# Basic properties of Fourier Transforms

$\mathcal{F}[f]$ $\quad \hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x)\,dx$

$\mathcal{F}^{-1}[\hat{f}]$ $\quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\,dk$

$f = f(\lambda)$ $\quad \hat{f}(k)$ $\quad k = \frac{2\pi}{\lambda}$

$x = x + \lambda$

$kx = kx + 2\pi$

$\omega$ = angular freq

$= 2\pi$ / freq

FT gives freq / wavelength

if $f(x)$ is real.

$\hat{f}(k)$ in general is complex.

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \cos(kx) f(x)\, dx - i \int_{-\infty}^{\infty} \sin kx \cdot f(x)\, dx$$

if $f(x)$ even $f(-x) = f(x)$

$\hat{f}(k)$ is real

" " odd $f(-x) = -f(x)$

$\hat{f}(k)$ is purely imaginary

# Linearity of F.T.

$$f(x) = a\,g(x) + b\,h(x)$$

$a, b$ constants (not fns of $x$)

$$\hat{f}(k) = \int_{-\infty}^{\infty} [a\,g(x) + b\,h(x)] e^{-ikx}\, dx$$

$$= a\int_{-\infty}^{\infty} g(x) e^{-ikx}\, dx + b\int_{-\infty}^{\infty} h(x) e^{-ikx}\, dx$$

$$= a\,\hat{g}(k) + b\,\hat{h}(k)$$

# Translation in x

$$f_a(x) = f(x-a)$$

~~transfted~~ translated version

$$\hat{f}_a(k) = \int_{-\infty}^{\infty} f(x-a)\, e^{-ik\cdot x}\, dx$$

$$u = x - a$$
$$du = dx$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-ik(u+a)}\, du$$

$$= e^{-ika}\, \hat{f}(k)$$

shift $x \to x - a$

$$FT \to \underline{FT \times e^{-ika}}$$

# Scale factor

$$f_b = f(bx)$$

$b =$ constant $\neq 0$

$$\hat{f_b}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(bx)\, dx$$

let $u = bx$

$du = b\, dx$

if $b > 0$

$$\hat{f_b}(k) = \int_{-\infty}^{\infty} e^{-iku/b} f(u) \frac{du}{b}$$

$$= \frac{1}{b} \hat{f}\left(\frac{k}{b}\right)$$

if $b < 0$

$$\hat{f}_b(k) = \int_{+\infty}^{-\infty} e^{-iku/b} f(u) \frac{du}{b}$$

$$= \int_{-\infty}^{\infty} e^{-iku/b} f(u) \frac{du}{|b|}$$

$$= \frac{1}{|b|} \hat{f}\left(\frac{k}{b}\right)$$

$$\Rightarrow \quad \hat{f}_b(k) = \frac{1}{|b|} \hat{f}\left(\frac{k}{b}\right)$$

for any $b \neq 0$

# FT of derivative of $f(x) = f'(x)$

$$f'(k) = \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx$$

$$dv = f'(x) \qquad u = e^{-ikx}$$

$$v = f(x) \qquad du = -ik e^{-ikx}$$

$$= \left[ f(x) e^{-ikx} \right]_{0}^{\infty} - \int_{-\infty}^{\infty} (-ik) f(x) e^{-ikx} dx$$

$$f(x) = 0 \ @ \pm\infty$$

$$f'(k) = ik \hat{f}(k)$$

FT of ~~A~~ $n^{th}$ derivative of a fn

$$\boxed{\hat{f^{(n)}}(k) = (ik)^n \hat{f}(k)}$$

Summarise

Basic of FT

- linearity
- shifting $x \to x-a$
- Scale $x \to bx$
- derivative FT.