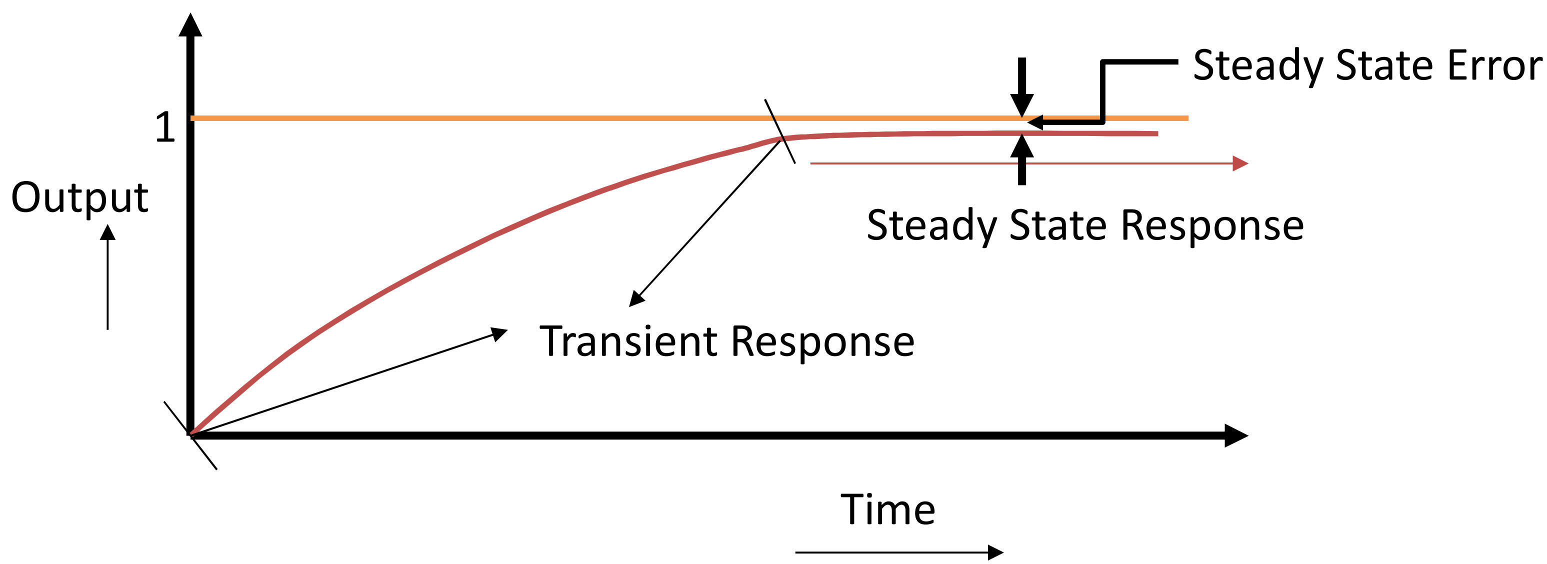
Automatic Control of Aircraft, Rockets and Spacecraft (AE649A)

Transient and Steady State Response

Control System responds to an input by undergoing a transient response before reaching a steady-state response.



Step response of an overdamed second order system

Stability of the System

1. A system is stable if its output is bounded for any bounded input

2. A system is asymptotically stable if in the absence of input, the output tends towards zero irrespective of initial condition.

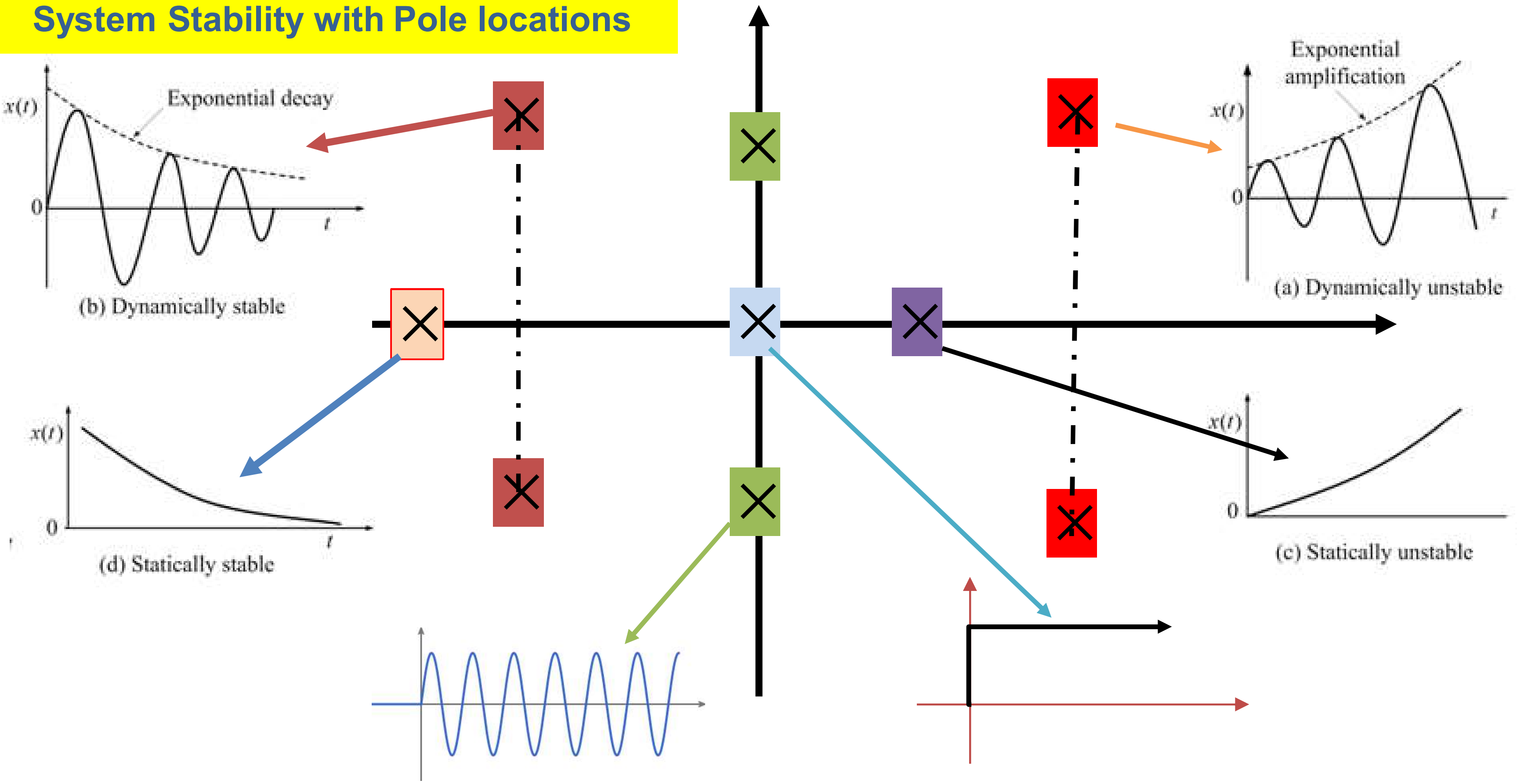
3. Stability depends on the pole location of the characteristic equation. If all the poles are located in the left half of s-plane, then the system is stable.

4. As the poles approaches origin, the stability of the overall system decreases.

5. When poles are located on the imaginary axis, the system is marginally stable.

6. If the poles located on the imaginary axis repeated in nature, then the system is unstable.

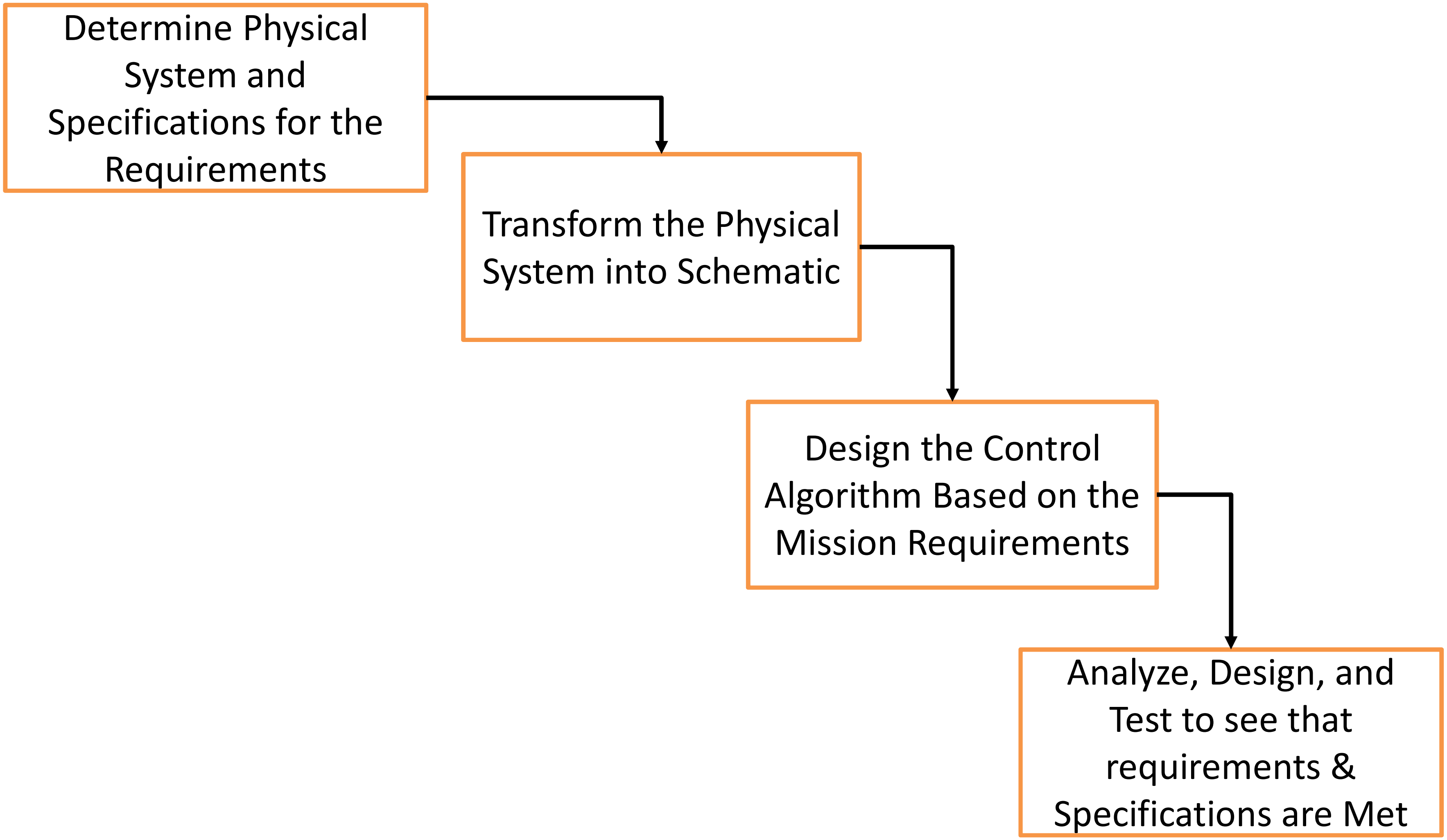
System Stability with Pole locations



Asymptotically Stable: The system response always decreases from their initial value and do not show permanent oscillations.

Marginally Stable: If the natural response neither nor grows but remains constant or oscillates with a bound. Marginal stable may goes to unstable, if the response goes further and further from the bounded state.

The design process

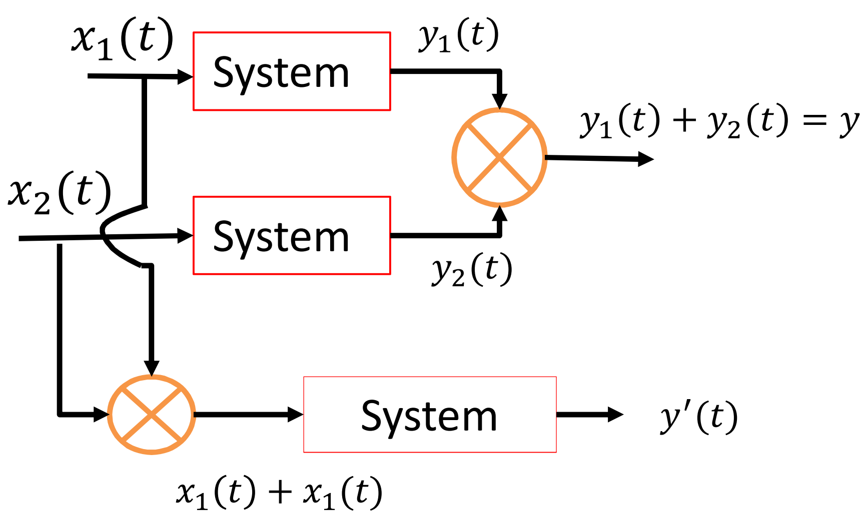


Linear System

The system which follows the principle of superposition is known as linear system.

Law of Additivity | **Law of Homogenity**

Law of Additivity



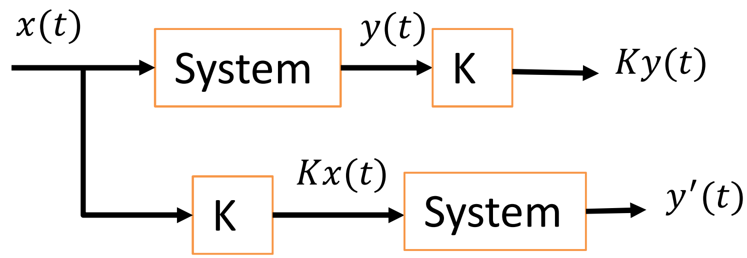
$$\boxed{y'(t) = y_1(t) + y_2(t)}$$

Follow the law of additivity

$$\boxed{y'(t) \neq y_1(t) + y_2(t)}$$

Not Follow the law of additivity

Law of Homogenity



$$\boxed{y'(t) = Ky(t)}$$

Follow the law of Homogenity

$$\boxed{y'(t) \neq Ky(t)}$$

Not Follow the law of Homogenity

Example $y(t) = x\sin(t)$

1. LOA $Let: y_1(t) = x_1 \sin(t); y_2(t) = x_2 \sin(t)$

If we add $y_1(t) + y_2(t) = x_1 \sin(t) + x_2 \sin(t)$

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow x_1 \sin(t) + x_2 \sin(t)$$

LOA is confirmed

2. LOH $y(t) = x\sin(t) \Rightarrow ky(t) = kx\sin(t)$

$$kx(t) \longrightarrow \boxed{\text{System}} \longrightarrow kx\sin(t)$$

LOH is confirmed

Static and Dynamic Systems

To discuss the static and dynamic systems, we need to know three types of inputs: PAST, PRESENT, & FUTURE.

Lets define them as: $\underbrace{y(t) = x(t-1);}_{\text{Previous/past input}}$ $\underbrace{y(t) = x(t);}_{\text{Present Input}}$ $\underbrace{y(t) = x(t+1)}_{\text{Future Input}}$

Static System: Output of the system depends only on the present values of input

Example:
$$\boxed{\begin{array}{l} y(t) = 2x(t) \quad \therefore\ y(t) = f(x(t)) \\ y(0) = 2x(0) \ \ at \ \ t = 0 \end{array}}$$

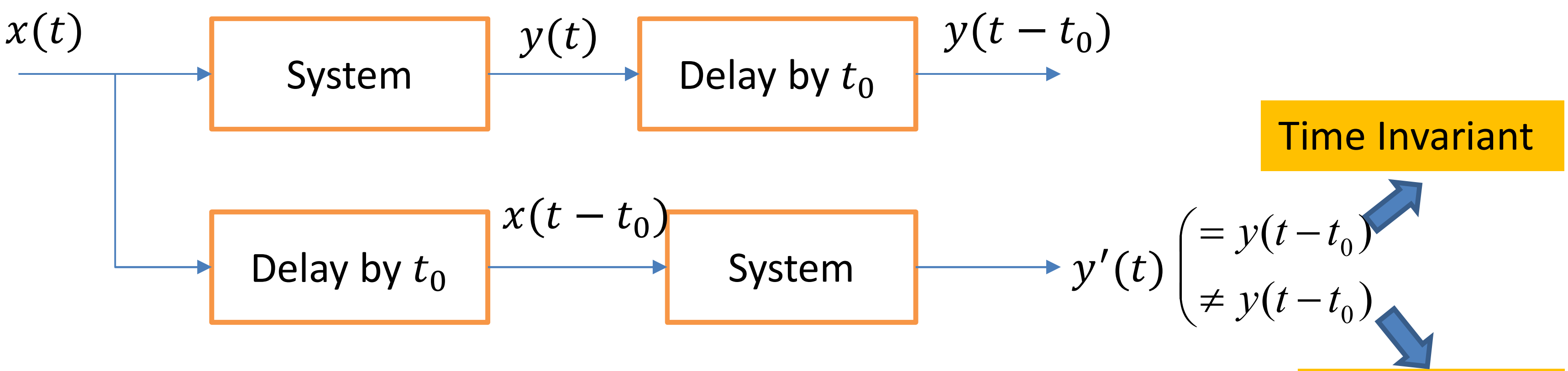
Dynamic System: Output of the system depends on past or future values of input at any instant of time.

Example: $y(t) = x(t) + x(t-1)$

Note: Static System-> Memory less
Dynamic System-> System with memory

Time-Invariant & Time-variant Systems

Time Variant System: Any delay applied in the input, must be reflected in the output.



Example: $y(t) = x(2t)$ $\qquad x(t) \rightarrow system \rightarrow y(t) = x(2t)$

Step1: $y(t) \xrightarrow{t_0} y(t-t_0) = x(2(t-t_0)) = x(2t-2t_0)$

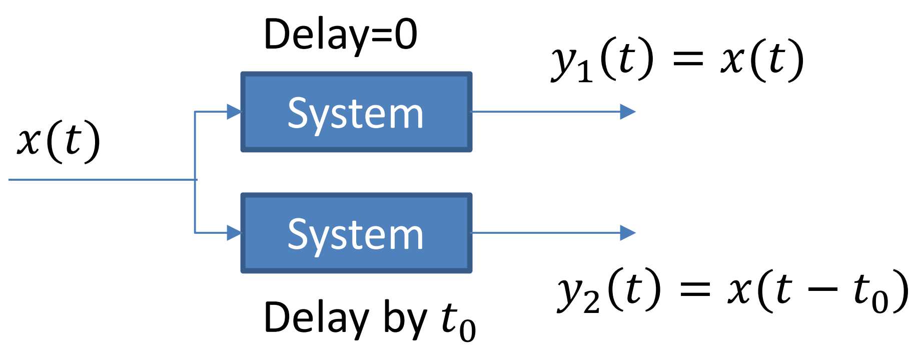
Step2: $x(t) \xrightarrow{t_0} x(t-t_0) \rightarrow system \rightarrow x(2t-t_0)$

$$\therefore x(2t-2t_0) \neq x(2t-t_0)$$

Time Variant

Causal and Non-Causal System

Causal System: Output of the system is independent of future values of the input. All practical and real life systems are causal system.



Example:

$$y(t) = x(t)$$

$$y(t) = x(t) + x(t-1) + x(t-2)$$

Non-Causal System: Output of the system depends on future values of input at any instant of time. It may depend on past or present, but it should depends on future.

Example: $y(t) = x(t+2); \quad y(t) = x(t) + x(t-1) + x(t+1)$

Anti-Causal System: Output of the is depends on the future values of the input only. Anti-causal system is always non-causal.

Example: $y(t) = x(t+2)$

Classification of Control System

$$G(s) = \frac{K(T_a s+1)(T_b s+1).......(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1).......(T_p s+1)}$$

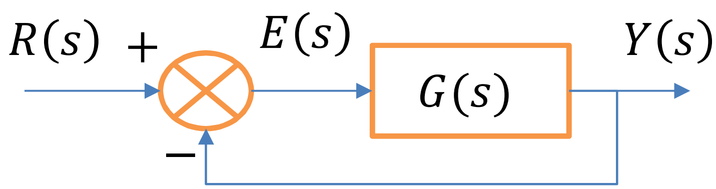
$s^N \rightarrow$ Pole of multiplicity N at the origin. The system type can be classified the number of integrators at the open loop T.F. Type 0 -> N=0; Type 1 -> N=1...

As the type number is increased, accuracy is improved; however, increasing the type number aggravates the stability problem.

A compromise between steady-state accuracy and relative stability is always necessary.

Steady-state Error

Changes in the reference input will cause unavoidable errors during transient periods and also cause steady state errors.



$$T.F. = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$E(s) = R(s) - Y(s) = R(s) - R(s)\frac{G(s)}{1+G(s)} = R(s)\left[1 - \frac{G(s)}{1+G(s)}\right]$$

$$\Rightarrow \frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s)$$

Final Value Theorem

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+G(s)}$$

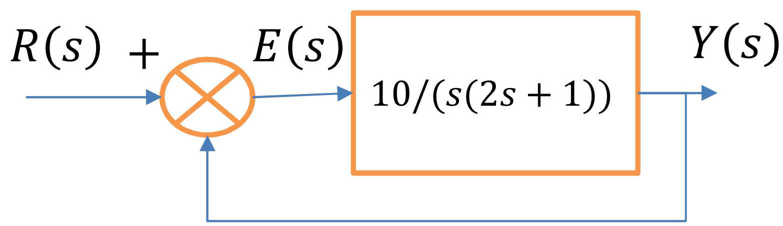
	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+k_p}$	∞	∞
Type 1 system	0	$\frac{1}{k_v}$	∞
Type 2 system	0	0	$\frac{1}{k_a}$

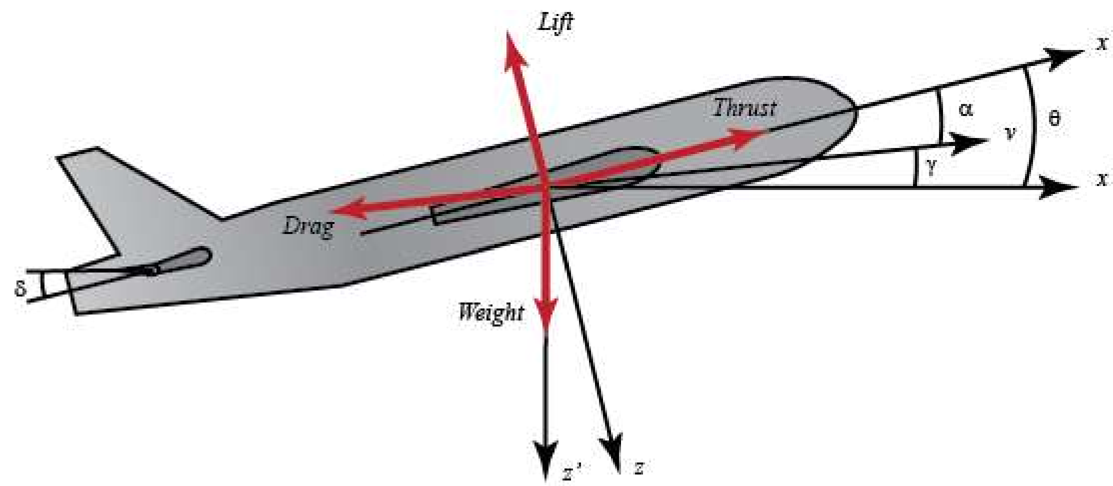
Steady-state Error

$r(t)$	e_{ss}	Error Coefficient	
$u(t)$	$1/(1+k_p)$	$k_p = \lim_{s\to 0} G(s)$	Position error constant
$tu(t)$	$1/k_v$	$k_v = \lim_{s\to 0} sG(s)$	Velocity error coefficient
$t^2u(t)$	$1/k_a$	$k_v = \lim_{s\to 0} s^2G(s)$	Acceleration error coefficient

Find the error coefficient and steady state of the following system

Example:





Aircraft pitch is governed by the longitudinal dynamics. In this example we will design an autopilot that controls the pitch of an aircraft.

We assume here: 1. the aircraft is in steady-cruise at constant altitude and velocity. 2. the thrust, drag, weight and lift forces balance each other in the *x*- and *y*-directions. 3. a change in pitch angle will not change the speed of the aircraft under any circumstance (unrealistic but simplifies the problem a bit). Under these assumptions, the longitudinal equations of motion for the aircraft can be written as follows.

$$\dot{\alpha} = \mu\Omega\sigma[-(C_L + C_D)\alpha + \frac{1}{(\mu - C_L)}q - (C_W \sin\gamma)\theta + C_L]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}[[C_M - \eta(C_L + C_D)]\alpha + [C_M + \sigma C_M(1 - \mu C_L)]q + (\eta C_W \sin\gamma)\delta]$$

$$\dot{\theta} = \Omega q$$

$\alpha \rightarrow Anle\ of\ Attack;\ q \rightarrow Pitch\ Rate;\ \theta \rightarrow Pitch\ Angle;\ \delta \rightarrow Elevator\ Deflection\ Angle$

For this system, the input will be the elevator deflection angle, δ and the output will be the pitch angle θ of the aircraft. Importing the data from one of Boeing's commercial aircraft, the simplified modeling equations yield

$$\dot{\alpha} = -0.313\alpha + 56.7q + 0.232\delta$$

$$\dot{q} = -0.0139\alpha - 0.426q + 0.0203\delta$$

$$\dot{\theta} = 56.7q$$

Transfer Function:

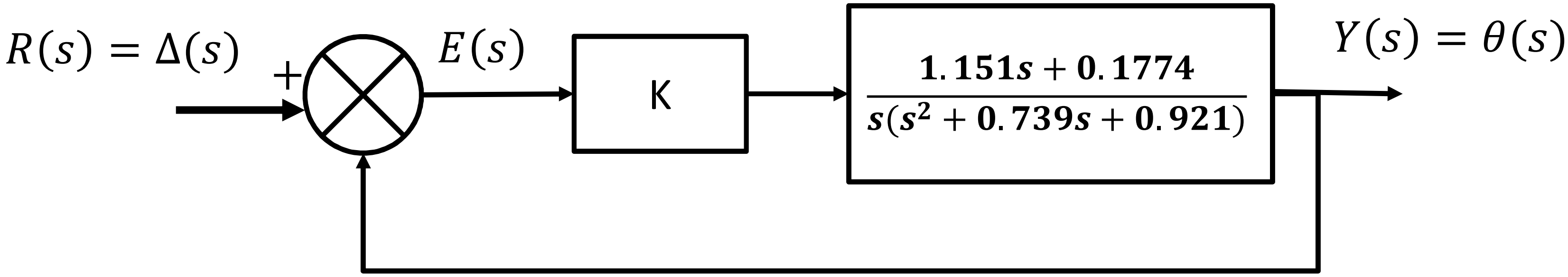
$$sA(s) = -0.313A(s) + 56.7Q(s) + 0.232\Delta(s)$$

$$sQ(s) = -0.0139A(s) - 0.426Q(s) + 0.0203\Delta(s)$$

$$s\Theta(s) = 56.7Q(s)$$

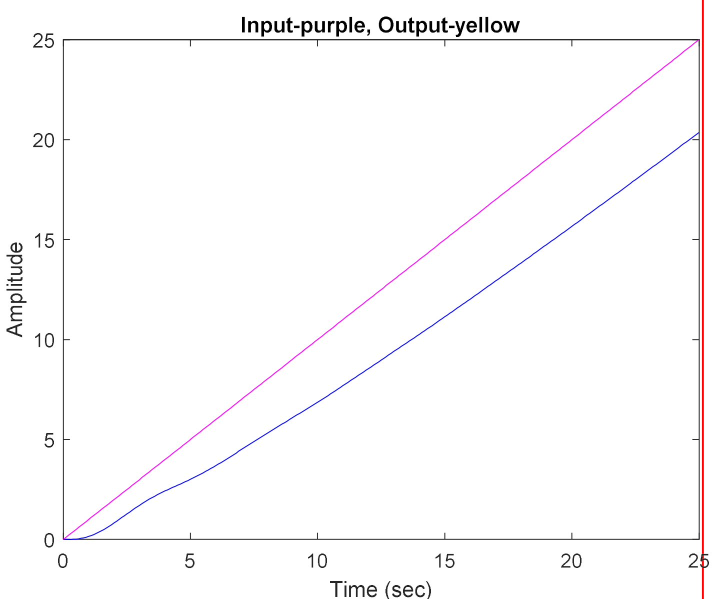
$$\Downarrow$$

$$\boxed{P(s) = \frac{\Theta(s)}{\Delta(s)} = \underbrace{\frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 0.921s}}_{\mathbf{G}(s)}}$$



Mission Requirements: **The value of K should be chosen in such a way that the error should be 0.1.**

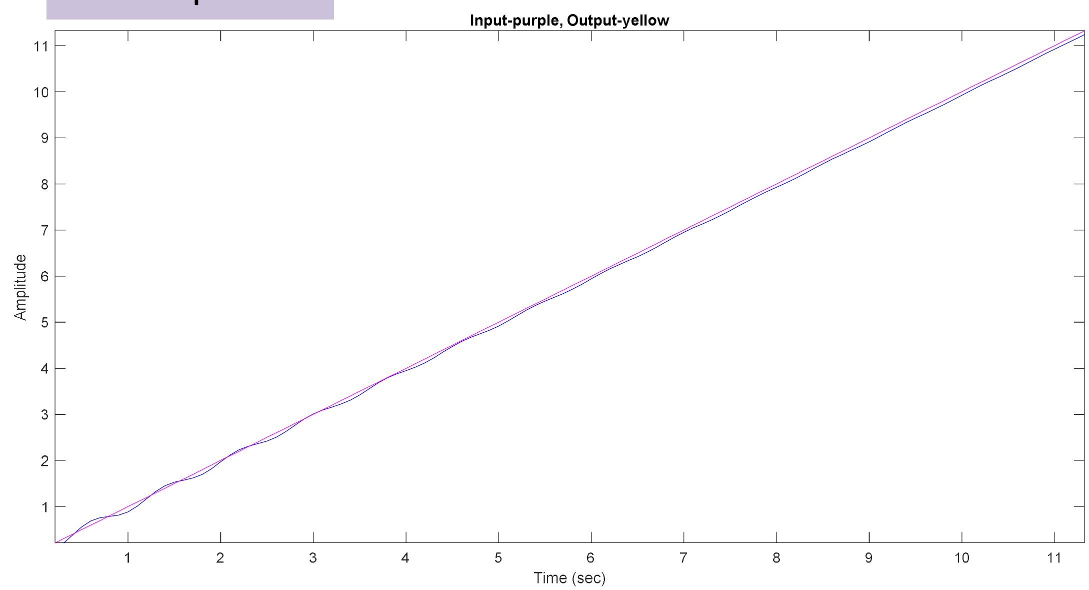
As the type of the system is 1, we have to consider ramp input. Lets choose K=1 (without control). The response:



$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{s^2(1+G(s))} = \frac{0.921}{K \times 0.174}$$

$$\Rightarrow 0.1 = \frac{5.319}{K} \Rightarrow K = 53.19$$

The response:



Transient Response

First Order System

The general form of the first-order differential equation is as follows

$$\dot{y} + ay = bu \quad \text{or} \quad \tau\dot{y} + y = k_{dc}u$$

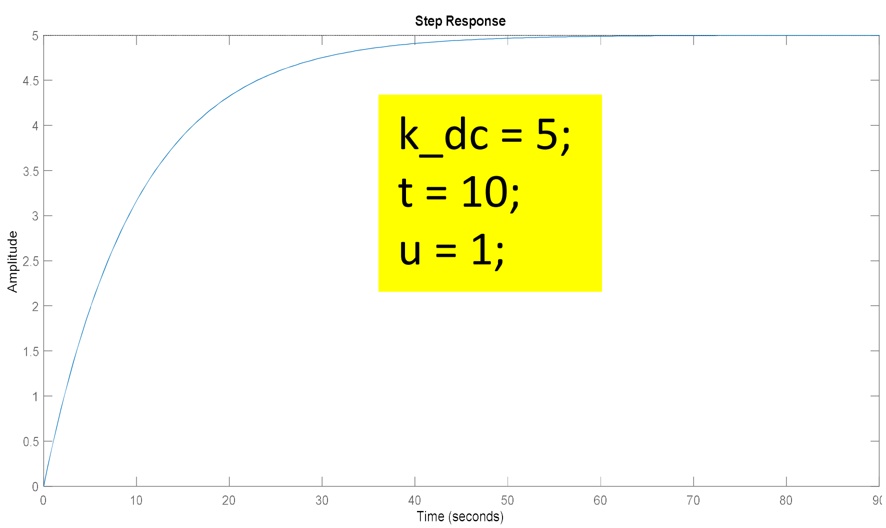
The form of a first-order transfer function is

$$G(s) = \frac{Y(s)}{R(s)} = \frac{k_{dc}}{\tau s + 1}$$

For unit-step: $R(s) = 1/s$

$$Y(s) = \frac{R(s)k_{dc}}{(\tau s+1)} = \frac{k_{dc}}{s(\tau s+1)} = k_{dc}\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right)$$

$$\Rightarrow y(t) = k_{dc}(1-e^{-t/\tau})$$



For Unit Ramp: $R(s) = \frac{1}{s^2}$

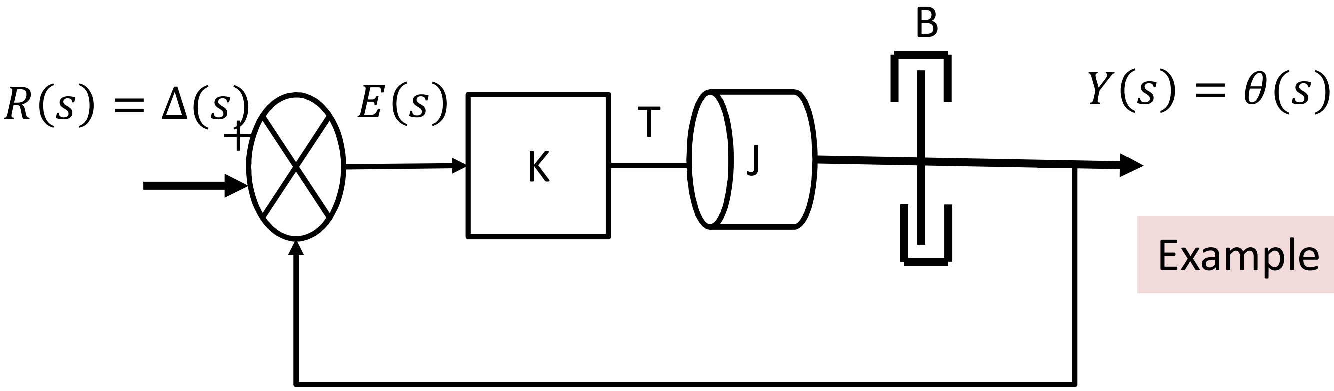
$$Y(s) = \frac{k_{dc}}{s^2(\tau s+1)} = k_{dc}\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}\right)$$

$$\Rightarrow y(t) = k_{dc}\left(t - \tau + \tau e^{-t/\tau}\right)$$

$$e(t) = r(t) - k_{dc}\left(t - \tau + \tau e^{-t/\tau}\right) = t - k_{dc}\left(t - \tau + \tau e^{-t/\tau}\right)$$

$$= t(1-k_{dc}) + k_{dc}\left(\tau - \tau e^{-t/\tau}\right)$$

Transient Response of Second-order System



Example of Servo System

$$J\ddot{y} + B\dot{y} = T \Rightarrow Js^2Y(s) + BsY(s) = T(s)$$

$$\Rightarrow \frac{Y(s)}{T(s)} = \frac{1}{s(Js+B)}$$

C.L.T.F

$$\frac{Y(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{K/J}{s^2 + (B/J)s + K/J}$$

In the transient response analysis, it is convenient to write $K/J = \omega_n^2, \quad B/J = 2\zeta\omega_n$

ω_n is the undammed natural frequency and ζ is the damping ratio of the system. The dynamic behaviour of system can be described in terms of ω_n and ζ. The standard form of the second order system can be written as

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n \rightarrow$ The frequency that the system will oscillate at when there is no damping ($\zeta = 0$)
$\zeta \rightarrow$ the rate at which an oscillation in the system's response decays due to effects such as viscous friction or resistance.

Transient Response of Second-order System

Underdamped Case ($0 < \zeta < 1$)

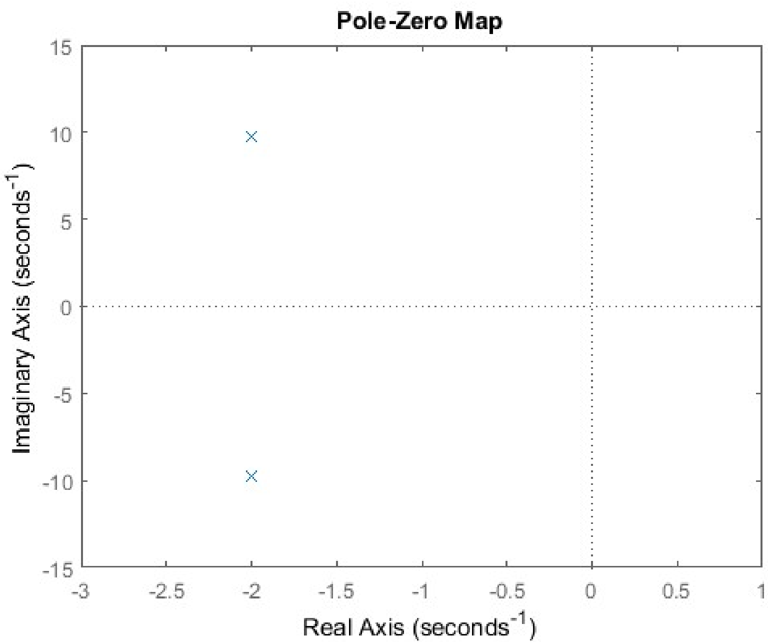
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

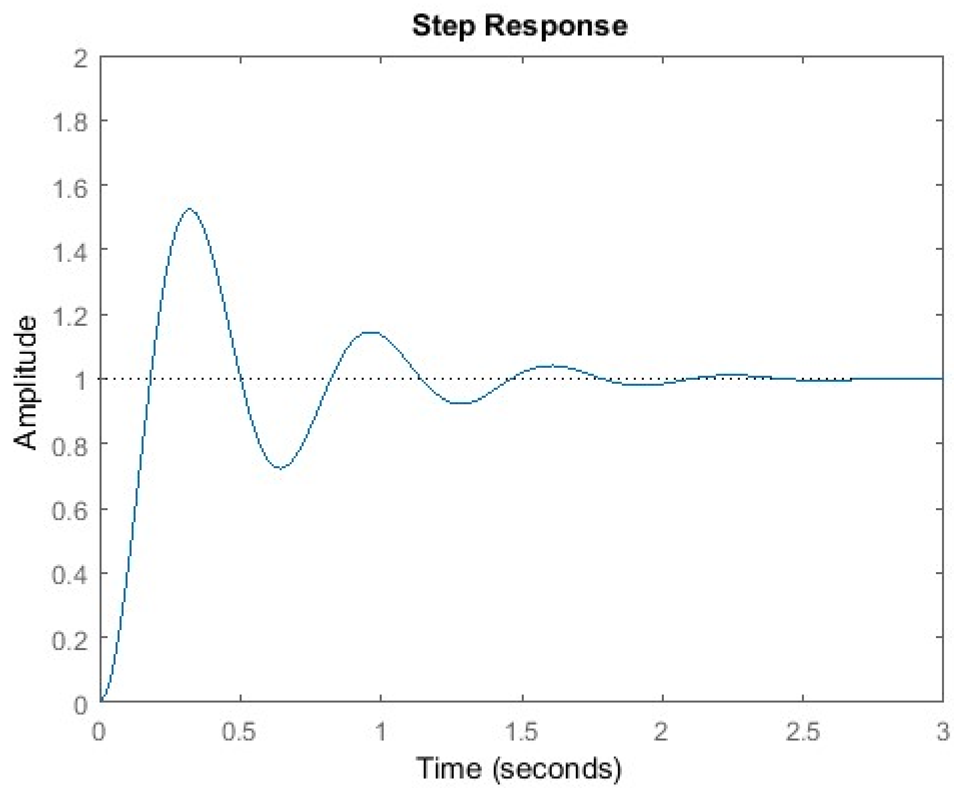
Step-response

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \quad t \geq 0$$

$$\omega_n = 10;\ \zeta = 0.2$$

$$s_{1,2} = -$$





Transient Response of Second-order System

Critically damped Case ($\zeta = 1$)

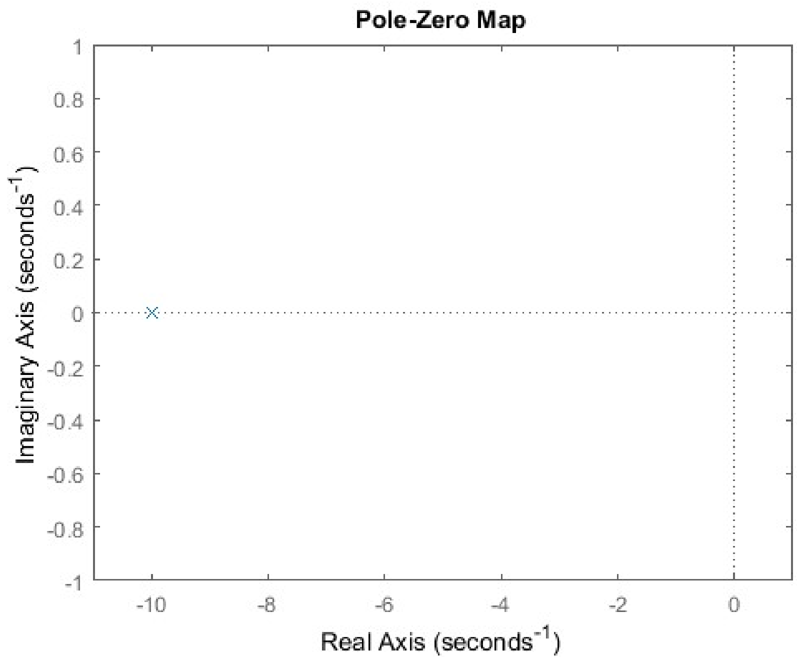
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

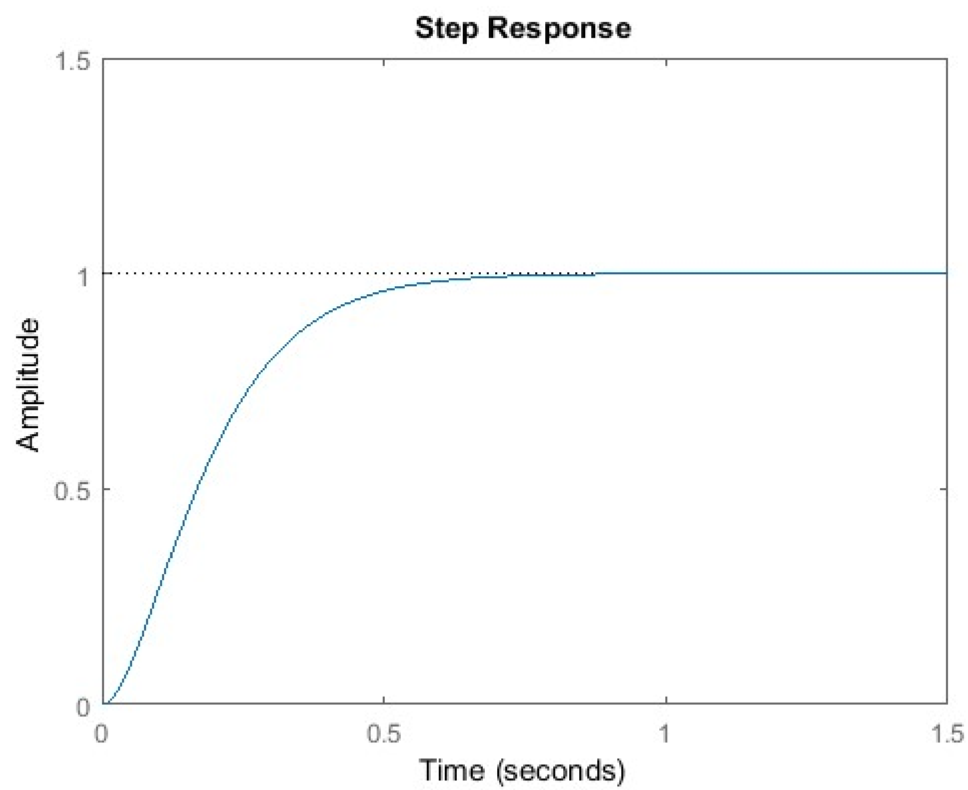
Step-response

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(1+\omega_n t\right), \quad t \geq 0$$

$$\omega_n = 10;\ \zeta = 1$$

$$s_{1,2} =$$





Transient Response of Second-order System

Overdamped Case ($\zeta > 1$)

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

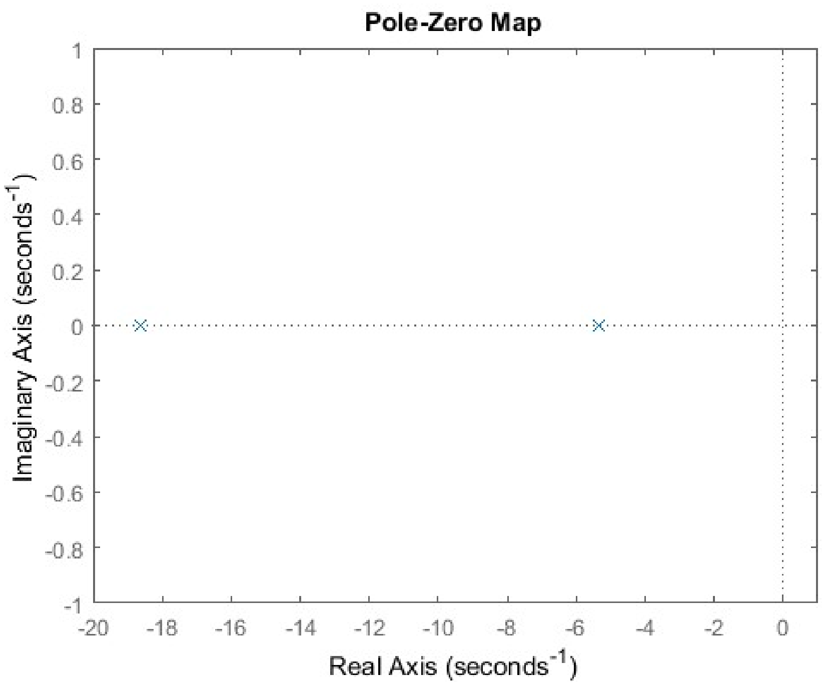
Step-response

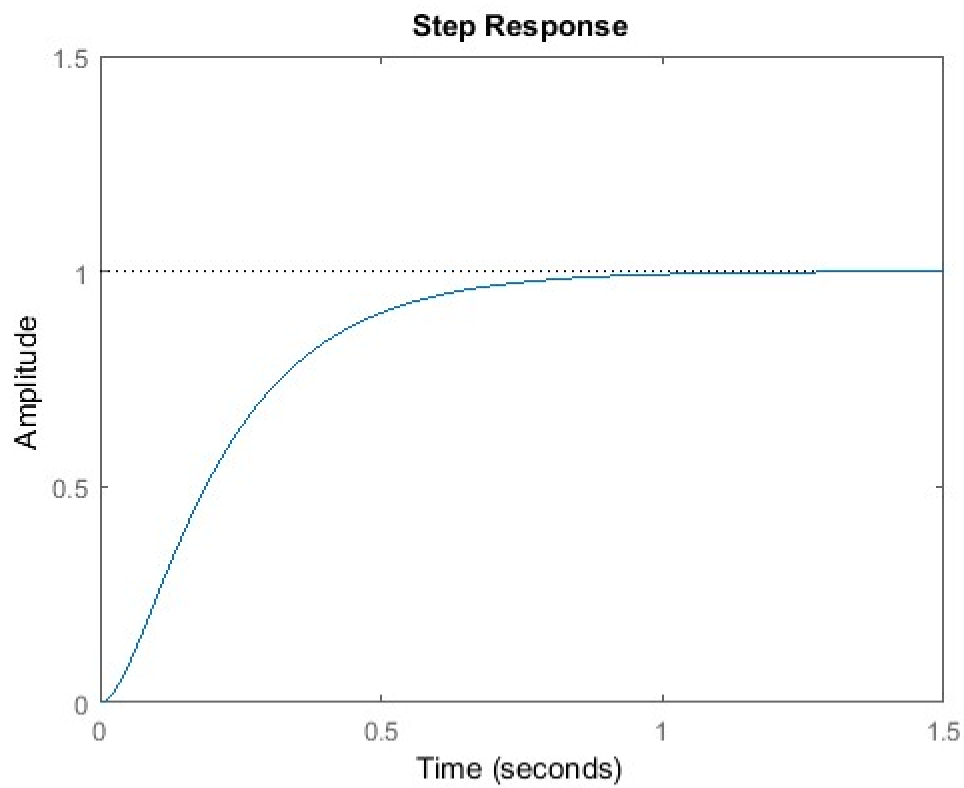
$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \quad t \geq 0$$

$$s_{1,2} = -\zeta \pm \left(\sqrt{\zeta^2 - 1}\right)\omega_n$$

$$\omega_n = 10; \; \zeta = 1.2$$

$$s_{1,2} =$$





Transient Response of Second-order System

Undamped Case ($\zeta = 0$)

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

Step-response

$$y(t) = 1 - \cos \omega_n t, \quad t \geq 0$$

$$\omega_n = 10;\ \zeta = 0$$

$$s_{1,2} = \pm j10$$