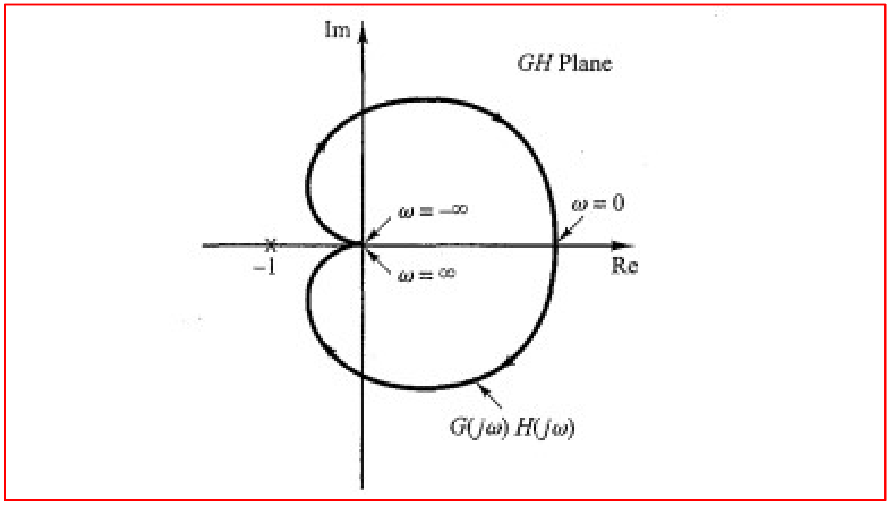
Examples

Consider a closed-loop system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)}$$

Examine the stability of the system.



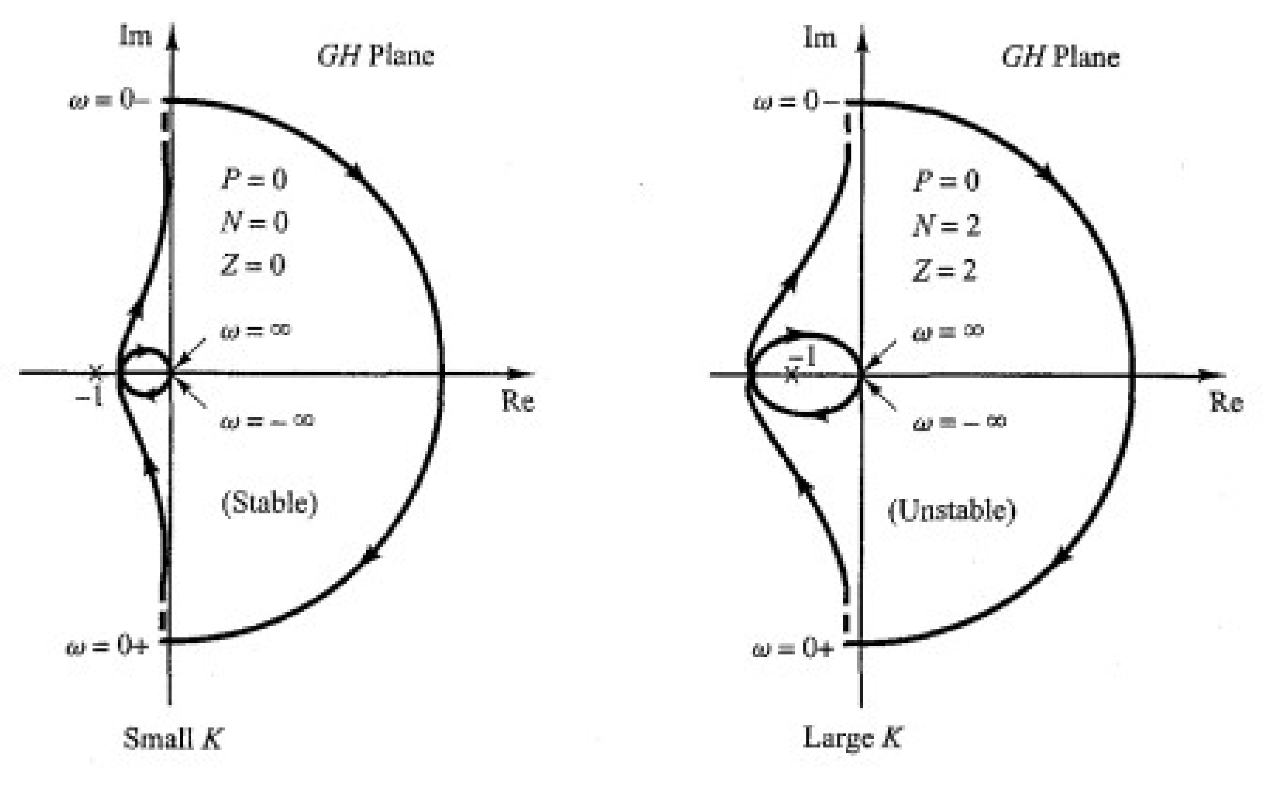
Since *G(s)H(s)* does not have any poles in the right-half s plane and the -1 + *jO* point is not encircled by the *G(jw)*H*(jw)* locus, this system is stable for any positive values of K, *T,,*and *T,.*

Examples

Consider the system with the following open-loop transfer function:

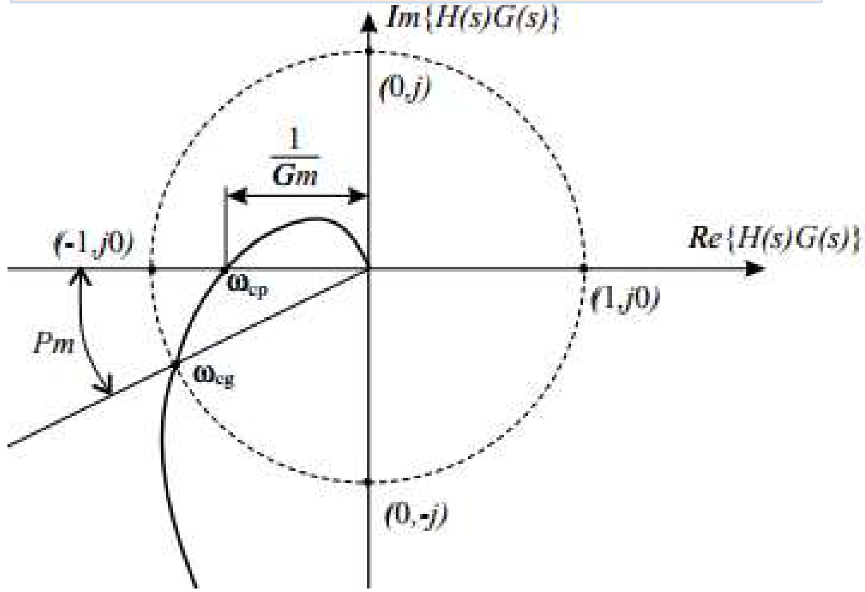
$$G(s)H(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}$$

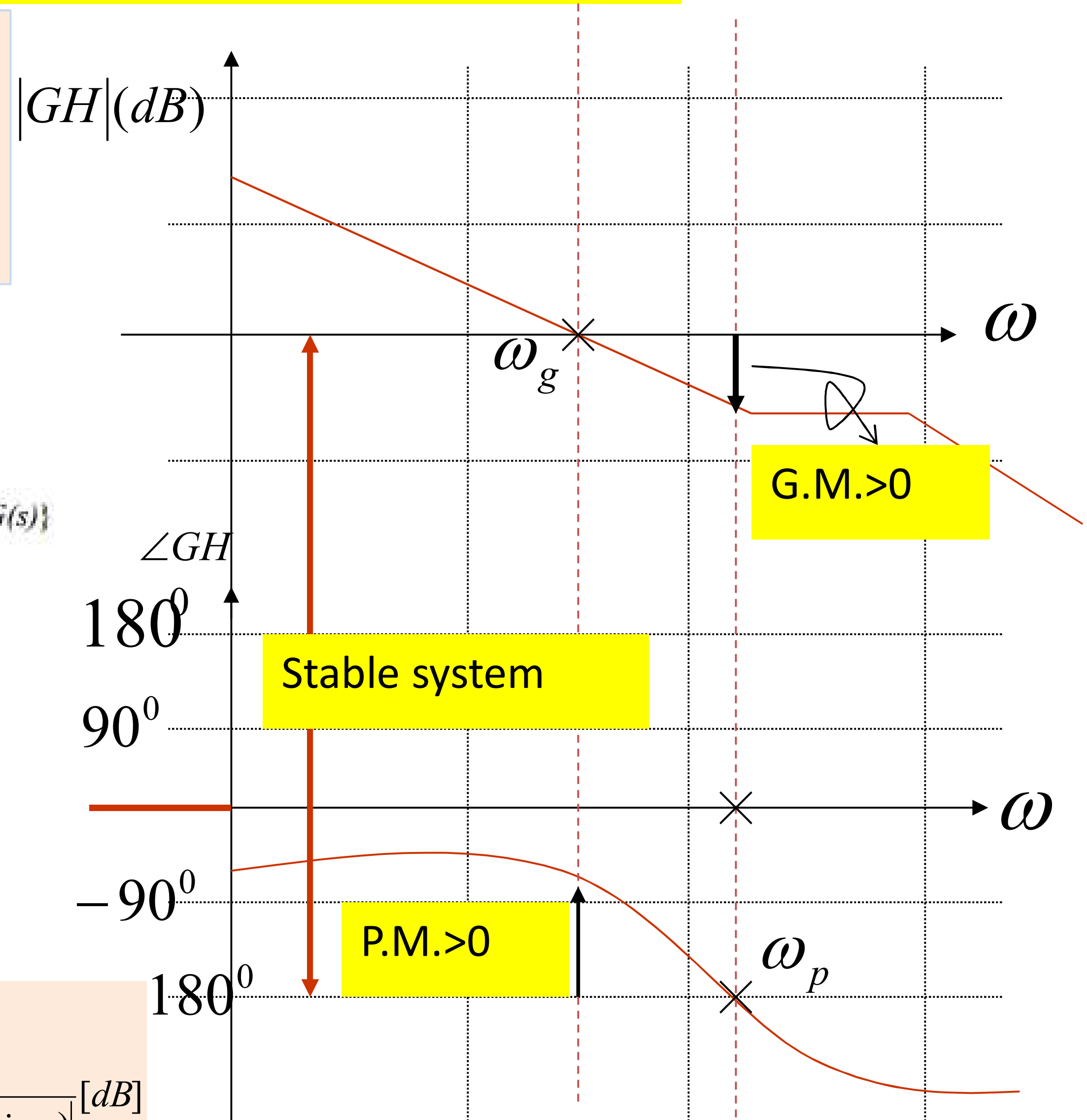
Determine the stability of the system for two cases: (1) the gain K is small and (2) K is large.



Relative Stability Analysis using Nyquist Plot- Gain and Phase Margins

They give the degree of relative stability; in other words, they tell how far the given system is from the instability region.





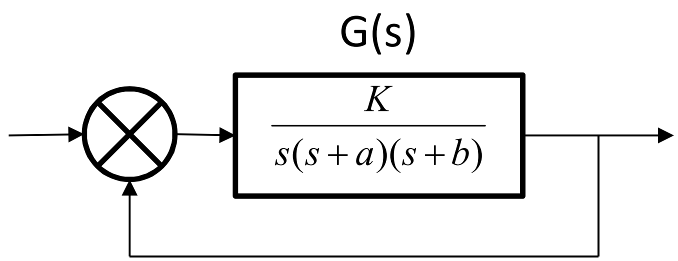
$$Pm = 180^0 + \arg\{G(j\omega_{cg})H(j\omega_{cg})\}$$

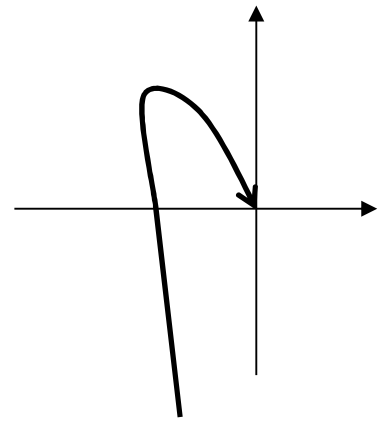
$$GM = \frac{1}{|G(j\omega_{cp})H(j\omega_{cp})|} \Rightarrow GM[dB] = 20\log\frac{1}{|G(j\omega_{cp})H(j\omega_{cp})|}[dB]$$

$$|G(j\omega_{cg})H(j\omega_{cg})| = 1 \Rightarrow \omega_{cg}$$

$$\arg\{G(j\omega_{cg})H(j\omega_{cg})\} = 180^0 \Rightarrow \omega_{cp}$$

Example- Gain Margin and Phase Margin



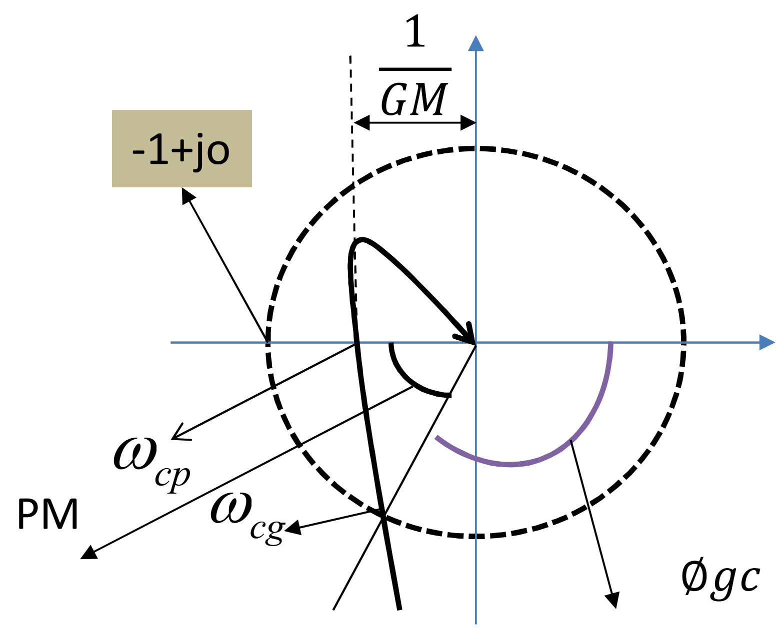


$$G(j\omega) = \frac{K}{j\omega(j\omega + a)(j\omega + b)}$$

$$\Rightarrow |G(j\omega)| = \frac{K}{\omega\sqrt{\omega^2 + a^2}\sqrt{\omega^2 + b^2}}$$

$$GM = \frac{\omega\sqrt{\omega^2 + a^2}\sqrt{\omega^2 + b^2}}{K}$$

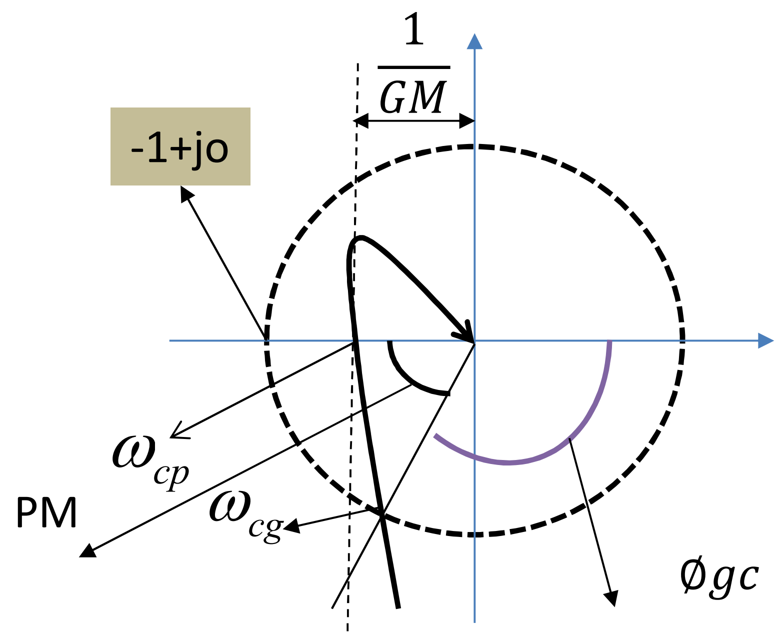
$$PM = 180^0 + \phi gc$$



Stable System Polar Plot

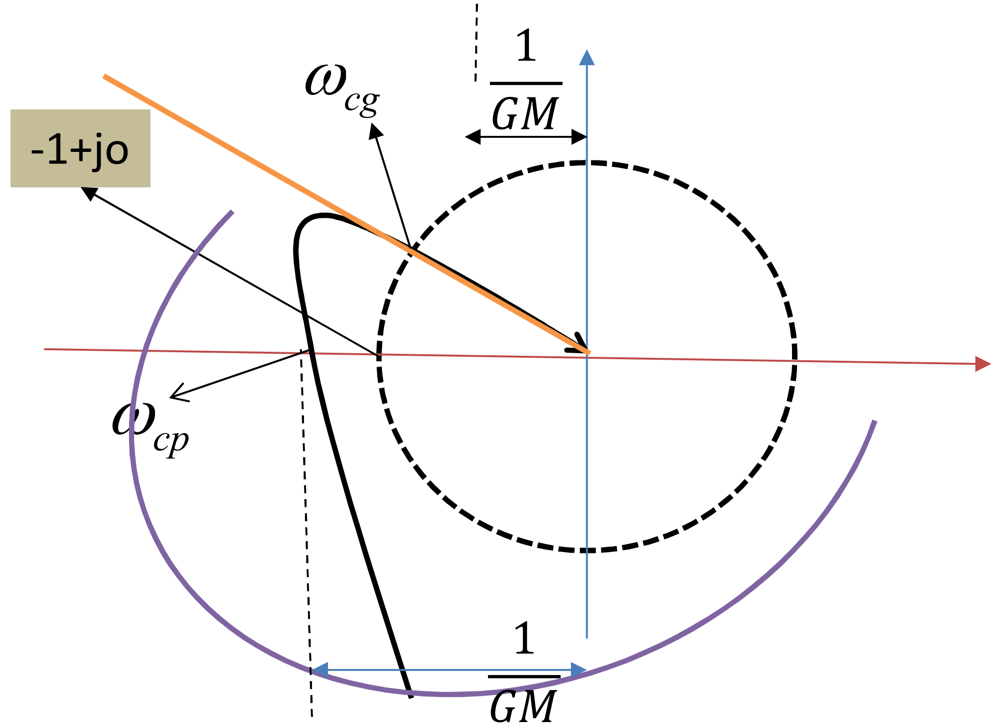
If both the phase and gain margins are Positive, the system is stable.

1. If $\boldsymbol{\omega_{cg}}$ exists below -180 Then PM is +ve.
2. If $\boldsymbol{\omega_{cp}}$ exists inside unit circle or magnitude corresponding to $\boldsymbol{\omega_{cp}}$ less than one, the gain margin is positive.

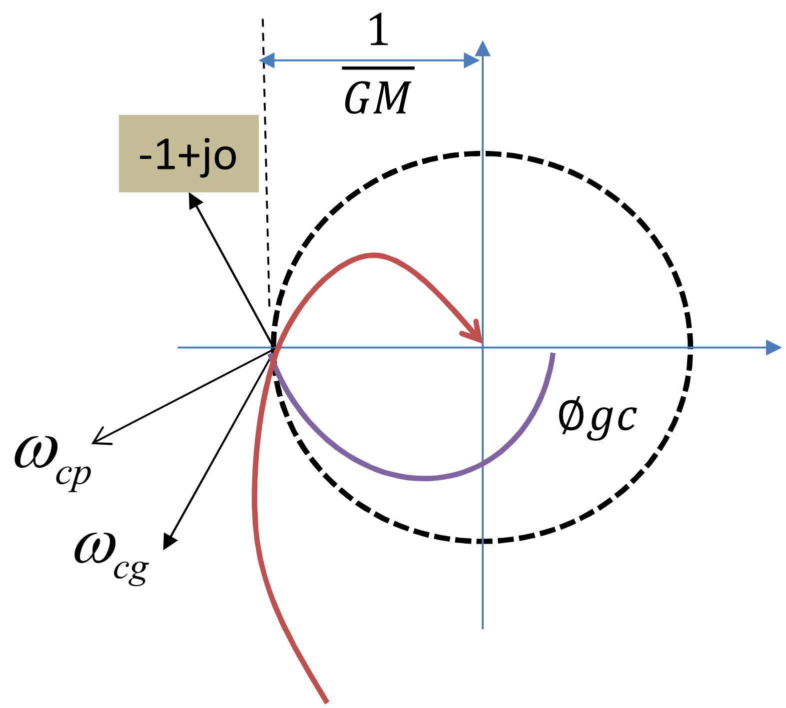


Unstable System Polar Plot

Gain and phase margins are Negative. Indirectly, the phase margin is greater than one and phase margin crosses -180 line.



Marginally System Polar Plot



$$\phi gc = -180^0$$

$$PM = 180^0 - 180^0 = 0^0$$

$$M = 1;$$

$$GM = 1/1 = 1; GM_{dB} = 20 \log 1 = 0 dB$$

Examples

Nyquist plot consider a feedback system where the OLTF is

$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)(1+sT_3)}$$

Draw Nyquist plot. Find also the range of K in terms of the crossover frequency ω_{pc} for stability.

Sol.

Given :

$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

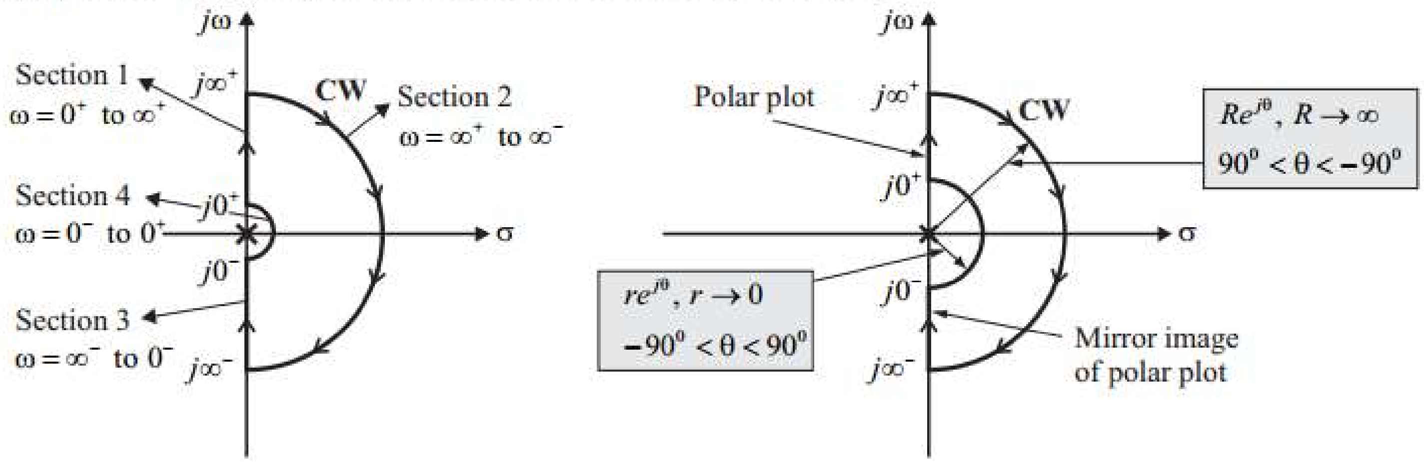
Magnitude can be written as,

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2T_1^2}\,\sqrt{1+\omega^2T_2^2}\,\sqrt{1+\omega^2T_3^2}}$$

Phase angle can be written as,

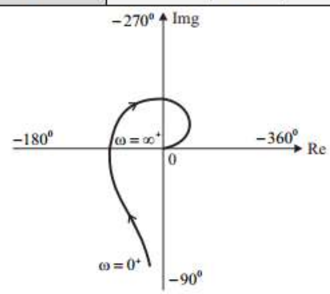
$$\angle G(j\omega) = -90^0 - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

Nyquist path for the given transfer function is shown below.



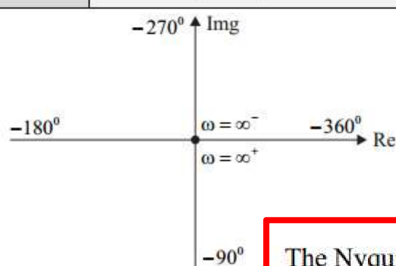
Section 1 : Polar plot

At $\omega = 0^+$	$\|G(j\omega)\| = \infty$	$\angle G(j\omega) = -90^0$
At $\omega = \infty^+$	$\|G(j\omega)\| = 0$	$\angle G(j\omega) = -360^0$



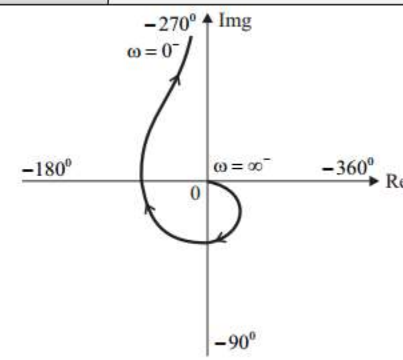
Section 2 : Semi-circle with radius tending to infinite.

At $\omega = \infty^+$	$\|G(j\omega)\| = 0$	$\angle G(j\omega) = -360^0$
At $\omega = \infty^-$	$\|G(j\omega)\| = 0$	$\angle G(j\omega) = 360^0$



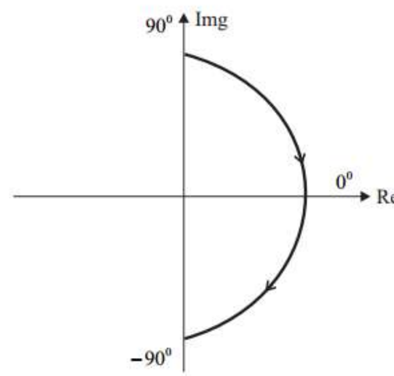
Section 3 : Mirror image of polar plot

At $\omega = \infty^-$	$\|G(j\omega)\| = 0$	$\angle G(j\omega) = 360^0$
At $\omega = 0^-$	$\|G(j\omega)\| = \infty$	$\angle G(j\omega) = 90^0$

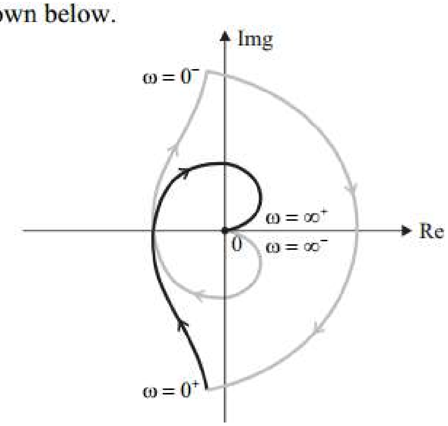


Section 4 : Semi-circle with radius tending to zero.

At $\omega = 0^+$	$\|G(j\omega)\| = \infty$	$\angle G(j\omega) = -90^0$
At $\omega = 0^-$	$\|G(j\omega)\| = \infty$	$\angle G(j\omega) = 90^0$



The Nyquist plot is shown below.



The frequency at which phase angle of $G(j\omega)H(j\omega)$ is -180^0 is called phase crossover frequency.

$$-180^0 = -90^0 - \tan^{-1}(\omega_{pc}T_1) - \tan^{-1}(\omega_{pc}T_2) - \tan^{-1}(\omega_{pc}T_3)$$

$$-90^0 = -\tan^{-1}(\omega_{pc}T_1) - \tan^{-1}(\omega_{pc}T_2) - \tan^{-1}(\omega_{pc}T_3)$$

$$\tan 90^0 = \left[\frac{\dfrac{\omega_{pc}T_1 + \omega_{pc}T_2}{1-\omega_{pc}^2 T_1 T_2} + \omega_{pc}T_3}{1 - \dfrac{\omega_{pc}T_1 + \omega_{pc}T_2}{1-\omega_{pc}^2 T_1 T_2} \cdot \omega_{pc}T_3}\right]$$

$$\infty \text{ or } \frac{1}{0} = \left[\frac{\dfrac{\omega_{pc}T_1 + \omega_{pc}T_2}{1-\omega_{pc}^2 T_1 T_2} + \omega_{pc}T_3}{1 - \dfrac{\omega_{pc}T_1 + \omega_{pc}T_2}{1-\omega_{pc}^2 T_1 T_2} \cdot \omega_{pc}T_3}\right]$$

$$1 - \omega_{pc}^2 \left[T_1T_2 + T_2T_3 + T_3T_1\right] = 0$$

$$\therefore \quad \omega_{pc} = \frac{1}{\sqrt{T_1T_2 + T_2T_3 + T_3T_1}}$$

$$\left|G(j\omega_{pc})\right| = \frac{K}{\omega_{pc}\sqrt{(\omega_{pc}^2 T_1^2 + 1)(\omega_{pc}^2 T_2^2 + 1)(\omega_{pc}^2 T_3^2 + 1)}}$$

Gain margin can be defined as reciprocal of the magnitude of the $G(j\omega)$ measured at phase crossover frequency. For the system to be stable, gain margin measured in dB should be positive or $G(j\omega)$ measured at phase crossover frequency should be less than 1.

$$\frac{K}{\omega_{pc}\sqrt{(\omega_{pc}^2 T_1^2 + 1)(\omega_{pc}^2 T_2^2 + 1)(\omega_{pc}^2 T_3^2 + 1)}} < 1$$

$$K < \omega_{pc}\sqrt{(\omega_{pc}^2 T_1^2 + 1)(\omega_{pc}^2 T_2^2 + 1)(\omega_{pc}^2 T_3^2 + 1)} \quad \text{where} \quad \omega_{pc} = \frac{1}{\sqrt{T_1T_2 + T_2T_3 + T_3T_1}}$$

Examples

A unity feedback system has open-loop transfer function

$$G(s) = \frac{1}{s(2s+1)(s+1)}$$

Sketch Nyquist plot for the system and from there obtain the gain margin and the phase margin.

Sol.

Given : $$G(s) = \frac{1}{s(2s+1)(s+1)}$$

Put $s = j\omega$, $$G(j\omega) = \frac{1}{j\omega(2j\omega+1)(j\omega+1)}$$

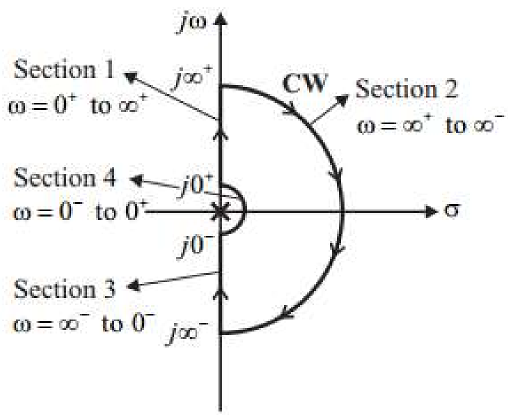
Magnitude can be written as,

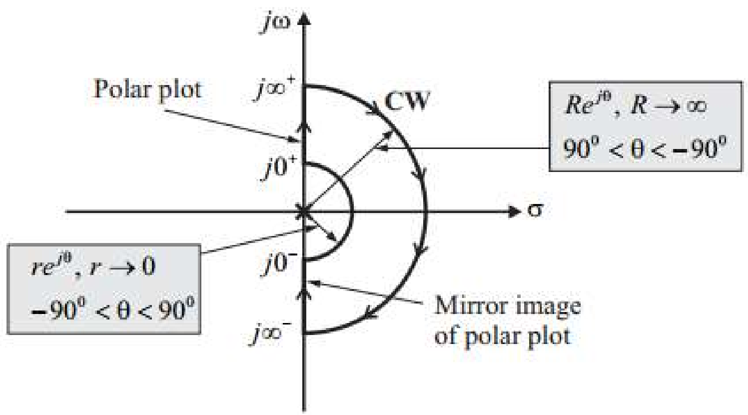
$$|G(j\omega)| = \frac{1}{\omega\sqrt{(4\omega^2+1)(\omega^2+1)}}$$

Phase angle can be written as,

$$\angle G(j\omega) = -90^0 - \tan^{-1} 2\omega - \tan^{-1}\omega$$

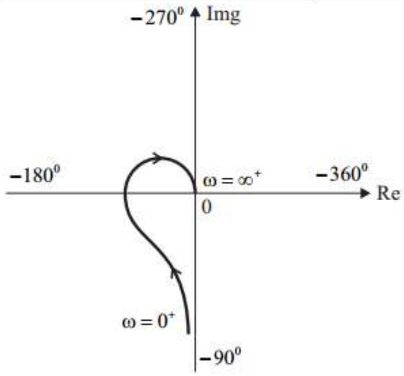
The Nyquist path for the transfer function is shown below.





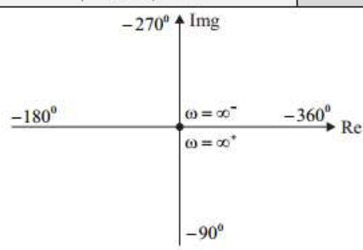
Section 1 : Polar plot

At $\omega = 0^+$	$\lvert G(j\omega)\rvert = \infty$	$\angle G(j\omega) = -90^0$
At $\omega = \infty^+$	$\lvert G(j\omega)\rvert = 0$	$\angle G(j\omega) = -270^0$



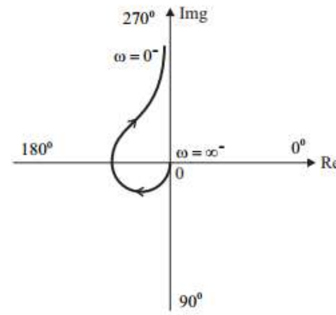
Section 2 : Semi-circle with radius tending to infinite.

At $\omega = \infty^+$	$\lvert G(j\omega)\rvert = 0$	$\angle G(j\omega) = -270^0$
At $\omega = \infty^-$	$\lvert G(j\omega)\rvert = 0$	$\angle G(j\omega) = 270^0$



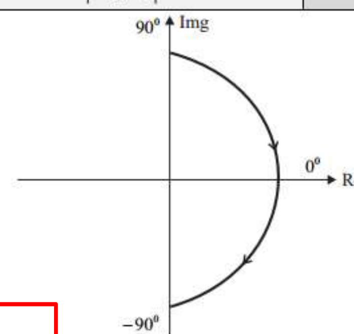
Section 3 : Mirror image of Polar plot

At $\omega = \infty^-$	$\lvert G(j\omega)\rvert = 0$	$\angle G(j\omega) = 270^0$
At $\omega = 0^-$	$\lvert G(j\omega)\rvert = \infty$	$\angle G(j\omega) = 90^0$

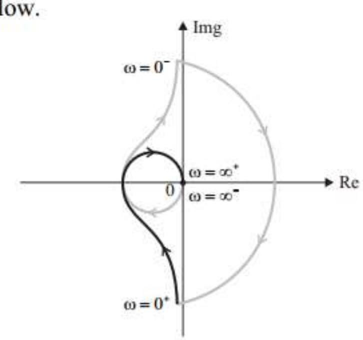


Section 4 : Semi-circle with radius tending to zero.

At $\omega = 0^-$	$\lvert G(j\omega)\rvert = \infty$	$\angle G(j\omega) = 90^0$
At $\omega = 0^+$	$\lvert G(j\omega)\rvert = \infty$	$\angle G(j\omega) = -90^0$



The Nyquist plot is shown below.



The frequency at which phase angle of $G(j\omega)H(j\omega)$ is -180^0 is called phase crossover frequency.

$$-180^0 = -90^0 - \tan^{-1} 2\omega_{pc} - \tan^{-1} \omega_{pc}$$

$$-90^0 = -\tan^{-1}(2\omega_{pc}) - \tan^{-1}(\omega_{pc})$$

$$90^0 = \tan^{-1}\left[\frac{2\omega_{pc} + \omega_{pc}}{1 - 2\omega_{pc}^2}\right]$$

$$\frac{1}{0} = \left[\frac{3\omega_{pc}}{1 - 2\omega_{pc}^2}\right]$$

$$1 - 2\omega_{pc}^2 = 0$$

$$\omega_{pc} = \frac{1}{\sqrt{2}} \text{rad/sec}$$

$$|G(j\omega_{pc})| = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{\left[4\times\left(\frac{1}{2}\right)+1\right]\left(\frac{1}{2}+1\right)}} = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{(3)(1.5)}}$$

$$|G(j\omega_{pc})| = 0.667$$

Gain margin can be defined as reciprocal of the magnitude of the $G(j\omega)H(j\omega)$ measured at phase crossover frequency.

$$\text{G.M.} = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{0.667} = 1.49$$

In dB, $\quad \text{G.M.} = 20\log\frac{1}{|G(j\omega_{pc})|} = 20\log 1.49 = 3.46 \text{ dB}$

The gain crossover is a point on the $G(j\omega)H(j\omega)$ plot at which the magnitude of $G(j\omega)H(j\omega)$ is equal to unity (1) and the corresponding frequency is known as gain crossover frequency ω_{gc}.

$$|G(j\omega_{gc})H(j\omega_{gc})| = 1$$

$$1 = \omega_{gc}\sqrt{(4\omega_{gc}^2+1)(\omega_{gc}^2+1)}$$

$$1 = \omega_{gc}^2(4\omega_{gc}^2+1)(\omega_{gc}^2+1)$$

$$(4\omega_{gc}^4+\omega_{gc}^2)(\omega_{gc}^2+1) = 1$$

$$4\omega_{gc}^6+\omega_{gc}^4+4\omega_{gc}^4+\omega_{gc}^2 = 1$$

$$4\omega_{gc}^6+5\omega_{gc}^4+\omega_{gc}^2 = 1$$

$$x = \omega_{gc}^2 \text{ say}$$

$$4x^3+5x^2+x = 1$$

$$4x^3+5x^2+x-1 = 0$$

$$x_1 = 0.326, \quad x_2 = -0.788+j0.379, \quad x_3 = -0.788-j0.379$$

$$\omega_{gc} = \sqrt{0.326}$$

$$\omega_{gc} = 0.57 \text{ rad/sec}$$

Phase margin is defined as,

$$\text{P.M.} = 180^0 + \angle G(j\omega_{gc})$$

$$\text{P.M.} = 180^0 - 90^0 - \tan^{-1}(2\omega_{gc}) - \tan^{-1}(\omega_{gc})$$

$$\text{P.M.} = 90^0 - \tan^{-1}(2\times 0.57) - \tan^{-1}(0.57)$$

$$\text{P.M.} = 11.57^0$$