Gamma Function

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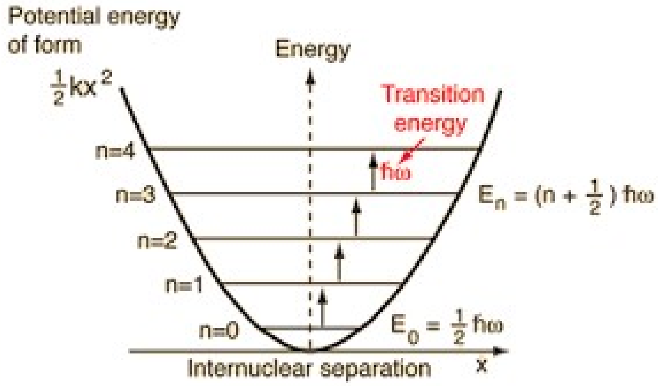
Overview

Last lecture:

- Quantum Harmonic Oscillators



This lecture:

- The gamma function

$$\boxed{\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt}$$

Reading: Chapter 8 of lecture notes

Euler Gamma Function

This is not a solution to the Sturm-Liouville equation

Defined for any x>0:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt$$

Gamma function

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$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

Value at x=1: $$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1\,.$$

Value at x+1: $$\Gamma(x+1) = \int_0^\infty e^{-t} t^x\, dt$$

and integrate by parts: with $u = t^x \qquad dv = e^{-t}\, dt$

$$\Gamma(x+1) \quad = -t^x e^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} x t^{x-1}\, dt = x\Gamma(x)$$

integral is over t

Gamma function

$$\Gamma(x+1) = x\Gamma(x)$$

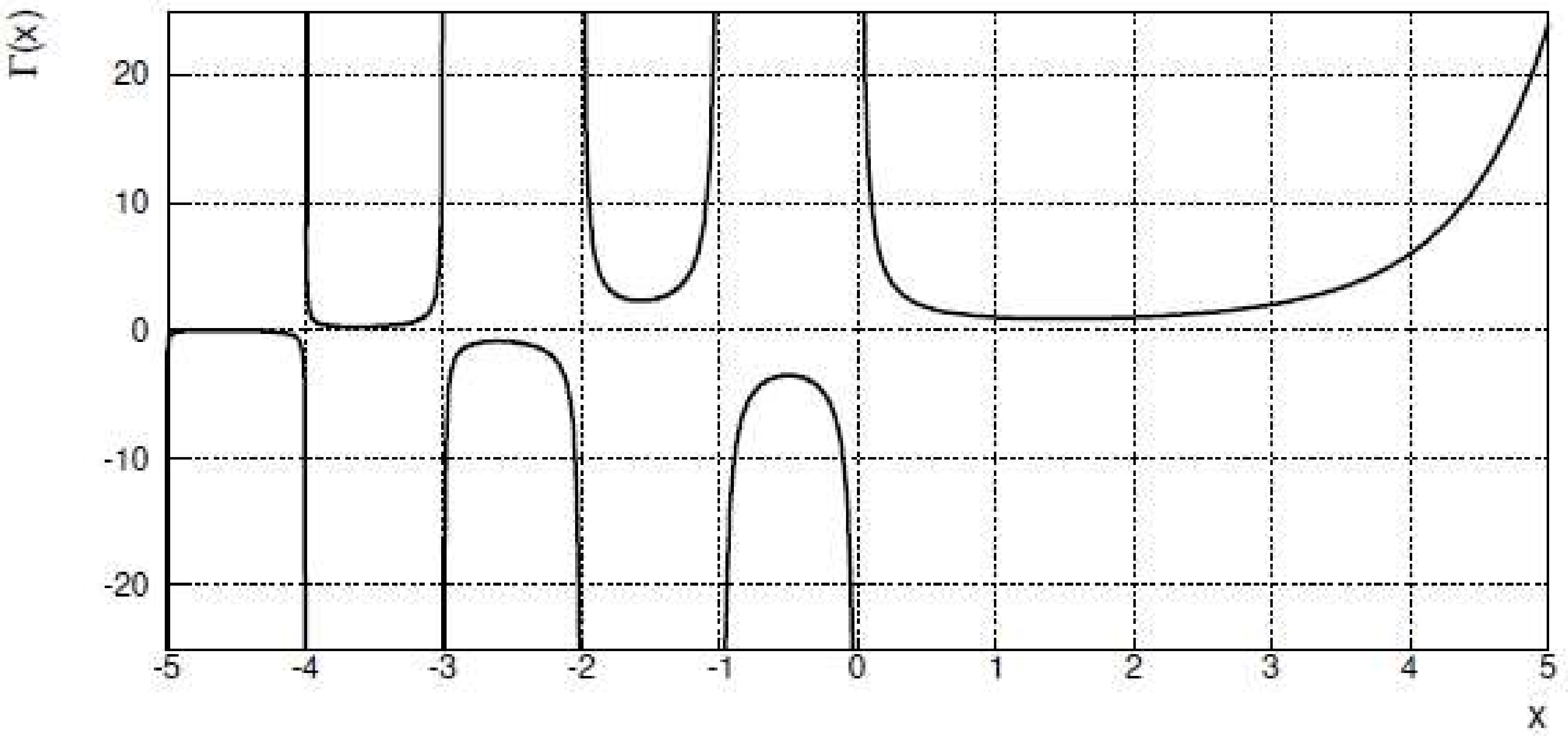
But: $$\Gamma(1) = 1\,.$$

So find....

$$\begin{aligned}
\Gamma(2) &= 1\Gamma(1) = 1\,,\\
\Gamma(3) &= 2\Gamma(2) = 2 \times 1\,,\\
\Gamma(4) &= 3\Gamma(3) = 3 \times 2 \times 1\,,\\
\Gamma(5) &= 4\Gamma(4) = 4 \times 3 \times 2 \times 1\,,
\end{aligned}$$

So for x>1: $$\Gamma(x) = (x-1)!\,.$$

Gamma function



Diverges to either $+\infty$ or $-\infty$ for all integer values of the argument less than or equal to zero

Example use of the gamma function

Consider Bessel's equation:

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \ ,$$

Suppose now, parameter v is not required to be an integer:
one finds the series solution :

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

This is the same as before for integer v,
but this series is valid for v equal to any real value.

Summary

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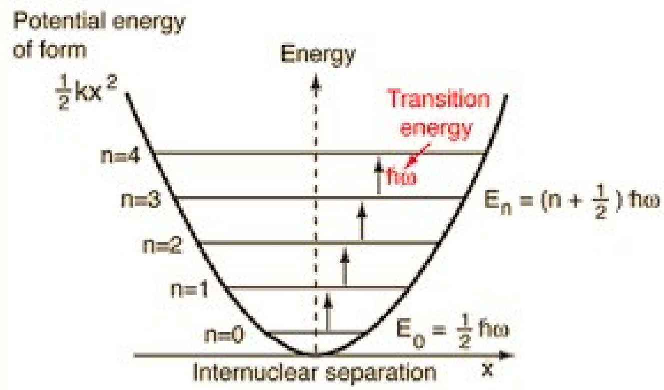

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