Series Solution of O.D.E

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Past methods

Apply Separation of Variables to Schrodinger Equ, Laplace, in cylindrical, spherical coordinates…

-> get O.D.E in corresponding coordinates

Usually non-constant co-efficients (like Euler)

Usually no simple variable sub to make to make constant coeff

Need new approaches…

1) Series Solution

2) Frobenius

Old method…

To solve..

$$y'' + y = 0 \ .$$

General solution is: (only because we know already..!)

$$y(x) = A\cos x + B\sin x \ .$$

For a more general solution: use a power series..

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Solve with a power series..

Substitute
$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Into the D.E equation.. $y'' + y = 0\,.$

$$y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}\,,$$
$$y''(x) = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}\,.$$

Gives:
$$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Solve with a power series..

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$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0\ .$$

Use the trick.. let m=n-2

Terms with m=-2,-1 are 0 anyway.

$$\sum_{m=-2}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{n=0}^{\infty} a_n x^n = 0\ .$$

So now let m=n and rewrite

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n\right] x^n = 0\ .$$

$$0 = \sum_{n=0}^{\infty} 0 \times x^n$$

All coefficients must = 0, for all terms.

Gives the recurrence relation for the coefficients:

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)} .$$

Relates odd and even series....

$$\begin{aligned}
a_2 &= -\frac{a_0}{2 \times 1} , \\
a_3 &= -\frac{a_1}{3 \times 2} , \\
a_4 &= -\frac{a_2}{4 \times 3} = \frac{a_0}{4 \times 3 \times 2 \times 1} , \\
a_5 &= -\frac{a_3}{5 \times 4} = \frac{a_1}{5 \times 4 \times 3 \times 2} .
\end{aligned}$$

Pattern is:

$$a_n = \begin{cases} \dfrac{(-1)^{n/2}}{n!} a_0, & n \text{ even}, \\ \dfrac{(-1)^{(n-1)/2}}{n!} a_1, & n \text{ odd}. \end{cases}$$

Relates odd and even series....

$$y(x) = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) .$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots .$$

Same solution as before!! ☺

Summary

This lecture:

Series Solutions to solve D.E

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Next lecture:

Frobenius Method