local laplacian : $L = \int_\Omega \nabla N_i \nabla N_j \, d\Omega$

Quad

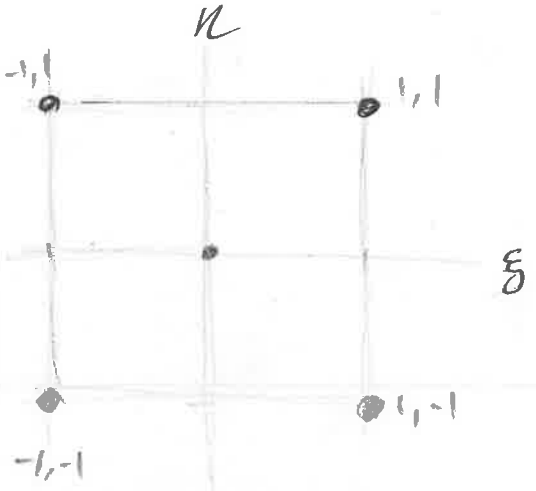
Where N_i is the basis function at the i^{th} point.

We rewrite the equation in complete form.

$$L = \int_\Omega (\nabla N)^T \cdot \nabla N \, d\Omega$$

Here N is the vector holding all entries of N_i, for x & y.

① We start on the reference element and we map back to the general element.



$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

$$L = \int_0^1 \int_0^1 (\nabla N(x,y))^T (\nabla N(x,y)) \, dx \, dy$$

With respect to η & ξ we can rewrite the above as:

$$L = \int_{-1}^1 \int_{-1}^1 (J^{-1} \nabla N(\xi,\eta))^T (J^{-1} \nabla N(\xi,\eta)) \, |J| \, d\xi \, d\eta$$

Where $J = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} dx & 0 \\ 0 & dy \end{bmatrix}$

*Assume $dx = dy = h$

$J = \frac{h}{2} I \rightarrow J^{-1} = \frac{2}{h} I$

$|J| = \frac{h^2}{4}$

$$L = (J^{-1})^2 |J| \int_{-1}^{1}\int_{-1}^{1} (\nabla N(\xi,\eta))^T (\nabla N(\xi,\eta))\, d\xi\, d\eta$$

$$L = \cancel{\frac{4}{h^2}\cdot\frac{h^2}{4}} \int_{-1}^{1}\int_{-1}^{1} (\nabla N(\xi,\eta))^T (\nabla N(\xi,\eta))\, d\xi\, d\eta$$

$$N = \frac{1}{4}\left[(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta)\right]$$

$$\nabla N = \nabla_{\xi\eta} N = \frac{1}{4}\begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix}\begin{bmatrix} N \end{bmatrix}$$

$$\nabla N = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}$$

$$L = \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} -(1-\eta) & -(1-\xi) \\ (1-\eta) & -(1+\xi) \\ (1+\eta) & (1+\xi) \\ -(1+\eta) & (1-\xi) \end{bmatrix} \cdot \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} d\xi\, d\eta$$

$$L = \frac{1}{16}\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$$

See next page for Calculations . . .

$$K_{11} = \int_{-1}^{1}\int_{-1}^{1} (1-\eta)^2 + (1-\xi)^2 \, d\xi \, d\eta$$

$$= \frac{(1-\eta)^3}{-3}\xi + \frac{(1-\xi)^3}{-3}\eta \Big|_{-1}^{+1}\Big|_{-1}^{+1}$$

$$= \left(\frac{(1-\eta)^3}{-3} + 0\right) - \left(\frac{(1-\eta)^3}{3} + \frac{2^3}{-3}\eta\right)\Big|_{-1}^{+1}$$

$$= (0 + 0) - \left(0 + \frac{2^3}{-3}\right) - \left[\left(\frac{2^3}{-3} + 0\right) - \left(\frac{2^3}{3} + \frac{2^3}{3}\right)\right]$$

$$= +\frac{2^3}{+3} + \frac{2^3}{3} + \frac{2^3}{3} + \frac{2^3}{3} = \frac{32}{3} \quad //$$

$$K_{12} = \int_{-1}^{1}\int_{-1}^{1} -(1-\eta)^2 + (1-\xi)(1+\xi)\, d\xi \, d\eta$$

$$= \frac{-(1-\eta)^3}{-3}\xi + \left(\xi - \frac{\xi^3}{3}\right)\eta \Big|_{-1}^{1}\Big|_{-1}^{1}$$

$$= \left(\frac{(1-\eta)^3}{3} + \left(1 - \frac{1}{3}\right)\eta\right) - \left(-\frac{(1-\eta)^3}{3} + \left(-1 + \frac{1}{3}\right)\eta\right)\Big|_{-1}^{1}$$

$$= \left(0 + \frac{2}{3}\right) - \left(0 + -\frac{2}{3}\right) - \left[\left(\frac{2^3}{3} + -\frac{2}{3}\right) - \left(-\frac{2^3}{3} + \frac{2}{3}\right)\right]$$

$$= \frac{4}{3} - \frac{2^3}{3} + \frac{2}{3} - \frac{2^3}{3} + \frac{2}{3} = -\frac{8}{3} \quad //$$

$$K_{13} = \int_{-1}^{+1}\int_{-1}^{1} -(1-\eta)(1+\eta) - (1-\xi)(1+\xi)\, d\xi \, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} -(1-\eta^2) - (1-\xi^2)\, d\xi \, d\eta =$$

$$= -\left(\eta - \frac{\eta^3}{3}\right)\xi - \left(\xi - \frac{\xi^3}{3}\right)\eta \Big|_{-1}^{+1}\Big|_{-1}^{+1}$$

$$= \left(-\left(\eta - \frac{\eta^3}{3}\right) - \frac{2}{3}\eta\right) - \left(\left(\eta - \frac{\eta^3}{3}\right) + \frac{2}{3}\eta\right)\Big|_{-1}^{+1}$$

$$K_{13} = \left(-\left(\eta - \eta^3/3\right) - \left(2/3\right)\eta \right) - \left(\left(\eta - \eta^3/3\right) + 2/3\,\eta \right) \Big|_{-1}^{+1}$$

$$= \left(-2/3 - \frac{2}{3} - 2/3 - 2/3\right) - \left(+2/3 + 2/3 + 2/3 + 2/3\right)$$

$$= -8/3 - 8/3 = -16/3$$

$$K_{14} = \int_{-1}^{1}\int_{-1}^{1} (1-\eta^2) - (1-\xi)^2 \, d\xi\, d\eta$$

$$= \left(\eta - \eta^3/3\right)\xi - \frac{(1-\xi)^3}{-3}\eta \Big|_{-1}^{+1} \Big|_{-1}^{+1}$$

$$= \left(\left(\eta - \eta^3/3\right) - 0 \right) - \left(-\left(\eta - \eta^3/3\right) + \frac{2^3}{+3}\eta \right) \Big|_{-1}^{+1}$$

$$= \left[\left(2/3 - 0\right) - \left(-2/3 + 2^3/3\right) \right] - \left[\left(-2/3 - 0\right) - \left(+2/3 - \frac{2^3}{3}\right) \right]$$

$$= 2/3 + 2/3 - 8/3 + 2/3 + 2/3 - 8/3$$

$$= 8/3 - 16/3 = -8/3$$

We know the local laplacian matrix for $dx = dy$ is equal along every diagonal. Thus:

$$L = \frac{1}{16}\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{11} & K_{12} & K_{13} \\ K_{13} & K_{12} & K_{11} & K_{12} \\ K_{14} & K_{13} & K_{12} & K_{11} \end{bmatrix}$$

We have shown

$K_{11} = 32/3$, $K_{12} = -8/3$, $K_{13} = -16/3$

$K_{14} = -8/3$

Mass Matrix : $M = \int_{\Omega} N_i N_j \, d\Omega$ **Quad**

As before we define the basis function of the refrence elements and map back to the generic element.

$$M = \int_0^1 \int_0^1 (N(x,y))^T (N(x,y)) \, dx \, dy$$

$$M = \int_{-1}^1 \int_{-1}^1 (N(\xi,\eta))^T (N(\xi,\eta)) \, |J| \, d\xi \, d\eta$$

Assuming $dx = dy = h$ $\quad J = h^2/4$

$$\therefore N = \frac{1}{4} \cdot \left[(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta) \right]$$

$$M = \frac{h^2}{4} \cdot \frac{1}{16} \int_{-1}^1 \int_{-1}^1 \begin{bmatrix} (1-\xi)(1-\eta) \\ (1-\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{bmatrix} \cdot \left[(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta) \right] d\xi \, d\eta$$

$$M = \frac{h^2}{64} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot \\ M_{41} & & & M_{44} \end{bmatrix}$$

See next page for calcs.

$$M_{11} = \int_{-1}^{1}\int_{-1}^{1} ((1-\xi)(1-\eta))^2 \, d\xi \, d\eta$$

$$= \frac{(1-\xi)^3}{-3} \cdot \frac{(1-\eta)^3}{-3} \Big|_{-1}^{1} \Big|_{-1}^{1} =$$

$$= \left(0 \cdot \frac{(1-\eta)^3}{-3} + \frac{2^3}{3} \cdot \frac{(1-\eta)^3}{-3}\right) \Big|_{-1}^{1}$$

$$= (0 + 0) - \left(\frac{2^3}{3} \cdot \frac{2^3}{-3}\right)$$

$$= 64/9 \;//$$

$$M_{12} = \int_{-1}^{1}\int_{-1}^{1} (1-\xi)(1-\eta) \cdot (1+\xi)(1-\eta) \, d\xi \, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} (1-\xi^2)(1-\eta)^2 \, d\xi \, d\eta$$

$$= 1/3\, \xi (\eta-1)^3 - 1/9\, \xi (\eta-1)^3 \Big|_{-1}^{+1} \Big|_{-1}^{+1}$$

$$= \left(\frac{(\eta-1)^3}{3} - \frac{(\eta-1)^3}{9} + \frac{(\eta-1)^3}{3} - \frac{(\eta-1)^3}{9}\right) \Big|_{-1}^{+1}$$

$$= -1\left(-\frac{2^3}{3} + \frac{2^3}{9} - \frac{2^3}{3} + \frac{2^3}{9}\right) = -1\left(-\frac{8}{3} + 8/9 - 8/3 + 8/9\right) = +\frac{16}{3} - \frac{16}{9} =$$

$$= \frac{48-16}{9} = \frac{32}{9} \;//$$

$$M_{13} = \int_{-1}^{1}\int_{-1}^{1} (1-\xi)(1-\eta)(1+\xi)(1+\eta)\, d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} (1-\xi^2)(1-\eta^2)\, d\xi\, d\eta$$

$$= \left(\xi - \xi^3/_2\right)\left(\eta - \eta^3/_3\right) \Big|_{-1}^{+1} \Big|_{-1}^{1} \qquad -1 + 2/3$$

$$= \left. 2/3\left(\eta - \eta^3/3\right) - \left(-2/3\left(\eta - \eta^3/3\right)\right)\right|_{-1}^{1}$$

$$= 2/3(2/3) - (-2/3(2/3)) - \left(2/3 \cdot -2/3 - \left(-2/3 \cdot -2/3\right)\right)$$

$$= 4/9 + 4/9 + 4/9 + 4/9$$

$$= 16/9 \,//$$

$$M_{14} = \int_{-1}^{+1}\int_{-1}^{+1} (1-\xi)(1-\eta)(1-\xi)(1+\eta)\, d\xi\, d\eta$$

$$= -(\xi-1)^3 \eta(\eta^2-3)\cdot 1/9 \Big|_{-1}^{1} \Big|_{-1}^{1}$$

$$= \left(0 + (-2)^3 \eta(\eta^2-3)\cdot 1/9\right) \Big|_{-1}^{1}$$

$$= -\frac{8}{9}\left(\overset{-2}{(1-3)} \overset{+}{-} \overset{-2}{-1(1-3)}\right)$$

$$= -\frac{8}{9} \cdot -4 = 32/9 \,//$$

We know the local mass matrix for $\Delta x = \Delta y = h$ is equal along ever diagonal.

$$M = \frac{h^2}{64}\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{11} & M_{12} & M_{13} \\ M_{13} & M_{12} & M_{11} & M_{14} \\ M_{14} & M_{13} & M_{12} & M_{11} \end{bmatrix}$$

$M_{11} = 64/9 \qquad M_{12} = 32/9$

$M_{13} = 16/9 \qquad M_{14} = 32/9$

local laplacian : $L = \int_{\Omega} \nabla N_i \nabla N_j \, d\Omega$. Triangl

The general basis function is :

$$N_1 = a_1 x_1 + b_1 y_1 + c_1$$
$$N_2 = a_2 x_2 + b_2 y_2 + c_2$$
$$N_3 = a_3 x_3 + b_3 x_3 + c_3$$

We solve for a, b, c such that N is equal to 1 at its own specif node

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

invert

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

See next for calculations

We now evaluate ~~the~~ the local laplacian going through every step...

$$N = \begin{bmatrix} -x-y+1 , & x , & y \end{bmatrix}$$

$$\nabla N = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \begin{bmatrix} -x-y+1 , & x , & y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L = \int_0^1 \int_0^x (\nabla N)^T (\nabla N) \, dy \, dx$$

$$L = \int_0^1 \int_0^x \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} dy \, dx.$$

$$L = \int_0^1 \int_0^x \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} dy \, dx$$

$$L = \int_0^1 \begin{bmatrix} 2x & -1x & -1x \\ -1x & 1x & 0 \\ -1x & 0 & 1x \end{bmatrix} dx$$

$$= \begin{bmatrix} \frac{2x^2}{2} & -1\frac{x^2}{2} & -1\frac{x^2}{2} \\ -x^2/2 & 1x^2/2 & 0 \\ -x^2/2 & 0 & x^2/2 \end{bmatrix} \Bigg|_0^1 = \begin{bmatrix} 1 & -1/2 & -1 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Mass Matrix : $M = \int_\Omega N_i \cdot N_i \, d\Omega$ Triangle.

We can calculate the mass matrix using the following formula.

$$M_{ij} = \int N_1^i N_2^j N_3^k = \frac{i!\,j!\,k!}{(i+j+k+d)!} d!\,V$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = \frac{2!}{4!} 2! \cdot 1/2 = \frac{2}{24} = 1/12$$

$$M_{12} = \frac{1}{4!} 2! \cdot 1/2 = 1/24$$

from the above we can see that all entries other than the diagonal are the same.

Thus:

$$M = \begin{bmatrix} 1/12 & 1/24 & 1/24 \\ 1/24 & 1/12 & 1/24 \\ 1/24 & 1/24 & 1/12 \end{bmatrix}$$