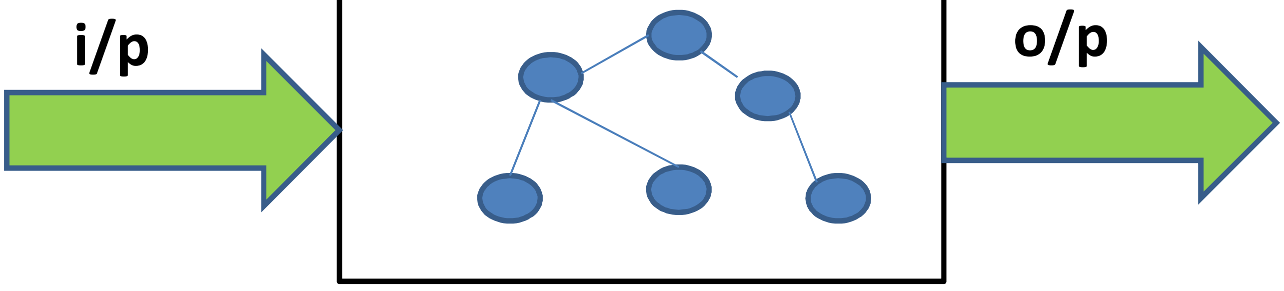
Controllability and Observablability

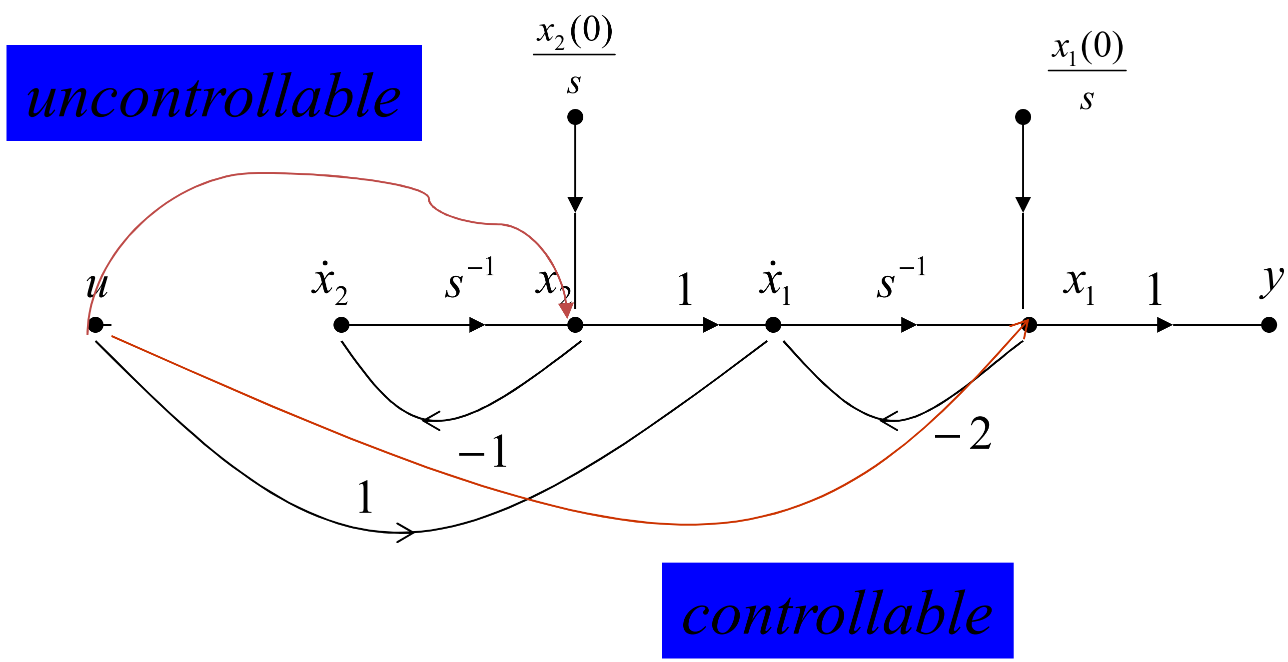
Motivation 1
Controllability and observability



Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Controllability

Plant:

$$\dot{x} = Ax + Bu, x \in R^n$$
$$y = Cx + Du$$

Definition of Controllability

A system is said to be (state) controllable at time t_0, *if there exists a finite* $t_1 > t_0$ *such that for any* $x(t_0)$ *and any* x_1 *, there exist an input* $u_{[t_0,t_1]}$ *that will transfer the state* $x(t_0)$ *to the state* x_1 *at time* $t_1 - t_0$*, otherwise the system is said to be uncontrollable at time* t_0 .

Controllability Matrix

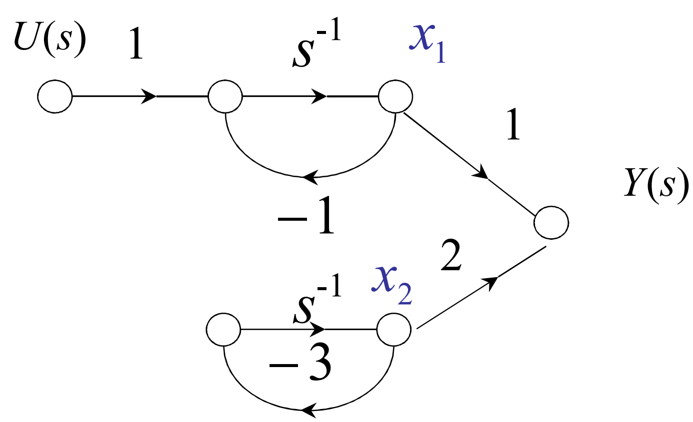
$$\left(A,B\right)\text{Controllable} \Leftrightarrow rank(U)=n,$$
$$U=\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix}$$

$$\Leftrightarrow \det(U)\neq 0 \quad \text{if } u\in R$$

Controllability Matrix $U=\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix}$

Example:

$$\dot{x}=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}x+\begin{bmatrix}1\\0\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 2\end{bmatrix}x$$



※ State x_2 is uncontrollable.

Proof of controllability matrix

$$x_{k+1} = Ax_k + Bu_k$$

$$x_{k+2} = Ax_{k+1} + Bu_{k+1}$$

$$x_{k+2} = A(Ax_k + Bu_k) + Bu_{k+1} = A^2 x_k + ABu_k + Bu_{k+1}$$

$$x_{k+n} = A^n x_k + A^{n-1}Bu_k + A^{n-2}Bu_{k+1} + \cdots + ABu_{k+(n-2)} + Bu_{k+(n-1)}$$

$$x_{k+n} - A^n x_k = A^{n-1}Bu_k + A^{n-2}Bu_{k+1} + \cdots + ABu_{k+(n-2)} + Bu_{k+(n-1)}$$

$$x_{k+n} - A^n x_k = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u_k \\ \vdots \\ u_{k+(n-2)} \\ u_{k+(n-1)} \end{bmatrix}$$

Initial condition

Example

Check the controllability and stabilizabilty of f the following system

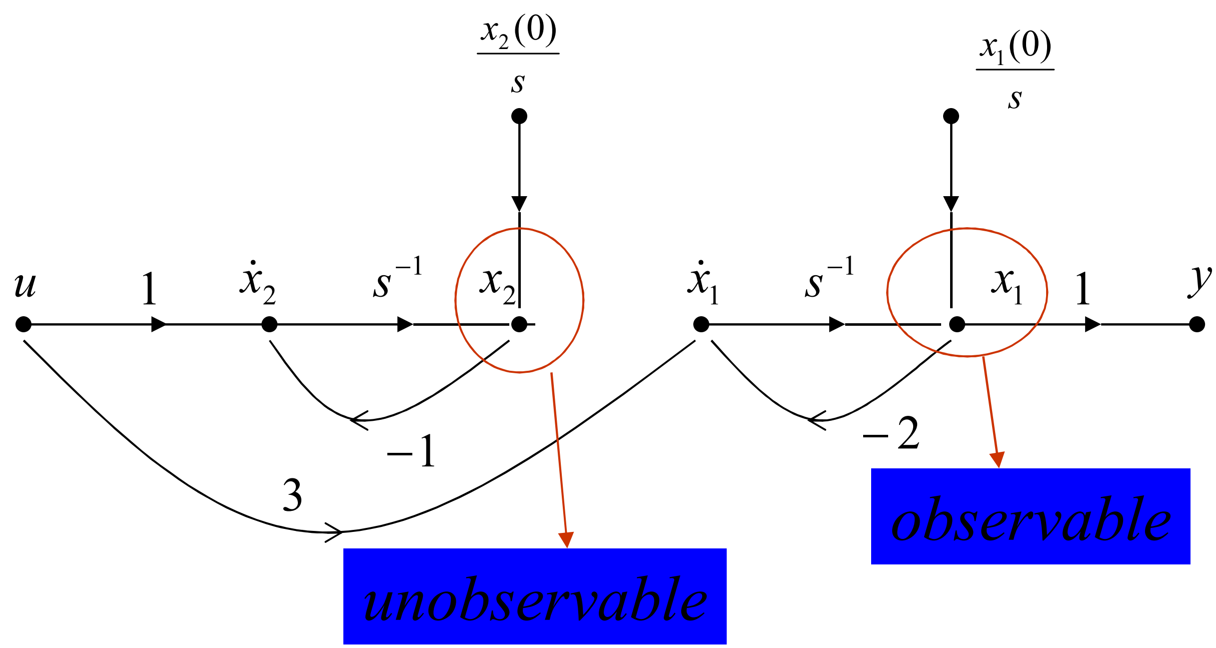
$$
\begin{aligned}
x_1^{(1)}(t) &= x_1(t) \\
x_2^{(1)}(t) &= -Kx_1(t)/3 - (2 + K/3)x_2(t) - Kx_3(t)/3 \\
&\quad - x_4(t)/3 + Ky_d(t)/3 \\
x_3^{(1)}(t) &= Kx_1(t)/2 + Kx_2(t)/2 + (-3 + K/2)x_3(t) \\
&\quad + x_4(t)/2 - Ky_d(t)/2 \\
x_4^{(1)}(t) &= -Kx_1(t) - Kx_2(t) - Kx_3(t) + Ky_d(t)
\end{aligned}
$$

The above system indicating an uncontrollable and unstable sub-system implies an unstabilizable system.

Observability Concept

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Definition of Observability

A system is said to be (completely state) observable at time t_0 *, if there exists a finite* $t_1 > t_0$ *such that for any* $x(t_0)$ *at time* t_0 *, the knowledge of the input* $u_{[t_0,t_1]}$ *and the output* $y_{[t_0,t_1]}$ *over the time interval* $[t_0, t_1]$ *suffices to determine the state* x_0*, otherwise the system is said to be unobservable at* t_0 *.*

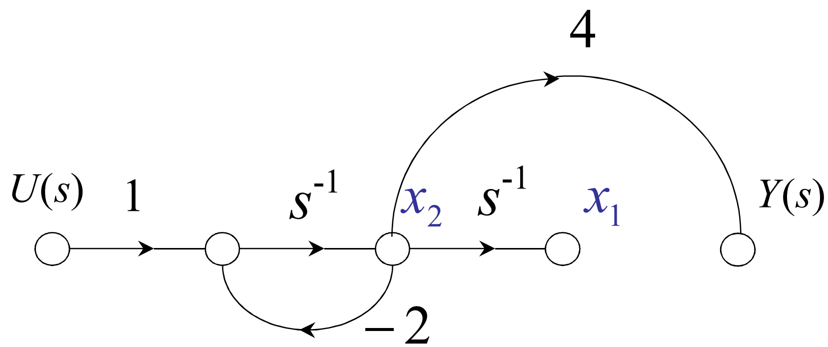
Observability Matrix

$$(A,C)\ \text{Observable} \Leftrightarrow rank(V)=n \quad \Leftrightarrow \det(V)\neq 0 \quad \text{if } y\in R$$

$$\text{Observability Matrix } V=\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Example:

$$\dot{x}=\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}x+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$
$$y=\begin{bmatrix} 0 & 4 \end{bmatrix}x$$



※ State x_1 is unobservable.

Proof of observability matrix

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k \cdots (1)$$

$$y_{k+1} = Cx_{k+1} + Du_{k+1}$$

$$y_{k+1} = C(Ax_k + Bu_k) + Du_{k+1} = CAx_k + CBu_k + Du_{k+1} \cdots (2)$$

$$y_{k+(n-1)} = CA^{n-1}x_k + CA^{n-2}Bu_k + CA^{n-3}Bu_{k+1} + \cdots + CBu_{k+(n-2)} + Du_{k+(n-1)} \cdots (n)$$

$$(1),(2),\cdots(n) \Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_k$$

$$= \begin{bmatrix} y_k - Du_k & y_{k+1} - CBu_k - Du_{k+1} & \cdots & \cdots & CABu_{k+(n-3)} - CBu_{k+(n-2)} - Du_{k+(n-1)} \end{bmatrix}$$

Inputs & outputs

Example of controllability and observability

Plant:

$$\dot{x} = Ax + Bu, x \in R^n$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Controllability Matrix $V = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Obervability Matrix $N = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$rank(V) = rank(N) = 2$$

Hence the system is both controllable and observable.

Theorem I

$$\dot{x}_c(t) = A_c x_c(t) + B_c u(t)$$

Controllable canonical form $\Rightarrow$ Controllable

Theorem II

$$\dot{x}_o(t) = A_o x_o(t) + B_o u(t)$$
$$y(t) = C_o x_o(t)$$

Observable canonical form $\Rightarrow$ Observable

Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

Controllable Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Canonical Controllable Form

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

$$a_1 = 3; a_2 = 2; b_0 = 0; b_1 = 1; b_2 = 3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

Observable Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_n \\ 0 & 0 & 0 & \cdots & -a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Observable Canonical Form Example

$$\frac{U(s)}{Y(s)} = \frac{s+3}{s^2+3s+2}$$

$$a_1 = 3; a_2 = 2; b_0 = 0; b_1 = 1; b_2 = 3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observable and Controllable Canonical Forms Example

Note the relationship between the observable and controllable forms:

$$A_{obs} = A_{cont}^{T}; B_{obs} = C_{cont}^{T}; C_{obs} = B_{cont}^{T}; D_{obs} = D_{cont}$$

example $\Rightarrow$ $T(s) = \dfrac{s+2}{(s+1)(s+2)}$

Controllable canonical form

$$\dot{x}_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x_c$$

$$U = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$
$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$$

$$rank[U] = 2 = n$$
$$rank[V] = 1 \neq n$$

Observable canonical form

$$\dot{x}_o = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x_o + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x_o$$

$$U = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$
$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$rank[U] = 1 \neq n$$
$$rank[V] = 2 = n$$

Diagonal Canonical Form

System in which the poles of the transfer function are arranged diagonally in the A matrix

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$\equiv b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \ldots + \frac{c_n}{s+p_n}$$

Diagonal Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

$$e^{At} = \begin{bmatrix} e^{-p_1 t} & & & \\ & e^{-p_2 t} & & \\ & & \ddots & \\ & & & e^{-p_n t} \end{bmatrix}$$

Example of Diagonal Canonical Form

$$\frac{U(s)}{Y(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

$$p_1 = 1;\ p_2 = 2;\ c_1 = 2;\ c_2 = 1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Jordan Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{(s+p_1)^3(s+p_4)(s+p_5)\ldots.)(s+p_n)}$$

This is a type of diagonal canonical form and in addition the transfer function involves multiple repeated poles

Jordan Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p_1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & \ldots & 0 \\ \hline 0 & \ldots & 0 & -p_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \ldots & 0 & 0 & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$