Lesson 7

Monday, September 25, 2023 5:42 PM

1. Determine if the integral converges or diverges. If the integral converges, then state its value.

(a) $\displaystyle\int_0^\infty \frac{1}{x+1}\,dx$

$$\lim_{t\to\infty}\int_1^t \frac{1}{x+1}\,dx = \lim_{t\to\infty} \ln|x+1|\Big|_1^t = \ln|\infty| - \ln|1| = \ln|\infty|$$

$$\boxed{\int_0^\infty \frac{1}{x+1}\,dx \text{ diverges}}$$

(b) $\displaystyle\int_{-\infty}^2 e^{2x}\,dx$

$$\lim_{t\to-\infty}\int_t^2 e^{2x}\,dx = \lim_{t\to-\infty} \frac{1}{2}e^{2x}\Big|_t^2 = \frac{e^4}{2} - \underbrace{\frac{e^{-\infty}}{2}}_{0} = \boxed{\frac{e^4}{2}}$$

$$\boxed{\int_{-\infty}^2 e^{2x}\,dx \text{ converges at } \frac{e^4}{2}}$$

(c) $\displaystyle\int_0^2 \frac{1}{x^3}\,dx$

$$\lim_{t\to0^+}\int_t^1 \frac{1}{x^3}\,dx = \lim_{t\to0^+} -\frac{1}{2}x^{-2}\Big|_t^1 = -\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{0} \text{ undefined}$$

$$\boxed{\int_0^2 \frac{1}{x^3}\,dx \text{ diverges}}$$

(d) $\displaystyle\int_{-1}^{7} \frac{1}{\sqrt[3]{x+1}}\,dx$

$$\lim_{t\to -1^+} \int_t^7 (x+1)^{-\frac{1}{3}}\,dx = \lim_{t\to -1^+} \frac{3}{2}(x+1)^{\frac{2}{3}}\Big|_t^7 = \frac{3}{2}(8)^{\frac{2}{3}} - \underbrace{\frac{3}{2}(t+1)^{\frac{2}{3}}}_{0}$$

$\sqrt[3]{64} = 4$

$\frac{3}{2}\cdot 4 = 6$

$$\boxed{\int_{-1}^{7} \frac{1}{\sqrt[3]{x+1}}\,dx \text{ converges at } 6}$$

(e) $\displaystyle\int_{-\infty}^{2} e^{2x}\,dx$

$$\lim_{t\to -\infty} \int_t^2 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_t^2 = \frac{e^4}{2} - \underbrace{\frac{1}{2e^{\infty}}}_{0}$$

$$\boxed{\int_{-\infty}^{2} e^{2x}\,dx \text{ converges at } \frac{e^4}{2}}$$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{4}{x^2+4}\,dx$

$$\lim_{\substack{t\to\infty \\ j\to -\infty}} \int_j^t \frac{4}{x^2+4}\,dx$$

$u = \frac{x}{2}$

$du = \frac{1}{2}\,dx$ $\qquad dx = 2\,du$

$$\int_j^t \frac{2}{u^2+1} = 2\arctan(u)\Big|_j^t = 2\arctan\left(\frac{x}{2}\right)\Big|_j^t = \underbrace{2\arctan(\infty)}_{\pi} - \underbrace{2\arctan(-\infty)}_{-\pi} = 2\pi$$

$$\boxed{\int_{-\infty}^{\infty} \frac{4}{x^2+4}\,dx \text{ converges at } 2\pi}$$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx$

$$\lim_{\substack{t\to\infty \\ q\to-\infty}} \int_q^t \frac{2x}{(x^2+1)^2}\,dx$$

$u = x^2+1$

$du = 2x\,dx \qquad dx = \frac{du}{2x}$

$$\int_q^t \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_q^t = 0-0$$

$$\boxed{\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx \text{ converges at } 0}$$

(h) $\displaystyle\int_{-32}^{32} \frac{1}{\left(\sqrt[5]{x}\right)^2}\,dx$

$$\lim_{t\to 0^-} \int_{-32}^{t} x^{-\frac{2}{5}}\,dx + \lim_{w\to 0^+} \int_{w}^{32} x^{-\frac{2}{5}}\,dx$$

$$\int x^{-\frac{2}{5}}\,dx = \frac{5}{3}x^{\frac{3}{5}} + C$$

$$\lim_{t\to 0^-} \frac{5}{3}x^{\frac{3}{5}}\Big|_{-32}^{t} + \lim_{w\to 0^+} \frac{5}{3}x^{\frac{3}{5}}\Big|_{w}^{32}$$

$$\underbrace{\frac{5}{3}(0)^{\frac{3}{5}}}_{0} - \underbrace{\frac{5}{3}(-32)^{\frac{3}{5}}}_{-13\frac{1}{3}} + \underbrace{\frac{5}{3}(32)^{\frac{3}{5}}}_{13\frac{1}{3}} - \underbrace{\frac{5}{3}(0)^{\frac{3}{5}}}_{0}$$

$$\boxed{\int_{-32}^{32} \frac{1}{\left(\sqrt[5]{x}\right)^2}\,dx \text{ converges at } 26\tfrac{2}{3} = \frac{80}{3}}$$

(i) $\displaystyle\int_{-\infty}^{0} x\,e^x\,dx$

$$\lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$xe^x - \int e^x dx = \lim_{t\to\infty} xe^x - e^x\Big|_t^0 = \lim_{t\to\infty}(0-1) - (-\infty\cdot 0 - 0) = -1$$

L'Hospital

$$\int \frac{x}{e^{-x}} = -e^x = 0$$

$$\boxed{\int_{-\infty}^0 xe^x\,dx = -1}$$

(j) $\displaystyle\int_0^1 2x\ln x\,dx$

$$\lim_{t\to0}\int_t^1 2x\ln x\,dx$$

$u = 2x \quad dv = \ln x\,dx$

$du = 2dx \quad v = x\ln(x) - x$

$$\lim_{t\to0}\int_t^1 2x\ln x\,dx = 2x(x\ln x - x) - \int 2x\ln x + \int 2x$$

$$2\lim_{t\to0}\int_t^1 2x\ln x\,dx = 2x(x\ln x - x) + x^2$$

$$\lim_{t\to0}\int_t^1 2x\ln x\,dx = x^2\ln x - \frac{x^2}{2}\Big|_0^1 = \left(0-\frac{1}{2}\right)-(0-0) = -\frac{1}{2}$$

$$\lim_{x\to0}\frac{\ln x}{\frac{1}{x^2}} = \frac{\frac{1}{x}}{-2\frac{1}{x^3}} = \frac{-x^3}{2x} = -\frac{x^2}{2} = 0$$

$$\boxed{\int_0^1 2x\ln x\,dx = -\frac{1}{2}}$$

(k) $\displaystyle\int_0^{\pi/2} \tan x\, dx$

$$\lim_{t\to\frac{\pi}{2}} \int_0^t \tan x\, dx$$

$$\tan x = \frac{\sin x}{\cos x}$$

$u = \cos x$

$du = -\sin x\, dx$

$dx = \frac{du}{-\sin x}$

$$\lim_{t\to\frac{\pi}{2}} \int_0^t -\frac{1}{u}\, du = \lim_{t\to\frac{\pi}{2}} -\ln|\cos x|\Big|_0^t = -\ln|0| + \ln|1|$$

undefined

$$\boxed{\int_0^{\frac{\pi}{2}} \tan x\, dx \quad \text{diverges}}$$

(l) $\displaystyle\int_2^{\infty} \frac{1}{x\ln x}\, dx$

$$\lim_{t\to\infty} \int_2^t \frac{1}{x\ln x}\, dx$$

$u = \ln x$

$du = \frac{1}{x} dx$

$dx = x\,du$

$$\lim_{t \to \infty} \int_2^t \frac{1}{u}\,du = \lim_{t \to \infty} \ln|u| \Big|_2^t = \ln|\ln|\infty|| - \ln|\ln|2||$$

undefined

$$\boxed{\int_2^\infty \frac{1}{x \ln x}\,dx \quad \text{diverges}}$$