Bessel Functions

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Overview

Last lectures:

- Frobenious method
- Special functions
- Bessel functions

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \; .$$

This lectures:

- Bessel functions

Reading: Chapter 8 of lecture notes

Recall: Bessel function

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Circular drum (the vibrations of this)

The wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} ,$$

Boundary conditions

$$\begin{aligned} u(a,\theta,t) &= 0 , \\ u(r,\theta,t) &= u(r,\theta+2\pi,t) , \\ |u(r,\theta,t)| &< \infty . \end{aligned}$$

Circular drum, so use cylindrical co-ordinates

Seek product solutions φ(r, θ, t)

$$\varphi(r,\theta,t) = R(r)Q(\theta)T(t) .$$

https://www.google.com/search?q=a+circular+drum+image&rlz=1C1CHBF_enGB822GB822&sxsrf=ALeKk03X6BSPe1K3gCkf88e9B4AdLFGmvw:1599052746578&source=lnms&tbm=isch&sa=X&ved=2ahUKEwjAgr_Xx8rrAhX5QhUIHYMxBEEQ_AUoAXoECAsQAw&biw=1274&bih=684#imgrc=dLrBGCicSbjVhM

Cylindrical Drum Solution

R equation $$r^2R'' + rR' + (\mu^2r^2 - n^2)R = 0\,.$$

Special case of SL

To solve R: e.g. let

$$\begin{aligned} x &= \mu r\,, \\ y(x) &= R(r) = R\left(\frac{x}{\mu}\right)\,. \end{aligned}$$

$$\boxed{x^2y'' + xy' + (x^2 - n^2)y = 0\,.}$$

Bessel's equation of "order" n

Goal: find y(x) -> R -> $\varphi(r,\theta,t) = R(r)Q(\theta)T(t)\,.$

The Bessel functions J_n and Y_n

Try series solution about x=0

Divide Bessel's eq by x^2-> **"standard form"**

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0\,.$$

P(x) and Q(x) both have singularities at x=0

Compare to…

$$y'' + P(x)y' + Q(x)y = 0\,,$$

$P(x) = 1/x$ and $Q(x) = 1 - n^2/x^2$ both have **singularities** at $x = 0$

The Bessel functions J_n and Y_n

$$\begin{aligned} xP(x) &= 1\,, \\ x^2Q(x) &= x^2 - n^2 \end{aligned}$$

are both well behaved and thus **analytic** at x = 0

therefore x = 0 is a **regular** **singular** point of the diff. equ.

So the **Frobenius method** will lead to at least one solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}$$

Sub into the Bessel equation…

The Bessel functions J_n and Y_n

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$$\boxed{y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}} \quad \Rightarrow \quad x^2y'' + xy' + (x^2 - n^2)y = 0$$

Required ingredients

$$n^2 y = \sum_{k=0}^{\infty} n^2 a_k x^{k+\alpha} ,$$

$$x^2 y = \sum_{k=0}^{\infty} a_k x^{k+\alpha+2} = \sum_{k=2}^{\infty} a_{k-2} x^{k+\alpha} ,$$

k->k-2, then rename k: k=0 and 1 terms go

$$xy' = \sum_{k=0}^{\infty} a_k (k+\alpha) x^{k+\alpha} ,$$

$$x^2 y'' = \sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha-1) x^{k+\alpha} .$$

The Bessel functions J_n and Y_n

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Substitute ingredients into the equation….

$$\sum_{k=0}^{1}\left[a_k(k+\alpha)(k+\alpha-1)+a_k(k+\alpha)-n^2a_k\right]x^{k+\alpha} \quad +$$

$$\sum_{k=2}^{\infty}\left[a_k(k+\alpha)(k+\alpha-1)+a_k(k+\alpha)+a_{k-2}-n^2a_k\right]x^{k+\alpha} \quad = \quad 0\,.$$

Equating the coefficients $x^{k-\alpha}$ to 0 gives

k=0 $$a_0(\alpha^2-n^2) \quad = \quad 0\,,$$

Gives indicial equation: α=+/-n

k=1 $$a_1\left[(1+\alpha)^2-n^2\right] \quad = \quad 0\,,$$

k>=2 $$a_k\left[(k+\alpha)^2-n^2\right]+a_{k-2} \quad = \quad 0\,.$$

The Bessel functions J_n and Y_n

$$
\begin{aligned}
a_0(\alpha^2 - n^2) &= 0\,, \\
a_1\left[(1+\alpha)^2 - n^2\right] &= 0\,, \\
a_k\left[(k+\alpha)^2 - n^2\right] + a_{k-2} &= 0\,.
\end{aligned}
$$

If α = +/- n, this can only be satisfied if α_1=0

So the **recurrence** coefficient relation is:

$$a_k = \frac{1}{n^2 - (k+\alpha)^2} a_{k-2}$$

Connects coeffs separated by 2: α_1=0, α_k=0 for all odd k

α_1 =n α_2= -n differ by an integer

Frobenius only gives 1 solution, take as $|\alpha|$ = n

The Bessel functions J_n and Y_n

In the circular drum, however, the **periodicity requirement** on the angular solution Q means n has to be an integer,

and so the two solutions α = ±n are **not linearly independent.**

May take one of the solutions to correspond to α = |n|, which is called a **Bessel function** of the first kind, $J_n(x)$.

Using the recurrence relation for the coefficients finds that $J_n(x)$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

(arbitrary) value of a_0 set to $a_0 = \frac{1}{2^n n!}$

chosen so that for all n > −1 $\int_0^\infty J_n(x)\,dx = 1\ .$

The Bessel functions J_n and Y_n

The second linearly independent solution to the Bessel Equ: use reduction of order.

These solutions are **Bessel functions** of the second kind:

$Y_\nu(x)$. (Neumann functions $N(x)$.)

The Bessel functions J_n and Y_n

General solution

$$y(x) = AJ_n(x) + BY_n(x)\ .$$

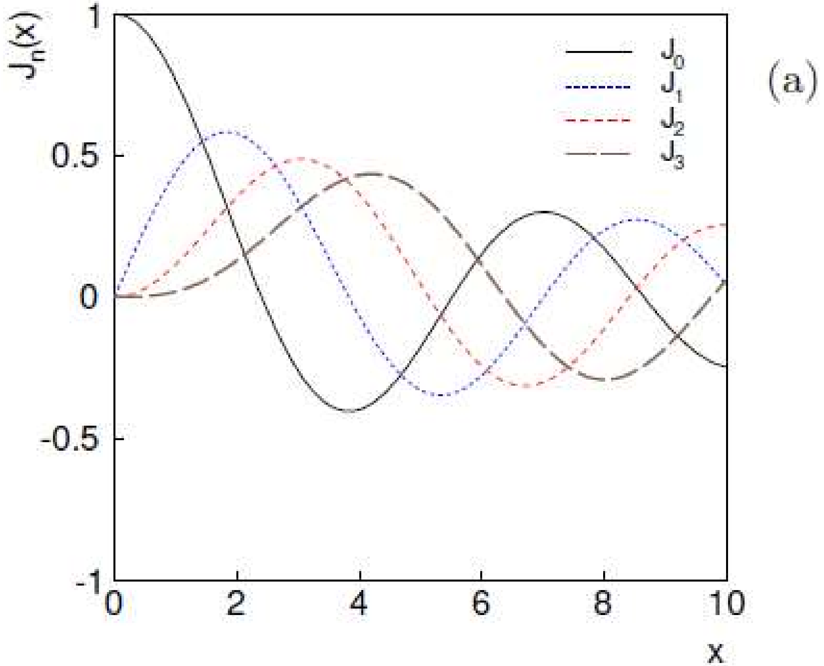
Diverges for x->o

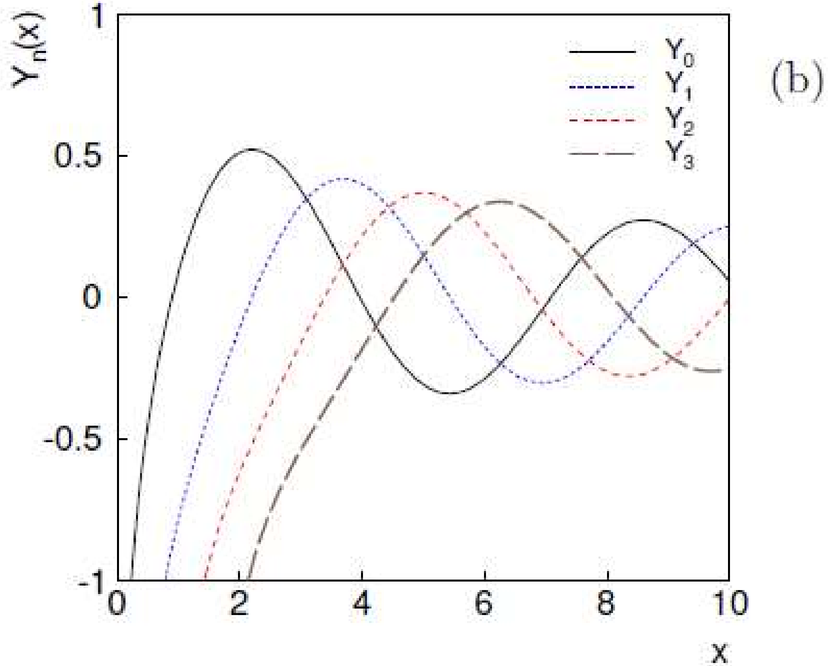
If BC require y(o), then B=o.
Often the case -> usually work with J_n not Y_n

The Bessel functions J_n and Y_n

Oscillations like sine and cosine

But spacing between oscillations is not constant.





first kind $J_n(x)$ second kind $Y_n(x)$

for integer order n.

It is not possible to solve explicitly for the values of x where the functions cross zero, but these can be determined numerically.

The Bessel functions J_n and Y_n

Table 8.1: Values of z_{nm}, the mth zero of J_n, for $n = 0, 1, 2, 3$ (computed with the GSL routine gsl_sf_bessel_zero_Jnu [12]).

m	z_{0m}	z_{1m}	z_{2m}	z_{3m}
1	2.40483	3.83171	5.13562	6.38016
2	5.52008	7.01559	8.41724	9.76102
3	8.65373	10.1735	11.6198	13.0152
4	11.7915	13.3237	14.796	16.2235
5	14.9309	16.4706	17.9598	19.4094
6	18.0711	19.6159	21.117	22.5827
7	21.2116	22.7601	24.2701	25.7482
8	24.3525	25.9037	27.4206	28.9084
9	27.4935	29.0468	30.5692	32.0649
10	30.6346	32.1897	33.7165	35.2187

More on Bessel functions

Bessel functions arise in many contexts and in a number of different forms.

E.g. cylindrical symmetry (like the circular drum).

Can express the **two linearly independent** solutions to **Bessel's equation** by **Hankel functions**

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) , \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) . \end{aligned}$$

Modified Bessel equation

$$x^2 y'' + xy' - (x^2 + \nu^2) y = 0 .$$

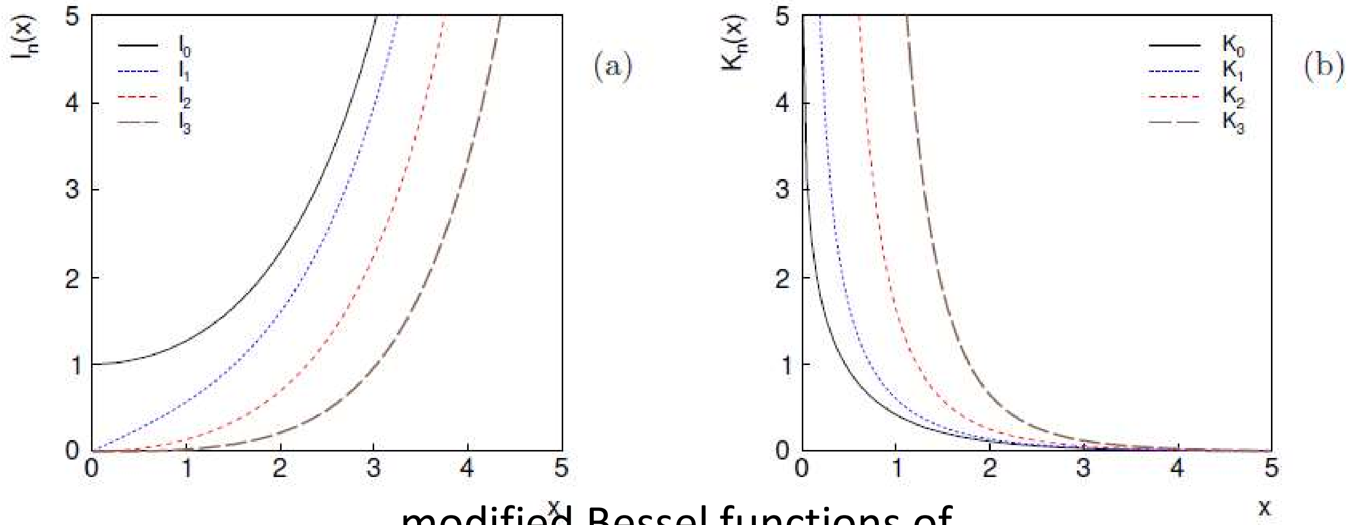
Modified Bessel functions

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0 \; .$$

2 linearly independent solutions

are called the **modified Bessel functions, I(x) and K(x).**

These functions are exponentially growing and decaying, in contrast to the oscillating behaviour of J(x) and Y(x).



modified Bessel functions of

(a) the first kind $I_n(x)$ (b) the second kind $K_n(x)$ for integer order n.

Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0\ .$$

Bessel's equation of order n,

"order" is not the same as that of ODE

In vibrating drum: n must be an **integer**

Bessel's equation appears in other contexts with **noninteger**

values for this parameter.

Use convention: **n** for an **integer**, **ν** if it is **noninteger**.

Notice that μ no longer appears explicitly in the equation when it is written in terms of x, and therefore only appears in the solution R(r) through the value of x, i.e., product μr.

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If the Helmholtz equation

$$\nabla^2\varphi + k^2\varphi = 0\ ,$$

is separated in spherical polar coordinates:
radial equation:

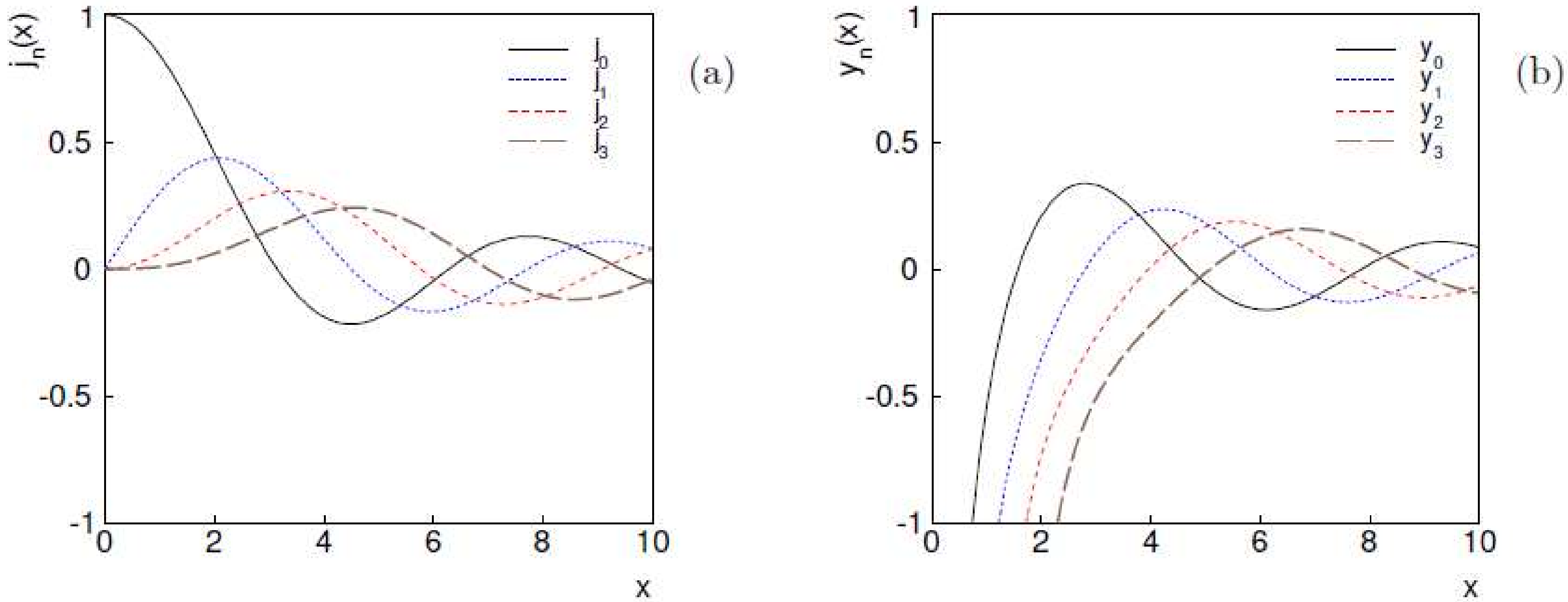
$$x^2y'' + 2xy' + \left[x^2 - n(n+1)\right]y = 0\ .$$

2 linearly independent solutions are called
spherical Bessel functions, $j_n(x)$ and $y_n(x)$:
related to the ordinary Bessel functions $J_n(x)$ and $Y_n(x)$ by

$$j_n(x) \;=\; \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x)\ ,$$

$$y_n(x) \;=\; \sqrt{\frac{\pi}{2x}}Y_{n+1/2}(x)\ .$$

Spherical Bessel functions, $j_n(x)$ and $y_n(x)$



Plots of spherical Bessel functions

(a) the first kind $j_n(x)$ (b) the second kind $y_n(x)$

for integer order n.

Summary

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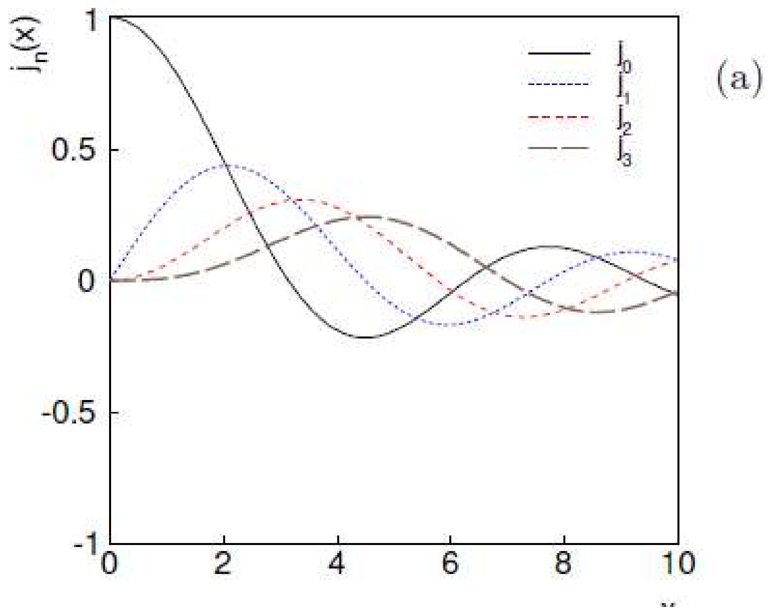
This lecture:

- Special functions
- Bessel functions

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \; .$$



Reading: Chapter 8 of lecture notes