PH2420
Electromagnetism
Problem Sheet 6 and Tutorial Sheet Week 12

1. Griffiths lists, in Table 4.1, the atomic polarisabilities of several elements. Also tabulated in Table 4.2 are some dielectric constants. Seeking out Helium, Neon and Argon ($\alpha = 0.205, 0.396, 1.64, \times 10^{-30}\mathrm{m}^3$ respectively) because they are noble gases and should be well behaved and well understood, we compare the dielectric constants measured at 1 atmosphere and 20° C (ϵ_r = 1.000065, 1.00013, 1.000517 respectively). We observe that they are not proportional. Calculate the expected dielectric constants from the atomic polarisabilities.

2. The work done W in charging a capacitor is $W = \frac{1}{2}CV^2$ where $C = \epsilon_r C_{vac}$, which can be used with the energy in an electric field, given by
$$W = \frac{\epsilon_0}{2} \int E^2 \; d(vol) \tag{1}$$
to calculate the energy in a capacitor with a dielectric
$$W = \frac{\epsilon_0}{2} \int \epsilon_r \; E^2 \; d(vol) = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \; d(vol) \tag{2}$$
the question is, why does only one of the electric fields become a displacement field? Shouldn't it be $\mathbf{D}^2$? Explain from the microscopic point of view what is going on.

3. (a) Write down in full the 'macroscopic' version of Maxwell's equations (that is, when matter is present) in the differential form derived in lectures. Derive the integral forms.
   (b) Make substitutions using the constitutive relations to rewrite the macroscopic differential form of Maxwell's equations in terms of $\mathbf{E}$ and $\mathbf{B}$. List any assumptions made in this process.
   (c) Hence show that Maxwell's equations in matter may be solved to find a wave equation. List any further assumptions that you may make.
   (d) Show that the wave equation may be subsequently solved with (for example) a plane wave solution and find an expression for the speed at which such a wave travels.

4. Maxwell's equations may be solved in the presence of conducting media following the recipe we used to derive the wave equations in free space (taking the $\nabla\times$ etc).

   Assume that the free charge density is zero (free charge would dissipate quickly) and use ohm's law to make a substitution for $\mathbf{J}$. Find the wave equations in $\mathbf{B}$ and $\mathbf{E}$ and show that an extra term arises in $\partial/\partial t$. (What sort of an effect does such a term have in a wave equation?)

   Then trial a plane wave solution (say, for $\mathbf{E}$) and show that the wave vector must be complex valued. Show, finally, that if the wavevector is complex rather than real, it may be factorised in the trial solution to give an exponential decay factor. (The distance to which em waves can penetrate a conducting medium is known as the skin depth.) If the wave is attenuated, where does the energy of the em wave go?

5. (a) Let,
$$V(\mathbf{r}, t) = 0$$
   and
$$\mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{qt}{r^2} \hat{\mathbf{r}}.$$
   Find the $\mathbf{E}$ and $\mathbf{B}$ fields.
   (b) Infer, and state, the charge and current distributions for the above fields.
   (c) Use the gauge function
$$\lambda = -\frac{1}{4\pi\epsilon_0} \frac{qt}{r},$$
   to (gauge) transform the potentials. [Hint, Griffiths, Sec 10.1.2]
   (d) Show that the $\mathbf{E}$ and $\mathbf{B}$ fields (and therefore the source configurations) are unchanged.
   (e) Suppose now that $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Find the fields and show that they satisfy M.E's in vacuum. What condition is imposed on ω and k?

6. Review the 'method of images' for a point charge Q "reflected" in a nearby conducting plane. (See Griffiths.) State the force that Q experiences and briefly explain why it experiences a force at all.