Example of Steady-state output

Consider the following system and determine the steady-state output $y_{ss}(t)$ for the sinusoidal input $x(s) = Asin(\omega t)$

$$G(s) = \frac{s + 1/T_1}{s + 1/T_2}$$

Solution:

$$G(j\omega) = \frac{j\omega + 1/T_1}{j\omega + 1/T_2} = \frac{T_2(T_1 j\omega + 1)}{T_1(T_2 j\omega + 1)}$$

$$\Rightarrow |G(j\omega)| = \frac{T_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}}$$

$$\phi = \angle G(j\omega) = \tan^{-1} T_1\omega - \tan^{-1} T_2\omega$$

The steady state output is given by

$$y_{ss}(t) = \frac{AT_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}} \sin(\omega t + \tan^{-1} T_1\omega - \tan^{-1} T_2\omega)$$

Key Concepts Learnt

- If input to a stable linear system is sinusoidal of frequency ω_i, then its output in the steady-state is also sinusoidal of frequency ω_i
  - Input an output may differ in amplitude and phase

- Gain of the linear system is the ratio between the output (steady-state) amplitude and the input amplitude
  - Gain varies with frequency

- Phase defines the delay between the input sinusoid and the output sinusoid
  - Phase varies with frequency

- If a system's transfer function is $G(s)$, then its gain and phase at ω_i are

$$M(\omega_i) = |G(j\omega_i)|, \quad \phi(\omega_i) = \angle G(j\omega_i)$$

- For a transfer function $G(s)$, the crossover frequencies are defined as

$$|G(j\omega_{cg})| = 1, \quad \angle G(j\omega_{cp}) = \pm 180°$$

Frequency Domain Specification of 2nd Order System

Lets consider a second order standard system as:

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Putting $s = j\omega$ yields

$$G(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n j\omega+\omega_n^2}$$

$$G(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n j\omega+\omega_n^2}=\frac{\omega_n^2}{-\omega^2+2j\zeta\omega_n\omega+\omega_n^2}$$

$$=\frac{\omega_n^2}{\omega_n^2-\omega^2+2j\zeta\omega_n\omega}=\frac{\omega_n^2}{\omega_n^2\left(\left(1-\omega^2/\omega_n^2\right)+2j\zeta\omega/\omega_n\right)} \qquad [Let, \quad \omega/\omega_n=x]$$

$$=\frac{\omega_n^2}{\left(1-x^2\right)+j2\zeta x}$$

$$\left|G(j\omega)\right|=\frac{1}{\sqrt{\left(1-x^2\right)^2+\left(2\zeta x\right)^2}}; \qquad \angle G(j\omega)=-\tan^{-1}\left(\frac{2\zeta x}{1-x^2}\right)$$

The Resonant Frequency

The frequency at which the value of $|G(j\omega)|$ is maximum. Or, we can write

$$\frac{|G(j\omega)|}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)^2 + (2\zeta x)^2\right]^{-1/2} = 0$$

$$\Rightarrow -\frac{1}{2}\left[(1-x^2)^2 + (2\zeta x)^2\right]^{-3/2}\left[2(1-x^2)(-2x) + 2(2\zeta x)2\zeta\right] = 0$$

$$\Rightarrow -4(1-x^2)x + 4(2\zeta^2 x) = 0$$

$$\Rightarrow (1-x^2) - 2\zeta^2 = 0 \Rightarrow x^2 = 1 - 2\zeta^2$$

$$Put\ \ x = \frac{\omega_r}{\omega_n}\ \ yields$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

The Resonant Peak

The peak (the value of maximum peak) at resonant frequency.

$$\left.\begin{aligned} x^2 &= 1-2\zeta^2 \\ 2\zeta^2 &= 1-x^2 \end{aligned}\right\}$$

$$\Rightarrow M_r = \frac{1}{\sqrt{(2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}} = \frac{1}{2\xi\sqrt{1-\zeta^2}}$$

$$0 \le \zeta \le 0.707$$

Bandwidth

It is the range of frequencies over which the magnitude of $|G(j\omega)|$ droops to 70.7% from its zero frequency value. Within the bandwidth the system works satisfactory. We know that

$$M = \frac{1}{\sqrt{\left(1-x^2\right)^2 + \left(2\zeta x\right)^2}} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

If we put, $\omega = 0 \Rightarrow$ $\Rightarrow$ $\left|G(j\omega)\right|_{\omega=0} = 1$

Now, $\omega = \omega_b$ $\Rightarrow$ $M = 0.707 \times (\omega = 0\, value) = 0.707 \times 1 = \dfrac{1}{\sqrt{2}}$

Hence,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-x^2)^2 + (2\zeta x)^2}} \Rightarrow 2 = (1-x^2)^2 + (2\zeta x)^2 \quad Let\ \ x^2 = u$$

$$1 + u^2 - 2u + 4\zeta^2 u = 2 \Rightarrow u^2 - 2u(1+2\zeta^2) - 1 = 0$$

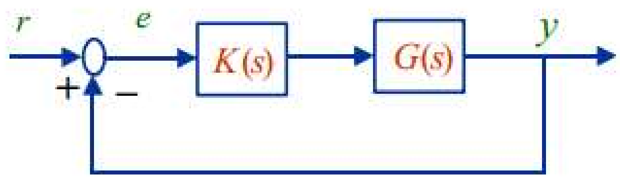
$$\Rightarrow u = \frac{2(1+2\zeta^2) \pm \sqrt{4(1+2\zeta^2)^2 + 4}}{2} = (1+2\zeta^2) \pm \sqrt{\left(1+2\zeta^2\right)}$$

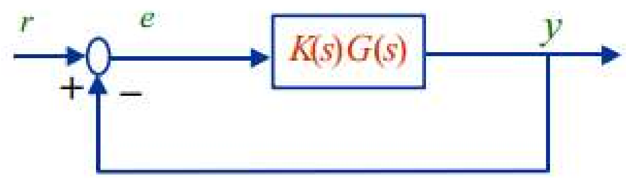
$$\Rightarrow x = \sqrt{(1+2\zeta^2) \pm \sqrt{\left(1+2\zeta^2\right)}}$$

$$\Rightarrow \omega = \omega_n\sqrt{(1+2\zeta^2) \pm \sqrt{\left(1+2\zeta^2\right)}} \Rightarrow \omega_b = \omega_n\sqrt{(1+2\zeta^2) \pm \sqrt{\left(1+2\zeta^2\right)}}$$

Bode Plots

Consider the following feedback control system,





Bode Plots are the magnitude and phase responses of the open-loop transfer function, i.e., $K(s)G(s)$, with s being replaced by $s = j\omega$.

The Bode diagram consists of two graphs: one is a plot of the logarithm of the magnitude of a sinusoidal transfer function and the other is a plot of the phase angle

The standard representation of the logarithm magnitude of $G(j\omega)$ is $20log|G(j\omega)|$, where the base of the logarithm is 10. The main advantage of using the Bode diagram is that multiplication of magnitudes can be converted into addition. Given a positive scalar a, its decibel is defined as $20\log(a)$. For example

$$a = 1 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 0 \quad \Rightarrow \quad a = 0\,\text{dB}$$

$$a = 10 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 20 \quad \Rightarrow \quad a = 20\,\text{dB}$$

$$a = 100 \quad \Rightarrow \quad 20 \cdot \log_{10}(a) = 40 \quad \Rightarrow \quad a = 40\,\text{dB}$$

$$a = \alpha \cdot \beta \quad \Rightarrow \quad 20 \cdot \log_{10}(\alpha \cdot \beta) = 20 \cdot \log_{10}(\alpha) + 20 \cdot \log_{10}(\beta) \quad \Rightarrow \quad a = \alpha \text{ in dB} + \beta \text{ in dB}$$

$$a = \alpha/\beta \quad \Rightarrow \quad 20 \cdot \log_{10}(\alpha/\beta) = 20 \cdot \log_{10}(\alpha) - 20 \cdot \log_{10}(\beta) \quad \Rightarrow \quad a = \alpha \text{ in dB} - \beta \text{ in dB}$$

Logarithmic coordinate

$$\omega_2 = 10\omega_1$$

$$\frac{\omega_2}{\omega_1} = 10$$

$$20\log_{10}\frac{\omega_2}{\omega_1} = 20dB/decade$$

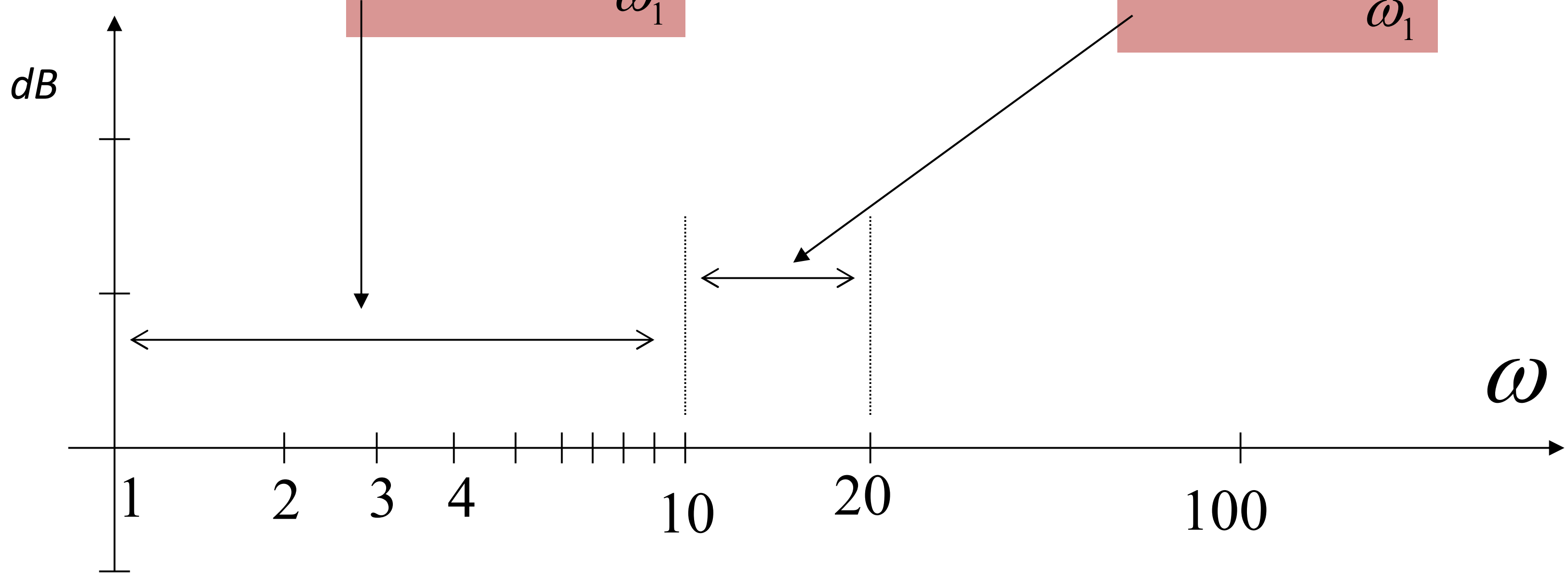
$$\omega_2 = 2\omega_1$$

$$\frac{\omega_2}{\omega_1} = 2$$

$$20\log_{10}\frac{\omega_2}{\omega_1} = 20\log_{10}2 = 6dB/octave$$

Decade : $dec = \log_{10}\frac{\omega_2}{\omega_1}$

Octave : $oct = \log_2\frac{\omega_2}{\omega_1}$



Bode plot of an Integrator

We start with finding the Bode plot asymptotes for a simple system characterized by

$$G(s)=\frac{1}{s} \quad \Rightarrow \quad G(j\omega)=\frac{1}{j\omega}=|G(j\omega)|\angle G(j\omega)=\frac{1}{|\omega|}\angle-90^{\circ}$$

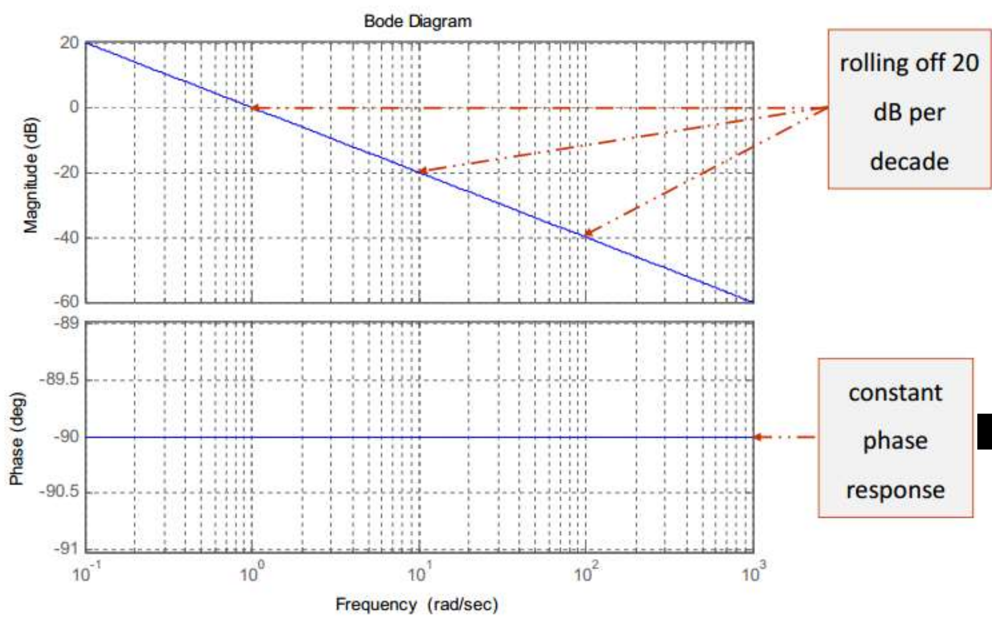
Examining the amplitude in dB scale, i.e.,

$$20\cdot\log_{10}|G(j\omega)|=20\cdot\log_{10}\frac{1}{|\omega|}=-20\cdot\log_{10}|\omega| \text{ dB}$$

$$\omega=1 \quad \Rightarrow \quad 20\cdot\log_{10}|G(j1)|=-20\cdot\log_{10}1=0 \text{ dB}$$

$$\omega=10 \quad \Rightarrow \quad 20\cdot\log_{10}|G(j10)|=-20\cdot\log_{10}10=-20 \text{ dB}$$

Thus, the above expressions clearly indicate that the magnitude is reduced by -20dB when the frequency is increased by 10 times. It is equivalent to say that the magnitude is rolling off 20dB per decade



The phase response of an integrator is -90 degrees, a constant.

Bode plot of a Diffrentiator

We start with finding the Bode plot asymptotes for a simple system characterized by

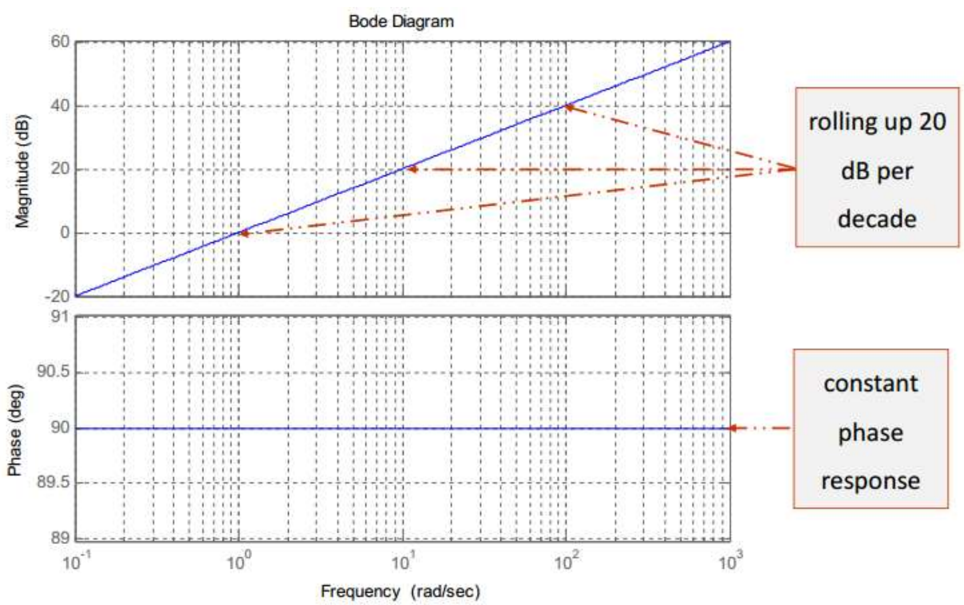
$$\boxed{G(s) = s \Rightarrow G(j\omega) = j\omega = |G(j\omega)|\angle G(j\omega) = |\omega|\angle 90^0}$$

Examining the amplitude in dB scale, i.e.,

$$\boxed{20\log_{10}|G(j\omega)| = 20.\log_{10}|\omega| = 20\log_{10}|\omega| dB}$$

$$\boxed{\begin{aligned}&\omega = 1 \Rightarrow 20\log_{10}|G(j1)| = 20\log_{10} 1 = 0dB\\&\omega = 10 \Rightarrow 20\log_{10}|G(j10)| = 20\log_{10} 10 = 20dB\end{aligned}}$$

Thus, the above expressions clearly indicate that the magnitude is increased by 20dB when the frequency is increased by 10 times. It is equivalent to say that the magnitude is rolling up 20dB per decade



The phase response of an integrator is 90 degrees, a constant.