PH2210 Quantum Mechanics

Problem Sheet 4

(To be submitted by 10am on Tuesday 24 November, 2020.)

1. Particle in a 3D Infinite Well

The eigenfunctions and the energy levels for a particle of mass m in a 2D rectangular well ($0 \le x \le a$ and $0 \le y \le b$) with infinite walls are given by[1]

$$\psi_{p,q}(x,y) = \sqrt{\frac{4}{ab}} \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \quad \text{and} \quad E_{p,q} = \frac{\hbar^2}{2m}\left\{\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}\right\},$$

where $p = 1, 2, 3, ...$ and $q = 1, 2, 3...$ are the two quantum numbers.

(a) Write down expressions for the eigenfunctions and eigenvalues (energies) for the same particle, but in a 3D rectangular box with dimensions $0 \le x, y \le a$ and $0 \le z \le 2a$.

(b) List the six lowest energy levels for the particle in units of $\frac{\hbar^2\pi^2}{8ma^2}$, determining the *degeneracy* of each level.

2. Angular Momentum Operators

(a) Starting from the classical definition of $\vec{L}$, derive the angular momentum operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in Cartesian coordinates.

(b) Show that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$; write down the other commutators obtained by performing the cyclic changes $x \to y \to z \to x...$ etc.

(c) A matrix representation of the angular momentum operator for a spin-$\frac{1}{2}$ particle is given by $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, where the Pauli spin matrices σ are given by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that the eigenvalues of S_x are $+\hbar/2$ and $-\hbar/2$, with associated eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively.

(d) Show that the spin-1/2 matrices have similar commutators to the orbital angular momentum $\hat{L}$, that is, $[S_x, S_y] = i\hbar S_z$ and $[S^2, S_x] = 0$.

(e) Find the eigenvalues and normalised eigenvectors for S_y and S_z.

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[1] Note that in lectures we covered the 1D (and the 3D) infinite square well "particle in a box" problems, for boxes centred on the origin, unlike the one here. For discussion of the case of boxes with the origin on the left (as is the case here) see eg Section II.D in the PH2210 Summary Lecture Notes by Dr Nicholls, available on moodle.