Monday, July 5, 2021 1:16 AM

Alirya Mohammadi
97163779

$u_{xx} + u_{yy} = 0$

$x, y \geq 0$

$u(x, 0) = 0$

$u_x(0, y) = e^{-y}$

$F_s\{u\} = U(x, \omega)$

حل با تبدیل فوریه سینوسی نسبت به متغیر y

$F_s\{u_{xx}\} = U_{xx}$

$F_s\{u_{yy}\} = -\omega^2 U$

$U_{xx} - \omega^2 U = 0$

$U = A(\omega) e^{-\omega x} + B(\omega) e^{\omega x}$

$\underset{\omega, x \to \infty}{U \ll \infty} \longrightarrow B(\omega) = 0$

$U = A(\omega) e^{-\omega x}$

شرایط اولیه

$$F_s\{u_x(0,y)\} = U_x(0,w) = -w\,A(w)$$

$$F_s\{e^{-y}\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin wy\, e^{-y}\,dy = \sqrt{\frac{2}{\pi}}\,\frac{w}{w^2+1}$$

$$\Rightarrow A(w) = \frac{-\sqrt{\frac{2}{\pi}}}{w^2+1}$$

$$u(x,y) = F_s^{-1}\left\{-\sqrt{\frac{2}{\pi}}\,\frac{1}{w^2+1}\,e^{-wx}\right\}$$

$$= \frac{-2}{\pi}\int_0^\infty \frac{1}{w^2+1}\,e^{-wx}\,\sin wy\,dw$$

$U(x, w)$ مكاسب سؤال

- $F_s\{u_{xx}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{\partial}{\partial x^2} u\right) \sin wy \, dy$

$$= \frac{\partial}{\partial x^2}\left(\sqrt{\frac{2}{\pi}} \int_0^\infty u \sin wy \, dy\right) = U_{xx}$$

- $F_s\{u_{yy}\} = -w F_c\{u_y\}$

$$= -w\left(-\sqrt{\frac{2}{\pi}}\, \underbrace{u(x,0)}_{0} + w F_s\{u\}\right)$$

$$= -w^2 U$$

$$\bullet \int_0^\infty \underbrace{\sin wt}_{u}\, \underbrace{e^{-st}\,dt}_{dv} = I$$

$$I = \frac{-1}{s} \sin wt\, e^{-st}\Big|_0^\infty + \frac{w}{s}\int_0^\infty \cos wt\, e^{-st}\,dt$$

$$\int_0^\infty \cos wt\, e^{-st}\,dt = \frac{-1}{s} \cos wt\, e^{-st}\Big|_0^\infty - \frac{w}{s}\int_0^\infty \sin wt\, e^{-st}\,dt$$

$$\frac{I + \frac{1}{s} \sin wt\, e^{-st}\Big|_0^\infty}{\frac{w}{s}} = \frac{-1}{s} \cos wt\, e^{-st}\Big|_0^\infty - \frac{w}{s} I$$

$$\lim_{\substack{t\to\infty \\ s>0}} e^{-st} = 0 \longrightarrow \frac{I}{\frac{w}{s}} = \frac{1}{s} - \frac{w}{s} I$$

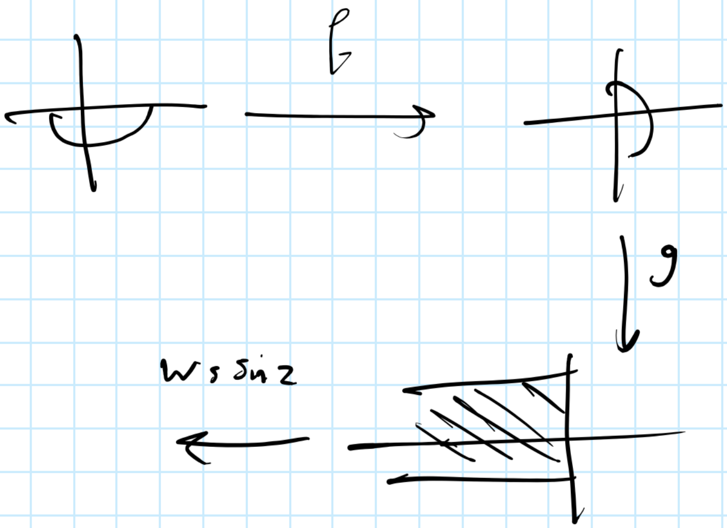
$$I\left(\frac{s}{w} + \frac{w}{s}\right) = \frac{1}{s} \longrightarrow I = \frac{w}{s^2 + w^2}$$

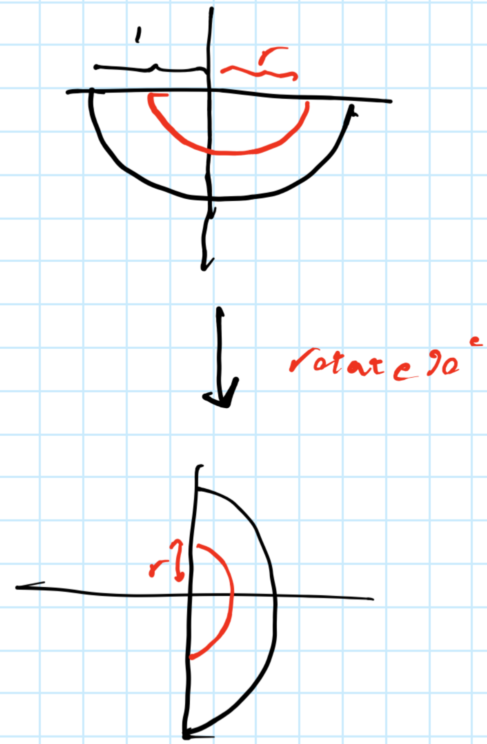
- $u_{xx} - w^2 u = 0$

$r^2 - w^2 = 0 \qquad r = \pm w \qquad u = A(w)\, e^{-wx} + B(w)\, e^{wx}$

2

Sunday, July 4, 2021 7:54 PM





$$f(z = re^{i\theta}) = re^{i(\theta+\frac{\pi}{2})}$$

ex $f(z) = az \longrightarrow re^{i(\theta+\frac{\pi}{2})} = are^{i\theta}$

$$\longrightarrow a = e^{i\frac{\pi}{2}} = i$$

$$f(z) = iz$$

تمرین: $g = \log z$

$$\log Re^{i\theta} = \log R + i\theta$$

$$\xrightarrow{R=1} \quad -\frac{i\pi}{2} \le \cancel{\log 1} + i\theta \le \frac{i\pi}{2}$$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

خط عمودی

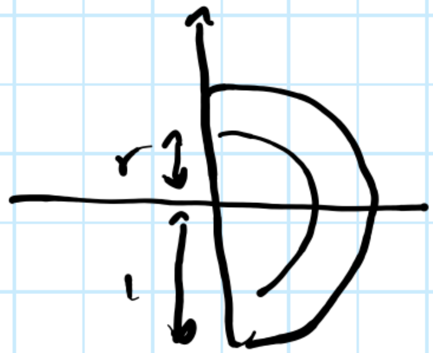
بین $-\frac{\pi}{2}i$ و $\frac{\pi}{2}i$ $(u=0 \,!)$

نیم دایره

ربع اول و

چهارم

pg

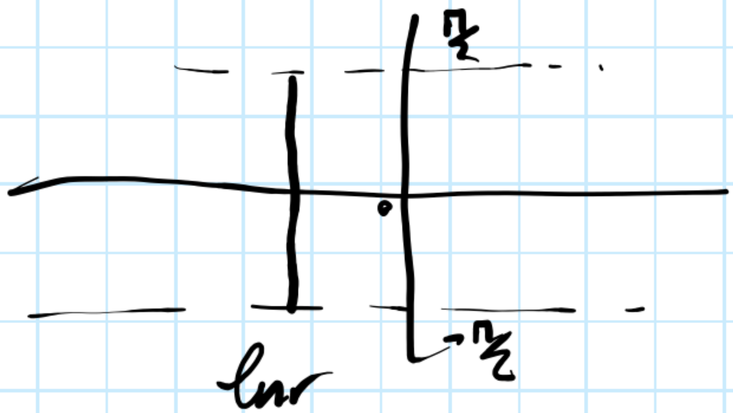


$0 \leq r \leq 1$

$\rightarrow$

$-\infty < \log r \leq 0$

$-\infty < u(x,y) \leq 0$



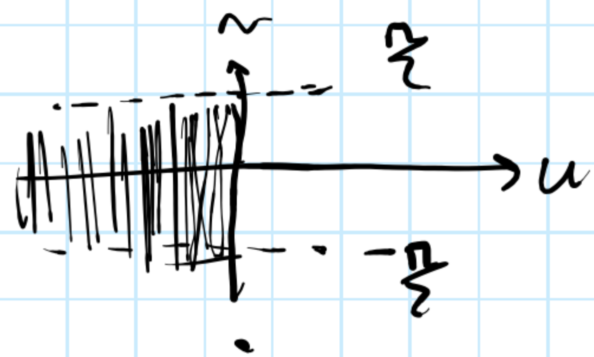
$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$\rightarrow -\frac{\pi i}{2} \leq i\theta \leq \frac{\pi i}{2}$

$-\frac{\pi}{2} i \leq iV(x,y) \leq \frac{\pi}{2} i$

$u \in (-\infty, 0]$

$V \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



انتقال خط

مثال: $z = a + bi \longrightarrow \sin(z) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)$

- $u(a,b) = \sin a \cosh b$
- $v(a,b) = \cos a \sinh b$
- $\sin^2 a + \cos^2 a = 1$

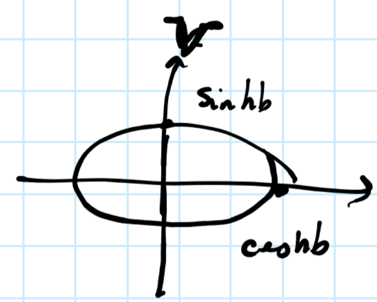
$$\xrightarrow[b \neq 0]{} \frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

بیضی

به مرکز $(0,0)$

$u = 0 \to v = \pm \sinh b$

$v = 0 \to u = \pm \cosh b$



از طرفی: $\cosh^2 b = 1 + \sinh^2 b$

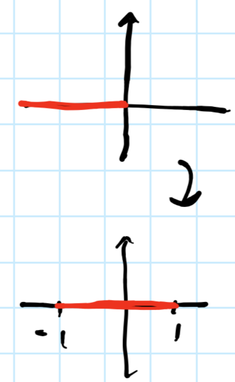
$\longrightarrow |\cosh b| > |\sinh b|$ بیضی افقی

$b = \frac{\pi}{2} \to u = \sin x \overset{+}{\overbrace{\cosh\frac{\pi}{2}}}$
$\quad v = \cos x \underset{+}{\underbrace{\sinh\frac{\pi}{2}}}$

$b = -\frac{\pi}{2} \to u = \sin x \overset{+}{\overbrace{\cosh\frac{\pi}{2}}}$
$\quad v = \cos x \underset{-}{\underbrace{\sinh\frac{\pi}{2}}}$
$\to$ تمام مقادیر بین سر و ته بازه را می‌گیرند

$x \in (-\infty, 0] \longrightarrow \sin x, \cos x \in [-1, 1]$

$-\frac{\pi}{2} < b < \frac{\pi}{2}$ if $b = 0 \longrightarrow u = \sin x \in [-1, 1]$
$v = 0$

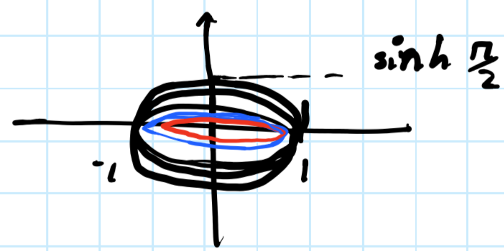


$\sinh x = \dfrac{e^{x} - e^{-x}}{2} \longrightarrow (\sinh)'(x) = \dfrac{e^{x} + e^{-x}}{2} \geq 1$ صعودی اکید

$\cosh x = \dfrac{e^{x} + e^{-x}}{2} \longrightarrow (\cosh)'(x) = \dfrac{e^{x} - e^{-x}}{2} = 0 \to x = 0$ صعودی برای $x>0$ نزولی برای $x<0$

$-\frac{\pi}{2} \leq b \leq \frac{\pi}{2} \longrightarrow \sinh -\frac{\pi}{2} \leq \sinh b \leq \sinh \frac{\pi}{2}$

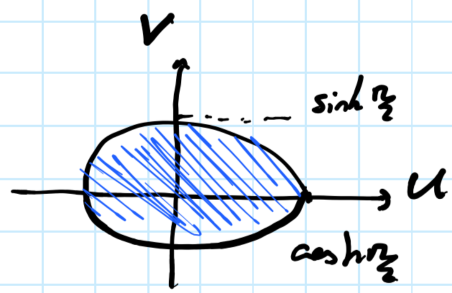
$1 = \cosh 0 \leq \cosh b \leq \cosh \frac{\pi}{2}$



بیضی های با طول ۱

بیضی های مشابه با طول متغیر بین ۱ و $\cosh\frac{\pi}{2}$ هستند. بزرگترین بیضی و شکل نهایی:

همان بیضی که مرزهای بالا و پایین به آن تبدیل می‌شوند.

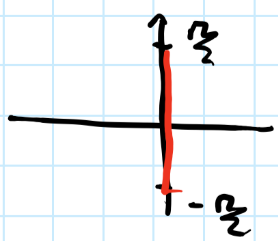


انتقال

خط عمودی : $z = a + yi$

$$\Rightarrow \sin z = \underbrace{\sin a \cdot \cosh y}_{u(x,y)} + \underbrace{\cos a \cdot \sinh y}_{v(x,y)}\, i$$

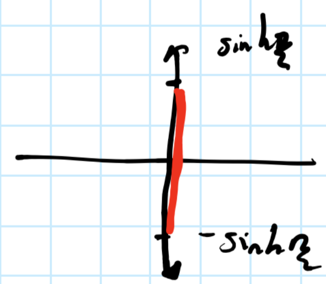
مرز راست :

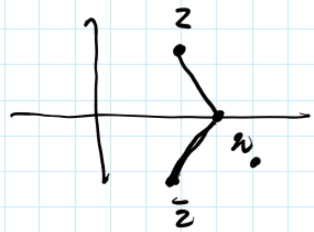


$z = yi \longrightarrow u = 0$

$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\quad v = \sinh y$

$\hookrightarrow \sin(z) \in [-\sinh\frac{\pi}{2}, \sinh\frac{\pi}{2}]$





$$\left\| \frac{z - x_0}{\bar z - x_0} \right\| = 1$$

درسنامه اشتباه

درس: $g = \dfrac{z - a}{\bar z - a}$

انتقال خط افقی:

$$z = x + bi \longrightarrow g(z) = \frac{(x - a) + bi}{(x - a) - bi}$$

$$\xrightarrow[\text{ضرب در مزدوج}]{\text{صورت مخرج}} \frac{1}{(x-a)^2 + b^2} \left((x-a)^2 + 2(x-a)bi - b^2 \right)$$

محاسبات ...

درس دو: $g = \sin(z)$

انتقال خطوط افقی:

$$z = x + bi \longrightarrow g(z) = g(x + bi) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)$$

$$\left.\begin{aligned} u(x,b) &= \sin x \cosh b \\ v(x,b) &= \cos x \sinh b \end{aligned}\right\} \longrightarrow \frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

$$\frac{-\pi}{2} \leq b \leq \frac{\pi}{2} \longrightarrow \cosh b = \frac{1}{2}\left(e^{b} + e^{-b}\right)$$

...

درس: $g = \dfrac{z}{\bar z}$

$$\frac{x + yi}{x - yi} = \frac{1}{x^2 + y^2} \left(x^2 + 2xyi - y^2 \right)$$

3)

الف

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

سری لوران تابع f

حول z_0

تحلیلی در D

$$\Rightarrow f(z) = g(z) + \frac{b_1}{z-z_0}$$

قطب رتبه یک

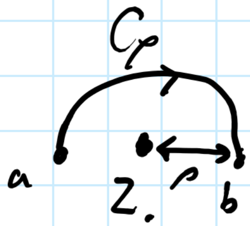
تفکیک f

به دو بخش تحلیلی

و غیر تحلیلی

در z_0

$$\int_{C_p} f(z)\,dz = \underbrace{\int_{C_p} g(z)\,dz}_{1} + \underbrace{\int_{C_p} \frac{b_1}{z-z_0}\,dz}_{2}$$



$$\int_{C_\rho} g(z)\,dz \leq ML = M\pi\rho$$

M: $\max|g(z)|$

L: $\int_{C_\rho} dz$ — طول نیم دایره به شعاع ρ

$$\Rightarrow \lim_{\rho\to 0}\int_{C_\rho} g(z)\,dz \leq \lim_{\rho\to 0} M\pi\rho = 0 \Rightarrow ① = 0$$

$$② = \int_{C_\rho} \frac{b_1}{z-z_0}\,dz = b_1 \int_{C_\rho} \frac{1}{z-z_0}\,dz$$

محاسبه انتگرال
با پیمایش خم
نیم دایره

$$z(t) = z_0 + \rho e^{it} \Longrightarrow z'(t) = \rho i e^{it}$$

$$\int_{C_\rho} \frac{1}{z-z_0}\,dz = \int_0^{\pi} \frac{1}{\rho e^{it} + z_0 - z_0}\,(\rho i e^{it})\,dt$$

$$= \int_0^{\pi} i\,dt = i t \Big|_0^{\pi} = i\pi$$

$$\Rightarrow ② = \pi i\, b_1 \longrightarrow \pi i \operatorname*{Res}_{z=z_0} f(z)$$

$$b_1 = \operatorname*{Res}_{@ z=z_0} f(z)$$

$$\int_{C_\rho} f(z)\, dz = ① + ② = \pi i \operatorname*{Res}_{z=z_0} f(z)$$

ب)

$$\int_C f(z)\,dz = \int_{C_\rho} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{-R}^{-2-\rho} f(x)\,dx + \int_{-2+\rho}^{R} f(x)\,dx$$

↳ مسیر روی محور ها

$z = x \rightarrow dz = dx$

$$f(z) = \frac{1}{z^3+8} = \frac{1}{(z+2)(z^2-2z+4)}$$

↳ $z = -2$ ↳ $z = \dfrac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3}$

اگر R به اندازه کافی بزرگ باشد C قطب $1+i\sqrt{3}$ را نیز در بر می‌گیرد اما $1-i\sqrt{3}$ و -2

زیر نیم دایره‌هاست و درون خم نمی‌افتند و تاثیری در محاسبات ندارند.

$$\int_C f(z)\,dz = 2\pi i \sum \operatorname{Res} f(z) = 2\pi i \operatorname*{Res}_{z=1+i\sqrt{3}} f(z) = \frac{-1}{24}(1+i\sqrt{3})(2\pi i) = \frac{-1}{12}\pi\left(-\sqrt{3}+i\right)$$

$$\operatorname*{Res}_{@z=1+\sqrt{3}i} f(z) = \left(\frac{1}{z^3+8}\left(z-(1+i\sqrt{3})\right)\right)^{(0)}\Bigg|_{z=1+\sqrt{3}i} = \frac{1}{(z+2)\left(z-(1-i\sqrt{3})\right)}\Bigg|_{z=1+i\sqrt{3}} = \frac{1}{(3+i\sqrt{3})(2i\sqrt{3})} = \frac{1}{6}\times\frac{1}{i\sqrt{3}-1} = \frac{-1}{24}(1+i\sqrt{3})$$

Monday, July 5, 2021 1:03 PM

ب)

• $$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = \lim_{\rho\to 0} -\int_{-C_\rho} f(z)\,dz = -\frac{\pi}{12} i$$

الف $\longrightarrow$ $$\lim_{\rho\to 0}\int_{-C_\rho} f(z)\,dz = \pi i \operatorname*{Res}_{z=-2} f(z) = \frac{\pi}{12} i$$

$$\operatorname*{Res}_{z=-2} f(z) = \frac{1}{(z+2)(z^2-2z+4)}(z+2)\Big|_{z=-2} = \frac{1}{12}$$

$$\bullet \lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty} \frac{\pi R}{R^3-8} = 0$$

$$\int_{C_R} f(z)\,dz \leq ML = \frac{\pi R}{R^3-8}$$

$$M:\ \max_{z\in C_R}\{f(z)\} \qquad L:\ \int_{C_R} dz$$

$$f(z) = \frac{1}{z^3+8} \rightarrow |f(z)| = \frac{1}{|z^3+8|} = \frac{1}{|R^3e^{3i\theta}+8|} = \frac{1}{\sqrt{(8+R^3)^2+(3\theta)}}$$

$$|z^3+8+(-8)| \leq |z^3+8| + |-8|$$

$$\longrightarrow |z^3+8| \geq \|z^3\| - 8 = R^3-8 \longrightarrow \frac{1}{\|z^3+8\|} \leq \frac{1}{R^3-8} \quad (R>2)$$

$$\Rightarrow \max\|f(z)\| \leq \frac{1}{R^3-8}$$

$$\bullet \quad \lim_{\rho \to 0} \left(\int_{-R}^{-2-\rho} f(x)\,dx + \int_{-2+\rho}^{R} f(x)\,dx \right) = \int_{-R}^{-2^-} f(x)\,dx + \int_{-2^+}^{R} f(x)\,dx = I$$

$$\Rightarrow I = \lim_{\substack{R \to \infty \\ \rho \to 0}} \left(\int_C f(z)\,dz - \int_{C_R} f(z)\,dz - \int_{C_\rho} f(z)\,dz \right)$$

$$\Longrightarrow I = \frac{-1}{12}\pi\left(-\sqrt{3}+i\right) - \left(-\frac{\pi}{12}i\right) - 0 = \frac{\sqrt{3}}{12}\pi$$