Reduced-Order Observer Design

Consider the n-dimensional dynamical equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

$$A \in R^{n\times n}, B \in R^{n\times p}, C \in R^{q\times n}$$

When some of the state variables of the plant cannot be measured- reduced order observer can be designed to estimate those are cannot be measured.

Suppose a state vector of a plant is defined as $x = [x_1(t) \quad x_2(t)]^T$, where $x_1(t)$ is measurable state and $x_2(t)$ is unmeasurable state.

The plant state equation can be partitioned as:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Let the order of the plant be n and the number of measured state be k and the reduced order observer to be designed to estimate n-k states.

The observer state vector is designed as

$$x_0 = \begin{bmatrix} x_1 \\ x_{02} \end{bmatrix} = \begin{bmatrix} C^{-1}y(t) \\ x_{02} \end{bmatrix}$$

Here, x_{02} is the estimation $x_2(t)$

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The estimation error between unobservable and estimates state is given by $e_{02} = x_2 - x_{02}$, which needs to be zero at steady state. A possible observer state equation for the reduced order observer can be written as

$$\dot{x}_0 = Ax_0 + Bu + L[y(t) - Cx_0]; \qquad L = observer \;\; gain$$

The error equation can be obtained as $\dot{e}_{02} = A_{22}e_{02}$

It can seen from the error equation that the error dynamics is unaffected by the observer gain matrix, L and only depends on the system sub-matrix, A_{22}. Thus the observer state equation considered here cannot be accepted for reduced observer design. To solve this issue, the reduced order dynamics can be considered as

$$x_{02} = Ly(t) + z(t)$$

Where the $z(t)$ is the following state equation $\dot{z} = Fz + Hu + Gy$

Now solving: $\dot{x}_2 - \dot{x}_{02}$ and substituting the expression $z(t) = x_{02} - Ly = x_2 - e_{02} - LCx_1$ yields

$$\begin{aligned}\dot{e}_{02} = Fe_{02} &+ (A_{21} - LCA_{11} + FLC - GC)x_1 + (A_{22} - LCA_{12} - F)x_2 \\ &+ (B_2 - LCB_1 - H)u\end{aligned}$$

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$$\dot{e}_{02} = Fe_{02} + (A_{21} - LCA_{11} + FLC - GC)x_1 + (A_{22} - LCA_{12} - F)x_2 + (B_2 - LCB_1 - H)u$$

From the above equation, the estimation error e_{02} to go zero in the steady state, irrespective of x_1, x_2, u and coefficient matrices of x_1, x_2, u must vanish. Hence, eigenvalues of F must lie in the LH plane. In this situation, the following condition must satisfy:

$$F = A_{22} - LCA_{12};\;\; H = B_2 - LCB_1;\; G = FL + (A_{21} - LCA_{11})C^{-1}$$

Example: Let us design a reduced order observer of the inverted pendulum represented as

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 10.78 & 0 & 0 \\ 0 & -0.98 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{\theta} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} u$$

Here the state vector is partitioned into measurable and unmeasurable states. Here, $x_1 = x$ is measurable and $x_2 = [\theta \;\; \dot{\theta} \;\; \dot{x}]^T$ is unmeasurable.

Solution

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 10.78 & 0 & 0 \\ 0 & -0.98 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \theta \\ \dot{\theta} \\ \dot{x} \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline 0 \\ -1 \\ 1 \end{array}\right] u$$

$$\Rightarrow$$

$$A_{11} = 0;\;\; A_{12} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix};\;\; B_1 = 0;$$

$$A_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};\;\; A_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 10.78 & 0 & 0 \\ -0.98 & 0 & 0 \end{bmatrix};\; B_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

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Since the measured output is $x_1 = x$, the output equation is $y = Cx$; $C = 1$. We have to select the gain, L, of the reduced order observer such that the eigenvalues of $F = A_{22} - LCA_{12}$ are lie in the LH plane. Let us choose the observer poles as: $s = -20$; $s = -20 \pm 20i$. Then the observer gain, L, can be obtained by the following program:

$$A_{12} = [0 \;\; 0 \;\; 1];\; A_{22} = [0\;1\;0;10.78\;0\;0;-0.98\;0\;0\;0];\; C = 1;$$
$$dp = [-20 \;\; -20-20i \;\; -20+20i]';$$
$$L = (place(A_{22}', A_{12}'{*}C', dp));$$

$$\Rightarrow \quad L = \begin{bmatrix} -1.643e^{+003} \\ -1.698e^{+004} \\ 6.000e^{+001} \end{bmatrix}$$

The reduced order observer dynamics can be calculated as

$$F = A_{22} - L*C*A_{12} = \begin{bmatrix} 0 & 1 & 1.643e^{+003} \\ 1.078e^{+001} & 0 & 1.698e^{+004} \\ -9.8e^{-001} & 0 & -600e^{+001} \end{bmatrix}$$

Lets check whether the eigenvalues of F are at the desired locations

$$eig(F) = \begin{bmatrix} -2.00e^{+001} + 2.00e^{+001}i \\ -2.00e^{+001} - 2.00e^{+001}i \\ -2.00e^{+001} \end{bmatrix}$$

Tracking Controller based on Reduced-order Compensator

The control law with reduced order compensator for tracking system can be expressed as

$$u = K[x_d - x_0] - K_d x_d = (K - K_d)x_d - K_1 x_1 - K_2 x_{02}$$

Here, K_d is the feedforward gain matrix and $K = [K_1; K_2]$ is the feedback gain matrix

The estimation error dynamics can be written as $\dot{e}_{02} = F e_{02}$

The dynamics of error system can be written as

$$\begin{aligned} \dot{e} &= \dot{x}_d - \dot{x} \\ &= A_d x_d - (Ax + Bu) \\ &= Ae + (A_d - A + BK_d)x_d - BK[x_d - x_0] \\ &= (A - BK)e + (A_d - A + BK_d)x_d - BK_2 e_{02} \end{aligned}$$

Hence the dynamics of the tracking system can be described by $\dot{e}_{02}$ and $\dot{e}$. To have tracking errors e_{02} and e to zero, irrespective of x_d, we must select feedforward gain K_d such that $(A_d - A + BK_2)x_d =$ 0 and K should be such that the eigenvalues of $(A - BK)$ are lie in the LH plane