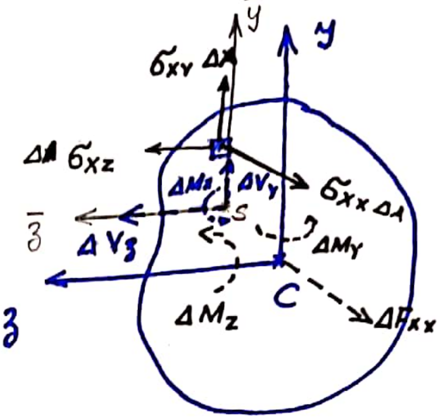
Review of Resultants

- Stresses on face:

  σ_{xx}, σ_{xy}, σ_{xz}



— Shift $\Delta F_{xx} = \sigma_{xx}\Delta A$ to C to get

ΔF_{xx}, $\Delta M_y|_c$, $\Delta M_z|_c$

~ C : point of zero axial displacement ≈ area centroid or modulus weighted centroid

— Shift ΔF_{xy}, ΔF_{xz} to S → point of zero twist induced displacement

This gives ΔV_z, ΔV_y, ΔM_x

— Combined effect of these shifts : * F_{xx}, $M_y|_c$, $M_z|_c$ at C.

* $\underbrace{V_y, V_z}$, $\underbrace{M_x|_s}$ at S.

decouple here !

V_y, V_z, $M_x|_s = 0$ + $V_y, V_z = 0$; $M_x|_s$

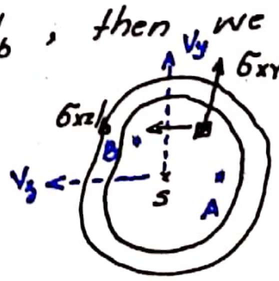
V_y, V_z → bending induced shear $(\sigma_{xy}|_b, \sigma_{xz}|_b)$

M_x → torsion induced shear $(\sigma_{xy}|_t, \sigma_{xz}|_t)$

or $\sigma_{xy} = \sigma_{xy}|_b + \sigma_{xy}|_t \Rightarrow q_{xs} = q_b + q_t$

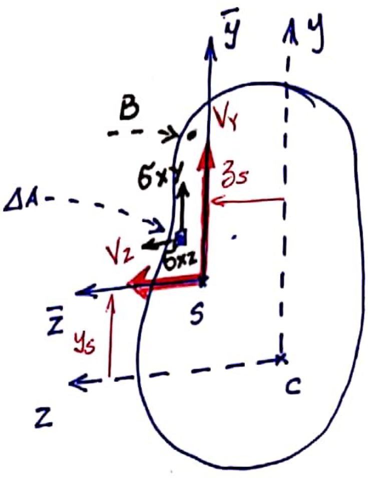
$\sigma_{xz} = \sigma_{xz}|_b + \sigma_{xz}|_t$

⇒ If somehow we get $\sigma_{xy}|_b$, $\sigma_{xz}|_b$, then we can find location of S as:



Take moment due to σ_{xy}, σ_{xz} about point B (chosen by me!)

Moment about B due to V_y, V_z



$$= V_y \cdot (z_B - z_s) - V_z (y_B - y_s)$$

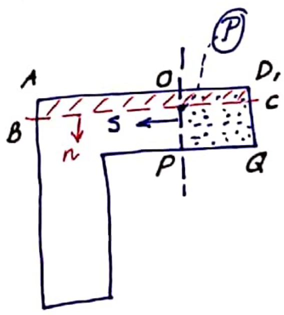
$$= \int_A \left(-\sigma_{xy} (z - z_B) + \sigma_{xz} (y + y_B) \right) dA$$

$$= \int_A (\sigma_{xz} y - \sigma_{xy} z) dA + \underbrace{z_B \int_A \sigma_{xy} dA}_{z_B V_y} \underbrace{- y_B \int_A \sigma_{xz} dA}_{- y_B V_z}$$

$$\Rightarrow \boxed{- V_y z_s + V_z y_s = \int_A (\sigma_{xz} y - \sigma_{xy} z) dA}$$

* $\sigma_{xy}|_b$, $\sigma_{xz}|_b$ are linear functions of V_y, V_z as

$$\left.\begin{array}{l} \sigma_{xy}|_b = \sigma_{xs} \text{ or } \sigma_{xn} \\ \sigma_{xz}|_b = \sigma_{xs} \text{ or } \sigma_{xn} \end{array}\right\} \text{depending whether } y \text{ or } z \text{ is the local } s, n$$



Cut area ABCD gives $\sigma_{xn} = \sigma_{xy}$ at P

" " OPQR given $\sigma_{xs} = \sigma_{xz}$ at P

$$\sigma_{xs}, \sigma_{xn} = V_z \frac{\left(-I_{yz} Q_z^{(1,2)} + I_{zz} Q_y^{(1,2)}\right)}{\Delta\, t_{loc}} + V_y \frac{\left(I_{yy} Q_z^{(1,2)} - I_{yz} Q_y^{(1,2)}\right)}{\Delta\, t_{loc}}$$

$$\left[\Delta = I_{yy} I_{zz} - I_{yz}^2 \right]$$

$Q^{(1)} \rightarrow$ 1st moment of area OPQR

$Q^{(2)} \rightarrow$ 1st moment of area ABCD

* We will elaborate on this later

S $\Rightarrow$ SHEAR CENTRE

C $\Rightarrow$ CENTROID ~ comes out of requirement of $F_x = 0$ due to $\sigma_{xx}|_b$.

REVIEW OF TORSION

→ Shifting the shear stress contribution to S gives us

V_y , V_z , $M_x|_S$

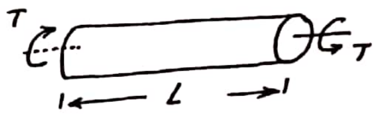
→ We have seen that this shift allows us to partition σ_{xy}, σ_{xz} in terms of that due to (V_y, V_z) alone, and that due to $M_x|_S$ $(= T)$.

We thus consider how to get $\sigma_{xy}|_t$, $\sigma_{xz}|_t$ ~ due to $M_x|_S$ only.

Case studied :

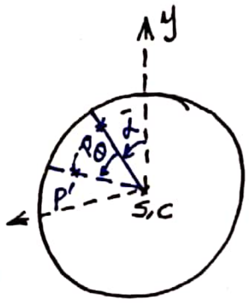
a) $M_x|_S$ = Constant, T ~ pure torsion

b) Ends are unconstrained ~



Known Result : Circular section

Point P at location $(r, \bar{\alpha})$ moves to new location P' with $(r, \bar{\alpha}+\theta)$ where θ is the twist induced due to T.



⟨θ is small⟩

Thus, in y-z coordinates

Initial location of P is : $y_p = r\cos\bar{\alpha}$, $z_p = r\sin\bar{\alpha}$

Final location of P is : $y_{p'} = r\cos(\bar{\alpha}+\theta)$, $z_{p'} = r\sin(\bar{\alpha}+\theta)$

∴ Total displacement of P, due to T, is

$$v = y_{p'} - y_p = r\cos(\bar{\alpha}+\theta) - r\cos\bar{\alpha} \approx -r\sin\bar{\alpha}\cdot\theta$$
$$\approx -z_p\cdot\theta$$

$$w = z_{p'} - z_p = r\sin(\bar{\alpha}+\theta) - r\sin\bar{\alpha} \approx r\cos\bar{\alpha}\,\theta$$
$$\approx y_p\cdot\theta .$$

$u = 0$ ← no out of plane deformation.

Thus, displacement due to torsion T, for a solid circular section is:

$$u = 0$$

$$v(x,y,z) = -z\,\theta(x)$$

$$w(x,y,z) = y\,\theta(x).$$

State of strain:

$$\varepsilon_{xx} = u_{,x} = 0\,; \quad \varepsilon_{yy} = v_{,y} = 0\,; \quad \varepsilon_{zz} = w_{,z} = 0$$

$$\gamma_{xy} = u_{,y} + v_{,x} = -z\,\theta_{,x} = -z\alpha\,; \quad \gamma_{xz} = u_{,z} + w_{,x} = y\,\theta_{,x} = y\alpha$$

$$\gamma_{yz} = v_{,z} + w_{,y} = 0 \qquad \langle\, \alpha = \theta_{,x} \rightarrow \text{rate of twist} \,\rangle$$

State of stress:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0\,; \quad \sigma_{xy} = G\gamma_{xy} = -Gz\alpha\,;$$

$$\sigma_{xz} = G\gamma_{xz} = Gy\alpha$$

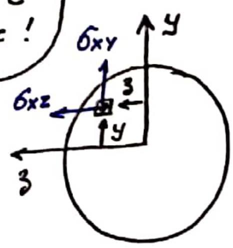
Resultant due to stresses $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$

$$\sigma_{xx} = 0 \Rightarrow F_x,\ M_y,\ M_z = 0$$

$$\sigma_{xy}, \sigma_{xz} \Rightarrow V_y = \int_A \sigma_{xy}\,dA = \int_A -G\alpha \cdot z\,dA = 0 \Rightarrow \int_A z\,dA = 0$$

$$V_z = \int_A \sigma_{xz}\,dA = G\alpha \int_A y\,dA = 0$$

S and C coincide !



$$M_x|_S = T = \int_A (\sigma_{xz}\,y - \sigma_{xy}\,z)\,dA$$

$$= G\alpha \int_A (y^2 + z^2)\,dA = G\alpha \cdot \underbrace{I_p}_{\text{polar moment of inertia}}$$

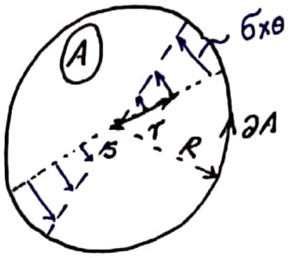
$$= G\alpha \int_{\bar{\alpha}=0}^{2\pi} \int_{r=0}^{R} r^2\, r\,dr\,d\bar{\alpha} = G\alpha \cdot 2\pi \frac{R^4}{4} = G\alpha \underbrace{\left(\frac{\pi R^4}{2}\right)}_{J}$$

$$= \underbrace{GJ}_{\text{torsional rigidity}}\,\alpha$$

$$\Rightarrow \boxed{\alpha = \frac{T}{GJ}} \Rightarrow \theta(x) = \theta(0) + \frac{Tx}{GJ}$$

CONSTANT!

In cylindrical coordinate system, the state of stress becomes: $\sigma_{xx} = 0$; $\sigma_{x\theta} = G\alpha r$; $\sigma_{xr} = 0$; all others also are zero.



S: Point of zero shear stress, zero displacement ~ Centre of rotation

Point of maximum $\sigma_{x\theta} \rightarrow r = R$ (outer periphery).

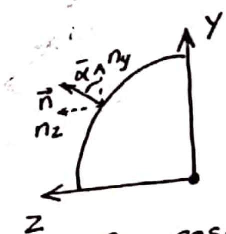
* How good is this constructed displacement, strain and stress field?

— "Exact", for the unconstrained shaft.

In A

$$\left\{ \begin{array}{l} \sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} = 0 \\ \sigma_{xy,x} + \sigma_{yy,y} + \sigma_{yz,z} = 0 \\ \sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0 \end{array} \right\}$$

satisfied as $\frac{d\alpha}{dx} = 0$ <CHECK!!>



On ∂A:

$$t_x = \sigma_{xx} \overset{0}{n_x} + \sigma_{xy} n_y + \sigma_{xz} n_z$$
$$= G\alpha\left[-z\,y/R + y\,z/R\right] = 0$$

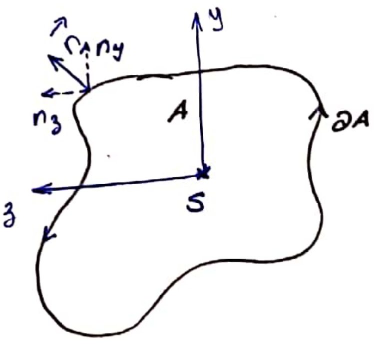
$$t_y = \sigma_{xy} \overset{0}{n_x} + \overset{0}{\sigma_{yy}} n_y + \overset{0}{\sigma_{yz}} n_z = 0$$

$$t_z = \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z = 0$$

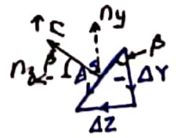
$n_y = \cos\alpha$
$= y/R$
$n_z = \sin\alpha =$
z/R

Thus both lateral B.C. and equilibrium equations are satisfied along with a resultant T on the longitudinal faces.

NON-CIRCULAR SOLID SECTION



Naively assuming that what worked for circular section, will also work for this arbitrary one also gives: $\sigma_{ij,j} = 0$ in A



$$n_y = \sin\beta = \frac{\partial z}{\partial s}$$

$$n_z = \cos\beta = -\frac{\partial y}{\partial s}$$

$$t_x = \sigma_{xx} \cancel{n_x} + \sigma_{xy} n_y + \sigma_{xz} n_z = G\alpha\left(-z\,\partial z/\partial s - y\,\partial y/\partial s\right)$$

$t_y, t_z = 0$

$$= -G\alpha\left(\frac{1}{2}\frac{\partial}{\partial s}(y^2+z^2)\right) \neq 0$$

$$\begin{bmatrix} C_{55} & C_{66} \\ C_{66} & C_{55} \end{bmatrix} \begin{Bmatrix} \hat{u}_{11,\hat{y}\hat{y}} \\ \hat{u}_{11,\hat{z}\hat{z}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$u_{11,\hat{y}\hat{y}} = 0$, $u_{11,\hat{z}\hat{z}} = 0$ $\longrightarrow$ $U_{11} \rightarrow$ linear in $\hat{y}, \hat{z}$

So, the inherited displacement field is not enough, We need to augment it $\rightarrow$ relax the assumption of $u = 0$

<u>Modified displacement field</u>

$$u(x,y,z) = \alpha\, \varphi(y,z)$$

$$v(x,y,z) = -z\, \theta(x) \quad ; \quad w(x,y,z) = y\, \theta(x) \qquad \langle \theta(x) = \theta(0) + \alpha x \rangle$$

$\Rightarrow \quad \varepsilon_{xx} = 0$ (as α is constant)

$$\varepsilon_{yy} = \varepsilon_{zz} = 0 \ ; \quad \gamma_{yz} = 0 \ ; \quad \gamma_{xz} = \alpha(\varphi_{,z} + y)\ ;$$

$$\gamma_{xy} = \alpha(\varphi_{,y} - z)$$

* Note here that y, z is with respect to centre of twist S.

Again,
$$\boxed{\begin{aligned} &\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0 \\ &\sigma_{xy} = G\alpha(\varphi_{,y} - z)\ ; \quad \sigma_{xz} = G\alpha(\varphi_{,z} + y) \end{aligned}}$$

$\varphi(y,z) \Rightarrow$ WARPING FUNCTION

How to get $\varphi(y,z)$? Put φ in the equilibrium eqn. and lateral B.C. $\Rightarrow (G\alpha\,(\varphi_{,y} - z))_{,y} + (G\alpha\,(\varphi_{,z} + y))_{,z} = 0 \leftarrow$ satisfied $\Rightarrow \boxed{\varphi_{,yy} + \varphi_{,zz} = 0}$ in A

$$\cancel{\sigma_{xx,x}} + \sigma_{xy,y} + \sigma_{xz,z} = 0$$

$$\left.\begin{aligned} &\cancel{\sigma_{yx,x}} + \cancel{\sigma_{yy,y}} + \cancel{\sigma_{yz,z}} = 0 \\ &\cancel{\sigma_{zx,x}} + \cancel{\sigma_{zy,y}} + \cancel{\sigma_{zz,z}} = 0 \end{aligned}\right\} \text{trivially satisfied}$$

On ∂A:

$$t_x = \sigma_{xy} n_y + \sigma_{xz} n_z = 0$$

$$\Rightarrow \quad G\alpha(\varphi_{,y} - z)\frac{\partial z}{\partial s} + G\alpha(\varphi_{,z} + y)\cdot\left(-\frac{\partial y}{\partial s}\right) = 0$$

$$\Rightarrow \quad \underbrace{\varphi_{,y} n_y + \varphi_{,z} n_z}_{\frac{\partial \varphi}{\partial n} = \varphi_{,n}} = \frac{\partial}{\partial s}\left(\frac{1}{2}(y^2 + z^2)\right) \quad \text{on } \partial A$$

Problem to solve:

$$\Delta \varphi = 0 \quad \text{in } A$$

$$\frac{\partial \varphi}{\partial n} = \frac{\partial}{\partial s}\left(\frac{1}{2}(y^2 + z^2)\right) \quad \text{on } \partial A \quad \longleftarrow \text{Neumann B.C.}$$

FROM COMPLEX ANALYSIS . . .

$\chi(y,z) = \phi(y,z) + i\psi(y,z)$ is harmonic

$\Delta\varphi = 0 \; ; \; \Delta\psi = 0$ in A

Then Cauchy - Reimann conditions give:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial z} \; ; \; \frac{\partial\varphi}{\partial z} = -\frac{\partial\psi}{\partial y}$$

$\therefore \; \Delta\varphi = \Delta\psi = 0$ and $\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial y} n_y + \frac{\partial\varphi}{\partial z} n_z$

$$= \frac{\partial\psi}{\partial z} n_y - \frac{\partial\psi}{\partial y} n_z = \frac{\partial\psi}{\partial z}\cdot\frac{\partial z}{\partial s} - \frac{\partial\psi}{\partial y}\left(-\frac{\partial y}{\partial s}\right) = \frac{\partial\psi}{\partial s}$$

$\therefore \; \frac{\partial\varphi}{\partial n} = \frac{\partial\psi}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{2}(y^2+z^2)\right)$ on ∂A

$\Rightarrow \; \psi = \frac{1}{2}(y^2+z^2) + C$ on ∂A $\longleftarrow$ Dirichlet B.C.

Taking $C = 0$ will not change anything (for simply - connected domains)

$\therefore \; \sigma_{xy} = G\alpha(\varphi_{,y} - z) = G\alpha(\psi_{,z} - z)$;

$\sigma_{xz} = G\alpha(\varphi_{,z} + y) = G\alpha(-\psi_{,y} + y) = -G\alpha(\psi_{,y} - y)$

Another transformation

$\bar{\psi} = \psi - \frac{1}{2}(y^2+z^2) \quad \Rightarrow \Delta\psi = \Delta\bar{\psi} + 2 = 0$

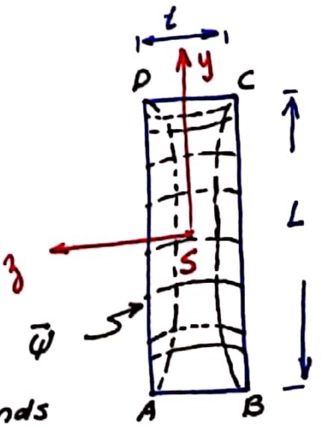
$$\Rightarrow \boxed{\begin{aligned} &\Delta\bar{\psi} = -2 \quad \text{in } A \\ &\bar{\psi}\big|_{\partial A} = \left(\psi - \tfrac{1}{2}(y^2+z^2)\right)\big|_{\partial A} = 0 \end{aligned}}$$

$\bar{\psi} \rightarrow$ PRANDTL STRESS FN.

$$\Rightarrow \boxed{\begin{aligned} &\sigma_{xy} = G\alpha(\psi_{,z} - z) = G\alpha\bar{\psi}_{,z} \\ &\sigma_{xz} = -G\alpha\bar{\psi}_{,y} \end{aligned}}$$

Where do we use it ?

$\rightarrow$ Long, thin rectangular strip $(t \ll L)$ {S is the shear centre}



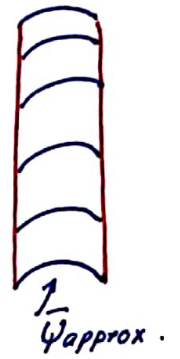
$\bar{\psi} = 0$ on ∂A

Approximation $\rightarrow$ ignore the far off ends AB, CD

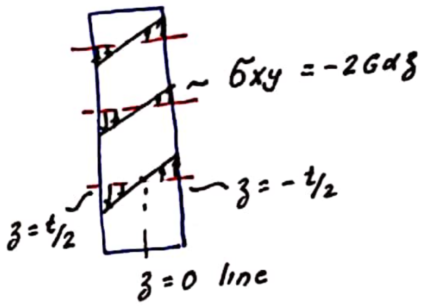
$\bar{\varphi}_{approx} \approx \bar{\varphi}(z)$ (independent of y)

$$\Delta\bar{\varphi} \approx \frac{\partial^2\bar{\varphi}}{\partial z^2} = -2 \Rightarrow \bar{\varphi} = -z^2 + A_1 z + A_2$$

$$\bar{\varphi}(\pm t/_2) = 0 \Rightarrow \bar{\varphi} = -z^2 + (t/_2)^2$$
$$= \{(t/_2)^2 - z^2\}$$



$$\sigma_{xy} = G\alpha\bar{\varphi}_{,z} = -2G\alpha z$$
$$\sigma_{xz} \approx 0$$



How to find α ?

$$T = \int_A (\sigma_{xz}\, y - \sigma_{xy}\, z)\, dA$$
$$= G\alpha \int_A (-\bar{\varphi}_{,y}\, y - \bar{\varphi}_{,z}\, z)\, dA$$
$$= G\alpha\left[+ \int_A 2\bar{\varphi}\, dA - \underbrace{\int_{\partial A}(\cancel{\bar{\varphi}}^{0} y\, n_y + \cancel{\bar{\varphi}}^{0} z\, n_z)\, ds}_{0} \right] = G\alpha \underbrace{\left(2\int_A \bar{\varphi}\, dA\right)}_{J}$$

$$\boxed{J = 2\int_A \bar{\varphi}\, dA}$$

For the long, thin rectangular strip,

$$J = 2\int_A \left((t/_2)^2 - z^2 \right) dA = \frac{Lt^3}{3}$$

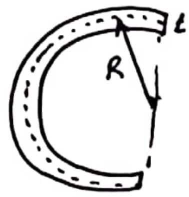
$$\Rightarrow \alpha = \frac{T}{GJ} = \frac{3T}{G \cdot L t^3} \Rightarrow \sigma_{xy} \approx -\frac{6Tz}{Lt^3}$$

$\sigma_{xy}|_{max}$ at $z = \pm t/_2 \Rightarrow |\sigma_{xy}|_{max} = \frac{3T}{Lt^2}$

* $\sigma_{xy}|_{average} = q_{xy}/t = 0$ in this case
  $\Rightarrow$ NO SHEAR FLOW !
* Only contributor to torque is the couples formed by σ_{xy} at $\pm z$.

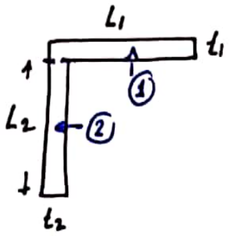
Scanned by CamScanner

Extension to slender open sections

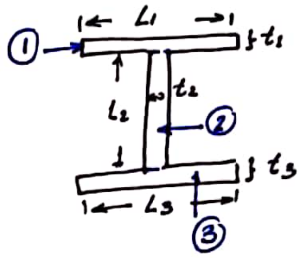


$L = \pi R$

$J = \dfrac{L t^3}{3} = \dfrac{\pi R t^3}{3}$



$J = J_1 + J_2$
$= \dfrac{L_1 t_1^3}{3} + \dfrac{L_2 t_2^3}{3}$

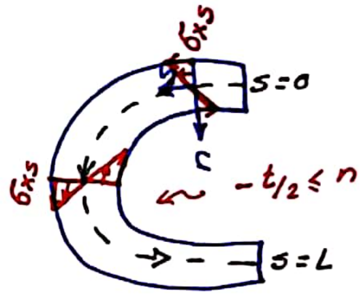


$J = J_1 + J_2 + J_3$
$= \dfrac{L_1 t_1^3}{3} + \dfrac{L_2 t_2^3}{3} + \dfrac{L_3 t_3^3}{3}$

$\boxed{\sigma_{xs} \approx -2G\alpha n \ ; \ \sigma_{xn} \approx 0}$

where s is tangential middle curve
n is local normal to tangent curve

* Tangential shear stress σ_{xs} is linear dominant shear stress, is linear in n and anti-symmetric



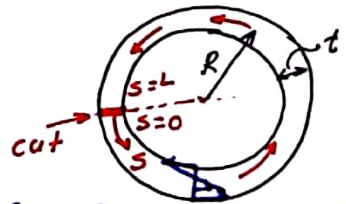
$\sim -t/2 \le n \le t/2$

$0 \le s \le L$

* At the ends $y = \pm L/2$, σ_{xn} will be large and it will have a significant contribution to T (Large moment arm !). } J has this contribution!

Cut - circular thin segment

$L = 2\pi R$, $J = \frac{1}{3} L t^3$
$= \frac{1}{3} \cdot (2\pi R) t^3$



$R_{mid} = \dfrac{R_{out} + R_{in}}{2}$

$\sigma_{xs} = \sigma_{x\theta}$
$\approx -2G\alpha n$
$\approx +2G\alpha(r - R_{mid})$

$\sigma_{xn} = \sigma_{xr} \approx 0$

$\sigma_{xs} = \sigma_{x\theta} \approx +2G\alpha(r - R_{mid})$

Closed - thin circular segment

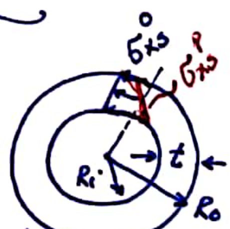
- Circular section result is valid or

$$\sigma_{xs} = \sigma_{x\theta} = G\alpha r = \underbrace{G\alpha R_{mid}}_{\sigma_{xs}^{0}} + \underbrace{G\alpha (r - R_{mid})}_{\sigma_{xs}^{p}}$$

σ_{xs}^{0} is fixed or constant with n or r

σ_{xs}^{p} is linear with r with $\sigma_{xs}^{p}(R_{mid}) = 0$

$|\sigma_{xs}^{p}|_{max} \sim G\alpha \cdot t/2$



$\langle t = R_o - R_i \rangle$

$$\frac{|\sigma^{p}_{xs}|_{max}}{\sigma^{0}_{xs}} \approx \frac{G\alpha \cdot t/2}{G\alpha \cdot R_{mid}} \rightarrow 0 \quad \text{as} \quad \frac{t}{R_{mid}} \rightarrow 0$$

$\therefore$ In the closed circular ring σ^{0}_{xs} is dominant

$\therefore$ $\sigma_{xs} \approx \sigma^{0}_{xs}$ is a GOOD APPROXIMATION

Shear flow $q_{xs} = \int_{-t/2}^{t/2} \sigma_{xs}\, dn \approx \sigma^{0}_{xs} \cdot t$

σ^{0}_{xs} is constant along the ring, i.e. as s varies

$\therefore$ $q_{xs}(s) = q_0 = \sigma^{0}_{xs} \cdot t$

How much is q_0? Moment due to $q_0 \Rightarrow 2A \cdot q_0 = T$

$$\Rightarrow q_0 = \frac{T}{2A} = \frac{T}{2(\pi R_{mid}^2)} = \frac{T}{2\pi R_{mid}^2}$$

$$\Rightarrow \sigma^{0}_{xs} \approx \sigma_{xs} = \frac{T}{2\pi R_{mid}^2 \cdot t} \approx G\alpha R_{mid} \Rightarrow \alpha = \frac{T}{\underbrace{2\pi R_{mid}^3 t G}_{GJ}}$$

$$\Rightarrow \boxed{J \approx 2\pi R_{mid}^3 t}$$

* Exact calculation:

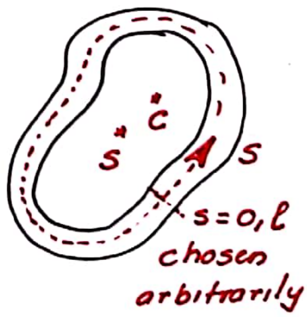
$$J = \frac{\pi}{2}\left(R_o^4 - R_i^4\right) \quad \text{CHECK!}$$

$$\approx 2\pi R_{mid}^3 \cdot t$$

Lesson: $\sigma_{xs} \approx \sigma^{0}_{xs}$ in a closed thin slender section

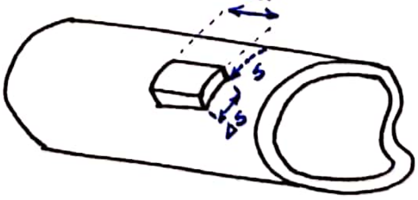
$\sigma_{xn} \approx 0$

$\Rightarrow q_{xs}(s) \approx \sigma^{0}_{xs}(s)\, t(s) = q_0(s)$



What about $q_0(s)$?

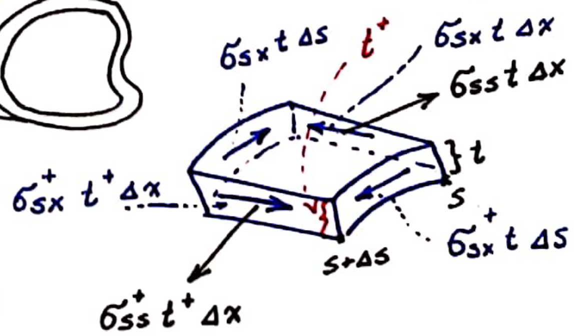
- Take an infinitesimal strip of size $\Delta s \times \Delta x \times t$



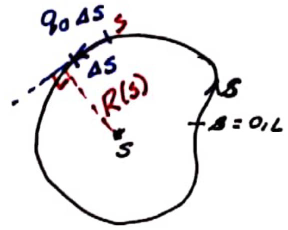
- $\sum F_x = 0 \Rightarrow$

$$\underbrace{\sigma^{+}_{sx}\, t^{+}}_{q^{+}_{sx}} \Delta x - \underbrace{\sigma_{sx}\, t}_{q_{sx}} \Delta x = 0$$

$$\Rightarrow \frac{\partial}{\partial s}(q_{sx}) = 0 \Rightarrow q_{sx} = q_0(s) \text{ is constant}$$



Twisting moment due to q_0



Take an infinitesimal element at s, of size Δs.

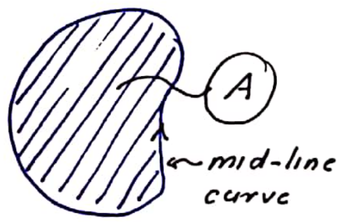
Tangential force due to q_0 = $q_0 \Delta s = \Delta F_s$

Torque at S, due to ΔF_s, $\Delta T = q_0 \Delta s \cdot R(s)$

$$\therefore \text{ Total torque } T = \int_{s=0}^{L} q_0 R(s)\, ds = q_0 \int_{s=0}^{L} R(s)\, ds$$

$R(s)\Delta s = 2\Delta A$, where ΔA is area of shaded triangle at S.

$\therefore T = 2A q_0$; A is the area enclosed by the mid-line.

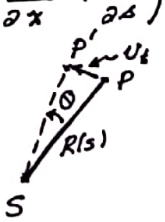


$$\Rightarrow \boxed{q_0 = \frac{T}{2A}}$$ ← BREDT - BATHO EQN.

Note that $q_0 = \sigma_{xs} t(s) \Rightarrow \sigma_{xs}(s) = \dfrac{q_0}{t(s)}$

$= G\gamma_{xs} = G\left(\dfrac{\partial v_t}{\partial x} + \dfrac{\partial u}{\partial s}\right)$

* $v_t(s)$ is the tangential displacement along the s- curve



Now $v_t(s) \approx R(s)\,\theta(x)$

$$\Rightarrow \frac{q_0}{G\,t(s)} = v_{t,x} + u_{,s} = R(s)\alpha + u_{,s}$$

$$\Rightarrow \oint \frac{q_0\, ds}{Gt} = \alpha \underbrace{\oint R(s)\, ds}_{2A} + \oint u_{,s}\, ds$$ → $u|_L - u|_0 = 0$ as u is continuous

$$\Rightarrow \boxed{\alpha = \frac{q_0}{2AG} \oint \frac{ds}{t}} = \frac{T}{4A^2 G} \oint \frac{ds}{t} \Rightarrow J = \frac{4A^2}{\oint \frac{ds}{t}}$$

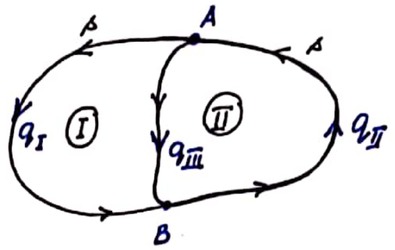
Rate of twist

Torsional rigidity coeff.

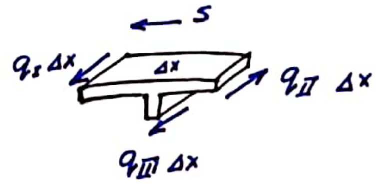
* Note that Shear Centre S was not needed in any calculation, but is fundamental to the derivations.

Torsion of multi-celled sections

shear flow constant in each arm



At A

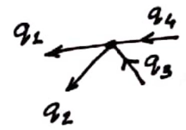


$\sum F_x = 0$

$\Rightarrow q_I + q_{III} - q_{II} = 0 \quad \Rightarrow \underbrace{q_I + q_{III}}_{\text{sum of outgoing shear flows}} = \underbrace{q_{II}}_{\text{sum of incoming shear flows}}$

$\Rightarrow \boxed{q_{III} = q_{II} - q_I}$

E.g.

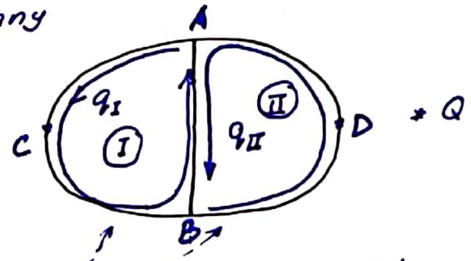


$q_1 + q_2 = q_3 + q_4$

2 cells, 2 independent (unknown) shear flows.

Need 2 eqns.

Eqn. 1: Moment balance: (About any point Q)



From cell I:

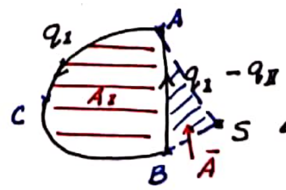
$\Delta T_1 = 2 A_I \cdot q_I$

ΔT_2 (for cell II) $= 2 A_{II} q_{II}$

$\therefore \quad T = \Delta T_1 + \Delta T_2 = \sum_{i=1}^{N} 2 A_i q_i \quad$ (Here $N = 2$)

Additional equations from geometric compatability:

In cell I:



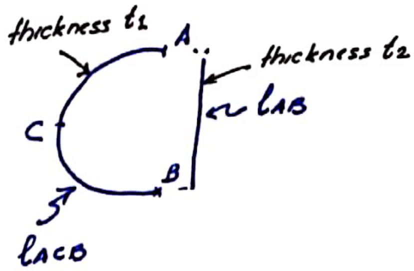
Due to curved part + st. part we have

$$\frac{q(s)}{tG} = \gamma_{xs} = u_{,s} + v_{t,x} = u_{,s} + R(s)\,\alpha$$

Integrate on full cell contour:

$$\underbrace{\oint_I \frac{q(s)\,ds}{tG}}_{\frac{q_I}{t_1 G} l_{ACB} + \frac{(q_I - q_{II})}{t_2 G} l_{AB}} = 0 + \alpha \oint_I R(s)\,ds = 2 A_I \cdot \alpha$$

$$\Rightarrow \quad \alpha = \frac{1}{2A_I}\left[\frac{q_I\, l_{ACB}}{t_1 G} + \frac{(q_I - q_{II})\, l_{AB}}{t_2 G}\right]$$



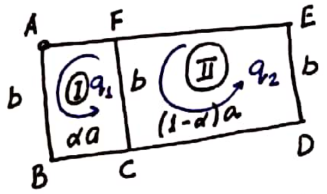
Similarly, for cell II :

$$\alpha = \frac{1}{2A_{II}}\left[\frac{q_{II}\, l_{BDA}}{t_3 G} + \frac{(q_{II} - q_I)\, l_{AB}}{t_2 G}\right]$$

Comparing expressions for cells I and II we get:

$$\alpha = \frac{1}{2A_I}\left[\frac{q_I}{t_1 G}\, l_{ACB} + \frac{(q_I - q_{II})}{t_2 G}\, l_{AB}\right] = \frac{1}{2A_{II}}\left[\frac{q_{II}\, l_{BDA}}{t_3 G} + \frac{(q_{II} - q_I)\, l_{AB}}{t_2 G}\right]$$

EXAMPLE :



All members of thickness t and shear modulus G :

$\therefore \quad A_I = \alpha b a \quad ; \quad A_{II} = (1-\alpha) b a \quad \leftarrow$ areas of the 2 cells

$$T = 2A_I q_1 + 2A_{II} q_2 = 2\alpha b a\, q_1 + 2(1-\alpha) b a\, q_2 \quad \text{— (1)}$$

$$\alpha = \frac{1}{\cancel{2\alpha b a G t}}\left[q_1(l_{AF} + l_{AB} + l_{BC}) + (q_1 - q_2)(l_{CF})\right] =$$

$$\frac{1}{\cancel{2(1-\alpha) b a G t}}\left[q_2(l_{CD} + l_{DE} + l_{EF}) + (q_2 - q_1)\, l_{CF}\right]$$

$$\Rightarrow \quad (1-\alpha)\big((2\alpha a + b) q_1 + b(q_1 - q_2)\big) = \alpha\big((2(1-\alpha)a + b) q_2 + b(q_2 - q_1)\big)$$

$$\Rightarrow \quad q_1\big((1-\alpha)(2\alpha a + 2b) + \alpha b\big) = q_2\big(\alpha(2(1-\alpha)a + 2b) + (1-\alpha) b\big) \quad \text{— (2)}$$

If $\alpha = 1/4$; $b = a$

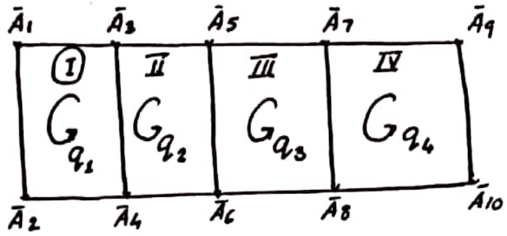
$$q_1\left[3/4\,(a/2 + 2a) + 1/4\, a\right] = \frac{17}{8} a\, q_1 = q_2\left(\frac{1}{4}\big(\overset{3/2}{\overbrace{2 \times 3/4\, a}} + 2a\big) + 3/4\, a\right)$$

$$= q_2 a\left(7/8 + 6/8\right) = \frac{13}{8} q_2 a$$

$$\Rightarrow \quad q_1/q_2 = 13/17 \qquad \Rightarrow T = 2 \times \frac{1}{4} \times a^2 \times q_1 + 2 \times 3/4 \times a^2 \times \frac{17}{13} q_1$$

$$= q_1 a^2\left(\frac{1}{2} + \frac{51}{26}\right) = \frac{64}{26} = \frac{32 q_1 a^2}{13}$$

N- CELLED STRUCTURE



$$\alpha = \frac{1}{2A_i}\left[q_i \left(\frac{l_{c_i}}{Gt}\right) - q_{i+1}\frac{l_{inter}}{Gt}\right]$$

l_{inter} = length of interface $\bar{A}_3\bar{A}_4$, $\bar{A}_5\bar{A}_6$, $\bar{A}_7\bar{A}_8$.

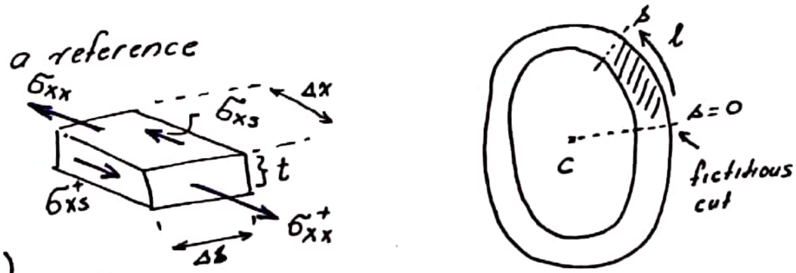
l_{c_i} = length of cell.

$\therefore$ $(N-1)$ equations come from this + Torque equation $\Rightarrow$ N eqns.

Resolving shear for bending of closed thin sections

Let $s = 0$ be chosen as a reference

We get



$\Sigma F_x = 0$

$$\Rightarrow \frac{\partial \sigma_{xx}}{\partial x}\cdot t + \frac{\partial(\sigma_{xs}t)}{\partial s} = 0$$

$$\Rightarrow \frac{\partial q_b}{\partial s} = -t\frac{\partial \sigma_{xx}}{\partial x} \Rightarrow \boxed{q_b(s) = q_b(0) - \int_{s=0}^{s} t\,\frac{\partial \sigma_{xx}}{\partial x}\,ds}$$

$$= q_b(0) + q_b^1(s)$$

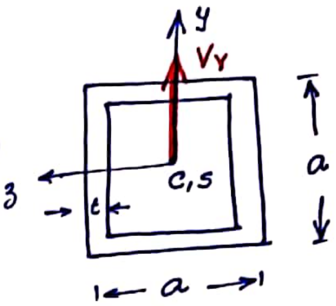
$$\left(q_b^1(s) = -\int_{s=0}^{s} t\cdot\frac{\partial \sigma_{xx}}{\partial x}\,ds \right)$$

$\therefore$ the shear flow is given in terms of $q_b(0)$ and V_y, V_z.

Taking twisting moment about S, due to $q_b(s)$, we should get $T = 0$, or NO TWIST.

$$\therefore \quad \alpha = \frac{1}{2A}\oint \frac{q_b(s)\,ds}{Gt} = 0 \Rightarrow q_b(0) = -\frac{\oint \frac{q_b^1(s)}{Gt}ds}{\oint \frac{ds}{Gt}}$$

EXAMPLE : Hollow square section of thickness t.

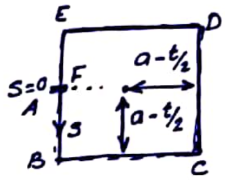


Centroid C (and S) is at centre of section

$$I_{YY} = I_{ZZ} = \frac{1}{12}\left[a^4 - (a-2t)^4\right]$$

$$\approx \frac{1}{12}\left(a^4 - a^4\left(1 - 4\cdot\frac{2t}{a}\right)\right)$$

$$\approx \frac{2}{3}a^3 t \qquad ; \qquad I_{YZ} = 0$$



$$\sigma_{xx} = -\frac{M_3\, y}{I_{ZZ}} = -\frac{M_3 \cdot y}{\frac{2}{3}a^3 t} \Rightarrow \sigma_{xx,x} = \frac{V_y \cdot y}{I_{ZZ}}$$

$$q_b^1(s) = -\int_{s=0}^{s} t\, \frac{\partial \sigma_{xx}}{\partial x}\, ds = -t\int_{s=0}^{s} \frac{V_y}{I_{33}}\, y\, ds = -\frac{V_y \cdot t}{I_{33}}\int_{s=0}^{s} y\, ds$$

In AB : $\displaystyle q_b^1(s) = -\frac{V_y t}{I_{ZZ}}\int_{s=0}^{s} -s\, ds = \frac{V_y \cdot t}{I_{ZZ}}\cdot\frac{s^2}{2}$

At B : $\displaystyle q_b^1(s) \approx \frac{V_y\cdot t}{I_{ZZ}}\cdot\left(\frac{a^2}{8}\right)$

In BC : $\displaystyle q_b^1(s) \approx \underbrace{\frac{V_y t}{I_{ZZ}}\left(a^2/8\right)}_{q_b^1(B)} \underbrace{- \frac{V_y t}{I_{ZZ}}\int_{s=0}^{s}(-a/2)\cdot ds}_{+\frac{V_y t}{I_{ZZ}}(a/2)\cdot s} \approx \frac{V_y t}{I_{ZZ}}\left(\frac{a^2}{8} + \frac{a}{2}s\right)$

At C : $\displaystyle q_b^1(s) \approx \frac{V_y t}{I_{ZZ}}\left(a^2/8\right) + \frac{V_y\cdot t}{I_{ZZ}}\left(a^2/2\right) \approx \frac{V_y t}{I_{ZZ}}\left(\frac{5a^2}{8}\right)$

In CD : $\displaystyle q_b^1(s) \approx q_b^1\big|_C - \frac{V_y t}{I_{ZZ}}\underbrace{\int_{s=0}^{s}(s - a/2)\, ds}_{\left[\frac{s^2}{2} - \frac{a}{2}s\right]_0^s} \approx \frac{V_y t}{I_{ZZ}}\left[\frac{5a^2}{8} + \frac{a}{2}s - \frac{s^2}{2}\right]$

At D : $\displaystyle q_b^1(s) \approx \frac{V_y t}{I_{ZZ}}\left[\frac{5a^2}{8} + \frac{a^2}{2} - \frac{a^2}{2}\right] \approx \frac{V_y t}{I_{ZZ}}\left(\frac{5a^2}{8}\right)$

In DE : $\displaystyle q_b^1(s) \approx q_b^1\big|_D - \frac{V_y t}{I_{ZZ}}\int_{s=0}^{s} a/2\, ds \approx \frac{V_y t}{I_{ZZ}}\left(\frac{5a^2}{8} - \frac{as}{2}\right)$

At E : $\displaystyle q_b^1(s) \approx \frac{V_y t}{I_{ZZ}}\left(\frac{5a^2}{8} - \frac{a^2}{2}\right) \approx \frac{V_y t}{I_{ZZ}}\left(a^2/8\right)$

In EF : $\displaystyle q_b^1(s) \approx q_b^1\big|_E - \frac{V_y t}{I_{ZZ}}\int_{s=0}^{s}(a/2 - s)\, ds \approx \frac{V_y t}{I_{ZZ}}\left[a^2/8 - a/2\, s + s^2/2\right]$

At F : $\displaystyle q_b^1(s) \approx \frac{V_y t}{I_{ZZ}}\left[a^2/8 - a^2/4 + a^2/8\right] = \underline{\underline{0}}$ ✓ CONSISTENT !!

$$\oint \frac{q_b^1\, ds}{Gt} = \frac{1}{Gt}\left[\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EF} q_b^1\, ds\right]$$

$$= \frac{1}{Gt}\cdot\frac{V_y t}{I_{zz}}\Big[(a/2)^3\cdot\frac{1}{6} + \Big(\frac{a^2}{8}\cdot(a) + \frac{a}{4}(a)^2\Big) + \Big(\frac{5a^2}{8}\cdot a + \frac{a}{4}\cdot a^2 - a^3/6\Big)$$

$$+ \Big(\frac{5a^2}{8}\cdot a - \frac{a}{4}\cdot a^2\Big) + \Big(a^2/8\cdot(a/2) - \frac{a}{4}(a/2)^2 + \frac{1}{6}(a/2)^3\Big)\Big]$$

$$= \frac{V_y}{G\,I_{zz}}\cdot a^3 \Big\{\frac{1}{48} + \frac{1}{8} + \frac{1}{4} + \frac{5}{8} + \cancel{\frac{1}{4}} - \frac{1}{6} + \frac{5}{8} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{16}} - \cancel{\frac{1}{16}} + \frac{1}{48}\Big\}$$

$$= \frac{2+6+12+30-8+30}{48} = \frac{\cancel{72}\,3}{\cancel{48}\,2}$$

$$= \frac{3}{2}\frac{V_y a^3}{G I_{zz}}$$

$$\oint \frac{ds}{Gt} = \frac{1}{Gt}\times 4a \quad\Rightarrow\quad q_b^0 = -\frac{\frac{3}{2}\frac{V_y a^3}{G I_{zz}}}{\frac{4a}{Gt}} = -\frac{3}{8}\frac{V_y\cdot a^2 t}{I_{zz}}$$

$$\Rightarrow \text{In } AB:\ q_b(s) = q_b^0 + q_b^1(s) = \frac{V_y t}{I_{zz}}\Big[-\frac{3}{8}a^2 + \frac{s^2}{2}\Big]$$

$-a^2/4$

$$\text{In } BC:\ q_b(s) = q_b^0 + q_b^1(s) = \frac{V_y t}{I_{zz}}\Big[-3/8\,a^2 + a^2/8 + a/2\, s\Big]$$

$a^2/4$

$$\text{In } CD:\ q_b(s) = q_b^0 + q_b^1(s) = \frac{V_y t}{I_{zz}}\Big[-3/8\,a^2 + 5a^2/8 + as/2 - s^2/2\Big]$$

$a^2/4$

$$\text{In } DE:\ q_b(s) = q_b^0 + q_b^1(s) = \frac{V_y t}{I_{zz}}\Big[-\frac{3}{8}a^2 + \frac{5a^2}{8} - \frac{as}{2}\Big]$$

$$\text{In } EF:\ q_b(s) = q_b^0 + q_b^1(s) = \frac{V_y t}{I_{zz}}\Big[-\frac{3}{8}a^2 + \frac{a^2}{8} - \frac{as}{2} + \frac{s^2}{2}\Big]$$

$-a^2/4$