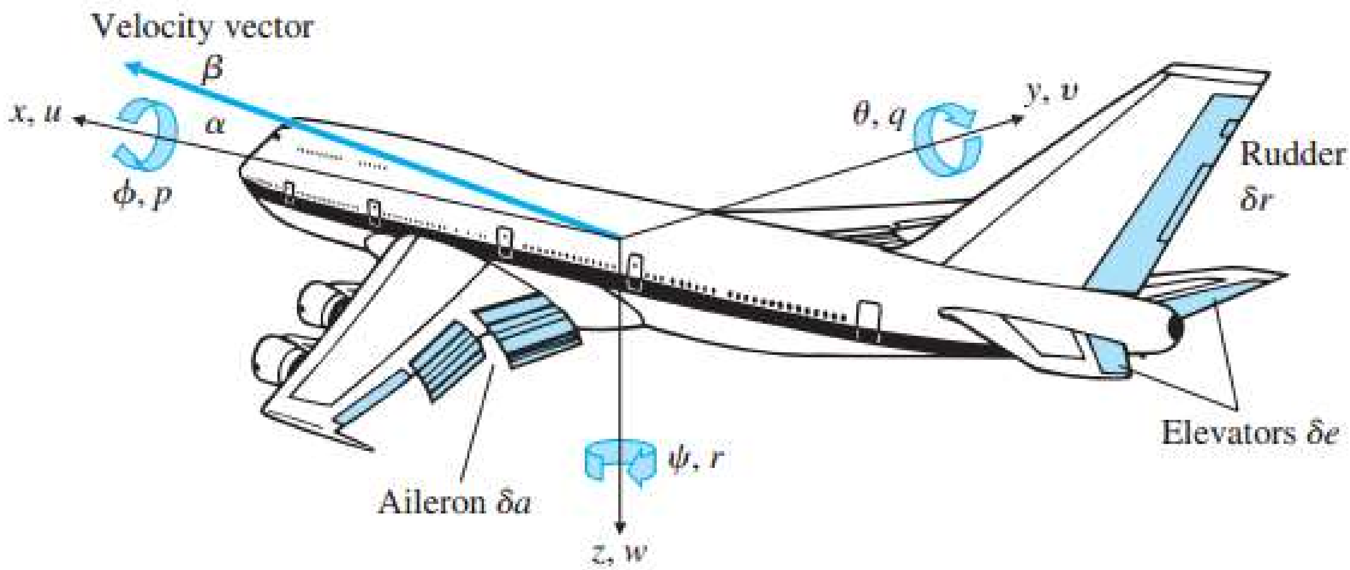
Aircraft Autopilot Design

Lateral and Longitudinal Autopilot Design



x, y, z = position coordinates
u, v, w = velocity coordinates
p = roll rate
q = pitch rate
r = yaw rate

ϕ = roll angle
θ = pitch angle
ψ = yaw angle
β = side-slip angle
α = angle of attack

The nonlinear rigid body equations of motion in body-axis coordinates, under proper assumptions, can be derived as

$$m(\dot{U} + qW - rV) = X - mg\sin\theta + \kappa T\cos\theta,$$
$$m(\dot{V} + rU - pW) = Y + mg\cos\theta\sin\phi,$$
$$m(\dot{W} + pV - qU) = Z + mg\cos\theta\cos\phi - \kappa T\sin\theta,$$

$$I_x\dot{p} + I_{xz}\dot{r} + (I_z - I_y)qr + I_{xz}qp = L,$$
$$I_y\dot{q} + (I_x - I_z)pr + I_{xz}(r^2 - p^2) = M,$$
$$I_z\dot{r} + I_{xz}\dot{p} + (I_y - I_x)qp - I_{xz}qr = N,$$

The state equations for the longitudinal motion, under proper assumption, can be written as

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}X_u & X_w & 0 & -g\\ Z_u & Z_w & u_0 & 0\\ M_u & M_w & M_q & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}0\\ Z_\delta\\ M_\delta\\ 0\end{bmatrix}[\Delta\delta_e]$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta$$

Here, A and B are the stability and control matrices and x and η are the state and control vectors. The eigenvalues of the system matrix, A, show the stability of the overall system. If the eigenvalues are not acceptable by the pilot, state feedback control design can be placed to provide the stability augmentation. The feedback control is chosen as:

$$\eta = -\mathbf{k}^T\mathbf{x} + \eta_p$$

Here, k and η_p are feedback gain and pilot input command to the system

With the above control, the closed loop system becomes

$$\dot{X} = (A - Bk^T)X + B\eta_p$$
$$= A^*X + B\eta_p \qquad Here,\ A^* = A - Bk^T$$

The augmented matrix becomes

$$\mathbf{A}^* = \begin{bmatrix}X_u & X_w & 0 & -g\\ Z_u - Z_\delta k_1 & Z_w - Z_\delta k_2 & u_0 - Z_\delta k_3 & -Z_\delta k_4\\ M_u - M_\delta k_1 & M_w - M_\delta k_2 & M_q - M_\delta k_3 & -M_\delta k_4\\ 0 & 0 & 1 & 0\end{bmatrix}$$

The characteristic equation for augmented matrix becomes

$$|\lambda\mathbf{I} - \mathbf{A}^*| = 0$$

The feedback gains can be determined once the desired longitudinal characteristics are specified. For example, if the desired characteristic roots are

$$\lambda_{1,2} = -\zeta_{sp}\omega_{n_{sp}} \pm i\omega_{n_{sp}}\sqrt{1-\zeta_{sp}^2}$$
$$\lambda_{3,4} = -\zeta_p\omega_{n_p} \pm i\omega_{n_p}\sqrt{1-\zeta_p^2}$$

And the desired characteristic equation becomes

$$\lambda^4 - [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)]\lambda^3 + [\lambda_1\lambda_2 + \lambda_3\lambda_4 + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)]\lambda^2 - [\lambda_1\lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4(\lambda_1 + \lambda_2)]\lambda + \lambda_1\lambda_2\lambda_3\lambda_4 = 0$$

By equating the coefficients of like powers of λ for the augmented and desired characteristic equations one obtains a set of four linear algebraic equations in terms of the unknown gains. The equations can be solved for the feedback gains.

Example

An airplane is found to have poor short-period flying qualities in a particular flight regime. To improve the flying qualities, a stability augmentation system using state feedback is to be employed. Determine the feedback gains so that airplane's short-period characteristics are $p_{1,2} = -2.1 \pm 2.14i$. Assume that the original short-period dynamics are given by

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} -0.334 & 1.0 \\ -2.52 & -0.387 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \end{bmatrix} + \begin{bmatrix} -0.027 \\ -2.6 \end{bmatrix} [\Delta\delta_e]$$

Solution

Augmented Matrix

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{k}^T$$

$$\mathbf{A}^* = \begin{bmatrix} -0.334 + 0.027k_1 & 1.0 + 0.027k_2 \\ -2.52 + 2.6k_1 & -0.387 + 2.6k_2 \end{bmatrix}$$

Eigenvalues

$$|\lambda\mathbf{I} - \mathbf{A}^*| = 0$$

$$\begin{vmatrix} \lambda + 0.334 - 0.027k_1 & -1.0 - 0.027k_2 \\ 2.52 - 2.6k_1 & \lambda + 0.387 - 2.6k_2 \end{vmatrix} = 0$$

Desired Characteristic Eq.

$$\lambda^2 + 4.2\lambda + 9 = 0$$

$$\lambda^2 + (0.721 - 0.027k_1 - 2.6k_2)\lambda + 2.65 - 2.61k_1 - 0.8k_2 = 0$$

Comparing the characteristic equation of the augmented system and desired system, yields

$$\boxed{k_1 = -2.03 \qquad k_2 = -1.318}$$

State Feedback control is given by

$$\boxed{\Delta\delta_e = 2.03\,\Delta\alpha + 1.318\,\Delta q}$$

Example

The longitudinal equations for an airplane having poor handling qualities follow. Use state feedback control to provide stability augmentation so that the augmented aircraft has the following short- and long- period (phugoid) characteristics:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}-0.01 & 0.1 & 0 & -32.2\\ -0.40 & -0.8 & 180 & 0\\ 0 & -0.003 & -0.5 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}0\\ -10\\ -2.8\\ 0\end{bmatrix}[\Delta\delta_e]$$

$$\zeta_{sp} = 0.6 \qquad \omega_{n_{sp}} = 3.0 \text{ rad/s}$$

$$\zeta_p = 0.05 \qquad \omega_{n_p} = 0.1 \text{ rad/s}$$

Solution

In this problem, we use Bass-Gura method, which lends itself to simple matrix manipulations. The feedback gains are found by solving the following equations

$$\mathbf{k} = [(\mathbf{VW})^T]^{-1}[\bar{\mathbf{a}} - \mathbf{a}]$$

Here, V is the controllability matrix, W is a transformation matrix, and $\bar{a}$ and a are vectors made up of the coefficients of the characteristic equation of the augmented or closed-loop system $(A - Bk^T)$ and the characteristic equation of the open-loop plant matrix A.

The desired characteristic equation can be written as:

$$(\lambda^2 + 2\zeta_{sp}\omega_{n_{sp}}\lambda + \omega_{n_{sp}}^2)(\lambda^2 + 2\zeta_p\omega_{n_p}\lambda + \omega_{n_p}^2) = 0$$

or

$$(\lambda^2 + 3.6\lambda + 9)(\lambda^2 + 0.01\lambda + 0.01) = 0$$

$$\lambda^4 + 3.61\lambda^3 + 9.05\lambda^2 + 0.126\lambda + 0.09 = 0$$

The vector $\bar{a}$ is created from the coefficients of the desired characteristic equation;

$$\lambda^4 + \bar{a}_1\lambda^3 + \bar{a}_2\lambda^2 + \bar{a}_3\lambda + \bar{a}_4 = 0$$

$$\bar{\mathbf{a}} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \end{bmatrix} = \begin{bmatrix} 3.61 \\ 9.05 \\ 0.126 \\ 0.09 \end{bmatrix}$$

The characteristic equation of the open-loop system is obtained by solving the equation:

$$|\lambda\mathbf{I} - \mathbf{A}| = 0$$ or $$\lambda^4 + 1.31\lambda^3 + 0.993\lambda^2 + 0.0294\lambda + 0.0386 = 0$$

The vector is created from the coefficient of the open-loop characteristic equation:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1.31 \\ 0.993 \\ 0.0294 \\ 0.0386 \end{bmatrix}$$

Now we need to determine the controllability matrix, V. For the fourth order system, the controllability matrix is

$$\mathbf{V} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \mathbf{A}^3\mathbf{B}]$$

The elements of V can be performed through appropriate matrix multiplications:

$$\mathbf{AB} = \begin{bmatrix} -0.01 & 0.1 & 0 & -32.2 \\ -0.40 & -0.8 & 180 & 0 \\ 0 & -0.003 & -0.5 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ -2.8 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.0 \\ -496 \\ 1.43 \\ -2.8 \end{bmatrix}$$

$$\mathbf{A}^2\mathbf{B} = \mathbf{A}[\mathbf{AB}] = \begin{bmatrix} -0.01 & 0.1 & 0 & -32.2 \\ -0.40 & -0.8 & 180 & 0 \\ 0 & -0.003 & -0.5 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix} \begin{bmatrix} -1.0 \\ -496 \\ 1.43 \\ -2.8 \end{bmatrix}$$

$$\mathbf{A}^2\mathbf{B} = \begin{bmatrix} 40.57 \\ 654.6 \\ 0.773 \\ 1.43 \end{bmatrix}$$

$$\mathbf{A}^3\mathbf{B} = \mathbf{A}[\mathbf{A}^2\mathbf{B}] = \begin{bmatrix} -0.01 & 0.1 & 0 & -32.2 \\ -0.40 & -0.8 & 180 & 0 \\ 0 & -0.003 & -0.5 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix} \begin{bmatrix} 40.57 \\ 654.6 \\ 0.773 \\ 1.43 \end{bmatrix}$$

$$\mathbf{A}^3\mathbf{B} = \begin{bmatrix} 19.008 \\ -400.77 \\ -2.35 \\ 0.773 \end{bmatrix}$$

Substituting the column matrices into the definition of V yields

$$\mathbf{V} = \begin{bmatrix} 0 & -1.0 & 40.57 & 19.008 \\ -10 & 496 & 654.6 & -400.77 \\ -2.8 & 1.43 & 0.773 & -2.35 \\ 0 & -2.8 & 1.43 & 0.773 \end{bmatrix}$$

The transformation matrix W is required if the plant matrix A is not in the companion form. In this example, the system matrix is not in companion form; therefore, the transformation matrix must be developed. The W is defined in terms of the coefficients of the characteristic equation of the plant matrix. For the present system, W is defined as

$$\mathbf{W} = \begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1.31 & 0.993 & 0.0294 \\ 0 & 1 & 1.31 & 0.993 \\ 0 & 0 & 1 & 1.31 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we have to find the matrix $(VW^T)^{-1}$. This will be accomplished in the following steps:

$$\mathbf{VW} = \begin{bmatrix} 0 & -1 & 40.57 & 19.008 \\ -10.0 & -496 & 654.6 & -400.781 \\ -2.8 & 1.43 & 0.773 & -2.35 \\ 0 & -2.8 & 1.43 & 0.773 \end{bmatrix} \begin{bmatrix} 1 & 1.31 & 0.993 & 0.0294 \\ 0 & 1.0 & 1.31 & 0.993 \\ 0 & 0 & 1.0 & 1.31 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\mathbf{VW} = \begin{bmatrix} 0 & -1.0 & 39.23 & 71.2 \\ -10.0 & -509.1 & -5.01 & -36.1 \\ -2.8 & -2.24 & -0.134 & 0 \\ 0 & -2.8 & -2.24 & -0.13 \end{bmatrix}$$

$$\Rightarrow (\mathbf{VW})^T = \begin{bmatrix} 0 & -10.0 & -2.8 & 0 \\ -1.0 & -509 & -2.234 & -2.8 \\ 39.3 & -5.1 & -0.13 & -2.24 \\ 71.2 & -36.1 & 0.0 & -0.13 \end{bmatrix}$$

The value of $(VW^T)^{-1}$:

$$([\mathbf{VW}]^T)^{-1} = \begin{bmatrix} 0.0008 & -0.0010 & 0.0004 & 0.0138 \\ 0.0014 & -0.0019 & 0.0025 & -0.0014 \\ -0.3622 & 0.0068 & -0.0089 & 0.0050 \\ 0.0321 & -0.0135 & -0.4446 & 0.2451 \end{bmatrix}$$

The state feedback gains matrix k now obtained as:

$$\begin{aligned} \mathbf{k} &= ([\mathbf{VW}]^T)^{-1}(\bar{\mathbf{a}} - \mathbf{a}) \\ &= \begin{bmatrix} 0.0008 & -0.0010 & 0.0004 & 0.0138 \\ 0.0014 & -0.0019 & 0.0025 & -0.0014 \\ -0.3622 & 0.0068 & -0.0089 & 0.0050 \\ 0.0321 & -0.0135 & -0.4446 & 0.2451 \end{bmatrix} \left(\begin{bmatrix} 3.61 \\ 9.05 \\ 0.126 \\ 0.09 \end{bmatrix} - \begin{bmatrix} 1.31 \\ 0.993 \\ 0.0294 \\ 0.0386 \end{bmatrix} \right) \\ &= \begin{bmatrix} -0.0055 \\ -0.0120 \\ -0.7785 \\ -0.0656 \end{bmatrix} \end{aligned}$$

The stability augmentation control law:

$$\begin{aligned} \Delta\delta_e &= -\mathbf{k}^T\mathbf{x} = -[-0.0055 \; -0.0120 \; -0.7785 \; -0.0656] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} \\ &= 0.0055\,\Delta u + 0.0120\,\Delta w + 0.7785\,\Delta q + 0.0656\,\Delta\theta \end{aligned}$$

Lateral Stability Augmentation - Autopilot

The lateral state equations are expressed in state-space form as follows:

$$\begin{bmatrix} \Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} Y_v & 0 & -u_0 & g \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix} + \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix}$$

Till now we have handled with single input system. In the present, we will designing the control for multiple input system. The modified control law can be expressed in terms of a constant row matrix, g, the gain vector, k, and the pilot's input, η_p:

$$\boldsymbol{\eta} = -\mathbf{g}\mathbf{k}^T\mathbf{x} + \boldsymbol{\eta}_p$$

Here the procedure is identical to that for the longitudinal equations. The constant vector g establishes the relationship between the aileron and rudder for augmentation. Either g_1 or g_2 is equal to 1, and the ratio $g_1/g_2 = {}^{\Delta\delta_a}/_{\Delta\delta_r}$, is specified by control deflection limits. Substituting the control vector into the state equation yields

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{g}\mathbf{k}^T)\mathbf{x} + \mathbf{B}\boldsymbol{\eta}_p$$
$$\dot{\mathbf{x}} = \mathbf{A}^*\mathbf{x} + \mathbf{B}\boldsymbol{\eta}_p$$

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{g}\mathbf{k}^T$$

Altitude Hold- Autopilot

Here, we demonstrate how the state feedback design approach can be used to design an altitude hold autopilot.

Example:

Use state feedback to design an autopilot to maintain a constant altitude. To simplify this problem, we will assume that forward speed of the airplane, u_0, is held fixed by a separate velocity control system and furthermore we neglect the control surface actuator dynamics. The airplane selected for this example is the STOL transport. Substituting the numerical values of the stability derivatives for the STOL transport yields:

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \\ \Delta\dot{h} \end{bmatrix} = \begin{bmatrix} -1.397 & 1 & 0 & 0 \\ -5.47 & -3.27 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -400 & 0 & 400 & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \\ \Delta\theta \\ \Delta h \end{bmatrix} + \begin{bmatrix} -0.124 \\ -13.2 \\ 0 \\ 0 \end{bmatrix} [\Delta\delta_e]$$

In state feedback design, the designer can specify the desired location of the eigenvalues. For this example we choose to locate the eigenvalues at

$$\lambda_{1,2} = -1.0 \pm 3.5i$$
$$\lambda_{3,4} = -2.0 \pm 1.0i$$