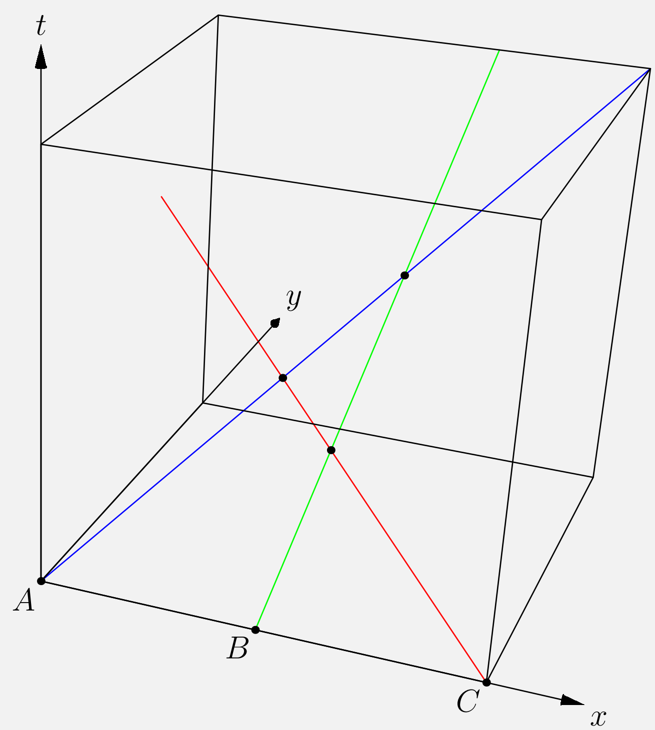
Measuring , we find that

$$
\begin{aligned}
|AB| &= 15.8 \text{ cm} \\
|AM| &= 7.9 \text{ cm} = vt \\
t &= 15.8 \text{ h} \\
|PM| &= 2.3 \text{ cm} = ut \\
u &= 14.55 \text{ km/h} \approx 15 \text{ km/h} \approx \boxed{4.2 \text{ m/s}} \\
\alpha &\approx \boxed{27°}
\end{aligned}
$$

Don't worry if you didn't set the same scale or measure the same distances as us; however, the ratios of your distances and the final answer should still remain (approximately) the same

pr 30. From the first two collisions, we can deduce that all three bodies lie on the same plane. For simplicity, let this be the x-y plane.

Additionally, we can assume that all three bodies lie on the x-axis, with body a at the origin O. We can then plot the motion of the bodies in three dimensions (with time as the third dimension) as follows:



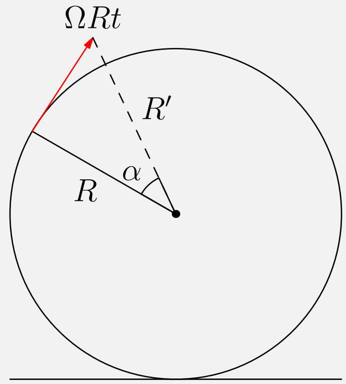
Since body a collides with body b, both the trajectories of a and b must lie on some unique plane $\mathcal{P}$ in our 3-D plot.

Since body a collides with body b, both the trajectories of a and c must lie on some unique plane $\mathcal{P}'$ in our 3-D plot.

However, both $\mathcal{P}$ and $\mathcal{P}'$ contain the trajectory of A and the x-axis, so they must be the same plane.

Therefore, $\boxed{\text{yes}}$, b and c would collide if a is missing.

pr 31.



In the free-falling frame, all the particles move with constant velocities; each particle had initial velocity equal to the wheel's velocity at the releasing point, i.e. tangential to the wheel and equal by modulus to ΩR. Hence the ensemble of particle expands as a circle, the radius of which can be calculated from the Pythagorean theorem.

$$R'^2 = R^2 + \Omega^2 R^2 t^2 \implies R' = R\sqrt{1+\Omega^2 t^2}$$

In the lab frame, the centre of the circle performs a free fall $d = \frac{1}{2}gt^2 - R$. A droplet reaching the point A corresponds to the expanding circle touching the ground. Therefore, setting $R' = d$ gives us

$$R\sqrt{1+\Omega^2 t^2} = \frac{1}{2}gt^2 - R$$
$$\frac{1}{4}g^2t^2 - gRt^2 + R^2 = R^2 + R^2\Omega^2 t^2$$
$$\frac{1}{4}g^2t^2 - gR = R^2\Omega^2$$
$$t^2 = \frac{4\left(\Omega^2 R^2 + gR\right)}{g^2} \implies \boxed{t = 2\sqrt{\frac{R}{g}\left(1 + \frac{R\Omega^2}{g}\right)}}$$

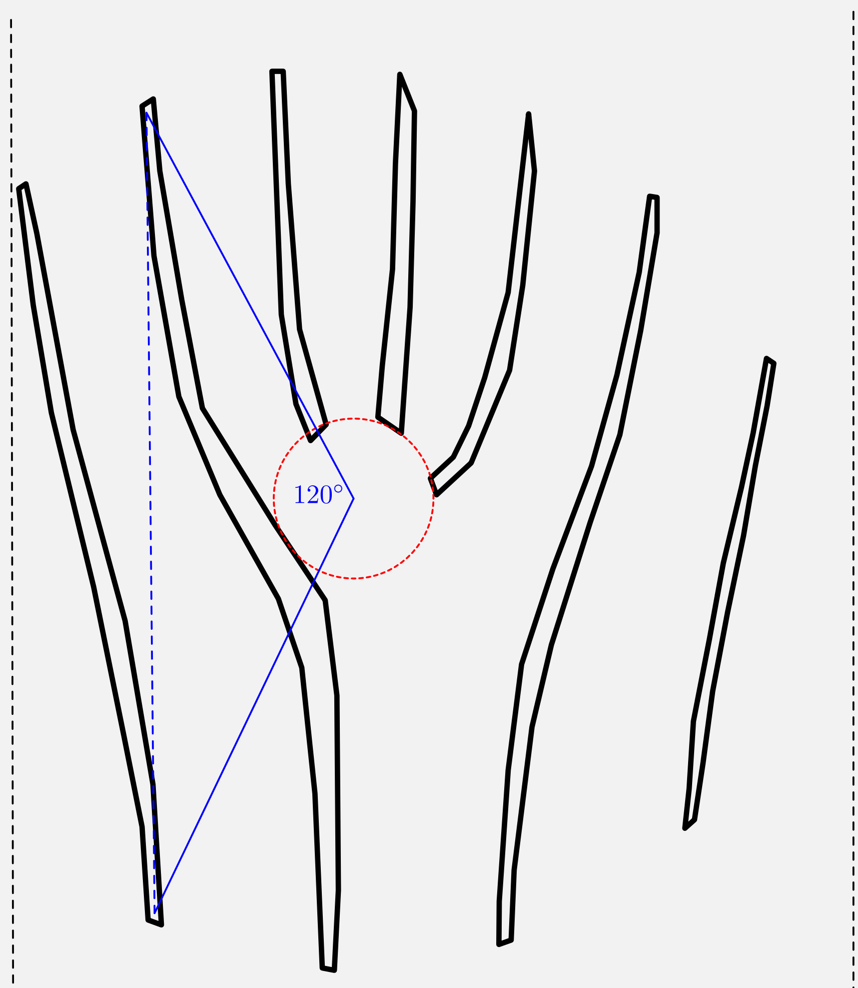
We can also tell from the given diagram that

$$\alpha = \arctan\left(\frac{\Omega Rt}{R}\right) \implies \boxed{\alpha = \arctan\left(\Omega t\right)}$$

pr 32. Let us make the assumption that the width of each propeller is much larger than a single pixel and the speed of the propellers aren't too fast. The reason we need to make this assumption is that this will prevent a propeller from being missed completely.

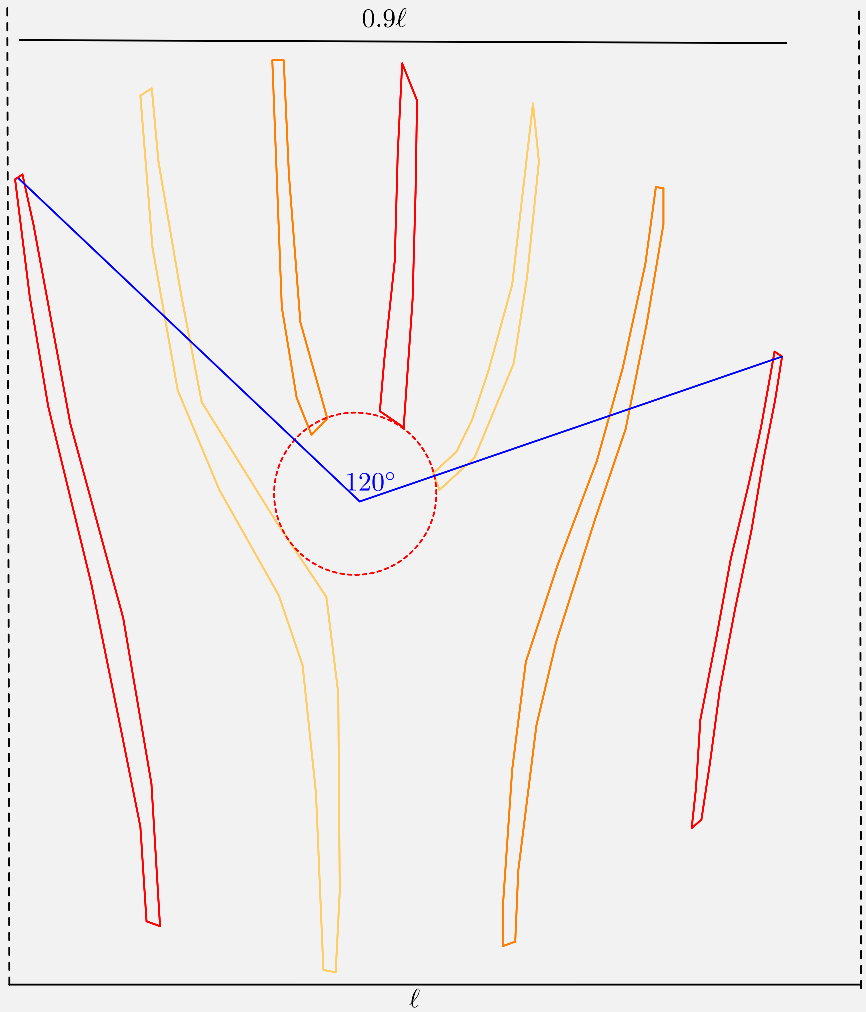
We can imagine a physical vertical line transverse the propellers as they spin. Every time a propeller passes this line, it will be caught on camera. We also make the assumption that the propellers are straight and equally spaced; this is what we should expect in a real life scenario.

a) To determine the direction of rotation of the blades, we note that the blades are more frequently spaced on the top side than the bottom side. This implies the blades have a higher speed relative to the moving line than the blades at the bottom. Although they have the same speed in the lab frame, they are moving in two separate directions. In order for the relative speed to be higher at the top, the blades need to be moving towards the scanning line, and thus, $\boxed{\text{counterclockwise}}$.



b) Refer to the diagram above. The figure shows one column with a maximum of 2 blades on top at a time. If there were 2 blades, only one blade should appear at a time. The blades themselves are marked in black on the figure. It is seen that the angle between the blades is near 120°. Using our assumption that they are equally spaced, this implies a $\boxed{\text{3-blade}}$ propeller.

Alternatively, we can label the seven blades at the top A, B, C, D, E, F, and G from left to right. From inspection, we know that the leftmost blade at the bottom must also be A. This comes from our assumption that A must be scanned before B, and so on. Furthermore, there couldn't be any blades in between or else it would have shown up in the scan. This follows that the second blade at the bottom must be B, the third must be C, and the fourth must be D. Due to similar reason, the rightmost propeller at the top must also be D. However, we have already stated that this propeller was G. Therefore, $G = D$. Continuing counterclockwise, we have $F = C$, then $E = B$ and $D = A = G$. With this systematic labeling, we have completed one loop and can confidently state that there are only three unique blades. In the diagram below, each unique blade has been coloured with a separate color.



c) We track the rotation of any arbitrary blade, the easiest being the red blade from its top left position to its top right position. This corresponds with a 600° rotation (one may be tempted to say a 240° rotation but note that the red blade can be seen when the scanning line is at the middle meaning at that point it has already passed the 240° mark.

Using a ruler, we can measure the horizontal distance between the starting and final location to be 0.9ℓ where ℓ is the width of the photo. If it takes 0.125 s to scan the entire photo, then it takes $0.9 \cdot 0.125 = 0.1125$ s for the blade to travel $1\frac{2}{3}$ revolutions. This corresponds with an angular frequency of:

$$f = \frac{5/3}{0.1125} = 14.8 \text{ Hz} \approx \boxed{15 \text{ Hz}}$$

pr 33. **i)** Assume that we have stuck paper strips with sinusoidal waves on both the combs such that the crests correspond with the teeth.

For the black comb, teeth exist when

$$\sin\left(\vec{k_1}\cdot\vec{r}\right) = 1,\ k_1 = \frac{2\pi}{\Delta l_1}$$

For the gray comb, teeth exist when

$$\sin\left(\vec{k_2}\cdot\vec{r} - \omega t\right) = 1,\ k_2 = \frac{2\pi}{\Delta l_2},\ \frac{\omega}{k_2} = v$$

($r = 0$ where the teeth coincide at $t = 0$, $k_1 > k_2$).

We will see dark regions when

$$\begin{aligned}\sin\left(\vec{k_1}\cdot\vec{r}\right)\sin\left(\vec{k_2}\cdot\vec{r} - \omega t\right) &= 1\\ \implies \vec{k_1}\cdot\vec{r} &= 2n\pi + \frac{\pi}{2},\\ \vec{k_2}\cdot\vec{r} - \omega t &= 2m\pi + \frac{\pi}{2}\\ \implies \left(\vec{k_1} - \vec{k_2}\right)\cdot\vec{r} &= 2k\pi - \omega t\end{aligned}$$

and

$$\begin{aligned}\vec{k_2}\cdot\vec{r} - \omega t &= 2m\pi + \frac{\pi}{2}\\ (\vec{k_1} - \vec{k_2})\cdot\vec{r} &= 2k\pi - \omega t\end{aligned}$$

following a constant phase (i.e. a constant k).

For the first part,

$$v_{\text{stripes}} = \frac{-\omega}{\left|\vec{k_1} - \vec{k_2}\right|} = \frac{\frac{-\omega}{k_2}}{\frac{k_1}{k_2} - 1} = \frac{-v}{\frac{l_2}{l_1} - 1} = -7v = \boxed{-7\text{ cm/s}}$$

Solution 2 to i) Let Δl_1 be the gap between the teeth of the black comb and let $\Delta l_2 =$ be the gap between the teeth of the gray comb.

From the gap between 2 transparent spaces in the given image we can observe that

$$7\Delta l_2 = 8\Delta l_1$$

If we observe the teeth of both the combs just before where they coincide, the gap between their centres will be

$$\Delta l_2 - \Delta l_1 = \frac{8}{7}\Delta l_1 - l_1 = \frac{\Delta l_1}{7}$$

When this gap is covered, the transparent, and hence dark, stripes move by an entire black comb's gap (i.e. Δl_1) backwards

$$\therefore v_{\text{stripes}} = -\frac{\Delta l_1}{\frac{\Delta l_1}{7v}} = -7v = \boxed{-7\text{ cm/s}}$$

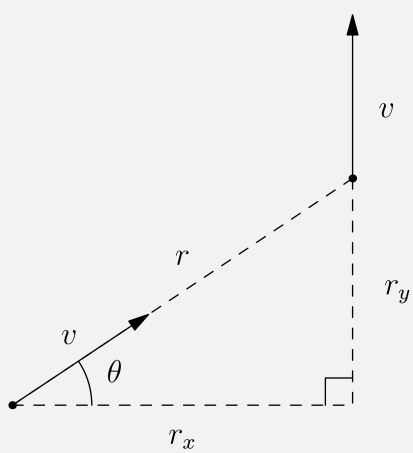
---

ii) For the second part,

$$\left|\vec{k_1} - \vec{k_2}\right| \approx \sqrt{(k_2\alpha)^2 + (k_1 - k_2)^2}$$

$$\begin{aligned}
\therefore v_{\text{stripes}} &= \frac{-\omega}{\left|\vec{k_1} - \vec{k_2}\right|} \\
&\approx \frac{\frac{-\omega}{k_2}}{\sqrt{\alpha^2 + \left(\frac{k_1}{k_2} - 1\right)^2}} \\
&= \frac{-v}{\sqrt{\alpha^2 + \frac{1}{49}}} = \boxed{-5.73 \text{ cm/s}}
\end{aligned}$$

pr 34. Consider the moment when the angle of the dog-fox line relative to the horizontal is θ.



By splitting the velocity of the dog into is x and y components, we can see that

$$\frac{dr_x}{dt} = -v\cos\theta = -\frac{vr_x}{r}$$
$$\frac{dr_y}{dt} = v - v\sin\theta = v - \frac{vr_y}{r}$$

By the Pythagorean Theorem, we have that

$$r_x^2 + r_y^2 = r^2$$

Differentiating with respect to time, we see that

$$\begin{aligned}
2r_x\frac{dr_x}{dt} + 2r_y\frac{dr_y}{dt} &= 2r\frac{dr}{dt} \\
r_x\left(-\frac{vr_x}{r}\right) + r_y\left(v - \frac{vr_y}{r}\right) &= r\frac{dr}{dt} \\
vr_y - v\cdot\frac{r_x^2 + r_y^2}{r} &= r\frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{vr_y}{r} - v
\end{aligned}$$

Note that

$$\frac{dr}{dt} + \frac{dr_y}{dt} = \left(\frac{vr_y}{r} - v\right) + \left(v - \frac{vr_y}{r}\right) = 0$$

Integrating with respect to time, we see that $r + r_y = C$, where C is a constant.

Taking the initial conditions of the system we can see that $C = L$.

Since $\frac{dr}{dt}$ is always nonnegative, we can consider the limit as $t \to \infty$, at which point $r = r_y$. Therefore,

$$r_{\text{min}} = \boxed{\frac{L}{2}}$$

Solution 2: Notice that at any moment,

$$-\frac{dr}{dt} = v(1 - \cos\alpha) = \frac{dx}{dt}$$

This is because the relative velocity of the fox with respect to the fox at any moment along the horizontal or along the curve the dog follows is the same.

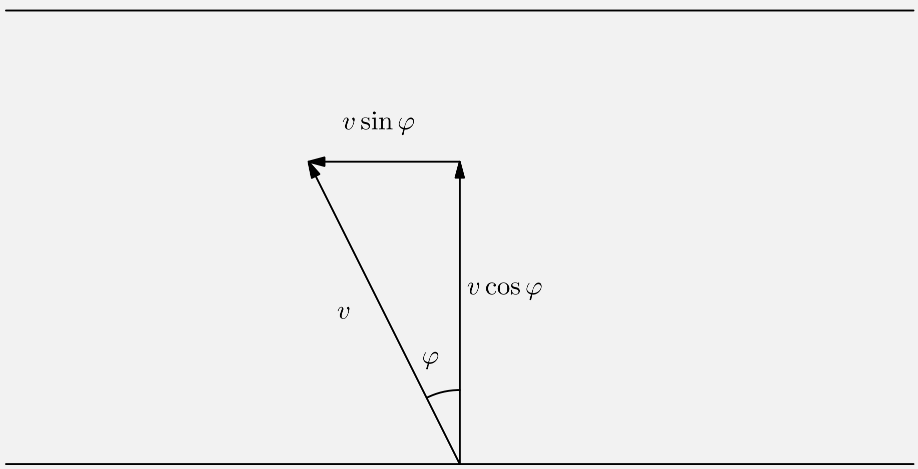
Hence, we have

$$\begin{aligned} -v\ dr &= v\ dx \\ dr &= -dx \\ \int_{\sqrt{L^2+0^2}}^{r_{\text{min}}} dr &= -\int_{r_{\text{min}}}^{0} dx \\ r_{\text{min}} - L &= -r_{\text{min}} \\ r_{\text{min}} &= \boxed{\frac{L}{2}} \end{aligned}$$

6 Solutions to Revision Problems

This section will contain problems 35-66 of the handout. Revision problems take concepts and ideas from earlier problems and place them in a new context. As a result, many of the problems in this section will seem familiar. This however, does not mean that all the problems in this section are easy. Some of the hardest questions originate in this section, which takes up nearly half of all the problems in the handout.

pr 35.



Since we have $u > v > v\sin\varphi\ \forall\ 90^\circ > \varphi \geq 0^\circ$, this means that the fast-flowing river carries the boy a lateral distance of $a = (u - v\sin\varphi)T$ (where T is the time it takes to reach the other shore) from point B.

Since the time taken to cross the river is simply $T = \dfrac{L}{v\cos\varphi}$, this means that

$$a = \frac{(u - v\sin\varphi)L}{v\cos\varphi} = \frac{(2 - \sin\varphi)L}{\cos\varphi}$$

where the last expression is achieved by substituting the given values.

Now, for minimising a, we have

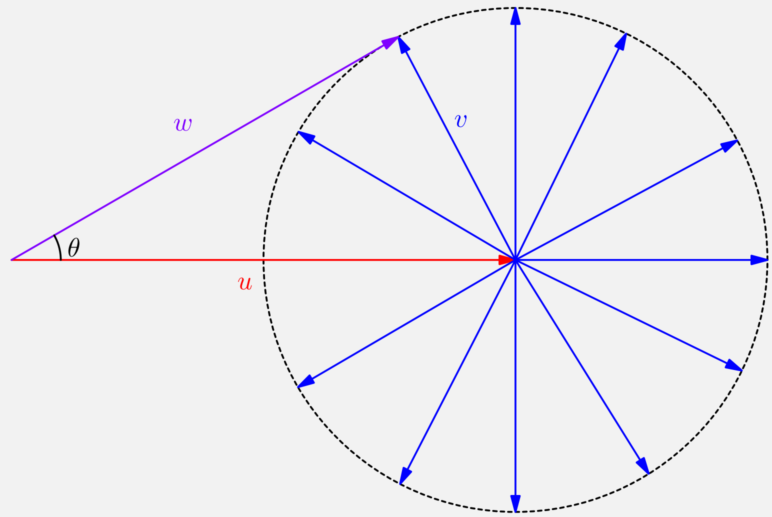
$$\frac{\mathrm{d}a}{\mathrm{d}\varphi} = L\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{2 - \sin\varphi}{\cos\varphi}\right) = \frac{L(2\sin\varphi - 1)}{\cos^2\varphi}$$

which clearly vanishes at $\sin\varphi = \frac{1}{2}$ or $\varphi = 30^\circ$.

Substituting this in the expression for a, we have

$$a_{\min} = \frac{L(2 - \frac{1}{2})}{\sqrt{3}/2} = \boxed{L\sqrt{3}}$$

Solution 2: Let the velocity of the boy with respect to the ground be $\vec{w} = \vec{u} + \vec{v}$. Since $\vec{u}$, the velocity of the water is fixed and the magnitude of $\vec{v}$ is fixed, we can only change the orientation.



The superposition of all possible orientations fill up a circle as shown. We want the velocity relative to the ground to make as large an angle as possible. To achieve this, w, u, and v must be the three sides of a right angled triangle such that:

$$w^2 = u^2 - v^2 \implies w = \sqrt{3} \text{ m/s}$$

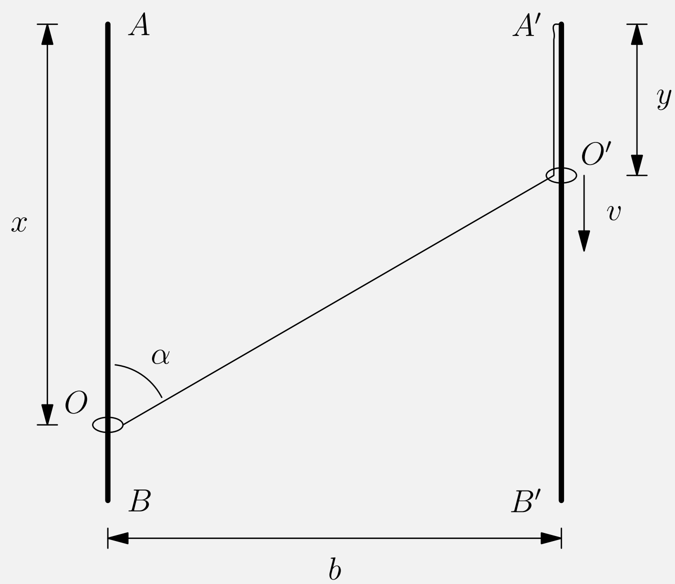
The time to cross is given by:

$$t = \frac{L}{w \sin\theta}$$

and the horizontal distance traveled during this time is:

$$a = w\cos\theta t = \frac{L}{\tan\theta} = \boxed{L\sqrt{3}}$$

pr 36. Let the length of the rope be L, let $AO = x$, and let $A'O' = y$ as shown in the diagram below.



Also let v_O be the velocity of ring O.

By the Pythagorean Theorem, we have that

$$(x-y)^2+b^2=(L-y)^2$$

Taking the derivative, we get that

$$\begin{aligned}\frac{d}{dt}(x-y)^2+\frac{d}{dt}b^2&=\frac{d}{dt}(L-y)^2\\ 2(x-y)\left(\frac{dx}{dt}-\frac{dy}{dt}\right)&=-2(L-y)\cdot\frac{dy}{dt}\\ (v_O-v)\left(\frac{x-y}{L-y}\right)&=-v\\ (v_O-v)\cos\alpha&=-v\\ v_O&=\boxed{v\left(1-\frac{1}{\cos\alpha}\right)}\end{aligned}$$

Note that this velocity is directed upwards along the rail, hence its negative value.

Note that

$$\begin{aligned}\cot\alpha&=\frac{x-y}{b}\\ -\frac{d\alpha}{dt}\csc^2\alpha&=b\cdot\frac{d}{dt}(x-y)\\ \frac{d\alpha}{dt}&=\frac{v}{b}\frac{\sin^2\alpha}{\cos\alpha}\end{aligned}$$

Since we have

$$\begin{aligned}\cot\alpha&=\frac{x-y}{b}\\ -\frac{d\alpha}{dt}\csc^2\alpha&=b\cdot\frac{d}{dt}(x-y)\\ \frac{d\alpha}{dt}&=\frac{v}{b}\frac{\sin^2\alpha}{\cos\alpha}\end{aligned}$$

We get the acceleration of point O as

$$a_O=\frac{dv_O}{dt}=v\sec\alpha\tan\alpha\frac{d\alpha}{dt}=\boxed{\frac{v^2}{b}\tan^3\alpha}$$

pr 37. The ball can escape the well if at the time at which the ball is at maximum height, it collides with the wall.

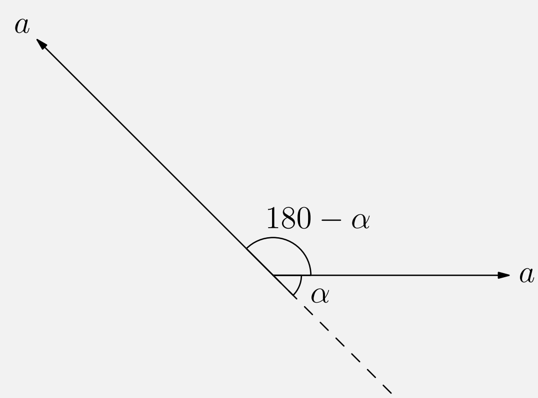
The time between any 2 collisions with the wall is $\dfrac{2R\cos\alpha}{v_0}$.

The total time of flight of 1 parabola is $2\sqrt{\frac{2H}{g}}$. This means that the required condition is then

$$2p\sqrt{\frac{2H}{g}} = \frac{2qR\cos\alpha}{v_0}, \text{ or}$$

$$\boxed{pv_0\sqrt{\frac{2H}{g}} = qR\cos\alpha, \quad p, q \in \mathbb{N}}$$

pr 38. The ball will have an acceleration a along the wedge to keep the length of the string constant. Therefore, by drawing the acceleration vectors, we can obtain the following diagram:



We then find from law of cosines that

$$\begin{aligned} a_{\text{net}} &= \sqrt{2a^2 - 2a^2\cos\alpha} \\ &= a\sqrt{2(1-\cos\alpha)} \\ &= a\sqrt{4\sin^2\left(\frac{\alpha}{2}\right)} \\ &= \boxed{2a\sin\left(\frac{\alpha}{2}\right)} \end{aligned}$$

pr 39. At that moment, the acceleration of the dog is equal to

$$\vec{a} = \frac{d\vec{v_2}}{dt} = v_2\frac{d\theta}{dt}\hat{r}$$

By Idea 37, we have the angular velocity of the relative position vector as

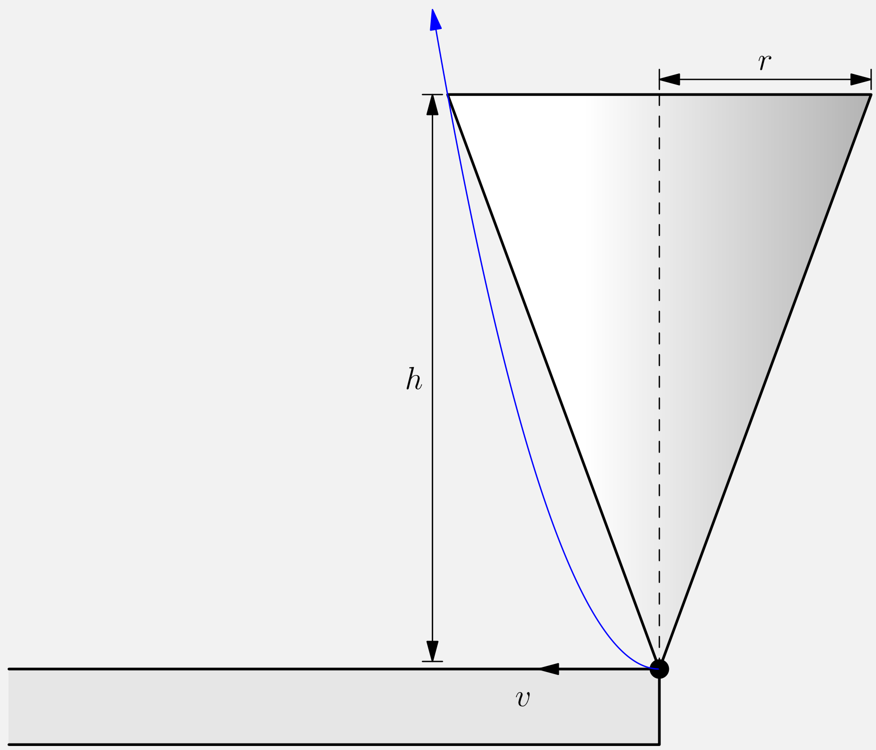
$$\frac{d\theta}{dt} = \frac{v_1\sin 90^\circ}{\ell} = \frac{v_1}{\ell}$$

Hence, from the two relations, the acceleration of the dog is

$$a = \boxed{\frac{v_1 v_2}{\ell}}$$

pr 40. Consider the moment when the cone is right on the edge and take the reference frame of the cone. In this reference frame the corner of the table is moving to the left with velocity v and has upward acceleration of g.

The corner of the table is essentially a projectile that is launched horizontally with velocity v and has to not touch the cone. The minimum value for v will then result in the trajectory where the corner touches the top of the cone.



The corner has to take time $\frac{r}{v}$ to reach the corner. In this time it has to travel

$$h = \frac{1}{2}g\left(\frac{r}{v}\right)^2$$

$$v = \boxed{r\sqrt{\frac{g}{2h}}}$$

pr 41. The rope will intuitively be something like a spiral.

Since the rope can go up to infinity, let's consider the last point instead. We set the point where all the shockwaves coincide at the origin and we use polar coordinates since we are going to be dealing with distances.

The rope lies along the curve $r(\theta)$, where $r(0)$ is the last point to be ignited. It takes time $r(0)/c$ for the shockwave from the last ignition point to reach the origin. If we go back an angle $d\theta$ along the rope, then it takes time

$$\frac{r(d\theta)}{c} - \frac{ds}{v}$$

for that shockwave to reach the origin, where ds is the infinitesimal arc length. Note that

$$ds = \sqrt{r^2 d\theta^2 + dr^2}.$$

We can set up our differential equation from this knowledge. For each $r(\theta)$, we want

$$\frac{r(\theta + d\theta) - r(\theta)}{c} = \frac{ds}{v}$$

Divide both sides by $d\theta$ to get

$$r'/c = \sqrt{r^2 + r'^2}/v$$

Square both sides to get

$$\frac{r'^2}{c^2} = \frac{r^2}{v^2} + \frac{r'^2}{v^2}$$

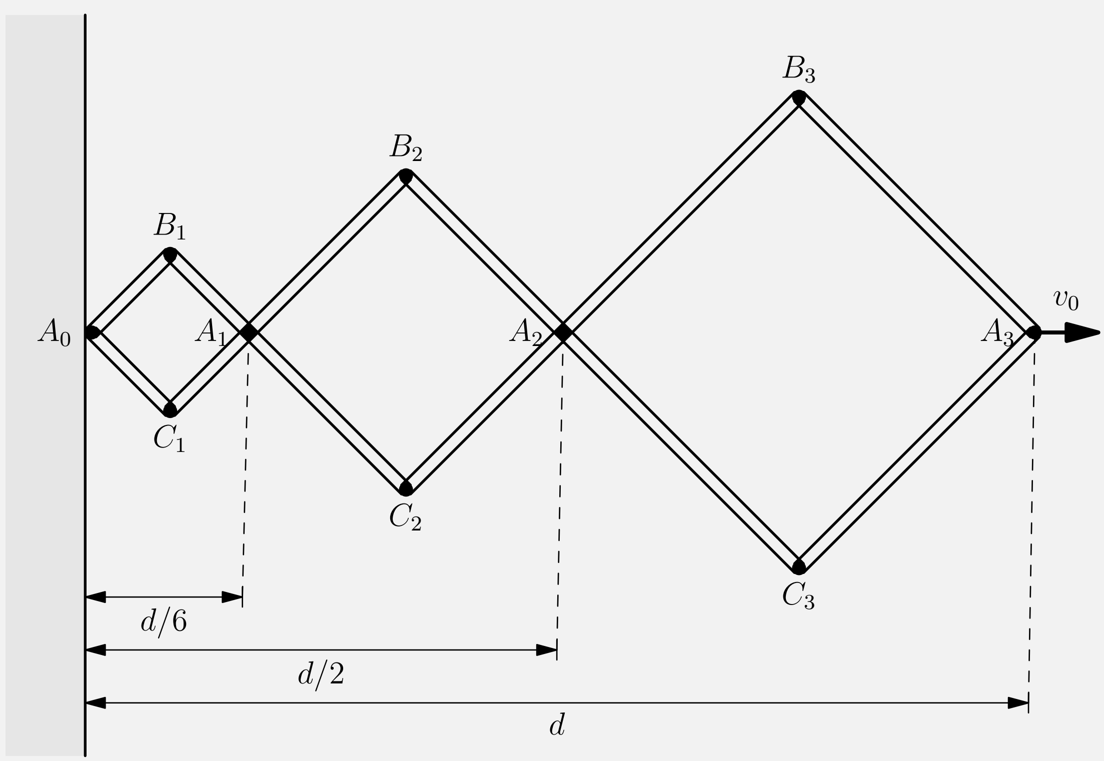
Combining like terms and simplifying gives us

$$\frac{dr}{d\theta} = r\sqrt{\frac{c^2}{v^2 - c^2}}$$

This is a separable differential equation which gives us solution

$$r = \boxed{Ce^{\sqrt{\frac{c^2}{v^2-c^2}}\theta}}$$

pr 42.



Since each of the rhombi deform at the same rate, we can apply homothetic transformations $\mathcal{H}(A_0, d/6)$

and $\mathcal{H}(A_0, d/2)$ along line A_0A_3 to see that

$$\boxed{v_{A_1} = \frac{v_0}{6},\ v_{A_2} = \frac{v_0}{2}}$$

Applying Idea 35 on lines A_1B_2 and A_2B_2, we see that v_{B_2} has a component of $\frac{v_0\cos(45°)}{2}$ along $\overrightarrow{B_2A_2}$ and a component of $\frac{v_0\cos(45°)}{6}$ along $\overrightarrow{A_1B_2}$.

Using the Pythagorean theorem, we see that

$$\begin{aligned} v_b &= \sqrt{\left(\frac{v_0\sqrt{2}}{4}\right)^2 + \left(\frac{v_0\sqrt{2}}{12}\right)^2} \\ &= \boxed{\frac{v_0\sqrt{5}}{6}} \end{aligned}$$

Since the horizontal velocities of all the points are constant, the acceleration of B_2 must be directed downwards.

Moving into the frame of reference of A_2, we can apply Idea 35 to see that $v_{B_2} = \frac{v_0\cos(45°)}{3}$ and is directed along $\overrightarrow{B_2A_1}$.

Thus, the centripetal acceleration of B_2 is

$$a_C = \frac{v_{B_2}^2}{2l} = \frac{v_0^2}{36l}$$

directed along $\overrightarrow{B_2A_2}$.

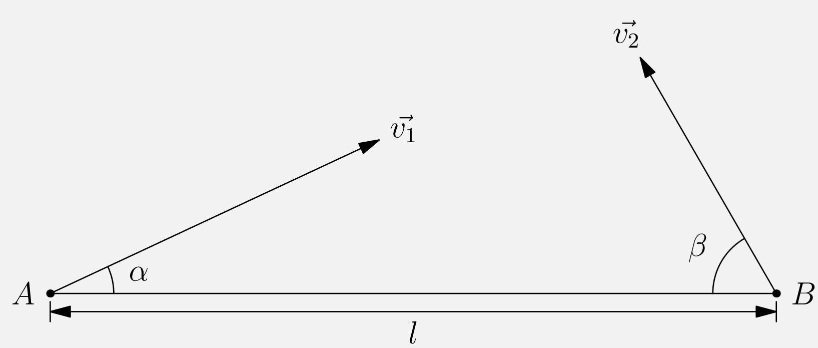
The tangential component must balance out the centripetal component and result in a net downwards acceleration, so

$$a_T = a_C = \frac{v_0^2}{36l}$$

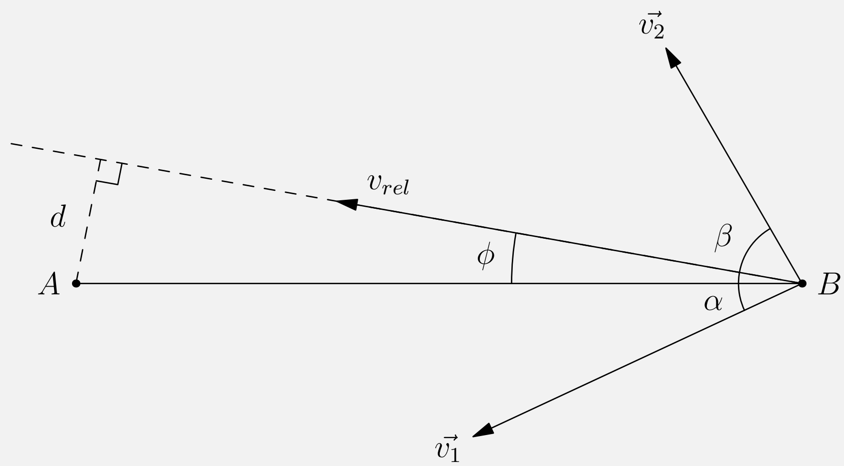
and

$$a_{B_2} = \sqrt{a_T^2 + a_C^2} = \boxed{\frac{v_0^2\sqrt{2}}{36l}}$$

pr 43. Consider the following setup:



Now, we move into the reference frame of the boat that departed from harbour A.



Since $\vec{v_{rel}} = \vec{v_2} - \vec{v_1}$, we can separate them into components to find that

$$\tan\phi = \frac{|v_1 \sin\alpha - v_2 \sin\beta|}{|v_1 \cos\alpha + v_2 \cos\beta|}$$

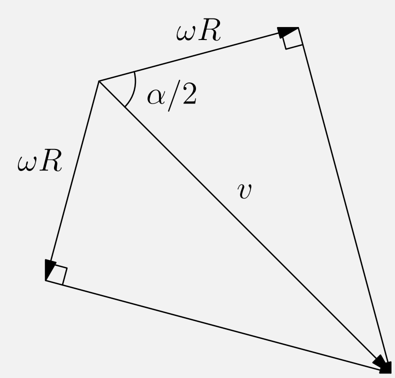
From this, we can find that

$$\sin\phi = \frac{|v_1 \sin\alpha - v_2 \sin\beta|}{\sqrt{v_1^2 + v_2^2 + 2v_1 v_2 \cos(\alpha+\beta)}}$$

We then have

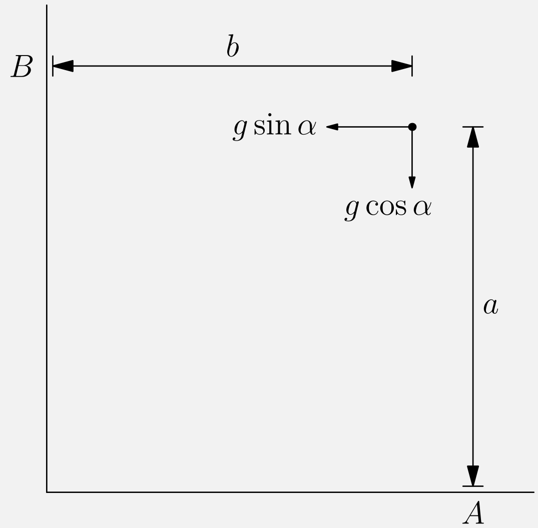
$$\begin{aligned} d &= l \sin\phi \\ &= \boxed{\frac{l \cdot |v_1 \sin\alpha - v_2 \sin\beta|}{\sqrt{v_1^2 + v_2^2 + 2v_1 v_2 \cos(\alpha+\beta)}}} \end{aligned}$$

pr 44. Because the ropes are constantly being unwinded at a rate of ωR , the disk has to move in the direction of the strings to keep the strings in tension the whole time. Using this information we can create a diagram



Now using trig we can see that $\cos\frac{\alpha}{2} = \frac{\omega R}{v}$ or $v = \boxed{\frac{\omega R}{\cos\frac{\alpha}{2}}}$.

pr 45. We tilt the plane by an angle α. This then gives us the following diagram



the ball is a distance b away from wall B and a distance a away from wall A. Let t_a and t_b be the jumping periods.

$$a = \frac{1}{2} g\cos\alpha \frac{t_a^2}{4} \implies t_a = 2\sqrt{\frac{2a}{g\cos\alpha}}$$

and

$$b = \frac{1}{2} g\sin\alpha \frac{t_b^2}{4} \implies t_b = 2\sqrt{\frac{2b}{g\sin\alpha}}.$$

On average, the ball bounces against A for each time it bounces against wall B in a ratio of

$$\frac{t_a}{t_b} = \boxed{\sqrt{\frac{a\tan\alpha}{b}}}$$

pr 46. First, note that $L = 2\pi rk$

By idea 21 (tension is perpendicular to direction of motion), the velocity v of the block remains constant throughout the motion.

Let l be the length of the portion of the string not in contact with the cylinder.

The angular velocity about the point of tangency with the cylinder is $\omega = \frac{v}{l}$.

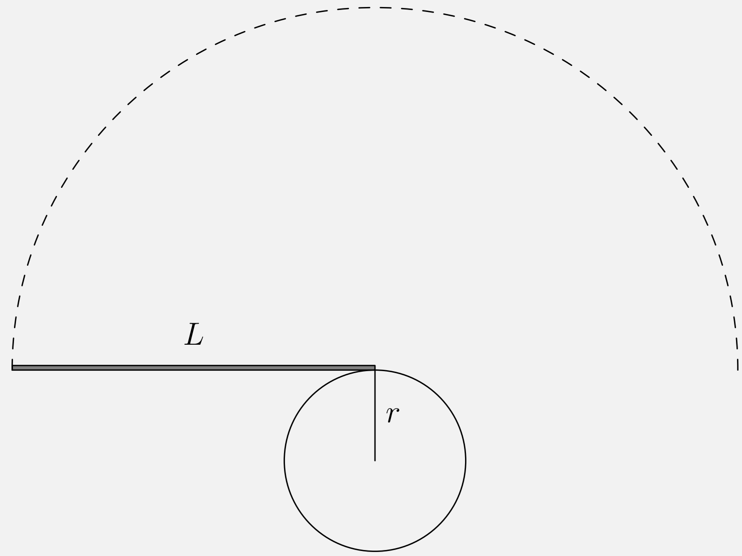
Note that $r\frac{d\theta}{dt} = \frac{dl}{dt}$.

$$r\omega = \frac{dl}{dt} \implies \frac{rv}{l} = \frac{dl}{dt} \implies rv = l\frac{dl}{dt}$$

$$rv = \frac{1}{2}\frac{d(l^2)}{dt} \implies l^2 = 2rvt \text{ since } l_0 = 0$$

$$t = \frac{l^2}{2rv} = \frac{2\pi^2 rk^2}{v}$$

Note that the string also completes an additional semicircle without changing length before starting to wrap back around again.

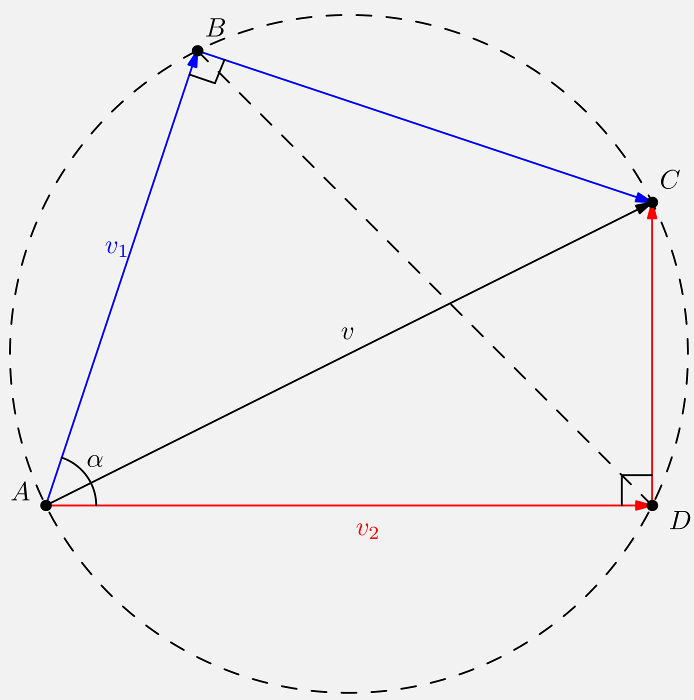


Therefore, our final time is $t = 2 \cdot \frac{2\pi^2 r k^2}{v} + \frac{\pi(2\pi r k)}{v}$, or

$$\boxed{t = \frac{2\pi^2 k r(2k+1)}{v}}$$

pr 47. If the velocity of the box is represented by the vector v_3, then the projection of v_3 onto v_1 must be equivalent to v_1 and the projection of v_3 onto v_2 must be equivalent to v_2.

Essentially this means that v_3 must be composed of the sum of v_1 and a vector perpendicular to v_1 and likewise for v_2. This gives us the following diagram:



We want to find the magnitude of AC. Since this quadrilateral is formed by two right triangles, it is a cyclic quadrilateral.

Since $\angle BDA$ and $\angle BCA$ are inscribed angles of the same arc, they are congruent. Using the law of sines, we get that

$$\frac{BD}{\sin\alpha} = \frac{AB}{\sin\angle BDA}, \ \frac{AB}{\sin\angle BCA} = AC$$

Since

$$\angle BDA = \angle BCA, \ \frac{BD}{\sin\alpha} = AC$$

Using the law of cosines, $BD = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha}$, so

$$AC = \boxed{\frac{\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha}}{\sin\alpha}}$$

Solution 2: Similar to above, but let us denote $\angle CAD = \theta$ such that we have $v_2 = v\cos\theta$ and $v_1 = v\cos(\alpha - \theta)$. We can rewrite v_1, using the cosine addition formula as:

$$v_1 = v\left(\cos\alpha\cos\theta - \sin\theta\sin\alpha\right) = v_2\cos\alpha - v\sin\theta\sin\alpha$$

We can solve for $\sin\theta$ to be:

$$\sin\theta = \frac{CD}{AD} = \frac{\sqrt{v^2 - v_2^2}}{v}$$

Substituting this in, we get:

$$v_1 = v_2\cos\alpha - v\left(\frac{\sqrt{v^2 - v_2^2}}{v}\right)$$

$$(v_2\cos\alpha - v_1)^2 = (v^2 - v_2^2)\sin^2\alpha$$

$$v_2^2\cos^2\alpha + v_1^2 - 2v_1v_2\cos\alpha = v^2\sin^2\alpha - v_2^2\sin^2\alpha$$

Rearranging and solving for v gives:

$$v = \boxed{\frac{\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha}}{\sin\alpha}}$$

pr 48. **i)** Let $\hat{j}$ be directed North and let $\hat{i}$ be directed East.

Then, the initial velocity is $v\,\hat{j}$ and the final velocity is $v\,\hat{i}$.

We then have that

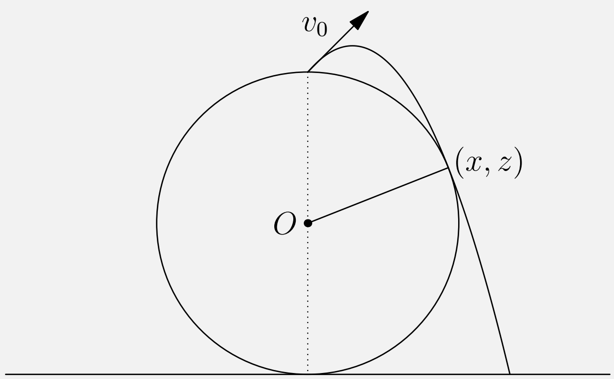
$$\begin{aligned} \Delta\vec{v} &= v\,\hat{i} + v\,\hat{j} \\ |\Delta\vec{v}| &= \sqrt{2}v \\ &= a\Delta t \end{aligned}$$

Since $a = \mu g$,

$$\Delta t = \boxed{\frac{\sqrt{2}v}{\mu g}}$$

ii) Since the optimal trajectory requires acceleration (in both magnitude and direction) to be constant, we see that the trajectory shape is simply that of a ball under free-fall (i.e. a $\boxed{\text{parabola}}$)

pr 49.



The equation for the trajectory of the ball (see problem 19) is

$$z = \frac{v_0^2}{2g} - \frac{g}{2v_0^2}x^2$$

By the equation of the circle,

$$x^2 + (z - R)^2 = R^2$$

$$x^2 + z^2 - 2xR = 0$$

$$x^2 + \left(\frac{v_0^2}{2g} - \frac{g}{2v_0^2}x^2\right)^2 - 2R\left(\frac{v_0^2}{2g} - \frac{g}{2v_0^2}x^2\right) = 0$$

$$\left(\frac{g^2}{4v_0^4}\right)x^4 + \left(\frac{v_0^2 - 2gR}{2v_0^2}\right)x^2 + \left(\frac{v_0^2 + 4gR}{4g^2}\right)v_0^2 = 0$$

We set the discriminant of the above equation to zero for the optimal trajectory to get

$$\left(\frac{v_0^2 - 2gR}{2v_0^2}\right)^2 - 4\left(\frac{v_0^2 + 4gR}{4g^2}\right)v_0^2 \cdot \frac{g^2}{4v_0^4} = 0$$

Solving this for v_0 gives

$$v_0 = \sqrt{\frac{gR}{2}}$$

By conservation of energy we have

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + 2mgR$$

$$v = \sqrt{v_0^2 + 4gR} = \boxed{\sqrt{\frac{9gR}{2}}}$$

[a]

[a]This problem was found in the 2012 IPhO

pr 50. We note that a time interval of 3 flashes occur for one full rotation of the disk.

This corresponds to a time interval of $3t = 0.3$ s

We also know that the distance between the top and the bottom of the lamp's trajectory is $2a = 9.0$ cm.

This corresponds (on the figure) to 6 squares of space.

The horizonal distance between two red flashes at the bottom of the trajectory is approximately 4 squares, which corresponds to a horizontal distance of 6.0 cm

Thus, the speed of the center of the disk is $v = \dfrac{6\text{ cm}}{0.3\text{ s}} = \boxed{20\text{ cm/s}}$

pr 51. **i)** Note that the total time is $\dfrac{2a}{v}$, so the cars can each only travel along 2 segments.

Since v_{dist} is never positive, the two cars are always approaching each other (aside from a brief instant at $t = \dfrac{a}{v}$).

From this, we note that both cars must end up at city O.

If the two cars started from cities A and B, then their initial v_{dist} would have been 0.

If the two cars started from cities B and C, then their initial v_{dist} would have been $v_0\sqrt{2}$.

This leaves only the option that $\boxed{\text{the two cars started from } A \text{ and } C \text{ and both ended at } O}$.

ii) Since the area under a velocity graph is just distance, the area under this velocity graph is the difference between the distance between the two cars at time $t = 0$ and time $t = \dfrac{a}{v}$.

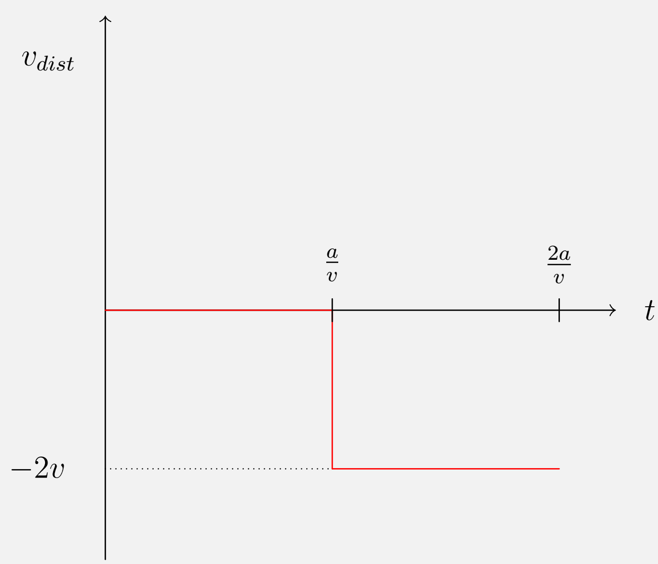
Thus, our answer is

$$2a - \sqrt{2}a = \boxed{(2-\sqrt{2})a}$$

iii) $A - B$:

For the first segment, the cars have the same velocity, so $v_{dist} = 0$.

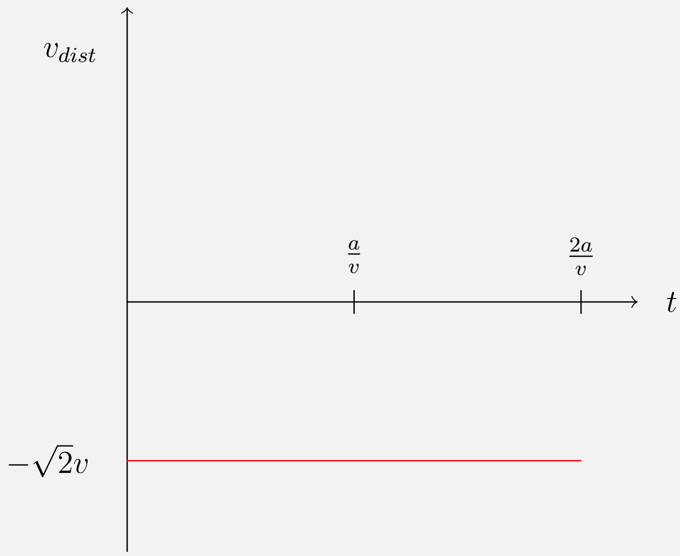
For the second segment, the cars face each other, so $v_{dist} = -2v$.



$B - C$:

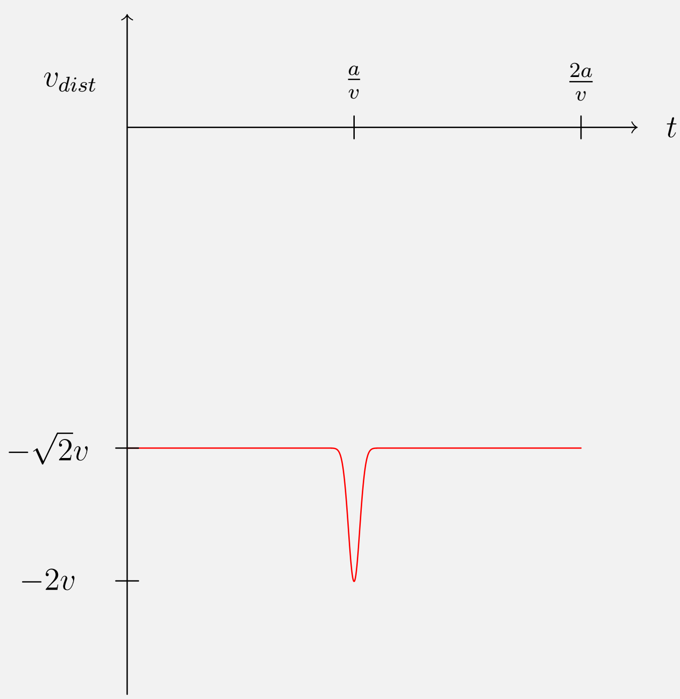
For the entire course of the motion, the velocity vectors of the two cars are perpendicular to each other and both cars approach each other, so

$$v_{dist} = -\sqrt{2}v$$

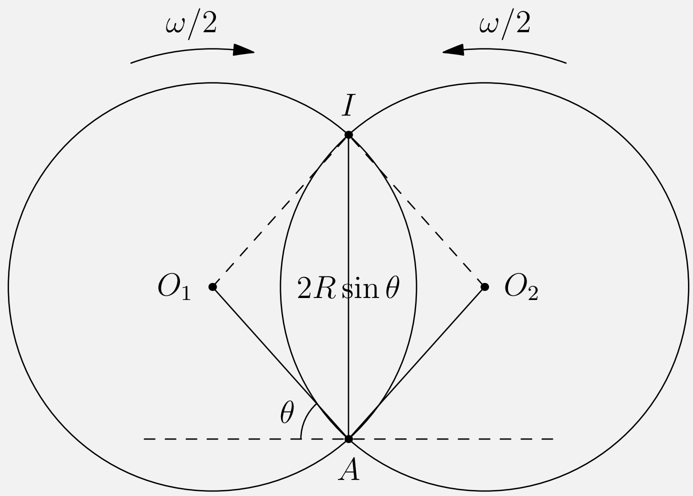


iv) $B - C$:

As they turn, the cars face each other and then turn to perpendicular again, so v_{dist} goes from $-\sqrt{2}v$ to $-2v$ and back to $-\sqrt{2}v$.



pr 52. If we shift into a reference frame rotating counterclockwise with angular velocity $\omega/2$ about point A, we can note that the intersection point I moves along a straight line in this reference frame.



We have that

$$\begin{aligned} AI &= 2R\sin\theta \\ \frac{d(AI)}{dt} &= 2R\cos\theta\cdot\frac{d\theta}{dt} \\ &= \omega R\cos\theta \end{aligned}$$

In the non-rotating reference frame, we have that

$$\begin{aligned} \vec{v}_{\text{ground}} &= \vec{v}_{\text{rotating}} + \frac{\vec{\omega}}{2}\times\vec{r} \\ &= \omega R\cos\theta\,\hat{j} - \frac{\omega}{2}\cdot 2R\sin\theta\,\hat{i} \\ &= \boxed{\omega R} \end{aligned}$$

Solution 2: Let the first ring be centered in $(0,0)$, so that its equation is $x^2+y^2=r^2$, and let the position of point O be (x_o, y_o).

We know that the second ring is centered at $(x_o + r\cos(\omega t), y_o + r\sin(\omega t))$, so its equation is

$$(x-(x_o+r\cos\omega t))^2 + (y-(y_o+r\sin\omega t))^2 = r^2$$

The two solutions to this system of equations are (x_o, y_o) and $(r\cos(\omega t), r\sin(\omega t))$, since $x_o^2+y_o^2=1$.

But then those are the coordinates of the second intersection point, and that means that the point moves in a circle of radius r with angular velocity ω and therefore its speed is constant and equal to $\boxed{\omega r}$.

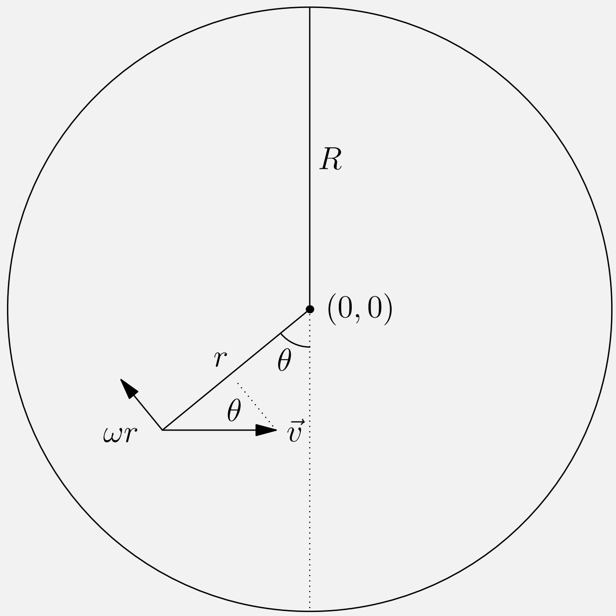
Solution 3: The point of intersection follows the arbitrary curve $\rho = 2rcos\theta$ with angular speed $\omega/2$ (we use ρ here to distinguish between the radius of the circle).

The speed of the point is

$$\begin{aligned} \frac{ds}{dt} &= \frac{ds}{d\theta}\cdot\frac{d\theta}{dt} \\ &= \frac{\omega}{2}\sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2} \\ &= \frac{\omega}{2}(2r)\sqrt{\cos^2\theta+\sin^2\theta} = \boxed{\omega r} \end{aligned}$$

pr 53. Note that a point on a spoke will appear motionless (i.e. sharp) if it's velocity is directed along (or parallel to) the spoke.

Let the horizontal velocity of the bike be $\vec{v}$, with $|\vec{v}| = \omega R$.



Since the tangential velocity is perpendicular to the radial distance vector, we need $\omega r = |\vec{v}|\cos\theta = \omega R\cos\theta$, or $r = R\cos\theta$

Labelling the center of the wheel as $(0,0)$ we can determine that the set of points can be represented as the parametric equation

$$x = -R\cos\theta\sin\theta,\ y = -R\cos^2\theta$$

Note that

$$x^2 + (y+R/2)^2 = R^2\cos^2\theta\sin^2\theta + R^2\cos^4\theta - R^2\cos^2\theta + \frac{R^2}{4} = \frac{R^2}{4}$$

Since we have $x^2 + (y+R/2)^2 = \left(\frac{R}{2}\right)^2$, the set of points on the spokes that appear sharp is described by $\boxed{\text{a circle of radius } \frac{R}{2} \text{ centered at } \frac{R}{2} \text{ below the center of the wheel}}$

pr 54. **a)** We notice that there is no image of the orange pulse, hence it must have taken place immediately before the shutter release. So the blue pulse is first, then red, then green, and finaly yellowl. As 4 pulses are recorded exposure time must be between 300 ms and 500 ms.

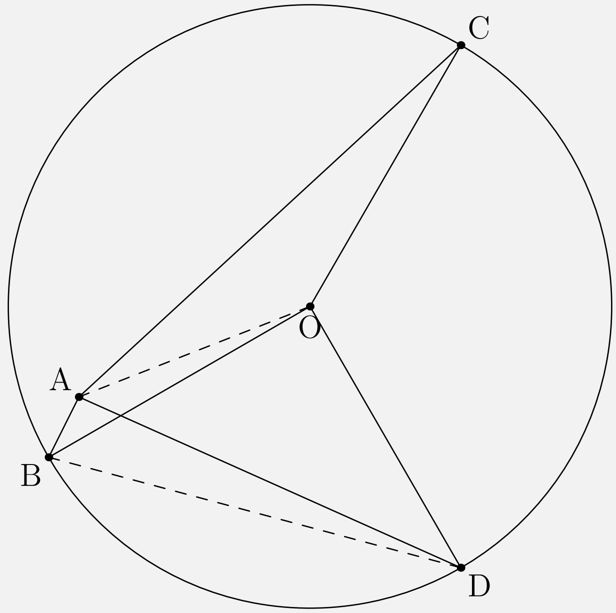
b) In the frame of disk's centre, the displacement vector $\vec{d}$ between neighbouring flashes has always the same modulus

$$d = 2R\sin(\omega\tau/2),$$

and neighbouring displacement vectors are always rotated by the same angle $\omega\tau$. In the lab frame, additional constant displacement vector $\vec{v}\tau$ is to be added due to the translational motion of the frame:

$$\vec{d'} = \vec{d} + \vec{v}\tau.$$

Because of that, if we bring all the displacement vectors to such positions that their starting points coincide, the endpoints will lie on a circle of radius d. So, we redraw the displacement vectors $\vec{br}$, $\vec{rg}$, and $\vec{gy}$ draw the circumcircle of the triangle formed by the endpoints of the vectors



From the figure we measure rotation angle

$$\omega\tau = 2.2 \text{ rad} \Rightarrow \omega = 22 \text{ rad/s}.$$

The constant displacement

$$a = v\tau = 6.55 \text{ cm} \Rightarrow v = 65.5 \text{ cm/s}$$

and the circle's radius

$$d = 2R\sin(\omega\tau/2) = 8.27 \text{ cm} \Rightarrow R = \boxed{4.6 \text{ cm}}.$$

pr 55. Let the distance between the minor gridlines provided be d.

We note that d is also (approximately) the diameter of the water jet.

We can use the points $(0,0)$, $(20d, -5.5d)$, and $(30d, -12.5d)$ to find the equation of the water trajectory, which we determine to be

$$y = -\frac{0.014}{d}x^2$$

The trajectory of the water is also given by the parametric equation $x = vt$, $y = -\frac{g}{2}t^2$, which gives

$$y = -\frac{g}{2v^2}x^2$$

Thus, we have

$$\frac{0.014}{d} = \frac{g}{2v^2} \implies v = \sqrt{\frac{gd}{0.028}}$$

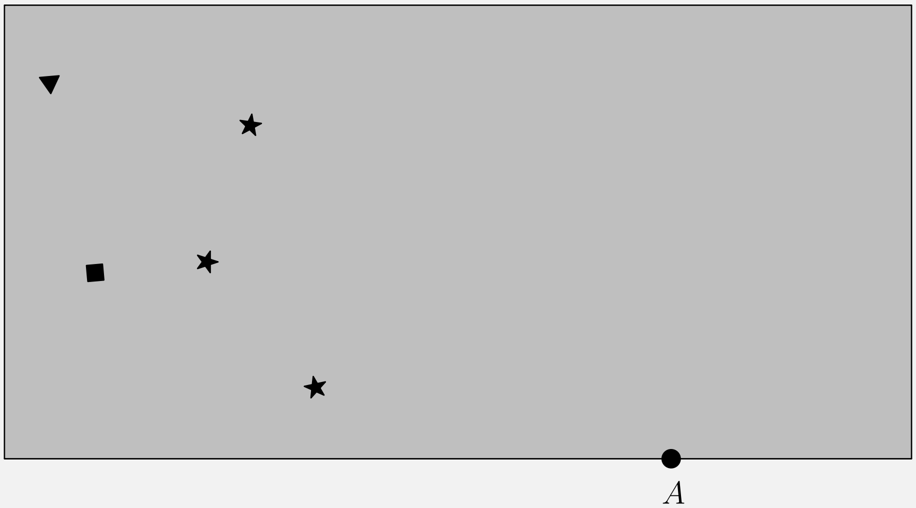
Since the pipe outflow rate must be the same as the bucket inflow rate,

$$\frac{\pi d^2 v}{4} = \frac{\pi d^2\sqrt{\frac{gd}{0.028}}}{4} = \frac{V}{t}$$

This equation gives

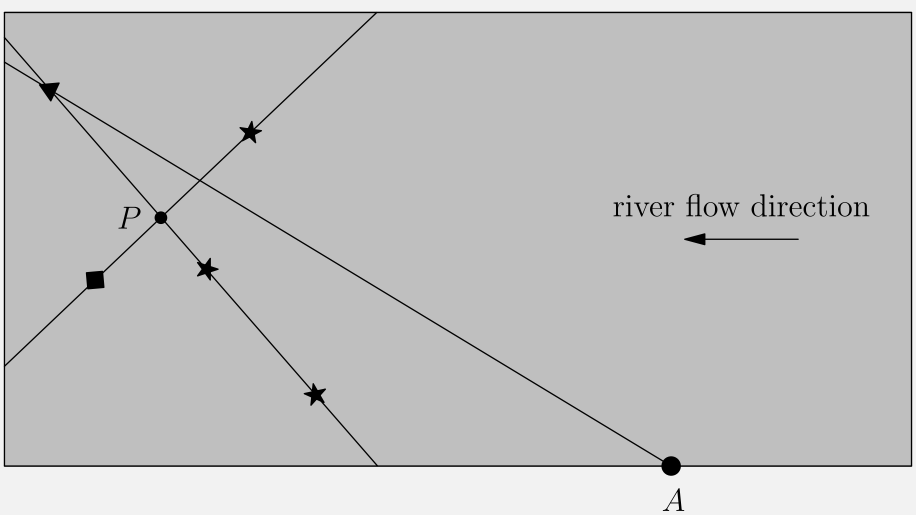
$$d = \left(\frac{4V}{\pi t}\sqrt{\frac{0.028}{g}}\right)^{2/5} = \boxed{1.03\text{ mm}}$$

pr 56. Consider the following diagram:

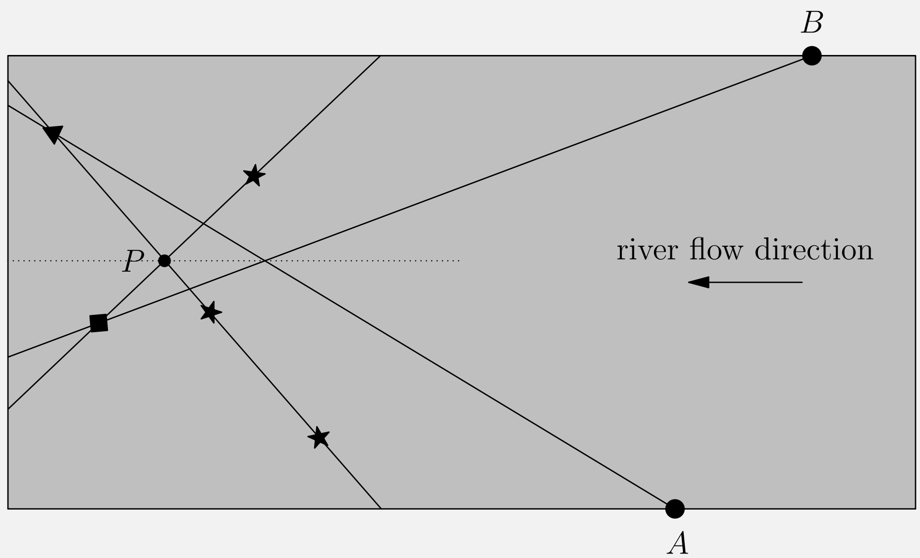


Since the velocity of both the boat and river are constant, the litter must lie on the same line as the boat it fell from, so we can deduce that the bottom two pieces of litter are from the boat marked with a triangle.

Thus, the boat marked with a triangle must have come from point A and the boat marked with a square must have come from the other side of the river (or else the other boat would not have been able to drop its litter above itself) Then, we can draw the following lines:



Since the river flow velocity is only directed horizontally, the boats meet at a point on the line parallel to the banks and passing through point P. Since we already have the path of the boat marked with a triangle, we can connect that intersection point with the square and extend it to the opposite bank to get the departure point of the boat marked with a square:



And so $\boxed{\text{Point } B}$ is the departure point of the second boat.

pr 57. Let d denote the common distance of separation between adjacent cars (it's the same for all lanes)

The flow rate (in cars/s) of the cars entering lane A is equal to $\frac{v_A}{d}$

The flow rate (in cars/s) of the cars entering lane B is equal to $\frac{v_B}{d}$

Note that $v_A = 3$ km/h and $v_B = 5$ km/h

By idea 39, the flow rate (in cars/s) of the cars entering lane C must be

$$\frac{v_A}{d} + \frac{v_B}{d}$$

This means that the velocity of the cars in lane C is simply

$$v_A + v_B = 8 \text{ km/h}$$

Thus, our final answer is

$$\frac{1 \text{ km}}{3 \text{ km/h}} + \frac{2 \text{ km}}{8 \text{ km/h}} = 0.583 \text{ h} = \boxed{35 \text{ min}}$$

pr 58. **a)** Consider a rectangular prism of length l, width w, and height h.

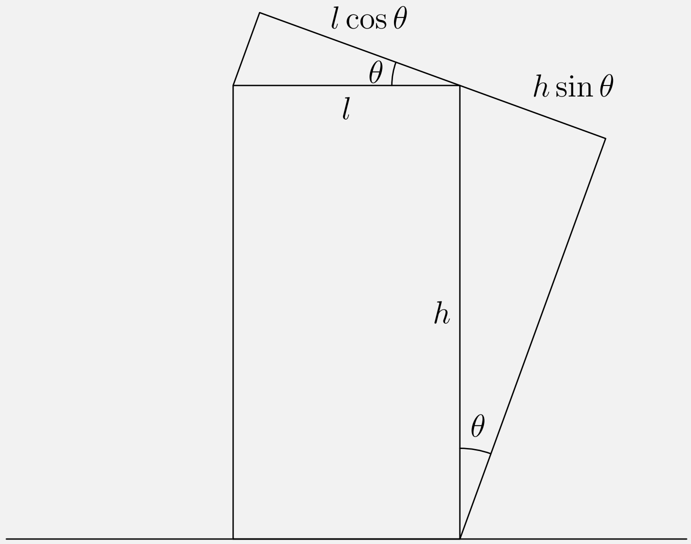
Assume that the volume of rain that the man receives per second is proportional to Av_r by some proportionality factor k, where A is the cross sectional area of where the rain strikes and where v_r is the velocity of the rain.

Let V be the critical volume of rain needed for the man to "get wet".

When the man is not moving, we find that

$$V = Akv_rt_1 = lwkv_rt_1$$

When the man is moving at speed v_m, in his frame of reference, the rain falls on him at an angle $\theta = \arctan\left(\frac{v_m}{v_r}\right)$ to the vertical at a speed of $\sqrt{v_r^2 + v_m^2}$, as shown in the following diagram:



Thus, we see that

$$V = Ak\sqrt{v_r^2 + v_m^2}t_2 = w(l\cos\theta + h\sin\theta)k\sqrt{v_r^2 + v_m^2}t_2$$

Since we have $\cos\theta = \frac{v_r}{\sqrt{v_r^2 + v_m^2}}$, $\sin\theta = \frac{v_m}{\sqrt{v_r^2 + v_m^2}}$, the expression is equivalent to

$$V = w(lv_r + hv_m)kt_2$$

We now have

$$lwkv_rt_1 = w(lv_r + hv_m)kt_2 \implies v_r = \frac{hv_mt_2}{l(t_1 - t_2)}$$

Plugging in $v_m = \frac{18}{3.6}$ m/s, $t_1 = 120$ s, $t_2 = 30$ s, we get that

$$v_r = \frac{5h}{3l} \text{ m/s}$$

This gives, for

$$v_m = 6 \text{ km/h}, t = \frac{lv_rt_1}{lv_r + hv_m} = \boxed{60 \text{ s}}$$

b) Consider a sphere of radius R.

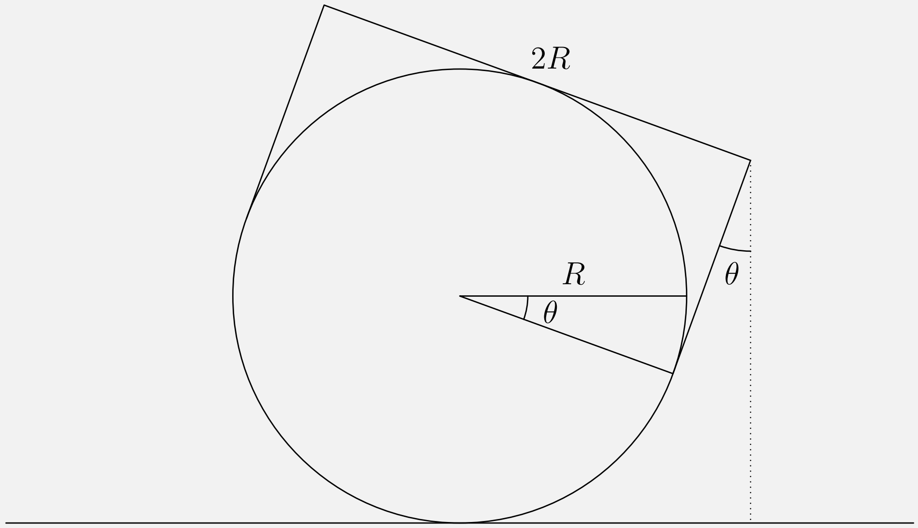
Assume that the volume of rain that the man receives per second is proportional to Av_r by some proportionality factor k, where A is the cross sectional area of where the rain strikes and where v_r is the velocity of the rain.

Let V be the critical volume of rain needed for the man to "get wet".

When the man is not moving, we find that

$$V = Akv_rt_1 = \pi R^2kv_rt_1$$

When the man is moving at speed v_m, in his frame of reference, the rain falls on him at an angle $\theta = \arctan\left(\frac{v_m}{v_r}\right)$ to the vertical at a speed of $\sqrt{v_r^2 + v_m^2}$, as shown in the following diagram:



Thus, we see that

$$V = Ak\sqrt{v_r^2 + v_m^2}t_2 = \pi R^2 k\sqrt{v_r^2 + v_m^2}t_2$$

We now have

$$\pi R^2 k v_r t_1 = \pi R^2 k\sqrt{v_r^2 + v_m^2}t_2 \implies v_r t_1 = \sqrt{v_r^2 + v_m^2}t_2$$

Solving the system of equations with $v_m = \frac{18}{3.6}$ m/s, $t_1 = 120$ s, $t_2 = 30$ s, we get

$$v_r = \frac{v_m t_2}{\sqrt{t_1^2 - t_2^2}} = \frac{\sqrt{15}}{3} \approx 1.29 \text{ m/s}$$

This gives, for $v_m = 6$ km/h,

$$t = 30\sqrt{6} \approx \boxed{73.5 \text{ s}}$$

pr 59. The length of the trails is defined by the time interval during which the droplet's image remains within the gap between the curtains.

This, in turn, is inversely proportional to that component of the image's relative velocity which is perpendicular to the curtain's edge.

In first case, the velocity of the curtains $\vec{v}$ and the velocity of the droplet's image $\vec{u}$ are parallel, in the second case antiparallel, and in the third case perpendicular. In the antiparallel case there are two possibilities as we don't know which is faster, the curtain or the image.

Thus, the time of appearance of a sufficient trail is $\frac{d}{|u \pm v|}$ and trace length $l = \frac{vd}{|u \pm v|}$

Let $u \geq v$; then

$$l_1 = \frac{vd}{u+v}, \quad l_2 = \frac{vd}{u-v}$$

By dividing the second equation by the first one, we get $\frac{u+v}{u-v} = \frac{l_2}{l_1} = \frac{5}{3}$, of which

$$3u + 3v = 5u - 5v \Rightarrow u = 4v$$

If the camera is in portrait position, then a sufficient image is present in the slit during d/u, so the trail length is

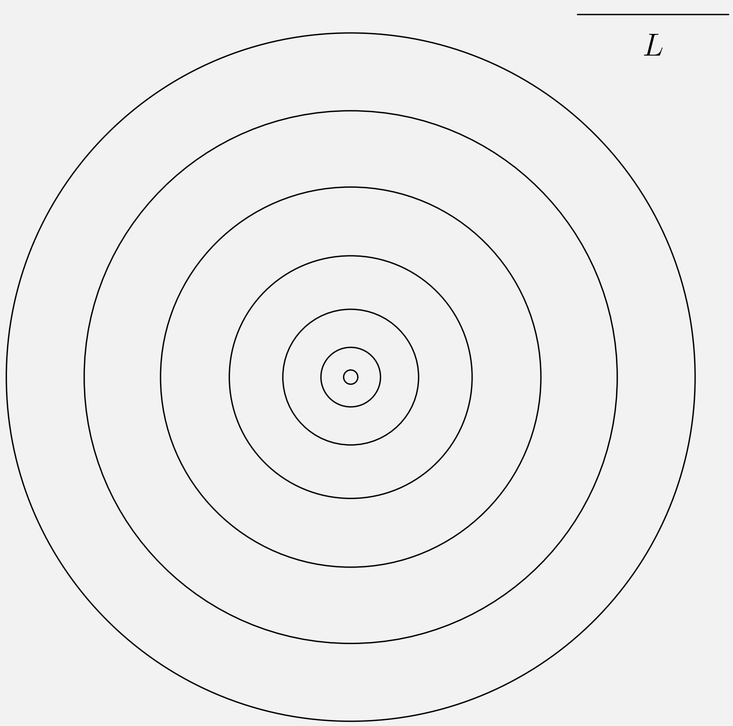
$$l_3 = \frac{vd}{u}$$

$$\frac{l_3}{l_1} = 1 + \frac{v}{u} = \frac{5}{4} \implies l_3 = \frac{5}{4} l_1 = \boxed{150 \text{ pixels}}$$

If $u < v$, only the second equation changes, $l_2 = \dfrac{vd}{v-u}$, so $3u + 3v = 5v - 5u$ and $u = v/4$, so

$$l_3 = 5l_1 = \boxed{600 \text{ pixels}}$$

pr 60.



Measuring the diameters of the wave fronts, we get approximately:

$$d_1 = 0.09L$$
$$d_2 = 0.39L$$
$$d_3 = 0.90L$$
$$d_4 = 1.61L$$
$$d_5 = 2.52L$$
$$d_6 = 3.53L$$
$$d_7 = 4.54L$$

Let the time between successive snapshots be Δt.

According to the question, the wavecrest initally moves out with acceleration $a = \dfrac{g}{\pi}$.

This gives

$$x = \frac{gt^2}{2\pi} \implies t = \sqrt{\frac{2\pi x}{g}}$$

Plugging in the first couple values of d, we find that

$$\Delta t \approx \sqrt{\frac{0.1\pi L}{g}}$$

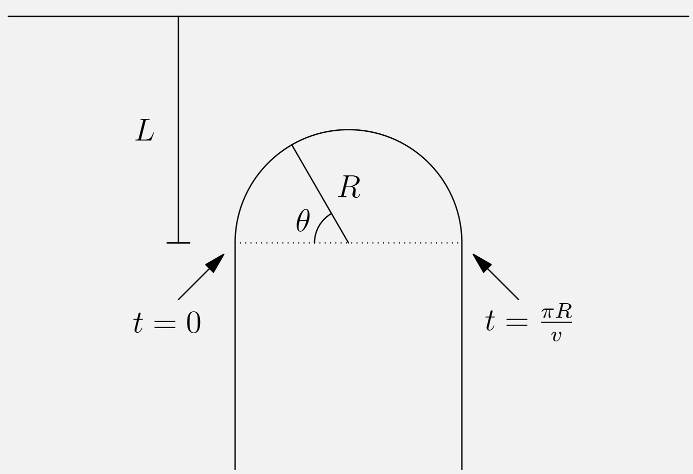
Also according to the question, the wavecrest approaches a terminal velocity $v_\infty = \sqrt{hg}$.

We can approximate this equation as

$$\frac{1.01L}{\Delta t} = \frac{1.01L}{\sqrt{\frac{0.1\pi L}{g}}} = \sqrt{hg}$$

Simplifying, this gives $\boxed{h \approx 3.2L}$.

pr 61. Consider the following diagram:



At time t, the boat has has swept out an angle

$$\theta = \frac{vt}{\pi R} \cdot \pi = \frac{tv}{R}$$

Thus, at time t, the boat is at a distance $L - R\sin\left(\frac{vt}{R}\right)$ away from the shore.

Therefore, at time t_2, the wavefront that was emitted at time t_1 is at a distance $L - R\sin\left(\frac{vt_1}{R}\right) - u(t_2 - t_1)$ away from the shore.

We wish to find the minimum possible t_2 such that the equation $L - R\sin\left(\frac{vt_1}{R}\right) - u(t_2 - t_1) = 0$ has a real solution for t_1.

To do this, we isolate t_2 to find that

$$t_2 = \frac{L - R\sin\left(\frac{vt_1}{R}\right)}{u} + t_1$$

Taking the derivative with respect to t_1 and setting it to zero, we get that $1 - \frac{v\cos\left(\frac{vt_1}{R}\right)}{u} = 0$, or

$$t_1 = \frac{R}{v}\arccos\left(\frac{u}{v}\right)$$

Since $u < v$, this is a valid solution, so we plug t_1 back into our expression for t_2 to find that

$$t_2 = \frac{L - R\sin\left(\arccos\left(\frac{u}{v}\right)\right)}{u} + \frac{R}{v}\arccos\left(\frac{u}{v}\right)$$

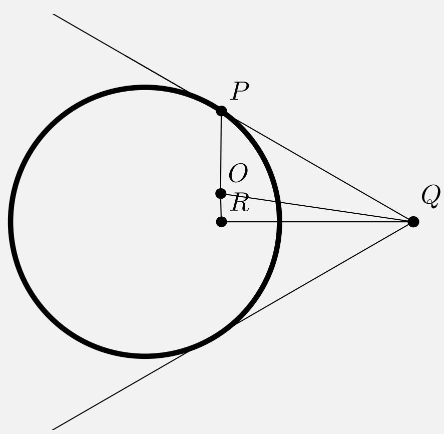
We have that

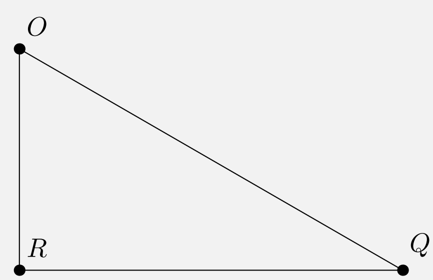
$$\sin\left(\arccos\left(\frac{u}{v}\right)\right) = \frac{\sqrt{v^2 - u^2}}{v}$$

Therefore,

$$\boxed{t_2 = \frac{R}{v}\arccos\left(\frac{u}{v}\right) + \frac{L}{u} + R\sqrt{\frac{1}{u^2} - \frac{1}{v^2}}}$$

pr 62. The waves move symmetrically about boat's trajectory in river's reference frame. In river's reference frame trajectory of boat is on angle bisector of the wave angle.





Direction of boat's speed relative to ground is from A to B. Let u be water speed. Then v, u and boat's speed relative to ground form a right triangle.

$$\sin(\angle RQO) = \frac{u}{v} \implies 0.26 = \frac{u}{v} \implies u = 1.82m/s$$

We also note that

$$OP = \omega t$$

and

$$OQ = vt$$

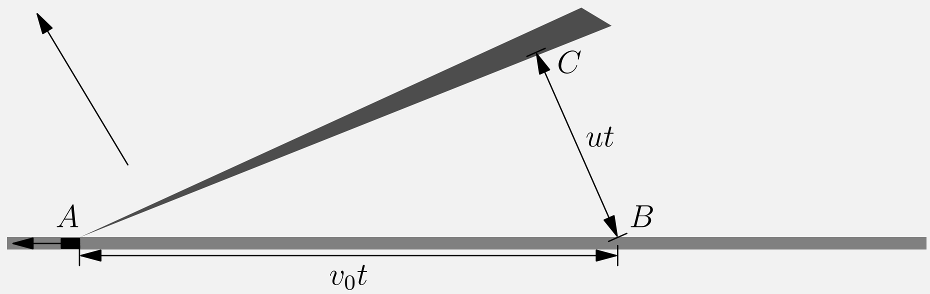
Therefore,

$$\frac{\omega}{v} = \frac{OP}{OQ} = 0.64 \implies \omega = \boxed{4.48 \text{ m/s}}$$

and

$$h = \frac{\omega^2}{g} = \boxed{2.05 \text{ m}}$$

pr 63. **a)** Let us examine an arbitrary point B on the road defined by the light gray color. The point of intersection of B with the gas trail represents C.



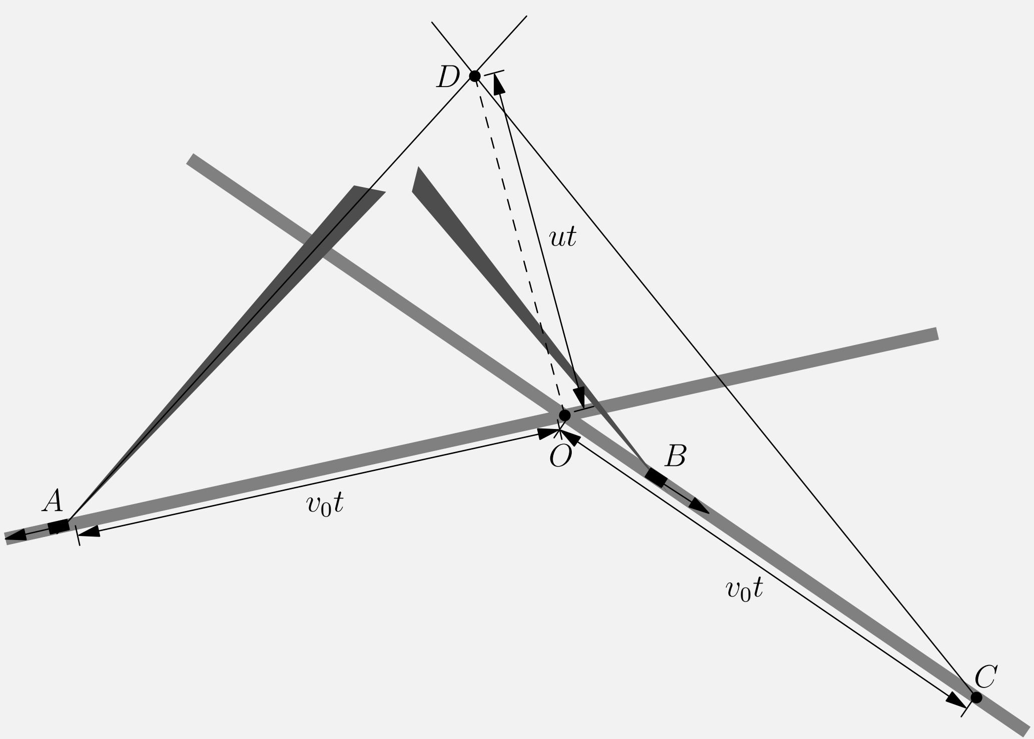
The smoke will have traveled a distance $|BC| = ut$, where u is the speed of the wind. In the time t, the tractor will have travelled a distance v_0t. By measuring we find that

$$|AB| = 7.1 \text{ cm}, \quad |BC| = 3.1 \text{ cm}.$$

We then find that

$$u = v_0\frac{ut}{v_0t} = v_0\frac{|BC|}{|AB|} = 30 \text{ km/h} \cdot \frac{3.1 \text{ cm}}{7.1 \text{ cm}} \approx \boxed{13 \text{ km/h}}$$

b) Label the initial point of the tractor on the left to be A, the origin to be O, and the initial point of the tractor on the right to be B.



We note that $|AO| = v_0 t$, thus if we extrapolate the line on the other road, we find that $|OC| = v_0 t$. Therefore, we find point C. The smoke trail of C can be found to be parallel to the smoke trail at B. If we extrapolate both smoke trails at A and C, we then find point D.

Therefore, we find that the meeting of the smoke trails of both tractors to be $|OD| = ut$.

Therefore,

$$u = v_0 \frac{ut}{v_0 t} = v_0 \frac{|OD|}{|OC|} = 30 \text{ km/h} \cdot \frac{4.9 \text{ cm}}{7.0 \text{ cm}} \approx \boxed{21 \text{ km/h}}$$

pr 64. In the freefalling frame, both balls move with a constant velocity. Let us define point Q as where both A and B are an equal distance from each other.

To find where Q is, we can connect line $|AB|$ and find the perpendicular bisector (which can be done using a compass and a straight-edge). We also draw a vertical line down P. Q is found where this line and $|AB|_\perp$ connects.

In the lab frame, Q is freefalling and thus after a time t it has fallen a distance $|PQ| = gt^2/2$. We know measure and see that

$$|PQ| = 8.6 \text{ cm}, \quad |AQ| = 26 \text{ cm}$$

we now find that

$$|PQ| = \frac{1}{2} g t^2 \implies t = 1.3 \text{ s}$$

and using $d = vt$ (because of a constant acceleration), we find that

$$|AQ| = 1.3v \implies v \approx \boxed{20 \text{ m/s}}$$

pr 65. We consider the boat in the frame of reference of the air. Since we have the wind measurements, we can find the displacement of the boat in the frame of reference of the air. We'll be taking south and east as positive.

During the first segment we have the wind blowing east, so in the frame of reference of the air, the boat is displaced east by

$$x_1 = 60 v_1 t_1$$

.

The 60 is there for a unit conversion of the time to seconds.

We do the same for the second leg, however we must account for the fact that the wind blows southeast, so there is both a southern displacement and an eastern displacement.

$$x_2 = 60 v_2 \cos(\pi/4) t_2 = 30\sqrt{2} v_2 t_2$$

$$y_2 = 60 v_2 \sin(\pi/4) t_2 = 30\sqrt{2} v_2 t_2$$

Similarly for the last leg, we have that

$$x_3 = 60 v_3 \cos(3\pi/4) t_3 = -30\sqrt{2} v_3 t_3$$

$$y_3 = 60 v_3 \cos(3\pi 4) t_3 = 30\sqrt{2} v_3 t_3$$

The total southern displacement is

$$y = 30\sqrt{2}v_2 t_2 + 30\sqrt{2}v_3 t_3 \approx 955\,\mathrm{m}$$

The total eastern displacement is

$$x = 60v_1 t_1 + 30\sqrt{2}v_2 t_2 - 30\sqrt{2}v_3 t_3 \approx 3018\,\mathrm{m}$$

Since the displacement in the lab frame is south 4000 m, we can find the displacement caused by the wind (in the lab frame) as $4000 - y$ south and $-x$ east. From there we just divide by the total time to find the wind speed as it's constant, so

$$v_{\text{wind}} = \frac{\sqrt{(4000-y)^2 + (-x)^2}}{60(t_1+t_2+t_3)} = \frac{\sqrt{3045^2 + 3018^2}}{60} \approx \boxed{12\,\mathrm{m/s}}$$

pr 66. Call the balls $a_1, a_2, a_3, \ldots a_N$.

When all the balls are released their comparative velocities will not change (i.e. if the velocity of a_i is initially less than that of a_j, then it will always be less than that of a_j at all times).

When two identical objects collide elastically, their velocities will switch. This is essentially the same as the two objects phasing through each other. So instead, let us assume that the balls never change direction and continue to phase through each other and instead we are trying to calculate the number of times a ball phases through another ball.

These ideas can be combined to conclude that each ball can at most only phase through each other ball exactly once, so the maximum would result from each ball phasing through every other ball exactly once for a final answer of

$$\boxed{\frac{N(N-1)}{2}}$$