Laplace's Equation in 2D polar coordinates

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ROYAL HOLLOWAY UNIVERSITY OF LONDON

Overview

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Last lecture:

- Laplace's Equation in Cartesian Co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

This lecture:

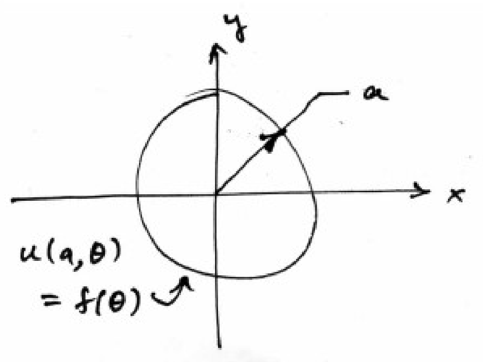
- Laplace's Equation in 2D Polar Co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \, .$$

- Laplace's Equation in 2D Polar Co-ordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\ .$$

Example: to find the electrical potential inside a circular tube of radius a which extends infinitely far in the z direction



2D Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\ .$$

Laplace's Equation in 2D Polar Co-ordinates

in polar coordinates

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

Boundary condition: $$u(a,\theta) = f(\theta)\ .$$

u(r, θ) must also satisfy $u(r,\theta) = u(r,\theta + 2\pi)$ for all θ

A final requirement is that the function should be finite everywhere in the interior of the circle, i.e.,

$$|u(r,\theta)| < \infty\ .$$

The Laplacian in 2D polar coordinates

Need to express the Laplace equation $\partial^2 u/\partial x^2$ & $\partial^2 u/\partial y^2$, in terms of r and θ.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\,.$$

Using the product rule

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)$$

use the chain rule again to find

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial r}{\partial x}\frac{\partial^2 u}{\partial r^2} + \frac{\partial \theta}{\partial x}\frac{\partial^2 u}{\partial \theta \partial r}$$

The Laplacian in 2D polar coordinates

Using the product rule

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)$$

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$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial r}{\partial x}\frac{\partial^2 u}{\partial r^2} + \frac{\partial \theta}{\partial x}\frac{\partial^2 u}{\partial \theta \partial r}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial r}{\partial x}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial^2 u}{\partial \theta^2}$$

Substituting these gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 r}{\partial x^2}\frac{\partial u}{\partial r} + \frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial^2 u}{\partial r^2} + \frac{\partial \theta}{\partial x}\frac{\partial^2 u}{\partial \theta \partial r}\right) + \frac{\partial^2 \theta}{\partial x^2}\frac{\partial u}{\partial \theta} + \frac{\partial \theta}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial^2 u}{\partial \theta^2}\right)$$

Required derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r},$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{y}{r^2},$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{r^3},$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{r^4}.$$

Goal: to express the result for the Laplacian in terms of r and θ, not containing x and y as, note can use $x^2 + y^2 = r^2$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{r^3}\frac{\partial u}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{2xy}{r^4}\frac{\partial u}{\partial\theta} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial\theta^2}\,.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3}\frac{\partial u}{\partial r} + \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial\theta\partial r} - \frac{2xy}{r^4}\frac{\partial u}{\partial\theta} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial\theta^2}\,.$$

Adding these terms together we find the two-dimensional Laplacian operator in polar coordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\,,$$

Or re-written..

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

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