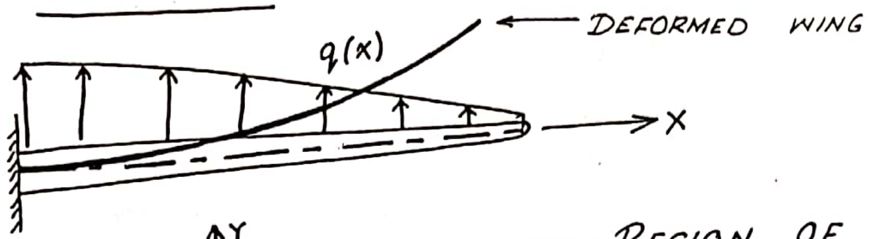
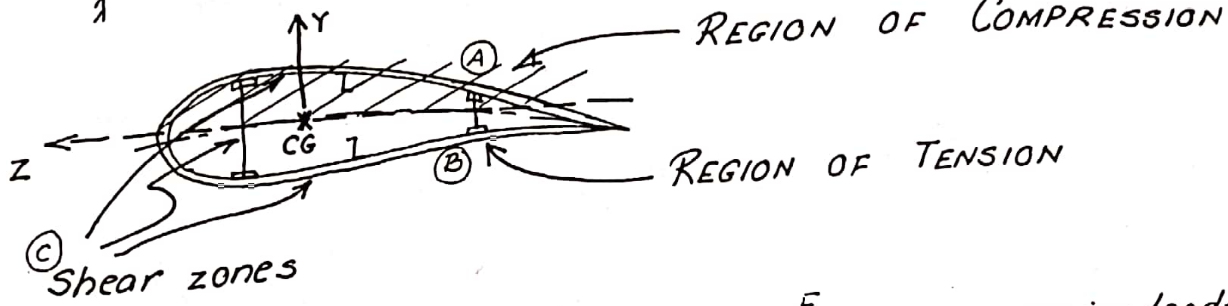
# BUCKLING

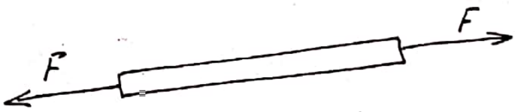



Region A :

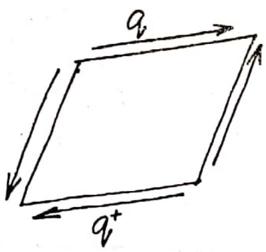

compressive loads (direct)

Region B :



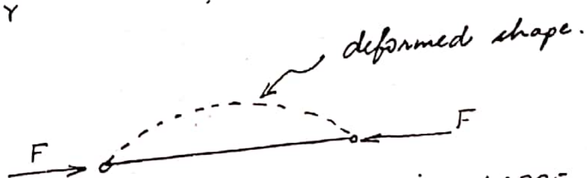

Tensile loads (direct)

Region C :




Shear load (dominant).

* Regions A & C buckle or loose form at a value of stress $\underbrace{\sigma_{rms}}_{\text{equivalent Mises stress}} < \sigma_Y$



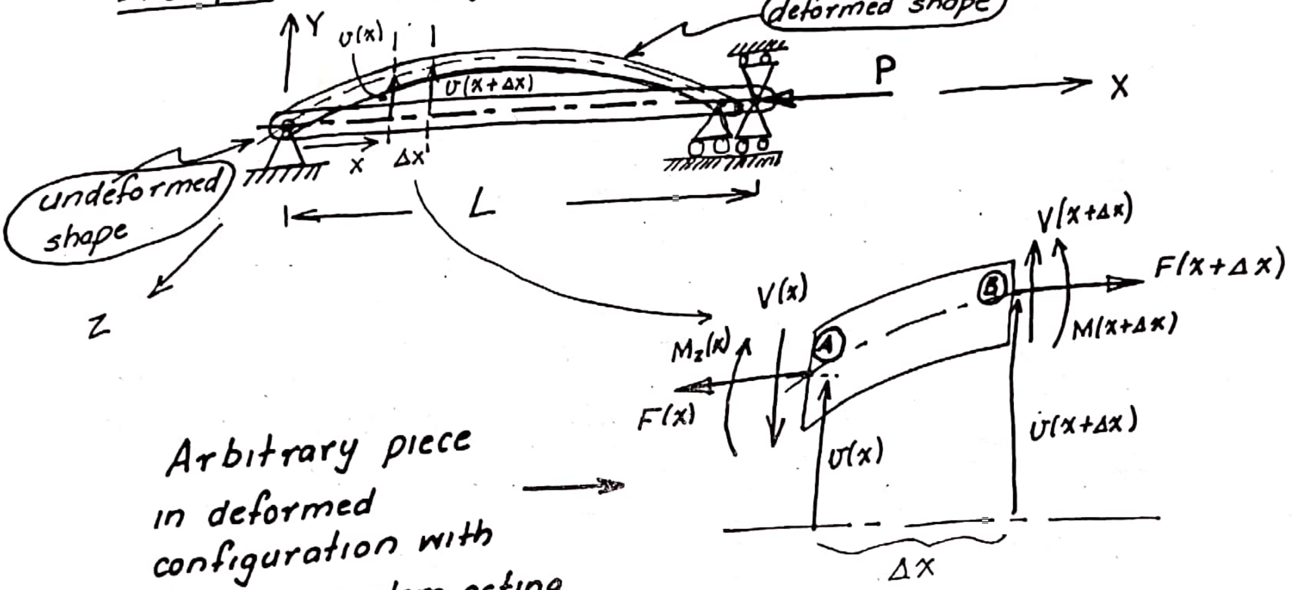

→ The deformation in transverse direction is LARGE, and hence cannot be treated as small deformation.

Q Infinitesimal assumption ⇒ equilibrium in undeformed coordinate system. How can axial load give rise to bending action ?

Here will be our first excursion into <u>finite elasticity</u>

i.e. we look at equilibrium in the <u>DEFORMED configuration</u>.

<u>Example</u>: Simply - supported beam under axial load




Arbitrary piece in deformed configuration with the force system acting

* We have <u>assumed</u> bending in the PLANE of the paper as the <u>only</u> possible alternate configuration

Now, we look at resultant force and moment balance here :

$$\sum F_x = 0 \Rightarrow F(x+\Delta x) - F(x) = 0 \Rightarrow \frac{dF(x)}{dx} = 0$$

$$\Rightarrow F(x) = \text{Constant} \equiv F_0$$

$$\text{But } F(L) = -P \ \langle \text{from B.C.} \rangle \Rightarrow \boxed{F(x) = -P} \quad \text{—ⓐ}$$

$$\sum F_y = 0 \Rightarrow V_y(x+\Delta x) - V_y(x) = 0 \Rightarrow \boxed{\frac{dV_y}{dx} = 0} \quad \text{—ⓑ}$$

* Note that $F(x) = EA\frac{du_0}{dx}$, with $u_0(x)$ is the stretch of centroidal line

$$\sum M_z \big|_A = 0$$

$$\Rightarrow M_z(x+\Delta x) - M_z(x) + V_Y(x+\Delta x)\cdot \Delta x$$

$$\underbrace{- F(x+\Delta x)\cdot\big(v(x+\Delta x) - v(x)\big)}_{\text{this is new, and arises due to the displacement}} = 0$$

$$\Rightarrow \frac{dM_z}{dx} + V_y - F\cdot\frac{dv}{dx} = 0 \quad \text{— (c1)}$$

$$\Rightarrow \frac{d^2 M_z}{dx^2} + \cancelto{0}{\frac{dV_y}{dx}} - \frac{d}{dx}\left(F\frac{dv}{dx}\right) = 0$$

$$\Rightarrow \boxed{\frac{d^2 M_z}{dx^2} + P\frac{d^2 v}{dx^2} = 0} \qquad 0 < x < L \quad \text{— (c)}$$

From (C1) $\boxed{V_y = -\left(\frac{dM_z}{dx} + P\frac{dv}{dx}\right)}$ — (d)

What are the boundary - conditions ?

At $x = 0$

$u_0(0) = 0 \longrightarrow$ axial stretch

$v(0) = 0 \longrightarrow$ transverse displacement constrained

$M_z(0) = 0 \longrightarrow$ unconstrained rotation or ZERO $M_z$

At $x = L$

$EA\frac{du_0}{dx} = -P \longrightarrow$ axial force specified

$v(L) = 0 \longrightarrow$ constrained transverse displacement

$M_z(L) = 0 \longrightarrow$ free rotation or zero $M_z$

(e)

* Note that ⓒ with boundary-conditions ⓔ is an EIGENVALUE problem, as neither $P$ is known nor $v(x)$ is known. We are looking for those values of $P$, i.e. $P_i$, for which the system of equations has a NON-TRIVIAL solution $v_i(x)$.

* There are infinite distinct $P_i$'s possible (eigenvalues) and the corresponding $v_i(x)$ (mode shapes).

How to get these?

Here, let $\quad v(x) = \sum_{i=1}^{\infty} a_i \sin\left(\frac{i\pi x}{L}\right)$

$$\Rightarrow v(0) = v(L) = 0$$
$$M_z(0) = M_z(L) = 0$$

$M_z = EI_{zz} \frac{d^2 v}{dx^2}$

$(I_{yz} = 0) \leftarrow$ ASSUME!

$$\Rightarrow \frac{d^2}{dx^2}\left(EI_{zz}\, v''\right) + P v'' = 0$$

$$\Rightarrow + EI_{zz} \sum_{i=1}^{\infty} a_i \left(\frac{i\pi}{L}\right)^4 \sin\left(\frac{i\pi x}{L}\right) - P \sum_{i=1}^{\infty} a_i \left(\frac{i\pi}{L}\right)^2 \sin\left(\frac{i\pi x}{L}\right) = 0$$

$\Rightarrow$ As each of $\sin\frac{i\pi x}{L}$ is independent, and $a_i = 0$ gives the trivial solution,

$$a_i\left(EI_{zz}\left(\frac{i\pi}{L}\right)^4 - P\left(\frac{i\pi}{L}\right)^2\right) = 0 \Rightarrow$$

$$P_i = \frac{EI_{zz}\left(\frac{i\pi}{L}\right)^4}{\left(\frac{i\pi}{L}\right)^2} = \frac{EI_{zz}\,\pi^2 i^2}{L^2}$$ are the non-trivial, distinct loads at which buckling can happen.

$$P_{min} = \min_i (P_i) = P_1 = P_{CR} \leftarrow \text{CRITICAL BUCKLING LOAD}$$

$$\Rightarrow P_{CR} = \frac{\pi^2 EI_{zz}}{L^2} \rightarrow v_{CR} = a_1 \sin\pi x/L$$

unknown! Can be anything

possible shapes

Here, we have solved the differential equation. We may not be so lucky always. Hence, we need a more generalized tool.

E.g. PRINCIPLE OF VIRTUAL WORK !

$$\int_0^L \left(\delta v \times \frac{d^2}{dx^2}\left(EI_{zz} v''\right) + P v'' \times \delta v\right) = \int_0^L (0 \times \delta v) \qquad 0 < x < L$$

$$\Rightarrow \int_0^L \left(\left(EI_{zz} v''\right)'' \delta v + P v'' \delta v\right) dx = 0$$

$$\Rightarrow \int_0^L \left(EI_{zz} v'' \delta v'' - P v' \delta v'\right) dx + \left(EI_{zz} v''\right)' \cdot \delta v \Big|_0^L + P v' \delta v \Big|_0^L$$
$$- \left(EI_{zz} v'' \cdot \delta v'\right)\Big|_0^L = 0$$

$$\Rightarrow \boxed{\int_0^L EI_{zz} v'' \cdot \delta v'' \, dx = \int_0^L P v' \delta v' dx} + \Big(\overbrace{\left(EI_{zz} v''\right)}^{M_z \to 0} \delta v'\Big)\Big|_0^L$$
$$- \Big\{ \underbrace{\left(\left(EI_{zz} v''\right)' + P v'\right)}_{-V_Y} \overset{0}{\cancel{\delta v}} \Big\}\Big|_0^L$$

Now, at $x = 0 \rightarrow v = 0 \Rightarrow \delta v = 0$

$M_z = 0$ :

$x = L \rightarrow v = 0 \Rightarrow \delta v = 0$

$M_z = 0$

Now use $v(x) = \sum_{i=1}^{\infty} a_i \sin\frac{i\pi x}{L} \Rightarrow \delta v(x) = \sum_{i=1}^{\infty} \delta a_i \sin\frac{i\pi x}{L}$

$$\Rightarrow \int_0^L EI_{zz} \sum_{i=1}^{\infty} a_i (i\pi/L)^2 \sin i\pi x/L \cdot \underbrace{(j\pi/L)^2 \sin j\pi x/L}_{\delta v \,=\, \sin j\pi x/L} dx = P \int_0^L \sum_{i=1}^{\infty} (i\pi/L) a_i \cos i\pi/L \cdot (j\pi/L) \cos j\pi/L \, dx$$

$$\Rightarrow a_j \int_0^L EI_{zz} (j\pi/L)^4 \sin^2 j\pi x/L \, dx = P \cdot a_j \int_0^L (j\pi/L)^2 \cos^2 j\pi x/L \, dx$$

$\{a_j \neq 0\}$

$$\Rightarrow \boxed{P_j = \frac{EI_{zz} j^2 \pi^2}{L^2}} \leftarrow \text{Same as before!}$$

In the generic case, with different boundary conditi
buckling load will change. Note that buckling is a geometry and constraint dependent failure phenomenon.

So, $$P_{CR} = \frac{\pi^2 EI}{l_e^2}$$ ← effective length

with $(l_e) = \beta l$ ← where $l$ is actual length.

| B.C. Type | $\beta$ | |
|---|---|---|
| a) Both ends pinned ($v = M_z = 0$) | 1 |  |
| b) Both fixed ($v = v' = 0$) | 1/2 |  |
| c) One fixed ($v = v' = 0$), other free ($M_z = V_y = 0$) | 2 |  |
| d) One fixed ($v = v' = 0$), other pinned ($v = M_z = 0$) | 0.6998 |  |

Now, let us look at this problem from the point of view of virtual work.

$$\delta v \left( \frac{d^2}{dx^2}\left(EI_{zz}\, v''\right) + \underbrace{P v''}_{-\frac{d}{dx}(F(x)\, v')} \right) = (0) \times \delta v$$

$$\Rightarrow \int_0^L \left\{ (EI_{zz}\, v'')'' - (F v')' \right\} \delta v \, dx = 0$$

$$\Rightarrow \int_0^L \left( EI_{zz}\, v'' \, \delta v'' + F v' \, \delta v' \right) dx = \underbrace{\left\{ -\left(EI_{zz}\, v''\right)' + F \right\}}_{V_y} \delta v \Big|_0^L$$

$$+ \underbrace{EI_{zz}\, v''}_{M_z} \, \delta v' \Big|_0^L = V_y \cdot \delta v \Big|_0^L + M_z \cdot \delta v' \Big|_0^L$$

In case no end-loads are applied : $V_y = M_z = 0$ at free / pinned ends

$\delta v , \delta v' = 0$ at pinned / fixed ends

RHS $= 0$ $\Leftarrow$

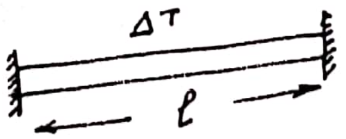
$$\Rightarrow \boxed{\int_0^L EI_{zz}\, v'' \, \delta v'' \, dx = -\int_0^L F \cdot v' \, \delta v' \, dx} \quad \text{— (a)}$$

For special case of end-load $F(x) = -P$ (a)

becomes

$$\int_0^L EI_{zz}\, v'' \, \delta v'' \, dx = P \int_0^L v' \, \delta v' \, dx \quad \text{— (b)}$$

For thermal loading of fixed column



$$\varepsilon_{xx}^{T} = \alpha \Delta T \Rightarrow \varepsilon_{xx} = \frac{1}{E}(\sigma_{xx}) + \varepsilon_{xx}^{T}$$

$\Rightarrow$ For $\varepsilon_{xx} = 0$, $\sigma_{xx} = -E \cdot \varepsilon_{xx}^{T} = -E \cdot \alpha \Delta T$

$$\Rightarrow \boxed{F(x) = -EA \cdot \alpha \Delta T}$$

$\therefore E\alpha \Delta T_{cr} A = P_{cr} = \frac{4\pi^2 E I_{zz}}{\ell^2}$

$$\Rightarrow \boxed{\Delta T_{CR} = \frac{4\pi^2 I_{zz}}{\alpha \cdot \ell^2 \cdot A}} \quad **$$

$\alpha_{Al} \approx 22.2 \times 10^{-6}\, m/(m^{\circ}K)$

$\Rightarrow$ Buckling problem is:

$$\boxed{\int_0^L E I_{zz}\, v''\, \delta v''\, dx = \Delta T \int_0^L E \cdot A \alpha\, v'\, \delta v'\, dx}$$

Now we are looking for $\Delta T_{CR} \rightarrow$ minimum eigenvalue of this problem!

Choice of basis functions:

For pinned - pinned : $\sin\frac{i\pi x}{L}$; $x(L-x)$, $x(L-x)(L/2-x)$, $x(L-x)(L/2-x)(L/4-x)\ldots$

$$\Rightarrow v(x) = \sum_{i=1}^{N} \alpha_i \sin\frac{i\pi x}{L}$$

$$\delta v(x) = \sum_{i=1}^{N} \delta\alpha_i \sin\frac{i\pi x}{L}$$

For fixed - fixed : Use hermite polynomials

e.g. $x^2(L-x)^2$ ; $x^2(L-x)^2 \cdot (L/2 - x)$ ; $x^2(L-x)^2(L/2-x)(L/4-x)$

.....

$\alpha_{Al} = 22.2 \times 10^{-6}\, m/m^{\circ}K$

** Let square section with $a = b = 10mm$, $L = 1m$ $\Rightarrow I_{zz} = \frac{1}{12} \times 10^4 \times 10^{-12}$

$$\Rightarrow \Delta T_{cr} \approx \frac{40 \times 10^{-4}}{12 \times 22.2 \times 10^{-6}} \approx \frac{1000}{66.6} \approx \frac{1000 \times 3}{2 \times 100} = \boxed{15^{\circ}K}$$

* Example of Fixed - Fixed B.C.

Let $v(x) \approx A \underbrace{x^2(L-x)^2}_{(L^2x^2 - 2Lx^3 + x^4)}$ $\Rightarrow$ $\delta v(x) \approx \delta A \cdot x^2(L-x)^2$

$\Rightarrow v' = A(2L^2x - 6Lx^2 + 4x^3)$

$v'' = A(2L^2 - 12Lx + 12x^2)$

$$\Rightarrow \int_0^L EI_{zz} \underbrace{(2L^2 - 12L\cdot x + 12x^2)^2}_{\substack{(4L^4 + 144L^2x^2 + 144x^4 \\ -48L^3x + 48L^2x^2 - 288Lx^3)}} dx = P\int_0^L \underbrace{(2L^2x - 6Lx^2 + 4x^3)^2}_{\substack{(4L^4x^2 + 36L^2x^4 + 16x^6 \\ -24L^3x^3 + 16L^2x^4 - 48Lx^5)}} dx$$

$$\Rightarrow EI_{zz}\left[4L^4\cdot x + \frac{144}{3}L^2x^3 + \frac{144}{5}x^5 - \frac{48}{2}L^3x^2 + \frac{48}{3}L^2x^3 - \frac{288Lx^4}{4}\right]_0^L$$

$$\downarrow$$

$$EI_{zz}\left[4 + \frac{144}{3} + \frac{144}{5} - 24 + \frac{48}{3} - 72\right]L^5$$

$$\downarrow$$

$$\frac{60 + 720 + 432 - 360 + 240 - 1080}{15}$$

$$= \frac{12}{15}$$

$$P\left[\frac{4}{3}L^4x^3 + \frac{36}{5}L^2x^5 + \frac{16}{7}x^7 - \frac{24}{4}L^3x^4 + \frac{16}{5}L^2x^5 - \frac{48}{6}Lx^6\right]_0^L$$

$$\downarrow$$

$$P\left[\frac{4}{3} + \frac{36}{5} + \frac{16}{7} - 6 + \frac{16}{5} - 8\right]L^7$$

$$\frac{140 + 756 + 240 - 630 + 336 - 840}{105}$$

$$= \frac{2}{105}$$

$$\Rightarrow P_{CR} \approx \frac{\cancel{105}^{\,7}}{\cancel{2}} \times \frac{\cancel{12}^{\,6}}{\cancel{15}} \frac{EI_{zz}}{L^2} \approx 42\frac{EI_{zz}}{L^2} > \frac{\overset{39.5}{4\pi^2} EI_{zz}}{L^2}$$

* Note that the approximation will give a value higher than the actual, but nevertheless, is close.

$v(x) \approx A\left(1 - \cos\frac{2\pi x}{L}\right) \Rightarrow v'(x) = A\left(\frac{2\pi}{L}\right).\sin\frac{2\pi x}{L};$

$v'' = A\left(\frac{2\pi}{L}\right)^2.\cos\frac{2\pi x}{L}$

$$\Rightarrow EI_{zz}\int_0^L (2\pi/L)^4 \underbrace{\cos^2\frac{2\pi x}{L}}_{\frac{1}{2}\left(1+\cos\frac{4\pi x}{L}\right)} dx = P\int_0^L (2\pi/L)^2 \underbrace{\sin^2\frac{2\pi x}{L}}_{\frac{1}{2}\left(1-\cos\frac{4\pi x}{L}\right)}$$

$$\Rightarrow EI_{zz}\left(\frac{2\pi}{L}\right)^4\cdot\frac{L}{2} = P\left(\frac{2\pi}{L}\right)^2\cdot\frac{L}{2}$$

$$\Rightarrow P_{CR} \approx \frac{4\pi^2 EI_{zz}}{L^2} \leftarrow \text{EXACT! (WHY?)}$$