Having a Ball: evaluating game excitement and scoring streaks using in-match trend estimation

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Abstract

Nu ved jeg godt nok intet om sport, men... 🏀!

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1 Introduction

We introduce the Execitement Trend Index (ETI) as an objective measure of spectator exicitement in a given match.

- Excitement defineret som skift af hvem, der er i føring
- Vurdering af om et hold trender lige pt.
- Identificere hvilket "hot periods" et hold har i løbet af en kamp til efterfølgende evaluering

Reference to Quantifying the Trendiness of Trends Jensen and Ekstrøm (2020).

Needs a reference and some kind of comparison to Chen, Dawson, and Müller (2020).

Vi kan overveje, om vi ikke udelukkende burde fokusere på Eddy i dette manus og foreslå dette som et objektivt excitement index i stedet for at gå ind i en diskussion om TDI også. Jeg tænker også i forhold til at dette værk måske ikke behøver være så voluminøst.

Materials to reproduce this manuscript can be found at Jensen and Ekstrøm (2019).

Philosophical question: What exactly is the sampling model for a single match?

2 Methods

Let m index a given match between teams a and b and let $D_m(t_{mi}) = S_a(t_{mi}) - S_b(t_{mi})$ be the difference in scores at times $t_{m1} < t_{m2} < t_{mi} < \ldots < t_{mJ_m}$ being the ordered event times when a score by either team a or b occurs during match m.

We use the model

$$
\begin{aligned}
(\boldsymbol{\beta}_m, \boldsymbol{\theta}_m, \sigma_m^2) \mid \boldsymbol{\Psi}_m, \mathbf{t}_m &\sim G(\boldsymbol{\Theta}_m \mid \boldsymbol{\Psi}_m, \mathbf{t}_m) \\
f_m(t) \mid \boldsymbol{\Theta}_m &\sim \mathcal{GP}(\mu_{\boldsymbol{\beta}_m}(t), C_{\boldsymbol{\theta}_m}(t, t')) \\
D_m(t_{mi}) \mid f_m(t_{mi}), t_{mi}, \boldsymbol{\Theta}_m &\overset{iid}{\sim} N\left(f_m(t_{mi}), \sigma_m^2\right)
\end{aligned} \tag{1}
$$

where f_m models a latent trajectory of score differences. We also have that

$$
\begin{bmatrix} f_m(s) \\ df_m(t) \\ d^2 f_m(u) \end{bmatrix} \mid \boldsymbol{\Theta}_m \sim \mathcal{GP}\left(\begin{bmatrix} \mu_{\boldsymbol{\beta}_m}(s) \\ d\mu_{\boldsymbol{\beta}_m}(t) \\ d^2\mu_{\boldsymbol{\beta}_m}(u) \end{bmatrix}, \begin{bmatrix} C_{\boldsymbol{\theta}_m}(s, s') & \partial_2 C_{\boldsymbol{\theta}_m}(s, t) & \partial_2^2 C_{\boldsymbol{\theta}_m}(s, u) \\ \partial_1 C_{\boldsymbol{\theta}_m}(t, s) & \partial_1 \partial_2 C_{\boldsymbol{\theta}_m}(t, t') & \partial_1 \partial_2^2 C_{\boldsymbol{\theta}_m}(t, u) \\ \partial_1^2 C_{\boldsymbol{\theta}_m}(u, s) & \partial_1^2 \partial_2 C_{\boldsymbol{\theta}_m}(u, t) & \partial_1^2 \partial_2^2 C_{\boldsymbol{\theta}_m}(u, u') \end{bmatrix} \right) \tag{2}
$$

where $d^k \mu_{\boldsymbol{\beta}}$ is the k'th derivative of $\mu_{\boldsymbol{\beta}}$ and ∂_j^k denotes the k'th order partial derivative with respect to the j'th variable

We then define the Excitement Trend Index (ETI) as the number of zero-crossings of df_m conditional on the observed score differences. Formally,

$$
\mathrm{ETI}_m(\boldsymbol{\Theta}_m) = \mathrm{E}\left[\# \left\{ t \in \mathcal{I}_m : df_m(t) = 0 \right\} \mid \mathbf{D}_m, \mathbf{t}_m, \boldsymbol{\Theta}_m \right]
$$

where $\mathcal{I}_m$ is the interval of the time duration of a match i.e., $\mathcal{I}_m = [0; 48]$ minutes without overtime. The ETI is given by the intergal of the local Excitement Trend Index

$$
\mathrm{ETI}_m(\boldsymbol{\Theta}_m) = \int_{\mathcal{I}_m} d\mathrm{ETI}_m(t \mid \boldsymbol{\Theta}_m) \mathrm{d}t
$$

where dETI is the local Excitement Trend Index given by

$$
d\mathrm{ETI}_m(t \mid \boldsymbol{\Theta}_m) = \lambda(t \mid \Theta) \phi\left(\frac{\mu_{df}(t \mid \boldsymbol{\Theta})}{\Sigma_{df}(t, t \mid \boldsymbol{\Theta})^{1/2}} \right) \left(2\phi(\zeta(t \mid \boldsymbol{\Theta})) + \zeta(t \mid \boldsymbol{\Theta}) \operatorname{Erf}\left(\frac{\zeta(t \mid \boldsymbol{\Theta})}{2^{1/2}} \right) \right)
$$

and $\phi\colon x \mapsto 2^{-1/2}\pi^{-1/2}\exp(-\frac{1}{2}x^2)$ is the standard normal density function, $\operatorname{Erf}\colon x \mapsto 2\pi^{-1/2}\int_0^x \exp(-u^2)\mathrm{d}u$ is the error function, and λ, ω and ζ are functions defined as

$$
\begin{aligned}
\lambda(t \mid \Theta) &= \frac{\Sigma_{d^2f}(t, t \mid \boldsymbol{\Theta})^{1/2}}{\Sigma_{df}(t, t \mid \boldsymbol{\Theta})^{1/2}} \left(1 - \omega(t \mid \boldsymbol{\Theta})^2 \right)^{1/2} \\
\omega(t \mid \boldsymbol{\Theta}) &= \frac{\Sigma_{df, d^2f}(t, t \mid \boldsymbol{\Theta})}{\Sigma_{df}(t, t \mid \boldsymbol{\Theta})^{1/2} \Sigma_{d^2f}(t, t \mid \boldsymbol{\Theta})^{1/2}} \\
\zeta(t \mid \boldsymbol{\Theta}) &= \frac{\mu_{df}(t \mid \boldsymbol{\Theta}) \Sigma_{d^2f}(t, t \mid \boldsymbol{\Theta})^{1/2} \omega(t) \Sigma_{df}(t, t \mid \boldsymbol{\Theta})^{-1/2} - \mu_{d^2f}(t \mid \boldsymbol{\Theta})}{\Sigma_{d^2f}(t, t \mid \boldsymbol{\Theta})^{1/2} \left(1 - \omega(t \mid \boldsymbol{\Theta})^2\right)^{1/2}}
\end{aligned}
$$

A derivation of this expression can be found in the supplementary material to Jensen and Ekstrøm (2020).

The posterior distribution of the hyper-parameters given the observed data is then. We define $\widetilde{\boldsymbol{\Theta}}_m \sim P(\boldsymbol{\Theta}_m \mid \mathbf{D}_m, \boldsymbol{\Psi}_m, \mathbf{t}_m)$ hence

$$
\widetilde{\boldsymbol{\Theta}}_m \sim \frac{G(\boldsymbol{\Theta}_m \mid \boldsymbol{\Psi}_m, \mathbf{t}_m) \int P(\mathbf{D}_m \mid f(\mathbf{t}_m), \boldsymbol{\Theta}_m, \boldsymbol{\Psi}_m, \mathbf{t}_m) dP(f_m(\mathbf{t}_m) \mid \boldsymbol{\Theta}_m, \boldsymbol{\Psi}_m, \mathbf{t}_m)}{\iint P(\mathbf{D}_m \mid f_m(\mathbf{t}_m), \boldsymbol{\Theta}_m, \boldsymbol{\Psi}_m, \mathbf{t}_m) dP(f_m(\mathbf{t}_m) \mid \boldsymbol{\Theta}_m, \boldsymbol{\Psi}_m, \mathbf{t}_m) dG(\boldsymbol{\Theta}_m \mid \boldsymbol{\Psi}_m, \mathbf{t}_m)}
$$

What we estimate is then the random variable $\widehat{\mathrm{ETI}}_m = \mathrm{ETI}_m(\widetilde{\boldsymbol{\Theta}}_m)$ which can be summarized by its moments or quantiles.

We need to argue that ETI for $S_a(t_{m_i}) - S_b(t_{m_i})$ is symmetric in a and b so that our choice of "reference group" in D_m is not important. The reason is that we look at both up- and down-crossings at 0 of df_m so the choice of sign in D_m is not relevant.

2.1 Estimation

We have implemented the model described in the previous section in Stan (Carpenter et al. 2017).

Prior mean and covariance:

$$\mu_{\boldsymbol{\beta}_m}(t) = \beta_m, \quad C_{\boldsymbol{\theta}_m}(t, t') = \alpha_m^2 \exp\left(-\frac{(t-t')^2}{2\rho_m^2}\right)$$

with $\alpha_m, \rho_m > 0$.

Hyper-parameters: We used independent priors on $\mathbf{\Theta}_m = (\beta_m, \alpha_m, \rho_m, \sigma_m)$ of the form

$$G(\mathbf{\Theta}_m \mid \mathbf{\Psi}_m, \mathbf{t}_m) = G(\beta_m \mid \Psi_{\beta_m})G(\alpha_m \mid \Psi_{\alpha_m})G(\rho_m \mid \Psi_{\rho_m})G(\sigma_m \mid \Psi_{\sigma_m})$$

where each prior is a heavy-tailed distribution with a moderate variance centered at the marginal maximum likelihood estimates. We used the following distributions

$$\beta_m \sim T\left(\widehat{\beta_m^{\text{ML}}}, 3, 3\right), \quad \alpha_m \sim T^+\left(\widehat{\alpha_m^{\text{ML}}}, 3, 3\right), \quad \rho_m \sim N^+\left(\widehat{\rho_m^{\text{ML}}}, 1\right), \quad \sigma_m \sim T^+\left(\widehat{\sigma_m^{\text{ML}}}, 3, 3\right)$$

where $T^+(\cdot, \cdot, \text{df})$ and $N^+(\cdot, \cdot)$ denotes the location-scale half T- and normal distribution functions with df degrees of freedom. For each match we ran four independent Markov chains for 25,000 iterations each with half of the iterations used for warm-up. Convergence was assessed by trace plots of the MCMC draws and the potential scale reduction factor, $\widehat{R}$, of Gelman and Rubin (1992).

3 Results

We use data from Sports Reference LLC (2020).

General idea: We estimate ETI for all matches in a given season and make a nice plot of the distribution of ETI_m. Then we can rank the matches according to increasing ETI and show the running score difference for e.g., the lowest, median and highest ranked matches. Maybe a large forest plot of ETI_m would look impressive.

Given the posteriors ETI_m we can summarize them by a posterior mean and variance and then do a **meta analysis** where we adjust for game-specific fixed effects such as number of spectators, location, stratify by season and so on.

4 Discussion

We could define a weighted Excitement Trend Index, ETI_m^W, so that zero crossings of the derivative of score differences are weighted higher towards the end of the game as in

$$\text{ETI}_m^W = \int_{\mathcal{I}_m} \text{ETI}_m(t) w_m(t) \text{d}t$$

where w_m is an increasing weight function.

Another approach would be to define team-specific excitement index nested with a match. Here we would only look at the **up**-crossings at zero of df_m and we would get two excitement indices for each match $(\text{ETI}_{am}, \text{ETI}_{bm})$. for teams a and b. This would somehow reflect how exciting each team were in match m with respect to chancing the sign of the score differences in their favor.

Acknowledgements

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Appendix (say what?)

the joint distribution of (f, df, d^2f) conditional on $\mathbf{Y}$, $\mathbf{t}$ and the hyper-parameters $\mathbf{\Theta}$ evaluated at any finite vector $\mathbf{t}^*$ of p time points is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \mathbf{\Theta} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$$

where $\boldsymbol{\mu} \in \mathbb{R}^{3p}$ is the column vector of posterior expectations and $\mathbf{\Sigma} \in \mathbb{R}^{3p \times 3p}$ is the joint posterior covariance matrix. Partitioning these as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_f(\mathbf{t}^* \mid \mathbf{\Theta}) \\ \mu_{df}(\mathbf{t}^* \mid \mathbf{\Theta}) \\ \mu_{d^2f}(\mathbf{t}^* \mid \mathbf{\Theta}) \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} \Sigma_f(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) & \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) & \Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) \\ \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta})^T & \Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) & \Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) \\ \Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta})^T & \Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta})^T & \Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) \end{bmatrix}$$

the individual components are given by

$$
\begin{aligned}
\mu_f(\mathbf{t}^* \mid \mathbf{\Theta}) &= \mu_{\boldsymbol{\beta}}(\mathbf{t}^*) + C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} (\mathbf{Y} - \mu_{\boldsymbol{\beta}}(\mathbf{t})) \\
\mu_{df}(\mathbf{t}^* \mid \mathbf{\Theta}) &= d\mu_{\boldsymbol{\beta}}(\mathbf{t}^*) + \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} (\mathbf{Y} - \mu_{\boldsymbol{\beta}}(\mathbf{t})) \\
\mu_{d^2f}(\mathbf{t}^* \mid \mathbf{\Theta}) &= d^2\mu_{\boldsymbol{\beta}}(\mathbf{t}^*) + \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} (\mathbf{Y} - \mu_{\boldsymbol{\beta}}(\mathbf{t})) \\
\Sigma_f(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) &= C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) - C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\
\Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) &= \partial_1 \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) - \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\
\Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) &= \partial_1^2 \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) - \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\
\Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) &= \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) - C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\
\Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) &= \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) - C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\
\Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \mathbf{\Theta}) &= \partial_1 \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) - \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \left(C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right)^{-1} \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*)
\end{aligned}
$$