Bessel Functions continued

Dr Tracey Berry

ROYAL HOLLOWAY UNIVERSITY OF LONDON

Overview

Last lectures:

- Bessel functions

$$x^2y'' + xy' + (x^2 - n^2)y = 0\ .$$

This lectures:

- Bessel functions continued..
- Justification of the sign of $-\mu^2$
- Eigenmodes of a circular drum

Reading: Chapter 8 of lecture notes

Bessel equation

In Bessel equation..

Now show it must be **negative**

and so called it $-\mu^2$.

Now show that **0** or **+ve values** would not allow us to satisfy the **initial** and **boundary conditions**.

$$x = \mu r ,$$
$$y(x) = R(r) = R\left(\frac{x}{\mu}\right) .$$

$$R' = \frac{dR}{dr} = \frac{dy}{dx}\frac{dx}{dr} = \mu\frac{dy}{dx} ,$$
$$R'' = \frac{dR'}{dr} = \frac{d}{dr}\left(\mu\frac{dy}{dx}\right) = \mu\frac{d^2y}{dx^2}\frac{dx}{dr} = \mu^2\frac{d^2y}{dx^2} .$$

$$\left(\frac{x}{\mu}\right)^2 \mu^2 y'' + \frac{x}{\mu}\mu y' + (x^2 - n^2)y = 0 ,$$

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 .$$

Bessel equation

If **$\mu = 0$**,

then the (r, θ) part of the wave equation:

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{Q''}{Q} = -\mu^2 r^2 \,.$$

reduces to the **Laplace equation**

Recall: extreme values to the Laplace equation always occur on the boundary,

eg here $u = 0$ at the edge of the drum.

Therefore have $u = 0$ everywhere in the interior of the drum as well, i.e., **the trivial solution**.

Bessel equation

If **μ = +ve** separation constant

i.e., μ^2 in place of $-\mu^2$, then radial equation would be

$$r^2 R'' + rR' - (\mu^2 r^2 + n^2)R = 0\ .$$

- the functions I_n and K_n are exponentially rising and falling, respectively.
- so these can't satisfy the B.C that the solution go to zero at the edge of the drum ….
- therefore conclude separation constant $-\mu^2$ must indeed be negative.

General solution to the radial equ $R(r) = y(x) = y(\mu r)$,

i.e. $R(r) = EI_n(\mu r) + FK_n(\mu r)$.

Eigenmodes of a circular drum

Vibrations of a circular drum

Can relate solutions $R(r)$ of the radial equation as $R(r) = y(x) = y(\mu r)$, to the solution $y(x)$ of Bessel's equation

$$R(r) = EJ_n(\mu r) + FY_n(\mu r)$$

where E and F are constants, fix using the boundary conditions.

Boundary condition states: solution must be finite at $r = 0$.
Since all of the Y_n diverge for $r \to 0$, we must have $F = 0$.

Eigenmodes of a circular drum

The other boundary condition says that

$$R(a) = EJ_n(\mu a) = 0$$

Cannot have E = 0 or else, left with R = 0 (trivial solution)

and therefore the eigenvalue μ must take on values such that μa is one of the zeros of J_n.

So the allowed values of μ are

mth zero of Jn,

$$\mu_{nm} = \frac{z_{nm}}{a}$$

Eigenmodes of a circular drum

ROYAL HOLLOWAY UNIVERSITY OF LONDON

Put altogether : RQT:

$$\begin{aligned} R(r) &= E J_n(z_{nm} r/a) \ , \\ Q(\theta) &= C\cos(n\theta) + D\sin(n\theta) \ , \\ T(t) &= A\cos(v z_{nm} t/a) + B\sin(v z_{nm} t/a) \ . \end{aligned}$$

If choose initial speed of the drum = zero,
but give it a nonzero displa
The **full solution** for the motion of
will therefore be a **liner combination** of the form

Each term in the sum corresponds to specific (n,m)th **eigenmodes** of vibration

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(z_{nm} r/a)\cos(z_{nm} v t/a)\left[a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)\right]$$

Time dependence:

$$\cos(vz_{nm}t/a) = \cos(2\pi\nu_{nm}t)$$

where the frequency of a specific eigenmode is given by

$$\nu_{nm} = \frac{vz_{nm}}{2\pi a} \, .$$

Because **spacing** of zeros of Bessel fns z_{nm} is **irregular**, **mixture of frequencies** from drum results in a **dissonant** sound.

Compare to a piano string of length L

Lateral displacement of a vibrating string

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \sin(2\pi\nu_n t) \ ,$$

where allowed frequencies v_n related to string lenght L
and the speed of waves v by

$$\nu_n = \frac{nv}{2L} \ .$$

Corresponding wavelengths λ_n for vibrations such that an **integer number** of halfwaves fits the length of the string, i.e., $\lambda_n/2 = L/n$.

Piano string of length L

For a vibrating string:

hear the combination of the fundamental tone n = 1 and the higher harmonics with n = 2. 3. . . ..

Depending on how the string is struck,

different combinations of harmonics are present.

overtones frequencies are always an

integer multiple x fundamental frequency,

Why our brains perceive a piano string as sounding

"harmonious".

For the drum

ROYAL HOLLOWAY UNIVERSITY OF LONDON

For the drum,

the **wavelengths** of the different eigenmodes are not related by ratios of small **integers**,

but by the **irregular spacing** of zeros of **Bessel functions**.

Why a drum does not play clear musical notes……..

Drum: mixture of eigenmodes produced depend on how struck.

If hits the drum in the centre:

by symmetry the solution cannot depend on θ,

so for the angular solution $Q(\theta)$ one must have $n = 0$.

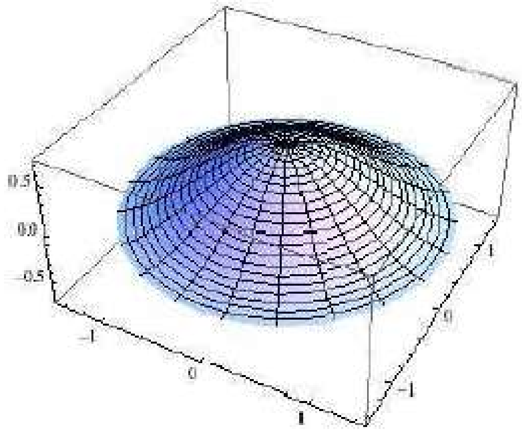
The resulting modes will have wavelengths corresponding to the zeros of J_0, i.e., z_{01}, z_{02}, ...

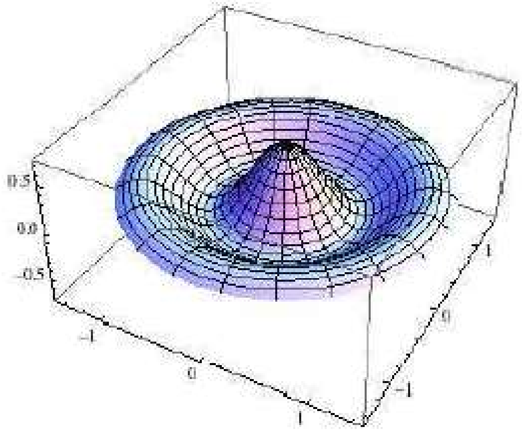
The eigenmodes for $n = 0$ and $m = 1, 2, 3$ are…

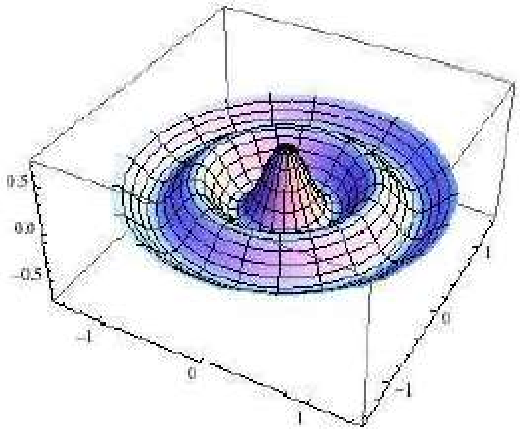
For the drum: struck at centre

ROYAL HOLLOWAY UNIVERSITY OF LONDON

The eigenmodes for n = 0 and m = 1, 2, 3 are…





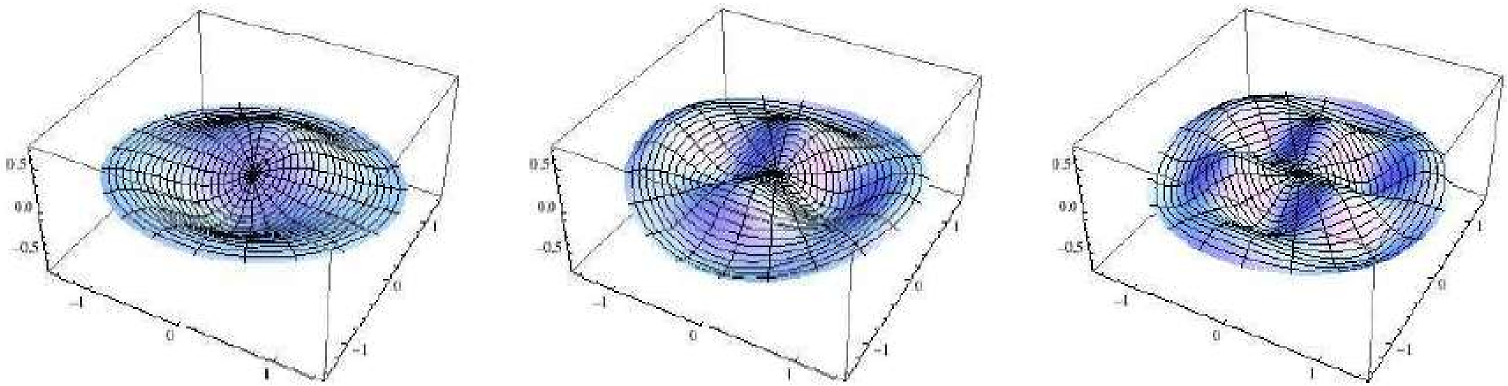


For the drum

Drum:

mixture of eigenmodes
produced depend on how struck.

If hit between **centre and the edge**
excites eigenmodes corresponding to
nonzero values of n.



Eigenmodes of a circular drum corresponding to m = 1 and n = 1, 2,

Orthogonality of Bessel functions

By specifying initial conditions: can determine coeffs a_{nm}

Which tell the drum's exact motion as a function of time

Suppose that the drum's membrane is initially at rest

$$\frac{\partial u}{\partial t}(r,\theta,0) = 0$$

initial position is given by a specified function of r and θ:

$$u(r,\theta,0) = f(r,\theta)$$

Already seen: radial eq. for R(r) is special case of the **Sturm-Liouville equation**,

So know solutions $R_{nm}(r) = J_n(z_{nm}r/a)$ corresponding to different eigenvalues are **orthogonal**.

Orthogonality of Bessel functions

For any given order of the Bessel functions n

and for any two zeros z_{nl} and z_{nm} : must have

$$\langle J_n(z_{nl}r/a), J_n(z_{nm}r/a)\rangle = \int_0^a J_n(z_{nl}r/a)J_n(z_{nm}r/a)r\,dr = \|J_n(z_{nm}r/a)\|^2\delta_{lm}\ .$$

Note: here inner product is defined on the interval $0 \le r \le a$ using the weight function $w(r) = r$

and that the value of n (the order of the Bessel function) is the same for both J_n terms in the inner product.

It is the eigenvalues of the corresponding Sturm-Liouville operator, $\mu_{nm} = z_{nm}/a$ and $\mu_{nl} = z_{nl}/a$, that are different.

Orthogonality of Bessel functions

ROYAL HOLLOWAY UNIVERSITY OF LONDON

For the case l = m,

i.e., where the two zeros of J_n are equal:

the integral gives the norm squared of the Bessel function:

$$\|J_n(z_{nm}r/a)\|^2 = \frac{a^2}{2} J_{n+1}^2(z_{nm}) \, .$$

Orthogonality of Bessel functions

Can use orthogonality relation for Bessel functions together with the corresponding formulae for sines and cosines to **solve for the coefficients a_{nm} and b_{nm}:**

Like the heated disc problem:

Orthogonality of Bessel functions

ROYAL HOLLOWAY UNIVERSITY OF LONDON

19

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(z_{nm}r/a)\cos(z_{nm}vt/a)\left[a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)\right] .$$

2 different types of orthogonal functions:

- trigonometric functions $\cos(n\theta)$ and $\sin(n\theta)$
- Bessel functions $J_n(z_{nm}r/a)$.

orthogonality relation for sines and cosines

$$\begin{aligned}
\int_0^{2\pi} \cos(n\theta)\cos(l\theta)\,d\theta &= \|\cos(n\theta)\|^2\delta_{nl} , \\
\int_0^{2\pi} \sin(n\theta)\sin(l\theta)\,d\theta &= \|\sin(n\theta)\|^2\delta_{nl} , \\
\int_0^{2\pi} \cos(n\theta)\sin(l\theta)\,d\theta &= 0 , \qquad (\text{all } n, l) ,
\end{aligned}$$

where $\|\cos(0)\|^2 = 2\pi$, $\|\sin(0)\|^2 = 0$,

$\|\cos(n\theta)\|^2 = \|\sin(n\theta)\|^2 = \pi$ for $n = 1, 2, \ldots$.

Solve for the coefficients

Can solve for the coefficients by using essentially the same approach as with the Fourier series.

First take the inner product of both sides of Eq. with cos(lθ)

and define the result as a new quantity $a_l(r)$;

$$
\begin{aligned}
a_l(r) &\equiv \langle \cos(l\theta), f(r,\theta)\rangle = \int_0^{2\pi} \cos(l\theta) f(r,\theta)\, d\theta \\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\left[a_{nm}\int_0^{2\pi}\cos(n\theta)\cos(l\theta)\, d\theta + b_{nm}\int_0^{2\pi}\sin(n\theta)\cos(l\theta)\, d\theta\right] .
\end{aligned}
$$

$$
\begin{aligned}
a_l(r) &\equiv \langle \cos(l\theta), f(r,\theta)\rangle = \int_0^{2\pi} \cos(l\theta) f(r,\theta)\, d\theta \\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}r}{a}\right)\left[a_{nm}\int_0^{2\pi}\cos(n\theta)\cos(l\theta)\, d\theta + b_{nm}\int_0^{2\pi}\sin(n\theta)\cos(l\theta)\, d\theta\right] .
\end{aligned}
$$

$$
b_l(r) \equiv \int_0^{2\pi} \sin(l\theta) f(r,\theta)\, d\theta = \|\sin(l\theta)\|^2 \sum_{m=1}^{\infty} b_{lm} J_l\left(\frac{z_{lm}r}{a}\right) .
$$

Now for the Bessel function

ROYAL HOLLOWAY UNIVERSITY OF LONDON

by multiplying both sides of by $J_l(zlkr/a)$

times the weight function r and integrating from o to a.

$$\int_0^a a_l(r) J_l\left(\frac{z_{lk}r}{a}\right) r\,dr = \|\cos(l\theta)\|^2 \sum_{m=1}^{\infty} a_{lm} \int_0^a J_l\left(\frac{z_{lm}r}{a}\right) J_l\left(\frac{z_{lk}r}{a}\right) r\,dr\,.$$

use the orthogonality relation for Bessel functions

which will gives a Kronecker delta δ_{km}

$$a_{lk} = \frac{2}{N_l \pi a^2} \frac{1}{J_{l+1}^2(z_{lk})} \int_0^a a_l(r) J_l\left(\frac{z_{lk}r}{a}\right) r\,dr\,, \qquad l = 0, 1, \ldots\,,$$

carry out the sum over m, which is nonzero only for m = k.
where defined $N_0 = 2$ and $N_l = 1$ for l = 1, 2, . . & $b_{lk} = 0$ for l = 0.

$$b_{lk} = \frac{2}{\pi a^2} \frac{1}{J_{l+1}^2(z_{lk})} \int_0^a b_l(r) J_l\left(\frac{z_{lk}r}{a}\right) r\,dr\,, \qquad l = 1, 2, \ldots\,,$$

l and k relabeled back to n and m.

Overview

Last lectures:

- Bessel functions $$x^2 y'' + xy' + (x^2 - n^2)y = 0\ .$$

This lectures:

- Bessel functions continued..
- Justification of the sign of $-\mu^2$
- Eigenmodes of a circular drum

Reading: Chapter 8 of lecture notes