PH2130 Mathematical Methods
Problem Sheet 8
Due 10 am Wednesday, 18th November, 2020

Approximate part marks shown in brackets.

Problem 1 Consider the electrostatic potential $u(r, \theta, \phi)$ in spherical coordinates and suppose its value on a sphere of radius a is given by a function $u(a, \theta, \phi) = f(\theta, \phi)$.

1(a) [3 marks] Starting from Eq. (8.166) in the notes, write down the general solution to Laplace's equation for the interior of the sphere as an expansion in r and in spherical harmonics. Justify why you are able to take certain coefficients equal to zero.

1(b) [3 marks] Suppose the potential at $r = a$ is given by

$$f(\theta, \phi) = \begin{cases} u_0 & 0 \leq \theta < \frac{\pi}{2} \, , \\ 0 & \frac{\pi}{2} \leq \theta \leq \pi \, . \end{cases}$$

That is, the two halves of the sphere are separated by a thin insulator at the equator, and each half is held at a different potential. Show that for these boundary conditions, the expansion for the potential from (a) reduces to

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \, ,$$

where the P_l are Legendre polynomials. What is the relation between the coefficients A_l here and the coefficients in the expansion from part (a) in terms of spherical harmonics?

1(c) [4 marks] By imposing the boundary condition above, show that the coefficients A_l are given by

$$A_l = \frac{u_0}{a^l} \frac{2l+1}{2} \int_0^1 P_l(x)\, dx \, .$$

1(d) [4 marks] Show that the solution is

$$u(r, \theta, \phi) = u_0 \left[\frac{1}{2} + \frac{3}{4} \frac{r}{a} P_1(\cos\theta) + \cdots \right]$$

and find the next nonzero term in the series. You can use the integral

$$\int_0^1 P_l(x)\, dx = \begin{cases} 1 & l = 0 \, , \\ \frac{1}{l+1} P_{l-1}(0) & l \geq 1 \, . \end{cases}$$

1(e) [6 marks] Suppose now the boundary condition at $r = a$ is given by

$$u(a, \theta, \phi) = u_0 \sin\theta \, \cos\phi \, .$$

Find the potential *outside* the sphere as an expansion in spherical harmonics, subject to the requirement $u(r) \to 0$ for $r \to \infty$. Find all of the nonzero coefficients. To do this, note that the function above can be written as a linear combination of exactly two of the Y_{lm}s.

Write down the solution for the potential in terms of sines and cosines of θ and ϕ.