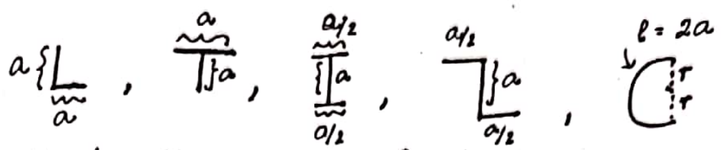
SOLVE :

(1) If an  cross-section is used such that thickness of section is $t = 2\,mm$. Further each section has the dimensions as shown. Determine the I_{yy}, I_{yz}, I_{zz} when $a = 10\,cm$.

(2) If a cantilever beam of length $L = 2m$ is subjected to $q_y = 1000(L - x)\ N/m$; $q_z = 100(L - x)\ N/m$, then determine the maximal bending stress distribution along the length of the beam, using the series with $N = 4, 6, 8$ ~~16~~

(3) Also determine the maximal shear and transverse normal stresses (using the post-processing).

* Write a MATLAB code to solve the problem and to display the data.

SUBMISSION DATE : 18th OCTOBER

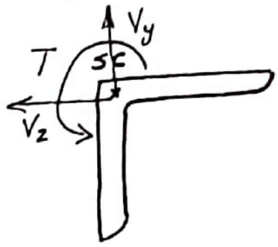
$E = 70\,GPa$; $\nu = 0.3$

→ What if the beam tapers with $a = a_0\left(1 - \frac{0.9x}{L}\right)$?

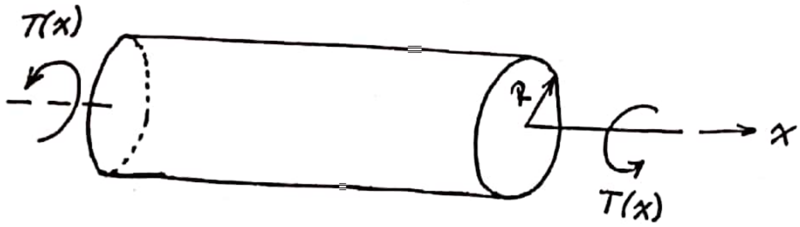
→ Solve this too !

TORSION

As discussed earlier, the resultant aerodynamic loads act at the centre of pressure of a section. When "shifted" to the shear centre, SC, the loads give rise to shear forces $V_y(x)$, $V_z(x)$ and a torque $T(x)$.

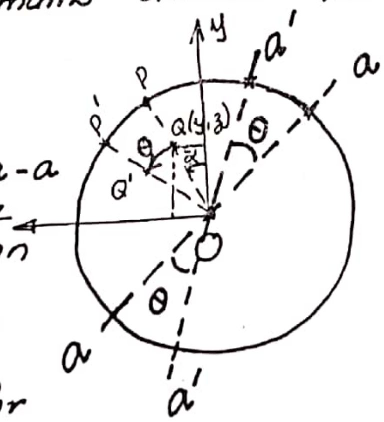


RECAP : Circular cross-section



Kinematics: Under a pure torque $T(x) = T$, the section deforms in its plane and remains circular. The deformation is given by :

The undistorted diameter a-a moves to a'-a', through an angular displacement θ. The "twist angle" θ is same for all diametric lines on a given section.



$Q \rightarrow (y, z)$

$Q' \rightarrow (y', z')$

$\Rightarrow$

$y = r\cos\bar{\alpha}$

$z = r\sin\bar{\alpha}$

$y' = r\cos(\bar{\alpha}+\theta)$

$z' = r\sin(\bar{\alpha}+\theta)$

$$\Rightarrow y' = r(\cos\bar{\alpha}\cos\theta - \sin\bar{\alpha}\sin\theta)$$

$$\approx r\cos\bar{\alpha}\cos\theta - r\sin\bar{\alpha}\cdot\theta. \approx y - z\theta$$

$$z' = r(\sin\bar{\alpha}\cos\theta + \cos\bar{\alpha}\sin\theta) \approx r\sin\bar{\alpha} + r\cos\bar{\alpha}\,\theta$$

$$\approx z + y\theta.$$

$$\Rightarrow \boxed{\begin{aligned} y' - y &= v = -z\theta \\ z' - z &= w = y\theta \end{aligned}}$$

$\leftarrow$ $\theta(x)$ is the sectional twist angle and is assumed to be small.

$u(x,y,z) = 0$ $\leftarrow$ no warping or out of plane deformation.

$\Rightarrow$ Strains: $\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0$; $\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$; $\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\frac{d\theta}{dx} \; ; 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = y\frac{d\theta}{dx} \; ;$$

$$2\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta - \theta = 0$$

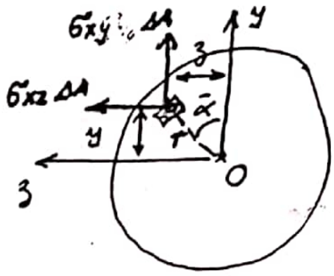
$\Rightarrow$ Non-zero strains: $\boxed{2\varepsilon_{xz} = y \cdot \alpha \; ; 2\varepsilon_{xy} = -z\alpha}$

$\Rightarrow$ State of stress : $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{yz} = 0$

$$\sigma_{xy} = -\frac{2\mu}{2} z\alpha \; ; \; \sigma_{xz} = \frac{2\mu}{2} \cdot y\alpha$$

$\leftarrow$ act on the x- face

$$= -\mu z\alpha \; ; \qquad = \mu y \alpha$$



<u>Resultants</u> : $F_{xx} = F_{xy} = F_{xz} = 0$?

F_{xx} $\leftarrow$ trivial

$$F_{xy} = \int_{\bar{\alpha}=0}^{2\pi} \int_{r=0}^{R} \sigma_{xy} \cdot dA = \int_{\bar{\alpha}=0}^{2\pi} \int_{r=0}^{R} -\mu \cdot \underbrace{z}_{r\sin\bar{\alpha}}\alpha \,(r\,dr\,d\bar{\alpha})$$

$$= 0$$

$$F_{xz} = \int_{\bar{\alpha}=0}^{2\pi} \int_{r=0}^{R} \sigma_{xz}\, dA = \int_{\bar{\alpha}=0}^{2\pi} \int_{r=0}^{R} \mu \underbrace{y}_{r\cos\bar{\alpha}}\alpha\,(r\,dr\,d\bar{\alpha}) = 0$$

Similarly; M_y, $M_z = 0$ and $M_x = \int_A (\sigma_{xz} \cdot y - \sigma_{xy} \cdot z)\, dA$

$$= \int_{\bar{\alpha}=0}^{2\pi} \int_{r=0}^{R} \mu\alpha\,(y^2 + z^2) \cdot r\,dr\,d\bar{\alpha}$$

$$\Rightarrow \quad M_x = \mu\alpha \int_A (y^2 + z^2)\, dA = \mu\alpha \left(I_{zz}\big|_0 + I_{yy}\big|_0 \right) = \left(I_p\big|_0 \right) \mu\alpha$$

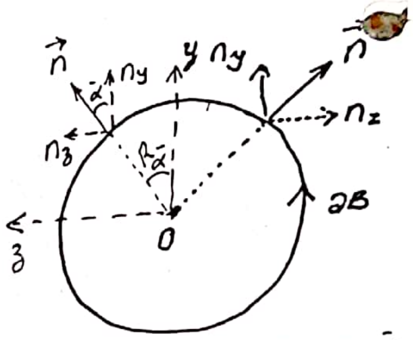
polar moment of inertia

$$= \mu \cdot \alpha \left(2\pi \cdot \frac{R^4}{4} \right) = \underbrace{\mu \cdot \left(\frac{\pi R^4}{2} \right)}_{\text{torsional rigidity}} \cdot \alpha$$

$$\mu = G \quad \Rightarrow \quad M_x = \underset{\substack{\downarrow \\ I_p|_0}}{GJ} \cdot \alpha$$

Interestingly, this state of stress satisfies the equilibrium equations, i.e. $\sigma_{ij,j} = 0$ (CHECK!) for a constant torque $T(x) = T$.

Also, on the boundary of the section



$n_y = \cos\alpha \;;\; n_z = \sin\alpha$

$= y/R \qquad = z/R$

$$T_x = \cancel{\sigma_{xx}\, n_x}^{\,0} + \sigma_{xy}\, n_y + \sigma_{xz}\, n_z$$

$$= \mu\alpha \left(-z \cdot y/R + y\, z/R \right) = 0$$

$$T_y = \sigma_{xy}\, \cancel{n_x}^{\,0} + \cancel{\sigma_{yy}}^{\,0}\, n_y + \cancel{\sigma_{yz}}^{\,0}\, n_z = 0$$

$$T_z = \sigma_{xz}\, \cancel{n_x}^{\,0} + \cancel{\sigma_{yz}}^{\,0}\, n_y + \cancel{\sigma_{zz}}^{\,0}\, n_z = 0$$

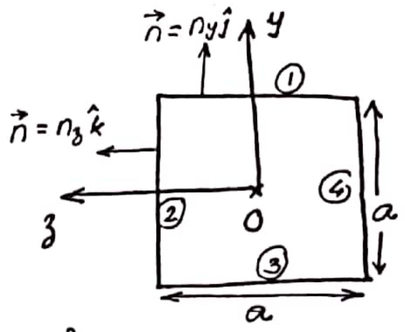
* This means that the traction vector $\vec{T}$ on the boundary of the section is ZERO or ∂B is traction free.

→ ALL consistency conditions are satisfied and the resultant is only the torque T

□ What will change if we use the same expressions for u, v, w and determine the state of stress for a NON-CIRCULAR section?

Scanned by CamScanner

Let's take the square section as shown. Then, on face ① $\vec{n} = \hat{j} = n_y \hat{j}$



$$\Rightarrow T_x = 0 + \sigma_{xy} \cdot 1 + 0$$
$$= -\mu z \big|_{y=a/2} \cdot \alpha \neq 0$$
$$T_y = 0 + \cancel{\sigma_{yy}}^{\,0} \cdot 1 + 0 = 0$$
$$T_z = 0 + \cancel{\sigma_{yz}}^{\,0} \cdot 1 + 0 = \cancel{\ldots} \neq 0$$

the boundary traction is not ZERO, but the problem definition demands that ∂B should be traction free. PROBLEM!

How to rectify the problem?

Note that $\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$; $\sigma_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\right)$

if these are non-zero then they can cancel the effect of v, w.

Want $u(x,y,z) \neq 0$, such that the resultants are consistent

$$\boxed{u(x,y,z) = \alpha\, \varphi(y,z)}$$ $\longrightarrow$ WARPING FUNCTION or CROSS-SECTION HAS OUT OF PLANE DEFORMATION

For T constant, α is a constant

$$\Rightarrow \varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \; ; \; \varepsilon_{yy} = 0 \; ; \; \varepsilon_{zz} = 0 ; \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$= \alpha \cdot \varphi_{,y} - \alpha z \; ; \; \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \alpha\, \varphi_{,z} + \alpha y$$
$$= \alpha(\varphi_{,y} - z) \; ; \qquad = \alpha(\varphi_{,z} + y)$$

$$\Rightarrow \boxed{\sigma_{xy} = G\alpha(\varphi_{,y} - z) \; ; \; \sigma_{xz} = G\alpha(\varphi_{,z} + y)}$$

How do you solve for $\varphi(y,z)$?

→ <u>From equilibrium</u>

$$\cancel{\sigma_{xx,x}}^{0} + \underset{\downarrow \atop G\alpha\,\varphi_{,yy}}{\sigma_{xy,y}} + \underset{\downarrow \atop G\alpha\,\varphi_{,zz}}{\sigma_{xz,z}} = 0 \quad \Rightarrow \quad \boxed{\varphi_{,yy} + \varphi_{,zz} = 0} \text{ — (1)}$$

$$\underset{\overset{\parallel}{0}}{\sigma_{xy,x}} + \cancel{\sigma_{yy,y}} + \cancel{\sigma_{yz,z}} = 0 \quad \text{trivial}$$

(α is constant)

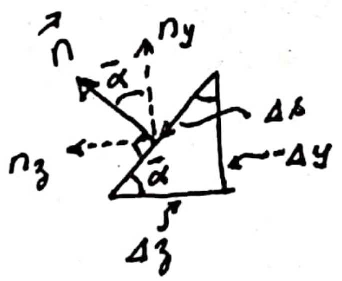
$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0 \quad \text{trivial}$$

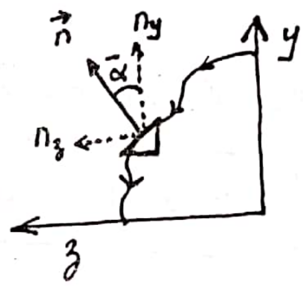
<u>B.C.</u>

$$T_x = \sigma_{xy} n_y + \sigma_{xz} n_z = G\alpha(\varphi_{,y} - z) n_y + G\alpha(\varphi_{,z} + y) n_z = 0$$

$$T_y = \cancel{\sigma_{yy}} n_y + \cancel{\sigma_{yz}} n_z = 0$$

$$T_z = \cancel{\sigma_{yz}} n_y + \cancel{\sigma_{zz}} n_z = 0$$





$$\cos\bar{\alpha} = \frac{\Delta z}{\Delta s} = n_y \quad ; \quad \sin\bar{\alpha} = -\frac{\Delta y}{\Delta s} = n_z$$

$$\Rightarrow n_y = \frac{\partial z}{\partial s} \quad ; \quad n_z = -\frac{\partial y}{\partial s}$$

$$\Rightarrow T_x = G\alpha\left[\varphi_{,y} n_y + \varphi_{,z} n_z - z \cdot \frac{\partial z}{\partial s} - y\frac{\partial y}{\partial s}\right] = 0$$

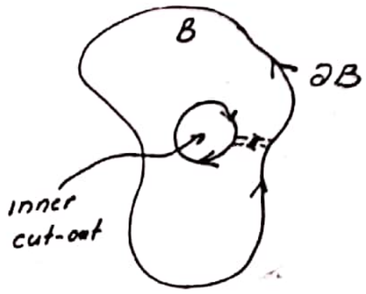
$$\Rightarrow \boxed{\varphi_{,n} = \frac{\partial}{\partial s}\left(\frac{1}{2}(y^2 + z^2)\right)} \quad \text{on } \partial B \quad \text{— (2)}$$

Boundary-value problem for $\varphi(y,z)$!

So,

the warping function is the solution of the following problem:

$$\varphi_{,yy} + \varphi_{,zz} = 0 \quad \text{on } B$$



$$\varphi_{,n} = \frac{\partial}{\partial s}\left(\frac{1}{2}(y^2+z^2)\right) \quad \text{on } \partial B$$

Solution unique upto a constant C.

OPTION 1 : Use $C = 0$!

Historically, this was inconvenient, and it was felt that a problem with Dirichlet B.C. would be more amenable. How do you achieve this ?

$$\Delta\varphi = 0 \rightarrow \text{Harmonic solution.}$$

$X = \varphi + i\psi$ ⇝ analytic function s.t.

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial z} \quad \text{and} \quad \frac{\partial\varphi}{\partial z} = -\frac{\partial\psi}{\partial y}$$

complex functions — Cauchy Reimann relationship

$$\Rightarrow \quad \Delta\psi = 0 \quad \text{and}$$

$$\frac{\partial\varphi}{\partial y}n_y + \frac{\partial\varphi}{\partial z}n_z = \frac{\partial\psi}{\partial z}\underbrace{n_y}_{\frac{\partial z}{\partial s}} - \frac{\partial\psi}{\partial y}\underbrace{n_z}_{-\frac{\partial y}{\partial s}} = \frac{\partial}{\partial s}\left(\frac{1}{2}(y^2+z^2)\right)$$

$$= \frac{\partial\psi}{\partial s} \Rightarrow \boxed{\psi = \frac{1}{2}(y^2+z^2) + C_1}$$

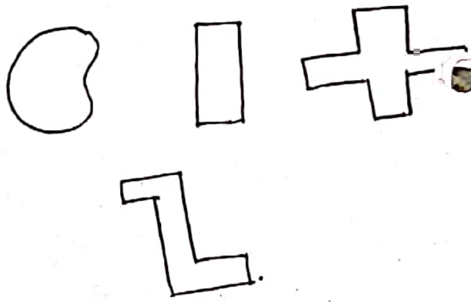
$$\Rightarrow \quad \bar{\psi} = \left(\psi - \frac{1}{2}(y^2+z^2)\right) = C_1$$

$$\Rightarrow \quad \frac{\partial \bar{\varphi}}{\partial y} = \frac{\partial \varphi}{\partial y} - y \; ; \quad \frac{\partial \bar{\varphi}}{\partial z} = \frac{\partial \varphi}{\partial z} - z$$

$$\Rightarrow \quad \frac{\partial^2 \bar{\varphi}}{\partial y^2} = \frac{\partial^2 \varphi}{\partial y^2} - 1 \; ; \quad \frac{\partial^2 \bar{\varphi}}{\partial z^2} = \frac{\partial^2 \varphi}{\partial z^2} - 1$$

$$\Rightarrow \quad \Delta \varphi = 0 \longrightarrow \Delta \bar{\varphi} + 2 = 0 \quad \text{or} \quad \boxed{-\Delta \bar{\varphi} = 2 \text{ in } B}$$

with $\boxed{\bar{\varphi} = C_1 \text{ on } \partial B}$ DIRICHLET PROBLEM!

Simply - connected domain $\Rightarrow$ 

$C_1 = 0$ can be taken without changing anything!

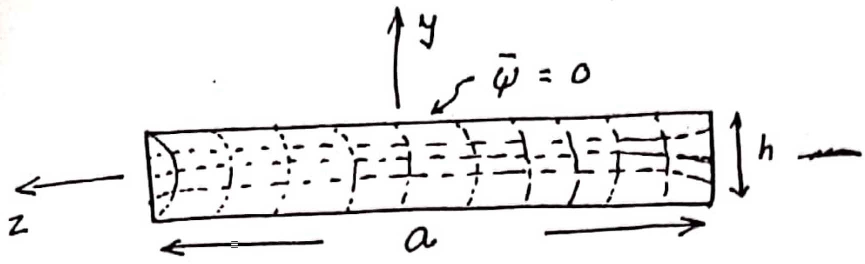
HOMOGENEOUS DIRICHLET PROBLEM

$$\therefore \quad \boxed{\sigma_{xy}} = G\alpha(\varphi_{,y} - z) = G\alpha(+\varphi_{,z} - z) = \boxed{G\alpha\, \bar{\varphi}_{,z}}$$

$$\boxed{\sigma_{xz}} = G\alpha(\varphi_{,z} + y) = G\alpha(-\varphi_{,y} + y) = \boxed{-G\alpha\, \bar{\varphi}_{,y}}$$

$\boxed{\bar{\varphi} \rightarrow \text{Prandtl Stress function!}}$

Nothing but application of complex variables and smart jugglary of terms!



$$\frac{h}{a} << 1.$$

Then $\dfrac{\partial \bar{\varphi}}{\partial z} \approx 0$; or $\bar{\varphi} \approx$ function of y only

$$\Rightarrow \quad \frac{\partial^2 \bar{\varphi}}{\partial y^2} = -2 \quad \Rightarrow \quad \bar{\varphi} = -2\frac{y^2}{2} + C_1 y + C_2$$

such that for $y = \pm h/2$, $\bar{\varphi} = 0$ or $C_1 = 0$, $C_2 = (h/2)^2$

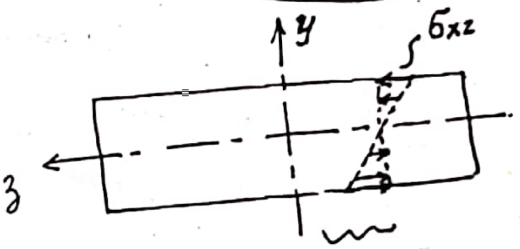
$$\Rightarrow \boxed{\bar{\varphi} \approx (h/2)^2 - y^2} \quad \Rightarrow \quad \frac{\partial \bar{\varphi}}{\partial z} \approx 0 ; \quad \frac{\partial \bar{\varphi}}{\partial y} = -2y$$

or

$$\sigma_{xy} = G\alpha \frac{\partial \bar{\varphi}}{\partial z} \approx 0$$

$$\sigma_{xz} = -G\alpha \frac{\partial \bar{\varphi}}{\partial y} \approx 2G\alpha y$$

IMPORTANT APPROXIMATION



Self-equilibrat and only gives rise to a torque as a resultant effect.

RESULTANT EFFECT

$$T = \int_A (\sigma_{xz}\, y - \sigma_{xy}\, z)\, dA = \int_A (-G\alpha \bar{\varphi}_{,y} \cdot y - G\alpha \bar{\varphi}_{,z} \cdot z)\, dA$$

$$= -G\alpha \int_A \underbrace{(\bar{\varphi}_{,y} \cdot y + \bar{\varphi}_{,z} \cdot z)}\, dA$$

$\int \rho^2 dA - \int \frac{R^2}{2} dA$

$$\frac{\partial}{\partial y}(\bar{\varphi} \cdot y) = \frac{\partial \bar{\varphi}}{\partial y} \cdot y + \bar{\varphi} \; ; \; \frac{\partial}{\partial z}(\bar{\varphi} z) = \frac{\partial \bar{\varphi}}{\partial z} \cdot z + \bar{\varphi}$$

for circular section

$\bar{\varphi} = -\frac{1}{2}(y^2 + z^2)$

$\bar{\varphi} = R^2 - \frac{1}{2}(y^2 + z^2)$

$\Rightarrow$

$$= -G\alpha \left[\int_A \left(\frac{\partial}{\partial y}(\bar{\varphi} \cdot y) + \frac{\partial}{\partial z}(\bar{\varphi} \cdot z) - 2\bar{\varphi} \right) dA \right]$$

$$= G\alpha \int_A 2\bar{\varphi}\, dA - G\alpha \int_{\partial A} (\cancelto{0}{\bar{\varphi} \cdot y} + \cancelto{0}{\bar{\varphi} \cdot z})\, dA$$

$$\therefore \quad \boxed{T = 2G\alpha \int_A \bar{\varphi}\, dA}$$

Note that $\int_A 2\bar{\varphi}\, dA = \underline{\underline{J}}$ — now.

torsional moment of inertia
or <u>torsional constant</u>

for rectangular section

$$J = 2\int_{y=-l/2}^{l/2} \int_{z=-t/2}^{t/2} \left((t/2)^2 - z^2 \right)$$

$$= 2 \cdot l \cdot \left[\frac{t^3}{4} - \frac{2}{3}\frac{t^3}{8} \right]$$

$$= \frac{2 \cdot l}{4} \cdot \frac{t^3}{3}$$

$$= \frac{lt^3}{3}$$

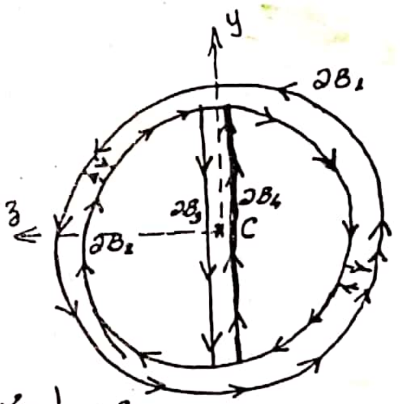
MEMBRANE ANALOGY WORKS !
(Oil film or soap film experiment).

* These are all OPEN SECTIONS. What will happen for a closed section?

Let us look at the following example:

thickness $= t$
Outer radius $= R$ } → Circular section

4 distinct boundary segments $\partial B_1, \partial B_2, \partial B_3, \partial B_4$.



On $\partial B_1, \partial B_2$: $\frac{\partial}{\partial s}\left(\frac{1}{2}(y^2+z^2)\right) = \frac{\partial}{\partial s}(r^2/2) = 0$

$\Rightarrow \quad \varphi(y,z) \approx 0$ in the circular part and only the form
$u = 0; \; v = -z\theta; \; w = y\theta$ will hold

On ∂B_3: $\frac{\partial}{\partial s}\left(\frac{1}{2}(y^2+z^2)\right) = -y$

On ∂B_4: $\frac{\partial}{\partial s}\left(\frac{1}{2}(y^2+z^2)\right) = +y$

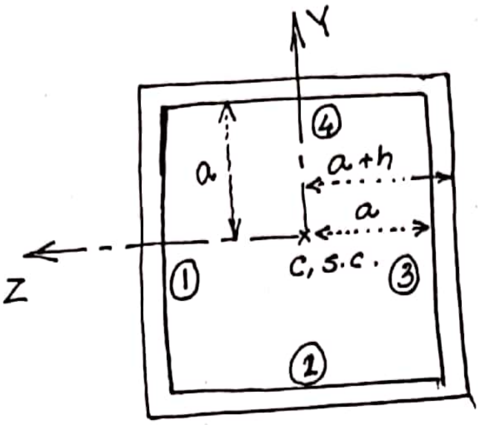
$\Bigg\} \; \frac{\partial \varphi(y,z)}{\partial n} = \begin{cases} -y & \text{on } \partial B_3 \\ +y & \text{on } \partial B_4 \end{cases}$

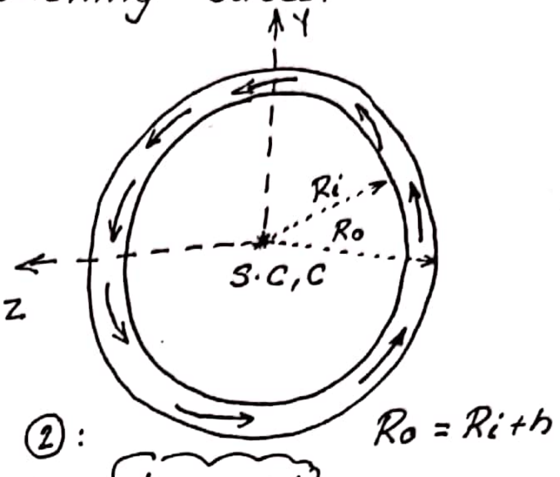
$$\Rightarrow \frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial z} = -y \quad \not\Rightarrow \quad \Delta\varphi = 0$$

This gives : $\sigma_{xy} = G\alpha(\varphi_{,y} - z) = -G\alpha z$; $\sigma_{xz} = G\alpha(\varphi_{,z} + y) = 0$

$\Rightarrow$ Shear is ALONG the length of the arm $\Rightarrow$
TANGENTIAL

Now, let us take the following cases:





1 : Thin, hollow square section

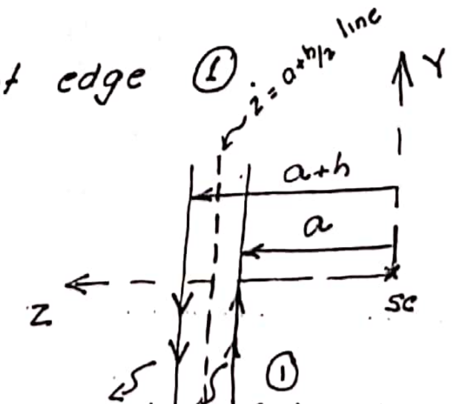
2 : Thin, $\{h/R_i \ll 1\}$ hollow circular section

For section 2 : The solution for solid circular section is valid and $\gamma_{\theta x} = \alpha r \Rightarrow \sigma_{\theta x} = G\gamma_{\theta x} = G\alpha r$.

$$\Rightarrow \sigma_{\theta x}\Big|_{R_i} = G\alpha R_i \; ; \quad \sigma_{\theta x}\Big|_{R_i+h} = G\alpha(R_i+h) = G\alpha R_i(1+h/R_i) \approx G\alpha R_i$$

$$\Rightarrow \sigma_{\theta x}(r) \approx \underline{G\alpha R_i}$$ a constant and not a self-equilibrating local variation!

For section 1 : Let us look at edge 1



Orientation of boundary (inner and outer have different orientations)

The boundaries are at $z = a$ and $z = a+h$.

Here $$\frac{\partial \varphi}{\partial n}\Big|_{bnd} = \frac{1}{2}\frac{\partial}{\partial s}(y^2+z^2) = -\frac{1}{2}\frac{\partial}{\partial y}(y^2+z^2)\Big|_{z=a+h}$$

$$\Rightarrow \frac{\partial \varphi}{\partial n}\Big|_{z=a+h} = -y\Big|_{z=a+h} = \frac{\partial \varphi}{\partial z}\Big|_{z=a+h}$$

Similarly, on edge ④:

$$\left.\frac{\partial \varphi}{\partial n}\right|_{y=a+h} = \left.\frac{\partial \varphi}{\partial y}\right|_{y=a+h} = \frac{1}{2}\frac{\partial}{\partial z}(y^2+z^2) = \frac{1}{2}\frac{\partial}{\partial z}(y^2+z^2)$$
$$= z$$

∴ on edge ④, following our analysis for a thin strip

$$\varphi(y,z)\Big|_{④} \approx yz + c_1 z \underset{+\text{h.o. terms?}}{} \approx (y+c_1)z + \text{h.o. terms?}$$

Similarly, on edge ① $\quad \varphi(y,z)\Big|_{①} \approx -y(z+c_2)$

∴ On edge ④ : $\sigma_{xy} = G\alpha(\varphi_{,y} - z) \approx G\alpha(z-z) \approx 0$

$\sigma_{xz} = G\alpha(\varphi_{,z} + y)$ ~~[illegible]~~

* Note that $\varphi(y,z)$ should be such that $\varphi(y,z) = 0$ at the centre of the strip, i.e.

$$\varphi(a+h/2, z)\Big|_{④} \approx 0 = (a+h/2 + c_1)z \Rightarrow \underline{\underline{c_1}} = -(a+h/2)$$

$$\Rightarrow \boxed{\varphi(x,y)\Big|_{④} \approx (y-(a+h/2))\cdot z} \Leftarrow \text{satisfies all other desired conditions}$$

$$\Rightarrow \frac{\sigma_{xz}}{G\alpha}\Bigg|_{④} \approx (y-(a+h/2)) + y \approx 2y - (a+h/2)$$
$$\approx 2(a+h/2) - (a+h/2)$$
$$\approx \boxed{(a+h/2)}$$

or $\sigma_{xz} \approx \underbrace{\bar{\sigma}_{xz}}_{\text{CONSTANT}} + \underbrace{\tilde{\sigma}_{xz}}_{\text{WIGGLY IN } n}$

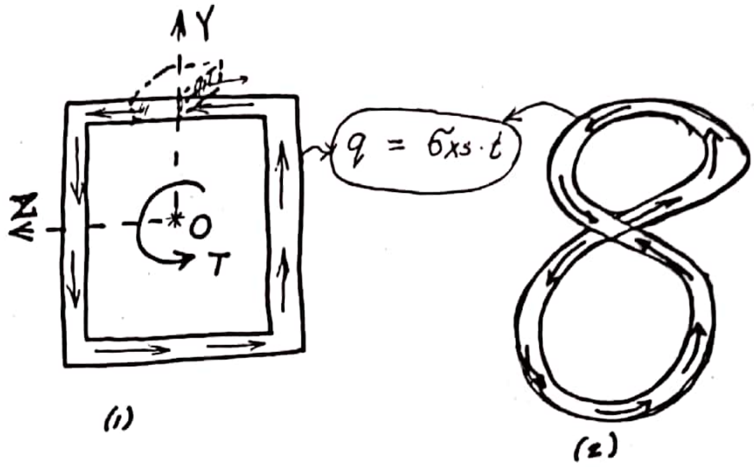
- Strip does not see existence of other strips, hence solution is with respect to its local coordinate system.
- Also, the constant in the solution can be arbitrary. Here, we choose it to give $\varphi(y,z)|_{\text{centre}} = 0$

⇒ Extension: For any arbitrary thin strip, the $\sigma_{xs} \approx$ constant (if thickness h remains unchanged)

* For open sections, c_1 adjusts to give the constant part of shear flow = 0
ONLY the WIGGLY PART REMAINS

Approximation : Shear flow is primarily tangential to the contour, for thin hollow sections.

* Since it is a closed section and only resultant is a torque T, we can take moment about any point O.

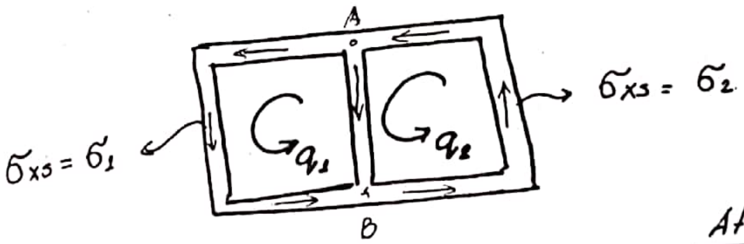


→ Going back to bending induced shear flow, we can see that again (if t is constant) $\sigma_{xx,x} + \sigma_{xs,s} = 0$ but $\sigma_{xx} = 0$

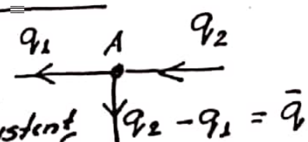
$\Rightarrow \sigma_{xs,s} = 0 \Rightarrow \sigma_{xs} = \text{Constant } \sigma_0$. (for fig 1).

$\Rightarrow q_s = q_0 = \sigma_{xs} \cdot t$

For fig. (2), this will be true piecewise, or



At A :

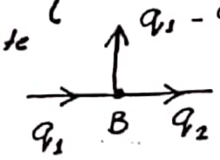


$q_2 - q_1 = \bar{q}$

consistent

At B :

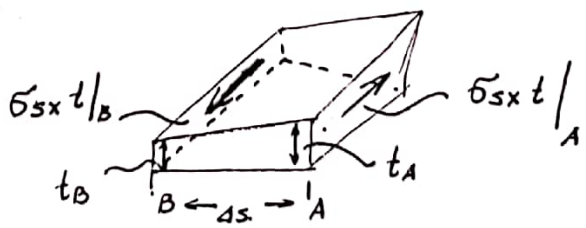
(equal and opposite pairs)



$q_1 - q_2 = -\bar{q}$

$q_i = \sigma_{xs}\big|_i \, t_i$

↑ shear flow satisfies the conservation law.



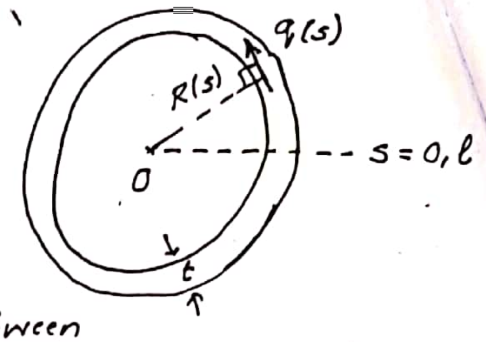
$\therefore \left(\sigma_{xs} t\big|_B - \sigma_{xs} t\big|_A \right) . \Delta x = 0$

$\Rightarrow \frac{\partial}{\partial s}(q) \, \Delta s \, \Delta x = 0$

$\Rightarrow \frac{\partial q}{\partial s} = 0 \Rightarrow q = \text{constant} = \bar{q}$

For a single-celled section:

$$q(s) = q_0 = \sigma_{xs} \cdot t(s)$$

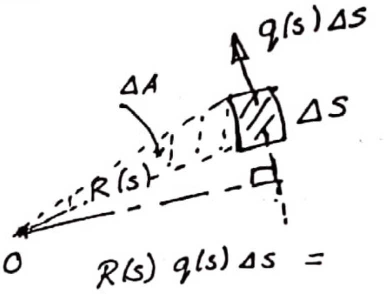


$\Rightarrow \Delta T$ = incremental torque due to

$q(s) = q_0 \cdot \underbrace{R(s)}\, \Delta s$

perpendicular distance between $q(s)$ and O.

$$T = \int_{s=0}^{\ell} q R\, ds = q_0 \int_{s=0}^{\ell} R\, ds = 2A q_0$$



$R(s)\, q(s)\, \Delta s$ = $2\,\Delta A$ = 2·area of enclosed triangle

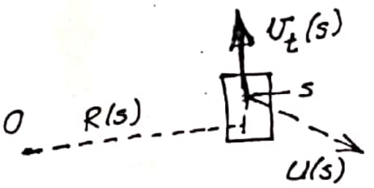
$$\Rightarrow \boxed{q_0 = \frac{T}{2A}} \leftarrow \text{BREDT-BATHO EQUATION}$$

How do we find rate of twist?

* Following where we left, $\sigma_{xs} \approx G(\gamma_{xs}) \approx G\left(\frac{\partial u}{\partial s} + \frac{\partial v_t}{\partial x}\right)$

$$\Rightarrow q_0 = \sigma_{xs} \cdot t = Gt\left(\frac{\partial u}{\partial s} + \underbrace{R\frac{d\theta}{dx}}\right)$$

$\frac{\partial}{\partial x}(R\theta)$, $\uparrow v_t$



$$\Rightarrow \frac{q_0}{Gt} = \frac{\partial u}{\partial s} + R \cdot \alpha$$

$$\Rightarrow \int_{s=0}^{s} \frac{q_0}{Gt}\, ds = u(s) - u(0) + \alpha \int_{s=0}^{s} R\, ds$$

we ignore it here

* In-class we had ignored the axial or warping part in derivation of α

$$= \boxed{\underbrace{u(s) - u(0)} + 2\underbrace{A(s)} \cdot \alpha} + \text{part from bending}$$

difference in axial displacement between s and 0. area upto point s.

$$\Rightarrow \oint \frac{q_0\, ds}{Gt} = \underbrace{u(\ell) - u(0)}_{=0} + 2A \cdot \alpha = 2A\alpha$$

$$\Rightarrow \alpha = \frac{1}{2A} \int_{s=0}^{\ell} \frac{q_0\, ds}{Gt} = \frac{q_0}{2A} \int_{s=0}^{\ell} \frac{ds}{Gt} = \frac{T}{4A^2} \int_0^{\ell} \frac{ds}{Gt}$$

outcome of B-B equation

$$\Rightarrow GJ = \frac{4A^2}{\int_0^{\ell} \frac{ds}{Gt}}$$

and for constant G; $\boxed{J = \dfrac{4A^2}{\int_0^{\ell} \frac{ds}{t}}}$ ← TORSIONAL CONSTANT

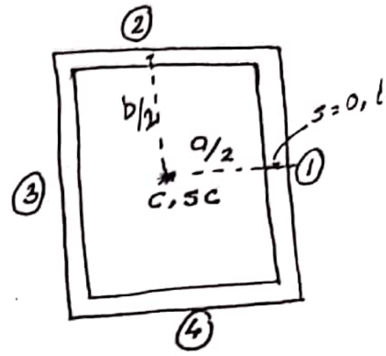
Thin circular $\Rightarrow \dfrac{4 \times (\pi r^2)^2 t}{2\pi r} = \dfrac{4\pi^2 r^4 t}{2\pi r} = \dfrac{4\pi r^3 t}{2} = 2\pi r^3 t$

$\frac{\pi}{2}(R_o^2 + R_i^2)(R_o + R_i)(R_o - R_i)$

$\frac{4\pi}{2} R^3 t$

$= 2\pi R^3 t$

Question in class : How do you show that for rectangular section $q(s) = q_0$?



$$q(s) = \sigma_{xs} t = G \gamma_{xs} t$$

If G, t are constant and $a \neq b$, then

$$\gamma_{xs}\Big|_{1} = \gamma_{xy}\Big|_{1} = \alpha(\varphi_{,y} - z) \quad \text{constant}$$

$$= \alpha(\varphi_{,y} + a/2). = \alpha\varphi_{,y} + \alpha\, a/2$$

$$q(s) = q_0 \Rightarrow \varphi_{,y} + a/2 = C \Rightarrow \varphi_{,y} = C - a/2 \quad \text{in } 1$$

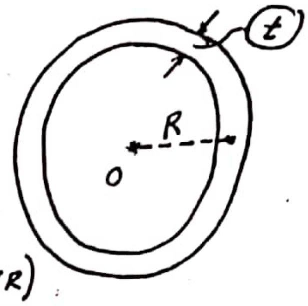
In arm 2 $\quad \gamma_{xs}\Big|_{2} = \gamma_{xz}\Big|_{2} = \alpha(\varphi_{,z} + y) = \alpha(\varphi_{,z} + b/2) = C$

$$\Rightarrow \varphi_{,z} = C - b/2 \quad \text{in } 2$$

So, the solution for the warping function ensures that γ_{xs} remains a constant in each arm.

But $\varphi(y,z)$ is a displacement function and hence is CONTINUOUS (WHY ?). Thus $\varphi(s=0) = \varphi(s=l)$ leading to the result relating α to q_0.

Ex.



Here, $A = \pi R^2$

$$\Rightarrow q_0 = \frac{T}{2\pi R^2};$$

$$\alpha = \frac{q_0}{2A}\int_0^{l}\frac{ds}{Gt} = \frac{T}{(2\pi R^2)}\cdot\frac{1}{(2\pi R^2)\,Gt}\cdot(2\pi R)$$

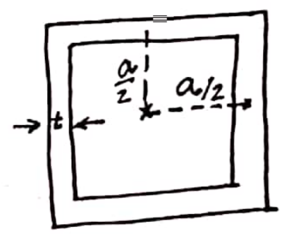
$$= \frac{T}{Gt\,(2\pi R^3)} \qquad \Rightarrow J = \boxed{2\pi R^3 t}$$

$$J_{actual} = \frac{\pi}{2}\left(R_0^4 - R_i^4\right) = \frac{\pi}{2}\left(R_0^2 - R_i^2\right)\left(R_0^2 + R_i^2\right)$$

$$\approx \frac{\pi}{2}\left(R_0 - R_i\right)\left(R_0 + R_i\right)\left(R_0^2 + R_i^2\right)$$

$$= \frac{\pi}{\cancel{2}}(t)(\cancel{2}R)\left(R^2 - \cancel{Rt} + t^2/4 + R^2 + \cancel{Rt} + t^2/4\right)$$

$$= \boxed{2\pi R^3 t} + h.o.t.$$



$$A = a^2 \Rightarrow 2A = 2a^2$$

$$\Rightarrow q_0 = \frac{T}{2a^2}$$

$$\alpha = \frac{T\oint ds}{4A^2\,Gt} = \frac{T \times 4a}{4 \times a^{4^3} \times Gt} = \frac{T}{\underbrace{a^3 t}_{J}\cdot G}$$

$$J = a^3 t$$

If $4a = 2\pi R \Rightarrow a = \frac{\pi R}{2}$

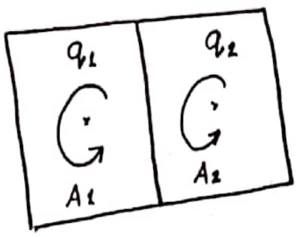
$$\Rightarrow a^3 = \frac{\pi^3 R^3}{8} \Rightarrow J = \frac{\pi^3 R^3}{8}t = \frac{\pi^3 R^3}{8}t = \frac{\pi^3}{8}\cdot(R^3 t)$$

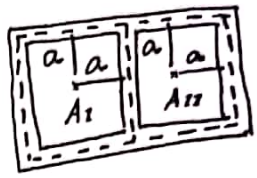
$< 2\pi$

Hence, with the same amount of material, the circular section gives higher torsional rigidity. ← IS THE CIRCULAR SECTION OPTIMAL WITH RESPECT TO J ?

SOLVE 18.2, 18.7, 18.8

MULTI-CELLED SECTIONS





* Rate of twist is preserved

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha$$

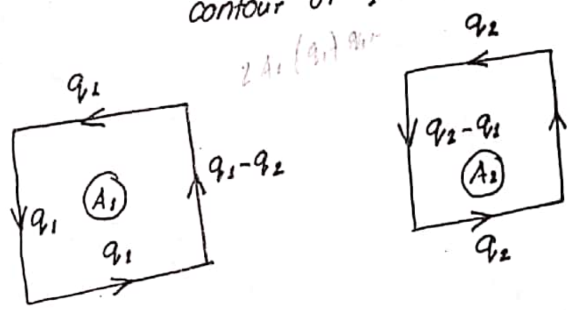
for cell 1   for cell 2   for full cross-section

different in each arm

But $\alpha_1 = \dfrac{1}{2A_1}\oint_{C_1}\dfrac{q\,ds}{Gt}$; $\alpha_2 = \dfrac{1}{2A_2}\oint_{C_2}\dfrac{q\,ds}{Gt}$

contour of 1st cell

Ex. 23.2 ;
Problems 23.2, 23.3, 23.4,



$A_1 = a^2 = A_2$

$$\Rightarrow \alpha_1 = \frac{1}{2a^2}\left[\frac{1}{Gt}\left(q_1 \cdot a + q_1 \cdot a + q_1 \cdot a + (q_1 - q_2)\cdot a\right)\right] = \frac{1}{2aGt}(4q_1 - q_2)$$

$$= \frac{1}{2a^2Gt}(4q_1 a - q_2 a)$$

$$\alpha_2 = \frac{1}{2a^2Gt}(4q_2 a - q_1 a) = \frac{1}{2aGt}(4q_2 - q_1)$$

$\therefore \alpha_1 = \alpha_2$ gives $4q_1 - q_2 = 4q_2 - q_1 \Rightarrow \boxed{q_1 = q_2}$

$$\therefore T = \sum_{i=1}^{2} 2A_i q_i = 2\left[a^2 q_1 + a^2 q_2\right] = 2a^2(q_1 + q_2)$$

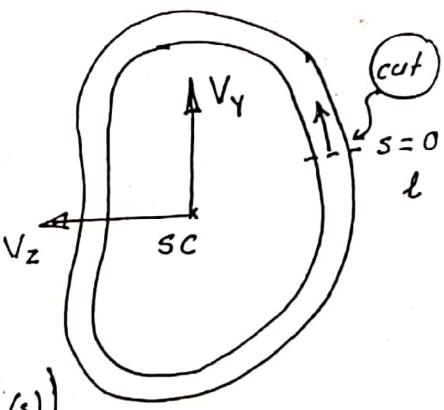
$$= 4a^2 q_1$$

$$\Rightarrow \boxed{q_1 = \frac{T}{4a^2}} = q_2 \Rightarrow \alpha = \frac{3T}{4a^2} \cdot \frac{1}{2aGt} = \frac{T}{\left(\underbrace{\tfrac{8}{3}a^3 t}_{J}\right)G}$$

Note: The section may have different material in different regions. Thus, take G of the arm (e.g. spar web and skin may have different materials).

We have conveniently assumed that all torsion induced displacements are with respect to the SHEAR CENTRE, as this point does not rotate and all moment arms are relative to this point. How do we get this point?

SHEAR CENTRE: Let us get back to the bending induced shear, due to shear forces V_y, V_z through the SC. Hence,



$$q(s) = q_b + q_0$$

q_0 ← shear flow at point where fictitious cut is made

$$q_b(s) = \frac{-V_z}{(I_{YY} I_{ZZ} - I_{YZ}^2)} \left(I_{YZ} Q_Z(s) + I_{ZZ} Q_Y(s) \right) + \frac{V_Y}{(I_{YY} I_{ZZ} - I_{YZ}^2)} \left(-I_{YY} Q_Z(s) + I_{YZ} Q_Y(s) \right)$$

$q_b(s)$: shear flow for "cut" section

Now, if V_Y, V_Z pass through SC, then they do not cause any twist $\Rightarrow \alpha = 0$

But $\alpha = \frac{1}{2A} \oint \frac{q(s)\, ds}{Gt} = 0 \Rightarrow \oint \frac{(q_b + q_0)}{Gt} ds = 0$ ← about centroid!

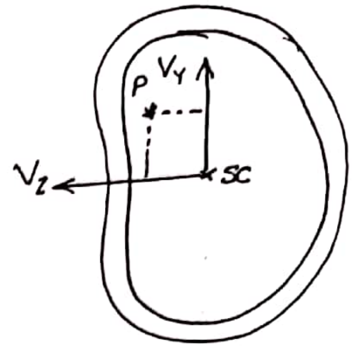
$$\Rightarrow q_0 = - \frac{\oint q_b/Gt \, ds}{\oint ds/Gt}$$

← The constant shear flow is now determined.

* Note that depending on where we cut, q_0 will be DIFFERENT as Q_Y, Q_Z will depend on cut.

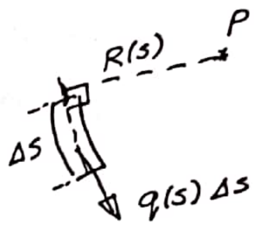
Let us now consider a point P about which we take the moment

Let P be located at distances (a,b) from SC.



Then resultant torque, about P, due to $q(s)$ is:

$$T/_P = \int_{s=0}^{\ell} R\, q\, ds$$



This has to equal the torque about P due to V_y, V_z or:

$$V_y\, a - V_z\, b = \int_{s=0}^{\ell} R\, q\, ds = \oint R(q_b + q_0)\, ds$$

From here we can find a, b either by equating coefficients of V_y (1 eqn.) and V_z (2nd eqn.) OR by

(a) Let $V_y = 1$, $V_z = 0$.

Find $q_b^{(1)}(s)$ due to this choice $\Rightarrow$ obtain $q_0^{(1)}$

Now $\boxed{1 \cdot a = \oint R(q_b^{(1)} + q_0^{(1)})\, ds}$ —— (i)

(b) Repeat with $V_y = 0$, $V_z = 1$

Find $q_b^{(2)}(s)$ due to this choice $\rightarrow$ obtain $q_0^{(2)}$

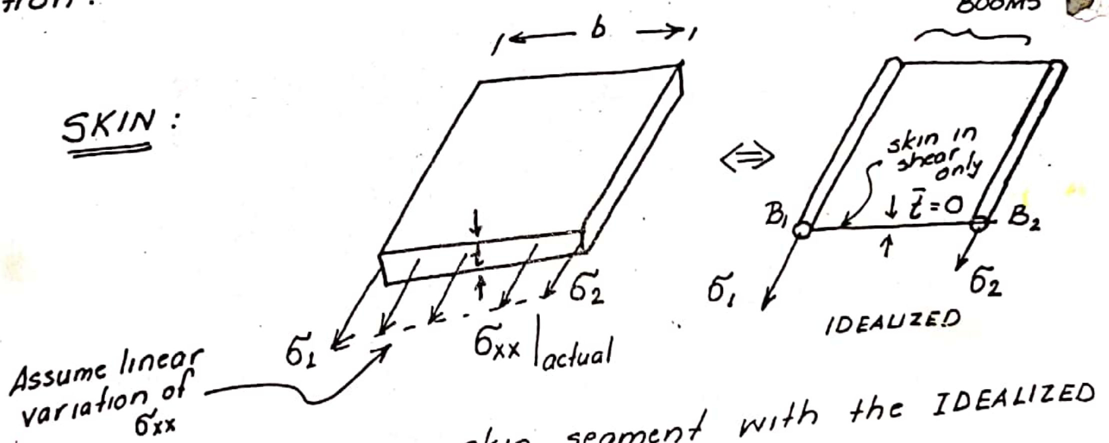
Then $\boxed{-1 \cdot b = \oint R(q_b^{(2)} + q_0^{(2)})\, ds}$ —— (ii)

STRUCTURAL IDEALIZATION

We can solve the shear problem, if we know $\phi(y,z)$ or $\bar{\psi}(y,z)$. For thin sections, we saw that we can get around this and get stress state σ_{xs} in terms of shear flow $q(s)$.

→ Yet, we want further simplification to get a solution which is not "so accurate", but is "easier" to obtain & captures all ESSENTIAL FEATURES of the solution.

SKIN :



* Replace the continuous skin segment with the IDEALIZED section, with the concentrated areas B_1, B_2 carrying all the DIRECT (or axial) stress and the intermediate skin only carrying shear load, i.e. $\sigma_{xs} \neq 0$.

EQUATE moments due to σ_{xx} ⇒ about B_2 :

$$\int_0^b \sigma_{xx}(b-x)\,dx.t = \int_0^b \left(\sigma_1 + (\sigma_2-\sigma_1)\left(\frac{x}{b}\right)\right)(b-x)t\,dx$$

$$= \int_0^b t\,\sigma_1(b-x)\,dx + \frac{1}{b}\int_0^b t(\sigma_2-\sigma_1)(bx - x^2)\,dx$$

$$= t\,\sigma_1\left(bx - \frac{x^2}{2}\right)\Big|_0^b + \frac{(\sigma_2-\sigma_1).t}{b}\left(\frac{bx^2}{2} - \frac{x^3}{3}\right)\Big|_0^b$$

$$= t\,\sigma_1\left(b^2/2\right) + \frac{(\sigma_2-\sigma_1)t}{b}\underbrace{\left(\frac{b^3}{2} - \frac{b^3}{3}\right)}_{\frac{b^3}{6}}$$

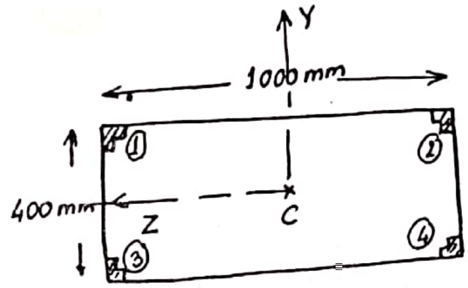
$\Rightarrow$ moment due to actual σ_{xx}, about B_2,

$$\Rightarrow \quad \sigma_1 t \frac{b^2}{2} + (\sigma_2 - \sigma_1) t \cdot \frac{b^2}{6} = \sigma_1 B_1 b$$

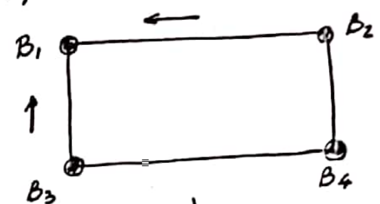
$$\Rightarrow \quad B_1 = \frac{tb}{6\sigma_1}\left\{3\sigma_1 + \sigma_2 - \sigma_1\right\} = \frac{tb}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right)$$

$$\Rightarrow \quad \boxed{B_1 = \frac{tb}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right)} \; ; \quad \boxed{B_2 = \frac{tb}{6}\left(2 + \frac{\sigma_1}{\sigma_2}\right)}$$

Ex



$\Rightarrow$ Spar cap area = 300 mm^2



Bending moment only in vertical plane (about z-axis only)

$t = 2\,mm$

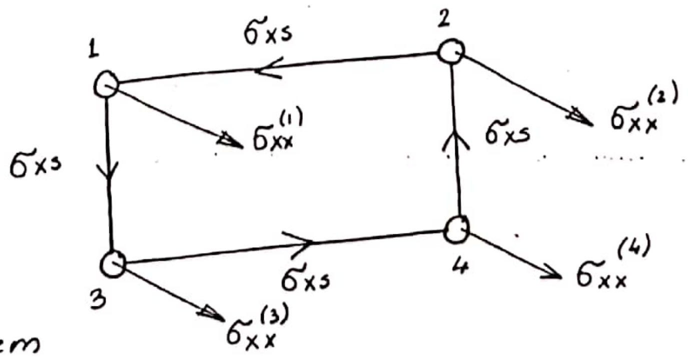
$$B_1 = 300 + \frac{2 \times 400}{6}\left(2 + \frac{\sigma_3}{\sigma_1}\right) + \frac{2 \times 1000}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right)$$

(-1) $\quad (1)$

$$= 300 + \frac{800}{6} \times 1 + \frac{2000}{6} \times 3$$

$$= 300 + \frac{400}{3} + 1000 = \frac{4300}{3}\ mm^2$$

$B_3 = B_1$; $\quad B_2 = B_4 = B_1$ (in this case).



* For pure torsion

$\sigma_{xx} = 0$

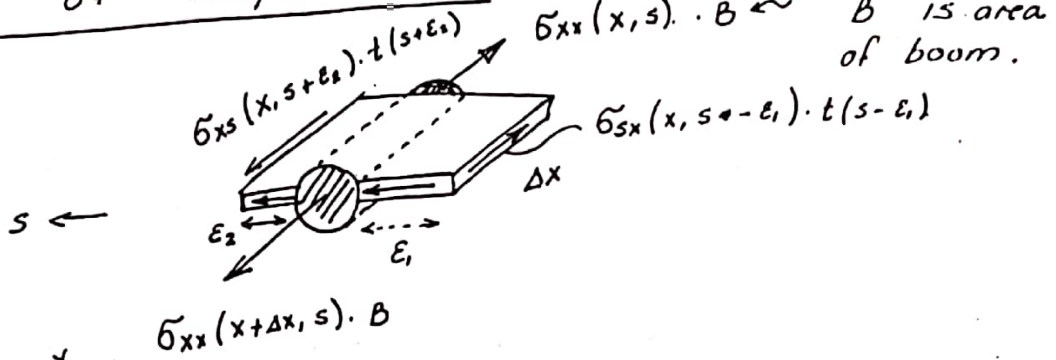
$\Rightarrow$ No effect

* For bending problem

$\rightarrow$ $\sigma_{xx}^{(i)}$ leads to JUMP in shear flow ACROSS the boom.

IDEALIZED CASE :

Effect of lumped area



$$\sum F_x = 0 \Rightarrow \left(q(x, s+\varepsilon_2) - q(x, s-\varepsilon_1) \right) \Delta x +$$

$$\left(\sigma_{xx}(x+\Delta x, s) - \sigma_{xx}(x,s) \right) B = 0$$

$$\Rightarrow \frac{\partial \sigma_{xx}}{\partial x} \cdot B + \left(q(s^+) - q(s^-) \right) = 0$$

$$\Rightarrow q(s^+) - q(s^-) = - \frac{\partial \sigma_{xx}}{\partial x} \cdot B$$

or $\boxed{[q]\big|_s = - \frac{\partial \sigma_{xx}}{\partial x}\Big|_s \cdot B\big|_s}$

JUMP IN SHEAR FLOW

Now $$\sigma_{xx,x}\big|_s = \frac{V_z \left(-y|_s I_{yz} + z|_s I_{zz} \right)}{\Delta} - \frac{V_y \left(-y|_s I_{yy} + z|_s I_{yz} \right)}{\Delta}$$

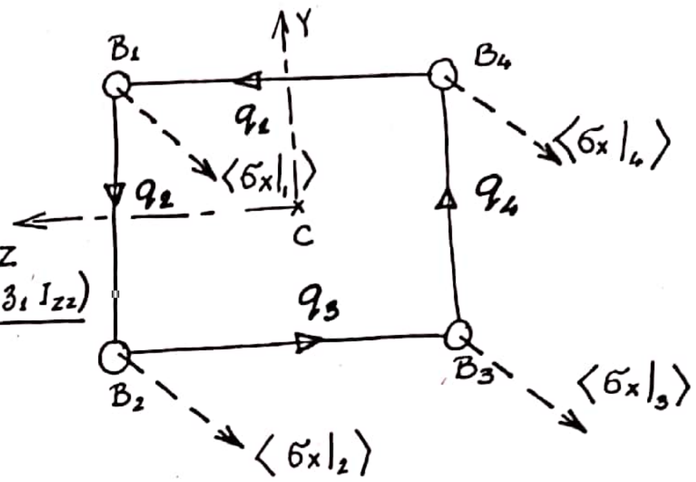
Compute this at the boom.

* Note : s is taken to be positive in the given direction. Reversal of direction will not change the result.

* Remember : $$\sigma_{xx}(x,y,z) = M_y \frac{(-y I_{yz} + z I_{zz})}{\Delta} + M_z \frac{(-y I_{yy} + z I_{yz})}{\Delta}$$

and $V_y = -M_{z,x}$; $V_z = M_{y,x}$

For the given example :



Location of boom B_i :

(y_i, z_i) $\{$ with $y_1 = y_4$; $y_2 = y_3$
$z_1 = z_2$; $z_3 = z_4$ $\}$

$$\therefore \quad q_2 - q_1 = -B_1\left\{ V_z \frac{(-y_1 I_{yz} + z_1 I_{zz})}{\Delta} - V_Y \frac{(-y_1 I_{yy} + z_1 I_{yz})}{\Delta} \right\}$$

$$q_3 - q_2 = -B_2\left\{ V_z \frac{(-y_2 I_{yz} + z_2 I_{zz})}{\Delta} - V_y \frac{(-y_2 I_{yy} + z_2 I_{yz})}{\Delta} \right\}$$

$$q_4 - q_3 = -B_3\left\{ V_z \frac{(-y_3 I_{yz} + z_3 I_{zz})}{\Delta} - V_y \frac{(-y_3 I_{yy} + z_3 I_{yz})}{\Delta} \right\}$$

$$q_1 - q_4 = -B_4\left\{ V_z \frac{(-y_4 I_{yz} + z_4 I_{zz})}{\Delta} - V_y \frac{(-y_4 I_{yy} + z_4 I_{yz})}{\Delta} \right\}$$

$$\Delta = I_{yy} I_{zz} - I_{yz}^2$$

Adding the 4 equations gives :

$$0 = -\frac{V_z}{\Delta}\left(\sum_{i=1}^{4}(-y_i B_i) I_{yz} + \sum_{i=1}^{4}(z_i B_i) I_{zz} \right) + \frac{V_y}{\Delta}\left(\sum_{i=1}^{4}(-y_i B_i) I_{yy} + \sum_{i=1}^{4}(z_i B_i) I_{yz} \right)$$

But note that by the definition of centroid,

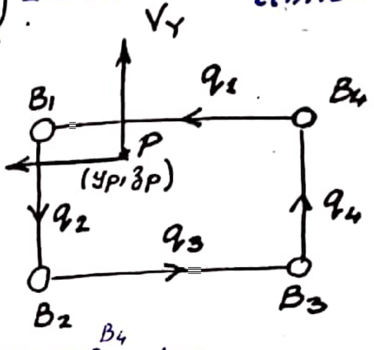
$$\sum_{i=1}^{4} y_i B_i = 0 \quad ; \quad \sum_{i=1}^{4} z_i B_i = 0 \quad \Rightarrow \text{RHS} = 0 \text{ also.}$$

So, the 1^{st} three equations are unique and the 4^{th} one is redundant (but consistent).

$\Rightarrow$ 4 UNKNOWNS : 3 EQUATIONS

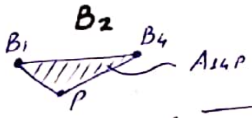
We need one more equation:

Let us say that the resultant is known and is given by V_y, V_z through point (y_P, z_P). ← E.g. aerodynamic centre!



∴ Taking moment, M_x, about point P (or any other point Q), we get:

$$V_y(0) + V_z(0) = \oint \underbrace{q(s) R_p(s)\, ds}_{2A_{14P}}$$



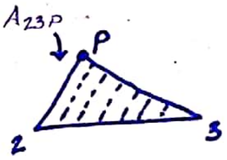
$$\Rightarrow 0 = q_1 \times (y_1 - y_P) \times (z_1 - z_4) + q_2 \times \underbrace{(z_1 - z_P) \times (y_1 - y_2)}_{2A_{12P}}$$
$$+ q_3 \times \underbrace{(y_P - y_2) \times (z_2 - z_3)}_{2A_{23P}} + q_4 \times \underbrace{(z_P - z_4) \times (y_4 - y_3)}_{2A_{34P}}$$

* Note that q_1 depends on point of application of V_y, V_z.

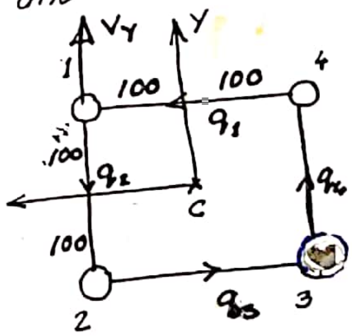
* The shear flow has contribution of both bending and torsion

* "Shifting" to SC would have given us, separately, the bending and torsion parts of q.

⇒ Obtain q_1 from this.



Ex. Let an external force $V_y = 10\ kN$ act through boom B_1, with $B_1 = B_2 = B_3 = B_4 = 200\ mm^2$ and each side of length $\ell = 200\ mm$.



$y_1 = y_4 = 100\ mm$; $z_1 = z_2 = 100\ mm$

$y_2 = y_3 = -100\ mm$; $z_3 = z_4 = -100\ mm$

< Note here shear flow develops "in response" to the resultant V_y >

$$I_{yy} = I_{zz} = 4 B_1 \times y_1^2 = 4 \times 200 \times 100^2 = 8 \times 10^6\ mm^4$$

$$I_{yz} = 0$$

$$\Rightarrow q_2 - q_1 = -200 \times (-10^4) \times \frac{100}{8 \times 10^6} = -\frac{V_y \cdot y_1}{I_{zz}} B_1$$

$$= -\frac{100}{4}\ N/mm$$

$$\Rightarrow q_3 - q_2 = \frac{100}{4}\ N/mm ; \quad q_4 - q_3 = \frac{100}{4}\ N/mm$$

$\Rightarrow q_2 = q_1 - 25 ; \quad q_3 = q_2 + 25 = q_1 ; \quad q_4 = q_3 + 25$
$= q_1 + 25$

Taking moment about boom 1 :

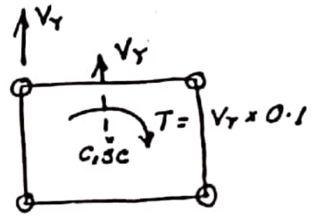
$0 = 0 \times q_1 + 0 \times q_2 + (q_3 + q_4) \times 200 \times 200 = 0$

$\Rightarrow q_3 + q_4 = 0 \Rightarrow 2q_1 + 25 = 0 \Rightarrow \boxed{q_1 = -12.5 \text{ N/mm}}$

$\Rightarrow q_2 = -37.5 \text{N/mm} ; \quad q_3 = -12.5 \text{ N/mm} ; \quad q_4 = 12.5 \text{N/mm}$

$\Rightarrow$ (Check) Total shear force $= 12.5 \times 200 - (-37.5 \times 200)$
$= 50 \times 200 = \boxed{10 \text{ kN}} = V_Y$

Check 2 : Shift V_y to sc (which is at c) :

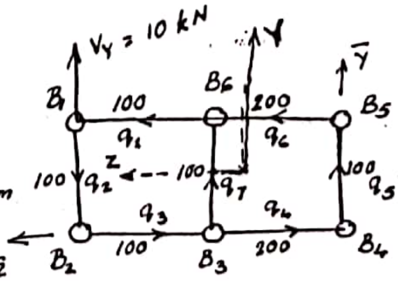


$\Rightarrow \sum M_{x,sc}$ due to q is :

$(q_1 + q_2 + q_3 + q_4) \times \underbrace{100}_{\text{moment arm}} \times \underbrace{200}_{\text{force arm}}$

$= (-12.5 - 37.5 - 12.5 + 12.5) \times 100 \times 200 = -50 \times 2 \times 10^4$ N.mm
$= -100 \times 10^4$ N.mm
$= \boxed{-1000 \text{ N.m}} = T$ ✓

Ex. 2



$B_i = 200$ mm^2
$i = 1, 2, \dots, 6$

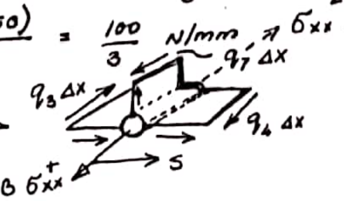
$y_c = \frac{100 \times 3B}{6B} = 50$ mm

$z_c = \frac{200 \times 2 + 300 \times 2}{6} = \frac{1000}{6} = 166.67$ mm

$q_2 - q_1 = -B_1 \cdot \frac{V_y \cdot y_1}{I_{zz}}$
$= -\frac{200 \times 10^4 \times 50}{3 \times 10^6} = -\frac{100}{3}$ N/mm
$= -\frac{100}{3}$ N/mm

$q_3 - q_2 = -\frac{200 \times 10^4 \times (-50)}{3 \times 10^6} = \frac{100}{3}$ N/mm

$q_4 + q_7 - q_3 = \frac{100}{3} \Longrightarrow$



$q_5 - q_4 = \frac{100}{3}$

$q_6 - q_5 = -\frac{100}{3}$

(5 equations)

$I_{zz} = 50^2 \times 6 \times 200$
$= 2500 \times 1200$
$= 3000000 = 3 \times 10^6$ mm^4

$I_{YY} = 2B_1 \times (133.33)^2 + 2B_3 \times (33.33)^2 + 2B_4 \times (166.67)^2$

7 unknowns $\Rightarrow$ 2 more equations required

Moment M_x about boom B_1 $\Rightarrow$

$$0 = q_3 \times 100 \times 100 + q_4 \times 200 \times 100 + q_5 \times 100 \times 300 + q_7 \times 100 \times 100$$

$$\Rightarrow \boxed{q_3 + 2q_4 + 3q_5 + q_7 = 0} \quad \text{— (IV)}$$

All cells twist by same amount or :

$$\alpha_1 = \frac{1}{2A_1 \cdot Gt}\left[q_1 \times 100 + q_2 \times 100 + q_3 \times 100 + q_7 \times 100 \right]$$

$$\alpha_2 = \frac{1}{2A_2 \cdot Gt}\left[q_4 \times 200 + q_5 \times 100 + q_6 \times 200 - q_7 \times 100 \right]$$

(Let G, t be same everywhere)

$\Rightarrow$ $\alpha_1 = \alpha_2$ gives

$$\frac{1}{100^2}(q_1 + q_2 + q_3 + q_7) = (2q_4 + q_5 + 2q_6 - q_7)\frac{1}{200 \times 100}$$

or $\boxed{2(q_1 + q_2 + q_3 + q_7) = 2q_4 + q_5 + 2q_6 - q_7}$ all eqns. are there ! (5 eqns., 5 unknowns)

$$\Rightarrow 2q_1 + 2q_2 + 2q_3 = 2q_4 + q_5 + 2q_6 - 3q_7 \quad \text{— (V)}$$

$\Rightarrow$ $q_2 = q_1 - 33.33$

$q_3 = q_2 + 33.33 = q_1$

$q_4 + q_7 - q_3 = 33.33 \Rightarrow q_4 + q_7 = q_1 + 33.33 \Rightarrow q_7 = q_1 - q_4 + 33.33$

$q_5 = q_4 + 33.33$; $q_6 = q_5 - 33.33 = q_4$

$$q_1 + 2q_4 + 3(q_4 + 33.33) + (q_1 - q_4 + 33.33) = 0 \quad \text{(from IV)}$$

$$\Rightarrow 2q_1 + 4q_4 = -133.33 \quad \text{— (a)}$$

$$2q_1 + 2(q_1 - 33.33) + 2q_1 = 2q_4 + (q_4 + 33.33) + 2q_4 - 3q_1 + 3q_4 - 33.33 \times 3$$

$$\Rightarrow 9q_1 - 8q_4 = 0$$

$$\Rightarrow q_4 = \frac{9}{8} q_1 \quad \text{— (b)}$$

$$\Rightarrow 2q_1 + \frac{9}{2}q_1 = -\frac{400}{3} \Rightarrow \frac{13}{2}q_1 = -\frac{400}{3} \Rightarrow q_1 = -\frac{[illegible]}{[illegible]} \text{ N/mm}$$

$= -20.[illegible]$ N/mm

$\Rightarrow$ $q_2 = -53.[illegible]$ N/mm ; $q_3 = -20.5$ N/mm ; $q_4 = -23.1[illegible]$ N/mm

$q_5 = 10.3$ N/mm ; $q_6 = -23.10$ N/mm ; $q_7 = 9.7$ N/mm

Check : $V_Y = (q_7 + q_5 - q_2) \times 100$ N $= 100(36.43 + 5.42 + 58.14)$

$= \underline{10\ kN}$

$V_z = 0$

Note : $q_7 \neq 0$ and is in-fact significant, leading to a reduction in q_5 $\rightarrow$ reduction in twist or ~~angle~~ rate of twist.

$\Rightarrow$ INCREASED TORSIONAL RIGIDITY

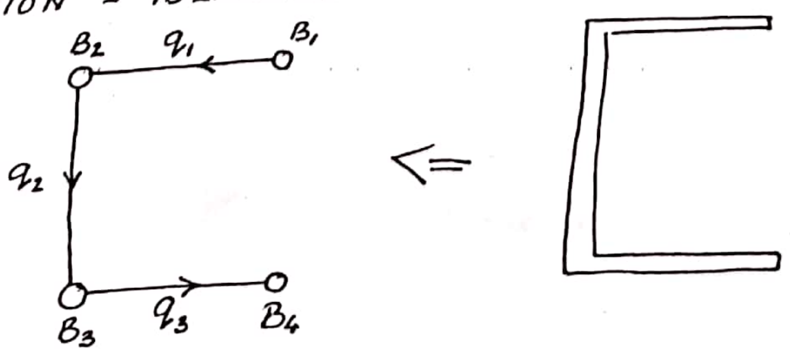
~ | Q What is shear flow if web joining B_3 to B_6 is removed.
Compare with the shear flow obtained in this example |

* In the general case of an n-celled section we have $(n-1)$ additional unknowns. The additional equations come from constraint on rate of twist, or $\alpha_1 = \alpha_2 = \dots = \alpha_n$

G	G	G	G

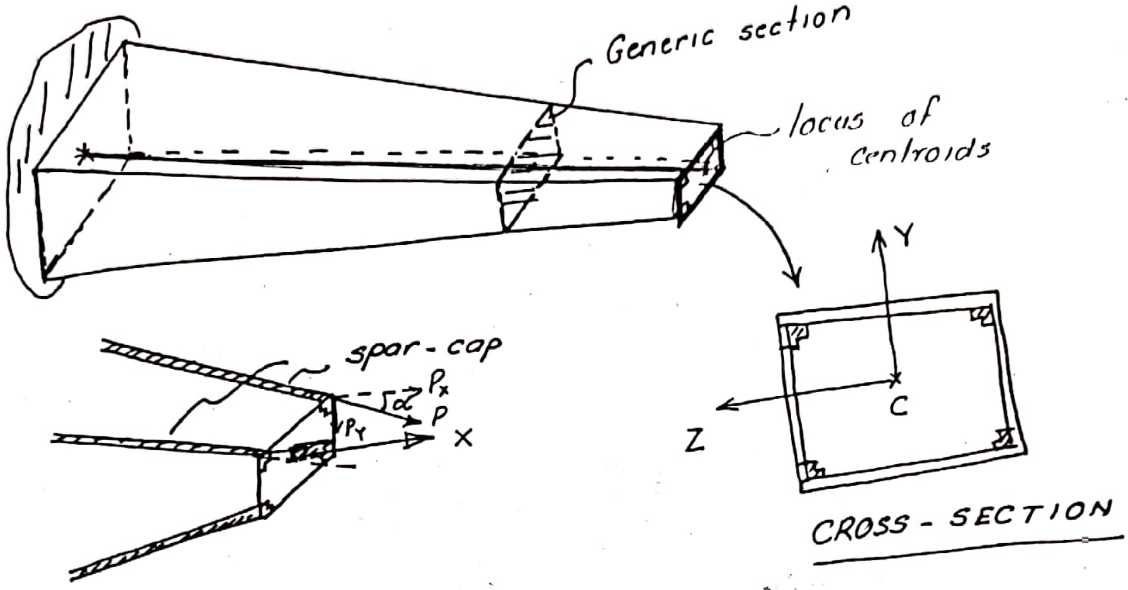
* For web/skin with different materials, the G of the element has to be accounted for via the α_i constraint.
(also $B_i = E_i B_i / E^*$)

* OPEN SECTION - IDEALIZED



Tapered Sections

(Only spanwise taper considered)

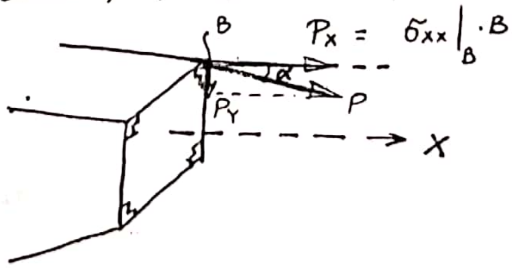


CROSS-SECTION

Is taper an artefact ?

No, since the tip is least loaded, it makes no sense to continue with the same cross-sectional dimensions as at the root. Material weight (and cost) reduction can be effective if the taper is used.

Tapered spar $\rightarrow$ bending stress σ_{xx} computed by usual procedure, but SHEAR has to done properly.



Bending analysis

~ gives the boom axial force ~

$\sigma_{xx}|_B \cdot B$ (B is area of boom).

$= P_x$

Since boom (or spar-cap) is an axial load carrying member

its axial load $= P$

such that $P \cos\alpha = P_x \Rightarrow P = \dfrac{P_x}{\cos\alpha}$

∴ the tangential (or shear) load due to P is:

$$P_z\Big|_B = -P\sin\alpha = -P_x\frac{\sin\alpha}{\cos\alpha} = \boxed{-P_x\tan\alpha}$$

known from bending analysis

∴ Total P_z is due to contribution of all longitudinals

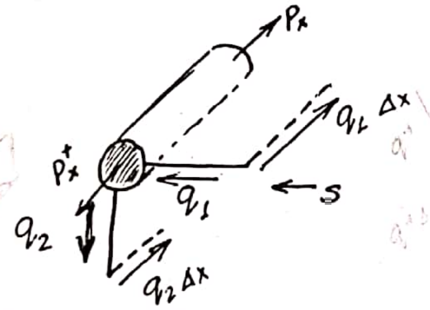
$$\bar{P}_z = \sum_{i=1}^{N} P_z\Big|_{B_i}$$

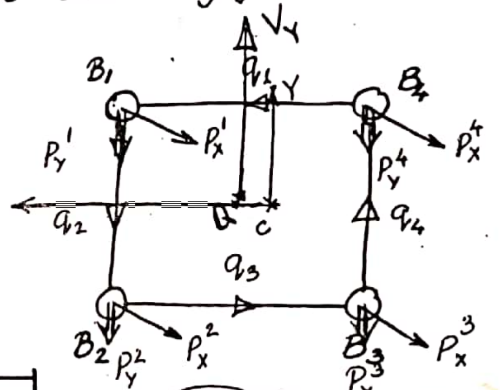
where N longitudinals are taken with effective area $\{B_i\}_{i=1}^{N}$.

∴ Shear force generated due to <u>web</u> and <u>skin</u> is:

$$\boxed{\bar{V} = V_Y - \bar{P}_z}$$

∴ the shear flow will change accordingly.





$$\boxed{q_2 - q_1 \approx \frac{d}{dx}P_x \approx -\frac{\bar{V}_Y\, y_i\, B_i}{I_{zz}}}$$

* P_y^i will contribute to the TORQUE balance equation (about Q, say)

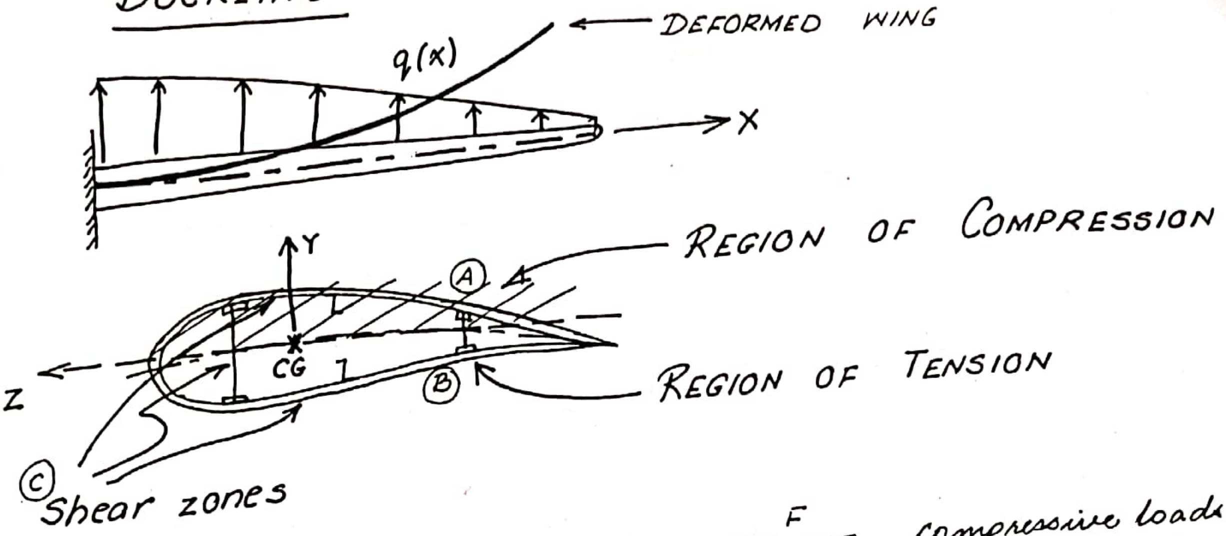
* Note that this is an IDEALIZATION, i.e. we have not bothered to take the effect of P_{zY}^i at the booms but have considered the resultant effect only.

Further, $\dfrac{\partial}{\partial x}\sigma_{xx} = -\dfrac{V_y \cdot y_i}{I_{zz}}$ where V_y is the total shear force.

Taking $\bar{V}$ is an approximation.

~ Better prediction ⇒ more sophisticated analysis.

BUCKLING



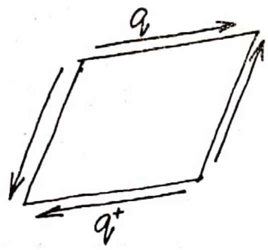
Region A :

compressive loads (direct)

Region B :

Tensile loads (direct)

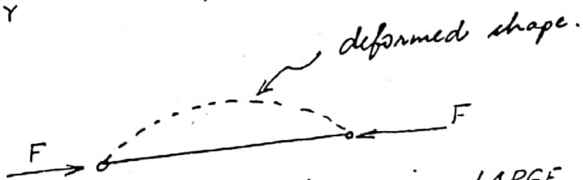
Region C :



Shear load (dominant).

* Regions A & C <u>buckle</u> or loose form at a value of stress $\sigma_{rms} < \sigma_Y$

equivalent Mises stress



→ The deformation in transverse direction is LARGE, and hence cannot be treated as small deformation.

<u>Q</u> <u>Infinitesimal assumption</u> ⇒ equilibrium in undeformed coordinate system. How can axial load give rise to bending action?