Laplace's Equation in Cartesian Co-ordinates

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Overview

This lecture:

- Laplace's Equation in Cartesian Co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

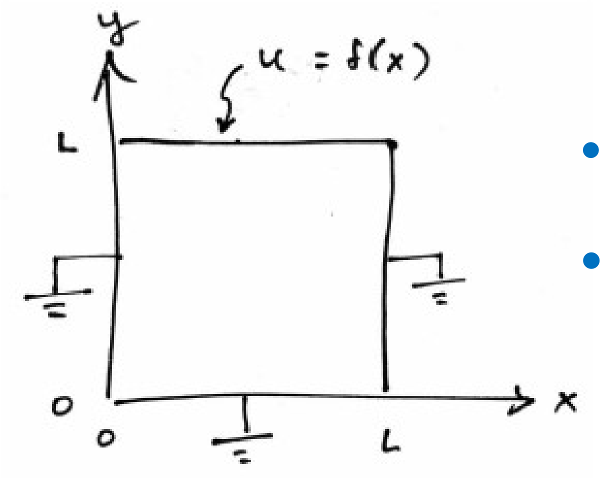
Reading: Chapter 6 of lecture notes

- Laplace's Equation in Cartesian Co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Aim: solve subject to boundary conditions that are easy to express in terms of x, y and z.

Infinitely long rectangular region with sides held at the potentials



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- In 2 dimensions:
- No variation in z we must have ∂u/∂z = 0

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\ .$$

- Boundary conditions

$$\begin{aligned} u(0,y) &= 0\,, \\ u(L,y) &= 0\,, \\ u(x,0) &= 0\,, \\ u(x,L) &= f(x)\,, \end{aligned}$$

Different to heated rod:

- Homogeneous for x
- But not for 1 y

$$\varphi(x, y) = X(x)Y(y) \ .$$

Find a family of such solutions φ_n and
Construct u(x, y) from a linear combination of them

Insist that they satisfy the homogeneou boundary conditions
i.e., φ(0, y) = φ(L, y) = φ(x, 0) = 0.

$$\varphi = XY$$

$$\nabla^2 \varphi = XY'' + X''Y = 0 \ .$$

Separation of variables in Cartesian coordinates

$$\nabla^2 \varphi = XY'' + X''Y = 0\ .$$

Divide by XY

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

λ is a separation constant

$$\frac{X''}{X} = \lambda\ ,$$

$$\frac{Y''}{Y} = -\lambda\ ,$$

Note that this is a special case of the Sturm-Liouville Equation

$$\mathcal{L}X = -\lambda X$$

$$\mathcal{L} = -d^2/dx^2.$$

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ w(x) &= 1 \end{aligned}$$

With eigenvalue of SL Eq = - separation constant λ.

The boundary conditions for X, are of the regular homogeneous and therefore L is a self-adjoint operator. The solutions X will thus have the properties we discussed (orthogonality, completeness, etc.).

Sturm-Liouville equation

$$
\begin{aligned}
u(0,y) &= 0\,, \\
u(L,y) &= 0\,, \\
u(x,0) &= 0\,, \\
u(x,L) &= f(x)\,,
\end{aligned}
$$

The boundary conditions for X, are of the regular homogeneous and therefore L is a self-adjoint operator. The solutions X will thus have the properties we discussed (orthogonality, completeness, etc.).

equation for Y , the boundary conditions are not homogeneous, and therefore the operator (which implicitly includes the boundary conditions) is not self-adjoint.

Solve for X equations

$$\frac{X''}{X} = \lambda \, ,$$

3 solutions

- $\lambda = 0$: lead to the trivial solution i.e., u = 0.
- $\lambda > 0$: lead to the trivial solution i.e., u = 0.
- For $\lambda < 0$ we define $\lambda = -k^2$ for real k, and find the solution to
  -

$$X = A\cos(kx) + B\sin(kx) \, .$$

Solve for X equations

$$X = A\cos(kx) + B\sin(kx)\ .$$

Do not consider n = 0: this gives trivial solution

nor n < 0, gives the negatives of the solutions with n > 0.

Use a subscript n to label the solution and its eigenvalue, e.g.,

$$\lambda_n = -k_n^2 = -\left(\frac{n\pi}{L}\right)^2$$

Y Equation

$$Y'' - k^2 Y = 0 \; .$$

General solution

$$Y(y) = Ae^{ky} + Be^{-ky} \; .$$

Final homogeneous boundary condition φ(x, 0) = 0 gives

$$\varphi(x, 0) = X(x)Y(0) = X(x)(A + B) = 0 \; ,$$

A = - B

Solution is $A(e^{ky} - e^{-ky}) = 2A\sinh(ky)$

$$Y = A\sinh(ky)$$

Y Equation

$$Y = A\sinh(ky)$$

Y equation co-efficients

$$\langle \sin(m\pi x/L), f\rangle = \int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x)\,dx = \sum_{n=1}^{\infty} b_n \sinh(n\pi) \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_{n=1}^{\infty} b_n \sinh(n\pi) \frac{L}{2} \delta_{mn}$$

$$= \frac{b_m L}{2} \sinh(m\pi) \ .$$

The coefficients b_n are

$$b_n = \frac{2}{L \sinh(n\pi)} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx \ ,$$

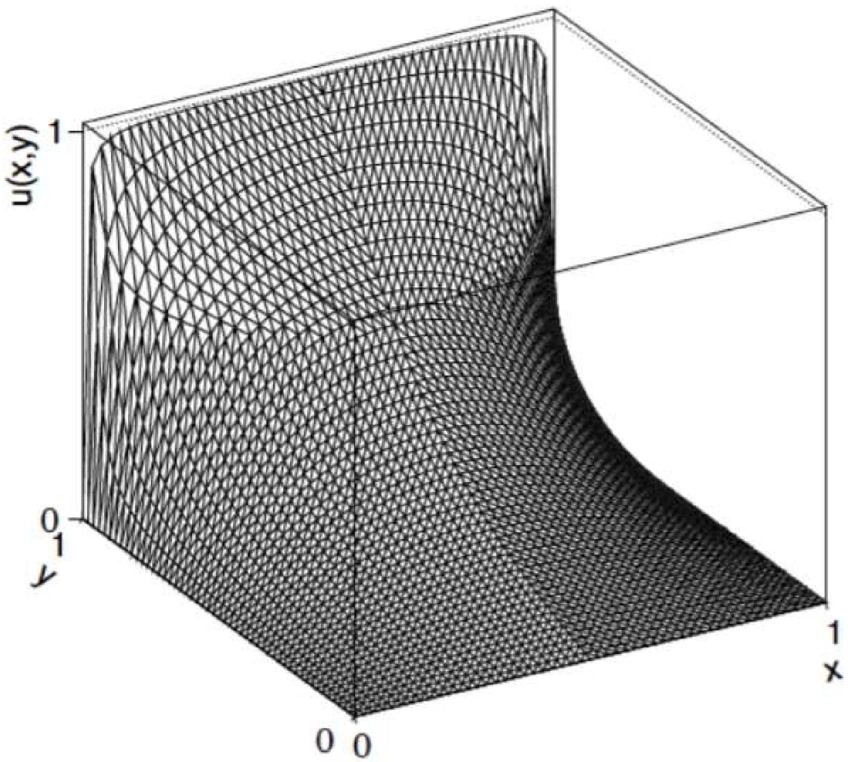
Example

Suppose f(x) = u_0 where u_0 = constant, units of electrical potential.

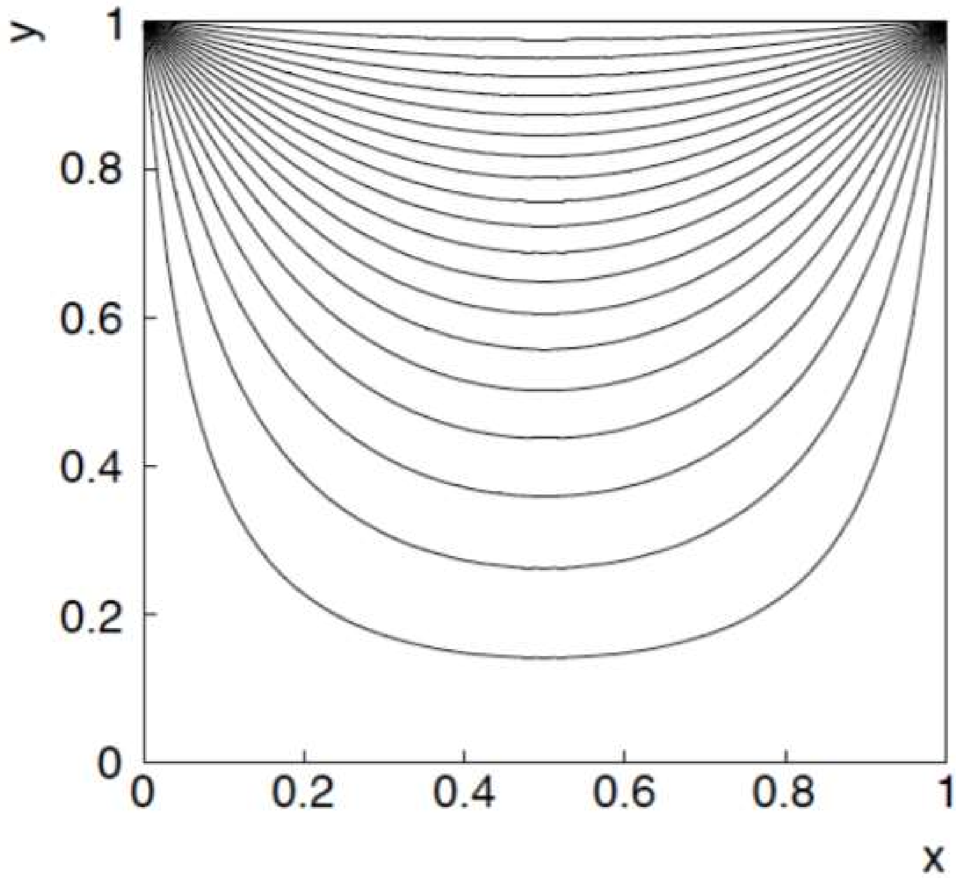
$$b_n = \frac{2}{L\sinh(n\pi)}\int_0^L u_0 \sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{2u_0}{n\pi\sinh(n\pi)}\left[1-\cos(n\pi)\right]$$

$$b_n = \begin{cases} 0 & n \text{ even,} \\ \frac{4u_0}{n\pi\sinh(n\pi)} & n \text{ odd.} \end{cases}$$

Potential in square region



Equipotential lines



Summary

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