PH2130 Mathematical Methods
Problem Sheet 7
Due 10 am Wednesday, 18th November, 2020

Approximate part marks shown in brackets.

Problem 1 Consider a long cylindrical rod of radius a and having thermal diffusivity α. Suppose the rod's temperature is described by a function $u(r,t)$ that depends only on the distance r from the rod's central axis and on the time t, but not on the azimuthal angle or on the position along the length of the rod. The temperature of the rod is governed by the heat equation

$$\nabla^2 u = \frac{1}{\alpha}\frac{\partial u}{\partial t} \ .$$

The rod begins at $t = 0$ with a temperature $u(r, 0) = f(r)$ where $f(r)$ is a given function. At $t = 0$, the rod is dropped into a large body of ice water, so that the temperature of the surface is $u(a, t) = 0$. Suppose that heat flows only through the cylindrical surface of the rod, not through its ends. Our goal is to find the temperature of the rod as a function of radius r and time t.

1(a) [**2**] Referring to the notes, write down the heat equation in terms of the coordinates r and t. (Use polar coordinates and remember we are assuming that u does not depend on the angle θ.)

1(b) [**2**] Use separation of variables to find product solutions of the form

$$\varphi(r,t) = R(r)T(t) \ .$$

By substituting $\varphi(r,t)$ into the heat equation, show that this leads to two ordinary differential equations of the form

$$\begin{aligned} r^2R'' + rR' + \mu^2 r^2 R &= 0 \ , \\ T' + \alpha\mu^2 T &= 0 \ , \end{aligned}$$

where μ^2 is a separation constant. (Here the sign of the separation constant has been chosen so that we will be able to satisfy the boundary conditions. For purposes of this problem you should simply use this choice.)

1(c) [**2**] Define a new variable $x = \mu r$. Using the chain rule, convert the equation for $R(r)$ into an equation for $y(x) = R(r)$. Show that this results in Bessel's equation of order zero,

$$x^2y'' + xy' + x^2y = 0 \ .$$

1(d) [**2**] Two linearly independent solutions to the equation in 1(c) are the zeroth-order Bessel functions of the first and second kind, $J_0(x)$ and $Y_0(x)$. Why can $Y_0(x)$ be ignored in the solution for the temperature of the rod?

1(e) [**2**] Show that the product solution $\varphi(r,t)$ has the form

$$\varphi(r,t) = J_0(\mu r)e^{-\alpha\mu^2 t} \; .$$

1(f) [**2**] Apply the boundary condition that the temperature at edge of the rod $r = a$ is zero. What does this imply for the allowed values of μ?

1(g) [**2**] The temperature of the rod can be written

$$u(r,t) = \sum_{m=1}^{\infty} a_m J_0(\mu_m r)e^{-\alpha\mu_m^2 t} \; .$$

where μ_m is related to the mth zero of J_0. Referring to Table 8.1 of the notes, find the values of μ_1, μ_2 and μ_3 assuming the rod has a diameter of 1 cm. (Do not forget the units.)

1(h) [**6**] Suppose the initial temperature of the rod is $u(r,0) = C$ where C is a constant. By using the orthogonality relation for Bessel functions given in the notes, show that the temperature as a function of r and t is

$$u(r,t) = \frac{2C}{a} \sum_{m=1}^{\infty} \frac{J_0(\mu_m r)}{\mu_m J_1(\mu_m a)} e^{-\alpha\mu_m^2 t} \; .$$

To do this you will need the integral

$$\int_0^x u J_0(u)\, du = x J_1(x) \; .$$