Consequences of strain derivation

Let $\overrightarrow{PQ_i} = \overrightarrow{\Delta X}^{(i)}$ and $\overrightarrow{P'Q_i'} = \overrightarrow{\Delta x}^{(i)}$, then

with $l_0^{(i)} = |\overrightarrow{\Delta X}^{(i)}|$; $l^{(i)} = |\overrightarrow{\Delta x}^{(i)}|$, we have

$$\frac{\Delta l^{(i)}}{l_0^{(i)}} = \frac{l^{(i)} - l_0^{(i)}}{l_0^{(i)}} = \frac{\overrightarrow{\Delta X}^{(i)\,T} [\varepsilon] \overrightarrow{\Delta X}^{(i)}}{l_0^{(i)\,2}}$$

where $[\varepsilon]\big|_P = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}_{\text{at } P}$ as the strain tensor at $\underline{P}$.

- Knowing $[\varepsilon]\big|_P$ we can get stretch of ANY infinitesimal line element $\overrightarrow{PQ_i}$ at P.

E.g. $\overrightarrow{\Delta X}^{(1)} = \Delta X\,\hat{i} \Rightarrow \dfrac{\Delta l^{(1)}}{l_0^{(1)}} = \dfrac{\{\Delta X \;\; 0 \;\; 0\} \begin{Bmatrix} \varepsilon_{11}\Delta X \\ \varepsilon_{21}\Delta X \\ \varepsilon_{31}\Delta X \end{Bmatrix}}{\Delta X^2}$

$$= \frac{\varepsilon_{11} \cdot \Delta X^2}{\Delta X^2} = \varepsilon_{11}$$

$\overrightarrow{\Delta X}^{(2)} = \dfrac{\Delta X}{\sqrt{2}}(\hat{i} + \hat{j}) \Rightarrow \dfrac{\Delta l^{(2)}}{l_0^{(2)}} = \dfrac{\Delta X^2 \{1 \;\; 1 \;\; 0\} \begin{Bmatrix} \varepsilon_{11} + \varepsilon_{12} \\ \varepsilon_{21} + \varepsilon_{22} \\ \varepsilon_{31} + \varepsilon_{22} \end{Bmatrix} \times \frac{1}{2}}{\Delta X^2}$

$$= \boxed{\varepsilon_{11} + 2\varepsilon_{12} + \varepsilon_{22}} \, / 2$$

and so on !!

To

Eigenvalue problem for strain tensor:

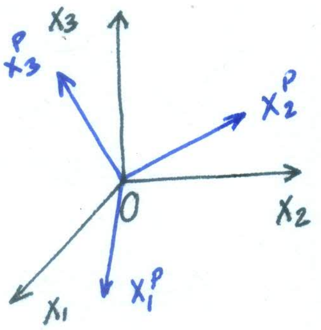
$$[\varepsilon]\vec{n} = \lambda \vec{n} \Rightarrow \text{ we get eigenvalues } \lambda_1, \lambda_2, \lambda_3$$

and corresponding eigenvectors $\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)}$.

- Since $[\varepsilon]$ is symmetric, $\{\lambda_i\}_{i=1}^{3}$ is real and

$$\vec{n}^{(1)} \perp \vec{n}^{(2)} \perp \vec{n}^{(3)} \Leftarrow \text{form an orthogonal coordinate system}$$

principal coordinates/axes. $\longrightarrow$ $x_1^p - x_2^p - x_3^p$



Now $[R] = [\vec{n}^{(1)} \ \vec{n}^{(2)} \ \vec{n}^{(3)}]$

$$\Rightarrow [\varepsilon][R] = [\lambda_1\vec{n}^{(1)} \ \lambda_2\vec{n}^{(2)} \ \lambda_3\vec{n}^{(3)}]$$

$$= [n^{(1)} \ n^{(2)} \ n^{(3)}]\begin{bmatrix}\lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3\end{bmatrix}$$

$$\Rightarrow [\varepsilon][R] = [R][\Lambda] \Rightarrow [\varepsilon] = [R][\Lambda][R]^T$$

* Note that $\{\Delta x^p\} = [R]^T\{\Delta x\} = [Q]\{\Delta x\}$ with

$$[Q] = [R]^T \Rightarrow [\varepsilon] = [Q]^T[\Lambda][Q]$$

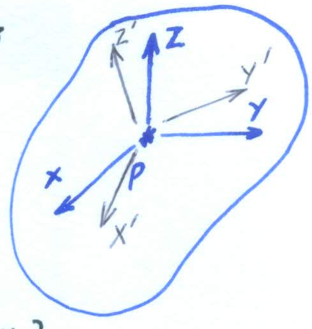
or $\boxed{[\Lambda] = [Q][\varepsilon][Q]^T}$

* Note that $\vec{\Delta x}' = \Delta x \, \vec{n}^{(i)}$ gives $\dfrac{\Delta l}{l_0} = \boxed{\lambda_i}$

* No shear in principal strain coordinate system.

Stress and strain transformation

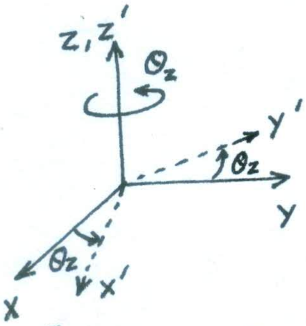
- Let point P have coordinates (x, y, z)



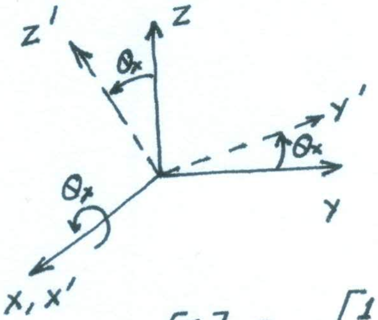
- Let (x', y', z') be coordinates in a new, rotated coordinate system obtained by:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \underset{\text{rotation matrix}}{[R]} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

Examples of $[R]$:



$$[R] = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}$$

$$[\sigma'] = [R][\sigma][R]^T \quad ; \quad [\varepsilon'] = [R][\varepsilon][R]^T$$

Principal stress directions:

$$[\sigma]\{n\} = \lambda\{n\} \Rightarrow \text{Eigenvalues (principal stresses)} \quad \lambda_1, \lambda_2, \lambda_3$$

Principal directions (or eigenvectors) $\vec{n}^1, \vec{n}^2, \vec{n}^3$ with $|\vec{n}^1| = |\vec{n}^2| = |\vec{n}^3| = 1$

and $\vec{n}^1 \perp \vec{n}^2 \perp \vec{n}^3$.

$$\begin{Bmatrix} \vec{n}^1 \\ \vec{n}^2 \\ \vec{n}^3 \end{Bmatrix} = \begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \\ n_1^3 & n_2^3 & n_3^3 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} \Rightarrow \{\vec{n}\} = \underbrace{[N]}_{[R^P]}\{\hat{i}\}$$

$$\Rightarrow \{x^P\} = [R^P]\{x\}$$

Similarly for $[\varepsilon]$ and $[\varepsilon^p] = [\Lambda^p]$ ← diagonal principal strains

If the state of stress at a point P is :

$$[\sigma] = \begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa.}$$ ← PLANE STRESS

$$\Rightarrow [\sigma]\{n\} = \lambda\{n\} \Rightarrow \det\left([\sigma] - \lambda[I]\right) = 0$$

$$\Rightarrow \left| \begin{bmatrix} 200-\lambda & -100 & 0 \\ -100 & 200-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \right| = 0 \Rightarrow (200-\lambda)(200-\lambda)(-\lambda)$$

$$-100 \times [0 - (-100)(-\lambda)] = 0$$

$$\Rightarrow -\lambda\left[(200-\lambda)^2 - 100^2\right] = 0$$

$$\Rightarrow \boxed{\lambda = 0,\ 100,\ 300 \text{ MPa}}$$

For $\lambda = 0$

$$\begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1^1 \\ n_2^1 \\ n_3^1 \end{Bmatrix} = \vec{0}$$

$\Rightarrow 200 n_1^1 = 100 n_2^1$

$\Rightarrow n_1^1 = \frac{1}{2} n_2^1$

$0 n_3^1 = 0 \Rightarrow n_3^1$ is anything

$100 n_1^1 = 200 n_2^1$

$\Rightarrow n_1^1 = n_2^1 = 0$

or $\vec{n}_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

For $\lambda = 100$

$$\begin{bmatrix} 100 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & -100 \end{bmatrix} \begin{Bmatrix} n_1^2 \\ n_2^2 \\ n_3^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$n_1^2 = n_2^2 = \frac{1}{\sqrt{2}}$

$n_3^2 = 0$

$\Rightarrow \vec{n}_2 = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}$

$$\lambda = 200 \Rightarrow \begin{bmatrix} -100 & -100 & 0 \\ -100 & -100 & 0 \\ 0 & 0 & -200 \end{bmatrix} \begin{Bmatrix} n_1^3 \\ n_2^3 \\ n_3^3 \end{Bmatrix} = \vec{0}$$

$\vec{n}_3 = \begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{Bmatrix}$

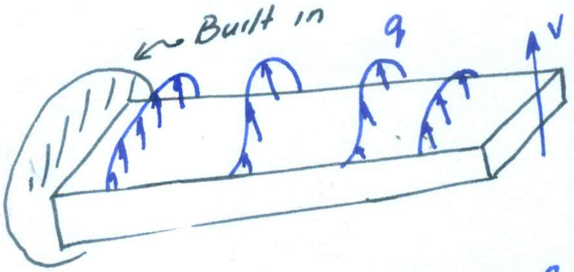
If $\vec{n}_2 = \hat{i}_p$; $\vec{n}_3 = \hat{j}_p$; $\vec{n}_{1} = \vec{k}_p$ then

$$\begin{Bmatrix} \hat{i}_p \\ \hat{j}_p \\ \hat{k}_p \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[R^p]} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

CHECK !

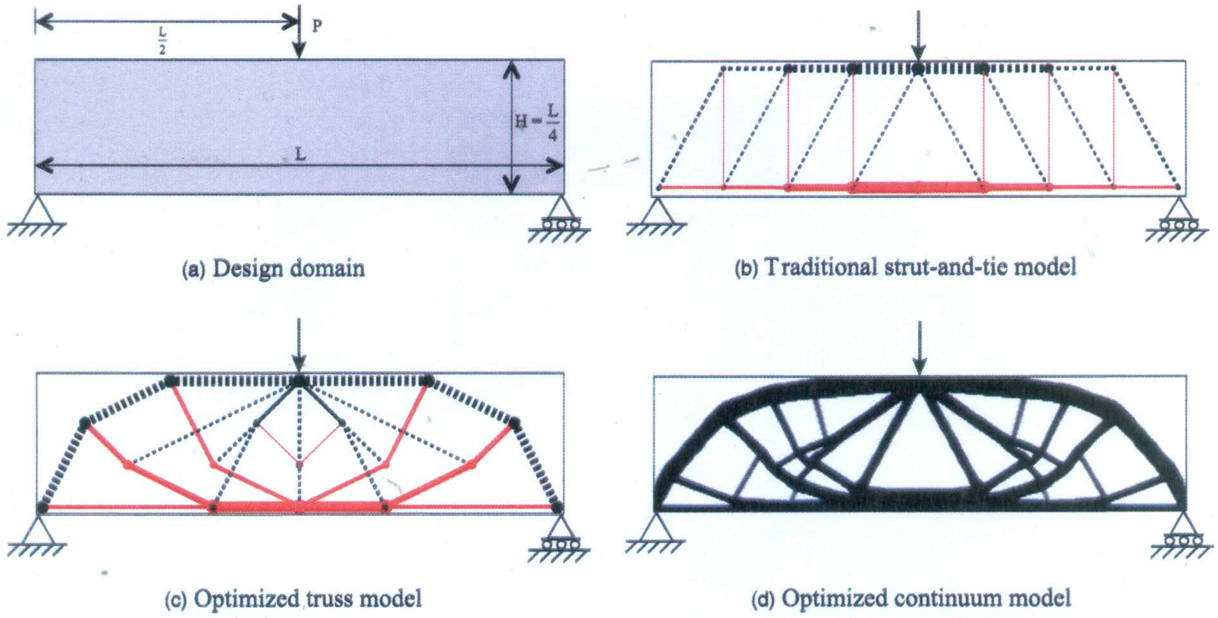
$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [R^p][\sigma][R^p]^T$$

What is the scenario for an actual structure ?



← If the structure is subjected to these loads then as a designer I have to figure out → profile / dimensions ; material used in order to get a <u>safe</u> structure.

– For this we need to obtain stress at <u>every point</u> and then CALCULATE σ_{RMS} or τ_{max} ; CHECK if point is yielding ; If yes, reinforce material to reduce local stress or change material to ensure no yielding ; If no, then remove USELESS material.



(a) Design domain

(b) Traditional strut-and-tie model

(c) Optimized truss model

(d) Optimized continuum model

→ Figure (a) ⇒ initial block of material

→ Figure (d) ⇒ optimized structure for this load i.e. point load P at top.