Non homogeneous D.E.

DE

- non homog. B.C.
  eg $u(0,t) = h \neq 0$
- non homo D.E.
  eg $u'' + au' = f(x) \neq 0$

Last lecture:

non homo B.C. converted to one with homo. B.C. & possibly a non-homog. D.E.

$$u(x,t) = w(x,t) + v(x,t)$$

homogeneous B.C. D.E. (transient)

steady state non. hom B.C.

This lecture

Non homogeneous D.E.

eg heat eq with a source

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = q(x,t)$$

heat generated

homogeneous B.C.

$$u(0,t) = 0$$

$$u(L,t) = 0$$

I.C. $u(x,0) = f(x)$

$$q(x,t)$$

Step 1 solve homogeneous DE.

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

separation of variables

$$u(x,t) = X(x)\,T(t)$$

leads to $\dfrac{d^2 X}{dx} = \lambda X$

SL Eq. homogeneous regular B.C.

$\Rightarrow$ self adjoint.

previous solved

eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

eigenvalues $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$

X_n = complete set of basis fns

so ~~f~~ any $f(x)$ on $[0, L]$ can be expressed as a linear combination of X_n

back to non-homo DE

Main idea: expand

1) solution to nonhomo DE $u(x,t)$

2) ~~n~~non-hom source term $q(x,t)$

at fixed t as a linear combination of $X_n(x)$

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t) X_n(x) \\ q(x,t) &= \sum_{n=1}^{\infty} q_n(t) X_n(x) \end{aligned}}$$

a_n q_n will depend on t ...

Find $q_n(t)$ usual way

$$\langle X_m, q \rangle = \sum_{n=1}^{\infty} q_n(t) \langle X_m, X_n \rangle$$

$$\|X_n^2\| \delta_{nm}$$

$$= \frac{L}{2} \underline{q_m(t)}$$

$$= \sum_{n=1}^{\infty} q_n(t) \frac{L}{2} \delta_{nm}$$

$$\boxed{\begin{aligned} q_m(t) &= \frac{2}{L} \langle X_m, q \rangle \\ &= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) q(x,t)\, dt \end{aligned}}$$

Similarly to find $a_n(t)$
at $t = 0$

$$u(x,0) = f(x)$$
$$= \sum_{n=1}^{\infty} a_n(0) X_n(x)$$

$$\langle X_m, f(x) \rangle = \sum_{n=1}^{\infty} a_n(0) \langle X_m, X_n \rangle$$

$$a_n(0) = \frac{\langle X_n, f \rangle}{\|X_n\|^2} = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

given

time dependence of a_n ....

sub expansion of $u(x.t)$

$q(x,t)$ into D.E.

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = q(x,t)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n}{dt} X_n(x)$$

$$\frac{\partial^2 u}{\partial x^2} \quad \sum_{n=1}^{\infty} a_n(t) \frac{d^2 X_n}{dx^2} \rightarrow \text{use eigenvalue \& eigenfn}$$

$$\frac{d^2 X_n}{dx^2} = \lambda_n X_n$$

$$= \sum_{n=1}^{\infty} a_n(t) \lambda_n X_n(x)$$

sub into D.E.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} X_n(x) - \alpha \sum_{n=1}^{\infty} a_n(t) \lambda_n X_n(x)$$

$$= \sum_{n=1}^{\infty} q_n(t) X_n(x)$$

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} - \alpha \lambda_n a_n(t) - q_n(t) \right] X_n(x) =$$

$$= 0$$

all coeff of $X_n = 0$
for all n.

$$\frac{da_n}{dt} - \alpha \lambda_n a_n = q_n$$

fn of t

$n = 1, 2, \ldots.$

Solve D.E in $a_n(t)$

$\times$ integrating factor

$e^{-\alpha\lambda_n t}$

$$e^{-\alpha\lambda_n t}\left(\frac{da_n}{dt} - \alpha\lambda_n a_n\right) = e^{-\alpha\lambda_n t} q_n$$

$$\Rightarrow \frac{d}{dt}\left(a_n e^{-\alpha\lambda_n t}\right) =$$

relabel $t \to t'$

integrate both sides $\int_0^t$

& rearrange.

$$\left[a_n e^{-\alpha\lambda_n t}\right]_0^t = \int_0^t q_n(t') e^{-\alpha\lambda_n t'} dt'$$

$$a_n(t) e^{-\alpha\lambda_n t}$$

$$- a_n(0) =$$

$$\boxed{\times \; e^{+\alpha\lambda_n t}}$$

$$a_n(t) = a_n(0) e^{\alpha\lambda_n t} +$$

$$e^{\alpha\lambda_n t} \int_0^t q_n(t') e^{-\alpha\lambda_n t'} dt'$$

heat Eq.

$$a_n(t) = a_n(0)\, e^{-\alpha\left(\frac{n\pi}{L}\right)^2 L}$$

$$+ \int_0^t q_n(t')\, e^{-\alpha\left(\frac{n\pi}{L}\right)(t-t')}\, dt$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$q_n(t) = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) q(x,t)\, dx$$

$$a_n(0) = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

Recap Main Ideas

① if needed convert to homogeneous B.C.

② homog D.E. & solve. using separation of variables try to find S.L.

③ Eigenfns of S.L.: use to expand
  (i) nonhomog Cn
  (ii) non hom source term

4) sub expansions into D.E

5) solve for coeff of this.

Done.

Example

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = Q_0$$

constant.

I.C. $u(x, 0) = f(x) = 0$

$$X_n = \sin\left(\frac{n\pi x}{L}\right)$$

$$q_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) Q_0 \, dx$$

See. 2.3.5.

we found that

$$q_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4Q_0}{n\pi} & n \text{ odd.} \end{cases}$$

$t = 0$

$$a_n(0) = \frac{\langle X_n, f \rangle}{\| X_n \|^2} = 0$$

0

$$a_n(t) = \int_0^t \frac{4Q_0}{n\pi} e^{-\alpha \left(\frac{n\pi}{L}\right)^2 (t - t')} \, dt'$$

$$= \frac{4Q_0}{\alpha n\pi} \left(\frac{L}{n\pi}\right)^2 \left[-e^{-\alpha \left(\frac{n\pi}{L}\right)^2 (t - t')} \right]_0^t$$

$$= \frac{4 q_0 L^2}{\alpha n^3 \pi^3} \left[1 - e^{-\alpha \left(\frac{n\pi}{L}\right)^2 t} \right]$$

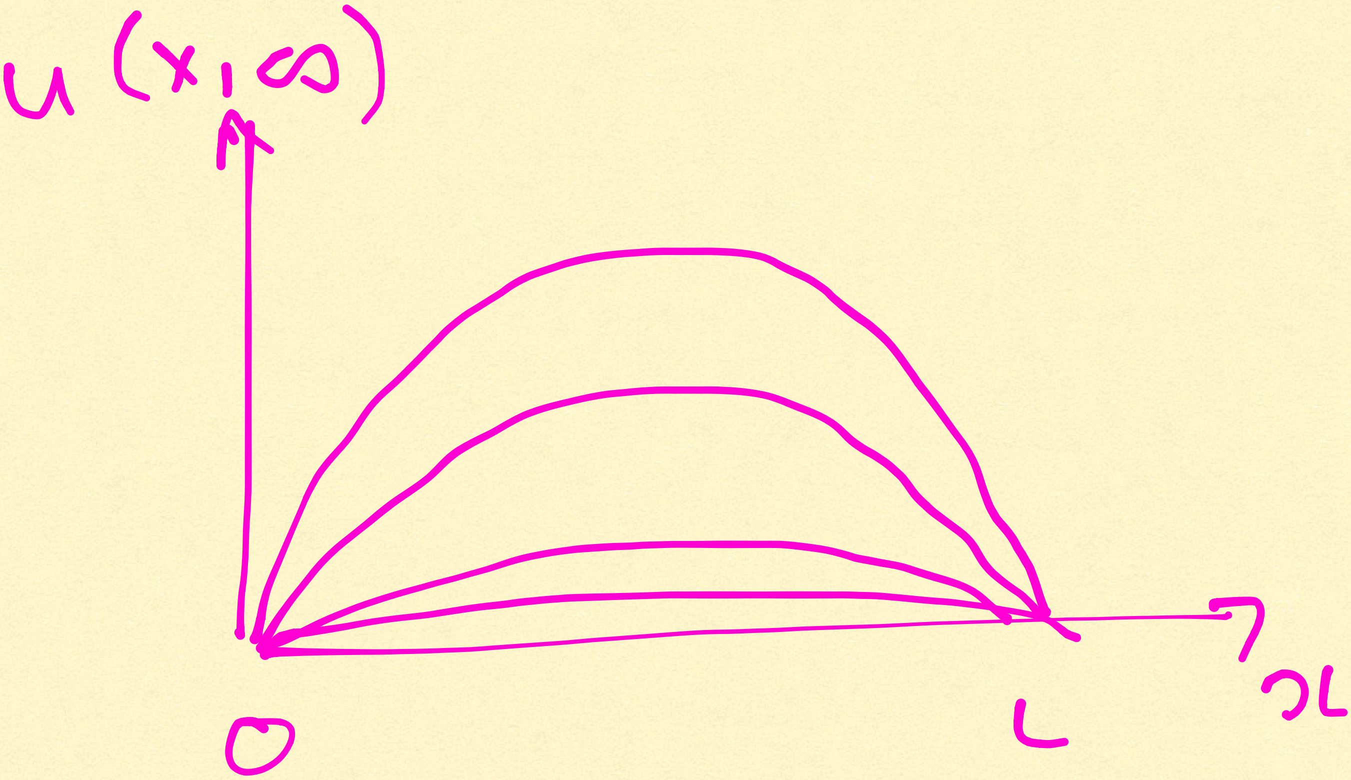
for n odd

& $a_n = 0$ for n = even

$$\boxed{u(x,t) = \sum_{1,3,5\ldots}^{\infty} \frac{4 q_0}{\alpha} \frac{L^2}{(n\pi)^3} \times \left[1 - e^{-\alpha (n\pi/L)^2 t} \right] \times \sin\left(\frac{n\pi x}{L}\right)}$$

as $t \to \infty$

$$u(x,\infty) = \sum_{n=1,3}^{\infty} \frac{4QL^2}{\alpha (n\pi)^3} \cdot \sin\left(\frac{n\pi x}{L}\right)$$



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