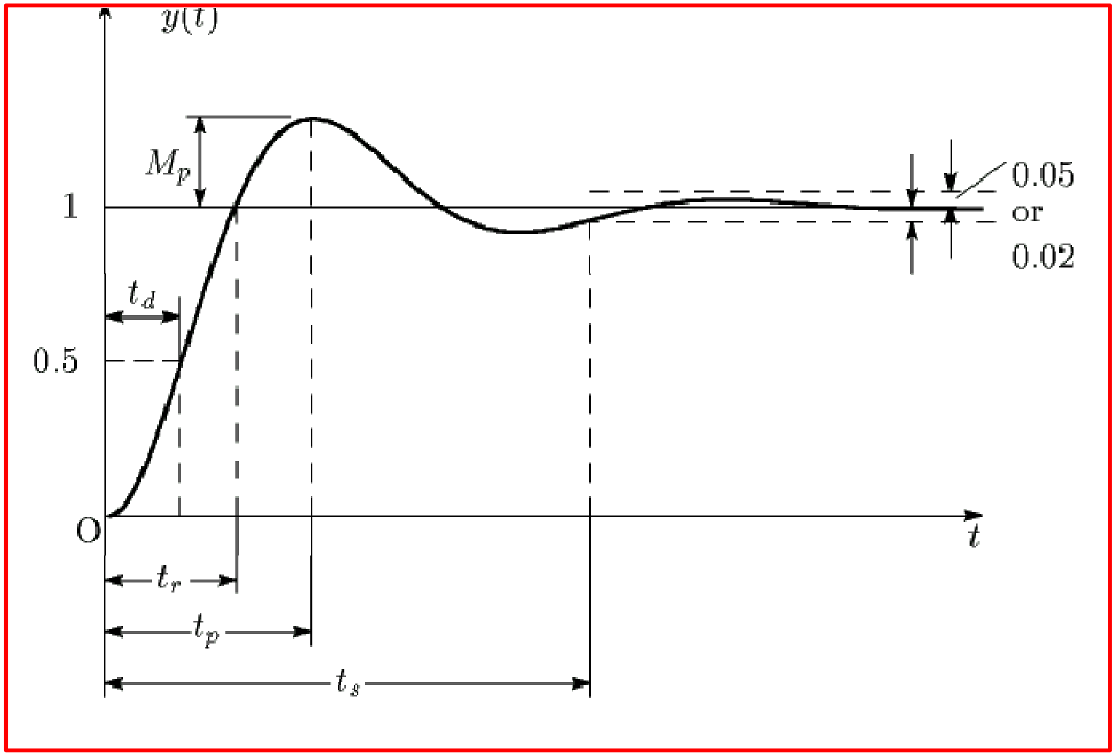
Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillation before reaching steady state. The transient and steady-state response analysis of a second order is shown in the following figure.



In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

Delay Time (t_d): The delay time is the time required for the response to reach half the final value of the very first time.

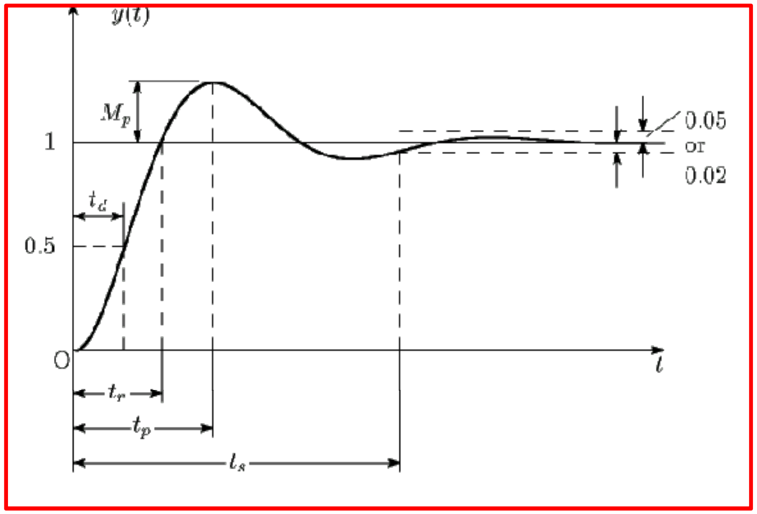
Transient-Response Specifications

Rise Time (t_r): The rise time is the time required for the response to rise from 10% to 90% or 5% to 95% or 0% to 100% of its final value. For underdamped second order system → $0\ to\ 100\%$ and overdamped → $10\%\ to\ 90\%$.

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

$$\Rightarrow \cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0 \Rightarrow \tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\Rightarrow t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$



Transient-Response Specifications

Peak Time (t_p): The peak time is the time required for the response to reach the first peak of the overshoot. We may obtain the peak time by differentiating $y(t)$ with respect to time and letting this derivative equal to zero.

$$\frac{dy(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

Cosine terms in the above equation cancel each other and $dy(t)/dt$ evaluated at $t = t_p$

$$\left.\frac{dy(t)}{dt}\right|_{t=t_p} = \left(\sin\omega_d t_p\right)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

$$\Rightarrow \sin\omega_d t_p = 0$$

$$\Rightarrow \omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots\ldots$$

$$\Rightarrow$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Since the peak time corresponds to the first peak overshoot.

Transient-Response Specifications

Maximum Overshoot (M_p): The maximum overshoot is the maximum peak value of the response curve measured from unity.

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$

$$\begin{aligned} M_p &= y(t_p) - 1 \\ &= -e^{-(\zeta\omega_n(\pi/\omega_d))}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) \\ &= e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} \end{aligned}$$

If the final value $y(\infty)$ of the output is not unity, then we need to use the following equation

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

Transient-Response Specifications

Settling Time (t_p): The settling time is the time required for the response curve to reach and stay with in a range about the final value of size specified by absolute percentage of the final value (usually, 2% or 5%).

The settling time corresponding to a $\pm 2\%$ or $\pm 5\%$ tolerance band may be measured in terms of the time constant $T = 1/\zeta\omega_n$.

For $0 < \zeta < 0.9$, if the $\pm 2\%$ criterion used, t_s is approximately four times the time constant of the system. If the $\pm 5\%$ criterion used, t_s is approximately three times the time constant of the system.

For convenience in comparing the response of systems, we commonly define the settling time to be

$$t_s = 4T = \frac{4}{\zeta\omega_n} \qquad (2\%)$$

$$t_s = 3T = \frac{3}{\zeta\omega_n} \qquad (5\%)$$

Transient-Response Specifications

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \cong \frac{1.8}{\omega_n}$$

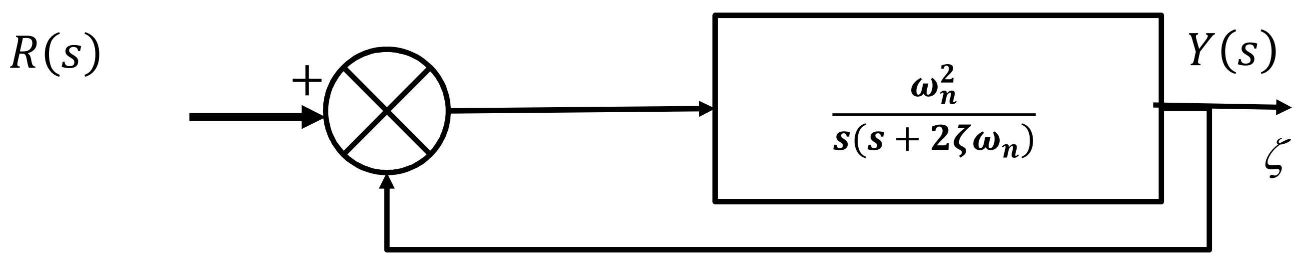
$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$M_p = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi}$$

$$t_s = 4T = \frac{4}{\zeta\omega_n} \qquad (2\%)$$

$$t_s = 3T = \frac{3}{\zeta\omega_n} \qquad (5\%)$$

Example:



$\zeta = 0.6;\ \omega_n = 5$

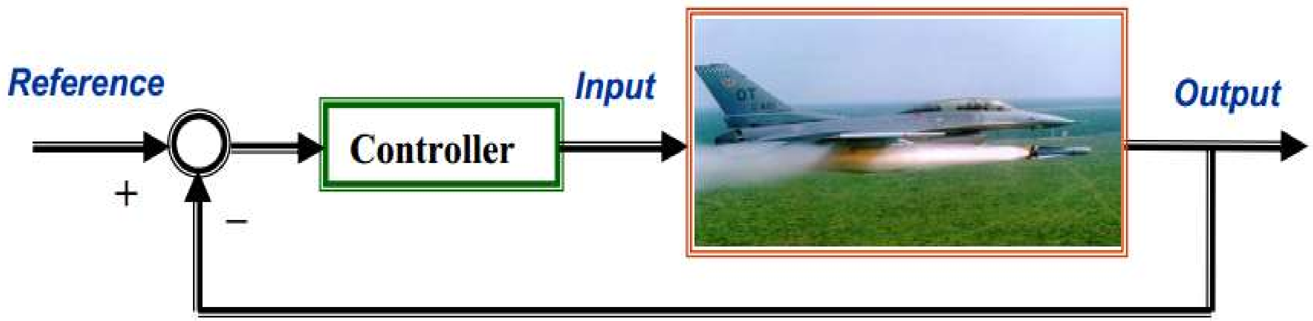
find t_r, t_p, t_s, M_p

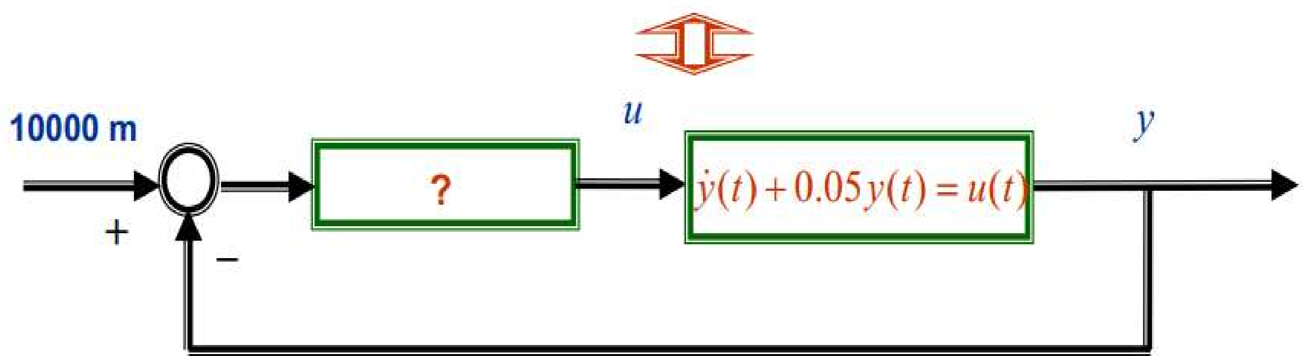
Control of Fighter Aircraft

The vertical position of a fighter aircraft can be approximated by the following equation:

$$\dot{y}(t) + 0.05y(t) = u(t)$$

$y(t)$ is the vertical position (in meters) and $u(t)$ is the thrust force

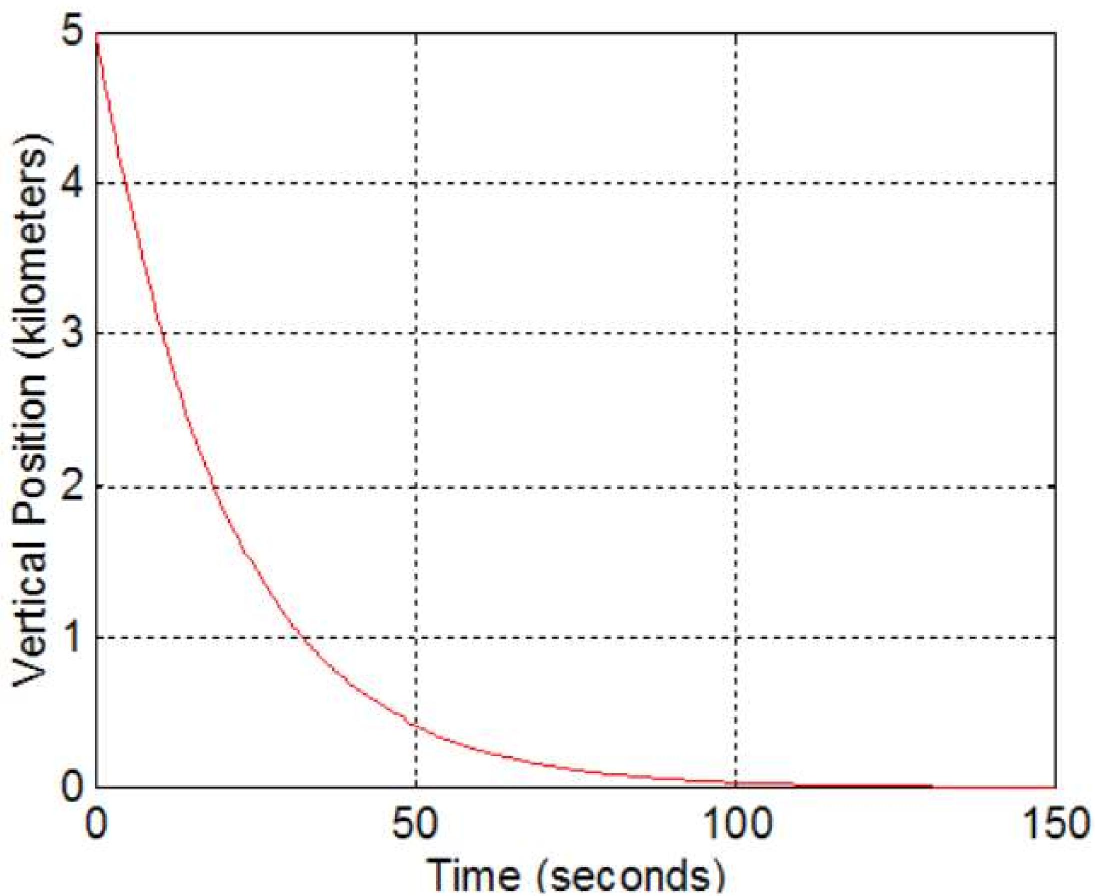




Control of Fighter Aircraft

For $u(t) = 0$, the equation becomes homogenous. For initial condition $y(0) = 5000$, the solution is given by

$$y(t) = 5000e^{-0.05t}$$



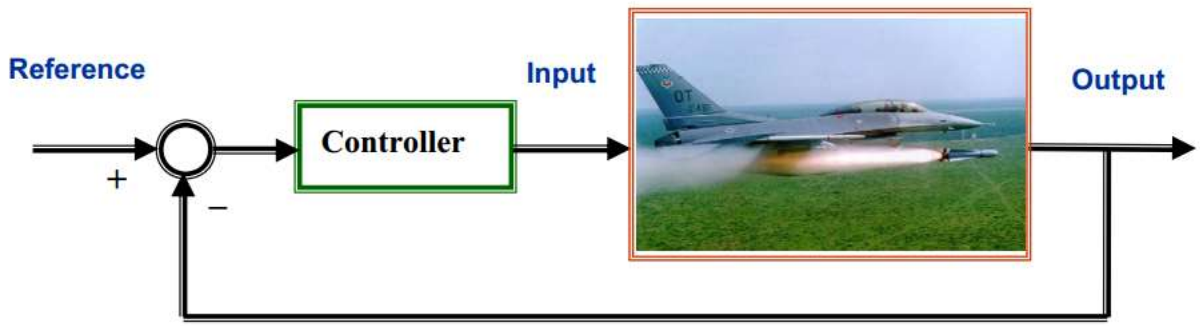
Control of Fighter Aircraft

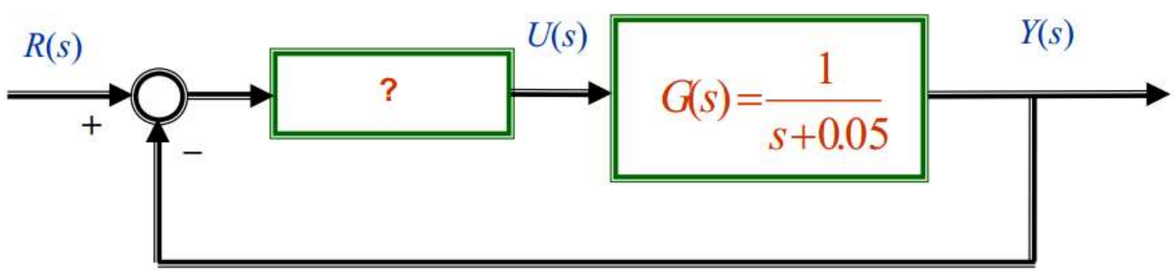
Now, if we consider non-homogeneous or apply some control to the aircraft, the equation of the aircraft becomes

$$\dot{y}(t) + 0.05y(t) = u(t)$$

$$L\{\dot{y} + 0.05y\} = L\{u(t)\} \Rightarrow sY(s) + 0.05Y(s) = U(s)$$

$$\Rightarrow (s + 0.05)Y(s) = U(s) \Rightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s + 0.05}$$





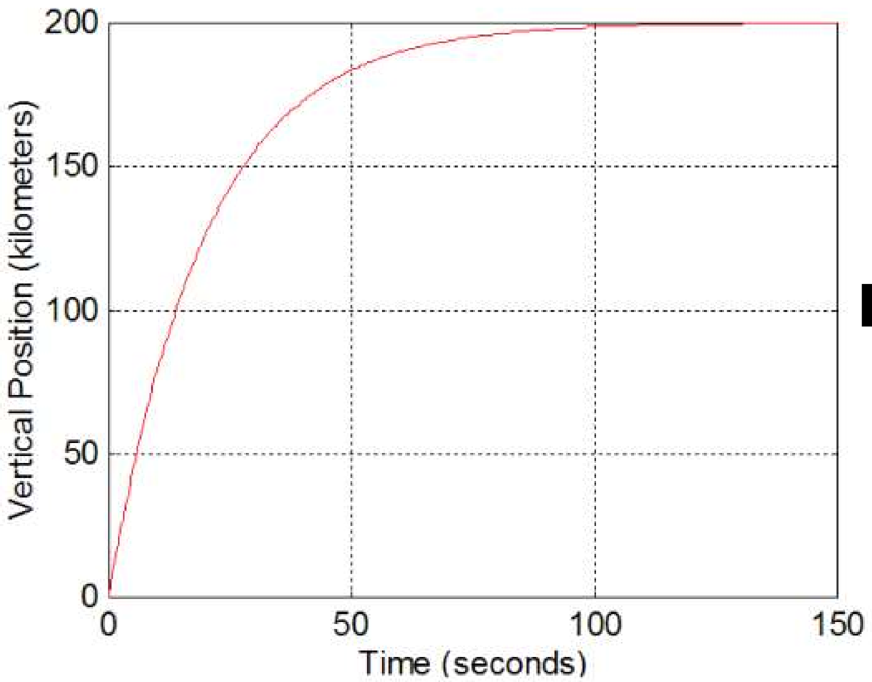
Control of Fighter Aircraft

If we want the aircraft to reach 10000 m above the sea level without a controller, one might guess that we need to apply an input force $u(t) = 10000m$. Let us try to see whether this works or not.

$$U(s) = L\{10000\} = \frac{10000}{s}$$

$$Y(s) = G(s)U(s) = \frac{1}{s+0.05}\frac{10000}{s} = 200000\left(\frac{1}{s} - \frac{1}{s+0.05}\right)$$

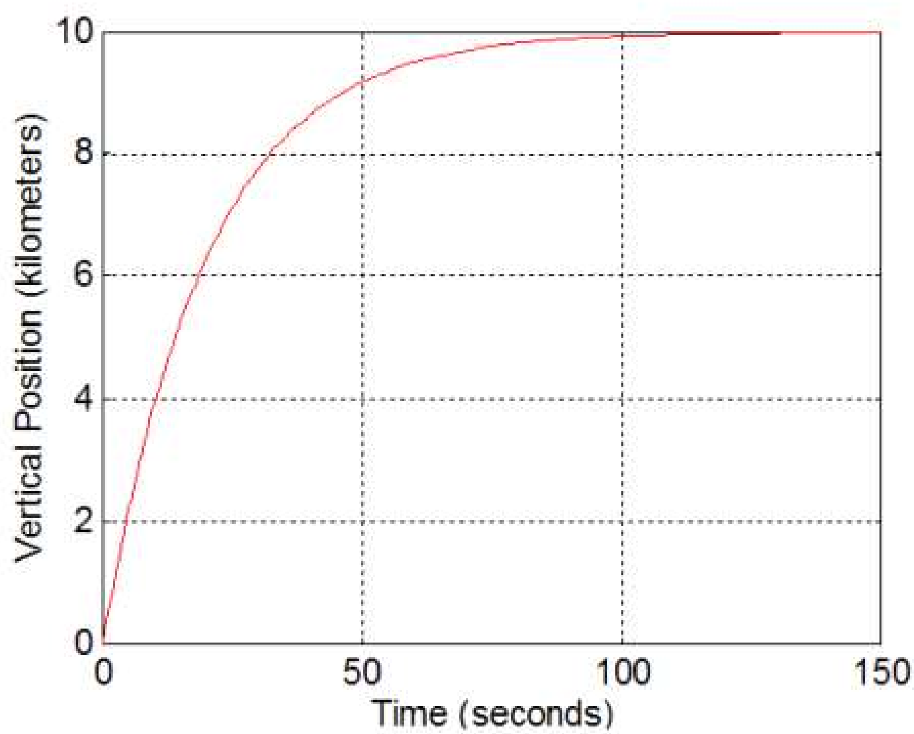
$$\Rightarrow y(t) = 200000\left(1 - e^{-0.05t}\right)$$



The vertical position of the aircraft: It reaches 200000 m instead of the desired 10000 m

Control of Fighter Aircraft

If we choose $u(t) = 100$, the resulting will be $y(t) = 10000\left(1 - e^{-0.05t}\right)$



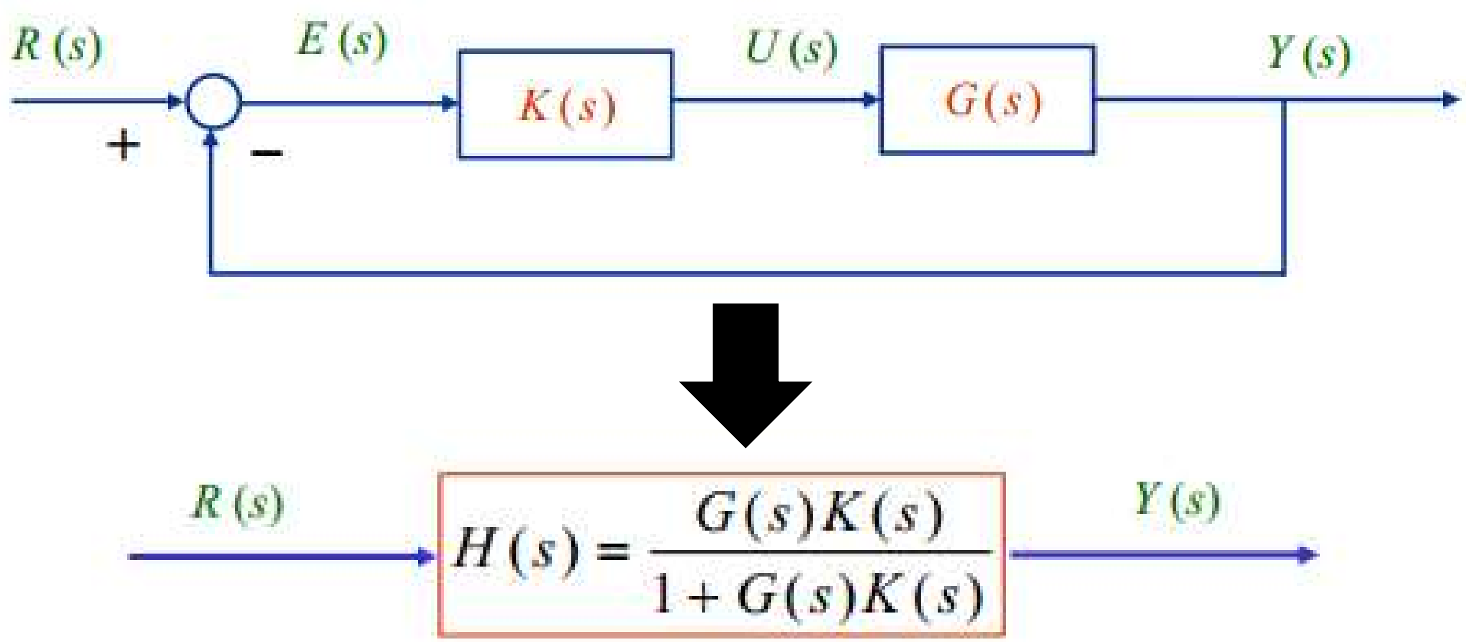
The vertical position of the aircraft: It reaches 10000m

It can be observed that the aircraft will reach the desired level in about 130 seconds. Can we improve this?

The answer is yes and the solution is to use a feedback controller.

Control of Fighter Aircraft

Feedback Controller: In general, a feedback control system can be represented by the following block diagram



Given a system represented by $G(s)$ and a reference $R(s)$, the objective of control system design is to find a control law (or controller) $K(s)$ such that the resulting output $Y(s)$ is as close to reference $R(s)$ as possible, or the error $E(s) = R(s) - Y(s)$ is as small as possible

Control of Fighter Aircraft

Lets replace the aircraft system T.F. to $G(s) = \frac{b}{s+b}$. The whole control problem becomes how to choose an appropriate $K(s)$ such that the resulting system would yield desired properties between $R(s)$ & $Y(s)$.

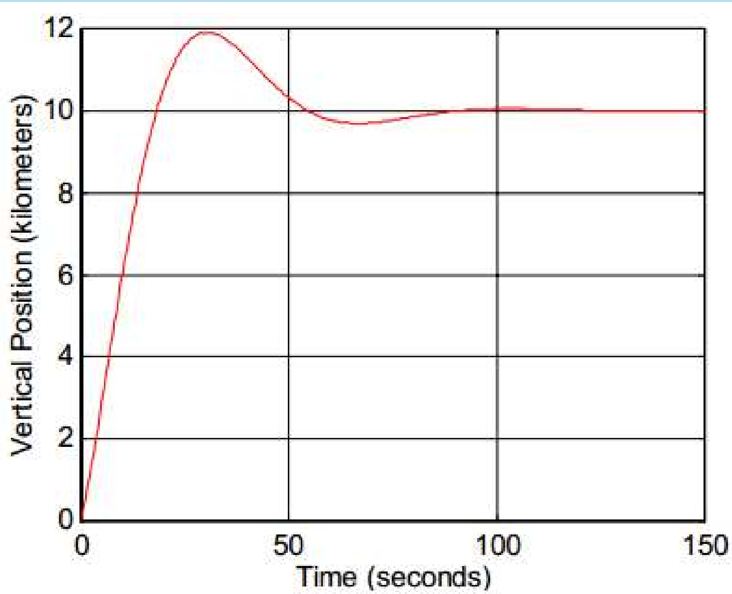
Lets consider a PI controller

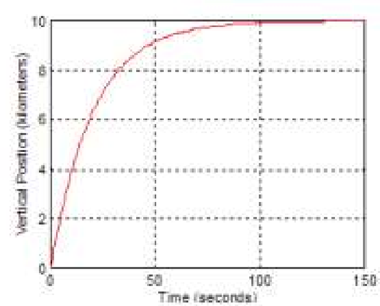
$$K(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

The resulting Transfer Function

$$H(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{bk_p s + bk_i}{s^2 + (a + bk_p)s + bk_i}$$

Results: The performance of the fighter aircraft with a PI controller ($k_p = 0.05, k_i = 0.01$)

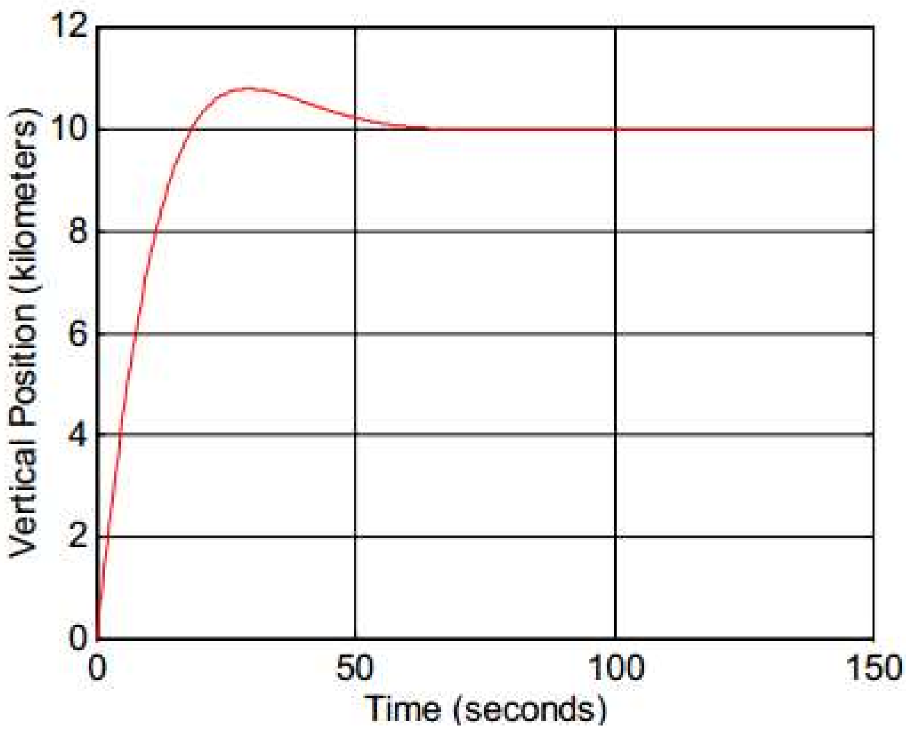


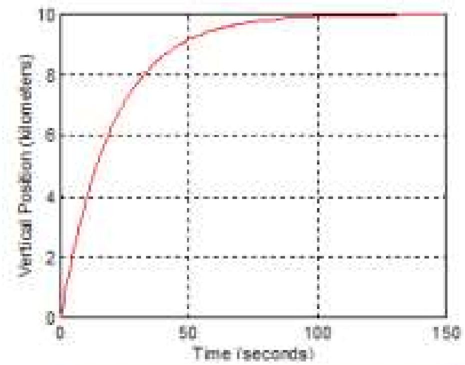


The response is much faster compared to the system without a controller. The drawback is that there is a 20% overshoot.

Control of Fighter Aircraft

Results: The performance of the fighter aircraft with a PI controller ($k_p = 0.05, k_i = 0.01$)

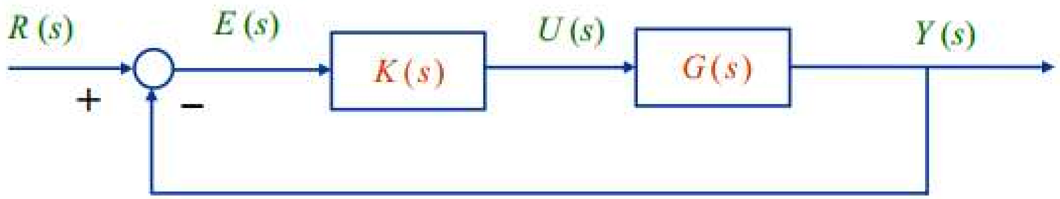




The response is faster and the overshoot is smaller. The controller does improve the overall system performance a great deal

Control System of Fighter Aircraft

Recall that



with $G(s) = \dfrac{b}{s+a}$ and $K(s) = k_p + \dfrac{k_i}{s} = \dfrac{k_p s + k_i}{s}$ results a closed-loop system:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{bk_p s + bk_i}{s^2 + (a + bk_p)\, s + bk_i}$$

Compare this with the standard 2nd order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\begin{aligned} 2\zeta\omega_n &= a + bk_p \\ \omega_n^2 &= bk_i \end{aligned} \rightarrow \begin{aligned} k_p &= \frac{2\zeta\omega_n - a}{b} \\ k_i &= \frac{\omega_n^2}{b} \end{aligned}$$

The key issue now is to choose parameters k_p and k_i such that the above resulting system has desired properties, such as prescribed settling time and overshoot

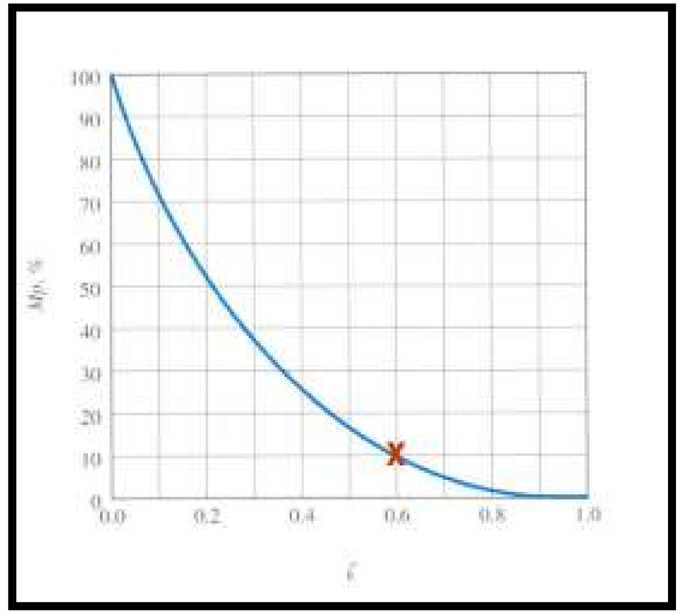
Control System of Fighter Aircraft

Mission Requirements: Let us design a PI controller for the above aircraft such that the aircraft will reach the desired vertical level in 30 seconds (i.e., the settling time is 30 sec) and the maximum overshoot is less than 10% .

To achieve an overshoot less than 10%, we obtain from the figure on the right that $\boxed{\zeta > 0.6}$

To be safe, we choose $\boxed{\zeta = 0.8}$

To achieve a settling time of 30 sec., we use



To achieve a settling time of 30 sec., we use

$$\boxed{t_s = \frac{4.6}{\zeta\omega_n} \quad \Rightarrow \quad \omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.8 \times 30} = 0.192}$$

Control System of Fighter Aircraft

Recall that the fighter aircraft has a transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s+a} = \frac{1}{s+0.05} \quad \Rightarrow \quad a = 0.05,\ b = 1$$

Then, using the formulae we have just derived, we obtain

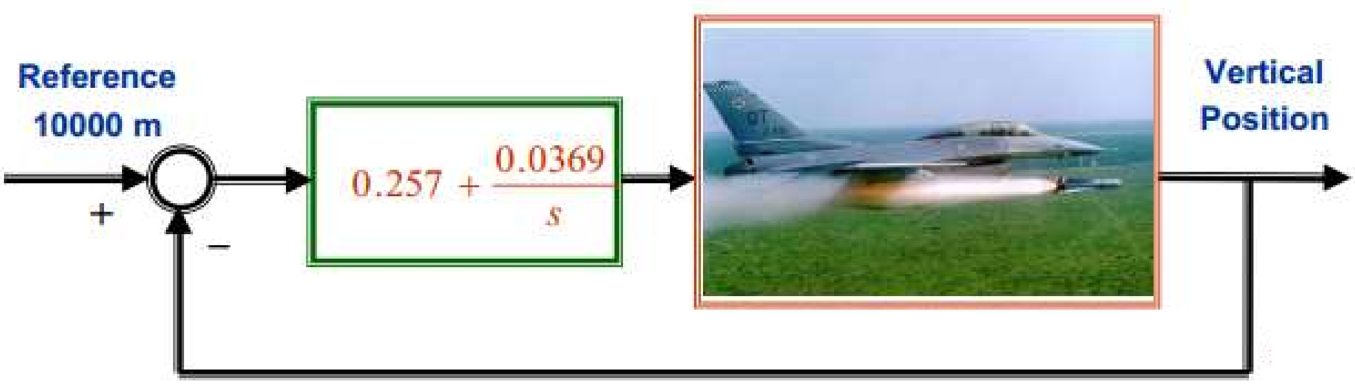
$$k_p = \frac{2\zeta\omega_n - a}{b}, \qquad k_i = \frac{\omega_n^2}{b}$$

$$\rightarrow$$

$$k_p = \frac{2\zeta\omega_n - a}{b} = \frac{2 \times 0.8 \times 0.192 - 0.05}{1} = 0.257$$

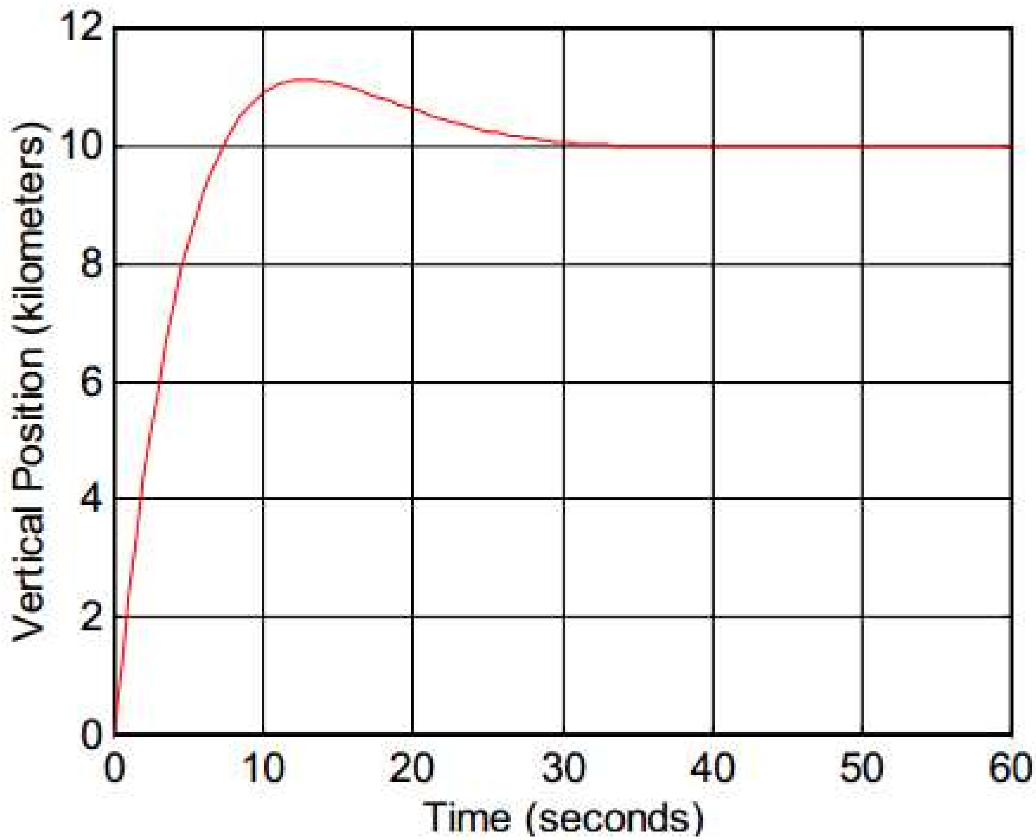
$$k_i = \frac{\omega_n^2}{b} = \frac{0.192^2}{1} = 0.0369$$

The final flight control system:



Control System of Fighter Aircraft

Simulation Result:



The resulting overshoot is about 10% and the settling time is about 30 seconds.

Thus, our design goal is achieved.