Solutions to Problems in Relativity Handout by Siim Ainsar *

With detailed diagrams and walkthroughs

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Preface

This solutions manual came as a sort of pilot project on the online database artofproblemsolving.com. Kalda's problems never did have solutions written for them, thus the idea of creating solutions to these problems were an interesting idea. The majority of the solutions seen here were written on a private forum given to those who wanted to participate in making solutions. It was an interesting idea and the forum was able to have a great start. International users from India, the US, and Canada came into the forums to give ideas and methods to create what we see today.

Structure of The Solutions Manual

Each chapter in this solutions manual will be directed towards a section given in Kalda's mechanics handout. There are three major chapters: statics, dynamics, and revision problems. At the final page, we have an acknowledgments chapter. Each solution will be as detailed as possible and will usually contain a remark located in the footnote of each page. If you are stuck on a problem, and come here for reference, look at only the starting of the problem. Looking at the entire solution wastes the problem for you and ruins an opportunity for yourself to improve. Some solutions will have more than one solution to the problem.

Contact Us

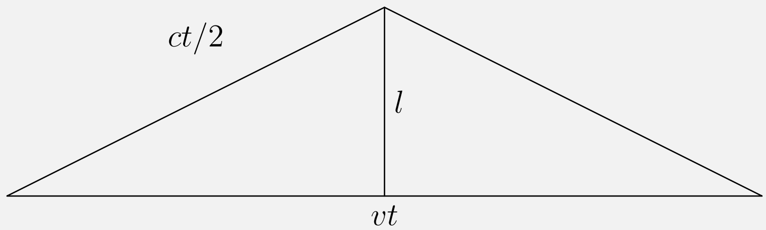
Despite editing, there is almost zero probability that there are *no* mistakes inside this book. If there are any mistakes amiss, you want to add a remark, have a unique solution, or know the source of a specific problem, then please contact us at solutionstokaldahandouts@gmail.com. Furthermore, please feel free to contact us at the same email if you are confused on a solution. Chances are that many others have the same confusion as you.

*All solutions will refer to the 1.2 English version given at https://www.ioc.ee/~kalda/ipho/meh_ENG2.pdf

1 Solutions to Statics Problems

This section will consist of the solutions to problems from problem 1-23 of the handout. Statics is typically the analysis of objects not in motion. However, objects travelling at constant velocity or with a uniform acceleration can be treated as a statics problem with a frame of reference change. This usually involves balancing forces, torques, and more to achieve equilibrium.

pr 1. The photon moves in a zig zag where it hits the ceiling then rebounds back down. If the distance between the floor and ceiling is l, and the total time is t, then we find that we have a triangle shaped as



Therefore, we find that we have a right triangle with legs l, and $vt/2$ and hypotenuse $ct/2$. Using Pythagorean theorem, we have that

$$\left(\frac{ct}{2}\right)^2 = \left(\frac{vt}{2}\right)^2 + l^2$$

Expanding both squares gives

$$\frac{c^2t^2}{4} + -\frac{v^2t^2}{4} = l^2$$

Simplifying gives

$$\boxed{t = \frac{4l^2}{c^2 - v^2}}$$

To verify this claim, we can consider what's happening in the first scenario. When the reference frame doesn't move with a velocity perpendicular to the velocity of the photon. The total distance the photon travels is $2l$, thus the total time in the first scenario is $t' = \frac{2l}{c}$. In this problem we found that

$$t^2 = \frac{4l^2}{c^2 - v^2}.$$

Substituting our value for t into this problem gives

$$t^2 = \frac{c^2t'^2}{c^2 - v^2} \implies t = t'\sqrt{\frac{c^2}{c^2 - v^2}} = \boxed{t'\frac{1}{\sqrt{1 - v^2/c^2}}}.$$

From fact 1, we see that this is the same as if the time interval between two events happening at a stationary point is t, then in a reference frame where the speed of the point is v the time interval is γt , where the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

pr 2. Imagine two people observing the the light clock. One person, person A, is standing inside the light clock, while another person, person B, stands outside of the light clock. In A's frame, the light clock has a total distance length l'. The round trip for the light beam in A's frame is

$$t_A = \frac{2l'}{c}.$$

In B's frame. the relative speed between the light and the clock in the first part of the trip (before the light beam hits the clock) is $c - v$. The relative speed after is $c + v$. Therefore, the toal time for round trip is then given by

$$t_B = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2}.$$

Substituting in $\frac{c}{c^2-v^2}$ in for $\frac{1}{c}\gamma^2$ gives us

$$t_B = \frac{2l}{c}\gamma^2.$$

From fact 1, we also know that the relationship between t_B and t_A is

$$t_b = \gamma t_A$$

Substituting in the results we found previously into this equation, gives us

$$\frac{2l}{c}\gamma^2 = \gamma\frac{2l'}{c} \implies \boxed{l = \frac{l'}{\gamma}}$$

pr 3. We see from the 3 dimensional distance formula that the distance between (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

We also know that from fact 1 that

$$t = t_2 - t_1 = \gamma\tau.$$

Squaring both sides gives us

$$t^2 = \frac{\gamma^2}{1 - v^2/c^2}$$
$$t^2(c^2 - v^2) = c^2\tau^2 \implies c^2t^2 - v^2t^2 = c^2\tau^2$$

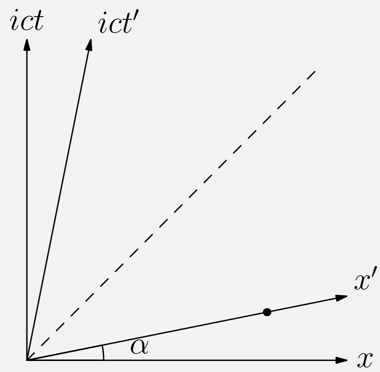
We know from basic kinematic relations that $v = dt \implies d^2 = v^2t^2$, therefore, by substituting this into our expression we find that

$$c^2t^2 - d^2 = c^2\tau^2$$
$$c^2\tau^2 = c^2t^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

Substituting our first expression for t gives

$$\boxed{c^2\tau^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}$$

pr 4.



Let us look at the dot given in the diagram. At this point, the dot has the coordinates

$$(x', ct') = (1, 0)$$

Lorentz transformations tells us that at this point

$$(x, ct) = (i\gamma, \beta\gamma)$$

where $\beta = v/c$. This then tells us that the angle betwee these two points are

$$\tan\alpha = \frac{ct}{ix} = \frac{v}{ic} \implies \alpha = \arctan\frac{\beta}{i}$$

$\boxed{\textbf{pr 5.}}$ We know that a Lorentz boost in the x-direction from standstill to a vertical velocity v corresponds to a rotation of the x and ict axes by an angle of

$$\alpha = \arctan\frac{v}{ic} = \arctan\frac{\beta}{i}.$$

To find $\cos\alpha$ and $\sin\alpha$, we first use the trigonometric identity

$$\tan^2\alpha + 1 = \sec^2\alpha \implies \cos\alpha = \frac{1}{\sqrt{\tan^2\alpha + 1}}$$

substituting our value of α in to $\tan^2\alpha$ gives

$$\cos\alpha = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma.$$

Now, since we now $\cos\alpha$, we can use the trig identity

$$\sin^2\alpha + \cos^2\alpha = 1$$

substituting in $\cos\alpha$ gives

$$\sin^2\alpha + \frac{1}{1-\beta^2} = 1$$
$$\implies \sin\alpha = \frac{\beta/i}{\sqrt{1-\beta^2}} \implies \sin\alpha \equiv \frac{\beta\gamma}{i}$$

$\boxed{\textbf{pr 6.}}$ We see that in the imaginary plane,

$$e^{\varphi} = \cos\alpha + i\sin\alpha$$
$$e^{-\varphi} = \cos\alpha - i\sin\alpha$$

In hyperbolic geometry,

$$\tan\varphi = \frac{\sinh\varphi}{\cosh\varphi} = \frac{e^{\alpha}-e^{-\alpha}}{e^{\alpha}+e^{-\alpha}} = \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha} = \frac{2i\sin\alpha}{2\cos\alpha}$$

From fact 7, we then find that

$$\tanh\varphi = \frac{\beta\gamma}{\gamma} = \beta.$$

In hyperbolic geometry,

$$\sinh\varphi = \frac{e^{\varphi}-e^{-\varphi}}{2} = \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{2}.$$

From fact 7, we then find that

$$\sinh\varphi = i\sin\alpha = \beta\gamma.$$

In hyperbolic geometry,

$$\cosh\varphi = \frac{e^{\varphi}+e^{-\varphi}}{2} = \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{2}$$

From fact 7, we find that

$$\cosh\varphi = \cos\alpha = \gamma$$

pr **7.**

pr **8.**

pr **9.**

pr 10. $\Delta x' = \gamma(\Delta x - v\Delta t)$ $\Delta t' = \gamma(\Delta t - \frac{\Delta x v}{c^2})$ $u' = \frac{u-v}{1-\frac{uv}{c^2}}$ If $u = c$, $u' = \frac{c-v}{1-\frac{v}{c}} = c$ (S' moves with $+v$ wrt S.)

pr **11.**

pr 12. If we place 2 clocks, synchronised in the stationary frame, near the positions of the events, we will observe the events when the clocks will show 0. In the moving frame, the clocks are moving with $-v$, so the clock to the right will be $\frac{\Delta x v}{c^2}$ ahead. Thus, the clock to the right will have ticked $\frac{\Delta x v}{c^2}$ between the 2 events $\equiv$ clocks showing 0. Since, time is dilated in the moving frame by the factor γ, the time gap, $\Delta t'$ in our frame will be:

$$\Delta t' = \frac{\gamma \Delta x v}{c^2}$$

Lorentz transformation approach: The two events correspond to $(x, 0), (x + \Delta x, 0)$ in the rest frame. Using $t' = \gamma(t - \frac{xv}{c^2})$

$$\Delta t' = \frac{\gamma \Delta x v}{c^2}$$

pr **13.**

pr **14.**

pr 15. We know that the position four vectors are given by

$$x^\mu = (ct, x, y, z).$$

From fact 15, the velocity four vector is given by (the particle is only moving in the x direction so y and z are constant)

$$v^\mu = \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\right)$$

where $\tau = \gamma t$. Considering small displacements of $\tau = \gamma t$ gives us

$$dt = \gamma d\tau \implies \gamma = \frac{dt}{d\tau}$$

Substituting this back into our four velocity vector gives

$$v^\mu = \left(\gamma c, \frac{dx}{d\tau}, 0, 0\right)$$

We have yet to prove that $\frac{dx}{d\tau} = \gamma v$. We can do this by looking back at the equation

$$\tau = \gamma dt.$$

Squaring both sides gives us

$$\tau^2 = \frac{\gamma^2}{1 - v^2/c^2} t^2$$

multiplying across gives

$$\tau^2(c^2 - v^2) = c^2 t^2 \implies c^2\tau^2 - v^2\tau^2 = c^2 t^2$$

We know from basic kinematic relations that $v = x\tau \implies v^2 = x^2\tau^2$, therefore, by substituting this into our expression we find that

$$c^2\tau^2 - x^2 = c^2 t^2.$$

Differentiating both sides with respect to τ yields in

$$c^2 \cdot 2\tau - 2x\frac{dx}{d\tau} = c^2 \cdot 2\gamma t$$

Moving terms

$$x\frac{dx}{d\tau} = c^2(\gamma t - \tau).$$

Dividing both sides by t,

$$v\frac{dx}{d\tau} = c^2\left(\gamma - \frac{1}{\gamma}\right) = c^2\left(\frac{\gamma^2 - 1}{\gamma}\right)$$

To get $\frac{dx}{d\tau}$, we first simplify the numerator

$$\gamma^2 - 1 = \frac{c^2}{c^2 - v^2} - 1 = \frac{v^2}{c^2 - v^2}$$

Going back into our previous equation, and dividing both sides by v gives us

$$\frac{dx}{d\tau} = \frac{1}{\gamma}\frac{c^2 v}{c^2 - v^2} = \frac{1}{\gamma}v\gamma^2 = \gamma v.$$

Going back and substituting γv gives us

$$\boxed{v^\mu = (\gamma c, \gamma v, 0, 0)}$$

pr 16. The Lorentz-invariant length of the four velocity-vector is given by

$$|v^\mu| = \sqrt{(v^t)^2 - (v^x)^2 - (v^y)^2 - (v^z)^2}$$

Substituting $v^y = v^z = 0$ gives us

$$|v^\mu| = \sqrt{(v^t)^2 - (v^x)^2}.$$

We remember from problem 15 that $v^t = \gamma c$ and $v^x = \gamma v$. Therefore, we substitute this to get

$$|v^\mu| = \sqrt{(\gamma c)^2 + (\gamma v)^2} \implies |v^\mu| = \gamma\sqrt{c^2 - v^2}$$

We note that because

$$\gamma = \frac{c}{\sqrt{c^2 - v^2}}$$

that

$$\sqrt{c^2 - v^2} = \frac{c}{\gamma}.$$

This then gives us the answer to be

$$|v^\mu| = \gamma\frac{c}{\gamma} = c$$

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pr 17. We know that

$$v^\mu = (\gamma c, \gamma, 0, 0)$$

from problem 16. Thus by differentiating with respect to τ we find that

$$a^\mu = \frac{dv^\mu}{d\tau} = \left(\frac{d}{d\tau}\gamma c, \frac{d}{d\tau}\gamma v, 0, 0\right)$$

From here we note

$$\frac{d}{d\tau}\gamma c = c\frac{d\gamma}{d\tau} = c\frac{d\gamma}{dv}\frac{dv}{d\tau}.$$

Since

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

then by differentiating this with respect to v will in turn give us

$$\frac{d\gamma}{dv} = -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot -\frac{2v}{c^2} = \frac{\gamma^3 v}{c^2}.$$

From here it follows that

$$\frac{d\gamma}{d\tau} = \frac{a\gamma^4 v}{c^2}.$$

We now look at the a_xth vector in the four acceleration vector. We see that

$$\frac{d}{d\tau}\gamma v = v\frac{d\gamma}{d\tau} + \gamma\frac{dv}{d\tau}$$

Since $\frac{dv}{d\tau} = a\gamma$, we can simplify the vector to be

$$\frac{d}{d\tau}\gamma v = v\frac{d\gamma}{d\tau} + a\gamma^2.$$

Also noting that from before that $\frac{d\gamma}{d\tau} = \frac{a\gamma^4 v}{c^2}$, we find that

$$v\frac{d\gamma}{d\tau} + \gamma\frac{dv}{d\tau} = \frac{a\gamma^4 v^2}{c^2} + a\gamma^2 = \gamma^4 a$$

Also noting that in the a_tth vector

$$c\frac{d\gamma}{d\tau} = c\frac{a\gamma^4 v}{c^2} = \beta\gamma^4 a.$$

This tells us that

$$\boxed{a^\mu = (\beta\gamma^4 a, \gamma^4 a, 0, 0)}$$

pr 18. From fact 21, we see that the kinetic energy is given by

$$E_k = (\gamma - 1)mc^2.$$

Replacing γ with $\frac{1}{\sqrt{1-v^2/c^2}}$ gives us

$$E_k = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) mc^2$$

Now, we use a taylor series approximation. We note that when $x \ll 1$, then

$$(1+x)^n = 1 + nx$$

holds true. Using this fact for γ gives us

$$\left(1 + \left(-\frac{v^2}{c^2}\right)\right)^{-1/2} = 1 + \frac{v^2}{2c^2}.$$

Substituting back in lets us get

$$E_k = \left(\left(1 + \frac{v^2}{2c^2}\right) - 1\right) mc^2 \equiv \frac{1}{2}mv^2$$

pr **19.**

pr **20.**

pr **21.**

pr **22.**

pr **23.**

pr 24. We know that from fact 1 that

$$t = t_2 - t_1 = \gamma\tau.$$

Squaring both sides gives us

$$t^2 = \frac{\gamma^2}{1 - v^2/c^2}$$
$$t^2(c^2 - v^2) = c^2\tau^2 \implies c^2t^2 - v^2t^2 = c^2\tau^2$$

We know from basic kinematic relations that $v = dt \implies d^2 = v^2t^2$, therefore, by substituting this into our expression (with $d^2 = 4\pi r^2$) we find that

$$c^2t^2 - 4\pi r^2 = c^2\tau^2 \implies \boxed{r = \frac{c\sqrt{t^2 - \tau^2}}{2\pi}}$$

pr 25. Kinematics tells us that

$$s = vt$$

Using fact 1, we find that

$$\boxed{s = \gamma v\tau}$$