3.1.1. $\frac{d}{dx}\left(\frac{d\Psi}{dx}\right)=0$

$\left(\frac{d\Psi}{dx}\right)=A \rightarrow \Psi(x)=Ax+B$

Please provide a few words that justifies each step. Imagine if you were explaining this to someone and they asked "why" for each step.

At $x=0,\ \Psi=0 \rightarrow 0=0+B \rightarrow B=0$

At $x=d,\ \Psi=V_0 \rightarrow V_0=Ad+0 \rightarrow A=V_0/d$

$\Psi=\left(V_0/d\right)x$

$\Delta\Psi=-\int_a^b E\cdot dl=-\frac{d\Psi}{dx}a_x=-\frac{V_0}{d}a_x$

$Q=\int\rho_s\,dS=\int\varepsilon_0 E\cdot a_n=\int\varepsilon_0\left(-\frac{V_0}{d}a_x\right)dS$

Here it looks like you are taking a dot product between two scalars.

$a_n=a_x$ on lower plate $\quad Q=-\frac{\varepsilon_0 V_0}{d}S$

$a_n=-a_x$ on top plate $\quad Q=\frac{\varepsilon_0 V_0}{d}S$

$C=\frac{|Q|}{V_0}=\frac{\varepsilon_0 S}{d}$

You have only one capacitance using a hybrid of both methods. The point of the problem was to show how two different approaches can be used to find C and they are the same.

3.1.2. $\Psi=\Psi(s),\ \nabla^2\Psi=0$

$\nabla^2\Psi=\frac{1}{s}\frac{d}{ds}\left(s\frac{d\Psi}{ds}\right)=0$

$s\frac{d\Psi}{ds}=A \rightarrow \frac{d\Psi}{ds}=\frac{A}{s} \rightarrow \Psi=A\ln(s)+B$

$\Psi(s=a)=0 \rightarrow 0=A\ln(a)+B \rightarrow B=-A\ln(a)$

$\Psi(s=b)=V_0 \rightarrow V_0=A\ln(b)+B=A\ln(b)-A\ln(a)=A\ln(b/a) \rightarrow A=\frac{V_0}{\ln(b/a)}$

$\Psi=\frac{V_0}{\ln(b/a)}\ln(s)+(-A\ln(a))=\frac{V_0}{\ln(b/a)}(\ln(s)-\ln(a))=\frac{V_0}{\ln(b/a)}\ln(s/a)$

$E=-\partial\Psi/\partial s\, a_s=\frac{-A}{s}a_s=\frac{-V_0}{s\ln(b/a)}a_s$

$Q=\int\varepsilon_0 E\cdot dS=\int_0^{2\pi}\int_0^{L}\frac{V_0\varepsilon_0 s}{s\ln(b/a)}\,dz\,d\Phi=\frac{2\pi L\varepsilon_0 V_0}{\ln(b/a)}$

$C=\frac{Q}{V_0}=\frac{2\pi L\varepsilon_0}{\ln(b/a)} \rightarrow C/L=\frac{2\pi\varepsilon_0}{\ln(b/a)}$

3.1.3. $\nabla^2\Psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = 0$

$$\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = 0 \rightarrow r^2\frac{d\Psi}{dr} = A \rightarrow \frac{d\Psi}{dr} = \frac{A}{r^2}$$

$$\Psi = -\frac{A}{r} + B$$

$$\Psi(r=b) = 0 \rightarrow 0 = -\frac{A}{r} + B \rightarrow B = A/b \qquad \Psi = \frac{-A}{r} + \frac{A}{b} = A\left(\frac{1}{b} - \frac{1}{r}\right)$$

$$\Psi(r=a) = V_0 \rightarrow V_0 = A(1/b - 1/a) \rightarrow A = \frac{V_0}{(1/b - 1/a)}$$

$$\Psi = \frac{V_0\left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$E = -\frac{d\Psi}{dr}a_r = -\frac{A}{r^2}a_r = \frac{V_0}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}a_r$$

$$Q = \int \varepsilon E \cdot dS = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\varepsilon_0\varepsilon_r V_0}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)} r^2 \sin\theta \, d\phi \, d\theta = \frac{4\pi\varepsilon_0\varepsilon_r V_0}{\left(\frac{1}{a} - 1/b\right)}$$

$$C = \frac{Q}{V_0} = \frac{4\pi\varepsilon_0\varepsilon_r}{(1/a - 1/b)}$$

Why is epsilon_r in this problem but not the other problems?

3.1.4.