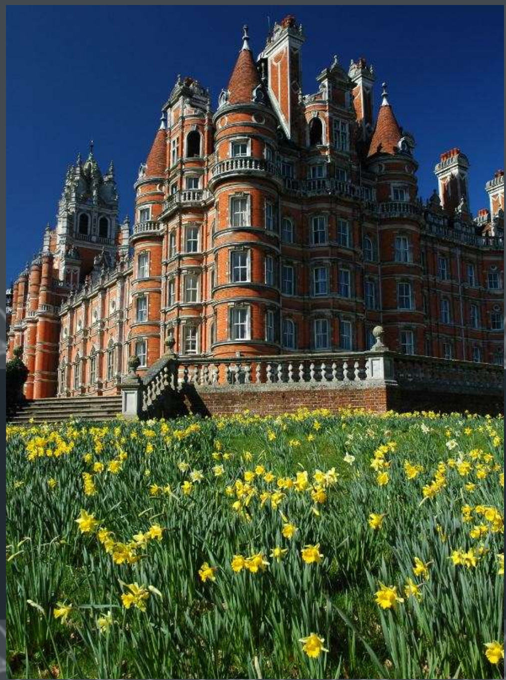
Singular Points
Euler Equation



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Overview

Last lecture:

- Series method to solve a differential equation

  (this does not always work)

This lecture:

- Regular singular point of a diff equation
- Example of the Euler equation

Reading: Chapter 7 of lecture notes

Terminology

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A function f(x) is said to be **analytic** at a point x_0 if:

it can be expanded in a **convergent Taylor series** about x_0,

i.e., a power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

in which all of the coefficients an are finite.

Define ordinary and singular points of a diff. equ.

For a **linear homogeneous second-order ODE** for a function y(x):
Can divide through by the coefficient of second derivative term

$$y'' + P(x)y' + Q(x)y = 0$$

Define ordinary and singular points of a diff. equ.

For a **singular** point to be **regular** we must be able to expand the functions $(x - x_0)P(x)$ and $(x - x_0)2Q(x)$ in series of the form.....

$$(x - x_0)P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$

$$(x - x_0)^2 Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

A problem with the series solution method arises if the point x_0 about which we want to expand is a singular point of the Equ.

Define ordinary and singular points of a diff. equ.

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A ***problem*** with the series solution method arises if the point x_0 about which we want to expand is a **singular** point of the Equ.

In this case one finds that the **infinite** series will **not Series Solutions converge**.

Either: expand about some **different point**,

but what if we particularly want to know the fn near x_0

Define ordinary and singular points of a diff. equ.

Define a new variable: $z = x - x_0$ such that corresponding solution written in terms of z is $\mathbf{g(z) = g(x - x_0) = y(x)}$

For the new function, the singular point is now at $z = 0$.

The Euler equation has a regular **singular** point

See that $x = 0$ is a singular point of the Euler equation!

Euler Equation

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$$P(x) = \frac{b}{ax},$$

$$Q(x) = \frac{c}{ax^2},$$

Rearrange to get:

$$xP(x) = \frac{b}{a},$$

$$x^2Q(x) = \frac{c}{a},$$

which are both finite and so x = 0 is a **regular** singular point.

Euler Equation

For the moment suppose x > 0.

It is easy to verify that a **solution** to the Euler equation

$$ax^2y'' + bxy' + cy = 0$$

is

$$y = x^{\alpha}$$

Try this at home.. Page 100 of the notes

Euler Equation Example…

$$x^2y'' + xy' - \tfrac{1}{9}y = 0\ .$$

Indical equation…

$$\alpha^2 - \frac{1}{9} = 0\ ,$$

Two real and distinct roots, $\alpha_1 = 1/3$ and $\alpha_2 = -1/3$.

Solution…

$$y(x) = Ax^{1/3} + Bx^{-1/3}\ .$$

Solution near a regular singular point

Suppose we have a differential equation of the form

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0 \,,$$

Suppose x = 0 is a regular singular point of the equation,

So one or both of the fns P(x) and Q(x) have a **singularity** at x = 0 but xP(x) and $x^2Q(x)$ are both **analytic** at zero.

Solution near a regular singular point

$$x^2 y'' + x\left(\sum_{n=0}^{\infty} p_n x^n\right) y' + \left(\sum_{n=0}^{\infty} q_n x^n\right) y = 0 \,,$$

Consider values of x close to the singular point of x = 0

Beyond the Euler Equation…

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The differential equations we want to consider are not, however, the Euler equation!

We want not only an **approximation** valid for **x near zero**

but the **exact** solution valid for **all** allowed values of x.

Summary

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This lecture:

- Regular singular point of a diff equation
- Example of the Euler equation

$$ax^2y'' + bxy' + cy = 0$$

Next Lecture

- Frobenius method for fractional and negative powers

Reading: Chapter 7 of lecture notes