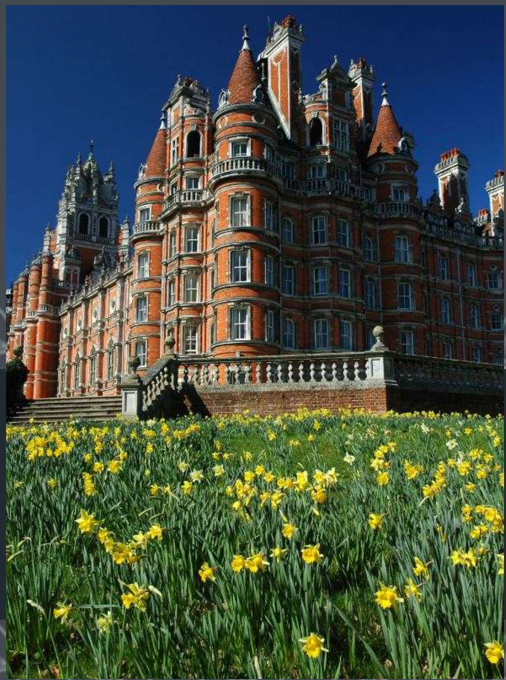
Special Functions



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Overview

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Last lecture:

- Frobenious method

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

This lecture:

- Special functions
- Bessel functions

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \ .$$

Reading: Chapter 8 of lecture notes

Introduction

Many of the important **differential equations** of mathematical physics contain the **Laplacian operator** and often other terms such as first or second derivatives with respect to time.

Solve: using separation of variables
choose a coordinate system that reflects the symmetry
so can easily satisfy the boundary conditions

Leads to Ordinary Differential Equations that often have
nonconstant coefficients and whose solutions therefore
cannot be written in terms of elementary functions.

Define ordinary differential equations

In all of the problems here:

the ODEs from separation of variables

are special cases of the **Sturm-Liouville equation**.

So know the solutions are: **mutually orthogonal**

form a complete basis.

Can write a solution as **a series**, named**…“special functions”**

- Bessel functions
- Legendre polynomials
- Spherical harmonics
- Hermite polynomials

Bessel function

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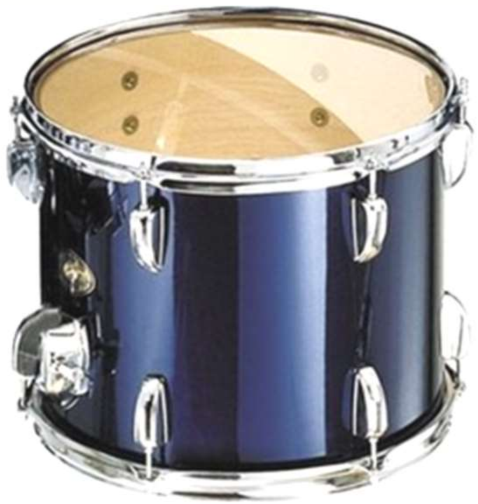
Circular drum (the vibrations of this)



The wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2},$$

speed with which waves
propagate across the surface

Circular drum, so use cylindrical co-ordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

Boundary conditions

$$\begin{aligned} u(a,\theta,t) &= 0\,, \\ u(r,\theta,t) &= u(r,\theta+2\pi,t)\,, \\ |u(r,\theta,t)| &< \infty\,. \end{aligned}$$

https://www.google.com/search?q=a+circular+drum+image&rlz=1C1CHBF_enGB822GB822&sxsrf=ALeKk03X6BSPe1K3gCkf88e9B4AdLFGmvw:1599052746578&source=lnms&tbm=isch&sa=X&ved=2ahUKEwjAgr_Xx8rrAhX5QhUIHYMxBEEQ_AUoAXoECAsQAw&biw=1274&bih=684#imgrc=dLrBGCicSbjVhM

Circular drum

Initial conditions

$$\begin{aligned} u(r,\theta,0) &= f(r,\theta) , \\ \frac{\partial u}{\partial t}(r,\theta,0) &= g(r,\theta) . \end{aligned}$$

Use separation of variables, seek product solutions φ(r, θ, t)

$$\varphi(r,\theta,t) = R(r)Q(\theta)T(t) .$$

Substituting this into the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} ,$$

$$QTR'' + \frac{1}{r}QTR' + \frac{1}{r^2}RTQ'' = \frac{1}{v^2}RQT'' \; .$$

Divide by QTR

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{Q''}{Q} = \frac{1}{v^2}\frac{T''}{T} \; .$$

LHS and RHS contain independent variables, = constant

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{Q''}{Q} = \frac{1}{v^2}\frac{T''}{T} = -\mu^2 \; .$$

So get 2 equations: the T equation is

$$T'' + v^2\mu^2 T = 0 \; ,$$

Cylindrical Drum Solution

T equation
$$T'' + v^2\mu^2 T = 0 ,$$

General solution has the form
$$T(t) = A\cos(v\mu t) + B\sin(v\mu t) .$$

Rθ equation

the Rθ equation can be separated further
$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{Q''}{Q} = -\mu^2 r^2 .$$

θ equation
$$\frac{Q''}{Q} = -n^2 ,$$
a constant

R equation
$$r^2\frac{R''}{R} + r\frac{R'}{R} = n^2 - \mu^2 r^2 ,$$

Cylindrical Drum Solution

Q equation

$$Q'' + n^2 Q = 0 \; ,$$

$$Q(\theta) = C\cos(n\theta) + D\sin(n\theta) \; .$$

cos(nθ) and sin(nθ) with negative n are not linearly independent from terms with positive n and so only need to consider n = 0, 1, 2, ...

Sturm-Liouville form of the radial equation

Radial equation $$r^2 R'' + rR' + (\mu^2 r^2 - n^2)R = 0\ .$$

similar to the Euler equation

Previously converted the Euler equation to one with constant coefficients by a clever variable transformation

In the present case this will not work:

it is not possible to express the solution in terms of usual elementary functions such as sines or exponentials,

try write the solution as an infinite power series.

Sturm-Liouville form of the radial equation

Sturm-Liouville

$$-\frac{d}{dx}\left(p\frac{d\varphi}{dx}\right) - q\varphi = \lambda w\varphi \; .$$

$$rR'' + R' - \frac{n^2}{r}R = -\mu^2 rR \; ,$$

Rewrite in the form

$$-\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{n^2}{r}R = \mu^2 rR \; .$$

Equate terms

$$\begin{aligned} p(r) &= r \; , \\ q(r) &= -\frac{n^2}{r} \; , \\ w(r) &= r \; . \end{aligned}$$

Sturm-Liouville form of the radial equation

Functions p(r) = r and w(r) = r

satisfy the requirements mentioned

for the SL operator to be **self-adjoint**:

on the interval $0 \le r \le a$: $p > 0$

$r > 0$, p' is continuous

$w > 0$ except at the isolated point $r = 0$.

$$p(r) = r\,,$$

$$q(r) = -\frac{n^2}{r}\,,$$

$$w(r) = r\,.$$

r = a

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle - \left[p\left(u^{*\prime}v - u^{*}v'\right)\right]\Big|_a^b\,.$$

r = 0

Furthermore, **homogeneous regular boundary condition**

at r=0, p(0) = 0, so BC do not matter

at r = a, R(a) = 0, and since p(0) = 0 regular homogeneous Dirichlet B.C

Solutions R(r) will be an infinite number of real eigenvalues $\lambda = \mu^2$

each corresponding to a distinct eigenfunction,

and eigenfunctions will be **orthogonal** to each other and constitute a **complete set**.

Now find a series solution for R(r).

Bessel's equation

$$x = \mu r \, ,$$

$$y(x) = R(r) = R\left(\frac{x}{\mu}\right) \, .$$

Derivatives of R' and R'', find using the chain rule

$$R' = \frac{dR}{dr} = \frac{dy}{dx}\frac{dx}{dr} = \mu\frac{dy}{dx} \, ,$$

$$R'' = \frac{dR'}{dr} = \frac{d}{dr}\left(\mu\frac{dy}{dx}\right) = \mu\frac{d^2y}{dx^2}\frac{dx}{dr} = \mu^2\frac{d^2y}{dx^2} \, .$$

Re-write the equation $\quad r^2R'' + rR' + (\mu^2r^2 - n^2)R = 0 \, .$

$$\left(\frac{x}{\mu}\right)^2\mu^2y'' + \frac{x}{\mu}\mu y' + (x^2 - n^2)y = 0 \, ,$$

Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0\,.$$

Bessel's equation of order n,

"order" is not the same as that of ODE

In vibrating drum: n must be an **integer**

Bessel's equation appears in other contexts with **noninteger** values for this parameter.

Use convention: **n** for an **integer**, **ν** if it is **noninteger**.
Notice that μ no longer appears explicitly in the equation when it is written in terms of x, and therefore only appears in the solution R(r) through the value of x, i.e., product μr.

Summary

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This lecture:

- Special functions
- Bessel functions

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \; .$$

Reading: Chapter 8 of lecture notes