PH2130 Mathematical Methods
Problem Sheet 5
Due 10 am Wednesday, Week 6

Approximate part marks shown in brackets.

Problem 1 Consider a rectangular drum extending from 0 to a in the x direction and 0 to b in the y direction. Our goal is to find the displacement of the drum $u(x, y, t)$ as a function of x, y and time t. The motion of the drum can be modeled by the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} ,$$

where v represents the velocity with which waves propagate on the surface of the drum. The boundary conditions are that the drum is clamped at its edges, i.e., $u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0$. Suppose that the initial speed of the drum $\partial u/\partial t$ is zero and that the its initial spatial configuration $u(x, y, 0)$ is equal to a given function $f(x, y)$.

1(a) [**3**] By seeking product solutions of the form $\varphi(x, y, t) = X(x)Y(y)T(t)$, show that the wave equation separates into three ordinary differential equations of the form

$$\begin{aligned} X'' &= \lambda_x X , \\ Y'' &= \lambda_y Y , \\ T'' &= v^2(\lambda_x + \lambda_y)T , \end{aligned}$$

where λ_x and λ_y are separation constants.

1(b) [**4**] By applying the boundary conditions, show that nontrivial solutions are found only if λ_x and λ_y are both less than zero. Therefore we can write $\lambda_x = -k_x^2$ and $\lambda_y = -k_y^2$ where k_x and k_y are real nonzero constants.

1(c) [**2**] Again by using the boundary conditions, find the allowed values of k_x and k_y.

1(d) [**3**] By applying the condition that the initial velocity of the drum is zero, show that the product solutions can be written in the form

$$\varphi_{nm}(x, y, t) = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \cos(\omega_{nm} t) ,$$

where n and m are positive integers. Find the corresponding values of the (angular) frequency ω_{nm} in terms of n, m, a, b and v.

1(e) [**3**] Show that the eigenfunctions φ_{nm} satisfy an orthogonality relation of the form

$$\int\int \varphi_{kl}(x, y, t)\varphi_{mn}(x, y, t)\, dx\, dy = C(t)\delta_{km}\delta_{ln} ,$$

where the region of integration is $0 \leq x \leq a$ and $0 \leq y \leq b$. Find $C(t)$.

1(f) [**3**] Suppose the dimensions of the drum are $a = 1$ and $b = 2$ (in some units), and we consider two initial spatial configurations:

$$f_{2,3}(x, y) = A \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right) , \tag{1}$$

$$f_{3,2}(x, y) = A \sin\left(\frac{3\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right) . \tag{2}$$

Each initial condition leads to a single nonzero eigenmode with frequencies $\omega_{2,3}$ and $\omega_{3,2}$, respectively. Find expressions for these two frequencies. Which one has the higher pitch?

1(g) [**2**] Suppose $a = 1$ and $b = 1$ (in some appropriate units). Show that the ratio of the frequencies of the eigenmode corresponding to φ_{nm} to that of $\varphi_{1,1}$ is

$$\frac{\omega_{nm}}{\omega_{1,1}} = \sqrt{\frac{n^2 + m^2}{2}} .$$

Bonus question: Comment on what this result implies for the musical quality of a drum.

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