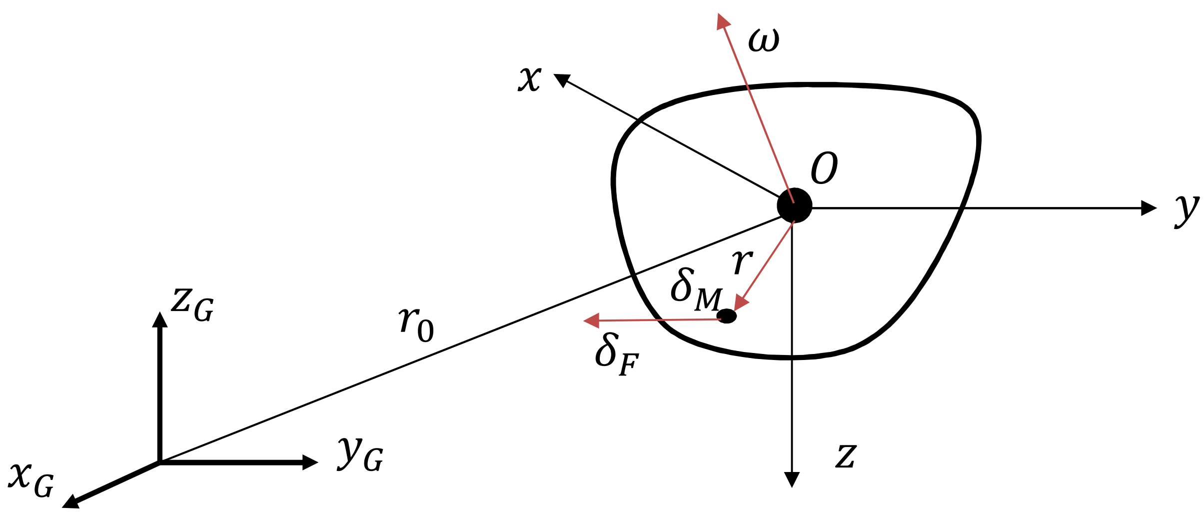
Let us consider a vehicle of mass M acted upon by an external force $F(t)$, then the individual mass of δm subject to the force δF with respect to the inertial frame can be written as

$$\sum \delta F = \frac{d}{dt} \sum v\delta m$$



The moment about the point ‘O’

$$M = \sum r \times dm \frac{dv}{dt}$$

The particle angular momentum

$$\partial H = r \times \partial m v \Rightarrow H = \int r \times v \partial m$$

Let consider the body has a constant mass

$$\frac{dH}{dt} = \int v \times v dm + \int r \times \frac{dv}{dt} dm = \int r \times \frac{dv}{dt} dm$$

Since the center of mass is a point fixed relative to the body, the magnitude of the vector ‘r’ is invariant with time. Hence, we can write the total velocity of an arbitrary point on the rigid-body located at ‘r’ relative to ‘O’ as follows:

$$v = v_0 + \omega \times r$$

Angular velocity of the reference coordinate frame

Velocity of the center of mass

Angular momentum yields

$$H = \int r \times (v_0 + \omega \times r) dm = \int r \times v_0 dm + \int r \times (\omega \times r) dm$$
$$= \int r \times (\omega \times r) dm$$

We choose to resolve all the vectors in the body frame with the axes ox, oy, oz along unit vectors i, j, k.

$$
\begin{aligned}
r &= xi + yj + zk \\
\omega &= \omega_x i + \omega_y j + \omega_z k \\
H &= H_x i + H_y j + H_k k \\
M &= M_x i + M_y j + M_z k
\end{aligned}
$$

$$
\begin{aligned}
\omega \times r &= (\omega_y z - \omega_z y)i + (\omega_z x - \omega_x z)j + (\omega_x y - \omega_y x)k \\
r \times \omega \times r &= \left[(y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z\right]i + \\
&= \left[-xy\omega_x + (x^2 + z^2)\omega_y - yz\omega_z\right]j + \\
&= \left[-xz\omega_x - yz\omega_y + (x^2 + y^2)\omega_{zy}\right]k
\end{aligned}
$$

Or,

$$
\begin{aligned}
&\int r \times (\omega \times r)dm = H_x i + H_y j + H_z k \\
&H_x = I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z;\ \ H_y = -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z; \\
&H_z = -I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z \\
&where\ \ I_x = \int_B (y^2 + z^2)\partial m;\ I_y = \int_B (x^2 + z^2)\partial m;\ I_z = \int_B (x^2 + y^2)\partial m \\
&I_{xy} = \int_B (xy)\partial m;\ I_{xz} = \int_B (xz)\partial m;\ I_{yz} = \int_B (yz)\partial m
\end{aligned}
$$

Or,

$$
H = I\omega; \qquad I = \begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{pmatrix}
$$

The equation of motion of the rigid body can be written as

$$\dot{H} + \omega \times H = M$$

$$I\dot{\omega} + \omega \times (I\omega) = M \Rightarrow M = I\dot{\omega} + S(\omega)\omega$$

$$where \quad S(\omega) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$and \quad M_x = I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z$$

$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x$$

$$M_z = I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y$$

$$or, \quad \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} M_x / I_x \\ M_y / I_y \\ M_z / I_z \end{bmatrix} - \begin{bmatrix} (I_z - I_y)\omega_y\omega_z / I_x \\ (I_x - I_z)\omega_z\omega_x / I_y \\ (I_y - I_x)\omega_x\omega_y / I_z \end{bmatrix}$$

Attitude Kinematics

The attitude kinematics of a body represents the orientation of the body fixed frame $(oxyz)$ w.r.t. $(OXYZ)$. The time dependence of the frame orientation relative to another frame is called rotational kinematics which can be represented by coordinate transformation between two frames. Let i, j, k be the unit vectors for body-fixed frame and I, J, K be the unit vectors for the reference frame, then the following condition holds:

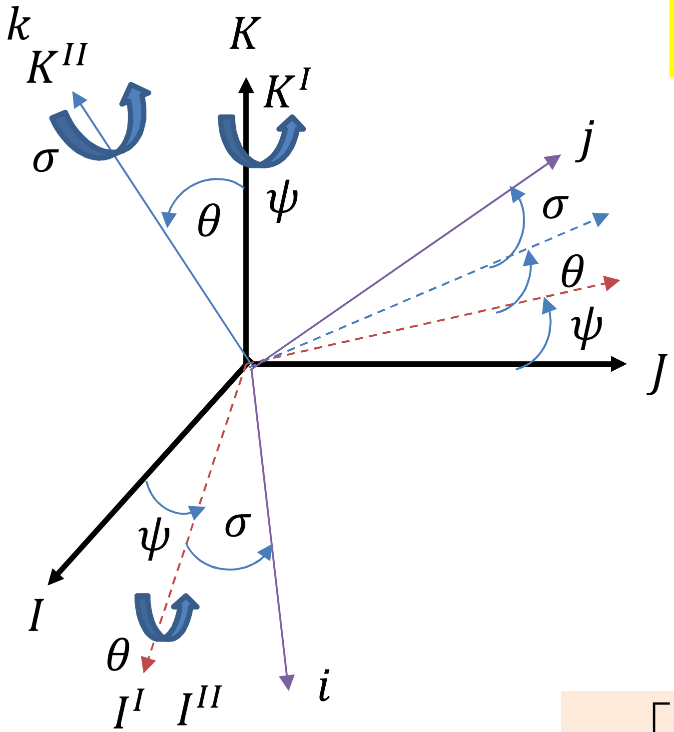
$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = C \begin{bmatrix} I \\ J \\ K \end{bmatrix} \qquad (1)$$

Where C is orthogonal, i.e., $C^T C = C C^T = I$ and $\det(C) = 1$.

The rotational kinematics can be obtained by differentiating Eq. (1)

$$\begin{pmatrix} \omega \times i \\ \omega \times j \\ \omega \times k \end{pmatrix} = \frac{dC}{dt} \begin{pmatrix} I \\ J \\ K \end{pmatrix} \Rightarrow -S(\omega) \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \frac{dC}{dt} C^T \begin{pmatrix} i \\ j \\ k \end{pmatrix} \Rightarrow \frac{dC}{dt} = -S(\omega) C$$

Represents the equation of attitude kinematics



Attitude Kinematics

$$C = C_3(\psi)C_1(\theta)C_3(\sigma)$$

$$C_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad C_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix};$$

$$C_3(\sigma) = \begin{bmatrix} \cos\sigma & \sin\sigma & 0 \\ -\sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On multiplication, yields

$$C = \begin{bmatrix} c\psi\, c\sigma - s\psi\, s\sigma\, c\theta & s\psi\, c\sigma + c\psi\, s\sigma\, s\theta & s\sigma\, s\theta \\ -(c\psi\, s\sigma + s\psi\, c\sigma\, c\theta) & -s\psi\, s\sigma + c\psi\, c\sigma\, c\theta & c\sigma\, s\theta \\ s\psi\, s\theta & -c\psi\, s\theta & c\theta \end{bmatrix}$$

Note that the above matrix becomes the following form when $\theta = 0$

$$C = \begin{bmatrix} \cos(\sigma \pm \psi) & \sin(\sigma \pm \psi) & 0 \\ \pm\sin(\sigma \pm \psi) & \pm\cos(\sigma \pm \psi) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

This matrix becomes a singular matrix at $\theta = n\pi (n = 1,2,3..)$ and the angles σ and ψ cannot be determined from the rotation matrix only their sum or difference can be determined.

Quaternion

A non-singular four parameters representation that is closely related Euler axis and principle angle combination is the quaternion representation. These four mutually dependent parameters can be expressed as: $q = [\underbrace{q_1 \quad q_2 \quad q_3}_{\text{Vector part}} \quad \underbrace{q_4}_{\text{Scaler part}}]^T$.

The quaternion for attitude representation can be derived from the Euler axis, e, and the principal rotation ϕ.

$$q_i = e_i \sin\frac{\phi}{2} \qquad (i = 1,2,3)$$

$$q_4 = \cos\frac{\phi}{2}$$

$$From\ this -> q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

The rotation matrix in terms of q can be written as

$$C = (q_4^2 - q_r^T q_r) + 2 q_r q_r^T - 2 q_4 S(q_r), where\ S(q_r) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

Quaternion

The quaternion can be written in unit vector form as

$$q = q_1 i + q_2 j + q_3 k + q_4$$

Quaternion Multiplication

$$
\begin{aligned}
a.b &= (a_1 i + a_2 j + a_3 k + a_4).(b_1 i + b_2 j + b_3 k + b_4) \\
&= (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1) i + (-a_1 b_3 + a_2 b_4 + a_3 b_1 + a_4 b_2) j \\
&\quad (a_1 b_2 - a_2 b_1 + a_3 b_4 + a_4 b_3) k + (-a_1 b_1 - a_2 b_2 - a_3 b_3 + a_4 b_4) \\
&= \begin{bmatrix} a_4 & -a_3 & a_2 & a_1 \\ a_3 & a_4 & -a_1 & a_2 \\ -a_2 & a_1 & a_4 & a_3 \\ -a_1 & -a_2 & -a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
\end{aligned}
$$

Quaternion conjugate can be defined as

$$q^* = \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ q_4 \end{bmatrix} = -q_1 i - q_2 j - q_3 k + q_4 \quad and \quad q^{-1} = q^* / \|q\|^2$$

Quaternion

Using quaternion, a rotation from coordinate system x to y coordinate can be accomplished using the following way:

$$y = q.x.q^*$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \end{bmatrix} = \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 - q_3q_4) & 2(q_1q_3 + q_2q_4) & 0 \\ 2(q_1q_2 + q_3q_4) & -q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(-q_1q_4 + q_2q_{34}) & 0 \\ 2(q_1q_3 - q_2q_4) & 2(q_1q_4 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 & 0 \\ 0 & 0 & q_1^2 + q_2^2 + q_3^2 + q_4^2 & \end{bmatrix}$$

The rotation matrix C_x^y which transform x to y cab be expressed as

$$\begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 - q_3q_4) & 2(q_1q_3 + q_2q_4) \\ 2(q_1q_2 + q_3q_4) & -q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(-q_1q_4 + q_2q_{34}) \\ 2(q_1q_3 - q_2q_4) & 2(q_1q_4 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

If q_x^y represents a rotation from x to y, and q_y^z represents a rotation from y to z, the rotation from x to z is given by:

$$q_x^z = q_y^z.q_x^y$$

Quaternion Dynamics or Attitude Kinematics

Express a quaternion that is changing with time as a rotation followed by an infinitesimal rotation

$$q(t+\Delta t) = (\Delta q).q(t)$$

$$\Delta q = \begin{bmatrix} e\sin\dfrac{\Delta\phi}{2} \\ \cos\dfrac{\Delta\phi}{2} \end{bmatrix} = \begin{bmatrix} e\dfrac{\Delta\phi}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}\omega\Delta t \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}\omega_x\Delta t \\ \dfrac{1}{2}\omega_y\Delta t \\ \dfrac{1}{2}\omega_z\Delta t \\ 1 \end{bmatrix}$$

The angular velocity of a frame in terms of Euler axis and incremental principal angle, $\Delta\phi$ is given by:

$$\omega(t) = \lim_{\Delta t -> 0} \frac{\Delta\phi e}{\Delta t}$$

Now, $q(t+\Delta t)$ can be expressed as:

$$q(t+\Delta t) = \begin{bmatrix} 1 & -\frac{1}{2}\omega_z\Delta t & \frac{1}{2}\omega_y\Delta t & \frac{1}{2}\omega_x\Delta t \\ \frac{1}{2}\omega_z\Delta t & 1 & -\frac{1}{2}\omega_x\Delta t & \frac{1}{2}\omega_y\Delta t \\ -\frac{1}{2}\omega_y\Delta t & \frac{1}{2}\omega_x\Delta t & 1 & \frac{1}{2}\omega_z\Delta t \\ -\frac{1}{2}\omega_x\Delta t & -\frac{1}{2}\omega_y\Delta t & -\frac{1}{2}\omega_z\Delta t & 1 \end{bmatrix} q(t);\ Let,\ R(\omega) \stackrel{\Delta}{=} \begin{bmatrix} 0 & -\omega_z & \omega_y & \omega_x \\ \omega_z & 0 & -\omega_x & \omega_y \\ -\omega_y & \omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}$$

$$q(t+\Delta t) = (I + \frac{1}{2}R(\omega)\Delta t)q(t) \Rightarrow \frac{q(t+\Delta t) - q(t)}{\Delta t} = \frac{1}{2}R(\omega)q(t)$$

$$\dot{q}(t) = \frac{1}{2}R(\omega)q(t)$$

Attitude Kinematics Equation

Flight Dynamics System

Before proceeding for the complete equation of motion of flight dynamics, lets define the follwoing terms:

$F_c(t), M_c(t) \rightarrow$ Contributing effectcs of control forces and moments
$F_E(t), t_E(t) \rightarrow$ Environmental Force and torque
$F_D(t), t_D(t) \rightarrow$ Disturbance forces & torques. The complete equations for flight dynamics system can be written as

$$v = \frac{dr}{dt} = \dot{r}_0 + \Omega \times r$$

$$F_c + F_D + F_E = m\frac{dv}{dt} = m(\dot{v}_0 + \Omega \times v)$$

$$t_c + t_E + t_D = I\dot{\omega} + S(\omega)I\omega$$

$$\frac{dC}{dt} = -S(\omega)C(t) \quad or \quad \dot{q}(t) = \frac{1}{2}R(\omega)q(t)$$

Space Flight Dynamics

The flight dynamics equation for space flight in vaccum environment yields $F_D = t_E = t_D = 0$ and $F_E = mg$. The attitude dynamics can be expressed as:

$$v = \frac{dr}{dt} = \dot{r}$$

$$F_c + mg = m\frac{dv}{dt} = m\ddot{r}$$

$$t_c = I\dot{\omega} + S(\omega)I\omega; \;\; \dot{C} = -S(\omega)C$$

Flight Control System

A typical flight control system can be modelled as

$$u(t) = \begin{Bmatrix} F_c(t) \\ t_C(t) \end{Bmatrix} \longrightarrow \boxed{\text{Flight Dynamics System}} \longrightarrow x(t) = \begin{Bmatrix} r_0(t) \\ v_0(t) \\ \omega(t) \\ q(t) \end{Bmatrix}$$

$$p(t) = \begin{Bmatrix} F_D(t) \\ M_D(t) \end{Bmatrix} \uparrow$$

We can express the flight dynamics by the following state equation

$$\frac{dx}{dt} = f(x,u,p,t); \quad y(t) = h(x,u,\omega,t)$$

State equation for nominal trajectory

$$\frac{dx}{dt} = f(x_n,0,0,t)$$

The linearized state equation for deviation from the nominal trajectory, $z = x - x_n$ is given by

$$\frac{dz}{dt} = A(t)z(t) + B(t)u(t)$$