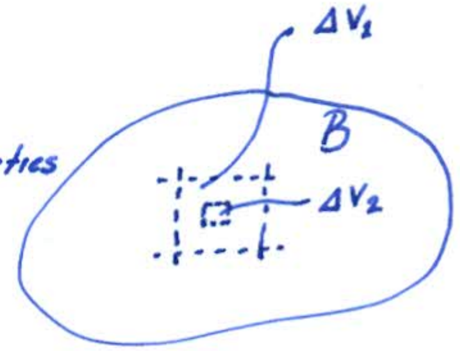
RECAP ON THEORY OF ELASTICITY

The continuum model:

For a body $\mathcal{B}$, the material properties (intrinsic) remain unchanged if we take volumes of shrinking size, i.e.

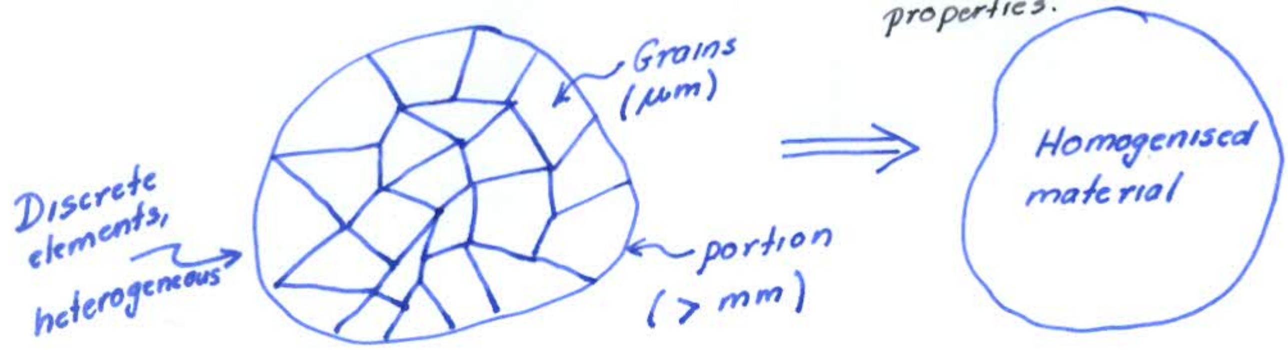


$$\text{Property} \quad p = \lim_{\Delta V \to 0} \frac{\Delta A}{\Delta V}$$

~ This means that our vanishingly small volume does not either see new physics or new material morphology, i.e. GRAINS, MOLECULES, etc.

Example: Density ρ; modulus of elasticity E, ν; coefficient of thermal expansion α, etc.

~ AVERAGED material response at MACRO-level
⟨statistical homogeneity⟩ → size of ensemble large enough to ensure small variance in properties.



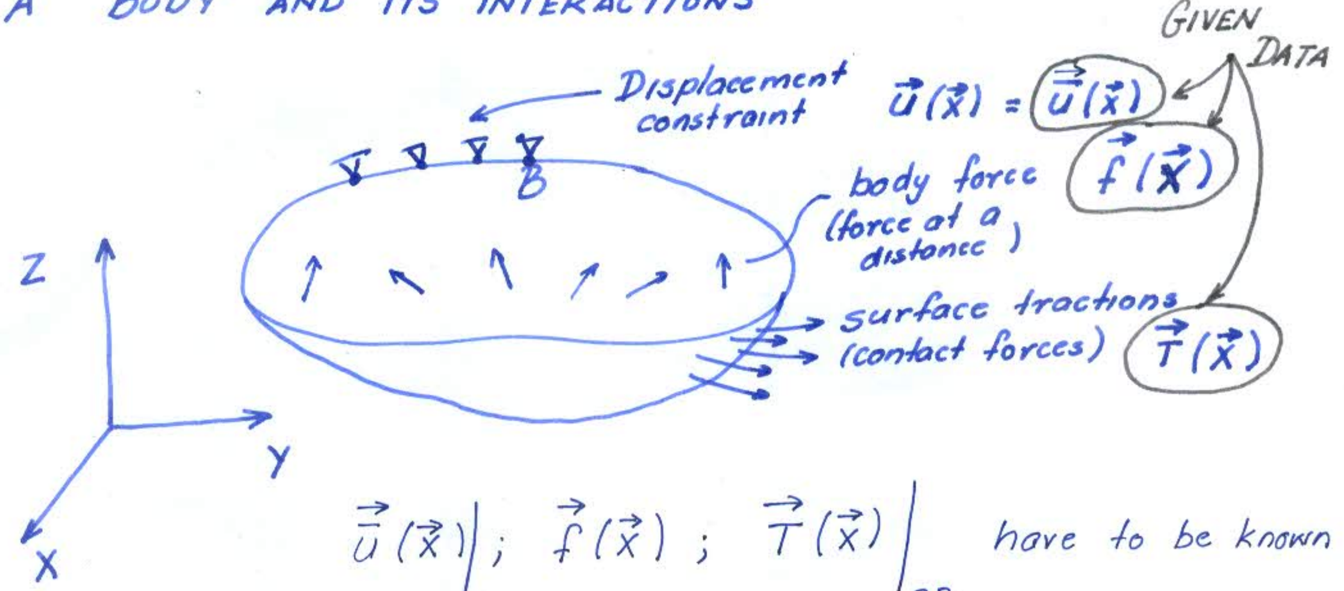
Very important to understand that interest at a given length-scale means corresponding material parameters are averaged to reflect OBSERVATIONS AT THIS LENGTH-SCALE

Material homogeneity does not mean macro-level uniformity. E.g. $E(x,y,z)$ could be there for a material (functionally graded material).

~ We will work mostly with uniform, isotropic materials.

~ Mechanical response given in terms of E, ν.

A BODY AND ITS INTERACTIONS

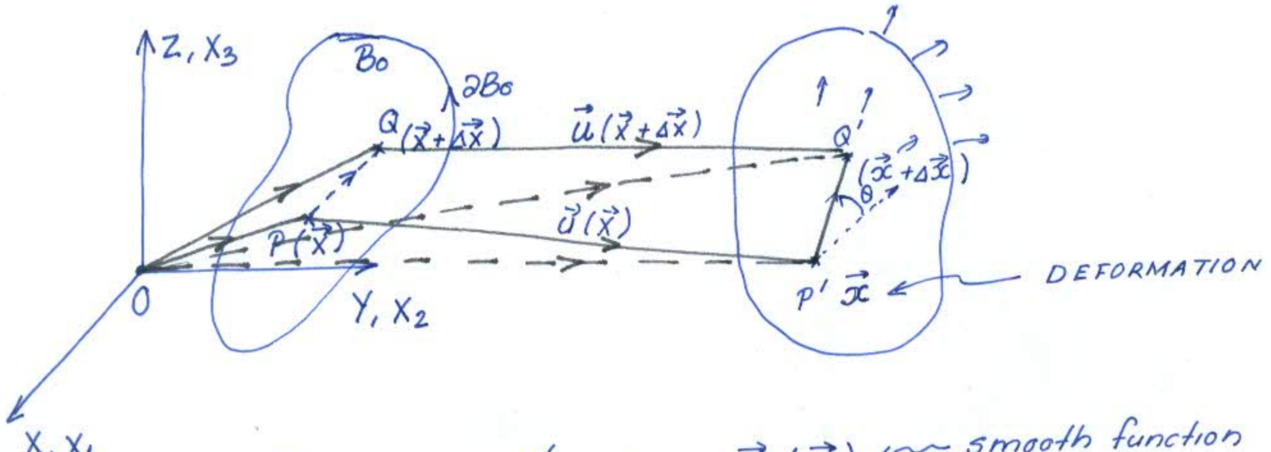


$\vec{\bar{u}}(\vec{x})\Big|_{\partial B_D}$; $\vec{f}(\vec{x})$; $\vec{T}(\vec{x})\Big|_{\partial B_F}$ have to be known for us to be able to say how the body responds.

Response of body : Forces, when applied, cause deformation of the body $\vec{x}(\vec{X})$. Deformation causes stretch of infimitesimal elements in the body. The stretched state is NOT the natural state of existence (equilibrium), hence the body develops resistance to the action of external influences by developing internal stress. The body attains a deformed equilibrium, i.e. no more stretch or stress happens, and the internal stresses BALANCE the action of the loads.

HOW DO WE STUDY THIS EQUILIBRIUM ?

KINEMATICS : Define the state of deformation



$$P \longrightarrow P' \Rightarrow \vec{x}(\vec{X})$$ ← smooth function

← depends on size of load

$$Q \longrightarrow Q' \Rightarrow \vec{x}(\vec{X} + \Delta\vec{X})$$

$$\Rightarrow \quad \vec{x}(\vec{X} + \Delta\vec{X}) \approx \vec{x}(\vec{X}) + \underbrace{\nabla_{\vec{X}}\vec{x}}_{} \cdot \Delta\vec{X}$$

gradient of $\vec{x}$ with respect to $\vec{X}$

OR $\nabla\vec{x} = [F]$ ← deformation gradient

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial (X_i + u_i)}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} = \delta_{ij} + u_{i,J}$$

$$\Rightarrow \quad \vec{PQ} = \Delta\vec{X} \quad ; \quad \vec{P'Q'} = \Delta\vec{x} = \vec{x}(\vec{X} + \Delta\vec{X}) - \vec{x}(\vec{X})$$

$$= [F]\,\Delta\vec{X}$$

STRETCH $\Rightarrow \quad |\vec{P'Q'}| \neq |\vec{PQ}|$

$$|\vec{P'Q'}|^2 = \vec{P'Q'}^{\,T}\,\vec{P'Q'} = \Delta\vec{X}^T [F]^T [F]\,\Delta\vec{X}$$

$$= \Delta\vec{X}^T\left([I] + \nabla\vec{u}^T\right)\left([I] + \nabla\vec{u}\right)\Delta\vec{X}$$

$$= \underbrace{\Delta\vec{X}^T[I]}_{l_o^2} + \left(\nabla\vec{u}^T + \nabla\vec{u} + \nabla\vec{u}^T \cdot \nabla\vec{u}\right)\Delta\vec{X}$$

$$= \left(\Delta X_i^2 + \Delta X_i\left(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}\right)\Delta X_j\right)$$

$$= l_o^2\left(1 + \frac{\Delta X_i\left(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}\right)\Delta X_j}{l_o^2}\right)$$

$$\Rightarrow \underbrace{|P'Q'|}_{\ell} \approx \ell_0 \sqrt{1 + \frac{\Delta X_i \left(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\right) \Delta X_j}{\ell_0^2}}$$

SMALL COMPARED TO 1

$$\approx \ell_0 \left[1 + \frac{1}{2} \frac{\Delta X_i \left(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\right) \Delta X_j}{\ell_0^2} \right]$$

$$\Rightarrow \frac{\ell - \ell_0}{\ell_0} \approx \frac{1}{2} \frac{\Delta X_i \left(\left(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\right)\right) \Delta X_j}{\ell_0^2 = \Delta X_i \delta_{ij} \Delta X_j}$$

(Right) Green strain E_{ij} ⟵ SYMMETRIC, NONLINEAR

SPECIAL CASE : $\vec{\Delta X} = \Delta X_1 \vec{e}_1 \Rightarrow \frac{\ell - \ell_0}{\ell_0} \approx \frac{1}{2} \frac{\left(2 u_{1,1} + u_{k,1} u_{k,1}\right) \Delta X_1^2}{\Delta X_1^2}$

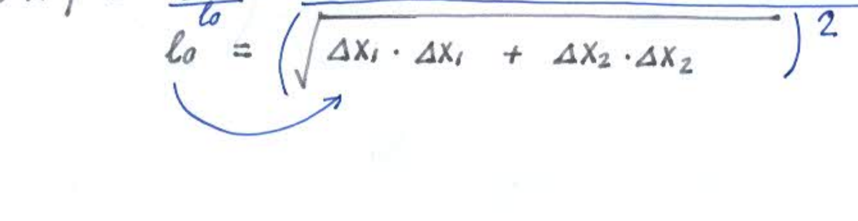
$$\approx u_{1,1} + \frac{1}{2}\left(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2\right)$$

$$= E_{11}$$

Similarly for $\vec{\Delta X} = \Delta X_2 \vec{e}_2 , \Delta X_3 \vec{e}_3$

- Thus E_{11}, E_{22}, E_{33} correspond to (relative) stretch of a line element that was initially aligned along the 3 cartesian directions.

- Note that E_{ij} also contains SHEAR STRAIN, which will work for stretch of line elements that are arbitrarily oriented.



$\vec{OR} = \Delta X_1 \hat{i} + \Delta X_2 \hat{j}$

comes in!

$$\left|\vec{O'R'}\right| = \frac{\ell - \ell_0}{\ell_0} = \frac{\Delta X_1 E_{11} \Delta X_1 + \Delta X_2 E_{22} \Delta X_2 + 2 E_{12} \Delta X_1 \cdot \Delta X_2}{\left(\sqrt{\Delta X_1 \cdot \Delta X_1 + \Delta X_2 \cdot \Delta X_2}\right)^2}$$

$$\ell_0 = \sqrt{\Delta X_1 \cdot \Delta X_1 + \Delta X_2 \cdot \Delta X_2}$$

FOR SMALL deformations $u_{k,i}\, u_{k,j} \approx 0$

$$\Rightarrow \boxed{\varepsilon_{ij} = \text{INFINITESIMAL STRAIN} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}$$

* Strain IS ALWAYS symmetric $\Rightarrow \boxed{E_{ij}\ (\overset{\text{or}}{\varepsilon_{ij}}) = E_{ji}\ (\overset{\text{or}}{\varepsilon_{ji}}).}$

Change in angle :

$\beta_1 + \beta_2 = \beta$

$$\Delta\vec{x} = [F]\,\Delta\vec{X} \quad \Rightarrow \text{For } \vec{O'A'} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \begin{Bmatrix} \Delta X \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} x_{1,1}\ \Delta X \\ x_{2,1}\ \Delta X \end{Bmatrix}$$

$$\vec{O'B'} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \begin{Bmatrix} 0 \\ \Delta X \end{Bmatrix} = \begin{Bmatrix} x_{1,2}\,\Delta X \\ x_{2,2}\,\Delta X \end{Bmatrix}$$

$\Rightarrow$

$$\vec{O'A'} \cdot \vec{O'B'} = \left(x_{1,1}\, x_{1,2} + x_{2,1}\, x_{2,2} \right) \Delta X^2$$

$$\left((1 + u_{1,1})(0 + u_{1,2}) + (u_{2,1})(1 + u_{2,2}) \right)$$

$$= \left(u_{1,1} \cdot u_{1,2} + u_{2,1} * u_{2,2} \right) + \left(u_{1,2} + u_{2,1} \right)$$

$$\approx \left(u_{1,2} + u_{2,1} \right) \Delta X^2$$

$$|O'A'| = |\Delta X| \sqrt{x_{1,1}^2 + x_{2,1}^2} = |\Delta X| \sqrt{(1 + u_{1,1})^2 + (u_{2,1})^2}$$

$$= |\Delta X| \sqrt{1 + 2u_{1,1} + u_{1,1}^2 + u_{2,1}^2}$$

$$|O'B'| = |\Delta X| \sqrt{x_{1,2}^2 + x_{2,2}^2} = |\Delta X| \sqrt{1 + 2u_{2,2} + u_{2,1}^2 + u_{2,2}^2}$$

$$\Rightarrow |O'A'|\,|O'B'| \approx |\Delta X|^2 (1 + u_{1,1})(u_{2,2} + 1) \approx |\Delta X|^2 (1 + u_{1,1} + u_{2,2})$$

$$\Rightarrow \cos\theta = \cos(90 - \beta) = \sin\beta \approx \beta = \frac{(u_{1,2} + u_{2,1})}{(1 + u_{1,1} + u_{2,2})}$$

$$\approx (u_{1,2} + u_{2,1})(1 - (u_{1,1} + u_{2,2})) \approx \boxed{(u_{1,2} + u_{2,1})} = \gamma_{12}$$

→ So, change in angle between 2 vectors is also given in terms of the strain components.

<u>Q</u> : Let $\vec{u} = (a_1 X_1^2 + a_2 X_1 X_3)\hat{i} + (b_1 X_1 X_2 + b_2 X_3^3)\hat{j} + (c_1 X_3^2 + c_2 X_1 X_2)\hat{k}$

where a_1, a_2, a_3 ; b_1, b_2, b_3 ; c_1, c_2, c_3 are constants.

(a) Determine the deformation gradient at (1, 1, 2), (3, 1, 2).

(b) Determine components of the Green strain and infinitesimal strain at (1, 1, 2), (3, 1, 2).

(c) Change in length of vector (0.003, 0.001, 0.0005) located at (1, 1, 2).

(d) Show that $\boxed{2[E] = [F]^T[F] - [I]}$

* Take $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 10^{-4}$