HW 25

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1. $\mathbf{r}(t)$ is the position of a particle in space at time t. Determine the velocity and acceleration of the particle at time t. Then find the speed and the direction of the particle at the specified value of t and write the velocity at that time as a product of its speed and direction.

   (a) $\mathbf{r}(t) = \langle t+1\,,\, t^2-1\,,\, 2t\rangle \qquad t=1$

   $v(t) = \langle 1, 2t, 2\rangle \quad a(t) = \langle 0, 2, 0\rangle \quad |\vec{v}| = \sqrt{1+4+4} = 3 \quad \hat{v} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle \quad \vec{v}(1) = 3\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$

   (b) $\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + (4t)\,\mathbf{k} \qquad t = \frac{\pi}{2}$

   $v(t) = \langle -2\sin t, 3\cos t, 4\rangle \quad a(t) = \langle -2\cos t, -3\sin t, 0\rangle \quad |\vec{v}| = \sqrt{20} \quad \hat{v} = \langle \frac{-2}{\sqrt{20}}, 0, \frac{4}{\sqrt{20}}\rangle \quad \vec{v}(\frac{\pi}{2}) = \sqrt{20}\langle \frac{-2}{\sqrt{20}}, 0, \frac{4}{\sqrt{20}}\rangle$

   (c) $\mathbf{r}(t) = (2\ln(t+1))\,\mathbf{i} + t^2\,\mathbf{j} + \frac{t^2}{2}\,\mathbf{k} \qquad t=1$

   $V(t) = \langle \frac{2}{t+1}, 2t, t\rangle \quad a(t) = \langle -\frac{2}{(t+1)^2}, 2, 1\rangle \quad |\vec{v}| = \sqrt{1+4+1} = \sqrt{6} \quad \hat{v} = \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle \quad \vec{v}(1) = \sqrt{6}\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$

2. The curves $\mathbf{r}_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2t\,\mathbf{k}$ and $\mathbf{r}_2(s) = (s+1)\,\mathbf{i} + \left(\frac{4}{s}\right)\mathbf{j} + (s^2+3)\,\mathbf{k}$ intersect at the point $(2,4,4)$.

   (a) What is t when $\mathbf{r}_1(t)$ is at the point $(2,4,4)$

   $2i = ti$
   $4j = t^2 j$
   $4k = 2tk$

   $\boxed{t=2}$

   (b) What is s when $\mathbf{r}_2(s)$ is at the point $(2,4,4)$

   $2 = s+1$
   $4 = \frac{4}{s}$
   $4 = s^2+3$

   $\boxed{s=1}$

   (c) Find the angle between $\mathbf{r}_1'$ and $\mathbf{r}_2'$ at the point $(2,2,4)$.

   $$\Theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$$

   $\vec{u} = r_1'(2) = \langle 1, 2\cdot 2, 2\rangle = \langle 1, 4, 2\rangle$

   $\vec{v} = r_2'(1) = \langle 1, -4, 2\rangle$

   $|\vec{u}| = \sqrt{21}$

   $|\vec{v}| = \sqrt{21}$

   $$\Theta = \cos^{-1}\left(\frac{1-16+4}{21}\right) = \boxed{\cos^{-1}\left(-\frac{11}{21}\right) \approx 2.12 \text{ radians}}$$

$|\vec{u}| = \sqrt{21}$

$|\vec{v}| = \sqrt{21}$

$\theta = \cos^{-1}\left(\frac{\ldots}{21}\right) = \boxed{\cos^{-1}\left(-\frac{\ldots}{21}\right) \approx 2.1 \text{ radians}}$

3. $\mathbf{r}(t)$ is the position of a particle in space at time t. Determine the angle between the velocity and acceleration of the particle at time $t = 0$.

(a) $\mathbf{r}(t) = \left\langle 3t+1\,,\ \sqrt{3}\,t\,,\ t^2 \right\rangle$

$\vec{v} = \langle 3, \sqrt{3}, 2t \rangle$

$|\vec{v}| = \sqrt{9 + 3 + 4t^2}$

$\vec{a} = \langle 0, 0, 2 \rangle$

$|\vec{a}| = 2$

$$\theta = \cos^{-1}\left(\frac{0+0+0}{2\sqrt{12}}\right) = \boxed{\frac{\pi}{2}}$$

(b) $\mathbf{r}(t) = \left\langle \frac{\sqrt{2}}{2}t\,,\ 9\,,\ \frac{\sqrt{2}}{2}t - 16t^2 \right\rangle$

$\vec{v} = \left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} - 32t \right\rangle$

$|\vec{v}| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2} - 32t\right)^2}$

$\vec{a} = \langle 0, 0, -32 \rangle$

$|\vec{a}| = 32$

$$\theta = \cos^{-1}\left(\frac{0+0-16\sqrt{2}}{32}\right) = \boxed{\frac{3\pi}{4}}$$

(c) $\mathbf{r}(t) = \left\langle \ln(t+1)\,,\ t\,,\ \frac{t^2}{2} \right\rangle$

$\vec{v} = \left\langle \frac{1}{t+1}, 1, t \right\rangle$

$|\vec{v}| = \sqrt{\left(\frac{1}{t+1}\right)^2 + 1 + t^2}$

$\vec{a} = \left\langle -\frac{1}{(t+1)^2}, 0, 1 \right\rangle$

$|\vec{a}| = \sqrt{\frac{1}{(t+1)^4} + 0 + 1}$

$$\theta = \cos^{-1}\left(\frac{-1+0+0}{\sqrt{2}\cdot\sqrt{2}}\right) = \boxed{\frac{2\pi}{3}}$$

4. Parameterize the tangent line to the smooth curve defined by $\mathbf{r}(t)$ at the indicated point.

(a) $\mathbf{r}(t) = \sin(t)\,\mathbf{i} + \left(t^2 - \cos t\right)\mathbf{j} + e^t\,\mathbf{k}$ $\quad (0, -1, 1)$

$t = 0$

$\vec{V} = \langle \cos t,\ 2t + \sin t,\ e^t \rangle$

$\langle 1, 0, 1 \rangle$

$x = at + x_0 = t + 0$

$y = bt + y_0 = 0 - 1$

$z = ct + z_0 = t + 1$

$$\boxed{\begin{aligned} x &= t \\ y &= -1 \\ z &= t + 1 \end{aligned}}$$

(b) $\mathbf{r}(t) = (\ln t)\,\mathbf{i} + \left(\dfrac{t-1}{t+2}\right)\mathbf{j} + (t \ln t)\,\mathbf{k}$ $\quad (0, 0, 0)$

$t = 1$

$\vec{V} = \left\langle \frac{1}{t},\ \frac{t+2-t+1}{(t+2)^2},\ \ln t + 1 \right\rangle$

$\frac{3}{(t+2)^2}$

$\langle 1, \frac{1}{3}, 1 \rangle$

$$\boxed{\begin{aligned} x &= t \\ y &= \tfrac{1}{3}t \\ z &= t \end{aligned}}$$

(c) $\mathbf{r}(t) = (4t \sin t)\,\mathbf{i} + (5 \cos t)\,\mathbf{j} + (\sin(2t))\,\mathbf{k}$ $\quad (2\pi, 0, 0)$

$t = \frac{\pi}{2}$

$\vec{V} = \langle 4\sin t + 4t\cos t,\ -5\sin t,\ 2\cos(2t) \rangle$

$\langle 4, -5, -2 \rangle$

$$\boxed{\begin{aligned} x &= 4t + 2\pi \\ y &= -5t \\ z &= -2t \end{aligned}}$$