Laplace's Equation in cylindrical & spherical co-ordinates

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Overview: Laplace's Equation

Introduce

- **Cylindrical Co-ordinates**
  - ➢ Laplace's Equation
  - ➢ Volume Element

- **Spherical Co-ordinates**
  - ➢ Laplace's Equation
  - ➢ Volume Element

Cylindrical and Spherical Co-ordinates

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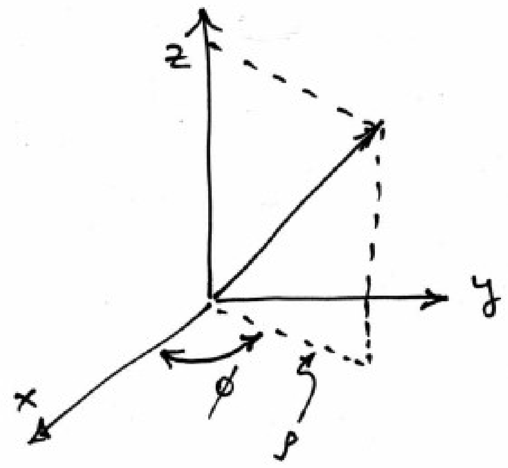
Cylindrical to Cartesian

$$\begin{aligned} x &= \rho\cos\phi\,, \\ y &= \rho\sin\phi\,, \\ z &= z\,. \end{aligned}$$



$$0 \le \rho < \infty,\ 0 \le \phi \le 2\pi \text{ and } -\infty < z < \infty.$$

Cartesian to cylindrical

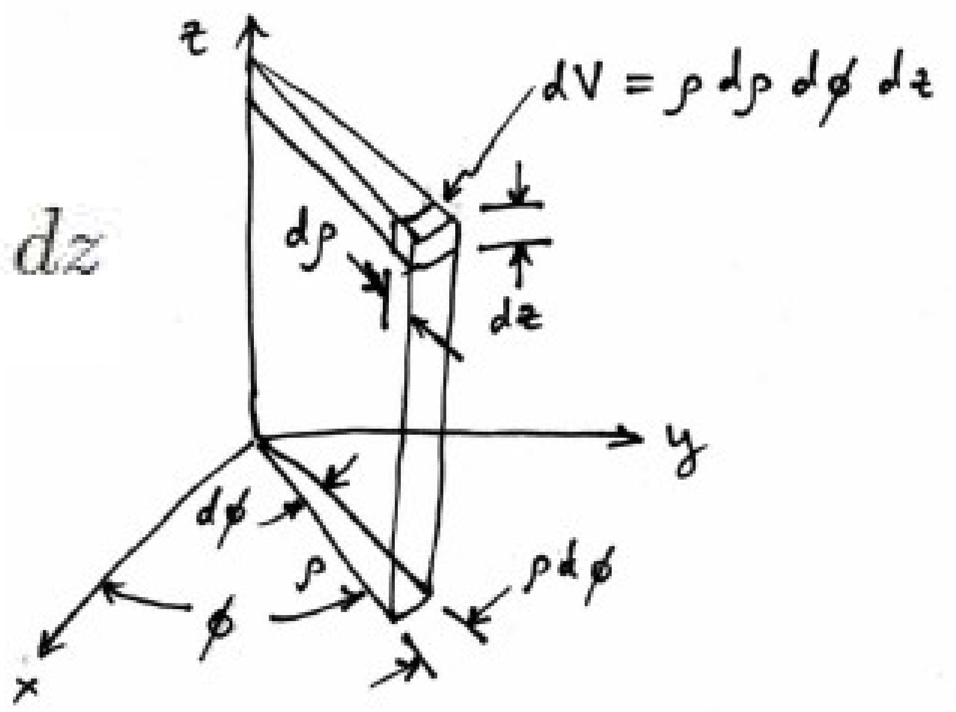
$$\begin{aligned} \rho &= \sqrt{x^2+y^2}\,, \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right)\,, \\ z &= z\,. \end{aligned}$$

Laplace's Equation in Cylindrical Coordinates

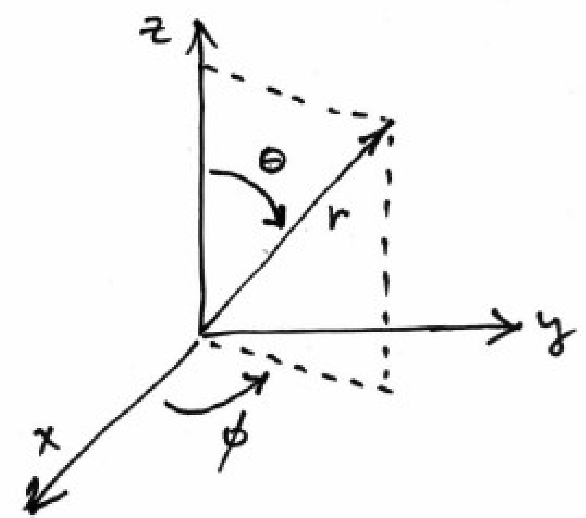
$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Volume Element

$$dV = \rho\, d\rho\, d\phi\, dz$$



Spherical Coordinates



$$x = r\sin\theta\cos\phi\,,$$
$$y = r\sin\theta\sin\phi\,,$$
$$z = r\cos\theta\,,$$

where $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$ and $r \ge 0$.

The inverse relations are

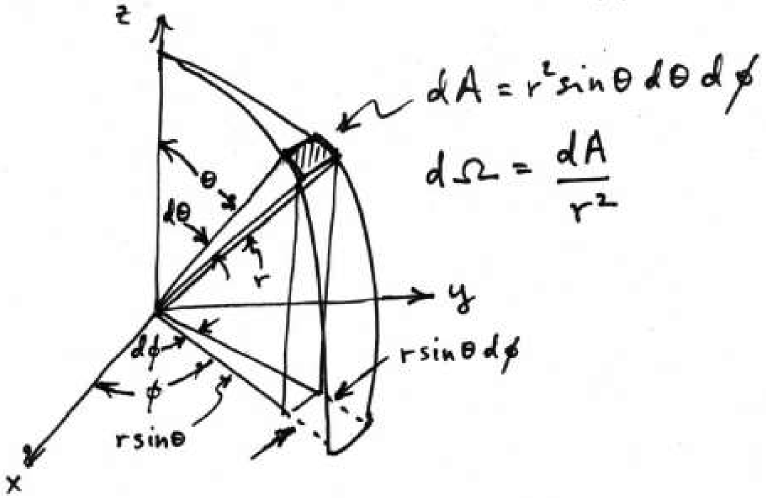
$$r = \sqrt{x^2 + y^2 + z^2}\,,$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}\,,$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)\,.$$

Spherical Laplace's Equation

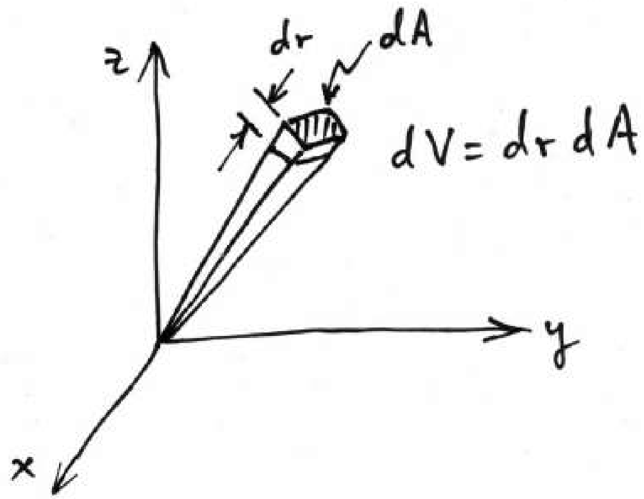
$$\nabla^2 u = \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 u}{\partial \phi^2}\right] .$$

$$
\begin{aligned}
dA &= r^2 \sin\theta \, d\theta \, d\phi \, , \\
dV &= dA \, dr = r^2 \sin\theta \, d\theta \, d\phi \, dr \, , \\
d\Omega &= \frac{dA}{r^2} = \sin\theta \, d\theta \, d\phi \, .
\end{aligned}
$$

(a)



(b)



Summary

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  - ➢ Laplace's Equation
  - ➢ Volume Element

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