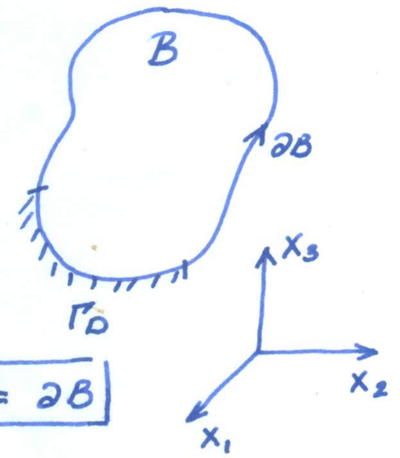
BOUNDARY VALUE PROBLEM

Conservation of momentum:



$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = 0 \quad \text{for } i = 1,2,3$$

< 3 EQUATIONS >

$\vec{u} = \vec{u}_0$ on Γ_D ← Displacement B.C.

$\sigma_{ji} n_j = T_i$ on Γ_N with $\boxed{\Gamma_N \cup \Gamma_D = \partial B}$

Force B.C.

UNKNOWNS : σ_{ij} ← 6 unknowns ; u_i ← 3 unknowns

CONSTITUTIVE RELATIONSHIP:

$$\sigma_{ij} = C_{ijkl} \underbrace{\varepsilon_{kl}}_{} \quad \leftarrow \text{6 EQUATIONS}$$

New unknowns ← 6 unknowns

KINEMATICS:

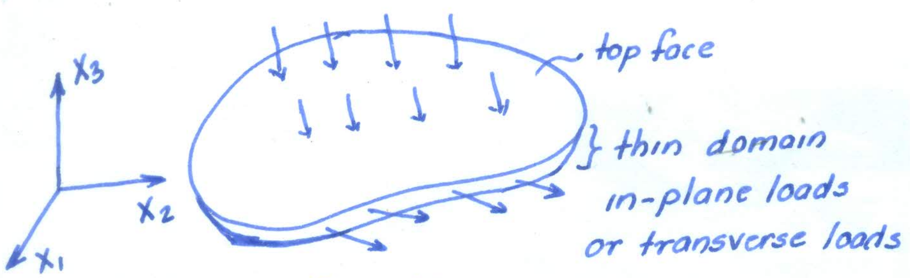
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \leftarrow \text{6 EQUATIONS}$$

$$\left.\begin{array}{l} \Rightarrow \text{UNKNOWNS} = 6 + 3 + 6 = 15 \\ \text{EQUATIONS} = 3 + 6 + 6 = 15 \end{array}\right\} \text{solvable !!}$$

Finally, we can solve for the displacement vector $\vec{u}(\vec{x})$ and recover everything else.

2^{nd} order PDE in $\vec{u}$ – 3 coupled equations

Not so trivial to solve for general domains. Can we IDEALIZE to reduce our effort?



PLANE STRESS

$\sigma_{3j} \approx 0$ as compared to other σ_{ij}

$\Rightarrow \sigma_{33} = 0$ (not ε_{33}, why?)

$\sigma_{31} = \sigma_{32} = 0$

$\Rightarrow \gamma_{31} = \gamma_{32} = 0 \quad (\gamma_{ij} = 2\,\varepsilon_{ij})$

$$\varepsilon_{33} = \frac{-\nu}{E}\left(\sigma_{11} + \sigma_{22}\right)$$

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\,\sigma_{22}\right)$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu\,\sigma_{11}\right)$$

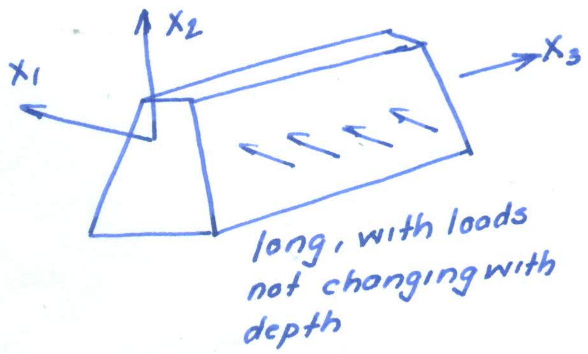
* For both cases find

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}$$

ENGINEERING STRESS "VECTOR"

ENGINEERING STRAIN "VECTOR"

(Only a convenient form of writing!)



PLANE STRAIN

$\varepsilon_{3j} \approx 0$ as compared to other directions.

$\varepsilon_{33} = 0$ (not σ_{33}!)

$\varepsilon_{31} = \varepsilon_{32} = 0$

$\Rightarrow \sigma_{31} = \sigma_{32} = 0$

$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right) = 0$$

$$\Rightarrow \boxed{\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})}$$

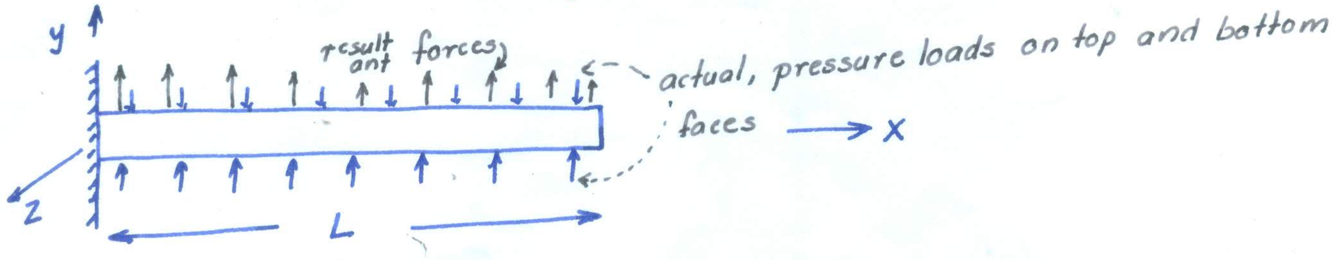
$$\Rightarrow \varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\,\sigma_{22} - \nu(\nu\,\sigma_{11} + \nu\,\sigma_{22})\right)$$

$$= \frac{1}{E}\left((1-\nu^2)\,\sigma_{11} - \nu(1+\nu)\,\sigma_{22}\right)$$

$$= \boxed{\frac{(1-\nu^2)}{E}\left(\sigma_{11} - \frac{\nu}{(1-\nu)}\,\sigma_{22}\right)}$$

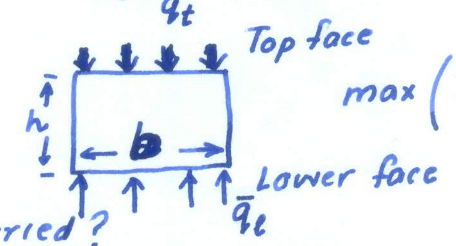
$$\Rightarrow \varepsilon_{22} = \frac{(1-\nu^2)}{E}\left(\sigma_{22} - \frac{\nu}{(1-\nu)}\,\sigma_{11}\right)$$

Let us now talk of long-slender members subjected to the transverse loads given:



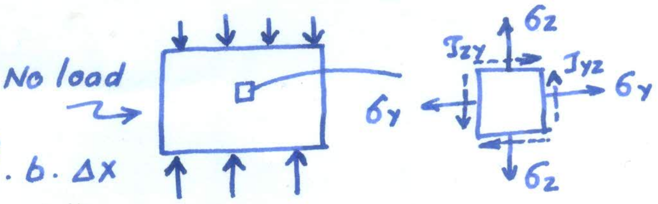
→ Pressure (and friction + drag) forces on the outer surface

→ CROSS-SECTION



$$\max\left(\frac{b}{L}, \frac{h}{L}\right) \ll 1$$

→ How will the load be carried?

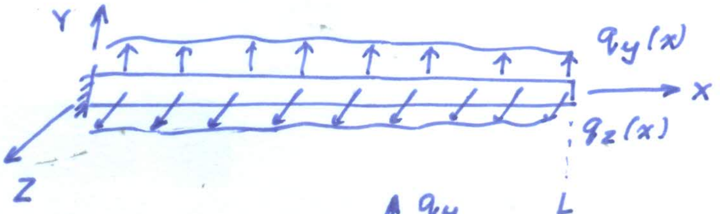


$$\bar{q}_t \cdot b \cdot \Delta x \approx \Delta F_{y}^{t}$$

$$\bar{q}_\ell \cdot b \cdot \Delta x \approx \Delta F_y^{\ell} \approx \sigma_z^{\ell} \cdot b \cdot \Delta x$$

$$\Rightarrow \sigma_z^{\ell} \approx \bar{q}_\ell \; ; \; \sigma_z^{t} \approx \bar{q}_t$$

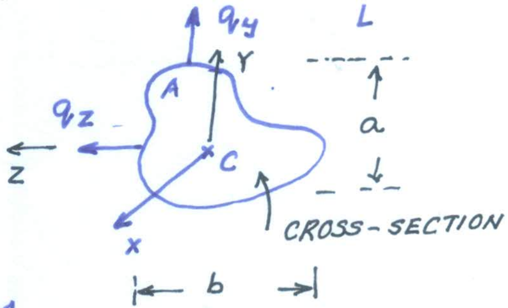
Bending of Arbitrary sections



Long slender beams

$\Downarrow$

Cross-sectional dimensions $a, b \ll L$



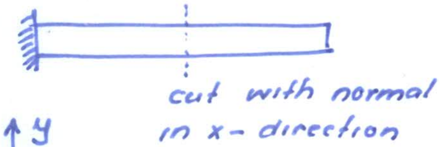
From 3D - consideration:

On boundary of cross-section A

- σ_{yy}, σ_{zz}, σ_{yz} ; σ_{yx}, σ_{zx} active (i.e. boundary tractions in terms of these).
- From figure $|\sigma_{yy}| \le q_y$; $|\sigma_{zz}| \le q_z$.

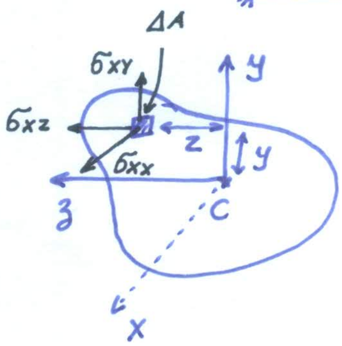
  $|\sigma_{yz}| \approx 0$
- Also σ_{yx}, σ_{zx} are varying in such a way that they vanish on the boundary of A.

→ CUT WITH NORMAL ALONG X-AXIS



Stresses active on this face:

σ_{xx}, σ_{xy}, σ_{xz}

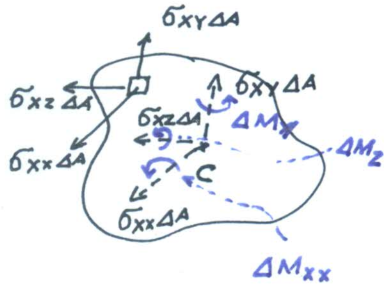


Resultants due to σ_{xx}, σ_{xy}, σ_{xz} about origin C of coordinate system:

$\Delta F_{xx} = \sigma_{xx} \Delta A$; $\Delta V_y = \sigma_{xy} \Delta A$;

$\Delta V_z = \sigma_{xz} \Delta A$; $\Delta M_x = (y \sigma_{xz} - \sigma_{xy} z) \Delta A$;

$\Delta M_y = \sigma_{xx} z \Delta A$; $\Delta M_z = - \sigma_{xx} y \Delta A$



Idea: Look at each resultant effect separately and add the final results up to get full picture (SUPERPOSITION)

Resultant Partition:

(a) AXIAL STRETCH: $F_{xx} = \int_A \sigma_{xx}\, dA$

(b) BENDING ACTION: $M_y = \int_A \sigma_{xx}\, z\, dA$; $M_z = -\int_A \sigma_{xx}\, y\, dA$;

$V_y = \int_A \sigma_{xy}\, dA$; $V_z = \int_A \sigma_{xz}\, dA$

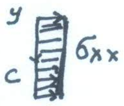
(c) TWISTING ACTION: $M_x = \int_A (y\,\sigma_{xz} - z\,\sigma_{xy})\, dA$

Separating out the influences $\Rightarrow$

(a) $F_{xx} \neq 0$; $\underbrace{V_y, V_z, M_y, M_z, M_x}_{\sigma_{xy},\ \sigma_{xz} \approx 0} = 0$

σ_{xx} symmetric about C
$\Downarrow$
σ_{xx} = constant with (y, z) is an option!

$\leadsto$ σ_{xx} uniform with y, z or
$\sigma_{xx}(x, y, z) \approx \sigma_{xx}(x)$

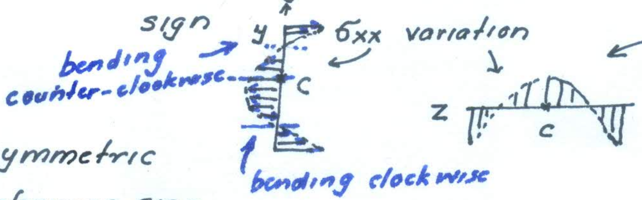


(c) $M_x \neq 0$; $F_{xx}, M_y, M_z, V_y, V_z = 0$

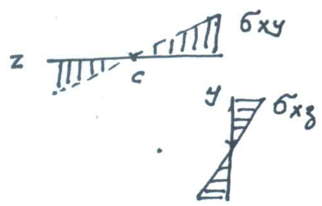
σ_{xy}, σ_{xz} linear in y, z

σ_{xx} symmetric but changes sign

σ_{xy}, σ_{xz} anti-symmetric with y, z



NEEDED FOR $F_{xx} = 0$



$\leadsto$ σ_{xx} symmetric but changes sign

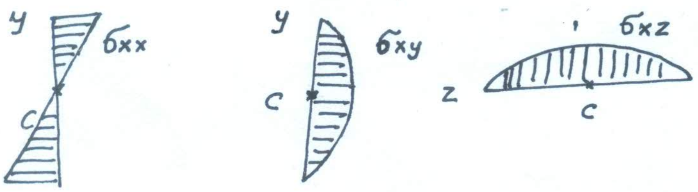
σ_{xy}, σ_{xz} linear in y, z

(b) $M_y, M_z, V_y, V_z \neq 0$; $F_{xx}, M_x = 0$

σ_{xx} atleast linear with y, z (changes sign, gives moments but no force)

σ_{xx} changes sign

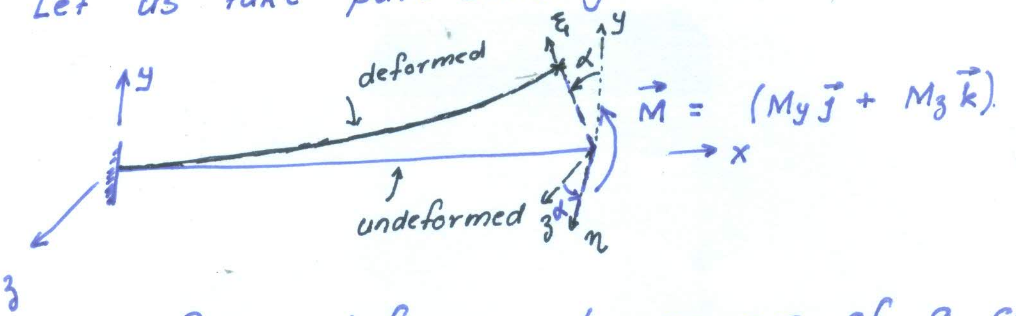
σ_{xy}, σ_{xz} symmetric with y, z



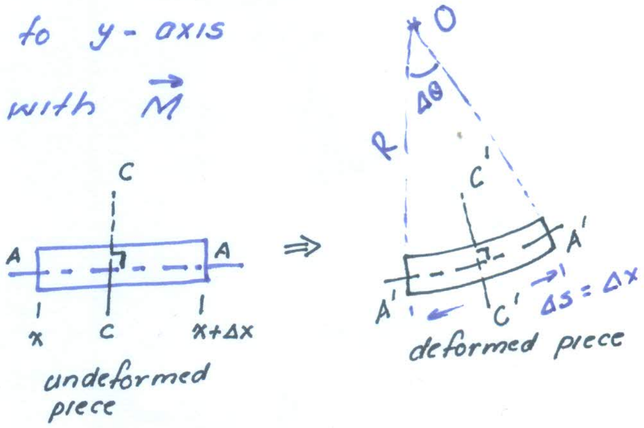
$\leftsquigarrow$ possible variations

BENDING CASE: We will look at bending (case b) first.

Let us take pure bending case:



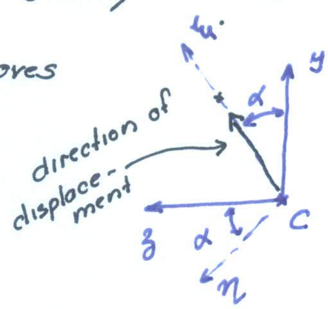
- Beam deforms into an arc of a circle with radius R (or ρ).
- Plane of deformation is $\xi - x$ plane with ξ inclined at $\alpha°$ to y-axis
- α will change with $\vec{M}$



In $\xi - x$ plane:

- Line A-A $\Rightarrow$ neutral line or line that does not stretch due to bending
- Line C-C $\Rightarrow$ perpendicular to A-A and remains $\perp$ in deformed configuration also (i.e. $\sigma_{xy}, \sigma_{xz} = 0$)

* In $\xi - \eta$ plane the neutral line moves by $(v_\xi, 0)$.



STRAIN?

$$\varepsilon_{xx} = \frac{l_{A'A'} - l_{AA}}{l_{AA}} = -\frac{\xi}{R}$$

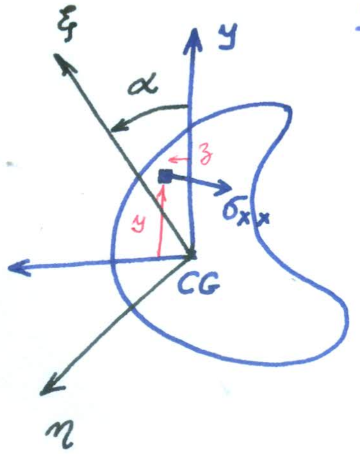
All other strains not important for this case ...

$\varepsilon_{yy}, \varepsilon_{zz} \approx 0$ assumed (even though Poisson's effect is there)

because v, w would vary primarily as a constant with (y, z) AND a QUADRATIC part in y, z which is negligible hence ignored.

$$\xi = y\cos\alpha + z\sin\alpha$$

$$\eta = -y\sin\alpha + z\cos\alpha$$



$$\sigma_{xx} = E\,\varepsilon_{xx} \leftarrow \sigma_{yy}, \sigma_{zz} \approx 0$$

$$\therefore \quad \sigma_{xx} = -\frac{E}{\rho}\left(y\cos\alpha + z\sin\alpha\right)$$

$$\therefore \quad M_y = \int_A \sigma_{xx}\, z\, dA \quad ; \quad M_z = -\int_A \sigma_{xx}\, y\, dA$$

What about layered material ?

$$\sigma_{yy}, \sigma_{zz} \approx 0 \quad \Rightarrow \quad \varepsilon_{xx} \approx \underbrace{S_{xx}}_{\text{What is this ?}} \sigma_{xx}$$

However, S_{xx} varies with lamina, hence use

$$\varepsilon_{xx} = S_{xx}^{(\ell)}\, \sigma_{xx}^{(\ell)} \quad \text{or} \quad \sigma_{xx}^{(\ell)} = C_{xx}^{*(\ell)}\, \varepsilon_{xx} \qquad -\frac{(y\cos\alpha + z\sin\alpha)}{\rho}$$

$$\therefore \quad \boxed{M_y = \sum_{\ell=1}^{N} \int_{A_\ell} \sigma_{xx}^{(\ell)}\, z\, dA} \quad ; \quad \boxed{M_z = -\sum_{\ell=1}^{N} \int_{A_\ell} \sigma_{xx}^{(\ell)} \cdot y\, dA}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11}^* & S_{12}^* & S_{13}^* & S_{16}^* \\ S_{12}^* & S_{22}^* & S_{23}^* & S_{26}^* \\ S_{13}^* & S_{23}^* & S_{33}^* & S_{36}^* \\ S_{16}^* & S_{26}^* & S_{36}^* & S_{66}^* \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \end{Bmatrix}$$

$\sigma_{yy}, \sigma_{zz}, \sigma_{xy} \approx 0$ as compared to σ_{xx}

$$\Rightarrow \varepsilon_{xx} \approx S_{11}^*\, \sigma_{xx}$$

$$\Rightarrow \boxed{\sigma_{xx} \approx \frac{1}{S_{11}^*}\, \varepsilon_{xx}}$$

1-3 plane

1-2 plane

* Find S_{11}^* for orientation θ !

be careful about θ

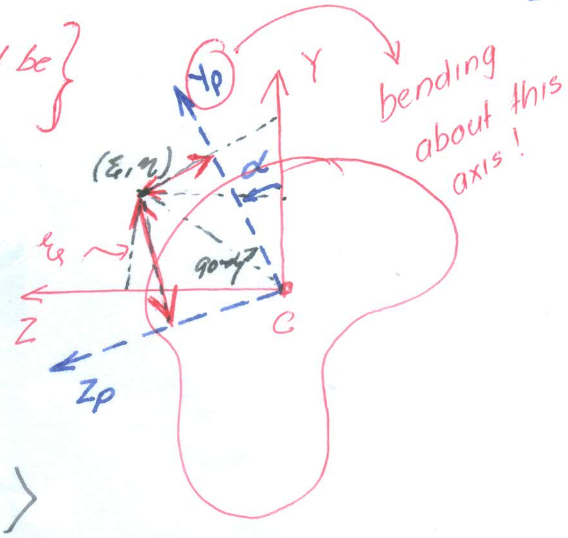
In 1-3 plane use γ_{xz} instead of γ_{xy}

$$\{\varepsilon^{(1)}\} = [S^{(1)}]\{\sigma^{(1)}\} \Rightarrow [T_2]\{\varepsilon^{(x)}\} = [S^{(1)}][T_1]\{\sigma^{(x)}\}$$

$$\Rightarrow \{\varepsilon^{(x)}\} = \underbrace{[T_2]^{-1}[S^{(1)}][T_1]}_{[S^*]}\{\sigma^{(x)}\}$$

LS-B5

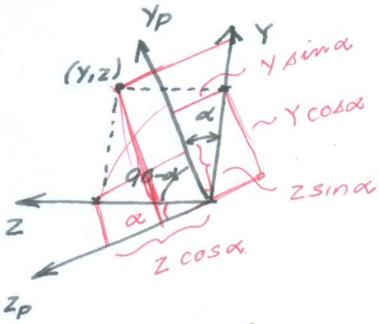
{ Note that bending axis y_p need not be a principal axis ! }



$$\Rightarrow \quad \xi = \cancel{\quad} \; Y\cos\alpha + Z\sin\alpha$$

$$\eta = -Y\sin\alpha + Z\cos\alpha$$

$$\Rightarrow \quad \sigma_x = -\frac{E\xi}{\rho_1} \quad \langle \text{due to moment } M_Z \rangle$$



$\left\langle -\frac{\xi}{\rho_1} \text{ is } \varepsilon_{xx} \right\rangle$
$$= -\frac{E\,(Y\cos\alpha + Z\sin\alpha)}{\rho_1}$$

$$\Rightarrow \left.\begin{aligned} M_Z &= -\int \sigma_x\, y\, dA \\ M_Y &= \int \sigma_x\, z\, dA \end{aligned}\right\} \text{due to a moment } \vec{M}$$

$$\Rightarrow \quad M_Y = -\frac{E}{\rho}\int_A (yz\cos\alpha + z^2\sin\alpha)\, dA = -\frac{E}{\rho}\left(I_{YZ}\cos\alpha + I_{YY}\sin\alpha\right)$$

$$M_Z = \frac{E}{\rho}\int_A (y^2\cos\alpha + yz\sin\alpha)\, dA = \frac{E}{\rho}\left(I_{ZZ}\cos\alpha + I_{YZ}\sin\alpha\right)$$

$$\boxed{I_{YY} = \int_A z^2\, dA \; ; \quad I_{YZ} = \int_A yz\, dA \; ; \quad I_{ZZ} = \int_A y^2\, dA}$$

$$\Rightarrow \begin{Bmatrix} M_Y \\ M_Z \end{Bmatrix} = \frac{E}{\rho}\begin{bmatrix} -I_{YZ} & -I_{YY} \\ I_{ZZ} & I_{YZ} \end{bmatrix}\begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix}$$

$$\Rightarrow \frac{E}{\rho}\begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} = \frac{1}{I_{YY}I_{ZZ} - I_{YZ}^2}\begin{bmatrix} I_{YZ} & I_{YY} \\ -I_{ZZ} & -I_{YZ} \end{bmatrix}\begin{Bmatrix} M_Y \\ M_Z \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_x = \frac{-y\{M_Y I_{YZ} + M_Z I_{YY}\} + z\{+M_Y I_{ZZ} + M_Z I_{YZ}\}}{I_{YY}I_{ZZ} - I_{YZ}^2}$$

$$= \boxed{\frac{M_Y\{-yI_{YZ} + z\,I_{ZZ}\}}{I_{YY}I_{ZZ} - I_{YZ}^2} + \frac{M_Z\{-yI_{YY} + z\,I_{YZ}\}}{I_{YY}I_{ZZ} - I_{YZ}^2}}$$

Where is $\underset{\sim}{C}$ located ?
origin of coordinates (y, z).

$$F_{xx} = 0 \Rightarrow \int_A \sigma_{xx}\, dA = 0 \Rightarrow \int_A E\, \varepsilon_{xx}\, dA = 0$$

$$\Rightarrow E \int_A \varepsilon_{xx}\, dA = 0 \quad \text{OR} \quad \left(\int_A y\, dA\right) \underbrace{\frac{\cos\alpha}{R}}_{} + \left(\int_A z\, dA\right) \underbrace{\frac{\sin\alpha}{R}}_{} = 0$$

depend on $\vec{M}$, hence not always zero

$\int_A y\, dA$, $\int_A z\, dA$ are geometric entities and do not change with load $\Rightarrow$

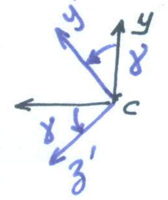
$$\int_A y\, dA = 0 \; ; \; \int_A z\, dA = 0$$

{ Note: If E changes with y, z then we get $\int_A E\, y\, dA = 0$, $\int_A E\, z\, dA = 0$ }

definition of area centroid

$\Rightarrow$ C is the centroid of cross-section.

* In the derivation of σ_{xx}, can we decouple the effects of the moment components? YES, if we resolve the moment and deformation in a rotated coordinate system $y'-z'$, such that $I_{y'z'} = 0$



$$I_{y'z'} = \int_A y'z'\, dA = \int_A (y\cos\gamma + z\sin\gamma) * (-y\sin\gamma + z\cos\gamma)\, dA$$

$$= \int_A \left(-y^2 \frac{\sin 2\gamma}{2} + yz\cos 2\gamma + z^2 \frac{\sin 2\gamma}{2}\right) dA$$

$$\Rightarrow \frac{1}{2}(I_{yy} - I_{zz})\sin 2\gamma + I_{yz}\cos 2\gamma = 0$$

$$\Rightarrow \boxed{\tan 2\gamma = \frac{2 I_{yz}}{(I_{zz} - I_{yy})}}$$

resolve everything in $y'-z'$ system

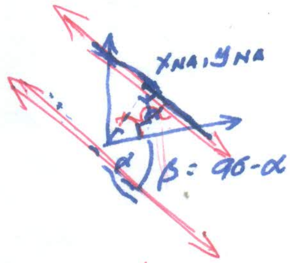
INDEPENDENT OF LOAD, PROPERTY OF SECTION GEOMETRY

* Not very convenient to use always

$\therefore$ In the principal coordinate system

$$\left\{ \sigma_x = \frac{M_{y'} z'}{I_{y'y'}} - \frac{M_{z'} y'}{I_{z'z'}} \right\} \leftarrow$$ all measured w.r.t. the principal coordinate system.

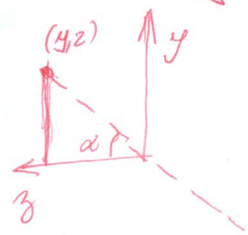
* Note that a moment about z-axis will give rise to a curvature about y-axis also!



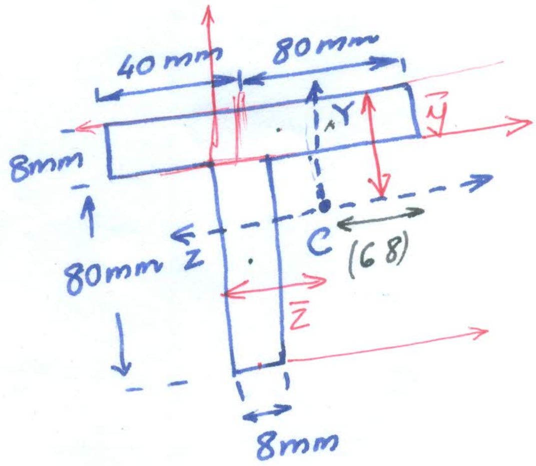
Where is the neutral axis?

$\sigma_{xx} = 0$ at y_{NA}, z_{NA}

$$\Rightarrow \tan\alpha = \frac{y_{NA}}{z_{NA}} = \frac{M_y I_{zz} + M_z I_{yz}}{M_y I_{yz} + M_z I_{yy}}$$



EXAMPLE



$M_z = 1500$ Nm ; $M_y = 0$

$$\Rightarrow \bar{z} = \frac{(A_1 z_1 + A_2 z_2)}{(A_1 + A_2)}$$

$$= \frac{120 \times 8 \times 24 + 80 \times 8 \times 4}{120 \times 8 + 80 \times 8}$$

$= 16$ mm

$\Rightarrow \bar{y} = 21.6$ mm ?

Find I_{yz}, I_{zz}, I_{yy} !

$$I_{zz} = \left(\frac{120 \times 8^3}{12} + (21.6-4)^2 \times 8 \times 120\right) + \left(\frac{8 \times 80^3}{12} + (21.6-48)^2 \times 8 \times 80\right)$$

$$= 1.09 \times 10^6 \, mm^4$$

$$I_{yy} = \left(\frac{8 \times 120^3}{12} + \left(120 \times 8 \times (68-60)^2\right)\right) + \left(\frac{80 \times 8^3}{12} + 8 \times 80 \times (80-68)^2\right)$$

$$= 1.31 \times 10^6 \, mm^4$$

$$I_{yz} = 120 \times 8 \times 17.6 \times (-8) + 80 \times 8 \times 12 \times (-26.4)$$

$$= -0.34 \times 10^6 \, mm^4$$

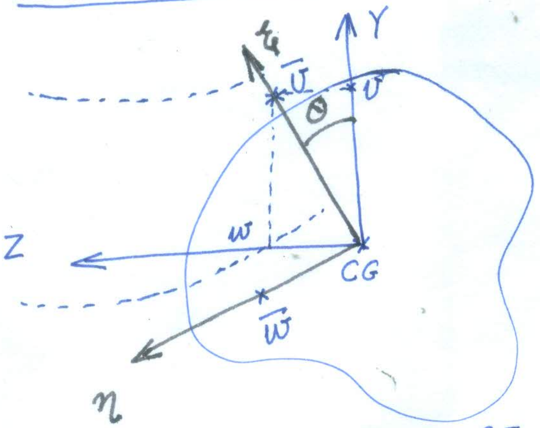
$$M_z = 1500 \; Nm$$

$$\Rightarrow \sigma_x = -(1.5y + 0.39z)$$

$\sigma_x|_{max}$ at $y = 8mm, \; z = -66.4mm$

$\sigma_x|_{max} = 96 \; N/mm^2$ ⟨ tensile ⟩

Bending - Curvature Relationship



In the ξ-η system, the transverse deflections are

$\bar{v} \neq 0$; $\bar{w} = 0$ ← no deflection in η direction

$\therefore \quad v = \bar{v}\cos\theta \; ; \quad w = +\bar{v}\sin\theta$

carefully! θ is also fn. of x!

$$\Rightarrow \frac{1}{R} \approx \frac{d^2\bar{v}}{dx^2} \quad \Rightarrow \frac{d^2 v}{dx^2} \approx \frac{d^2\bar{v}}{dx^2}\cos\theta \; ; \; \frac{d^2 w}{dx^2} \approx \frac{d^2\bar{v}}{dx^2}\sin\theta$$

$$\Rightarrow \boxed{\frac{1}{R}\cos\theta \approx \frac{d^2 v}{dx^2} \; ; \; \frac{1}{R}\sin\theta \approx \frac{d^2 w}{dx^2}}$$

$v = \bar{v}\cos\theta$

$v_{,x} = \bar{v}_{,x}\cos\theta + \bar{v}(-\sin\theta)\theta_{,x}$

$v_{,xx} = \bar{v}_{,xx}\cos\theta + \bar{v}_{,x}(-\sin\theta)\theta_{,x}$
$+ \bar{v}_{,x}(-\sin\theta)\theta_{,x}$
$- \bar{v}_{,x}\cos\theta\,\theta_{,x}^2$
$+ \bar{v}_{,x}(-\sin\theta)\theta_{,xx}$

The above is true only for θ constant, i.e. pure bending!

$$\Rightarrow E \begin{Bmatrix} v'' \\ w'' \end{Bmatrix} = \frac{1}{(\Delta)} \begin{bmatrix} I_{YZ} & I_{YY} \\ -I_{ZZ} & -I_{YZ} \end{bmatrix} \begin{Bmatrix} M_Y \\ M_Z \end{Bmatrix}$$

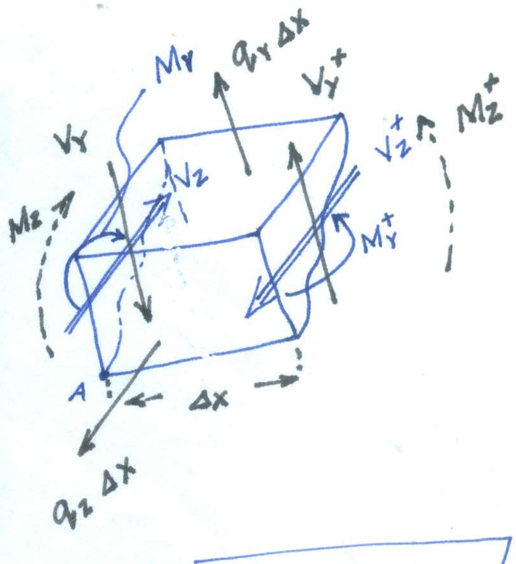
or

$$\begin{Bmatrix} M_Y \\ M_Z \end{Bmatrix} = E \begin{bmatrix} -I_{YZ} & -I_{YY} \\ I_{ZZ} & I_{YZ} \end{bmatrix} \begin{Bmatrix} v'' \\ w'' \end{Bmatrix}$$

← CONSTITUTIVE RELATION FOR M_Y, M_Z IN TERMS OF CURVATURES

- Note that curvature also is like a vector entity
  - This can be done because we are talking of <u>small deformations</u>, or θ can be taken to be fixed at a point.

Equilibrium Equations



$$\sum M_{Z,A} = 0$$

$$\Rightarrow M_Z^+ - M_Z + V_Y^+ \Delta x + q_Y \Delta x \cdot \frac{\Delta x}{2} = 0$$

$$\Rightarrow \frac{dM_Z}{dx} + V_Y + h.o.t. = 0$$

$$\Rightarrow \boxed{\frac{dM_Z}{dx} + V_Y \approx 0} \quad \text{— (i)}$$

$$\sum F_Y = 0 \Rightarrow V_Y^+ - V_Y + q_Y \Delta x = 0 \Rightarrow \boxed{\frac{dV_Y}{dx} + q_Y = 0} \quad \text{— (ii)}$$

From (i) and (ii) :

$$\boxed{\frac{d^2 M_Z}{dx^2} = q_Y} \quad \text{— (A)}$$

Similarly,

$$\sum M_{Y,A} = 0 \Rightarrow M_Y^+ - M_Y - V_Z^+ \Delta x = 0$$

$$\Rightarrow \boxed{\frac{dM_Y}{dx} - V_Z = 0} \quad \text{— (iii)}$$

$$\sum F_Z = 0 \Rightarrow V_Z^+ - V_Z + q_Z \Delta x = 0 \Rightarrow \boxed{\frac{dV_Z}{dx} + q_Z = 0} \quad \text{— (iv)}$$

From (iii) and (iv) :

$$\boxed{\frac{d^2 M_Y}{dx^2} + q_Z = 0} \quad \text{— (B)}$$

$\therefore$ The final equations, in terms of v and w, are :

$$\frac{d^2}{dx^2}\left[E\left(I_{zz}\,v'' + I_{yz}\,w''\right)\right] = q_y \qquad \text{— (A)}$$

$$\frac{d^2}{dx^2}\left[E\left(I_{yz}\,v'' + I_{yy}\,w''\right)\right] = q_z \qquad \text{—— (B).}$$

$\Uparrow$

4th Order differential equations ~ Boundary Conditions?

At each end 2 + 2 boundary conditions to be specified, i.e.

1	$V_Y\big\vert_{end}$ given	OR	$v\big\vert_{end}$ given
+		AND	
1	$M_Z\big\vert_{end}$ given	OR	$\frac{dv}{dx}\big\vert_{end}$ given
+		+	
1	$V_Z\big\vert_{end}$ given	OR	$w\big\vert_{end}$ given
+		AND	
1	$M_Y\big\vert_{end}$ given	OR	$-\frac{dw}{dx}\big\vert_{end}$ given
$\underline{\underline{4}}$			

$\Uparrow$

① How Do You Solve It ?

② What Happens When $\varepsilon_{ij}^T = \alpha\,\Delta T\,\delta_{ij}$ also exists ?

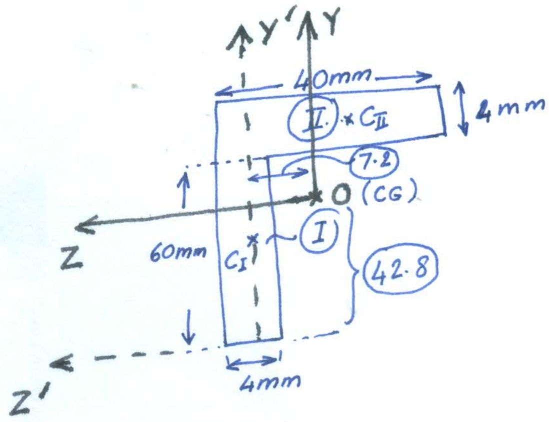
$$\varepsilon_{11} = \varepsilon_{11}^E + \alpha\,\Delta T \quad\Rightarrow\quad \varepsilon_{11}^E = \left(\varepsilon_{11} - \alpha\,\Delta T\right)$$

$$\Rightarrow \sigma_{11} = E\,\varepsilon_{11}^E = E\left(\varepsilon_{11} - \alpha\,\Delta T\right) = E\left(-\frac{\xi}{R} + u_{0,x} - \alpha\,\Delta T\right)$$

↳ Work from here !!

Thermal stretch + Bending !

EXAMPLES



Find I_{YY}, I_{ZZ}, I_{YZ}

First find centroid.

$$A \cdot y_c' = A_I \cdot y_1' + A_{II}\, y_2'$$

$$= 60 \times 4 \times 30 + 40 \times 4 \times 62$$

$$\Rightarrow y_c' = \frac{60 \times 30 \times 4 + 40 \times 62 \times 4}{60 \times 4 + 40 \times 4} = 42.8\ mm$$

$$z_c' = \frac{60 \times 4 \times 0 - 40 \times 4 \times 18}{400} = -7.2\ mm$$

$$I_{YY} = \int_A z^2 dA = \int_A (\bar{z} + z_c)^2 dA = z_c^2 A + \int_A \bar{z}^2 dA$$

$$\Rightarrow \underbrace{(7.2)^2 \times 60 \times 4 + \frac{1}{12} \cdot 60 \times 4^3}_{I} + \underbrace{(18 - 7.2)^2 \times 40 \times 4 + \frac{1}{12} \times 4 \times 40^3}_{II}$$

$$I_{ZZ} = \int_A y^2 dA = \int_A (\bar{y} + y_c)^2 dA = y_c^2 A + \int_A \bar{y}^2 dA$$

$$= \underbrace{(42.8 - 30)^2 \times 60 \times 4 + \frac{1}{12} \times 4 \times 60^3}_{I} + \underbrace{(62 - 42.8)^2 \times 40 \times 4 + \frac{1}{12} \times 40 \times 4^3}_{II}$$

$$I_{YZ} = \int_A yz\, dA = \int_A (y_c + \bar{y})(z_c + \bar{z})\, dA = y_c z_c \cdot A + \int_A \bar{y}\bar{z}\, dA$$

$$\cancel{+\ y_c \int_A \bar{z}\, dA + z_c \int_A \bar{y}\, dA}^{\,0}$$

$$= \underbrace{(-42.8 + 30) \times (7.2) \times 60 \times 4 + 0}_{I} + \underbrace{(62 - 42.8) \times (-18 + 7.2) \times 40 \times 4 + 0}_{II}$$

Now, given $M_y = 100\ N\cdot m$; $M_z = -20\ N\cdot m$

what is σ_{xx}? $\left\{ \sigma_{xx} = \dfrac{M_y\{-y I_{yz} + z I_{zz}\}}{I_{yy} I_{zz} - I_{yz}^2} + \dfrac{M_z\{-y I_{yy} + z I_{yz}\}}{I_{yy} I_{zz} - I_{yz}^2} \right\}$

Provided $E = \cancel{200}\ 70\ GPa$; what is the value of $\dfrac{d^2 v}{dx^2}$; $\dfrac{d^2 w}{dx^2}$

at this point?

$$\begin{Bmatrix} v'' \\ w'' \end{Bmatrix} = \frac{1}{E\left(I_{yy} I_{zz} - I_{yz}^2\right)} \begin{bmatrix} I_{yz} & I_{yy} \\ -I_{zz} & -I_{yz} \end{bmatrix} \begin{Bmatrix} M_y \\ M_z \end{Bmatrix}$$

Now, let us imagine that the top cap is made of steel ($E = 200\,GPa$).

How will you find local v'', w''?

Here, the deformation pattern is assumed to be same, i.e.

$$\varepsilon_{xx} = -\frac{\xi}{R}\ ; \quad \sigma_{xx} = -\frac{E_1 \xi}{R} \text{ in material (1)}$$

$$= -\frac{E_2 \cdot \xi}{R} \text{ in material (2)}$$

Then, $\xi = y\cos\theta + z\sin\theta$

$$\Rightarrow F_{xx} = 0 = -\frac{E_1}{R}\int_{A_1} (y\cos\theta + z\sin\theta)\, dA - \frac{E_2}{R}\int_{A_2} (y\cos\theta + z\sin\theta)\, dA$$

$$= -\frac{\cos\theta}{R} \underbrace{\left[+ E_1 \int_{A_1} y\, dA + E_2 \int_{A_2} y\, dA \right]}_{=0} - \frac{\sin\theta}{R} \underbrace{\left[E_1 \int_A z\, dA + E_2 \int_A z\, dA \right]}_{=0}$$

Let us take any coordinate system. We can find

MODOLAS WEIGHTED CENTROID