PH2130 Mathematical Methods
Problem Sheet 10
Due 10am Monday 11th January 2021

Problem 1 In Problem Sheet 7 the temperature u of a rod of radius a as a function of the distance r from the central axis and the time t was found by solving the heat equation, where it was assumed that the temperature depended only on r and t and not on the azimuthal angle or on the position along the rod. In this question it will now be assumed that there is a source of heat in the rod, which could be produced by an electric current flowing through it. The heat equation is therefore

$$\frac{\partial u}{\partial t} - \alpha \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) = q(r,t) \ ,$$

where the nonhomogeneous term $q(r,t)$ represents the heat produced in the rod. As in Problem Sheet 7, assume that the rod is very long, so one can consider heat only flowing through the cylindrical surface and not through the ends. Also, as before take the boundary conditions to be $u(a) = 0$ and $|u(0)| < \infty$.

Recall that for the corresponding homogeneous problem where $q(r,t) = 0$, product solutions of the form $\varphi(r,t) = R(r)T(t)$ were sought, and from separation of variables it was found that R and T followed the differential equations

$$\begin{aligned} \frac{dT}{dt} &= -\alpha\mu^2 T \ , \\ \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right) R &= -\mu^2 R \ , \end{aligned}$$

where $-\mu^2$ is a separation constant. The solutions were found to be

$$\begin{aligned} T(t) &= e^{-\alpha\mu^2 t} \ , \\ R(r) &= J_0(\mu r) \ , \end{aligned}$$

where J_0 is the Bessel function of order zero. By application of the boundary conditions above it was found that possible values of μ are restricted to

$$\mu_n = \frac{z_{0n}}{a} \ ,$$

where z_{0n} is the nth positive zero of J_0. Furthermore it was shown that Bessel's equation is a special case of the Sturm-Liouville equation, and therefore the eigenfunctions $J_0(\mu_n r)$, $n = 1, 2, \ldots$, from a complete set of eigenfunctions on the interval $0 \le r \le a$.

1(a) [2 marks] Following the general procedure in Sec. 9.2 of the notes, express the solution $u(r,t)$ as a superposition of eigenfunctions $J_0(\mu_n r)$ with time-dependent coefficients $a_n(t)$ (i.e.,

$J_0(\mu_n r)$ plays the role of $X_n(x)$ in Sec. 9.2). Do the same for the source term $q(r,t)$ with time-dependent coefficients $q_n(t)$.

1(b) [2 marks] By exploiting the orthogonality of the eigenfunctions $J_0(\mu_n r)$, $n = 1, 2, \ldots$, show that the coefficients $q_n(t)$ are

$$q_n(t) = \frac{2}{a^2 J_1^2(z_{0n})} \int_0^a J_0(\mu_n r) q(r,t) r \, dr \; .$$

Note you will need the orthogonality relation for Bessel functions from Sec. 8.1.8.

1(c) [2 marks] Assume that the initial conditions are given by $u(r,0) = f(r)$ for a specified function $f(r)$. Show that the coefficients $a_n(0)$ are

$$a_n(0) = \frac{2}{a^2 J_1^2(z_{0n})} \int_0^a J_0(\mu_n r) f(r) r \, dr \; .$$

1(d) [5 marks] By substituting the expansions for $u(r,t)$ and $q(r,t)$ into the heat equation (follow the general procedure of Sec. 9.2 in the notes) show that one obtains

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} J_0(\mu_n r) + \alpha \sum_{n=1}^{\infty} a_n(t) \mu_n^2 J_0(\mu_n r) = \sum_{n=1}^{\infty} q_n(t) J_0(\mu_n r) \; ,$$

and that this implies that the a_n obey

$$\frac{da_n}{dt} + \alpha \mu_n^2 a_n = q_n \; , \quad n = 1, 2, \ldots \; .$$

1(e) [3 marks] By using an appropriate integrating factor, show that the coefficients $a_n(t)$ are

$$a_n(t) = a_n(0) e^{-\alpha \mu_n^2 t} + \int_0^t e^{-\alpha \mu_n^2 (t - t')} q_n(t') \, dt' \; .$$

1(f) [6 marks] As an example, suppose $q(r,t) = q_0$ (a constant) and consider the initial condition $f(r) = 0$, i.e., the rod is already in ice water before the electric current is switched on. Show that the temperature of the rod as a function of r and t is

$$u(r,t) = \sum_{n=1}^{\infty} \frac{2a^2 q_0}{\alpha z_{0n}^3 J_1(z_{0n})} \left[1 - e^{-\alpha z_{0n}^2 t / a^2} \right] J_0(\mu_n r) \; .$$

You will need the integral $\int_0^x J_0(u) u \, du = x J_1(x)$.