Ackermann Formula for controller design

Plant: $\dot{x} = Ax + Bu \qquad x \in R^{n\times 1} \qquad u \in R$

$y = Cx \qquad y \in R$

State Feedback: $u(t) = -Kx(t) + r(t)$ $\quad K \in R^{1\times n}$

Desired poles : $\lambda_1, \cdots, \lambda_n$ Desired Char. Poly. : $\prod_{i=1}^{n}(s-\lambda_i) = s^n + a_{cn}s^{n-1} + \cdots + a_{c2}s + a_{c1}$

The Matrix Polynomial $\quad \alpha_c(A) = \prod_{i=1}^{n}(A-\lambda_i) = A^n + a_{cn}A^{n-1} + \cdots + a_{c2}A + a_{c1}I$

Then the state feedback gain matrix is

$$K = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}}_{\in R^{1\times n}} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}^{-1} \alpha_c(A)$$

Ackermann Formula for controller design

$$\dot{x}(t)=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}x(t)+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} & \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$K=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^{-1}\left(\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}^{2}+11\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}+30I\right)$$

$$=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}43 & 14\\ 14 & 57\end{bmatrix}=\begin{bmatrix}14 & 57\end{bmatrix}$$

Steady State Error

$$\dot{x}(t) = Ax(t) + Br(t)$$
$$y(t) = Cx(t)$$

Error Variable

$$e(t) = r(t) - y(t)$$

Lapalce Transform of the Error Variable

$$E(s) = R(s) - Y(s)$$

$$E(s) = R(s)[1 - C(sI - A)^{-1}B]$$

By Final Value Theorem

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$
$$= \lim_{s\to 0} sR(s)[1 - C(sI-A)^{-1}B]$$

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Can I make a state space controller without using observers?

Question Asked March 14, 2014

I'm working nowadays on a project in which I want to control a robot using state feedback techniques. I want to make a full state feedback, but I read that in order to do this I have to have an observer that estimates the values of the system states. I have a couple of questions:

1) In case I have enough sensors to measure all the states of the system (I have 4 states), can I perform the state feedback without an observer?

2) Does the observer have something special? Does it have a noise filtering property for example? Does it improve something in the system? Or just it is for estimating the unknown states?

Thank you in advance.

https://www.researchgate.net/post/Can_I_make_a_state_space_controller_without_using_observers

Víctor Campos added an answer

March 14, 2014

Note that this changes from case to case.

Theoretically if you want to employ a state-feedback controller you could simply use the state measurements in your controller (Given that you have enough sensors and that their quality is reasonably good). For example, if you are controlling a robot manipulator, it is common to have all of the states available to the controller with a reasonable quality. Therefore, there should be no need for an observer.

Recommend Share

12 Recommendations

Víctor Campos added an answer

March 14, 2014

However, also note that, in case you have noisy measurements, your controller will generate a noisy control action (which will be amplified in the system and appear again in the sensors). So, if you are considering a noisy environment (and a linear system and controller) I'd suggest looking up the LQG controller since they will give you an optimal solution (though you have to think of the cost you are minimizing). In this case, your controller will be an LQR (which is a state feedback law) using a Kalman filter (to find the optimal state estimates) and you will have a better response using a Kalman filter than using the measurements directly.

Recommend Share

9 Recommendations

Observers

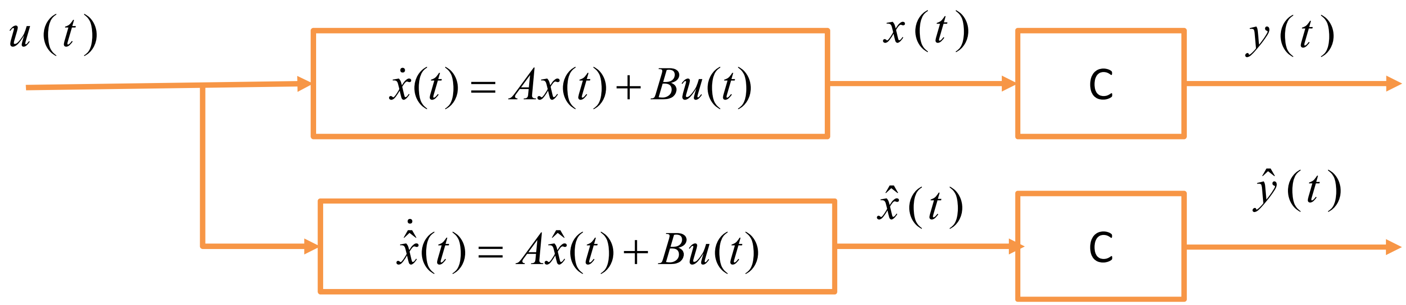
1. While designing full state based feedback control, it was assumed that we can measure and feedback all the states of the plant using sensors.
2. Some state variables can be expensive and measurements of some can be so noisy that a control system based on such method would be unsuccessful. Hence, it is required to estimate the sate rather than measure the state vector of a system.
3. The mathematical model that estimate the states from the measured output vector is called the observer or state estimator.
4. Due to the excess numbser of sensors or error in the measurement, most of practical solutions are not to the user requirement.
5. Only subset of state variables are available for the measurement.
6. The relationship between observability and controllability are dual in nature in terms of input and output state vector.

Thus, as estimator or observer is obvious which can estimate all the state variables which observing input and output.

Full Order Observer: If the state observer estimates all the state variables, it is called a full order observer.

Reduced Order Observer: A observer estimates fewer than the “n” states of the system is called reduced order observer.

Open Loop Estimator



The dynamics of the estimators: $\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t); \hat{y}(t) = C\hat{x}(t)$

where $\hat{x}$ is the estimaton of x and $\hat{y}$ is the estimation of y. Let $\tilde{x} = \hat{x} - x$ be the estimation errror and the error dynamics can be wrriten as

$$\begin{aligned} \dot{\tilde{x}} = \dot{\hat{x}} - \dot{x} &= A\hat{x}(x) + Bu(t) - Ax(t) - Bu(t) \\ &= A(\hat{x}(t) - x(t)) = A\tilde{x} \end{aligned}$$

If the eigenvalues of A are in the L.H. of S-plane, then the convergence of $\tilde{x} \rightarrow 0$ can be achieved. But the rate of convergence of $\tilde{x} \rightarrow 0$ can be controlled with the application of control law

Block diagram of Full-order Observer

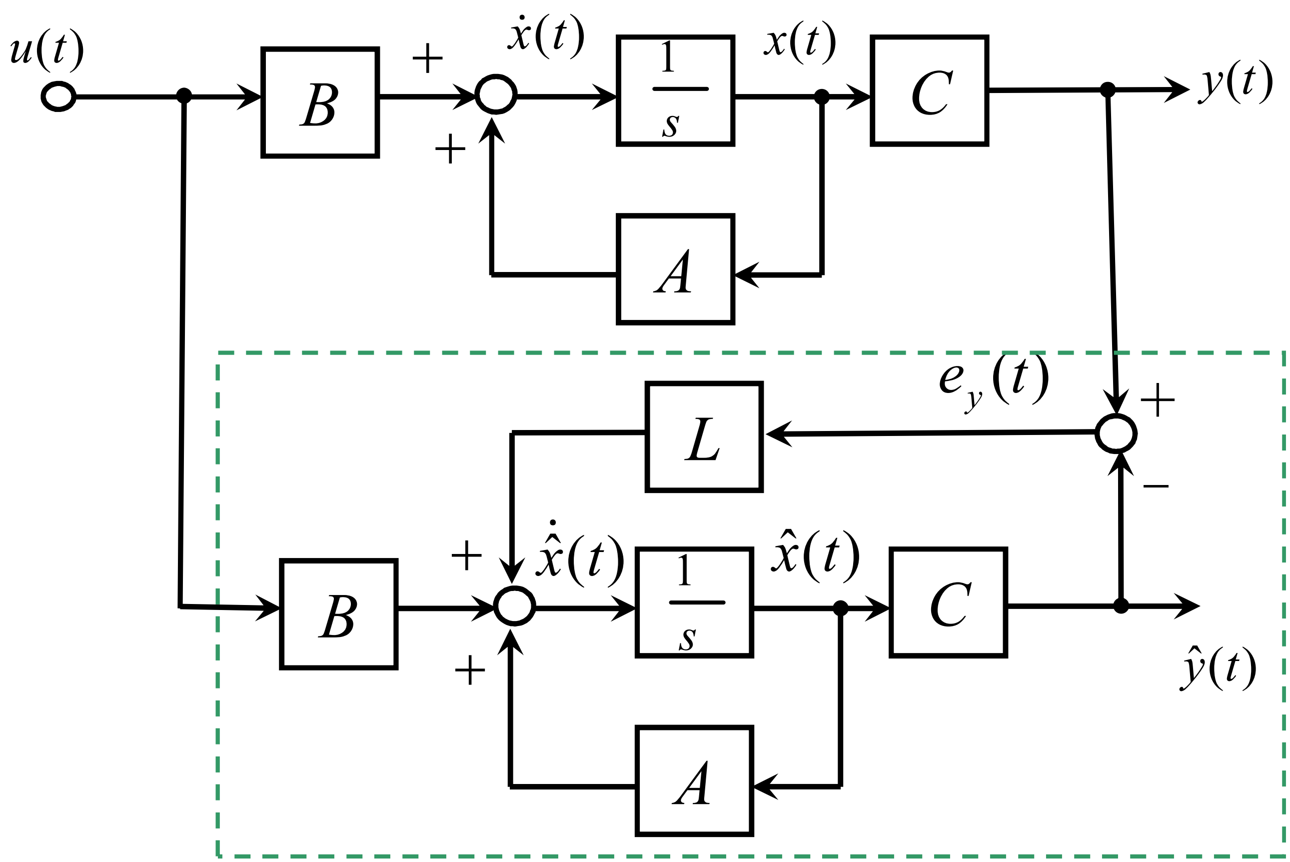


Fig. Full-Order Observer

Full-Order Observer Design

Plant:

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x_0$$
$$y = Cx$$

Suppose $\hat{x}$ is the observer state

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + Bu + Ly\end{aligned}$$

L: Observer gain

Estimation error: $e = x - \hat{x}$

Error Dynamics Equation:

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - (A - LC)\hat{x} - Bu - Ly \\ &= (A - LC)e \qquad (\because y = Cx)\end{aligned}$$

Hence if all the eigenvalues of (A-LC) lie in LHP, then the error system

$$\dot{e} = (A - LC)e$$

is asy. stable and $e \to 0, \quad \text{as } t \to \infty$

How to obtain observer gain matrix, L

1. **Identify the state variable matrices: A,B,C,D**
2. **Calculate the matrix, $\boldsymbol{SI}-(\boldsymbol{A}-\boldsymbol{LC})$**
3. **Find the determinant of the compensated system, $\boldsymbol{\Delta}(\boldsymbol{s})=\det[\boldsymbol{sI}-(\boldsymbol{A}-\boldsymbol{LC})]$**
4. **Determine the desired characteristic equation to satisfy design constraints,**

$$\boldsymbol{\Delta_d(s)=s^n+d_n s^{n-1}+\cdots+d_1 s+d_0}$$

5. **Equate coefficients to find observer gains, L**

By duality between controllable from and obeservable form

$$A_o = A_c^T \qquad B_o = C_c^T \qquad C_o = B_c^T$$

Example

The control system of a system given below has been designed for a 4Sec settling time and 20.8% overshoot. Design a observer to respond 10 times faster than a closed loop design

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+5)}$$

Soln.

Finding state equation in controllable canonical form

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \; y = [4 \;\; 1 \;\; 0] X$$

From the example

Controller poles: $-1 \pm j2, -4$
Observer Poles: $-10 \pm j20, -40$

By duality between controllable from and obeservable form

$$A_o = A_c^T \qquad B_o = C_c^T \qquad C_o = B_c^T$$

$$A_0 = A_c^T = \begin{bmatrix} 0 & 0 & -10 \\ 1 & 0 & -17 \\ 0 & 1 & -8 \end{bmatrix}; \; B_0 = C_c^T = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}; \; C_0 = B_C^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\Delta_d(s) = (s+40)(s+10+j20)(s+10-j20) = s^3 + 60s^2 + 1300s + 20000$$

Example

$$A_0 - LC_0 = \begin{bmatrix} 0 & 0 & -10-l_1 \\ 1 & 0 & -17-l_2 \\ 0 & 1 & -8-l_3 \end{bmatrix}; \; sI-(A_0-LC_0) = \begin{bmatrix} s & 0 & 10+l_1 \\ 0 & s & 17+l_2 \\ 0 & 0 & s+8+l_3 \end{bmatrix}$$

$$\Delta(s) = \left| sI-(A_0-LC_0) \right| = s^3 + (8+l_3)s^2 - (17+l_2)s + 10 + l_1$$

Comparing the above characteristic equation with

$$\Delta_d(s) = (s+40)(s+10+j20)(s+10-j20) = s^3 + 60s^2 + 1300s + 20000$$

The observer gains yield

$$l_1 = 19990; \; l_2 = -1317; \; l_3 = 52$$

$$L = \begin{bmatrix} 19990 \\ -1317 \\ 52 \end{bmatrix}$$

Example 2

Design a observer for the following control to have 0.18sec settling time and 16.3% overshoot

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \quad y = [0 \quad 1]x$$

Observer Based Controller Design and Separation Principle

Plant:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

State Feedback Law using estimated state: $u = -K\hat{x}$

State Equation:

$$\dot{x} = Ax + Bu = Ax - BK\hat{x} \quad = Ax - BKx + BKe$$
$$= (A - BK)x + BKe$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = A\hat{x} - BK\hat{x} + LC(x - \hat{x})$$
$$= (A - BK - LC)\hat{x} + Ly$$

Error Dynamics:

$$e = x - \hat{x}$$
$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e$$

We obtain the overall state equation

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = \widetilde{A} \begin{bmatrix} x \\ e \end{bmatrix}$$

Eigenvalues of the overall state equation

$$\left|\lambda I - \widetilde{A}\right| = \left|\begin{bmatrix} \lambda I - (A-BK) & -BK \\ 0 & \lambda I - (A-LC) \end{bmatrix}\right| = \left|\lambda I - (A-BK)\right| \cdot \left|\lambda I - (A-LC)\right|$$

The eigenvalues of the observer-based state feedback system is consisted of eigenvalues of (*A-BF*) and (*A-LC*). Hence, the design of state feedback and observer gain can be done independently.

Selection of Controller and Observer Poles

1. Observer poles ten times greater than controller poles, hence, observers are faster than plant controlled system.
2. Controller poles are dominant poles as they are situated near to the imaginary axis compared to the controller poles.

Design Process

1. Matlab Code- for the SISO system we use acker command and for MIMO we use place command.
2. For finding the controller gains: we have to use system matrix A and B and the desired poles. K=acker(A,B,dp) or K=place(A,B,dp).
3. When we are designing controller for a system, first step is to check the controllability of the system (det(A,B)).
4. For plant Controller: K=acker(A,B,dp)- actual pole placement controller.
   For observer controller: L=acker(A’,C’,dp).

Fig. Observer-based control system

$u(t) = -K\hat{x}(t) + r(t)$