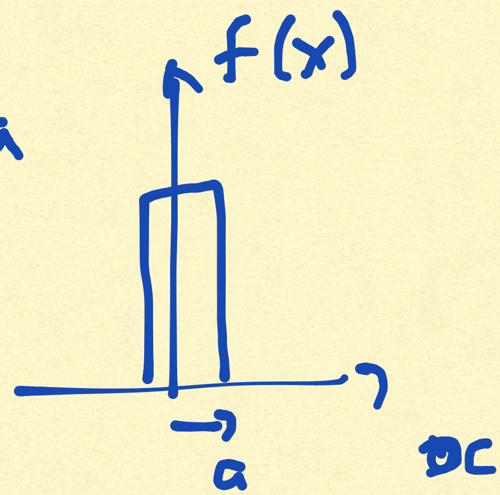
Examples of F.T.

- Box fn
- Gaussian fn
- δ fn

Box fn

$$f(x) = \frac{1}{2a} \qquad -a \le x \le a$$

$$= 0 \qquad \text{otherwise}$$

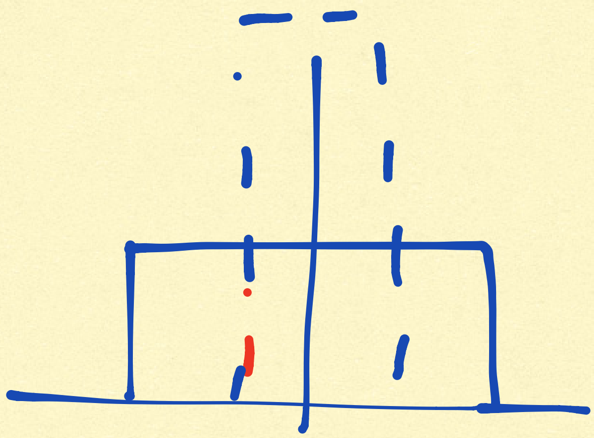


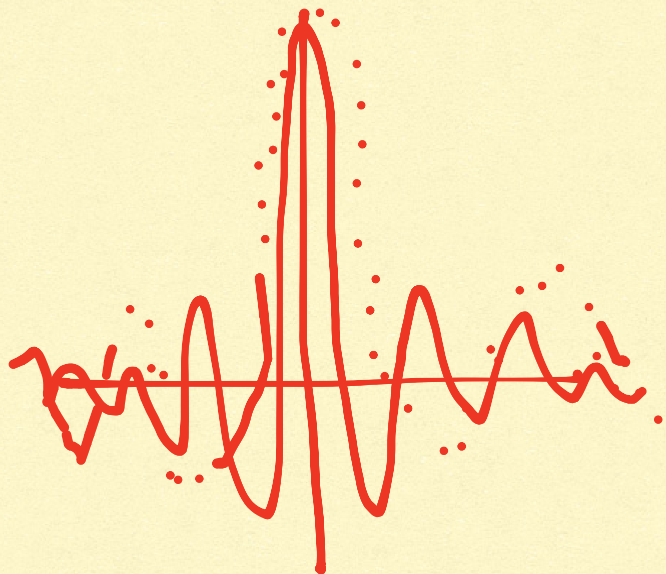
FT

$$\hat{f}(k) = \int_{-a}^{a} e^{-ikx} \frac{1}{2a}\, dx = \frac{1}{-2ika}\left(e^{-ika} - e^{ika}\right)$$

$$= \frac{\sin(ka)}{ka} = \operatorname{sinc}(ka)$$

$$\mathrm{sinc}(x) = \frac{\sin x}{x}$$





narrower box

→ wider distribution

FT of Gaussian

$$f(x) = e^{-ax^2}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} e^{-ax^2} dx$$

$$= \int_{\infty}^{+\infty} e^{-a\left(x^2 + \frac{ikx}{a} - \frac{k^2}{4a^2}\right)} e^{-k^2/4a} dx$$

$$= \int_{-\infty}^{\infty} e^{-a\left(x+\frac{ik}{2a}\right)^2} e^{-k^2/4a}\, dx$$

variable transformation

$$y = \sqrt{a}\left(x+\frac{ik}{2a}\right)$$

$$dy = \sqrt{a}\, dx$$

$$\hat{f}(k) = e^{-k^2/4a}\, \frac{1}{\sqrt{a}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\, dy}_{\text{Gaussian integral}}$$

Trick

$$I^2 = \int_{-\infty}^{\infty} e^{-u^2}\, du \int_{-\infty}^{\infty} e^{-v^2}\, dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)} \, du \, dv$$

$$u = r\cos\theta \qquad v = r\sin\theta$$

$$r^2\cos\theta + r^2\sin^2\theta = r$$

$$u \quad v \quad \rightarrow \quad r\,dr\,d\theta$$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= 2\pi\int_0^{\infty} e^{-r^2} r\,dr$$

Let $s = r^2$

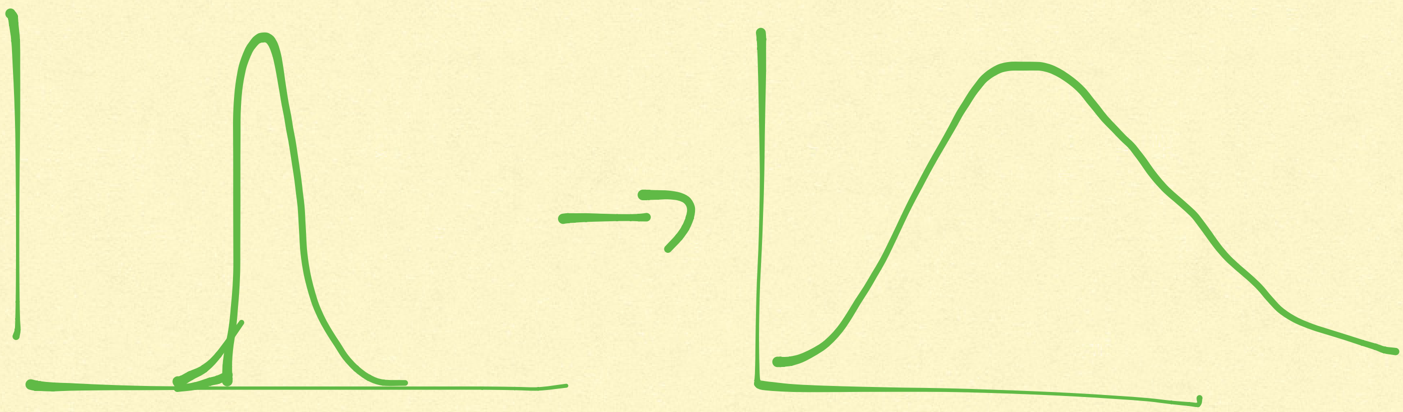
$$I^2 = 2\pi\int_0^{\infty} r e^{-s} \frac{ds}{dr} = 2\pi\left(-\frac{1}{2}\right)e^{-s}\Big]_0^{\infty}$$

$$= \pi$$

$$I = \sqrt{\pi}$$

$$\hat{f}(k) = \sqrt{\frac{\pi}{a}}\, e^{-k^2/4a}$$

FT of a Gaussian fn.



see notes p181

FT of δ fn

$$f(x) = \delta(x-a)$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} \delta(x-a)\, e^{-ikx}\, dx = e^{-ika}$$

δ at $x = 0$ $\quad \hat{f}(k) = 1$

inverse F.T.

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-a)}\, dx$$

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delta function

<u>FT</u>: 3 examples

- Box
- Gaussian
- Delta.