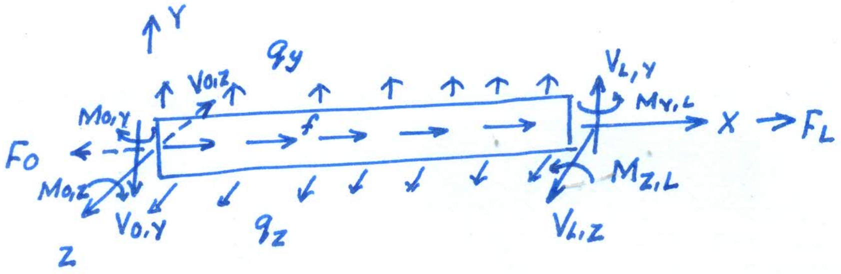
Principle of Virtual Work



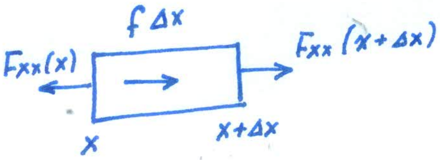
Distributed external forces: f, q_y, q_z

End forces/moments:

$x = 0$: F_0, $V_{0,y}$, $V_{0,z}$, $M_{0,z}$, $M_{0,y}$

$x = L$: F_L, $V_{L,y}$, $V_{L,z}$, $M_{L,y}$, $M_{L,z}$

Equilibrium equations:



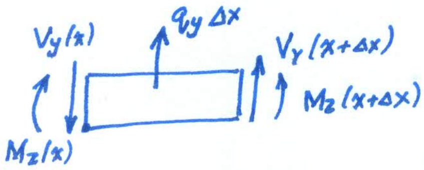
$\sum F_x = 0 \Rightarrow$

$$(F_{xx}^{+} - F_{xx}) + f\Delta x = 0$$

$\Downarrow$

$$\frac{dF_{xx}}{dx} + f = 0 \quad — (1)$$

$\Downarrow$



$\sum F_y = 0$; $\sum M_z|_x = 0$

$$(V_y^{+} - V_y) + q_y\Delta x = 0$$

$$(M_z^{+} - M_z) + V_y^{+}\Delta x = 0$$

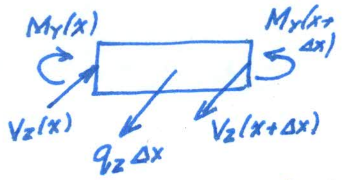
$\Downarrow$

$$\frac{dV_y}{dx} + q_y = 0$$

$$\frac{dM_z}{dx} + V_y = 0$$

$\Downarrow$

$$-\frac{d^2M_z}{dx^2} + q_y = 0 \quad — (2)$$



$\sum F_z = 0$; $\sum M_y|_x = 0$

$$(V_z^{+} - V_z) + q_z\Delta x = 0$$

$$(M_y^{+} - M_y) - V_z^{+}\Delta x = 0$$

$\Downarrow$

$$\frac{dV_z}{dx} + q_z = 0$$

$$\frac{dM_y}{dx} - V_z = 0$$

$\Downarrow$

$$\frac{d^2M_y}{dx^2} + q_z = 0 \quad — (3)$$

$(1) \times \delta u_0 \Rightarrow$

$$\int_0^L \left(\frac{dF_{xx}}{dx}\delta u_0 + f\,\delta u_0\right)dx = 0$$

$(2) \times \delta v \Rightarrow$

$$\int_0^L \left(-\frac{d^2M_z}{dx^2}\delta v + q_y\,\delta v\right)dx = 0$$

$(3) \times \delta w \Rightarrow$

$$\int_0^L \left(\frac{d^2M_y}{dx^2} + q_z\right)\delta w\, dx = 0$$

What are δu_0, δv, δw?

- These are "virtual" displacements, a figment of our imagination, which are "imposed" on the structure in equilibrium (under the action of the forces).
- $\delta u_0 = 0$ wherever u_0 is SPECIFIED; $\delta v = 0$ wherever v is specified; $\delta w = 0$ wherever w is specified.

* Now we want to "remove" derivatives from F_{xx}, M_y, M_z in the integral expressions. This can be done by INTEGRATION BY PARTS (once for F_{xx}, twice for M_y, M_z) to get:

(1) $$\int_0^L \frac{dF_{xx}}{dx}\,\delta u_0\,dx = \underbrace{\int_0^L \frac{d}{dx}(F_{xx}\,\delta u_0)\,dx}_{F_{xx}|_L\,\delta u_0|_L - F_{xx}|_0\,\delta u_0|_0} - \int_0^L F_{xx}\frac{d\delta u_0}{dx}\,dx$$

or (on rearranging terms):

$$\int_0^L F_{xx}\frac{d\delta u_0}{dx}\,dx = \int_0^L f\cdot\delta u_0\,dx + F_{xx}|_L\,\delta u_0|_L - F_{xx}|_0\,\delta u_0|_0 \quad \text{— (A)}$$

(2) $$\int_0^L \frac{d^2M_z}{dx^2}\,\delta v\,dx = \int_0^L \frac{d}{dx}\left(\frac{dM_z}{dx}\delta v\right)dx - \underbrace{\int_0^L \frac{dM_z}{dx}\cdot\frac{d\delta v}{dx}\,dx}_{\shortparallel}$$

$$-\left[\int_0^L \frac{d}{dx}\left(M_z\frac{d\delta v}{dx}\right)dx - \int_0^L M_z\frac{d^2\delta v}{dx^2}\,dx\right.$$

$$= \left(\frac{dM_z}{dx}\delta v\right)\Big|_L - \left(\frac{dM_z}{dx}\delta v\right)\Big|_0 - \left[\left(M_z\frac{d\delta v}{dx}\right)\Big|_L - \left(M_z\frac{d\delta v}{dx}\right)\Big|_0\right]$$
$$+ \int_0^L M_z\frac{d^2\delta v}{dx^2}\,dx$$

Using $V_y = -\frac{dM_z}{dx}$ and re-arranging terms we get:

$$\int_0^L M_z\frac{d^2\delta v}{dx^2}\,dx = \int_0^L q_y\,\delta v\,dx + V_y|_L\,\delta v|_L - V_y|_0\,\delta v|_0$$
$$+ M_z|_L\frac{d\delta v}{dx}\Big|_L - M_z|_0\frac{d\delta v}{dx}\Big|_0 \quad \text{— (B)}$$

Similarly, (using the above derivation):

$$\int_0^L \frac{d^2M_y}{dx^2}\,\delta w\,dx = \int_0^L M_y\frac{d^2\delta w}{dx^2}\,dx + \left(\frac{dM_y}{dx}\delta w\right)\Big|_L - \left(\frac{dM_y}{dx}\delta w\right)\Big|_0$$
$$- \left[\left(M_y\frac{d\delta w}{dx}\right)\Big|_L - \left(M_y\frac{d\delta w}{dx}\right)\Big|_0\right]$$

Using $V_z = \frac{dM_y}{dx}$ and re-arranging terms, we get:

$$-\int_0^L M_y \frac{d^2 \delta w}{dx^2}\, dx = \int_0^L q_z\, \delta w\, dx + V_z|_L\, \delta w|_L - V_z|_0\, \delta w|_0 + M_y|_L \left(-\frac{d\delta w}{dx}\right)\Big|_L - M_y|_0 \left(-\frac{d\delta w}{dx}\right)\Big|_0 \quad - \text{(C)}$$

* Note that a positive M_y will rotate the beam to give a NEGATIVE slope - hence $-\frac{d}{dx}\delta w$ in the expression (C)

* Each of expressions (A), (B), (C) define a "work" relationship i.e. internal virtual work (due to F_{xx}, M_y, M_z)

= external virtual work (due to $f(x)$, $q_y(x)$, $q_z(x)$, F_L, F_0, $V_{0,y}$, $V_{L,y}$, $V_{0,z}$, $V_{L,z}$, $M_{0,y}$, $M_{L,y}$, $M_{0,z}$, $M_{L,z}$)

Since work is "A NUMBER" or a scalar defined for the body, (A) + (B) + (C) defines the TOTAL <u>principle of virtual work</u> for the body

— The expressions above are general, i.e. specific choices of data can be used to define specific problems.

E.g. in our case the beam is CANTILEVERED, i.e

$u|_0 = 0$; $\frac{dv}{dx}\Big|_0 = 0$; $\frac{dw}{dx}\Big|_0 = 0$

$v|_0 = 0$

$w|_0 = 0$

$x = 0$ $\qquad$ $x = L$

$F_L = 0$; $\quad M_y|_L = 0$

$V_y|_L = 0$; $\quad M_z|_L = 0$

$V_z|_L = 0$;

This implies that:

$\delta u|_0 = 0$; $\delta v|_0 = 0$; $\delta w|_0 = 0$;

$\frac{d\delta v}{dx}\Big|_0 = 0$; $\frac{d\delta w}{dx}\Big|_0 = 0$

Now, we are in a position to solve the problem, i.e. use a SERIES representation for $u_0(x)$, $v(x)$, $w(x)$ as:

$$u_0(x) = \sum_{i=0}^{\infty} a_i x^i \; ; \quad v(x) = \sum_{i=0}^{\infty} b_i x^i \; ; \quad w(x) = \sum_{i=0}^{\infty} c_i x^i$$

(assuming the series is convergent)

or a <u>truncated</u> series, i.e. an approximation as:

$$u_0^N(x) = \sum_{i=0}^{N} a_i x^i \; ; \quad v^N(x) = \sum_{i=0}^{N} b_i x^i \; ; \quad w^N(x) = \sum_{i=0}^{N} c_i x^i$$

<u>NOTE</u>: We chose a polynomial series. This is not sacrosanct. We could have chosen trigonometric series, special polynomial forms too. $\Rightarrow$ $u_0(x) \approx \sum_{i=0}^{N} a_i \varphi_i(x)$; $v(x) \approx \sum_{i=0}^{N} b_i \psi_i(x)$; $w(x) \approx \sum_{i=0}^{N} c_i g_i(x)$

* <u>The tricky part !!</u>

We <u>choose</u> δu_0, δv, δw to be of the SAME FORM as u_0, v, w respectively. Thus,

$$\delta u_0^N(x) = \sum_{i=0}^{N} \delta a_i x^i \; ; \quad \delta v^N(x) = \sum_{i=0}^{N} \delta b_i x^i \; ;$$

$$\delta w^N(x) = \sum_{i=0}^{N} \delta c_i x^i$$

* This assumption gives us what is called the GALERKIN form.

- Note δa_i, δb_i, δc_i are coefficients that we <u>CHOOSE</u>
- a_i, b_i, c_i are coefficients that we <u>SOLVE</u> for

* Note that in our case Ⓐ is ONLY in terms of u_0, δu_0 so whatever happens with v, w (hence also δv, δw) does not affect u_0. So <u>CHOOSING</u> $\delta v^N, \delta w^N = 0$, we get from Ⓐ

$$\int_0^L F_{xx} \left(\sum_{i=0}^{N} \delta a_i \frac{d}{dx}(x^i) \right) dx = \int_0^L f \cdot \left(\sum_0^N \delta a_i x^i \right) dx + 0 \qquad \text{— (A1)}$$

Now $u_0(0) = 0 \Rightarrow a_0 = 0$; also this means $\delta u_0(0) = 0$

$\Rightarrow$ $\underline{\underline{\delta a_0 = 0}}$ has to be forced

Also, $\quad F_{xx} = E(x)\,A(x)\dfrac{du_0}{dx} \approx E(x)\,A(x)\Big(\sum_{i=1}^{N} a_i \underbrace{\dfrac{d}{dx}(x^i)}_{\varphi_i(x)}\Big)$

$\therefore$ (A1) becomes:

$$\int_0^L E(x)\,A(x)\Big(\sum_{j=1}^{N} a_j \varphi_{j,x}\Big)\Big(\sum_{i=1}^{N}\delta a_i \varphi_{i,x}\Big)\,dx = \int_0^L f(x)\cdot\sum_{i=1}^{N}\delta a_i \varphi_i$$

OR

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_j\,\delta a_i\Big(\int_0^L E(x)\,A(x)\,\varphi_{j,x}\,\varphi_{i,x}\,dx\Big) = \sum_{i=1}^{N}\delta a_i\int_0^L f\,\varphi_i\,dx$$

$$\Downarrow$$

$$\{\delta a\}^T [K]\{a\} = \{a\}^T\{F\} \quad \text{— (A2)} \qquad \Bigg|\quad K_{ij} = \int_0^L EA\,\varphi_{i,x}\,\varphi_{j,x}\,dx, \quad F_i = \int_0^L f\,\varphi_i\,dx$$

with δa_i, K_{ij}, a_j, F_i

$[K]$ is: (a) SYMMETRIC (b) POSITIVE DEFINITE (hence invertible)

$\approx \varphi_i(x) = x^i$

$$\Rightarrow \boxed{K_{ij} = i\cdot j\int_0^L EA\,x^{i+j-2}\,dx}$$

(A2) is true for ANY $\{\delta a\}$ $\Rightarrow$

$$\boxed{[K]\{a\} = \{F\}} \quad \text{— (A3)}$$

(B) and (C) have to be taken together as $M_y(x)$, $M_z(x)$ are in terms of <u>both</u> v'', w'' if $I_{yz} \neq 0$. [If $I_{yz} = 0$ then (B) and (C) can be taken independently.]

Now $\quad M_z(x) = E\left(I_{zz}\,v'' + I_{yz}\,w''\right)$; $\quad M_y(x) = -E\left(I_{yz}\,v'' + I_{yy}\,w''\right)$

Let $\quad v^N(x) \approx v(x) \approx \sum_{i=0}^{N} b_i\,\varphi_i(x)$; $\quad w^N(x) \approx \sum_{i=0}^{N} c_i\,\varphi_i(x)$

For our problem: $\quad v(0) = \dfrac{dv}{dx}\Big|_0 = 0 \Rightarrow b_0, b_1 = 0$

$\qquad\qquad\qquad\quad \delta v(0) = \dfrac{d\delta v}{dx}\Big|_0 = 0 \Rightarrow \delta b_0, \delta b_1 = 0$

Similarly,

$$w(0) = \frac{dw}{dx}\Big|_0 = 0 \Rightarrow C_0, C_1 = 0$$

$$\delta w(0) = \frac{d}{dx}\delta w\Big|_0 = 0 \Rightarrow \delta C_0, \delta C_1 = 0$$

For the cantilevered beam, we thus get:

$$\int_0^L E\left(I_{zz}\sum_{j=2}^N b_j \varphi_{j,xx} + I_{yz}\sum_{j=2}^N c_j \varphi_{j,xx}\right)\sum_{i=2}^N \delta b_i \varphi_{i,xx}\, dx$$

$$= \int_0^L q_y \cdot \sum_{i=2}^N \delta b_i \varphi_i \, dx \quad \text{—— (B1)}$$

$$\int_0^L E\left(I_{yz}\sum_{j=2}^N b_j \varphi_{j,xx} + I_{yy}\sum_{j=2}^N c_j \varphi_{j,xx}\right)\sum_{i=2}^N \delta c_i \varphi_{i,xx}\, dx$$

$$= \int_0^L q_z \sum_{i=2}^N \delta c_i \varphi_i \, dx \quad \text{—— (C1)}$$

Putting $d_{2j-1} = b_j$; $d_{2j} = c_j$; $\delta d_{2j-1} = \delta b_j$, $\delta d_{2j} = \delta c_j$

we get:

$$\{\delta d\}^T_{1\times 2\tilde{N}} [\bar{K}] \{d\}_{2\tilde{N}\times 1} = \{\delta d\}^T \{\bar{F}\}_{2\tilde{N}\times 1} \qquad (\tilde{N} = N-1 \text{ here}).$$

with:

$$\bar{K}_{(2i-1),(2j-1)} = \int_0^L E\, I_{zz}\, \varphi_{j,xx}\, \varphi_{i,xx}\, dx$$

$$\bar{K}_{(2i-1),(2j)} = \int_0^L E\, I_{yz}\, \varphi_{j,xx}\, \varphi_{i,xx}\, dx$$

$$\bar{K}_{2i,(2j-1)} = \int_0^L E\, I_{yz}\, \varphi_{j,xx}\, \varphi_{i,xx}\, dx = \bar{K}_{(2i-1),(2j)}$$

$$\bar{K}_{2i,2j} = \int_0^L E\, I_{yy}\, \varphi_{j,xx}\, \varphi_{i,xx}\, dx$$

$$\bar{F}_{2i-1} = \int_0^L q_y \varphi_i \, dx \; ; \quad \bar{F}_{2i} = \int_0^L q_z \varphi_i \, dx$$

* This is true for ANY $\{\delta d\}$ $\Rightarrow$ $\boxed{[\bar{K}]\{d\} = \{\bar{F}\}}$ —— (B2)

— Note that:

(a) $[\bar{K}]$ is also symmetric

(b) $[\bar{K}]$ is positive definite and hence invertible

— Solve (A3) for $\{a\}$ and (B2) for $\{d\}$ to get the desired coefficients of $u_0^N(x)$, $v^N(x)$, $w^N(x)$.

— Now $\quad \varepsilon_{xx} = u_{0,x} - y\, v_{,xx} - z\, w_{,xx}$

$$\boxed{\sigma_{xx} = E\varepsilon_{xx} = E\left(u_{0,x} - y\, v_{,xx} - z\, w_{,xx}\right)}$$

$$\approx E\left(\sum_{i=1}^{N} a_i \varphi_{i,x} - y \sum_{i=2}^{N} b_i \varphi_{i,xx} - z \sum_{i=2}^{N} c_i \varphi_{i,xx}\right)$$

* Note that $\quad \frac{1}{2}\{a\}^T[K]\{a\} + \frac{1}{2}\{d\}^T[\bar{K}]\{d\} = U$

where U is the STRAIN ENERGY of the beam.

* Note also that $\{a\}$ and $\{d\}$ can be seen as the MINIMIZER of $\Pi = U - V$ with

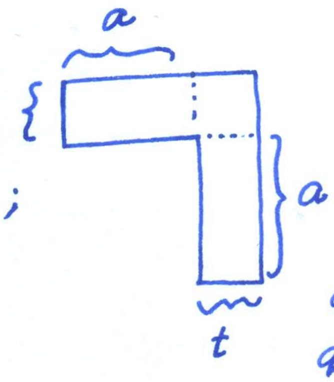
$$V = \{a\}^T\{F\} + \{d\}^T\{\bar{F}\} \quad \Longleftarrow \text{ PROVE THIS!!}$$

$\Pi \Rightarrow$ total potential energy

$V \Rightarrow$ work done by external forces

EXAMPLE :

Let cross-section be



$t = 2mm$

$a = 10t$

$f(x) = 100\ N/m$

$q_y(x) = 1000\ N/m$

$q_z(x) = 100\ N/m$

$L = 2m.$

→ Find A, I_{YY}, I_{ZZ}, I_{YZ} (about O); where O is centroid.

→ Let $N = 2, 4, 6$.

Solve for $u_0^N(x)$, $v^N(x)$, $w^N(x)$

→ At $x = 0.01$ find the variation of $\sigma_{xx}(x, y, z)$ on the cross-section

→ Where, on the cross-section, is σ_{xx} a maximum?

→ Plot $u_0(x)$, $v(x)$, $w(x)$.