First order theories

(Chapter 1, Sections 1.4 – 1.5, DP)

First order logic

- A first order theory consists of
  - Variables
  - Logical symbols: $\wedge$ $\vee$ $\neg$ $\forall$ $\exists$ \`(' \`)'
  - Non-logical Symbols Σ: Constants, predicate and function symbols
  - Syntax

Quantifiers

existential quantifier: $\exists x. F(x)$ "there exists an x such that $F(x)$"

universal quantifier: $\forall x. \underbrace{F(x)}$ "for all x, $F(x)$"

Quantified variable

Scope of quantified variable

A variable is **bound** if there exists an occurrence in the scope of some quantifier

A variable is **free** if there exists an occurrence not bound by any quantifier

A variable may be both bound and free!

In a given formula

Closed/Ground formula:
no free variables

Open formula: some free variables

Ground, quantifier-free formula:
no variables

Example, scope

$$\forall x.\ \underbrace{p(f(x),x)\ \rightarrow\ (\exists y.\ \underbrace{p(f(g(x,y)),g(x,y)))}_{G}\ \wedge\ q(x,f(x))}_{F}$$

The scope of $\forall x$ is F.

The scope of $\exists y$ is G.

The formula reads:

"for all x,
if $p(f(x),x)$
then there exists a y such that
$p(f(g(x,y)),g(x,y))$ and $q(x,f(x))$"

Examples

- Σ = {0,1, '+', '>'}
  - '0','1' are constant symbols
  - '+' is a binary function symbol
  - '>' is a binary predicate symbol

- An example of a Σ-formula:

$$\exists y\ \forall x.\ x > y$$

Examples

- $\Sigma = \{1, \text{'>'}, \text{'<'}, \text{'isprime'}\}$
  - '1' is a constant symbol
  - '>', '<' are binary predicates symbols
  - 'isprime' is a unary predicate symbol
- An example Σ-formula:

$$\forall n\ \exists p.\ n > 1 \rightarrow \mathrm{isprime}(p) \wedge n < p < 2n.$$

- Are these formulas valid ?
- So far these are only symbols, strings. No meaning yet.

Interpretations

- Let $\Sigma = \{0,1, \text{'+'}, \text{'='}\}$ where 0,1 are constants, '+' is a binary function symbol and '=' a predicate symbol.
- Let $\phi = \exists x.\ x + 0 = 1$
- Q: Is ϕ true in $\mathcal{N}_0$?
- A: Depends on the interpretation!

Structures

- A structure is given by:
  1. A domain
  2. An interpretation of the nonlogical symbols: i.e.,
     - Maps each predicate symbol to a predicate of the same arity
     - Maps each function symbol to a function of the same arity
     - Maps each constant symbol to a domain element
  3. An assignment of a domain element to each free (unquantified) variable

Similar definitions

An interpretation $I : (D_I, \alpha_I)$ consists of:

- Domain D_I
  non-empty set of values or objects
  cardinality $|D_I|$ finite (eg, 52 cards),
  countably infinite (eg, integers), or
  uncountably infinite (eg, reals)
- Assignment α_I
  - each variable x assigned value $x_I \in D_I$
  - each n-ary function f assigned

    $$f_I : D_I^n \to D_I$$

    In particular, each constant a (0-ary function) assigned value $a_I \in D_I$
  - each n-ary predicate p assigned

    $$p_I : D_I^n \to \{\underline{\text{true}}, \underline{\text{false}}\}$$

In particular, each propositional variable P (0-ary predicate) assigned truth value (true, false)

Structures

- Remember $\phi = \exists x.\ x + 0 = 1$
- Consider the structure S:
  - Domain: $\mathcal{N}_0$
  - Interpretation:
    - ‘0’ and ‘1’ are mapped to 0 and 1 in $\mathcal{N}_0$
    - ‘=’ $\mapsto$ = (equality)
    - ‘+’ $\mapsto$ * (multiplication)
- Now, is ϕ true in S ?

Satisfying structures

- Definition: A formula is satisfiable if there exists a structure that satisfies it

- Example: $\phi = \exists x.\ x + 0 = 1$ is satisfiable
- Consider the structure S':
  - Domain: $\mathcal{N}_0$
  - Interpretation:
    - '0' and '1' are mapped to 0 and 1 in $\mathcal{N}_0$
    - '=' $\mapsto$ = (equality)
    - '+' $\mapsto$ + (addition)
- S' satisfies ϕ. S' is said to be a model of ϕ.

What happens to Qunatifiers

x variable.
<u>x-variant</u> of interpretation I is an interpretation $J : (D_J, \alpha_J)$ such that

- $D_I = D_J$
- $\alpha_I[y] = \alpha_J[y]$ for all symbols y, except possibly x

That is, I and J agree on everything except possibly the value of x

Denote $J : I \triangleleft \{x \mapsto \mathsf{v}\}$ the x-variant of I in which $\alpha_J[x] = \mathsf{v}$ for some $\mathsf{v} \in D_I$. Then

- $I \models \forall x.\ F$ iff for all $\mathsf{v} \in D_I$, $I \triangleleft \{x \mapsto \mathsf{v}\} \models F$
- $I \models \exists x.\ F$ iff there exists $\mathsf{v} \in D_I$ s.t. $I \triangleleft \{x \mapsto \mathsf{v}\} \models F$

I is an interpretation of ∀x. F iff

- all x-variants of I are interpretations of F .
- I is an interpretation of ∃x. F iff some x-variant of I is an interpretation of F .

Example

Example

For $\mathbb{Q}$, the set of rational numbers, consider

$$F_I : \forall x.\ \exists y.\ 2 \times y = x$$

Compute the value of F_I (F under I):

Let

$$J_1 : I \lhd \{x \mapsto \mathsf{v}\} \qquad\qquad J_2 : J_1 \lhd \{y \mapsto \tfrac{\mathsf{v}}{2}\}$$

x-variant of I $\qquad$ y-variant of J_1

for $\mathsf{v} \in \mathbb{Q}$.

Then

1.	J_2	$\models$	$2 \times y = x$	since $2 \times \frac{\mathsf{v}}{2} = v$
2.	J_1	$\models$	$\exists y.\ 2 \times y = x$	
3.	I	$\models$	$\forall x.\ \exists y.\ 2 \times y = x$	since $\mathsf{v} \in \mathbb{Q}$ is arbitrary

Semantic Judgements for proving

F is <u>satisfiable</u> iff there exists I s.t. $I \models F$
F is <u>valid</u> iff for all I, $I \models F$

F is valid iff $\neg F$ is unsatisfiable

<u>Semantic rules</u>: given an interpretation I with domain D_I,

$$\frac{I \models \forall x.\ F[x]}{I \triangleleft \{x \mapsto \mathsf{v}\} \models F[x]} \quad \text{for any } v \in D_I$$

$$\frac{I \not\models \forall x.\ F[x]}{I \triangleleft \{x \mapsto \mathsf{v}\} \not\models F[x]} \quad \text{for a } \underline{\text{fresh}}\ v \in D_I$$

$$\frac{I \models \exists x.\ F[x]}{I \triangleleft \{x \mapsto \mathsf{v}\} \models F[x]} \quad \text{for a } \underline{\text{fresh}}\ v \in D_I$$

$$\frac{I \not\models \exists x.\ F[x]}{I \triangleleft \{x \mapsto \mathsf{v}\} \not\models F[x]} \quad \text{for any } v \in D_I$$

Contradition Rule

Contradiction rule

A contradiction exists if two variants of the original interpretation I disagree on the truth value of an n-ary predicate p for a given tuple of domain values:

$$\frac{\begin{array}{l} J : I \lhd \cdots \models p(s_1, \ldots, s_n) \\ K : I \lhd \cdots \not\models p(t_1, \ldots, t_n) \end{array}}{I \models \bot} \quad \text{for } i \in \{1, \ldots, n\}, \alpha_J[s_i] = \alpha_K[t_i]$$

<u>Intuition</u>: The variants J and K are constructed only through the rules for quantification. Hence, the truth value of p on the given tuple of domain values is already established by I. Therefore, the disagreement between J and K on the truth value of p indicates a problem with I.

Example

Example: $F : \ (\forall x.\ p(x)) \ \leftrightarrow \ (\neg\exists x.\ \neg p(x))$ valid?

Suppose not. Then there is I s.t.

0. $I \not\models (\forall x.\ p(x)) \ \leftrightarrow \ (\neg\exists x.\ \neg p(x))$

First case

1.	I	$\models$	$\forall x.\ p(x)$	assumption
2.	I	$\not\models$	$\neg\exists x.\ \neg p(x)$	assumption
3.	I	$\models$	$\exists x.\ \neg p(x)$	2 and $\neg$
4.	$I \triangleleft \{x \mapsto \mathsf{v}\}$	$\models$	$\neg p(x)$	3 and $\exists$, for some $\mathsf{v} \in D_I$
5.	$I \triangleleft \{x \mapsto \mathsf{v}\}$	$\models$	$p(x)$	1 and $\forall$

4 and 5 are contradictory.

Example

Second case

$$
\begin{array}{lrcll}
1. & I & \not\models & \forall x.\ p(x) & \text{assumption} \\
2. & I & \models & \neg\exists x.\ \neg p(x) & \text{assumption} \\
3. & I \triangleleft \{x \mapsto \mathsf{v}\} & \not\models & p(x) & 1 \text{ and } \forall,\ \text{for some } \mathsf{v} \in D_I \\
4. & I & \not\models & \exists x.\ \neg p(x) & 2 \text{ and } \neg \\
5. & I \triangleleft \{x \mapsto \mathsf{v}\} & \not\models & \neg p(x) & 4 \text{ and } \exists \\
6. & I \triangleleft \{x \mapsto \mathsf{v}\} & \models & p(x) & 5 \text{ and } \neg
\end{array}
$$

3 and 6 are contradictory.

Both cases end in contradictions for arbitrary $I \quad \Rightarrow \quad F$ is valid.

Example

Example: Prove

$$F:\ p(a) \rightarrow \exists x.\ p(x) \quad \text{is valid.}$$

Assume otherwise.

1.	I	$\not\models$	F	assumption
2.	I	$\models$	$p(a)$	1 and $\rightarrow$
3.	I	$\not\models$	$\exists x.\ p(x)$	1 and $\rightarrow$
4.	$I \triangleleft \{x \mapsto \alpha_I[a]\}$	$\not\models$	$p(x)$	3 and $\exists$

2 and 4 are contradictory. Thus, F is valid.

First-order theories

- First-order logic is a framework.
- It gives us a generic syntax and building blocks for constructing restrictions thereof.
- Each such restriction is called a first-order theory.

- A theory defines
  - the signature Σ (the set of nonlogical symbols) and
  - the interpretations that we can give them.

Definitions

- Σ – the signature. This is a set of nonlogical symbols.
- Σ-formula: a formula over Σ symbols + logical symbols.
- A variable is free if it is not bound by a quantifier.
- A sentence is a formula without free variables.
- A Σ-theory T is defined by a set of Σ-sentences.

Definitions…

- Let T be a Σ-theory
- A Σ-formula ϕ is T-satisfiable if there exists a structure that satisfies both ϕ and the sentences defining T.
- A Σ-formula ϕ is T-valid if all structures that satisfy the sentences defining T also satisfy ϕ.

Example

- Let Σ = {0,1, '+', '='}
- Recall $\phi = \exists x.\ x + 0 = 1$
- ϕ is a Σ-formula.
- We now define the following Σ-theory:
  - $\forall x.\ x = x$ // '=' must be reflexive
  - $\forall x,y.\ x + y = y + x$ // '+' must be commutative
- Not enough to prove the validity of ϕ !

Theories through axioms

- The number of sentences that are necessary for defining a theory may be large or infinite.
- Instead, it is common to define a theory through a set of axioms.
- The theory is defined by these axioms and everything that can be inferred from them by a sound inference system.

Example 1

- Let Σ = {'='}
  - An example Σ-formula is $\phi = ((x = y) \wedge \neg (y = z)) \rightarrow \neg(x = z)$
- We would now like to define a Σ-theory T that will limit the interpretation of '=' to equality.
- We will do so with the equality axioms:
  - $\forall x.\ x = x$ (reflexivity)
  - $\forall x,y.\ x = y \rightarrow y = x$ (symmetry)
  - $\forall x,y,z.\ x = y \wedge y = z \rightarrow x = z$ (transitivity)
- Every structure that satisfies these axioms also satisfies ϕ above.
- Hence ϕ is T-valid.

Example 2

- Let $\Sigma = \{\text{'}<\text{'}\}$
- Consider the Σ-formula ϕ: $\forall x\ \exists y.\ y < x$
- Consider the theory T:
  - $\forall x,y,z.\ x < y \wedge y < z \rightarrow x < z$ (transitivity)
  - $\forall x,y.\ x < y \rightarrow \neg(y < x)$ (anti-symmetry)

Example 2 (cont'd)

- Recall: ϕ: ∀x ∃y. y < x

- Is ϕ T-satisfiable?
- We will show a model for it.
  - Domain: $\mathcal{Z}$
  - '<' $\mapsto$ <
- Is ϕ T-valid ?
- We will show a structure to the contrary
  - Domain: $\mathcal{N}_0$
  - '<' $\mapsto$ <

Fragments

- So far we only restricted the nonlogical symbols.
- Sometimes we want to restrict the grammar and the logical symbols that we can use as well.
- These are called logic fragments.
- Examples:
    - The quantifier-free fragment over Σ = {'=', '+',0,1}
    - The conjunctive fragment over Σ = {'=', '+',0,1}

Fragments

- Let $\Sigma = \{\}$
  - (T must be empty: no nonlogical symbols to interpret)
- Q: What is the quantifier-free fragment of T ?
- A: propositional logic

- Thus, propositional logic is also a first-order theory.
  - A very degenerate one.

Theories

- Let $\Sigma = \{\}$
  - (T must be empty: no nonlogical symbols to interpret)
- Q: What is T ?
- A: Quantified Boolean Formulas (QBF)

- Example:
  - $\forall x_1 \, \exists x_2 \, \forall x_3 . \; x_1 \rightarrow (x_2 \vee x_3)$

Some famous theories

- Presburger arithmetic: Σ = {0,1, '+', '='}
- Peano arithmetic: Σ = {0,1, '+', '*', '='}
- Theory of reals
- Theory of integers
- Theory of arrays
- Theory of pointers
- Theory of sets
- Theory of recursive data structures
- …

The algorithmic point of view...

- It is also common to present theories NOT through the axioms that define them.

- The interpretation of symbols is fixed to their common use.
  - Thus ‘+’ is plus, ...

- The fragment is defined via grammar rules rather than restrictions on the generic first-order grammar.

The algorithmic point of view...

- Example: equality logic (= "the theory of equality")
- *Grammar:*

formula : *formula* $\vee$ *formula* | $\neg$ *formula* | *atom*

atom : term-variable = term-variable

| term-variable = constant | Boolean-variable

- Interpretation:

'=' is equality.

The algorithmic point of view...

- This simpler way of presenting theories is all that is needed when our focus is on decision procedures specific for the given theory.

- The traditional way of presenting theories is useful when discussing generic methods (for any decidable theory T)
  - Example 1: algorithms for combining two or more theories
  - Example 2: generic SAT-based decision procedure given a decision procedure for the conjunctive fragment of T.

Expressiveness of a theory

- Each formula defines a language:
  the set of satisfying assignments (‘models’) are the words accepted by this language.

- Consider the fragment ‘2-CNF’

formula : (*literal* $\vee$ *literal*) | *formula* $\wedge$ *formula*

literal: Boolean-variable | $\neg$Boolean-variable

$$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_2)$$

Expressiveness of a theory

- Now consider a Propositional Logic formula

  ϕ: $(x_1 \vee x_2 \vee x_3)$.

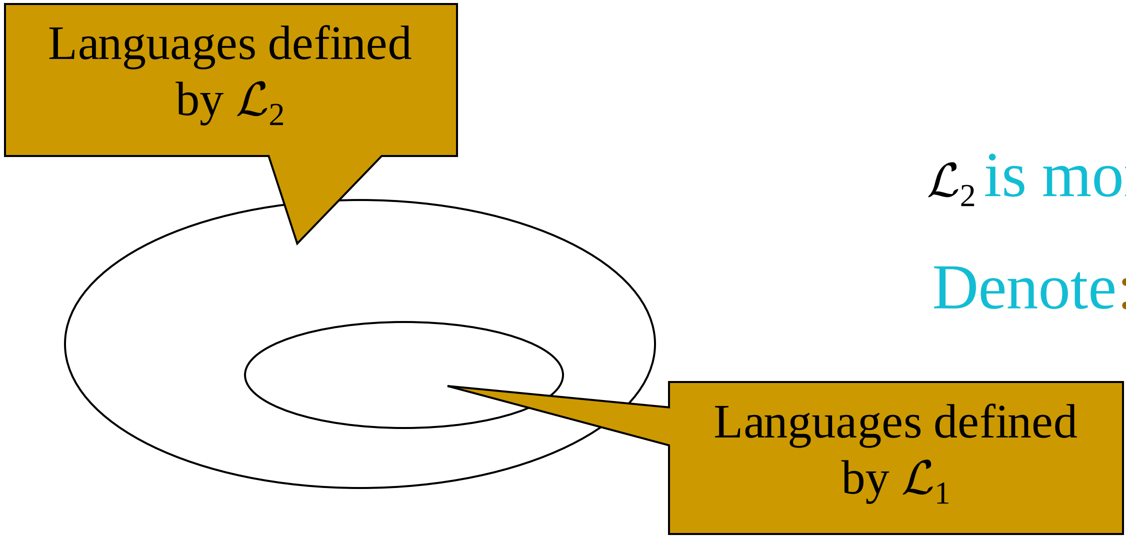
- Q: Can we express this language with 2-CNF?

- A: No.

  Proof:

  - The language accepted by ϕ has 7 words: all assignments other than $x_1 = x_2 = x_3 = F$.
  - The first 2-CNF clause removes ¼ of the assignments, which leaves us with 6 accepted words. Additional clauses only remove more assignments.

Expressiveness of a theory



$\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$.

Denote: $\mathcal{L}_1 \prec \mathcal{L}_2$

- *Claim*: 2-CNF $\prec$ Propositional Logic
- Generally there is only a partial order between theories.

The tradeoff

- So we see that theories can have different expressive power.
- Q: why would we want to restrict ourselves to a theory or a fragment ? why not take some ‘maximal theory’…
- A: Adding axioms to the theory may make it harder to decide or even undecidable.

Example: Hilbert axiom system ($\mathcal{H}$)

- Let $\mathcal{H}$ be (M.P) + the following axiom schemas:

$$\frac{}{A \rightarrow (B \rightarrow A)} \quad (H1)$$

$$\frac{}{((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))} \quad (H2)$$

$$\frac{}{(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)} \quad (H3)$$

- $\mathcal{H}$ is sound and complete
- This means that with $\mathcal{H}$ we can prove any valid propositional formula, and only such formulas. The proof is finite.

Example

- But there exists first order theories defined by axioms which are not sufficient for proving all T-valid formulas.

Example: First Order Peano Arithmetic

- Σ = {0,1,'+', '*', '='}
- Domain: Natural numbers
- Axioms ("semantics"):

Undecidable!

1. $\forall x : (0 \neq x + 1)$
2. $\forall x : \forall y : (x \neq y) \rightarrow (x + 1 \neq y + 1)$
3. Induction
4. $\forall x : x + 0 = x$
5. $\forall x : \forall y : (x + y) + 1 = x + (y + 1)$
6. $\forall x : x * 0 = 0$
7. $\forall x \forall y : x * (y + 1) = x * y + x$

+ (axioms 4–5)

* (axioms 6–7)

These axioms define the semantics of '+'

Example: First Order Presburger Arithmetic

- $\Sigma = \{0, 1, \text{'+'}, \cancel{\text{'*'}}, \text{'='}\}$
- Domain: Natural numbers

decidable!

- Axioms ("semantics"):
  1. $\forall x : (0 \neq x + 1)$
  2. $\forall x : \forall y : (x \neq y) \rightarrow (x + 1 \neq y + 1)$
  3. Induction
  4. $\forall x : x + 0 = x$
  5. $\forall x : \forall y : (x + y) + 1 = x + (y + 1)$
  6. $\cancel{\forall x : x * 0 = 0}$
  7. $\cancel{\forall x \forall y : x * (y + 1) = x * y + x}$

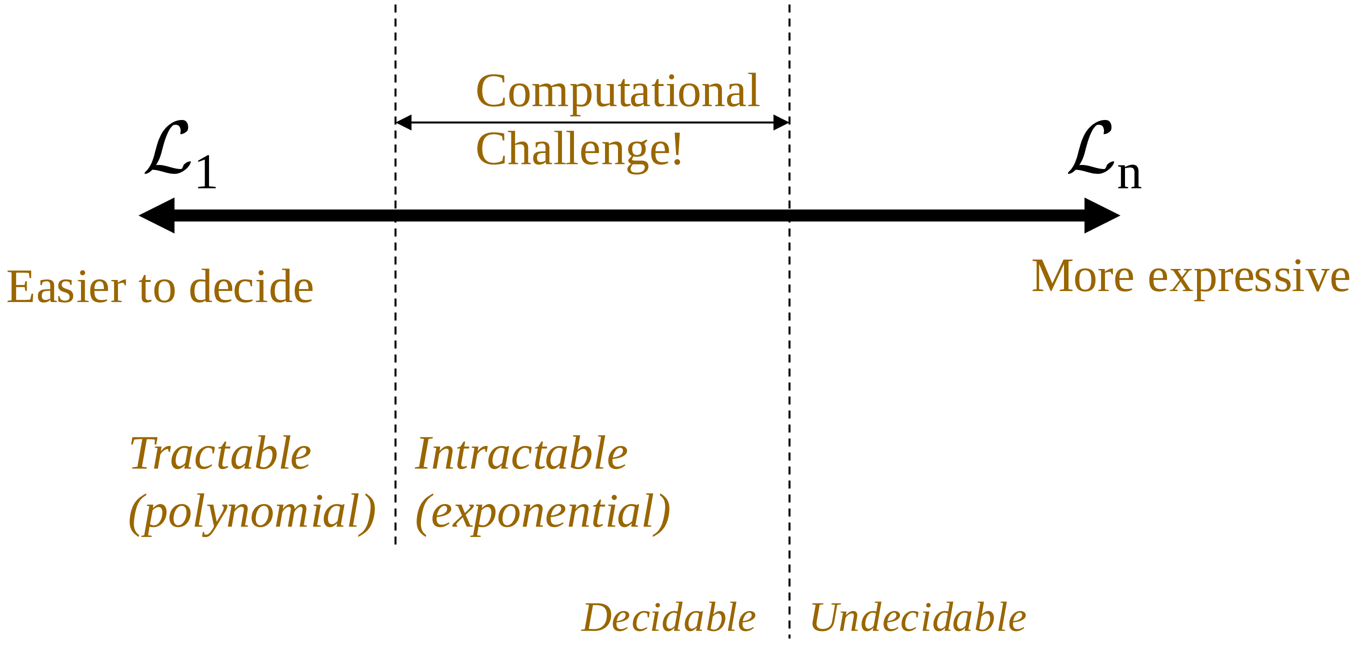
+ (axioms 4–5)

* (axioms 6–7)

These axioms define the semantics of '+'

Tradeoff: expressiveness/computational hardness.

- Assume we are given theories $\mathcal{L}_1 \prec \ldots \prec \mathcal{L}_n$



When is a specific theory useful?

1. Expressible enough to state something interesting.
2. Decidable (or semi-decidable) and more efficiently solvable than richer theories.
3. More expressible, or more natural for expressing some models in comparison to ‘leaner’ theories.

Expressiveness and complexity

- Q1: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two theories whose satisfiability problem is decidable and in the same complexity class. Is the satisfiability problem of an $\mathcal{L}_1$ formula reducible to a satisfiability problem of an $\mathcal{L}_2$ formula?

- Q2: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two theories whose satisfiability problems are reducible to one another. Are $\mathcal{L}_1$ and $\mathcal{L}_2$ in the same complexity class ?