Gain Cross-over Frequency

For any system, there may exist one or more frequencies at which the gain of the system is unity. This frequency is called gain cross-over frequency.

$$\left|G(j\omega_{cg})\right| = 1$$

Find the gain-cross over frequency ➡ $G(s) = \dfrac{10}{s^2 + 3s + 9}$

Solution:

$$\left|G(j\omega_{cg})\right| = 1$$

$$\Rightarrow \left|\frac{10}{(j\omega_{cg})^2 + 3j\omega_{cg} + 9}\right| = 1 \Rightarrow \left|\frac{10}{9 - \omega_{cg}{}^2 + j3\omega_{cg}}\right| = 1$$

$$\Rightarrow \frac{10}{\sqrt{\left(9 - \omega_{cg}{}^2\right)^2 + (3\omega_{cg})^2}} = 1 \Rightarrow \left(9 - \omega_{cg}{}^2\right)^2 + (3\omega_{cg})^2 = 100$$

$$\Rightarrow \omega_{cg}{}^2 = 10.765 \Rightarrow \omega_{cg} = 3.281 rad/s$$

Gain Cross-over Frequency - Example

Find the gain-crossover frequency of the following transfer function

$$G(s) = 100\frac{(s+10)}{(s+1)(s+50)}$$

Use MATLAB to verify your result.

Solution:

$$|G(j\omega)|^2 = 100^2 \frac{|j\omega+10|^2}{|j\omega+1|^2 \cdot |j\omega+50|^2}$$

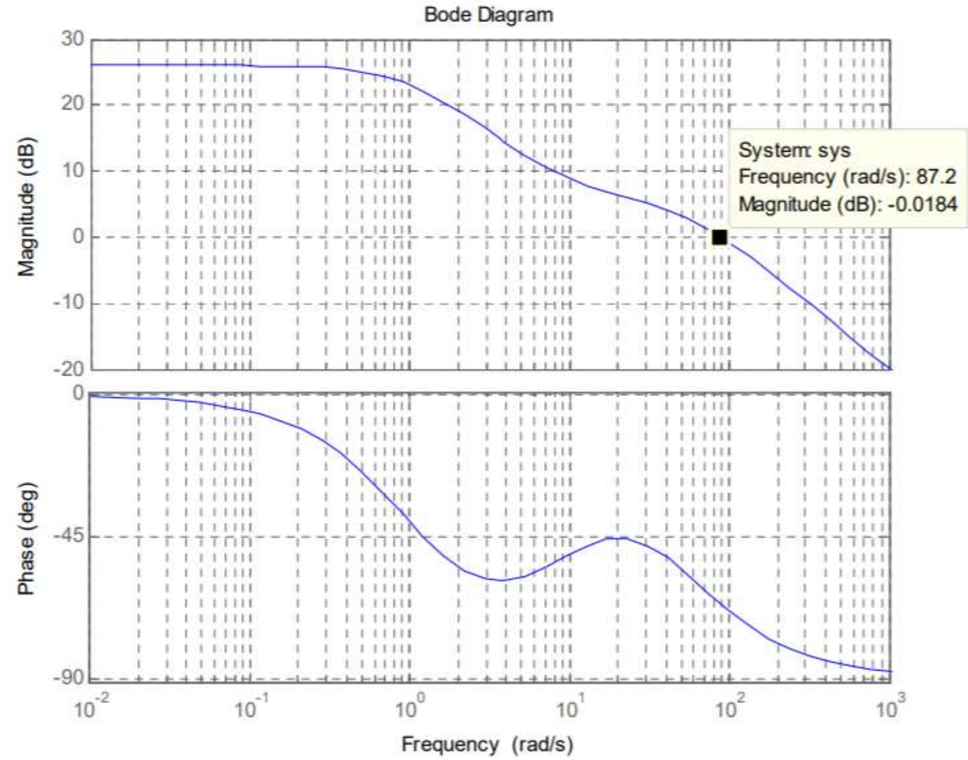
$$= 100^2 \frac{\omega^2+10^2}{(\omega^2+1)\cdot(\omega^2+50^2)}$$

$$= 1$$

$$\Rightarrow \quad (\omega^2)^2 - 7499\omega^2 - 997500 = 0$$

$$\Rightarrow \quad \omega^2 = 7629.7384$$

$$\Rightarrow \quad \omega = 87.35 \text{ rad/s}$$

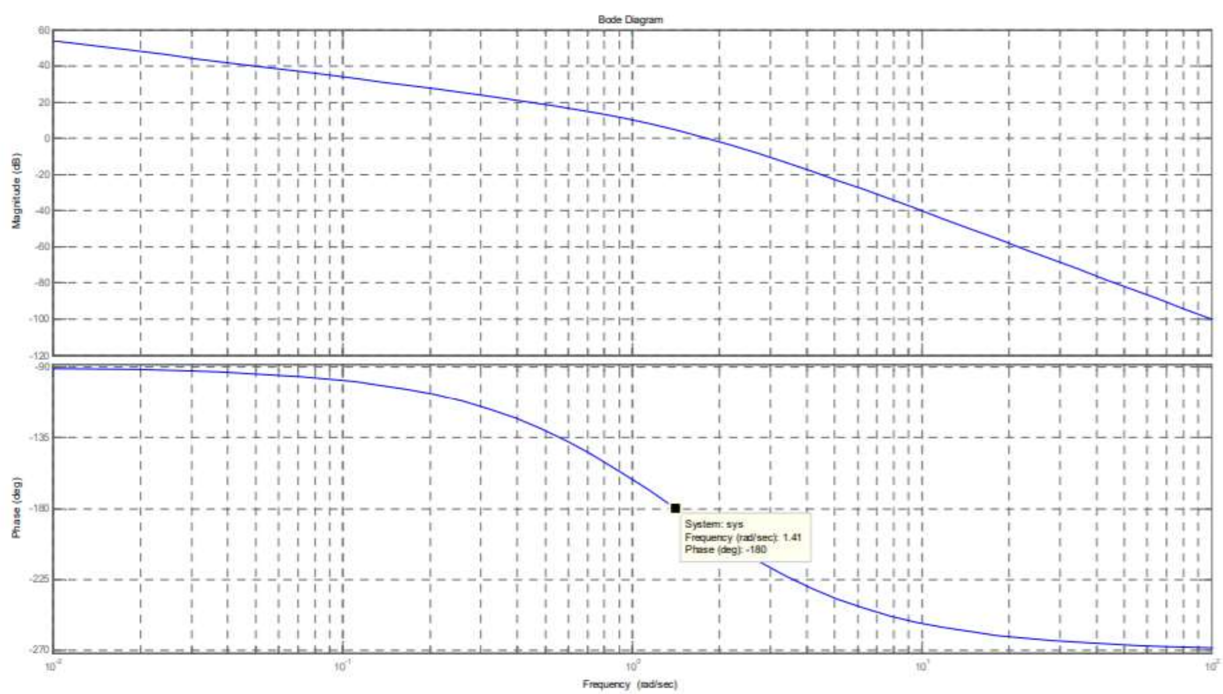


Phae Cross-over Frequency

For any system, there may exist one or more frequencies at which the phase of the system is $\pm 180°$. Such a frequency is called the **phase-crossover frequency**. We'll use the symbol ω_{cp} to represent it, i.e.,

$$\angle G(j\omega_{cp}) = \pm 180°$$

Exercise 1-3: Find the phase-crossover frequency ω_{cp} of $G(s) = \dfrac{10}{s(s+1)(s+2)}$



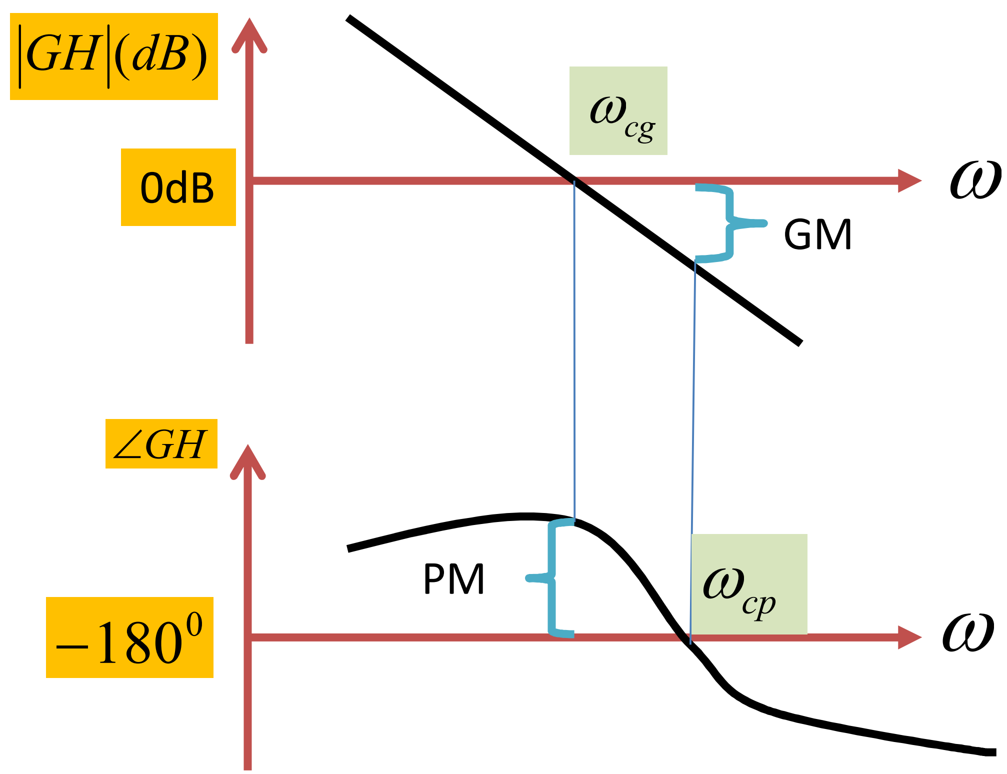
Frankly, it is really troublesome to calculate the crossover frequencies (gain and frequency) using pen and paper…

Relative stability

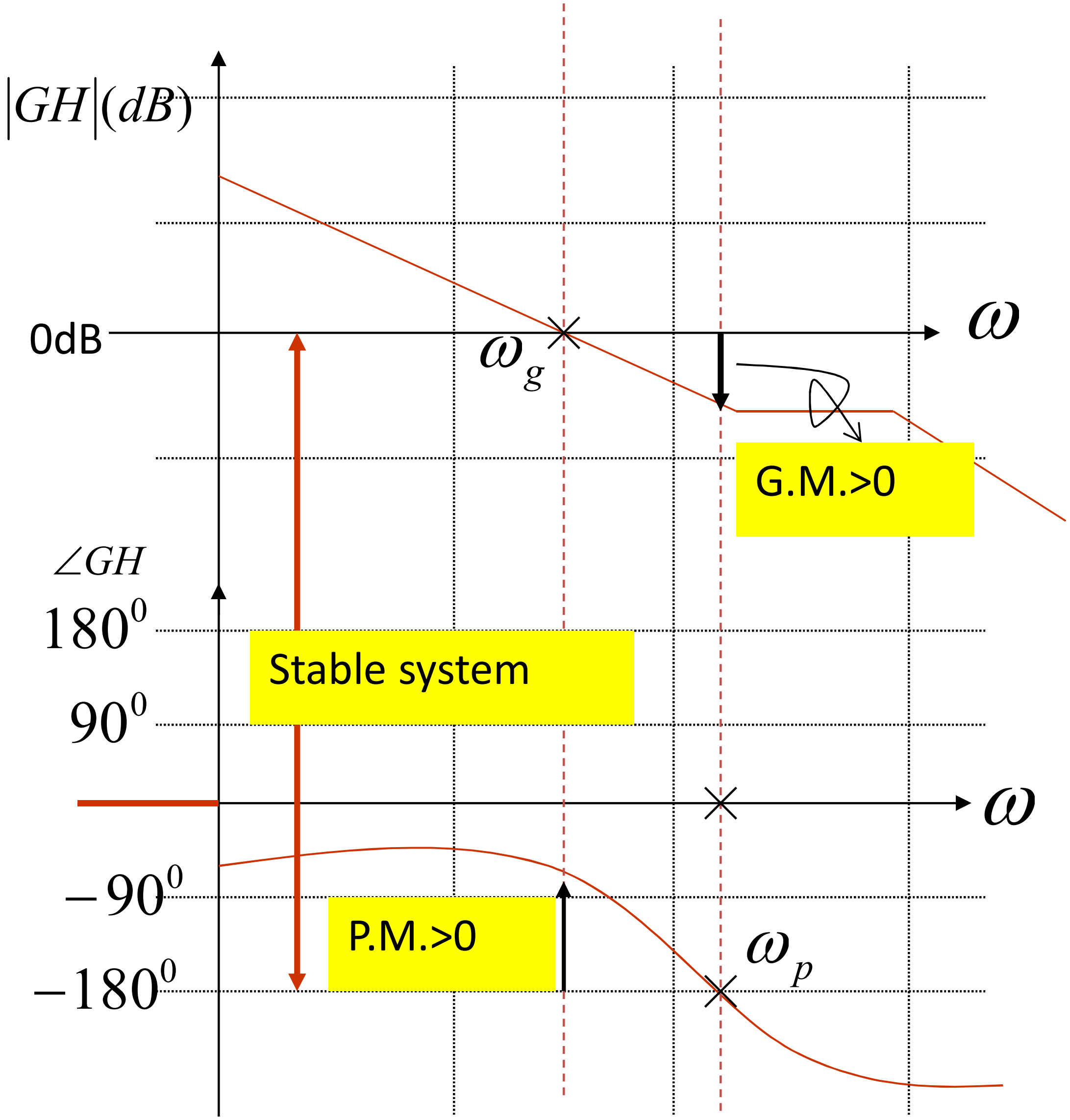
A transfer function is called **minimum phase** when all the poles and zeros are LHP and **non-minimum-phase** when there are RHP poles or zeros.

Minimum phase system → Stable

The gain margin (GM) is the distance on the bode magnitude plot from the amplitude at the phase crossover frequency up to the 0 dB point. GM=-(dB of GH measured at the phase crossover frequency)



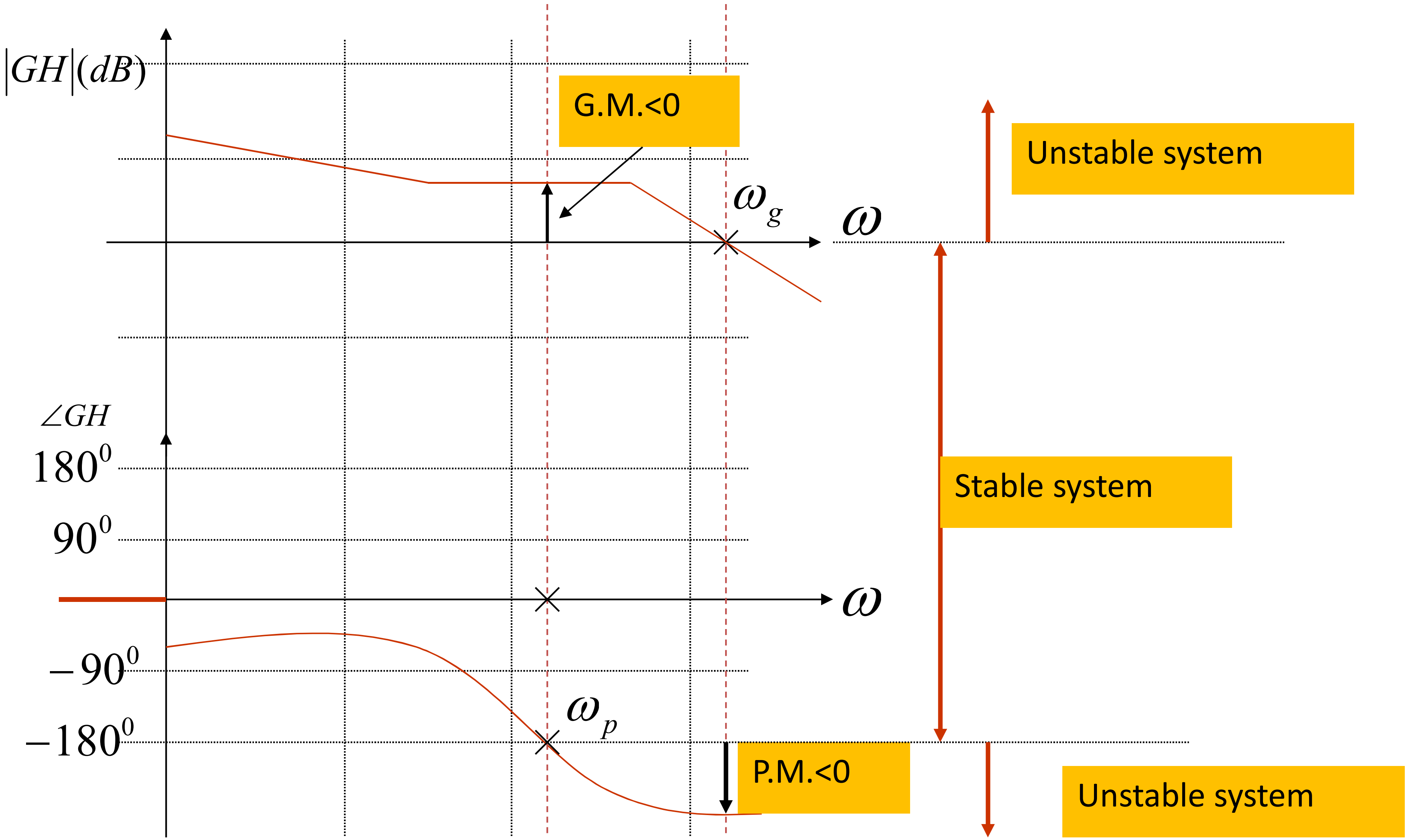
The phase margin (PM) is the distance from -180 up to the phase at the gain crossover frequency. PM=180+phase of GH measured at the gain crossover frequency



$$(-1,0) \Rightarrow \begin{cases} 0dB \\ -180^0 \end{cases}$$

Gain crossover frequency: ω_g

phase crossover frequency: ω_p

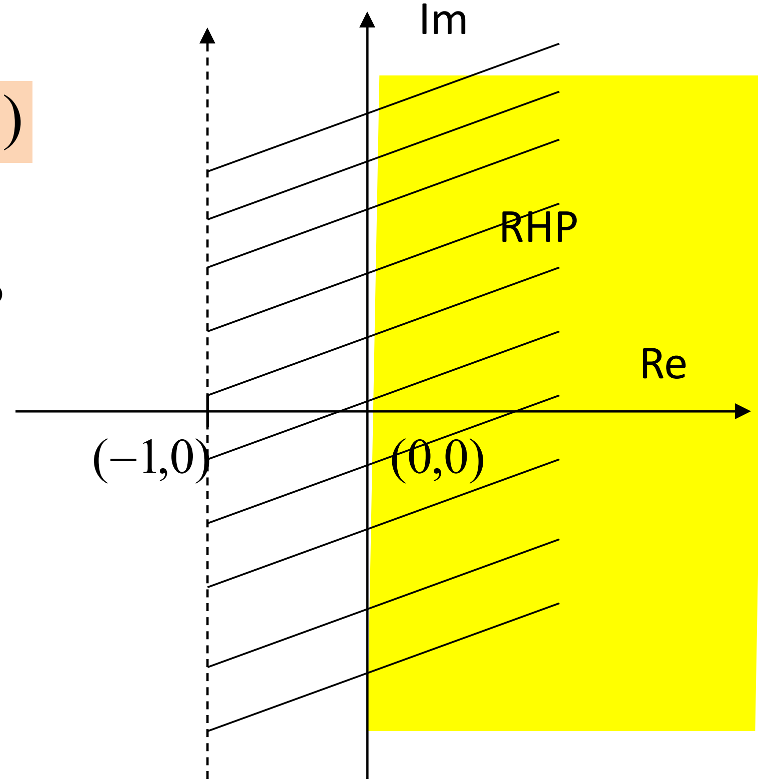

$|GH|(dB)$
G.M.<0
ω_g
ω
Unstable system
Stable system
$\angle GH$
180^0
90^0
ω
-90^0
ω_p
-180^0
P.M.<0
Unstable system

Open loop transfer function : $G(s)H(s)$

Closed-loop transfer function : $1+G(s)H(s)$

Open loop Stability → poles of $G(s)H(s)$ in LHP

Closed-loop Stability :
poles of $G(s)H(s)$ in left side of (-1,0)



Aircraft Pitch Dynamics: Frequency Domain Methods for Controller Desgign *(Ref. Matlab Control System Tutorials)*

T.F. of Pitch Dynamics

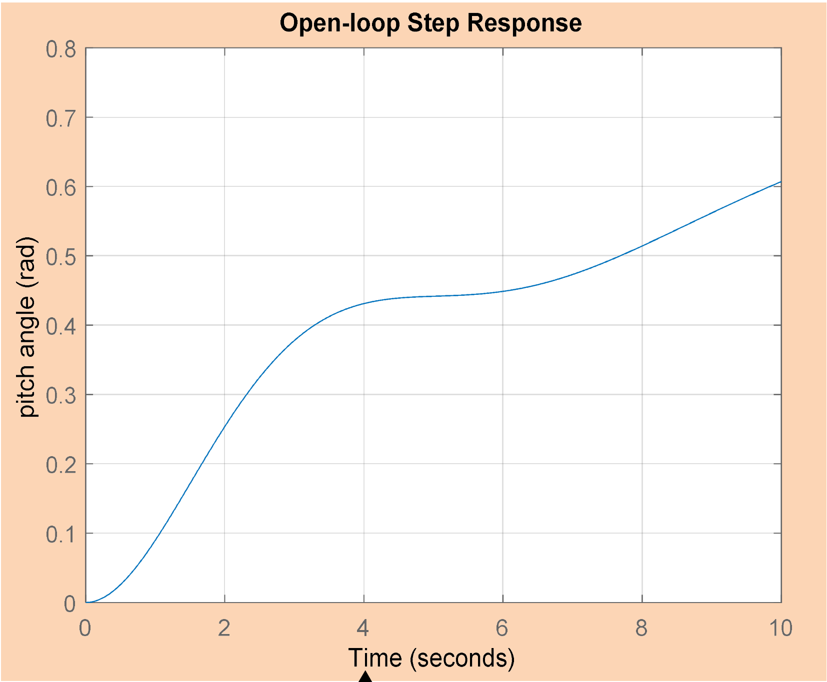
$$P(s) = \frac{\theta(s)}{\Delta(s)} = \frac{1.151s + 0.1774}{s^3 + 0.739s^2 + 0.921}$$

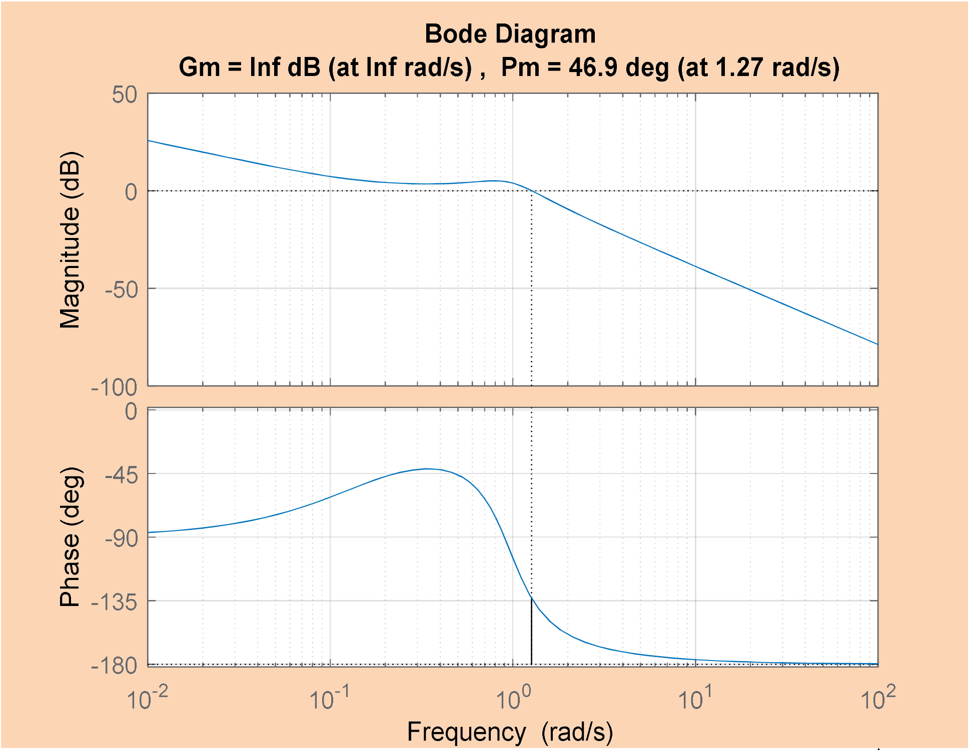
Mission Requirements:

For a step reference of 0.2 radians, the design criteria are the following:

1. Overshoot less than 10%
2. Rise time less than 2 seconds
3. Settling time less than 10 seconds
4. Steady-state error less than 20%

Open Loop Response:

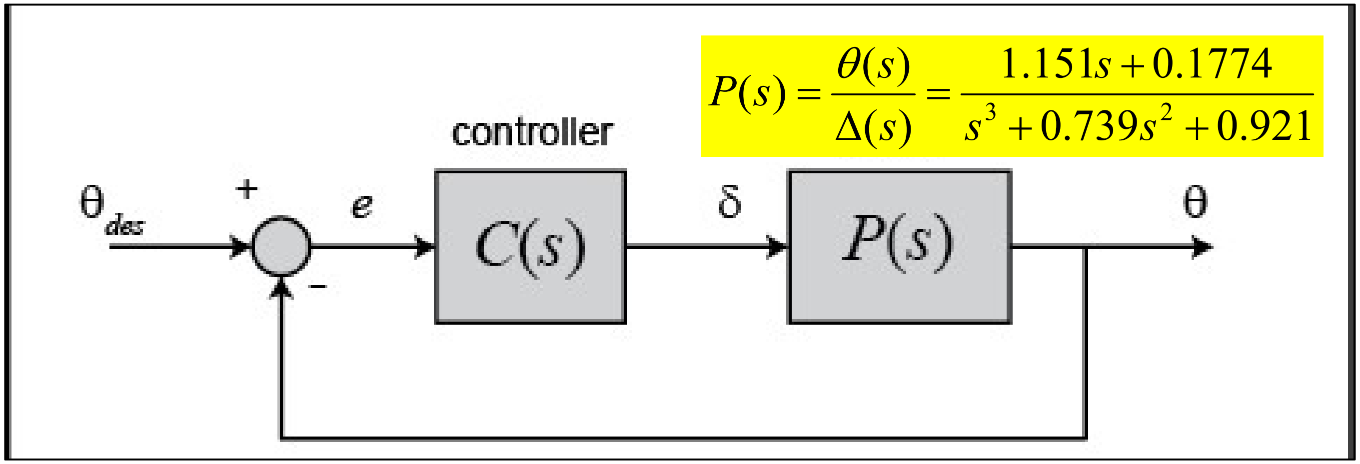


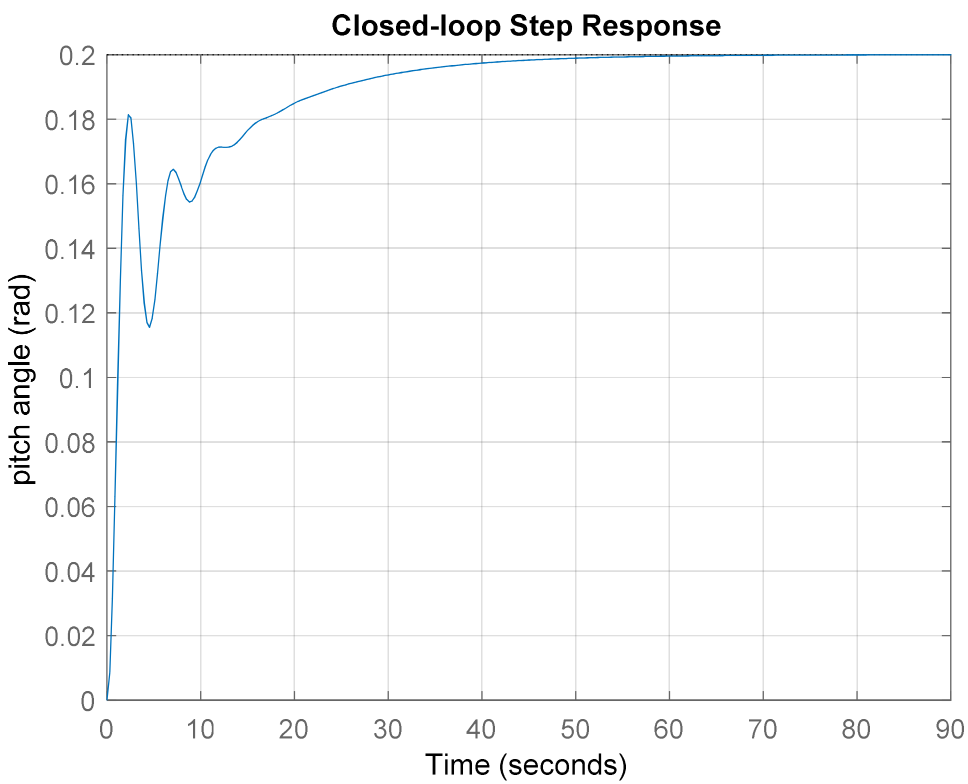


Examination of the above plot indicates that the open-loop system is unstable for a step input, that is, its output grows unbounded when given a step input. This is due to the fact that the transfer function has a pole at the origin.

Examination of the above demonstrates that the closed-loop system is indeed stable since the phase margin and gain margin are both positive. Specifically, the phase margin equals 46.9 degrees and the gain margin is infinite

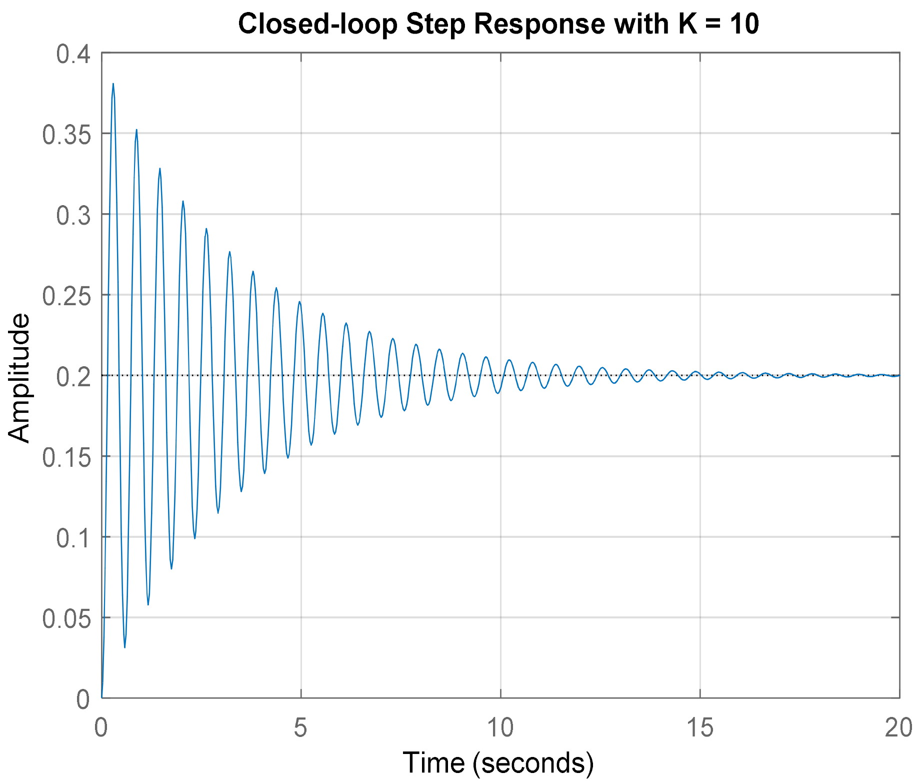
Closed-loop response

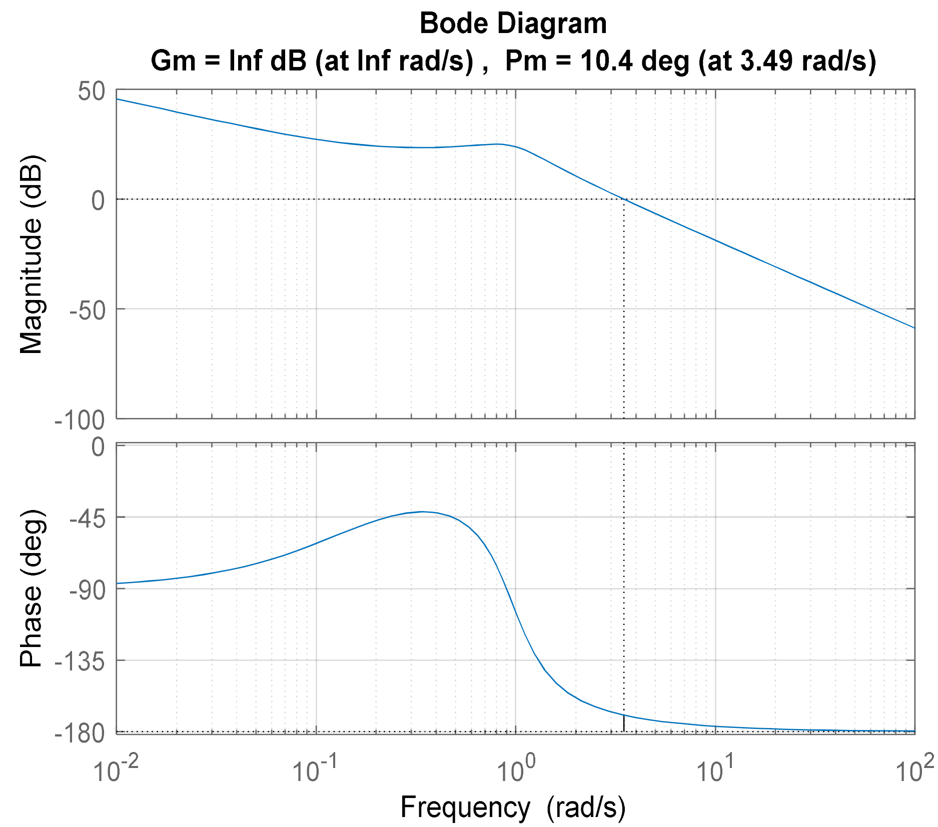




settle time requirement of 10 seconds is not close to being met.

Effect of Gain in time and frequency response





It can be observed:

1. the system's magnitude at all frequencies and have pushed the gain crossover frequency higher.
2. the addition of the K has also reduced the system's phase margin as evidenced by the increased overshoot in the system's step response.

Compensator

One way to address this is to make the system response faster, but then the overshoot shown above will likely become a problem. Therefore, the overshoot must be reduced in conjunction with making the system response faster. We can accomplish these goals by adding a compensator to reshape the Bode plot of the open-loop system.

The Bode plot of the open-loop system indicates behavior of the closed-loop system.

1. the gain crossover frequency is directly related to the closed-loop system's speed of response, and
2. the phase margin is inversely related to the closed-loop system's overshoot.

$$\boxed{C(s) = K\frac{Ts+1}{\alpha Ts+1} \quad (\alpha < 1)}$$

$$\boxed{C(s) = K_c\frac{(s-z_0)}{(s-p_0)}}$$

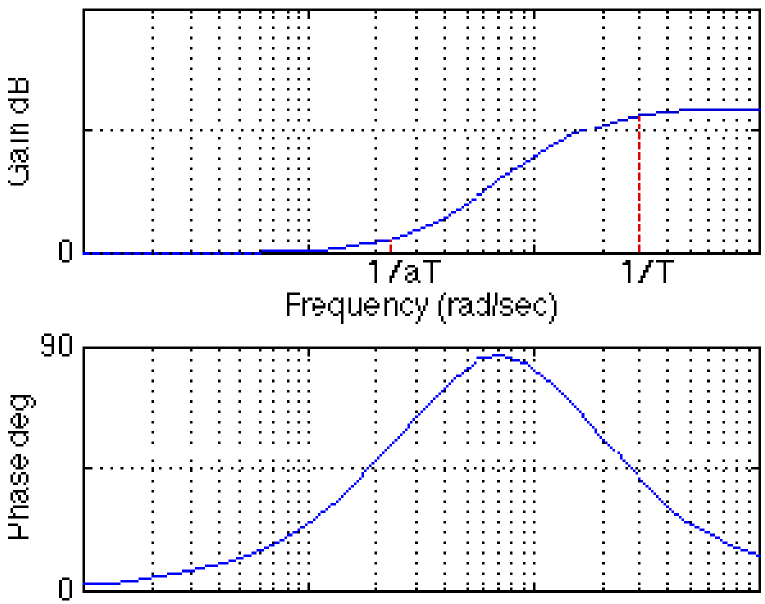
Phase lear or lead compensator

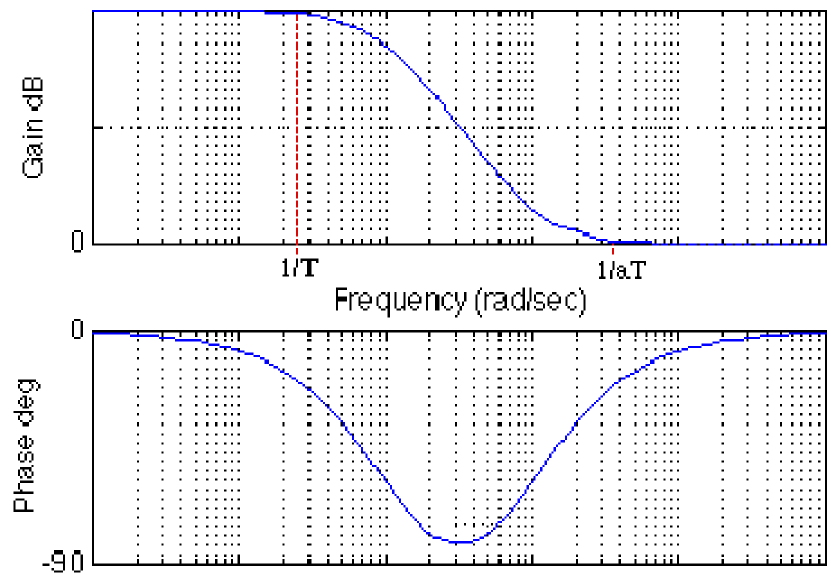
Adds positive phase to system->increases phase marging->increases damping
Increases magnitude of OLTF at higher freq. Increases gain cross-over frequency and thus speed->settling time would reduce

Lag-Lead compensator

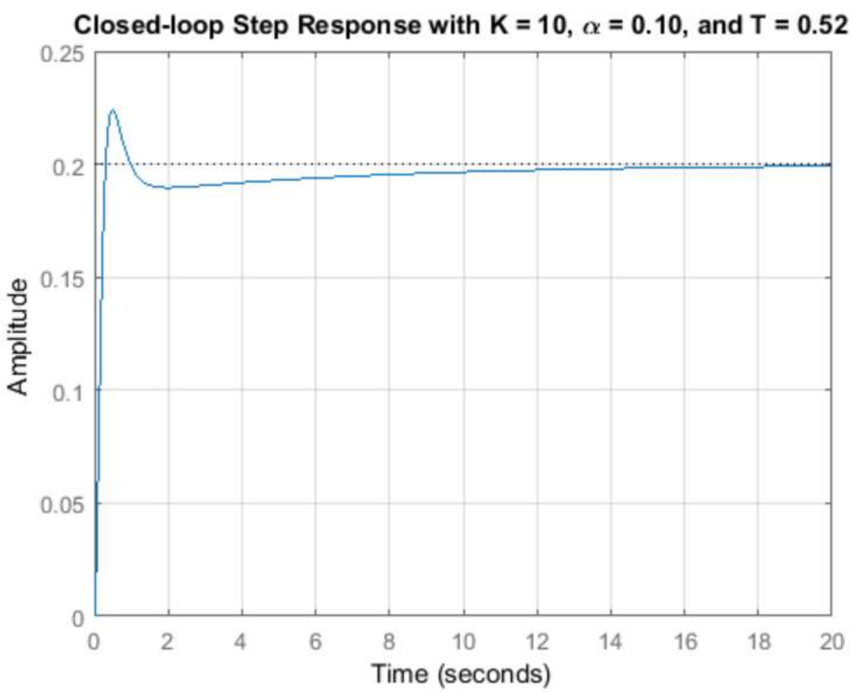
$$C(s) = \frac{1}{a}\left(\frac{1+aTs}{1+Ts}\right) \qquad [a<1]$$

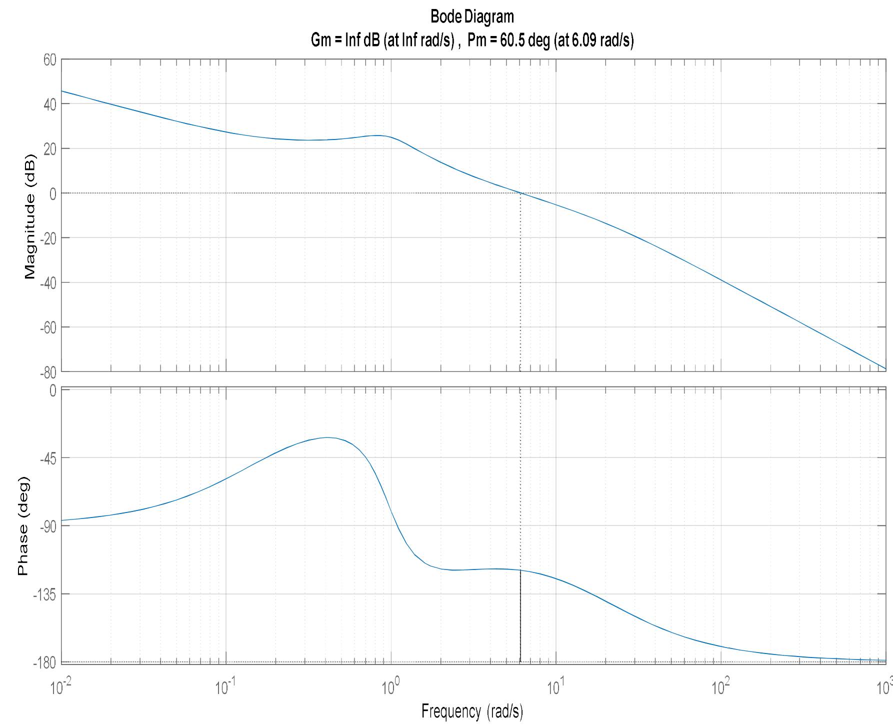
$$C(s) = \frac{1+aTs}{1+Ts} \qquad [a>1]$$





System with compensator





Examination of the above demonstrates that we are close to meeting our requirements. However, overshoot which is a bit larger than the requirement of 10%.

this compensator is able to satisfy all of our design requirements

$$C(s) = 10\frac{0.55s+1}{0.022s+1}$$