State Feedback Controller Design

State Feedback and Stabilization

Stabilization by State Feedback: Regulator Case

Plant: $\dot{x} = Ax + Bu, \quad x(t_0) = x_0$

State Feedback Law: $u = -Kx$

Closed-Loop System:
$$\dot{x} = Ax + Bu = Ax - BKx = \left(A - BK\right)x$$

Theorem

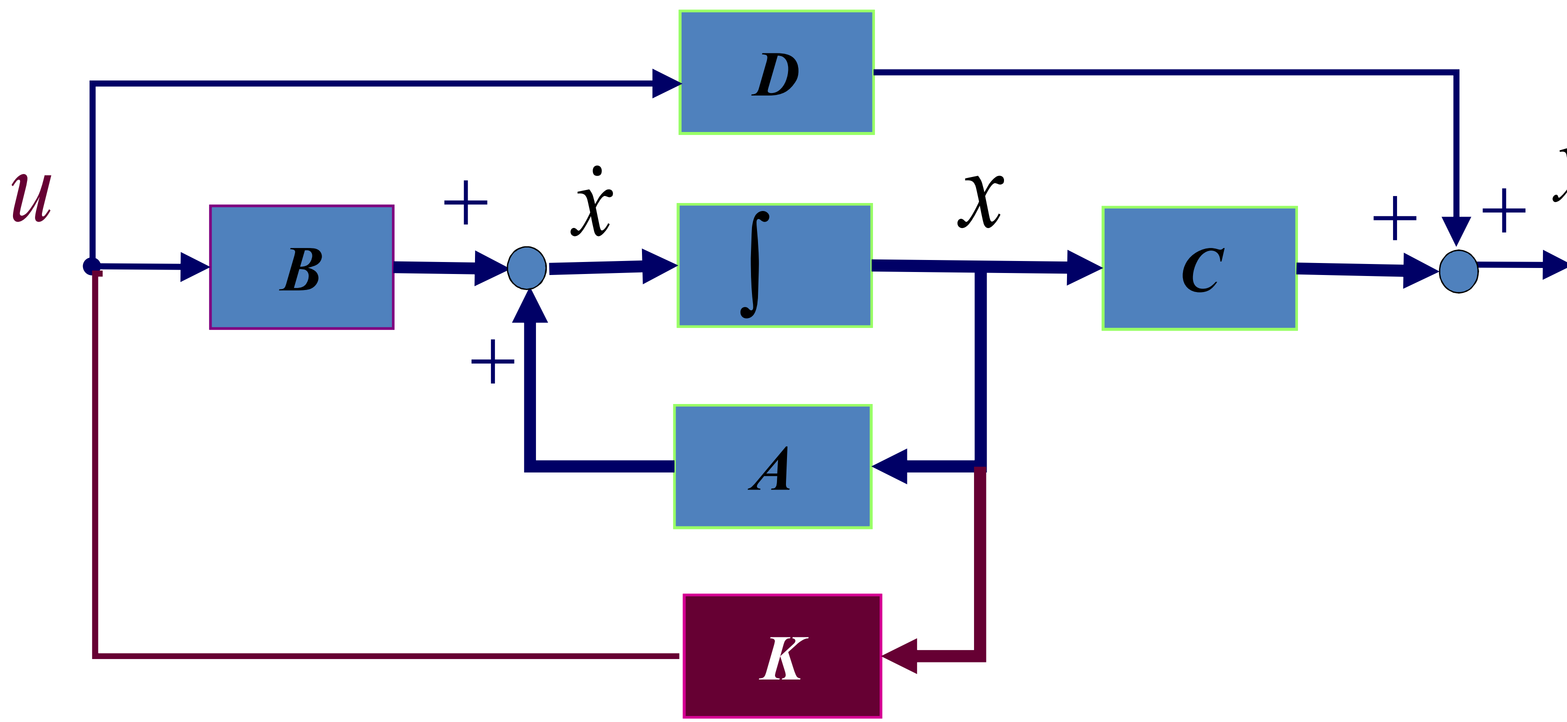
Given $\lambda_i, \quad i = 1, \cdots, n$

(A,B) Controllable $\Leftrightarrow$

There exists a state feedback matrix, F, such that

$$\det\left(\lambda I - \left(A - BK\right)\right) = \left(\lambda - \lambda_1\right)\left(\lambda - \lambda_2\right)\cdots\left(\lambda - \lambda_n\right)$$

State Feedback System (Regulator Case)



State Feedback Design in Controllable Form

$$A_c = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$u = -Kx = \begin{bmatrix} -k_1 & -k_2 & \cdots & -k_n \end{bmatrix} x$$

$$A_c - B_c K = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(a_1 + k_1) & -(a_2 + k_2) & \cdots & -(a_n + k_n) \end{bmatrix}$$

$$\det(sI - (A_c - B_c k)) = s^n + (a_n + k_n)s^{n-1} + \cdots + (a_1 + k_1)$$

Suppose the desired characteristic polynomial

$$\prod_{i=1}^{n}(s-\lambda_i)=s^n+a_{cn}s^{n-1}+\cdots+a_{c1}$$

Comparing

$$k_n=a_{cn}-a_n$$
$$k_{n-1}=a_{cn-1}-a_{n-1}$$
$$\vdots$$
$$k_1=a_{c1}-a_1$$

$$K=\begin{bmatrix}a_{c1}-a_1 & a_{c2}-a_2 & \cdots & a_{cn}-a_n\end{bmatrix}$$

Example: State Feedback System (Regulator Case)

Objective: Pick K so that A_{cl} has the design property
1. A is unstable, we want A_{cl} is stable
2. Put 2 poles at -2±2i

Consider the system $\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$$\det(sI - A) = (s-1)(s-2) - 1 = s^2 - 3s + 1 = 0 \Rightarrow system\ is\ unstable$$

$$Define: u = -[k_1 \;\; k_2]x(t) = -Kx(t), then$$

$$A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \;\; k_2] = \begin{bmatrix} 1-k_1 & 1-k_2 \\ 1 & 2 \end{bmatrix}$$

$$\det(sI - A_{cl}) = s^2 + (k_1 - 3)s + (1 - 2k_1 + k_2)$$

Thus, by choosing k_1 & k_2, we can put $\lambda_i(A_{cl})$ anywhere in the complex plane. Now let put the poles at s=-5,-6 in the complex plane. The desired characteristic equation is

$$(s+5)(s+6) = s^2 + 11s + 30 = 0$$

Example: State Feedback System (Regulator Case)

Compare the above equation with the closed loop charateristic equation $s^2 + (k_1 - 3)s + (1 - 2k_1 + k_2) = 0$

Yelds

$$\left.\begin{aligned} k_1 - 3 &= 11 \\ 1 - 2k_1 + k_2 &= 30 \end{aligned}\right\} => k_1 = 14, k_2 = 57$$

So K=[14 57] is the desired values of F coming through the concept of pole placement design

Example2: State Feedback System (Regulator Case)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

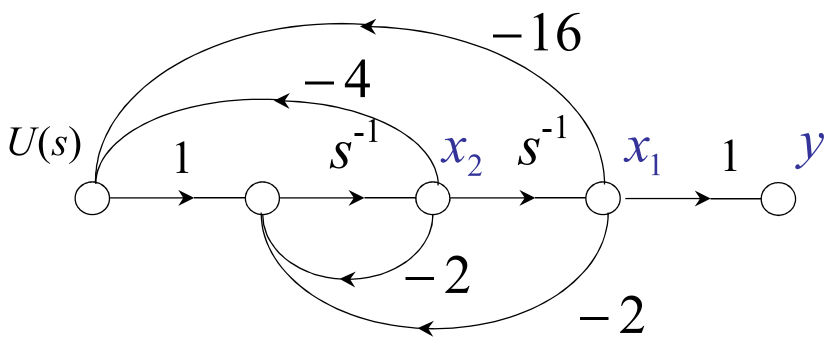
Desired poles : $-3 \pm 3j$

Desired Char. Polynomial : $(s+3)^2 + 3^2 = s^2 + 6s + 18$

(A, B) is in controllable from, we can derive the state feedback gain as

$$K = \begin{bmatrix} 18-2 & 6-2 \end{bmatrix} = \begin{bmatrix} 16 & 4 \end{bmatrix}$$

$$u = -Kx = -\begin{bmatrix} 16 & 4 \end{bmatrix} x$$



State Feedback Design with Transformation to Controllable Form

$$\text{Desired poles}: \lambda_1, \cdots, \lambda_n$$

$$\text{Desired Char. Poly.}: \prod_{i=1}^{n} (s - \lambda_i)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$x = Tz$$

Controllable From:

$$\dot{z} = A_c z + B_c u$$
$$y = C_c z$$

$$A_c = T^{-1}AT$$
$$B_c = T^{-1}B$$
$$C_c = CT$$

$$u = -K_c z$$

$$u = -K_c z = -K_c T^{-1} x$$

$$K_c = \begin{bmatrix} a_{c1} - a_1 & a_{c2} - a_2 & \cdots & a_{cn} - a_n \end{bmatrix}^T$$

$$\det(sI - (A - BF)) = \prod_{i=1}^{n} (s - \lambda_i) = s^n + a_{cn} s^{n-1} + \cdots + a_{c2} s + a_{c1}$$

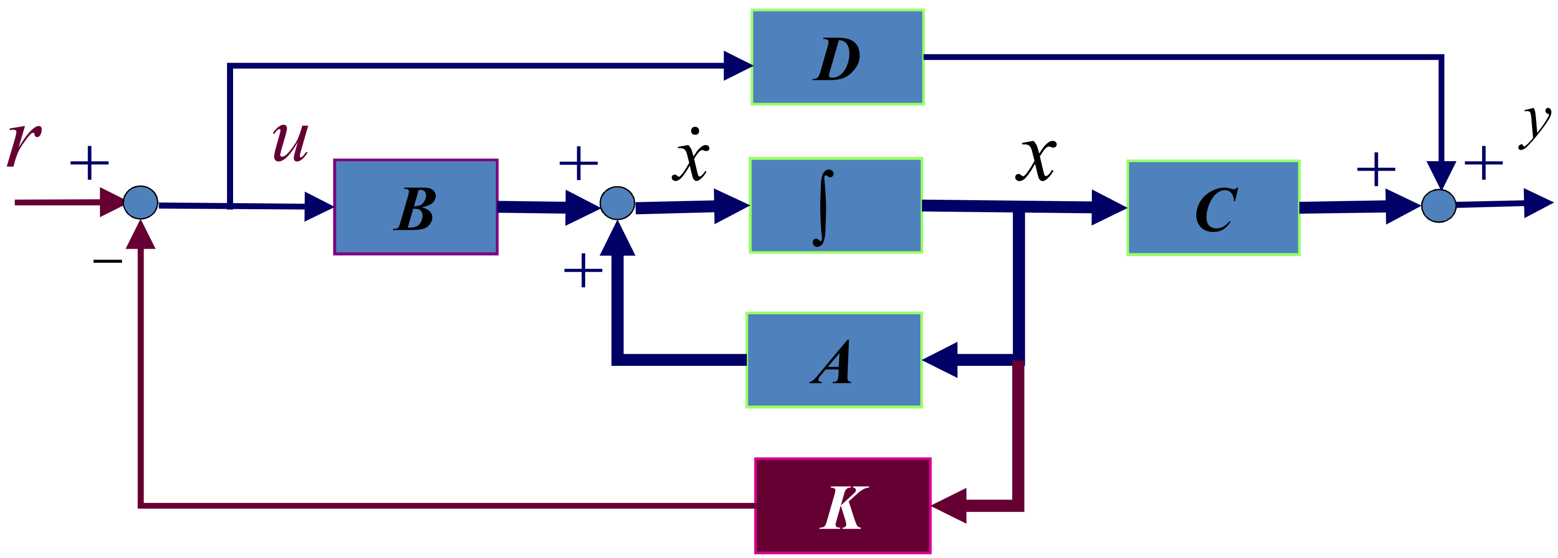
$$A_c = \begin{bmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

State Feedback: General Case (Non-Zero Input Case)

$$u = -Fx + r$$

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}$$



State Feedback Control System

Obtain the State Feedback Matrix by Comparing Coefficients

Plant:

$$\dot{x} = Ax + Bu \qquad x \in R^{n\times 1} \qquad u \in R$$
$$y = Cx \qquad y \in R$$

State Feedback: $u(t) = -Kx(t) + r(t)$ $\quad K \in R^{1\times n}$

Closed Loop System: $\dot{x} = (A - BK)x + Br$

Char. Equation: $|sI - A + BK| = 0$

Suppose that the system is controllable, i.e. $\mathrm{rank}\,[B \;\; AB \;\; A^2B \cdots A^{n-1}B] = n$

Then, for any desired pole locations: $\lambda_1, \cdots, \lambda_n$

We can obtain the desired char. polynomial $(s-\lambda_1)\cdots(s-\lambda_n)$

By controllability, there exists a state feedback matrix K, such that

$$\left|sI - A + BK\right| = (s-\lambda_1)\cdots(s-\lambda_n)$$

From the abve equation we can solve for the state feedback gain K.

Example

Plant: $\frac{Y(s)}{U(s)} = \frac{8}{s(s+1)(s+10)}$

State Feedback: $u(t) = -Kx(t)$ $K = [k_1 \quad k_2 \quad k_3]$

$$\frac{Y(s)}{R(s)} = \frac{-8k_1}{s^3 + (11+8k_3)s^2 + (10+8k_2)s - 8k_1}$$

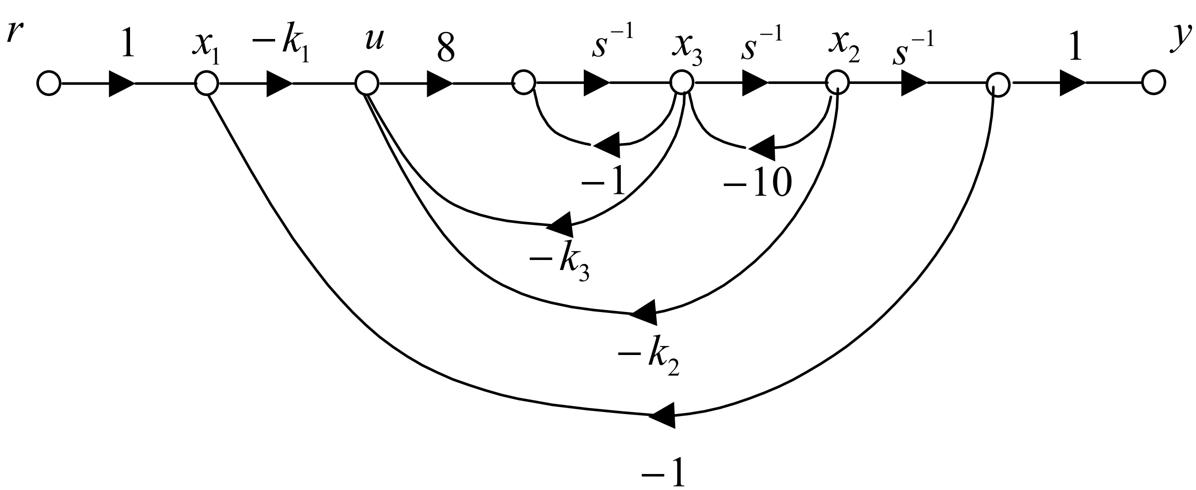


Fig. State Feedback Design Example

Spec. for Step Response:

Percent Overshoot 5%, Settling Rise time 5 sec.

$$\zeta\omega_n = 8, \quad \zeta = 0.707$$

Desired pole locations:

$$s_{1,2} = -8 \pm j8 \text{ (dominant ploes)}$$

$$s_3 = -40$$

Comparing with the desired dynamics

$$s^3 + (11 + 8k_3)s^2 + (10 + 8k_2)s - 8k_1 = (s + 8 - j8)(s + 8 + j8)(s + 40)$$

By comparing coefficients on the both sides, we obtain

$$k_1 = \text{-}640 \qquad k_2 = 94.75 \qquad k_3 = 5.625$$

$$\Rightarrow \quad K = \begin{bmatrix} \text{-}640 & 94.75 & 5.625 \end{bmatrix}$$

Simulation Results

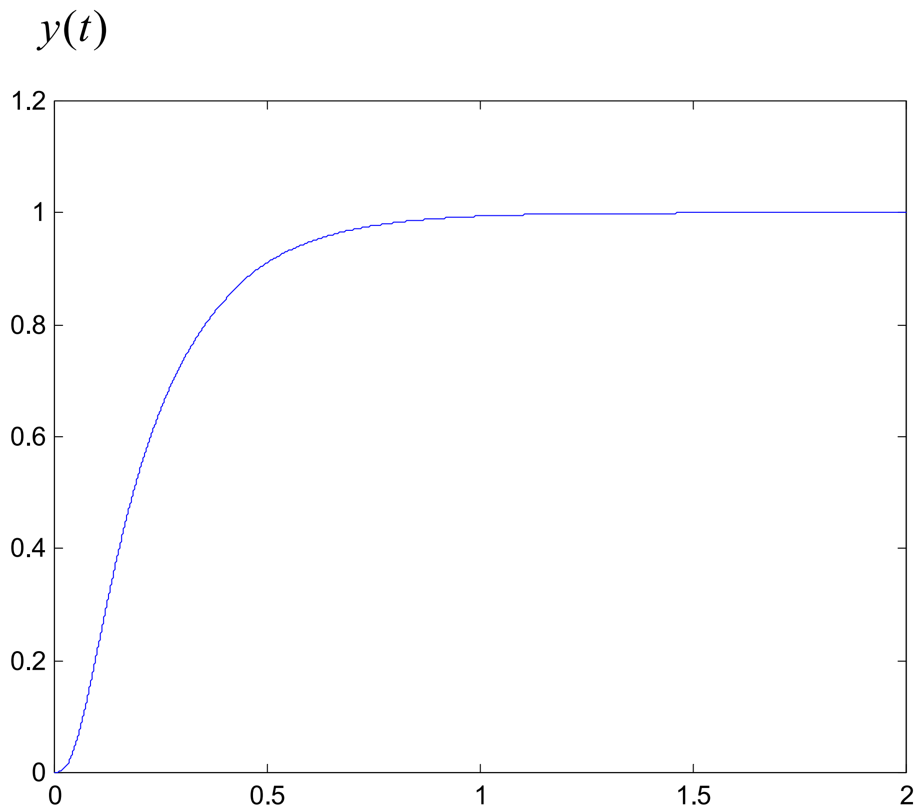


Fig. Step response of above example

Ackermann Formula for controller design

Plant:
$$\dot{x} = Ax + Bu \qquad x \in R^{n\times 1} \qquad u \in R$$
$$y = Cx \qquad y \in R$$

State Feedback: $u(t) = -Kx(t) + r(t)$ $\quad K \in R^{1\times n}$

Desired poles : $\lambda_1, \cdots, \lambda_n$ Desired Char. Poly. : $\prod_{i=1}^{n}(s-\lambda_i) = s^n + a_{cn}s^{n-1} + \cdots + a_{c2}s + a_{c1}$

The Matrix Polynomial
$$\alpha_c(A) = \prod_{i=1}^{n}(A-\lambda_i) = A^n + a_{cn}A^{n-1} + \cdots + a_{c2}A + a_{c1}I$$

Then the state feedback gain matrix is

$$K = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}}_{\in R^{1\times n}} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}^{-1} \alpha_c(A)$$

Ackermann Formula for controller design

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{2} + 11 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 30I \right)$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 43 & 14 \\ 14 & 57 \end{bmatrix} = \begin{bmatrix} 14 & 57 \end{bmatrix}$$