## Examples and Solutions

**Example** Write the following differential equations in state-space form

$$\frac{d^2c_1}{dt^2} + 5\frac{dc_1}{dt} + 4c_2 = r_1$$

$$\frac{dc_2}{dt} + \frac{dc_1}{dt} + c_1 + 3c_2 = r_2$$

**Solution** Let the states be

$$x_1 = c_1;\ x_2 = \frac{dc_1}{dt};\ x_3 = c_2$$

$$\dot{x}_1 = \frac{dc_1}{dt} = x_2$$

$$\dot{x}_2 = \frac{d^2c_1}{dt^2} = -5\frac{dc_1}{dt} - 4c_2 + r_1$$

$$\dot{x}_2 = -5x_2 - 4x_3 + r_1$$

$$\dot{x}_3 = \frac{dc_2}{dt} = -\frac{dc_1}{dt} - c_1 - 3c_2 + r_2$$

$$\dot{x}_3 = -x_2 - x_1 - 3x_3 + r_2$$

These equations can now be put into state-space form $\dot{x} = Ax + Bu$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -5 & -4 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

The output eqution: y= $Cx + Du$:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example

Given the following state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Determine the response of the system if u is a step function

Solution

First we need to determine the state transition matrix. The Laplace transform of the state transition matrix yields

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

$$\mathbf{\Phi}(s) = \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right)^{-1}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}$$

$$\mathbf{\Phi}(s) = \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ \dfrac{-2}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix}$$

Or

$$\mathbf{\Phi}(s) = \begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2[e^{-t} - e^{-2t}] & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Response

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\,\mathbf{B}u\,d\tau$$

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix} + \begin{bmatrix} 2\int_0^t (e^{-(t-\tau)} - e^{-2(t-\tau)})\,d\tau \\ 2\int_0^t (2e^{-2(t-\tau)} - e^{-(t-\tau)})\,d\tau \end{bmatrix}$$

The integrals of the second term can be obtained as

$$
\begin{aligned}
2\int_0^t [e^{-(t-\tau)} - e^{-2(t-\tau)}]\, d\tau &= 2e^{-t}\int_0^t e^{\tau}\, d\tau - 2e^{-2t}\int_0^t e^{2\tau}\, d\tau \\
&= 2e^{-t}e^{\tau}\Big|_0^t - (2e^{-2t})\frac{e^{2\tau}}{2}\Big|_0^t \\
&= 2e^{-t}[e^{t} - 1] - 2e^{-2t}\left[\frac{e^{2t}}{2} - \frac{1}{2}\right] \\
&= 1 - 2e^{-t} + e^{-2t}
\end{aligned}
$$

$$
2\int_0^t (2e^{-2(t-\tau)} - e^{-(t-\tau)})\, d\tau = 2e^{-t} - 2e^{-2t}
$$

Substituting the integral solution back into x(t) and combining terms yields

$$
\mathbf{x}(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}
$$

The output of the system is given by

$$
y = \mathbf{C}\mathbf{x}
$$

$$
y = [3 \quad 1]\begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix} = 3 - 2e^{-t}
$$

**Example**

Given the following state equations, determine the transformation matrix P so that the new state equations are in the state canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}[u]$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Solution**

First we need to find the eigenvalues of A

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

$$\left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0$$

$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -2 \quad \text{and} \quad \lambda = -1$$

The eigenvector for $\lambda = -1$

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{P}_i = 0$$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right)\begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} = 0$$

$$-P_{11} - P_{21} = 0$$

$$2P_{11} + 2P_{21} = 0$$

Both equations yield the same relationship between $p_{11}$ *and* $p_{22}$. We will arbitrarily selection $p_{11} = 1$ then $p_{22} = -1$

$$\mathbf{P}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvector for $\lambda = -2$

$$\left(\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right)\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} = 0$$

$$-2P_{12} - P_{22} = 0$$

$$2P_{12} + 1P_{22} = 0$$

$$\mathbf{P}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The transformation matrix P now can be constructed by stacking the eigenvectors as follows:

$$\mathbf{P} = [\mathbf{P}_1 \mathbf{P}_2]$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

To find the new state equation we need the following terms

$$\mathbf{P}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$\mathbf{\Lambda} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\bar{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}$$

$$\bar{\mathbf{B}} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\bar{\mathbf{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\bar{\mathbf{C}} = \mathbf{C}\mathbf{P}$$

$$\bar{\mathbf{C}} = [3 \quad 1]\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\bar{\mathbf{C}} = [2 \quad 1]$$

The new state equations are

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix}[u]$$

$$y = [2 \quad 1]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \mathbf{P}^{-1}\mathbf{x}(0)$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In this example, the transformed plant matrix is purely diagonal matrix having the eigenvalues of the original A matrix along the diagonal. For this particular case, the state transition matrix cab be shown to be the following

$$\mathbf{\Phi}(t) = e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

The solution of the transformed state equations would be

$$\mathbf{z}(t) = \mathbf{\Phi}(t)\mathbf{z}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\,\bar{\mathbf{B}}\boldsymbol{\eta}(\tau)\,d\tau$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \int_0^t 2e^{-(t-\tau)}\,d\tau \\ -\int_0^t 2e^{-2(t-\tau)}\,d\tau \end{bmatrix}$$

$$= \begin{bmatrix} 2 - e^{-t} \\ -1 \end{bmatrix}$$

The output equation is given by

$$y = \bar{\mathbf{C}}\mathbf{z}$$

$$y = [2 \quad 1]\begin{bmatrix} 2 - e^{-t} \\ -1 \end{bmatrix}$$

$$y = 3 - 2e^{-t}$$

**Example**

The state equations for a system follow:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$
$$\mathbf{A} = \begin{bmatrix} -3 & 8 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$\mathbf{C} = [1 \quad 0]$$

Use state feedback control so that the closed-loop system has the following properties

$$\omega_n = 25 \text{ rad/s}$$
$$\zeta = 0.707$$

**Solution**

First we have to check the system controllability

$$\mathbf{V} = [\mathbf{B} \quad \mathbf{AB}]$$
$$\mathbf{AB} = \begin{bmatrix} -3 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 32 \\ 0 \end{bmatrix}$$
$$\mathbf{V} = \begin{bmatrix} 0 & 32 \\ 4 & 0 \end{bmatrix}$$
$$\det[\mathbf{V}] = -128$$

The rank is 2 and non-zero determinant, hence the system is state controllable

The desired characteristic equation for the closed-loop system can be written as

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

Substituting the numerical values of $\xi$ *and* $\omega_n$

$$\lambda^2 + 35.35\lambda + 625 = 0$$

The augmented matrix with state feedback control

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{k}^T$$

$$\mathbf{A}^* = \begin{bmatrix} -3 & 8 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \end{bmatrix} [k_1 \quad k_2]$$
$$= \begin{bmatrix} -3 & 8 \\ -4k_1 & -4k_2 \end{bmatrix}$$

The eigenvalues of the augmented matrix can be determined as follow

$$|\lambda \mathbf{I} - \mathbf{A}^*| = 0$$
$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 8 \\ -4k_1 & -4k_2 \end{bmatrix} \right| = 0$$
$$\begin{vmatrix} \lambda+3 & -8 \\ 4k_1 & \lambda+4k_2 \end{vmatrix} = 0$$
$$\lambda^2 + (3 + 4k_2)\lambda + 12k_2 + 32k_1 = 0$$

Comparing this with the desired characteristic equation

$$3 + 4k_2 = 35.35$$
$$12k_2 + 32k_1 = 625$$

$$k_1 = 16.5$$
$$k_2 = 8.09$$

**Example**

The state equations for a system follow:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\eta$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} -3 & 8 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Design a state observer to estimate the system states

**Solution**

First we have to check the observability matrix

$$\mathbf{U} = [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T]$$

$$\mathbf{A}^T\mathbf{C}^T = \begin{bmatrix} -3 & 0 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & -3 \\ 0 & 8 \end{bmatrix}$$

$$\det \mathbf{U} = \begin{vmatrix} 1 & -3 \\ 0 & 8 \end{vmatrix} = 8$$

$\Rightarrow$ The rank is 2 and non-zero determinant, hence the system is state observable

The observer is determined by solving the equation

$$|\lambda \mathbf{I} - (\mathbf{A} - \mathbf{k}_e\mathbf{C})| = 0 \Rightarrow \begin{vmatrix} \lambda + 3 + k_{e_1} & -8 \\ k_{e_2} & \lambda \end{vmatrix} = 0$$

$$\lambda^2 + (3 + k_{e_1})\lambda + 8k_{e_2} = 0$$

The dynamics of the observer must be faster than the system to be controlled. In this example, we assume that the observer roots are four times faster as desired closed loop performance.

Roots of the closed-loop system are

$$\lambda_{1,2} = -17.68 \pm 17.68i$$

Thus, the roots of the observer are chosen as

$$\lambda_{1,2_{OB}} = -70.72 \pm 70.72i$$

The desired characteristic equation for the observer is given as

$$(\lambda - \lambda_{1_{OB}})(\lambda - \lambda_{2_{OB}}) = 0$$

$$\lambda^2 + 141.44\lambda + 10{,}003 = 0$$

This allows us to select the observer gains by equating the desired characteristic equation to the observation characteristic equation in terms of observer gain

$$3 + k_{e_1} = 141.44 \Rightarrow k_{e_1} = 138.4$$

$$8k_{e_2} = 10{,}003 \Rightarrow k_{e_2} = 1250$$

**Example**

Find the control law that minimizes the performance index

$$J = \int_0^{\infty} (x_1^2 + x_2^2 + u^2)\, dt$$

For the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

**Solution**

First, we need to determine the weighting matrices Q and R. The performance index is expresses as

$$J = \int_0^{\infty} (\mathbf{x}^T \mathbf{Q} \mathbf{x} + \boldsymbol{\eta}^T \mathbf{R} \boldsymbol{\eta})\, dt$$

The first part of integration yields:

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} = [x_1 \quad x_2] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= x_1^2 Q_{11} + x_2 x_1 Q_{21} + x_2 x_1 Q_{12} + x_2^2 Q_{22}$$

IF we compare this with the original cost, $Q_{11} = Q_{22} = 1, Q_{12} = Q_{21} = 0$. In a similar manner, R=1. Thus the weighting matrices are

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = [1]$$

The Riccati equations: $\mathbf{A}^T\mathbf{S} + \mathbf{S}\mathbf{A} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{Q} = 0$

The terms of the Ricaati Equation are

$$\mathbf{A}^T\mathbf{S} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S_{11} & S_{12} \end{bmatrix}$$

$$\mathbf{S}\mathbf{A} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & S_{11} \\ 0 & S_{21} \end{bmatrix}$$

$$SBR^{-1}B^TS = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1][0 \quad 1]\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$
$$= \begin{bmatrix} s_{12}s_{21} & s_{12}s_{22} \\ s_{22}s_{21} & s_{22}^2 \end{bmatrix}$$

Riccati equation becomes

$$\begin{bmatrix} 0 & 0 \\ S_{11} & S_{12} \end{bmatrix} + \begin{bmatrix} 0 & S_{11} \\ 0 & S_{21} \end{bmatrix} - \begin{bmatrix} S_{12}S_{21} & S_{12}S_{22} \\ S_{22}S_{21} & S_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$-S_{12}S_{21} + 1 = 0$$
$$S_{11} - S_{12}S_{22} = 0$$
$$S_{12} + S_{21} - S_{22}^2 + 1 = 0$$
$$S_{12} = S_{21}$$
$$-S_{12}^2 + 1 = 0 \Rightarrow S_{12} = \pm\sqrt{1} = \pm 1$$
$$2S_{12} - S_{22}^2 + 1 = 0$$

For $s_{12} = 1$

$$-S_{22}^2 + 3 = 0 \Rightarrow S_{22} = \pm\sqrt{3}$$
$$S_{11} - S_{12}S_{22} = 0 \Rightarrow S_{11} = \pm\sqrt{3}$$

$$\mathbf{S} = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$\mathbf{k} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}$$
$$\mathbf{k} = [1][0 \quad 1]\begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} = [1 \quad \sqrt{3}]$$

$$\boldsymbol{\eta} = -\mathbf{k}^T\mathbf{x} = -[1 \quad \sqrt{3}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= -x_1 - \sqrt{3}x_2$$