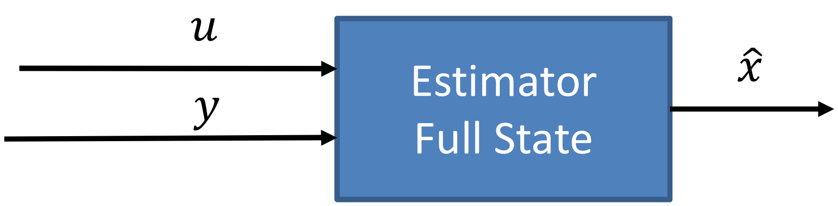
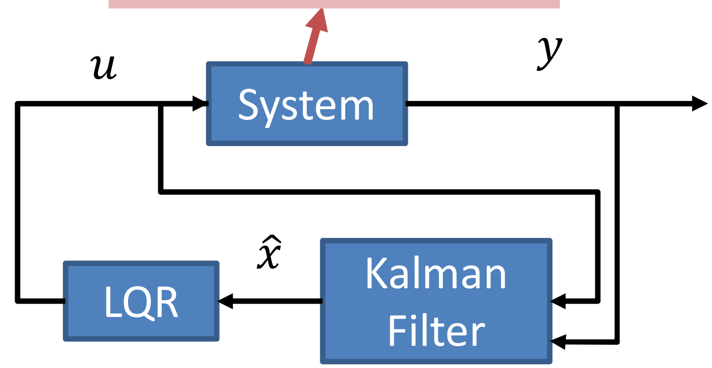
Full State Estimator



$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$



Dynamics of estimated equation can be written as

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y}); \quad \hat{y} = C\hat{x}$$
$$\dot{\hat{x}} = A\hat{x} + Bu + K_e y - K_e C\hat{x}$$
$$= (A - K_e C)\hat{x} + \begin{bmatrix} B & K_e \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix}$$

If the matrix $(A - K_e C)$ is stable, then $\hat{x}$ steadily converge to x.

Lets define an error, $e = x - \hat{x}$

$$\dot{e} = Ax + Bu - A\hat{x} - Bu + K_e C\hat{x} - K_e y$$
$$= A(x - \hat{x}) + K_e C(\hat{x} - x)$$
$$= (A - K_e C)e$$

Kalman Filter for LTI System

Consider a LTI System

$$\dot{x} = Ax + Bu + v(t)$$
$$y = Cx + w(t)$$

$v(t) \rightarrow \textit{Gaussian input noise},$
$w(t) \rightarrow \textit{Gaussian Measurement Noise}$

Assume

1. (A,C) is observable
2. $w(t)$ and $v(t)$ are independent white noises with the following properties

$$E[v(t)] = 0;\ E[w(t)] = 0$$
$$E[v(t)v^T(t)] = Q\partial(t-\tau),\ Q = Q^T \geq 0$$
$$E[w(t)w^T(t)] = R\partial(t-\tau),\ R = R^T \geq 0$$

The problem of Kalman Filter is to design a state feedback estimator to estimate the state $x(t)$ by $\hat{x}(t)$ such that the estimator error covariance is minimized, i.e., the following index is minimized

$$J_e = E[\{x(t)-\hat{x}(t)\}^T\{x(t)-\hat{x}(t)\}]$$

Construction of Steady State Kalman Filter

Kanmal filter is a state observer with a specially selected observer gain (or Kalman filter gain). It has the dynamic equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + k_e(y - \hat{y})$$
$$\hat{y} = c\hat{x}; \quad u = -k_e\hat{x};$$

With the Kalman filter gain K_e being given as

$$k_e = P_e C^T R^{-1}$$

where P_e is the positive definite of the following function Riccati Equation

$$P_e A^T + AP_e - P_e C^T R^{-1} C P_e + Q = 0$$

Let $e = x - \hat{x}$. The above Kalman Filter has the following properties:

$$\lim_{t->\infty} E[e(t)] = \lim_{t->\infty} E[x - \hat{x}] = 0$$
$$\lim_{t->\infty} J_e = \lim_{t->\infty} E[e^T e] = trace(P_e)$$

Kalman Filter and LQR- they are dual

LQR

$$K = R^{-1}B^T P$$

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

$$J_{optimal} = trace(P)$$

Kalman Filter

$$K_e = P_e C^T R^{-1}$$

$$P_e A^T + AP_e - P_e C^T R^{-1} C P_e + Q = 0$$

$$J_{optimal} = trace(P_e)$$

$$A^T <-> A$$

$$B^T <-> C$$

$$K^T <-> K_e$$

$$P <-> P_e$$

Kalman Filter - Example

Design a Kalman filter for the following aircraft system

$$A = \begin{bmatrix} -1.7 & 50 & 260 \\ 0.22 & -1.4 & -32 \\ 0 & 0 & -12 \end{bmatrix}; \; B = \begin{bmatrix} -272 \\ 0 \\ 14 \end{bmatrix}; \; F = \begin{bmatrix} 0.02 & 0.1 \\ -0.0035 & 0.04 \\ 0 & 0 \end{bmatrix};$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \; D = 0$$

Solution: See the example 7.6 in “Modern Control Engineering” by Ashish Tewari.

Linear Quadratic Gaussian (LQG) Control

It is often in control system design for a real life problem that one cannot measure all the state variables of the given plant. Thus, LQR, although it has a very impressive gain and phase margins, is impractical as it utilizes all state variables in the feedback, i.e., $u = -Kx$. In most practical situations, only partial information of the state of the given plant is accessible or can be measured for feedback. The natural questions one would ask:

1. Can we recover or estimate the state variables of the plant through the partially measurable information? The answer is yes. The solution is Kalman Filter.
2. Can we replace x the control law in LQR, i.e., $u = -Kx$, by the estimated state to carry out a meaningful control system design? The answer is yes. The solution is called LQG.
3. Do we still have impressive propeties associated with LQG? The answer is no. Any solution? Yes. Its called loop transfer recovery (LTR) technique.

Solution of LQG Problem – Separation Principle

Step-1: Design an LQR control law $u = -Kx$ which solves the following problem

$$\dot{x} = Ax + Bu \quad J(x,u,Q,R) = \int_0^\infty (x^T Q x + u^T R u)dt \qquad i.e., compute$$

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \; P > 0, \; K = R^{-1}B^T P$$

Step-2: Design a Kalman Filter for the given plant:

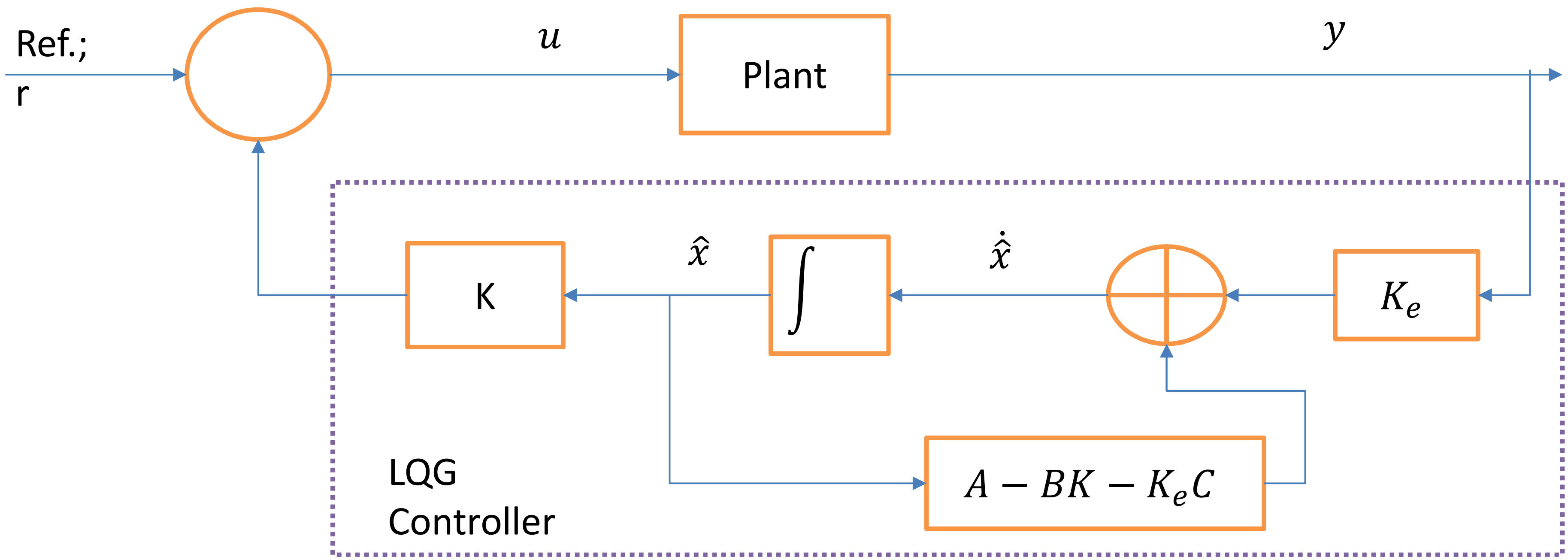
$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - \hat{y})$$

$$\hat{y} = c\hat{x} \quad where \quad K_e = P_e C^T R^{-1}, \quad P_e A^T + AP_e - P_e C^T R_e^{-1} C P_e + Q_e = 0, \; P_e > 0$$

Step-3: The LQG control law is given by $u = -K\hat{x}$

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_e(y - c\hat{x}) \\ u &= -K\hat{x} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \dot{\hat{x}} &= (A - BK - K_e C)\hat{x} + K_e y \\ u &= -K\hat{x} \end{aligned}\right.$$

Solution of LQG Problem



Closed-loop dynamics of the given plant together with LQG controller:

$$\dot{\hat{x}} = (A - BK - K_eC)\hat{x} + K_e y$$

$$u = -K\hat{x} + r$$

Solution of LQG Problem

Let Define an error variable $e = x - \hat{x}$

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{\hat{x}} \\
&= Ax + Bu + v(t) - (A - BK - K_e C)\hat{x} - K_e y \\
&= Ax + B(-K\hat{x} + r) + v(t) - (A - BK - K_e C)\hat{x} - K_e (Cx + w(t)) \\
&= Ax - BK\hat{x} + Br + v(t) - A\hat{x} + BK\hat{x} + K_e C\hat{x} - K_e Cx - K_e w(t) \\
&= A(x - \hat{x}) - K_e C(x - \hat{x}) + Br + v(t) - K_e w(t) \\
&= (A - K_e C)e + Br + v(t) - K_e w(t)
\end{aligned}$$

And

$$\begin{aligned}
\dot{x} &= Ax + Bu + v(t) = Ax - BK\hat{x} + Br + v(t) \\
&= Ax - BK(x - e) + Br + v(t) \\
&= (A - BK)x + BKe + Br + v(t)
\end{aligned}$$

Clearly, the above closed loop system is characterised by the following state-space equation

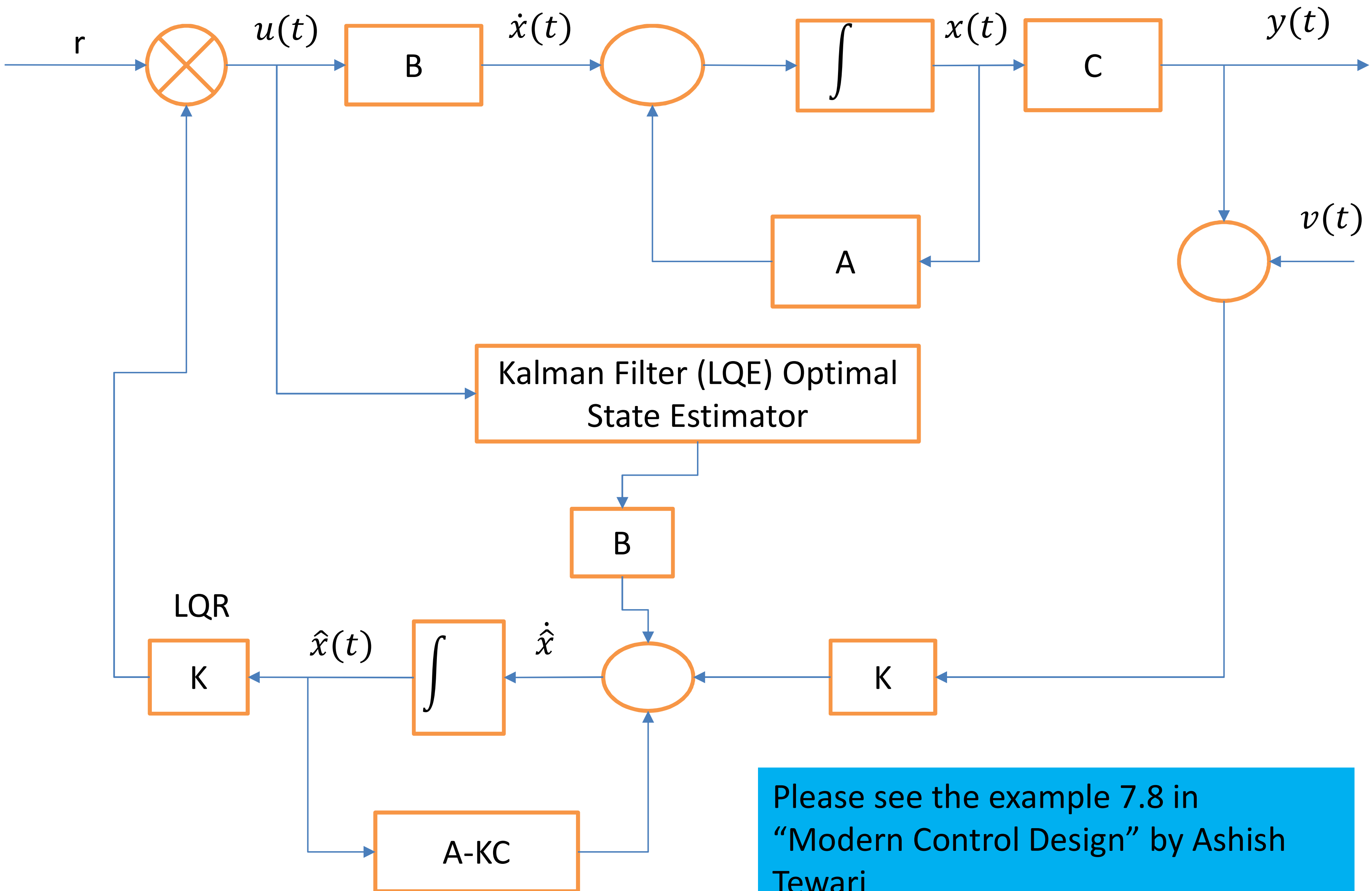
$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - K_e C \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r + \begin{bmatrix} v \\ v - K_e w \end{bmatrix}$$

$$y = [c \quad 0] \begin{bmatrix} x \\ e \end{bmatrix} + w$$

Kalman Filter Eigenvalues

LQR Eigenvalues

Complete Diagram of LQG Contrlol



Please see the example 7.8 in "Modern Control Design" by Ashish Tewari