PDE Lab - Assignment 5

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1 Prework - Calculations

1. Find f :

   To solve for f we will find the laplacian of u_0 :

$$u_0 = xe^{-(y-1)^2}$$

$$-f = 2xe^{(-(y-1)^2)}(2y^2 - 4y + 1)$$
$$f = -2xe^{(-(y-1)^2)}(2y^2 - 4y + 1)$$

2. Verify that the exact solution satisfies the homogeneous Neumann conditions at $y = 1$

$$\nabla u = \nabla xe^{-(y-1)^2} = [e^{-(y-1)^2}, -2x(y-1)e^{-(y-1)^2}]|_{y=1} = [1, 0]$$

   At $y = 1$ the normal vector is only in the y direction. Thus from the above it equals to 0. Thus $\frac{\partial u}{\partial n} = 0$

2 Solution of Poisson's equation

The code was implemented using Matlab OOP, thus function inputs may differ.

2.1 Mesh generation

```
function this = quad(this)
        m=this.n;
        n=this.n;
        delta_x =this.dx;
        delta_y =this.dy;
        Elements = zeros(n*m,4);
        Points = zeros((n+1)*(m+1),3);
        PointMarker = zeros((n+1)*(m+1),1);
        j = 0;i = 0;
        for row = 1:(n+1)
            for col = 1:(m+1)
                if(row<=n && col<=m)
                    j=j+1;
                    Elements(j,:) = [
                        (row-1)*(m+1)+(col-1),
                        (row-1)*(m+1)+(col-1)+1,
                        (row-0)*(m+1)+(col-1)+1,
                        (row-0)*(m+1)+(col-1),
                        ];
                end

                i=i+1;
                Points(i,:) = [(col-1)+(col-1)*(delta_x-1),(row-1)+(row-1)*(delta_y-1),0];
                %Find edge points of grid
                if (Points(i,2)==1)

                    PointMarker(i,1) = 3;

                elseif(Points(i,2)==0)

                    PointMarker(i,1) = 1;

                elseif (Points(i,1)==1)

                    PointMarker(i,1) = 2;

                elseif (Points(i,1)==0)

                    PointMarker(i,1) = 4;
                end

            end
        end
        Elements=Elements+1;
        this.Points=Points;
        this.Elements=Elements;
        this.PointMarker=PointMarker;
        this = this.descProblem();
    end
```

2.2 Finite Element - Hand Calcuations

Refer to calculation at the end of the document.

2.3 Enforce boundary conditions

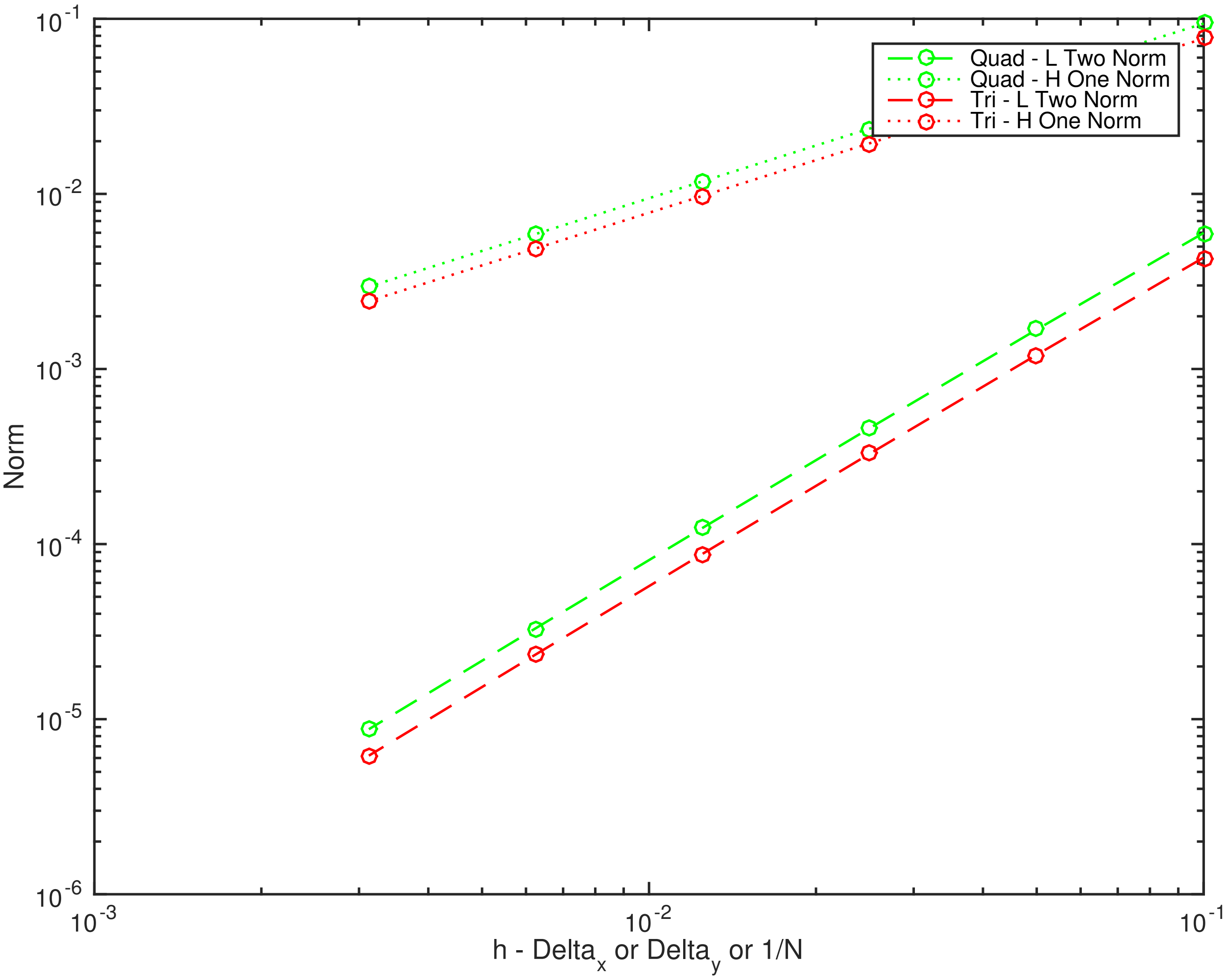
Describe the necessary modifications to the code so that you enforce homogeneous Neumann conditions on the discrete operators you have assembled.
No major modification were made for the homogeneous Neumann conditions. Indeed homogeneous Neumann conditions are byproduct of solving the weak formulation. Thus we only need to enforce Dirichlet conditions and leave the homogeneous Neumann conditions untouched.

2.4 Solution on triangular grid

```
function this = tri(this)
            this = this.quad();

            QPoints=this.Points;
            QElements=this.Elements;
            QPointMarker=this.PointMarker;


            elSize = size(QElements,1)*2;
            Elements  = ones(elSize,3);

            for i = 1 : size(QElements,1)
                Elements(i,:) = [QElements(i,1),QElements(i,2),QElements(i,3)];
                Elements(elSize-i+1,:)=[QElements(i,3),QElements(i,4),QElements(i,1)];
            end

            this.Elements=Elements;
            this = this.descProblem();
        end
```

3 Convergence Study

The L_2 and H_1 norm for both element types are plotted below.

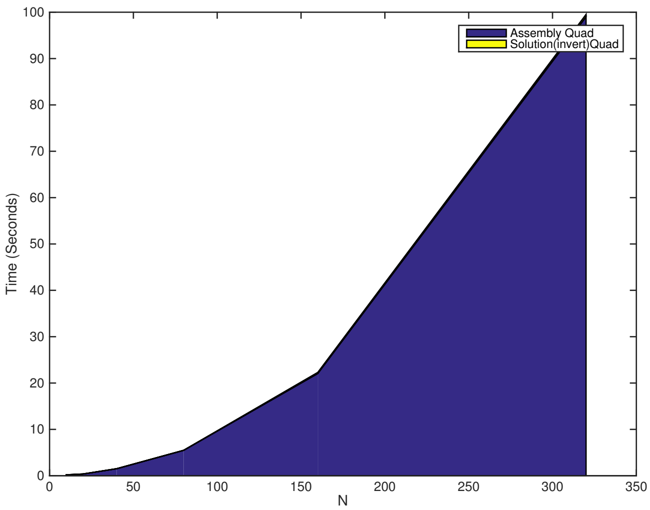


4 Performance study

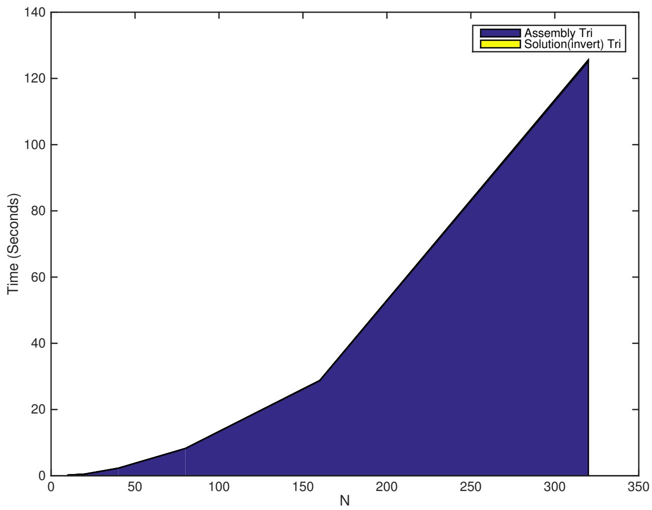
The performance (time to completion) of the assembly and solution of the Quad elements can be observed in the plot below. **Note Assembly refers to all tasks other than inverting the matrix.**

Compared to the last time the new matrix assembly routine has greatly improved the overall assembly time. Between the quad and triangle it can be observed that the triangle take more time to assembly due the higher number of elements.

Figure 1: Performance Study



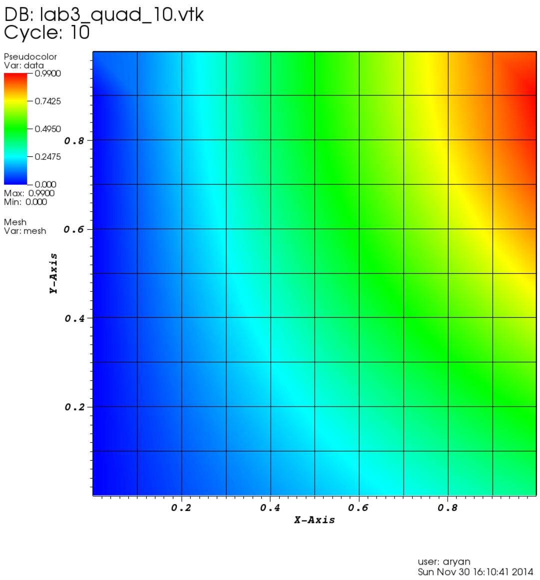
(a) Quad



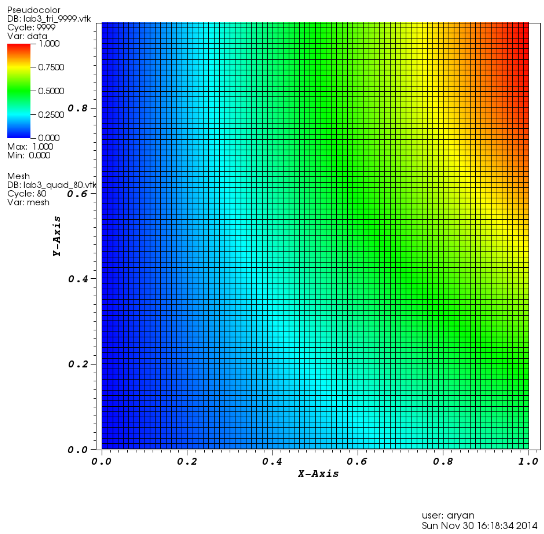
(b) Triangle

5 Visualization

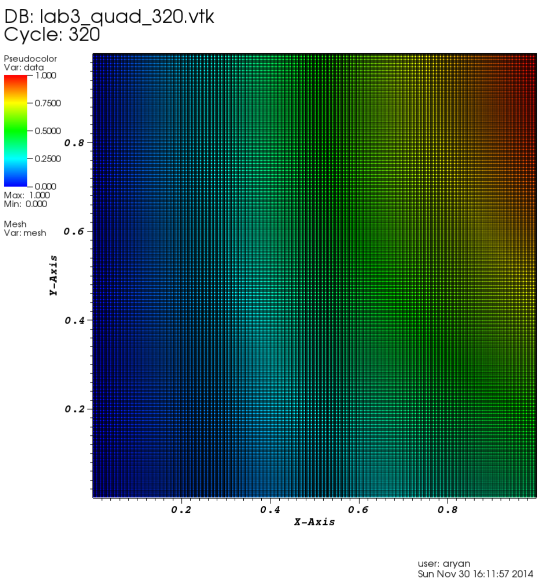
Figure 2: Visualization Quadrilateral Elements - X[0,1], Y[0,1]



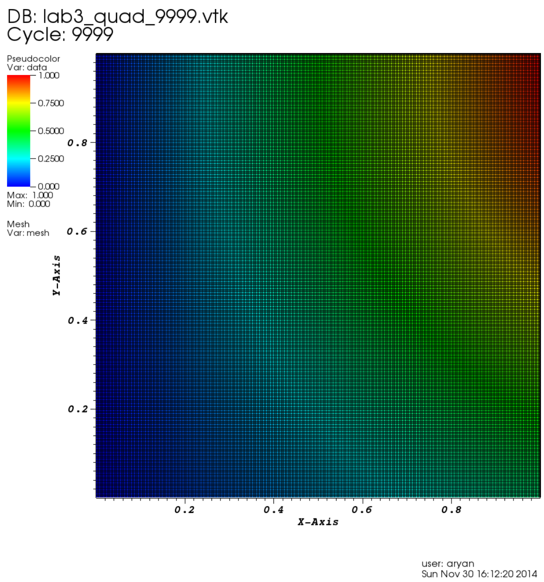
(a) N = 10



(b) N=80

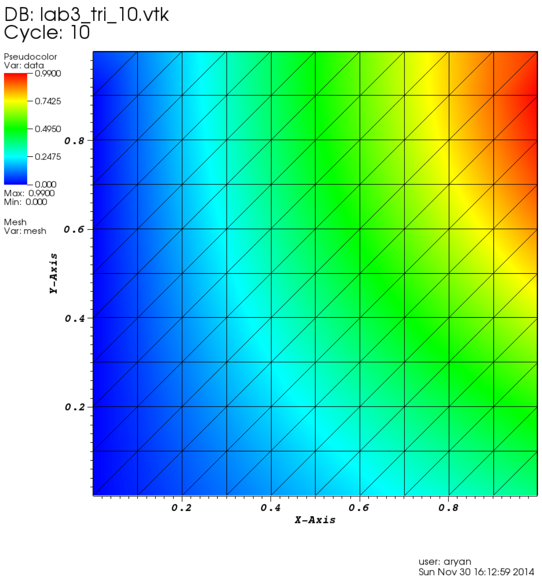


(c) N=320

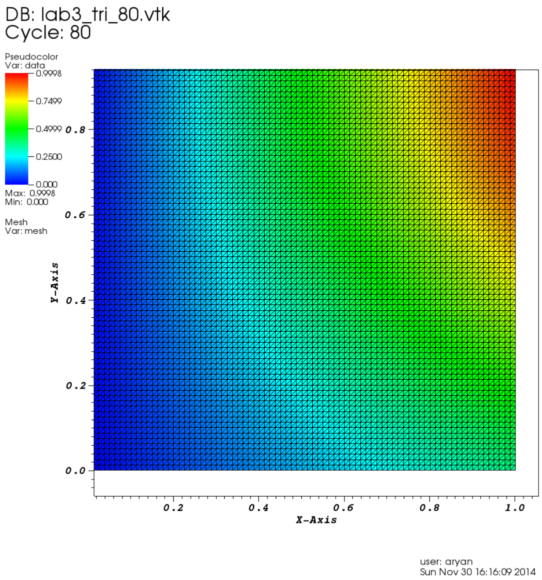


(d) Exact Solution

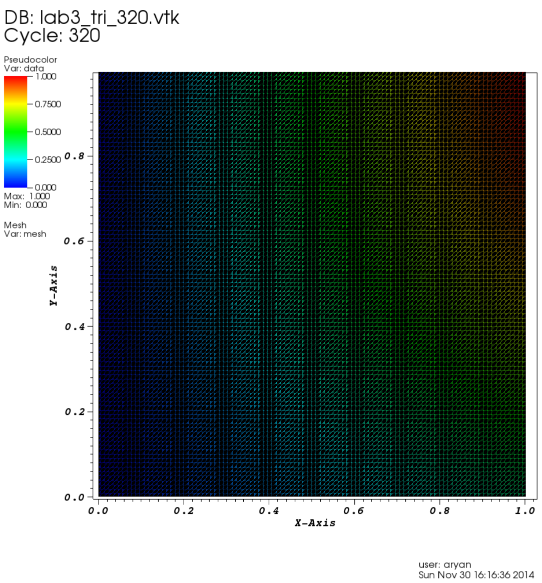
Figure 3: Visualization Triangular Elements - X[0,1], Y[0,1]



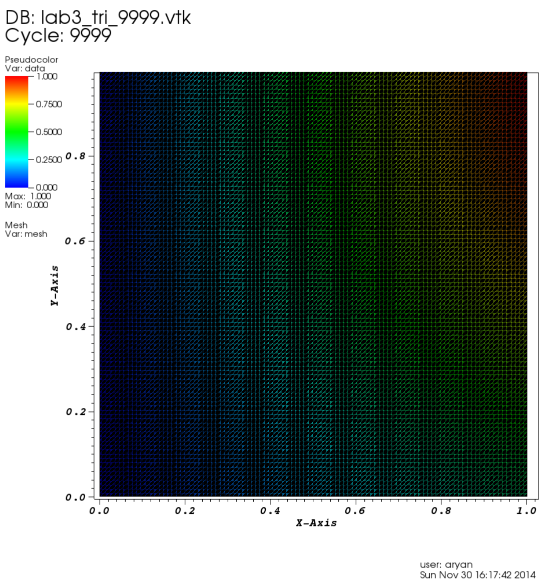
(a) N = 10



(b) N=80



(c) N=320



(d) Exact Solution

Local laplacian : $L = \int_\Omega \nabla N_i \nabla N_j \, d\Omega$

Quad

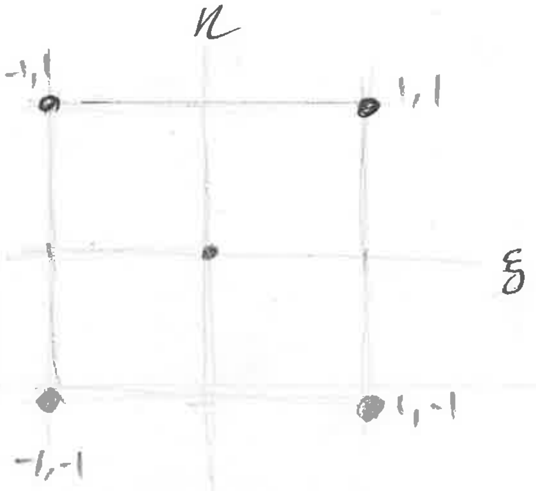
Where N_i is the basis function at the i^{th} point.

We rewrite the equation in complete form.

$$L = \int_\Omega (\nabla N)^T \cdot \nabla N \, d\Omega$$

Here N is the vector holding all entries of N_i, for x & y.

1) We start on the reference element and we map back to the general element.



$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

$$L = \int_0^1 \int_0^1 (\nabla N(x,y))^T (\nabla N(x,y)) \, dx \, dy$$

With respect to η & ξ we can rewrite the above as:

$$L = \int_{-1}^1 \int_{-1}^1 (J^{-1} \nabla N(\xi,\eta))^T (J^{-1} \nabla N(\xi,\eta)) \, |J| \, d\xi \, d\eta$$

Where $J = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} dx & 0 \\ 0 & dy \end{bmatrix}$

*Assume $dx = dy = h$

$J = \frac{h}{2} I \rightarrow J^{-1} = \frac{2}{h} I$

$|J| = \frac{h^2}{4}$

$$L = (J^{-1})^2 |J| \int_{-1}^{1}\int_{-1}^{1} (\nabla N(\xi,\eta))^T (\nabla N(\xi,\eta))\, d\xi\, d\eta$$

$$L = \frac{4}{h^2} \cdot \frac{h^2}{4} \int_{-1}^{1}\int_{-1}^{1} (\nabla N(\xi,\eta))^T (\nabla N(\xi,\eta))\, d\xi\, d\eta$$

$$N = \tfrac{1}{4}\begin{bmatrix} (1-\xi)(1-\eta) & (1+\xi)(1-\eta) & (1+\xi)(1+\eta) & (1-\xi)(1+\eta) \end{bmatrix}$$

$$\nabla N = \nabla_{\xi\eta} N = \tfrac{1}{4}\begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix}\begin{bmatrix} N \end{bmatrix}$$

$$\nabla N = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}$$

$$L = \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} -(1-\eta) & -(1-\xi) \\ (1-\eta) & -(1+\xi) \\ (1+\eta) & (1+\xi) \\ -(1+\eta) & (1-\xi) \end{bmatrix} \cdot \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} d\xi\, d\eta$$

$$L = \frac{1}{16}\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$$

See next page for Calculations . . .

$$K_{11} = \int_{-1}^{1}\int_{-1}^{1} (1-\eta)^2 + (1-\xi)^2 \, d\xi \, d\eta$$

$$= \frac{(1-\eta)^3}{-3}\xi + \frac{(1-\xi)^3}{-3}\eta \Bigg|_{-1}^{+1}\Bigg|_{-1}^{+1}$$

$$= \left(\frac{(1-\eta)^3}{-3} + 0\right) - \left(\frac{(1-\eta)^3}{3} + \frac{2^3}{-3}\eta\right)\Bigg|_{-1}^{+1}$$

$$= (0 + 0) - \left(0 + \frac{2^3}{-3}\right) - \left[\left(\frac{2^3}{-3} + 0\right) - \left(\frac{2^3}{3} + \frac{2^3}{3}\right)\right]$$

$$= +\frac{2^3}{+3} + 2^3/3 + 2^3/3 + 2^3/3 = 32/3 \;//$$

$$K_{12} = \int_{-1}^{1}\int_{-1}^{1} -(1-\eta)^2 + (1-\xi)(1+\xi) \, d\xi \, d\eta$$

$$= \frac{-(1-\eta)^3}{-3}\xi + \left(\xi - \xi^3/3\right)\eta \Bigg|_{-1}^{1}\Bigg|_{-1}^{1}$$

$$= \left(\frac{(1-\eta)^3}{3} + (1 - 1/3)\eta\right) - \left(-\frac{(1-\eta)^3}{3} + (-1 + 1/3)\eta\right)\Bigg|_{-1}^{1}$$

$$= (0 + 2/3) - (0 + -2/3) - \left[\left(-2^3/3 + -2/3\right) - \left(-2^3/3 + 2/3\right)\right]$$

$$= 4/3 - 2^3/3 + 2/3 - 2^3/3 + 2/3 = -8/3 \;//$$

$$K_{13} = \int_{-1}^{+1}\int_{-1}^{1} -(1-\eta)(1+\eta) - (1-\xi)(1+\xi) \, d\xi \, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} -(1-\eta^2) - (1-\xi^2) \, d\xi \, d\eta =$$

$$= -\left(\eta - \frac{\eta^3}{3}\right)\xi - \left(\xi - \frac{\xi^3}{3}\right)\eta \Bigg|_{-1}^{+1}\Bigg|_{-1}^{+1}$$

$$= \left(1 \cdot -(\eta - \eta^3/3) - 2/3\,\eta\right) - \left(1\,(\eta - \eta^3/3) + 2/3\,\eta\right)\Bigg|_{-1}^{+1}$$

$$K_{13} = \left(-\left(\eta - \eta^3/_3\right) - \left(2/_3\right)\eta \right) - \left(\left(\eta - \eta^3/_3\right) + 2/_3\,\eta \right) \Big|_{-1}^{+1}$$

$$= \left(-2/_3 - \frac{2}{3} - 2/_3 - 2/_3\right) - \left(+2/_3 + 2/_3 + 2/_3 + 2/_3\right)$$

$$= -8/_3 - 8/_3 = -16/_3 \,//$$

$$K_{14} = \int_{-1}^{1}\int_{-1}^{1} (1-\eta^2) - (1-\xi)^2 \, d\xi\, d\eta$$

$$= \left(\eta - \eta^3/_3\right)\xi - \frac{(1-\xi)^3}{-3}\eta \Big|_{-1}^{+1} \Big|_{-1}^{+1}$$

$$= \left(\left(\eta - \eta^3/_3\right) - 0\right) - \left(-\left(\eta - \eta^3/_3\right) + \frac{2^3}{+3}\eta\right) \Big|_{-1}^{+1}$$

$$= \left[\left(2/_3 - 0\right) - \left(-2/_3 + 2^3/_3\right)\right] - \left[\left(-2/_3 - 0\right) - \left(+2/_3 - \frac{2^3}{3}\right)\right]$$

$$= 2/_3 + 2/_3 - 8/_3 + 2/_3 + 2/_3 - 8/_3$$

$$= 8/_3 - 16/_3 = -8/_3 \,//$$

We know the local laplacian matrix for $dx = dy$ is equal along every diagonal. Thus:

$$L = \frac{1}{16}\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{11} & K_{12} & K_{13} \\ K_{13} & K_{12} & K_{11} & K_{12} \\ K_{14} & K_{13} & K_{12} & K_{11} \end{bmatrix}$$

We have shown

$K_{11} = 32/_3$, $K_{12} = -8/_3$, $K_{13} = -16/_3$

$K_{14} = -8/_3$

Mass Matrix : $M = \int_\Omega N_i N_j \, d\Omega$

Quad

As before we define the basis function of the refrence elements and map back to the generic element.

$$M = \int_0^1 \int_0^1 (N(x,y))^T (N(x,y)) \, dx\, dy$$

$$M = \int_{-1}^1 \int_{-1}^1 (N(\xi,\eta))^T (N(\xi,\eta)) \, |J| \, d\xi\, d\eta$$

Assuming $dx = dy = h$ $\quad J = h^2/4$

$$\therefore N = \frac{1}{4} \cdot \left[(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta) \right]$$

$$M = \frac{h^2}{4} \cdot \frac{1}{16} \int_{-1}^1 \int_{-1}^1 \begin{bmatrix} (1-\xi)(1-\eta) \\ (1-\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{bmatrix} \cdot \left[(1-\xi)(1-\eta) \quad (1+\xi)(1-\eta) \quad (1+\xi)(1+\eta) \quad (1-\xi)(1+\eta) \right] d\xi\, d\eta$$

$$M = \frac{h^2}{64} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot \\ M_{41} & & & M_{44} \end{bmatrix}$$

See next page for calcs.

$$M_{11} = \int_{-1}^{1}\int_{-1}^{1} \left((1-\xi)(1-\eta)\right)^2 d\xi\, d\eta$$

$$= \frac{(1-\xi)^3}{-3} \cdot \frac{(1-\eta)^3}{-3} \Big|_{-1}^{1} \Big|_{-1}^{1} =$$

$$= \left(0 \cdot \frac{(1-\eta)^3}{-3} + \frac{2^3}{3} \cdot \frac{(1-\eta)^3}{-3}\right) \Big|_{-1}^{1}$$

$$= (0 + 0) - \left(\frac{2^3}{3} \cdot \frac{2^3}{-3}\right)$$

$$= 64/9 \;//$$

$$M_{12} = \int_{-1}^{1}\int_{-1}^{1} (1-\xi)(1-\eta) \cdot (1+\xi)(1-\eta)\, d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} (1-\xi^2)(1-\eta)^2 d\xi\, d\eta$$

$$= \tfrac{1}{3}\xi(\eta-1)^3 - \tfrac{1}{9}\xi(\eta-1)^3 \Big|_{-1}^{+1} \Big|_{-1}^{+1}$$

$$= \left(\frac{(\eta-1)^3}{3} - \frac{(\eta-1)^3}{9} + \frac{(\eta-1)^3}{3} - \frac{(\eta-1)^3}{9}\right) \Big|_{-1}^{+1}$$

$$= -1\left(-\frac{2^3}{3} + \frac{2^3}{9} - \frac{2^3}{3} + \frac{2^3}{9}\right) = -1\left(-\frac{8}{3} + \frac{8}{9} - \frac{8}{3} + \frac{8}{9}\right) = +\frac{16}{3} - \frac{16}{9} =$$

$$= \frac{48-16}{9} = \frac{32}{9} \;//$$

$$M_{13} = \int_{-1}^{1}\int_{-1}^{1} (1-\xi)(1-\eta)(1+\xi)(1+\eta)\, d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} (1-\xi^2)(1-\eta^2)\, d\xi\, d\eta$$

$$= \left(\xi - \xi^3/_2\right)\left(\eta - \eta^3/_3\right) \Big|_{-1}^{+1} \Big|_{-1}^{1} \qquad -1 + 2/_3$$

$$= 2/_3\left(\eta - \eta^3/_3\right) - \left(-2/_3\left(\eta - \eta^3/_3\right)\right)\Big|_{-1}^{1}$$

$$= 2/_3\left(2/_3\right) - \left(-2/_3\left(2/_3\right)\right) - \left(2/_3 \cdot -2/_3 - \left(-2/_3 \cdot -2/_3\right)\right)$$

$$= 4/9 + 4/9 + 4/9 + 4/9$$

$$= 16/9 \,//$$

$$M_{14} = \int_{-1}^{+1}\int_{-1}^{+1} (1-\xi)(1-\eta)(1-\xi)(1+\eta)\, d\xi\, d\eta$$

$$= -(\xi-1)^3 \eta(\eta^2-3)\cdot 1/9 \Big|_{-1}^{1} \Big|_{-1}^{1}$$

$$= \left(0 + (-2)^3 \eta(\eta^2-3)\cdot 1/9\right)\Big|_{-1}^{1}$$

$$= -\frac{8}{9}\left(\overset{-2}{(1-3)} - \overset{+}{-}1\overset{-2}{(1-3)}\right)$$

$$= -\frac{8}{9} \cdot -4 = 32/9 \,//$$

We know the local mass matrix for $\Delta x = \Delta y = h$ is equal along ever diagonal.

$$M = \frac{h^2}{64}\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{11} & M_{12} & M_{13} \\ M_{13} & M_{12} & M_{11} & M_{14} \\ M_{14} & M_{13} & M_{12} & M_{11} \end{bmatrix}$$

$M_{11} = 64/9 \qquad M_{12} = 32/9$

$M_{13} = 16/9 \qquad M_{14} = 32/9$

local laplacian : $L = \int_\Omega \nabla N_i \nabla N_j \, d\Omega$. Triangle

The general basis function is :

$N_1 = a_1 x_1 + b_1 y_1 + c_1$
$N_2 = a_2 x_2 + b_2 y_2 + c_2$
$N_3 = a_3 x_3 + b_3 x_3 + c_3$

We solve for a, b, c such that N is equal to 1 at its own specif node

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

invert

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

See next for calculations

We now evaluate ~~the~~ the local laplacian going through every step...

$$N = [-x-y+1 \ , \ x \ , \ y]$$

$$\nabla N = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} [-x-y+1 \ , \ x \ , \ y]$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L = \int_0^1 \int_0^x (\nabla N)^T (\nabla N) \, dy \, dx$$

$$L = \int_0^1 \int_0^x \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} dy \, dx.$$

$$L = \int_0^1 \int_0^x \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} dy \, dx$$

$$L = \int_0^1 \begin{bmatrix} 2x & -1x & -1x \\ -1x & 1x & 0 \\ -1x & 0 & 1x \end{bmatrix} dx$$

$$= \begin{bmatrix} \frac{2x^2}{2} & -1\frac{x^2}{2} & -1\frac{x^2}{2} \\ -x^2/2 & 1x^2/2 & 0 \\ -x^2/2 & 0 & x^2/2 \end{bmatrix} \Bigg|_0^1 = \begin{bmatrix} 1 & -1/2 & -1 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Mass Matrix : $M = \int_\Omega N_i \cdot N_i \, d\Omega$ Triangle.

We can calculate the mass matrix using the following formula.

$$M_{ij} = \int N_1^i N_2^j N_3^k = \frac{i!\,j!\,k!}{(i+j+k+d)!} d!\,V$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = \frac{2!}{4!} 2! \cdot 1/2 = \frac{2}{24} = 1/12$$

$$M_{12} = \frac{1}{4!} 2! \cdot 1/2 = 1/24 \quad M$$

from the above we can see that all entries other than the diagonal are the same.

Thus:

$$M = \begin{bmatrix} 1/12 & 1/24 & 1/24 \\ 1/24 & 1/12 & 1/24 \\ 1/24 & 1/24 & 1/12 \end{bmatrix}$$