States Space form of Linear Systesm

The time derivative of state equations in terms of state variables and system inputs can be written in general form as:

$$\begin{aligned} \dot{x}_1 &= f_1(x, u, t) \\ \dot{x}_2 &= f_2(x, u, t) \\ &\dots \\ \dot{x}_n &= f_n(x, u, t) \end{aligned}$$

or the the above equation can be written as

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n + b_{11}u_1 + \dots\dots\dots + b_{1r}u_r \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n + b_{21}u_1 + \dots\dots\dots + b_{2r}u_r \\ &. \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n + b_{n1}u_1 + \dots\dots\dots + b_{nr}u_r \end{aligned}$$

In matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ . & & & . \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1r} \\ b_{21} & \dots & b_{2r} \\ . & & . \\ b_{n1} & \dots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ u_r \end{bmatrix}$$

which can be summerized:

$$\dot{x} = Ax + Bu$$

Standard State Space Form

The output variable variables by the linear combination of the state variables and system inputs can be written as

$$
\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + ...+ c_{1n}x_n + d_{11}u_1 + ...+ d_{1r}u_r \\
y_2 &= c_{21}x_1 + c_{22}x_2 + ...+ c_{2n}x_n + d_{21}u_1 + ...+ d_{2r}u_r \\
. & \\
y_n &= c_{m1}x_1 + c_{m2}x_2 + ...+ c_{mn}x_n + d_{m1}u_1 + ...+ d_{mr}u_r
\end{aligned}
$$

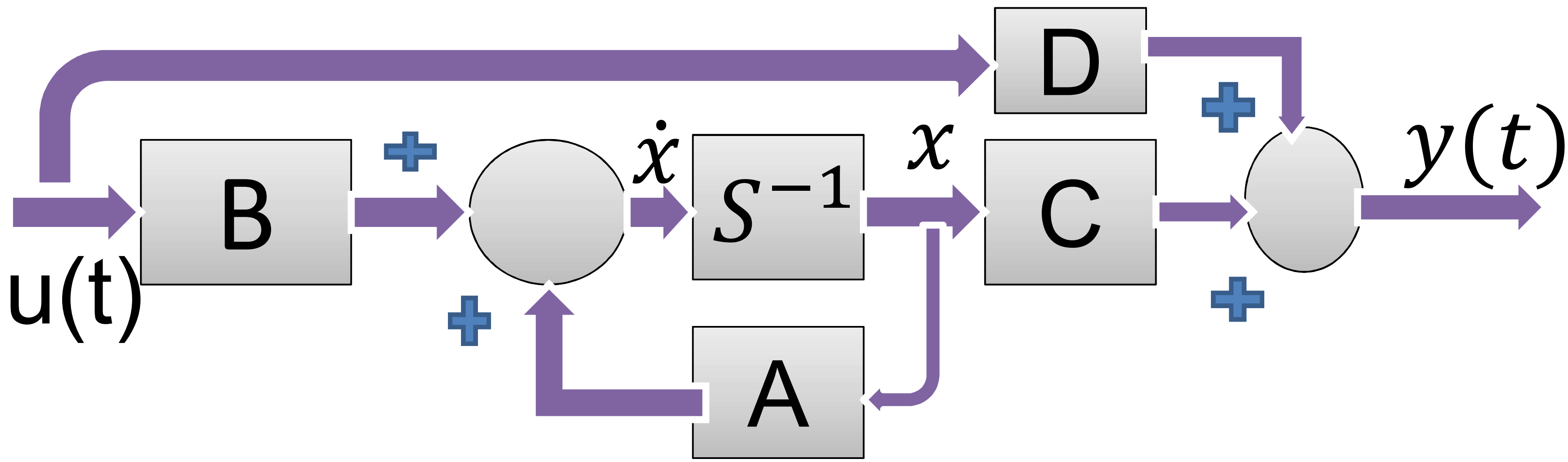
or in matrix form

$$
\begin{bmatrix} y_1 \\ y_2 \\ . \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ . & & & . \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \dots & d_{1r} \\ d_{21} & \dots & d_{2r} \\ . & & . \\ d_{m1} & \dots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ u_r \end{bmatrix}
$$

which can be summerized in compact form as:

$$y = Cx + Du$$

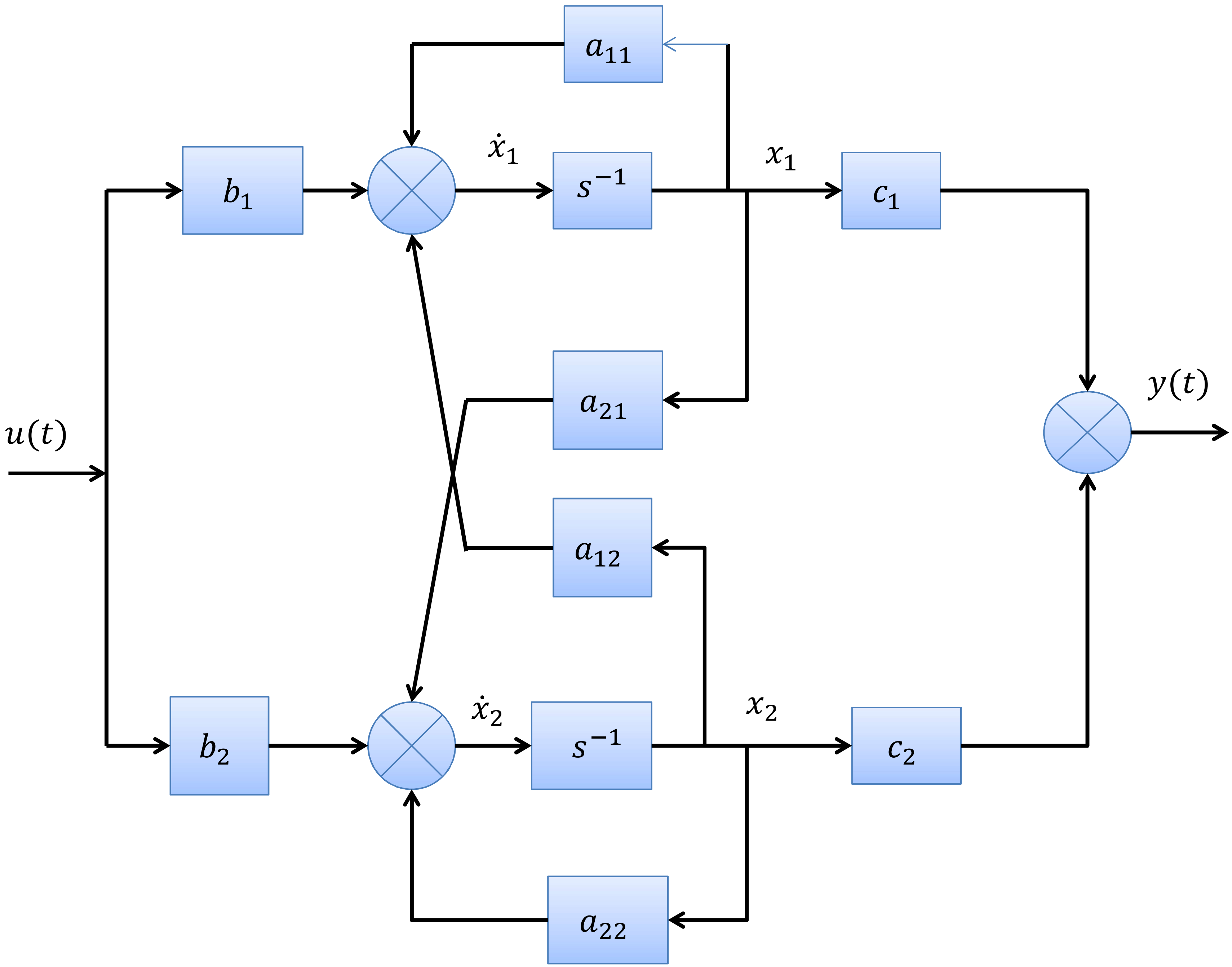
Block Diagram Representation of State Equations



Draw a block diagram for the gneral second, single-input single output system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t)$$



$u(t)$
b_1
b_2
a_{11}
$\dot{x}_1$
s^{-1}
x_1
c_1
a_{21}
a_{12}
$y(t)$
$\dot{x}_2$
s^{-1}
x_2
c_2
a_{22}

Solution of Linear Systems

- Solution of Homogeneous State Equation- The state response of a system described by $\dot{x} = Ax + Bu$ with zero input, $u(t) = 0$, and arbitrary set of initial conditions $x(0)$ is the solution of the set of n homogeneous first-order differential equations: $\dot{x} = Ax$
- Solution of Non-homogeneous State Equation- We consider the complete response of the system with arbitrary set of initial conditions $x(0)$.

Solution of state Equation:

Homogeneous

$$\dot{x} = Ax$$

$$x_h(t) = e^{At}x(0)$$

State-transition Matrix

$$e^{At} = (1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ...)$$

Non-homogeneous

$$\dot{x} = Ax + Bu$$

$$e^{-At}\dot{x} = e^{-At}Ax + e^{-At}Bu$$

$$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$$

$$\frac{d}{dt}(e^{-At}x) = e^{-At}Bu$$

$$\int_0^t \frac{d}{dt}(e^{-A\tau}x(\tau))d\tau = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}x(t) - e^{-A.0}x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{Convolution Integral of Linear System}}$$

Solution of Output Equation:

Homogeneous

$$y_h(t) = Ce^{At}x(0)$$

Non-homogeneous

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(\tau)$$

Example:

Find the response of the two state variables of the system: $\dot{x}_1 = -2x_1 + u; \dot{x}_2 = x_1 - x_2$ to a constant input $u(t) = 5$ for $t > 0$; if $x_1(0) = 0$ & $x_2(0) = 0$.

Soln.

State matrix: $A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$

$$e^{At} = (I + At + \frac{A^2 t^2}{2!} + \ldots\ldots..)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \frac{t^2}{2!} + \ldots\ldots$$

$$= \begin{bmatrix} 1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \ldots\ldots & 0 \\ 0 - t + \frac{3t^2}{2!} + \frac{7t^3}{3!} + \ldots\ldots & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\ldots. \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

For zero initial condition:

$$x(t) = e^{At} \int_0^t e^{-A\tau} Bu(\tau) d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} \int_0^t 5e^{2\tau} d\tau \\ \int_0^t (5e^{\tau} - 5e^{2\tau}) d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} - \frac{5}{2} e^{-2t} \\ \frac{5}{2} - 5e^{-t} + \frac{5}{2} e^{-2t} \end{bmatrix}$$

The Output Response:

$$y(t) = 2x_1(t) + x_2(t) = \frac{15}{2} - \frac{5}{2} e^{-2t} - 5e^{-t}$$

State Transition Matrix (Laplace Domain) of Homogeneous Solution

$$\dot{x}(t) = Ax(t)$$
$$sX(s) - x(0) = AX(s)$$
$$X(s) = (SI - A)^{-1}x(0)$$

$$x(t) = L^{-1}[(SI - A)^{-1}]x(0)$$
$$= e^{At}x(0)$$

State Transition Matrix

$$\Phi(t) \overset{\Delta}{=} e^{At} = L^{-1}[(SI - A)^{-1}]$$

$$x(t_0) = e^{At_0}x(0)$$
$$x(0) = e^{-At_0}x(t_0)$$
$$x(t) = e^{At}e^{-At_0}x(t_0) = e^{A(t-t_0)}x(t_0) = \Phi(t-t_0)x(t_0)$$

State Transition Matrix (Laplace Domain) of Non-Homogeneous Solution

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$sX(s) - x(0) = AX(s) + BU(s)$$

For Homogeneous System:

$$sX(s) - x(0) = AX(s)$$
$$X(s)(SI - A) = x(0)$$
$$X(s) = (SI - A)^{-1} x(0)$$
$$x(t) = L^{-1}(SI - A)^{-1} x(0)$$
$$= \Phi(t) x(0)$$

For forced or Non-Homogeneous System

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s)[SI - A] = x(0) + BU(s)$$
$$X(s) = [SI - A]^{-1}(x(0) + BU(s))$$
$$x(t) = L^{-1}\left\{(SI - A)^{-1} x(0)\right\} +$$
$$\underbrace{L^{-1}\left\{(SI - A)^{-1}\right.}\left. BU(s)\right\}$$
$$= \underbrace{\Phi(t)x(0)}_{\text{Homogeneous}} + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau$$

State-transition Matrix

Homogeneous

$$x(t) = L^{-1}\left\{(SI - A)^{-1} x(0)\right\} + L^{-1}\left\{(SI - A)^{-1} BU(s)\right\}$$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$x(t) = \Phi(t-t_0)x(t_0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$y(t) = Cx(t) + Du(t)$$

$$= \underbrace{C\Phi(t-t_0)x(t_0)}_{\text{Zero-input response}} + \underbrace{\int_0^t C\Phi(t-\tau)Bu(\tau)d\tau + Du(t)}_{\text{Zero-state response}}$$

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$let \qquad x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$\Phi(t) = L^{-1}[(sI - A)^{-1}] = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-1} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 2e^{-t} - \frac{3}{2}e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$L^{-1}[(sI - A)^{-1}BU(s)]$$

$$X(s) = (SI - A)^{-1} x(0)$$

$$X(s) = [SI - A]^{-1} (x(0) + BU(s))$$

Solution of the above equation is called the eigenvalue, s, and $X(s)$ is called eigen vectors.
The characteristic equation can be written as

$$|SI - A| = 0$$

Hence, the eigenvalues of matrix A are the roots of the above equation. If we take an Eigenvalue as

$$s_1 = a + jb$$

Growth of Amplitude

Frequency of oscillation about equilibrium point

Diagonalization via Coordinate Transfromation

Plant: $$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad A \in R^{n \times n}$$

Let eigenvalues of A are λ_i, where $i = 1, 2, \dots. n$, then eigenvalues satisfyng the condition $Av_i = \lambda_i v_i$. Assume that all the eigenvalues of A are disticnt, i.e., $\lambda_1 \neq \lambda_2 \neq \cdots . \neq \lambda_n$. The Eigenvectors, $v_1, v_2, \dots.. v_n$ are independent.

Coordinate transformation matrix: $T = [v_1,\ v_2, \dots. v_n]$

Λ=$T^{-1}AT$ is a diagonal matrix

State Transition matrix using diagonalization

Diagonal Matrix

$$T^{-1}AT = \Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

State-transition matrix

$$e^{At} = Te^{\Lambda t}T^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

Details of Coordinate Transformation

New Coordinate

$$\xi = T^{-1}x \Rightarrow \dot{\xi} = T^{-1}AT\xi \Rightarrow \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0\cdots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$$

$$\dot{\xi}_i(t) = \lambda_i \xi_i(t), \; i = 1, \cdots, n \Rightarrow \xi_i(t) = e^{\lambda_i t}\xi_i(0), i = 1, \cdots, n$$

$$x(t) = T\xi(t) = v_1 e^{\lambda_1 t}\xi_1(0) + v_2 e^{\lambda_2 t}\xi_2(0) + \cdots + v_n e^{\lambda_n t}\xi_n(0), \quad \xi(0) = T^{-1}x(0)$$

The above expansion of $x(t)$ is called modal decomposition. Hence, system is assymptotically Stable – when all the eigenvalues of A lie in LHP.

Example

System $\dot{x} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigenvalues $\lambda_1 = -1, \lambda_2 = -3$

Eigenvector

$$(\lambda_1 I - A)v_1 = \begin{bmatrix} -1+2 & -1 \\ -1 & -1+2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(\lambda_2 I - A)v_2 = \begin{bmatrix} -3+2 & -1 \\ -1 & -3+2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad T^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \quad T^{-1}AT = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

Coordinate transformation matrix

$$\boxed{\dot{\xi} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \xi} \Rightarrow \boxed{\begin{aligned} \dot{\xi}_1 &= -1\xi_1 \\ \dot{\xi}_2 &= -3\xi_2 \end{aligned}}$$

$$\boxed{\xi(0) = T^{-1}x(0), \;\; \xi(0) = \begin{bmatrix} \xi_1(0) \\ \xi_2(0) \end{bmatrix}}$$

Solution

$$\boxed{x(t) = T\xi(t) = v_1 e^{\lambda_1 t}\xi_1(0) + v_2 e^{\lambda_2 t}\xi_2(0)} \quad \boxed{\xi(0) = T^{-1}x(0) \Rightarrow \xi(0) = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}}$$

$$x(t) = \frac{3}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{bmatrix}$$

Example:

Consider a system with the following state-dynamics matrix:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the state transition matrix and initial response. Initial condition is $x(0) = [0; 0; 1]^T$.

Example: The eigenvalues of the system are calculated as

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} (\lambda+2) & -1 & -5 \\ 0 & \lambda & 3 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda+2) = 0$$

From the above equation it follows that the eigenvalues of the system are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$. Since the first two eigenvalues are repeated, the system cannot be decoupled, and the approach discussed before cannot be applied to obtain diagonal state transition matrix

The resolvent $(sI - A)^{-1}$ is calculated as

$$(s\mathbf{I} - \mathbf{A})^{-1} = \mathrm{adj}((s\mathbf{I} - \mathbf{A})/|s\mathbf{I} - \mathbf{A}| = 1/[s^2(s+2)]$$
$$\times \begin{bmatrix} s^2 & 0 & 0 \\ s & s(s+2) & 0 \\ (5s+3) & -3(s+2) & s(s+2) \end{bmatrix}^T$$
$$= \begin{bmatrix} 1/(s+2) & 1/[s(s+2)] & (5s+3)/[s^2(s+2)] \\ 0 & 1/s & -3/s^2 \\ 0 & 0 & 1/s \end{bmatrix}$$

Taking the inverse Laplace transform with the help of partial fraction expansions for the elements of $(sI - A)^{-1}$, we get the state-transient matrix as

$$\mathrm{e}^{\mathbf{A}t} = \begin{bmatrix} \mathrm{e}^{-2t} & (1-\mathrm{e}^{-2t})/2 & 7(1-\mathrm{e}^{-2t})/4 + 3t/2 \\ 0 & 1 & -3t \\ 0 & 0 & 1 \end{bmatrix}; \quad (t > 0)$$

The initial response is calculated as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} 7(1-\mathrm{e}^{-2t})/4 + 3t/2 \\ -3t \\ 1 \end{bmatrix}; \quad (t > 0)$$