Frobenius Method

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Overview

Last lecture:

- Regular singular point of a diff equation
- Example of the Euler equation

This lecture:

- Frobenius method for fractional and negative powers

Reading: Chapter 7 of lecture notes

Example of the Frobenius method

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$$4xy'' + 2y' + y = 0 \, .$$

Want a series solution about x=o

Putting this in the standard form of Eq

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0 \, .$$

$$P(x) = \frac{1}{2x} \, ,$$

$$Q(x) = \frac{1}{4x} \, .$$

Both P(x) and Q(x) **diverge** at x = 0, so this is a **singular** point.

But we also find

$$xP(x) = \frac{1}{2} \, ,$$

$$x^2Q(x) = \frac{x}{4} \, ,$$

both **analytic** at x = 0 so this is a **regular** **singular** point.

Example of the Frobenius method

Leading coeff p_0 and q_0 in the expansions of xP(x) and x2Q(x)

$$p_0 = \lim_{x\to 0} xP(x) = \frac{1}{2} ,$$

$$q_0 = \lim_{x\to 0} x^2Q(x) = 0 .$$

Let us try to solve using the **Frobenius** method…

Frobenius method

Let us try to solve using the **Frobenius** method.

Seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

for some power α.

For the case $\alpha = 0$ this reduces to our previous series method

In the neighbourhood of the regular singular point,

i.e., where $x \to 0$, the diff eq is ~ an Euler equation,

so in this limit we need to have $y(x) \approx a_0 x^{\alpha}$.

Therefore need the first coefficient a_0 to be nonzero to obtain the correct limiting behaviour for small x.

Frobenius method

$$4xy'' + 2y' + y = 0\,.$$

Look for solutions of the form.. $$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

Required ingredients are..

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}\,,$$

$$2y' = 2\sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1}$$

let m= n-1

$$= 2\sum_{n=-1}^{\infty} a_{n+1}(n+\alpha+1) x^{n+\alpha}\,,$$

relabel m back to n

$$4xy'' = 4\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-1}$$

$$= 4\sum_{n=-1}^{\infty} a_{n+1}(n+\alpha+1)(n+\alpha) x^{n+\alpha}\,.$$

Frobenius method

$$4xy'' + 2y' + y = 0\,.$$

Look for solutions of the form.. $$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

Substituting the ingredients gives.

$$[4a_0(\alpha-1)\alpha + 2a_0\alpha]\, x^{\alpha-1} \quad +$$

$$\sum_{n=0}^{\infty} [4a_{n+1}(n+\alpha)(n+\alpha+1) + 2a_{n+1}(n+\alpha+1) + a_n]\, x^{n+\alpha} \quad = \quad 0\,.$$

all of the coefficients of x must equal zero

Frobenius method

$$
\begin{aligned}
& \left[4a_0(\alpha-1)\alpha + 2a_0\alpha\right] x^{\alpha-1} & + \\
\sum_{n=0}^{\infty} \left[4a_{n+1}(n+\alpha)(n+\alpha+1) + 2a_{n+1}(n+\alpha+1) + a_n\right] x^{n+\alpha} & = & 0\,.
\end{aligned}
$$

General solution has the form...

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha_1} + \sum_{n=0}^{\infty} b_n x^{n+\alpha_2}$$

where two of the coefficients a_n and b_n will be fixed from the **initial** or **boundary** conditions and
the rest are determined from **recurrence relations**.....

Recurrance relations

Equating the coefficient of $x^{n+\alpha}$

$$4a_{n+1}(n+\alpha)(n+\alpha+1) + 2a_{n+1}(n+\alpha+1) + a_n = 0\ ,$$

Gives recurrence relation…

$$a_{n+1} = -\frac{1}{2}\frac{1}{(2n+2\alpha+1)(n+\alpha+1)}a_n\ .$$

Providing $\alpha_1 \neq \alpha_2$ will have two different recurrence relations, one for each root of the indicial equation

Recurrance relations

$$a_{n+1} = -\frac{1}{2}\frac{1}{(2n+2\alpha+1)(n+\alpha+1)}a_n \, .$$

Start with $\alpha_1 = 1/2$

$$a_{n+1} = -\frac{1}{4}\frac{1}{(n+1)(n+\frac{3}{2})} \, .$$

Summary of Frobenius Method

$$a_n = \frac{(-1)^n}{4^n} \times \frac{1}{(1 \times 2 \times 3 \times \cdots n) \times \left(\frac{3}{2} \times \frac{5}{2} \times \frac{7}{2} \times \cdots \frac{2n+1}{2}\right)} a_0 \, .$$

Simplify the products

$$2 \times 4 \times 6 \times \cdots \times 2n \quad = \quad 2^n \times \left(\frac{2}{2} \times \frac{4}{2} \times \frac{6}{2} \times \cdots \times \frac{2n}{2}\right)$$

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Series solution based on the root $\alpha_1 = 1/2$

$$
\begin{aligned}
y(x) &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+\frac{1}{2}} \\
&= a_0 \left[x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \cdots \right] \\
&= a_0 \left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \cdots \right]
\end{aligned}
$$

Frobenius Method

$$b_{n+1} = -\frac{1}{2}\frac{1}{(2n+1)(n+1)}\,b_n\ .$$

For arbitrary b_0 the series is

$$\begin{aligned}
b_1 &= -\frac{1}{2}\times\frac{1}{1\times 1}\,b_0 = -\frac{1}{2}b_0\ ,\\
b_2 &= -\frac{1}{2}\times\frac{1}{3\times 2}\,b_1 = \frac{(-1)^2}{2^2}\frac{1}{3\times 1\times 2\times 1}\,b_0\ ,\\
b_3 &= -\frac{1}{2}\times\frac{1}{5\times 3}\,b_2 = \frac{(-1)^3}{2^3}\frac{1}{5\times 3\times 1\times 3\times 2\times 1}\,b_0\ .
\end{aligned}$$

Frobenius Method

$$
\begin{aligned}
y(x) &= b_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n \\
&= b_0 \left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots \right] \\
&= b_0 \left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots \right] \\
&= b_0 \cos\left(\sqrt{x}\right) \ .
\end{aligned}
$$

Summary of the Frobenius method

A **second order linear differential equation** will has

two linearly independent solutions: $y_1(x)$ and $y_2(x)$.

Fuchs's theorem

If $x = 0$ is an **ordinary** point or **regular** **singular** point: then the Frobenius method will lead to at least one of these solutions as an expansion about $x = 0$.

3 cases: 1: Unequal roots and $\alpha_1 - \alpha_2$ not equal to an integer

2: Equal roots: $\alpha_1 = \alpha_2 = \alpha$

3: Unequal roots differing by an integer: $\alpha_1 - \alpha_2 = N$

1. Unequal roots and $\alpha_1 - \alpha_2$ not equal to an integer

$$y_1(x) = |x|^{\alpha_1} \sum^{\infty} a_n x^n \,,$$

$$y_2(x) = |x|^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n \,.$$

2. Equal roots: $\alpha_1 = \alpha_2 = \alpha$.

Coefficients a_n and b_n are not the same,

even though there is only one root α.

$$y_1(x) = |x|^\alpha \sum_{n=0}^{\infty} a_n x^n ,$$

$$y_2(x) = y_1(x) \ln|x| + |x|^\alpha \sum_{n=0}^{\infty} b_n x^n .$$

3. Unequal roots differing by an integer: $\alpha_1 - \alpha_2 = N$

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One solution corresponding to the greater of the two roots α_1

$$y_1(x) = |x|^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n \,.$$

second solution is given by

$$y_2(x) = a y_1(x) \ln|x| + |x|^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n$$

a_0 and b_0, are arbitrary, and fixed by the **initial conditions**.

Summary

This lecture:

- Frobenius method for fractional and negative powers

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

Reading: Chapter 7 of lecture notes