PH2130 Mathematical Methods
Problem Sheet 9
Due 10am Wednesday 2nd December 2020

Part marks shown in brackets.

In a previous lecture on the Heat Equation it was proven that the product solutions to the heat equation with **homogeneous Dirichlet boundary conditions problem**:

$$u_t = ku_{xx}$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = f(x)$$

had the form

$$u(x,t) = B_n e^{-(n\pi/l)^2 kt} sin(\frac{n\pi x}{l})$$

for n = 1, 2, 3, ...

Taking linear combinations of these (over each n) gives a general solution to the above problem.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{(-n\pi/l)^2 kt} sin(\frac{n\pi x}{l}).$$

Setting t = 0, this implies that we must have

$$f(x) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{l}).$$

In other words, the coefficients in the general solution for the given initial condition are the Fourier Sine coefficients of $f(x)$ from $(0,l)$, which are given by

$$B_n = \frac{2}{l}\int_0^1 f(x) sin(\frac{n\pi x}{l})dx.$$

So to solve a homogeneous heat equation problem, one begins by identifying the type of boundary conditions one has.

Problem 1 Solve the initial heat equation value problem:

$$u_t = 3u_{xx} \qquad\qquad 0 < x < 2, t > 0$$

$$u(0,t) = u(2,t) = 0$$

$$u(x,0) = 20.$$

[To help your revision: confirm this.] Separation of variables ($u = X(x)T(t)$) and the boundary conditions can be used to show that the general solution $u(x,t)$ to the problem is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-3(n\pi/2)^2 t} sin(\frac{n\pi x}{2}).$$

1 (a) [5 marks] Show that the coefficients for the particular solution are the Fourier Sine coefficients of $u(x,0) = 20$ are:

$$B_n = \frac{40}{n\pi}(1 + (-1)^{n+1}).$$

1 (b) [2 marks] Write the solution to the problem: $u(x,t)$.

Nonhomogeneous Dirichlet Conditions Now consider the Dirichlet boundary conditions, which fix the value of u at the endpoints $x = 0$ and $x = l$. For the heat equation, this corresponds to fixing the temperature at the ends of the rod. Above the homogeneous conditions are considered where the ends of the rod had fixed temperature 0. Now consider the **nonhomogeneous** Dirichlet conditions $u(0,t) = T_1, u(l,t) = T_2$.

This problem is slightly more difficult than the homogeneous Dirichlet condition problem above. Recall that for separation of variables to work, the differential equations and the boundary conditions must be homogeneous. When we have nonhomogeneous conditions we need to try to split the problem into one involving homogeneous conditions, which we know how to solve, and another dealing with the nonhomogeneity.

How can we separate the core homogeneous problem from what is causing the nonhomogeneity? Consider what happens as t goes to infinity. We should expect that, since we fix the temperatures at the endpoints and allow free heat flux at the boundary, at some point the temperature will stabilize and will be at equilibrium. Such a temperature distribution would clearly not depend on time, and one can write the limit as t goes to ∞ of $u(x,t) = v(x)$.

Notice that $v(x)$ must still satisfy the boundary conditions and the heat equation, but we should not expect it to satisfy the initial conditions (since for large t we are far from where we initially started). A solution such as $v(x)$ which does not depend on t is called a steady-state or equilibrium solution.

For a steady-state solution the boundary value problem becomes

$$0 = kv'' \qquad v(0) = T_1 \qquad v(l) = T_2.$$

1 (c) [4 marks] Show that the solutions to this second order differential equation are

$$v(x) = c_1 x + c_2$$

and applying the boundary conditions, it is

$$v(x) = T_1 + \frac{(T_2 - T_1)}{l}x.$$

Now, let

$$w(x,t) = u(x,t) - v(x)$$

so that

$$u(x,t) = w(x,t) + v(x).$$

This function $w(x,t)$ represents the transient part of $u(x,t)$ (since $v(x)$ is the equilibrium part). Taking derivatives we have

$$u_t = w_t + v_t = w_t$$

and

$$u_{xx} = w_{xx} + v_{xx} = w_{xx}$$

Using the fact that $v(x)$ is independent of t and must satisfy the differential equation. Also, using the equilibrium equation

$$v'' = v_{xx} = 0.$$

Thus $w(x,t)$ must satisfy the heat equation, as the relevant derivatives of it are identical to those of $u(x,t)$, which is known to satisfy the equation. What are the boundary and initial conditions?

$$w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0$$

$$w(l,t) = u(l,t) - v(l) = T_2 - T_2 = 0$$

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x)$$

where $f(x) = u(x,0)$ is the given initial condition for the nonhomogeneous problem. Now, even though our initial condition is slightly messier, we now have homogeneous boundary conditions, since $w(x,t)$ must solve the problem

$$w_t = kw_{xx}$$

$$w(0,t) = w(l,t) = 0$$

$$w(x,0) = f(x) - v(x)$$

This is just a homogeneous Dirichlet problem. We know the general solution is

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{(-n\pi/l)^2 kt} sin(\frac{n\pi x}{l})$$

where the coefficients are given by

$$B_n = \frac{2}{l}\int_0^1 (f(x) - v(x)) sin(\frac{n\pi x}{l}) dx$$

Notice that in the limit that t goes to infinity $w(x,t) = 0$, so that $w(x,t)$ is transient.

Thus, the solution to the original nonhomogeneous Dirichlet problem

$$u_t = ku_{xx}$$

$$u(0,t) = T_1, u(l,t) = T_2$$

$$u(x,0) = f(x)$$

is $u(x,t) = w(x,t) + v(x)$, or

$$u(x,t) = T_1 + \frac{(T_2 - T_1)}{l} x + \sum_{n=1}^{\infty} B_n e^{(-n\pi/l)^2 kt} sin(\frac{n\pi x}{l}) \qquad (1)$$

(Equation (1)) with coefficients

$$B_n = \frac{2}{l} \int_0^1 (f(x) - T_1 - \frac{(T_2 - T_1)}{l} x) sin(\frac{n\pi x}{l}) dx.$$

Problem 2. Solve the following heat equation problem

$$u_t = 3u_{xx}$$

$$u(0,t) = 20, u(40,t) = 100$$

$$u(x,0) = 40 - 3x$$

2 (a) [4 marks] Start by writing

$$u(x,t) = v(x) + w(x,t),$$

where

$$v(x) = 20 + 2x.$$

Write the differential equation which w must satisty *i.e.* of the form

$$w_t = aw_{xx},$$

where a is a constant to be determined. Write the boundary and initial conditions satisfied by w.

$$w(0,t) = ?, w(40,t) = ?$$

$$w(x,0) = 40 - 3x + ???$$

2 (b) [2 marks] This is a homogeneous Dirichlet problem, so the general solution for $w(x,t)$ will have the form:

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{(-n\pi/l)^2 kt} sin(\frac{n\pi x}{l}),$$

what are k and l for this case?

2 (c) [5 marks] Show that the coefficients in this case are given by:

$$B_n = \frac{40}{n\pi}((-1)^n + 1).$$

2 (d) [3 marks] Write down the complete solution (of the form in Equation (1) above).