Frequency Response

The frequency response of system is defined as the steady state response of the system to a sinusoidal response. The sinusoid is a unique input signal and the resulting output signal for a linear system, is sinusoidal at the steady state. It defines from the input waveform only in amplitude and phase

When a stable linear system is subject to sinusoidal input of frequency ω_i rad/s

the output and all intermediate signals are sinusoidal of frequency ω_i rad/s in the steady state

the output and the intermediate signals differ from the input in amplitude and phase.

$$y_{ss}(t) = B\sin(\omega_i t + \phi)$$

Frequency Example

Example: Response of a 1st-order system to sinusoidal input

$$G(s)=\frac{1}{s+1},\quad u(t)=\sin(10t) \quad\Rightarrow\quad U(s)=\frac{10}{s^2+100}$$

Solution: $Y(s)=\frac{1}{(s+1)}\frac{10}{(s^2+100)}$

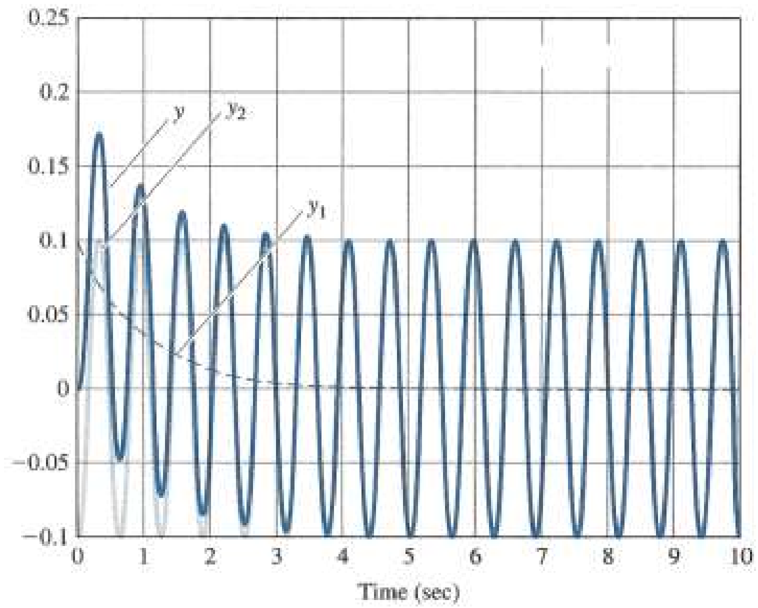
$$=\frac{A}{(s+1)}+\frac{Bs+C}{(s^2+100)}$$

$$10=A(s^2+100)+Bs(s+1)+C(s+1)$$
$$=(A+B)s^2+(B+C)s+(100A+C)$$

$$A+B=0,\quad B+C=0,\quad 100A+C=10$$
$$A=\frac{10}{101},\quad B=-\frac{10}{101},\quad C=\frac{10}{101}$$

$$Y(s)=\frac{10/101}{(s+1)}-\frac{10}{101}\frac{s}{(s^2+10^2)}+\frac{1}{101}\frac{10}{(s^2+10^2)}$$
$$y(t)=\frac{10}{101}e^{-t}-\frac{10}{101}\cos(10t)+\frac{1}{101}\sin(10t)$$

$$y(t)=\frac{10}{101}e^{-t}+\frac{1}{\sqrt{101}}\sin(10t-84^\circ)$$



$$y_1=\frac{10}{101}e^{-t}$$
$$y_2=\frac{1}{\sqrt{101}}\sin(10t-84^\circ)$$

Frequency Response

$$u(t) = A\sin(\omega_i t) \Longrightarrow \boxed{\text{Linear system } G(s)} \Longrightarrow y(t) = y_{tr}(t) + y_{ss}(t)$$

$$y_{ss}(t) = B\sin(\omega_i t + \phi)$$

Gain of the system: $M = \dfrac{B}{A}$ [amplitude scaling]

Phase of the system: ϕ [shift in time]

Gain & phase vary with the frequency of the input sinusoid

$M(\omega)$ and $\phi(\omega)$ define the **frequency response** of the linear system

Frequency Response

$G(s)$: Transfer function of a linear system

$M(\omega_i)$: Gain of the system at frequency ω_i rad/s

$\phi(\omega_i)$: Phase of the system at frequency ω_i rad/s

Then,

$$M(\omega_i) = |G(j\omega_i)|, \quad \phi(\omega_i) = \angle G(j\omega_i)$$

Example: Find the gain and the phase at ω = 4 rad/s for $G(s) = \dfrac{10}{s^2 + 3s + 9}$

$$G(j4) = \frac{10}{(j4)^2 + 3(j4) + 9} = \frac{10}{-7 + j12} = \frac{10}{\sqrt{7^2 + 12^2} \angle \tan^{-1}\left(\dfrac{12}{-7}\right)} = 0.72\angle -120.25°$$

Example

$$G(s) = \frac{0.2}{s+1} \quad \Rightarrow \quad G(j\omega) = \frac{0.2}{j\omega+1} = |G(j\omega)| \cdot \angle G(j\omega) = \frac{0.2}{\sqrt{1+\omega^2}} \angle -\tan^{-1}\omega$$

$$\omega = 0.01 \quad \Rightarrow \quad |G(j\omega)| = 0.2, \ \angle G(j\omega) = -0.57^\circ$$

$$\omega = 0.1 \quad \Rightarrow \quad |G(j\omega)| = 0.199, \ \angle G(j\omega) = -5.71^\circ$$

$$\omega = 0.2 \quad \Rightarrow \quad |G(j\omega)| = 0.196, \ \angle G(j\omega) = -11.31^\circ$$

$$\omega = 0.5 \quad \Rightarrow \quad |G(j\omega)| = 0.179, \ \angle G(j\omega) = -26.57^\circ$$

$$\omega = 1 \quad \Rightarrow \quad |G(j\omega)| = 0.141, \ \angle G(j\omega) = -45^\circ$$

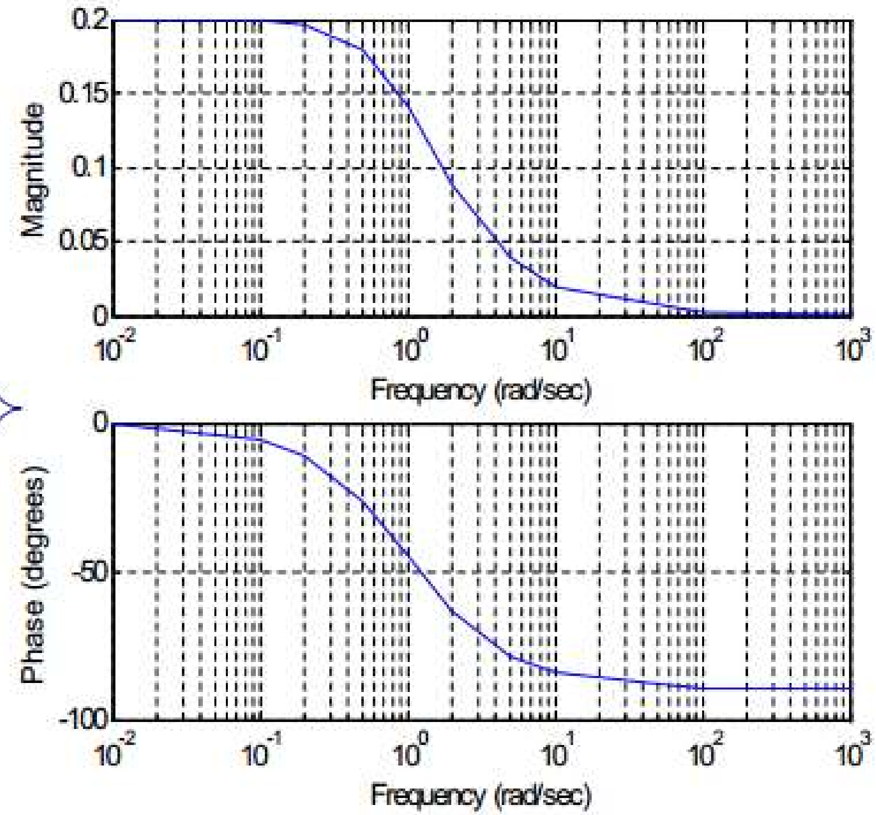
$$\omega = 2 \quad \Rightarrow \quad |G(j\omega)| = 0.089, \ \angle G(j\omega) = -63.43^\circ$$

$$\omega = 5 \quad \Rightarrow \quad |G(j\omega)| = 0.039, \ \angle G(j\omega) = -78.69^\circ$$

$$\omega = 10 \quad \Rightarrow \quad |G(j\omega)| = 0.020, \ \angle G(j\omega) = -84.29^\circ$$

$$\omega = 100 \quad \Rightarrow \quad |G(j\omega)| = 0.002, \ \angle G(j\omega) = -89.43^\circ$$

$$\omega = 1000 \quad \Rightarrow \quad |G(j\omega)| = 0.0002, \ \angle G(j\omega) = -89.94^\circ$$



Frequency Response

Modeling transfer function from the real data

Closed loop time response behavior can be predicted from the open loop frequency response.

Effect of noise and disturbance can be easily visualized. Time domain can be easily predicted using frequency domain specifications.

Gain Cross-over Frequency

For any system, there may exist one or more frequencies at which the gain of the system is unity. This frequency is called gain cross-over frequency.

$$\left|G(j\omega_{cg})\right| = 1$$

Find the gain-cross over frequency ➡ $G(s) = \dfrac{10}{s^2 + 3s + 9}$

Solution:

$$\left|G(j\omega_{cg})\right| = 1$$

$$\Rightarrow \left|\frac{10}{(j\omega_{cg})^2 + 3j\omega_{cg} + 9}\right| = 1 \Rightarrow \left|\frac{10}{9 - \omega_{cg}{}^2 + j3\omega_{cg}}\right| = 1$$

$$\Rightarrow \frac{10}{\sqrt{\left(9 - \omega_{cg}{}^2\right)^2 + (3\omega_{cg})^2}} = 1 \Rightarrow \left(9 - \omega_{cg}{}^2\right)^2 + (3\omega_{cg})^2 = 100$$

$$\Rightarrow \omega_{cg}{}^2 = 10.765 \Rightarrow \omega_{cg} = 3.281\, rad/s$$

Gain Cross-over Frequency - Example

Find the gain-crossover frequency of the following transfer function

$$G(s) = 100\frac{(s+10)}{(s+1)(s+50)}$$

Use MATLAB to verify your result.

Solution:

$$|G(j\omega)|^2 = 100^2\frac{|j\omega+10|^2}{|j\omega+1|^2\cdot|j\omega+50|^2}$$

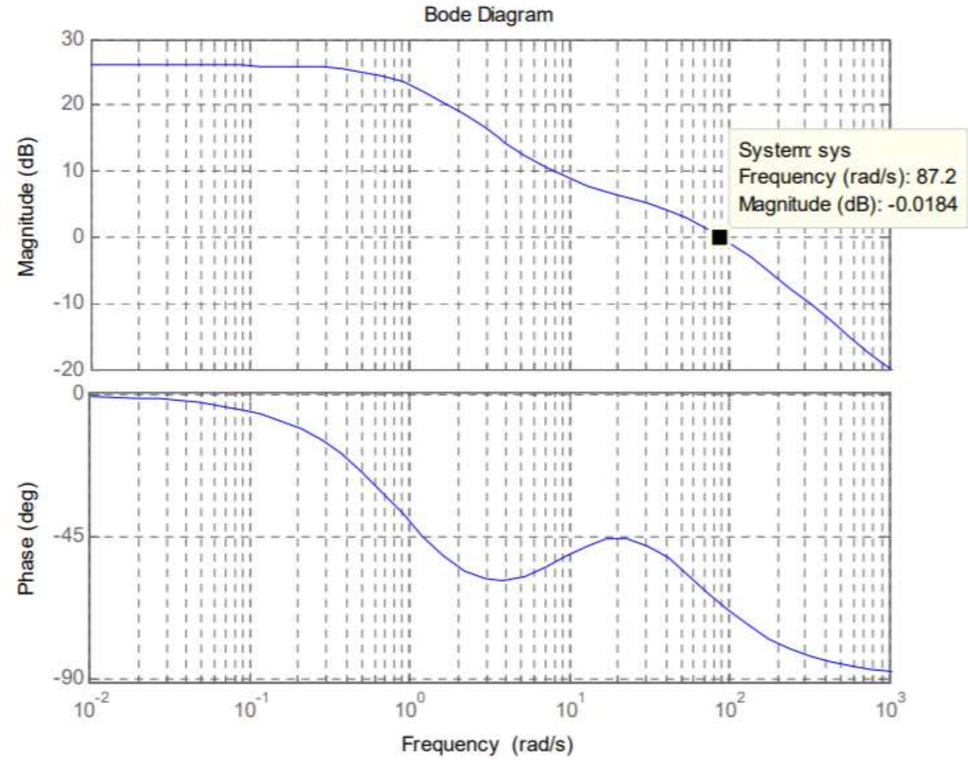
$$= 100^2\frac{\omega^2+10^2}{(\omega^2+1)\cdot(\omega^2+50^2)}$$

$$= 1$$

$$\Rightarrow \quad (\omega^2)^2 - 7499\omega^2 - 997500 = 0$$

$$\Rightarrow \quad \omega^2 = 7629.7384$$

$$\Rightarrow \quad \omega = 87.35 \text{ rad/s}$$

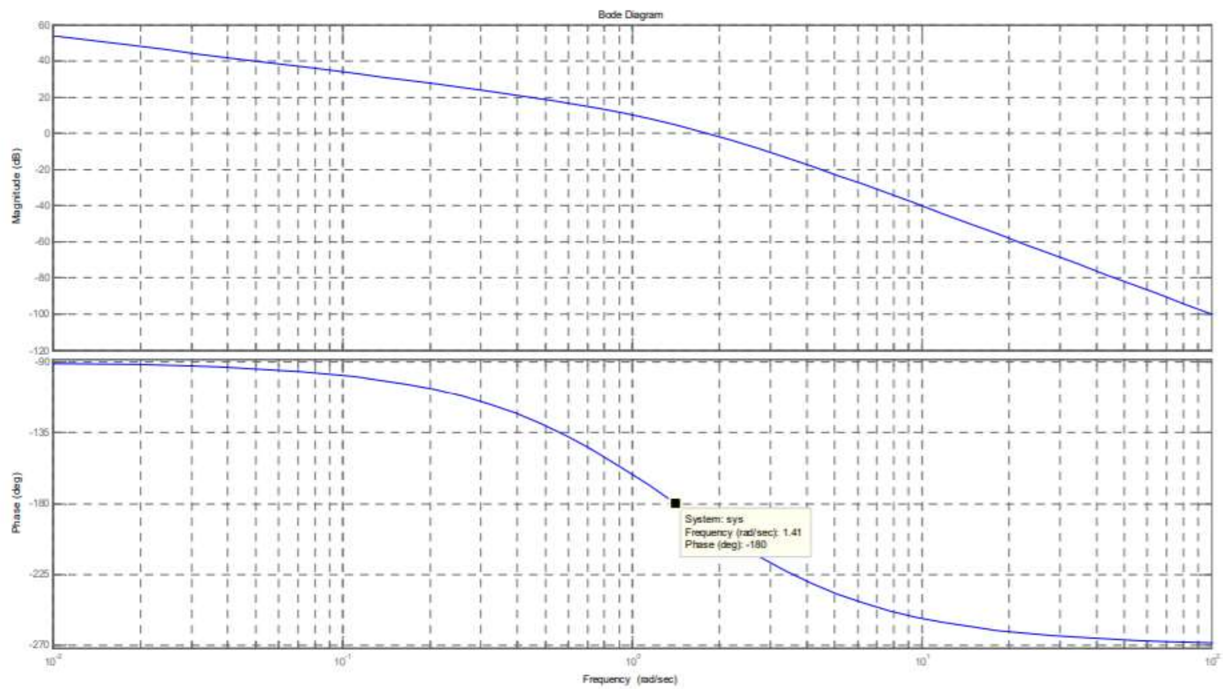


Phae Cross-over Frequency

For any system, there may exist one or more frequencies at which the phase of the system is $\pm 180°$. Such a frequency is called the **phase-crossover frequency**. We'll use the symbol $\boldsymbol{\omega_{cp}}$ to represent it, i.e.,

$$\angle G(j\omega_{cp}) = \pm 180°$$

Exercise 1-3: Find the phase-crossover frequency ω_{cp} of $G(s) = \dfrac{10}{s(s+1)(s+2)}$



Frankly, it is really troublesome to calculate the crossover frequencies (gain and frequency) using pen and paper…