Tetrahedron Element Formalism

Ahmed Elgailani - January, 8, 2020

Last update:

January 10, 2020

1 Tetrahedron shape functions

The current position of a point inside a tetrahedron can be interpolated from the known current positions of the four nodes of the tetrhedron. Then, the x,y and z in the current configuration for an interior point with coordinates X,Y and Z in the reference configuration can be given as:

$$x(X,Y,Z) = a_0 + a_1 X + a_2 Y + a_3 Z \tag{1}$$

$$y(X,Y,Z) = a_4 + a_5 X + a_6 Y + a_7 Z \tag{2}$$

$$z(X,Y,Z) = a_8 + a_9 X + a_{10} Y + a_{11} Z \tag{3}$$

Where a_i are constants to be determined.

Note: Throughout this document the lower case x,y and z represent the position vectors in the **current** configuration while the capitalized X,Y and Z are the coordinates in the **reference** configuration.

Consider the x poistion. The four nodes of the tetrahedron equation 1 gives:

$$\begin{aligned} x_1 &= a_o + a_1 X_1 + a_2 Y_1 + a_3 Z_1 \\ x_2 &= a_o + a_1 X_2 + a_2 Y_2 + a_3 Z_2 \\ x_3 &= a_o + a_1 X_3 + a_2 Y_3 + a_3 Z_3 \\ x_4 &= a_o + a_1 X_4 + a_2 Y_4 + a_3 Z_4 \end{aligned}$$

In matrix form

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{4}$$

Or, to find the unknown coefficients vector, simply:

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix}^{-1} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \tag{5}$$

Similiarly for y and z coefficients:

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix}^{-1} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{6}$$

$$\begin{Bmatrix} a_8 \\ a_9 \\ a_{10} \\ a_{11} \end{Bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix}^{-1} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \tag{7}$$

Writing equation 1 in matrix form and inserting the results from equation 5 we get:

$$x(X,Y,Z) = \begin{bmatrix} 1 & X & Y & Z \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & X & Y & Z \end{bmatrix} \begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix}^{-1}}_{[C]} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \tag{8}$$

Where $[C] = \{N_1, N_2, N_3, N_4\}$ is the row vector of "shape functions". Notice that [C] is the same for the other positions y(X,Y,Z) and z(X,Y,Z). The summary equations are:

$$x(X,Y,Z) = [C] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \tag{9}$$

$$y(X,Y,Z) = [C] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \tag{10}$$

$$z(X,Y,Z) = [C] \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} = N_1z_1 + N_2z_2 + N_3z_3 + N_4z_4 \tag{11}$$

2 Deformation gradient tensor

Before working out the **F** tensor, let us first look at the gradients of any interpolated quantity. Motiveated by the analysis in section 1 we can see that any field quantity $\phi(X, Y, Z)$ with known values $(\phi_1, \phi_2, \phi_3, \phi_4)$ at the nodes of the tetrahedral can be linearly interpolated as

$$\phi(X, Y, Z) = a + bX + cY + dZ \tag{12}$$

And the constants are given by

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix}^{-1}}_{[A]} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{13}$$

Doing the math we get:

$$a = \frac{1}{J_o} \begin{Bmatrix} -X_4Y_3Z_2 + X_3Y_4Z_2 + X_4Y_2Z_3 - X_2Y_4Z_3 - X_3Y_2Z_4 + X_2Y_3Z_4 \\ X_4Y_3Z_1 - X_3Y_4Z_1 - X_4Y_1Z_3 + X_1Y_4Z_3 + X_3Y_1Z_4 - X_1Y_3Z_4 \\ -X_4Y_2Z_1 + X_2Y_4Z_1 + X_4Y_1Z_2 - X_1Y_4Z_2 - X_2Y_1Z_4 + X_1Y_2Z_4 \\ X_3Y_2Z_1 - X_2Y_3Z_1 - X_3Y_1Z_2 + X_1Y_3Z_2 + X_2Y_1Z_3 - X_1Y_2Z_3 \end{Bmatrix}^T \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{14}$$

$$b = \underbrace{\frac{1}{J_o} \begin{Bmatrix} Y_4(-Z_2 + Z_3) + Y_3(Z_2 - Z_4) + Y_2(-Z_3 + Z_4) \\ Y_4(Z_1 - Z_3) + Y_1(Z_3 - Z_4) + Y_3(-Z_1 + Z_4) \\ Y_4(-Z_1 + Z_2) + Y_2(Z_1 - Z_4) + Y_1(-Z_2 + Z_4) \\ Y_3(Z_1 - Z_2) + Y_1(Z_2 - Z_3) + Y_2(-Z_1 + Z_3) \end{Bmatrix}^T}_{[G_X]} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{15}$$

$$c = \underbrace{\frac{1}{J_o} \begin{Bmatrix} X_4(Z_2 - Z_3) + X_2(Z_3 - Z_4) + X_3(-Z_2 + Z_4) \\ X_4(-Z_1 + Z_3) + X_3(Z_1 - Z_4) + X_1(-Z_3 + Z_4) \\ X_4(Z_1 - Z_2) + X_1(Z_2 - Z_4) + X_2(-Z_1 + Z_4) \\ X_3(-Z_1 + Z_2) + X_2(Z_1 - Z_3) + X_1(-Z_2 + Z_3) \end{Bmatrix}^T}_{[G_Y]} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{16}$$

$$d = \underbrace{\frac{1}{J_o} \begin{Bmatrix} X_4(-Y_2 + Y_3) + X_3(Y_2 - Y_4) + X_2(-Y_3 + Y_4) \\ X_4(Y_1 - Y_3) + X_1(Y_3 - Y_4) + X_3(-Y_1 + Y_4) \\ X_4(-Y_1 + Y_2) + X_2(Y_1 - Y_4) + X_1(-Y_2 + Y_4) \\ X_3(Y_1 - Y_2) + X_1(Y_2 - Y_3) + X_2(-Y_1 + Y_3) \end{Bmatrix}^T}_{[G_Z]} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{17}$$

Where J_o is the jacobian of the [A] matrix above in equation 13. The subscript "o" is to indicate that J_o is related to the reference configuration.
And the gradients clearly are:

$$\frac{\partial \phi}{\partial X} = b, \quad \frac{\partial \phi}{\partial Y} = c, \quad \frac{\partial \phi}{\partial Z} = d$$

Note: We can obtain the shape functions of the tetrahedron element by a trivial rearrangment to equation 12 with the help of the obtained values of the constants ($N_i(X, Y, Z)$ = the sum of the coeffients of ϕ_i)

$$\begin{aligned} J_o = 6V_o = X_3Y_2Z_1 - X_4Y_2Z_1 - X_2Y_3Z_1 + X_4Y_3Z_1 + X_2Y_4Z_1 - X_3Y_4Z_1 \\ - X_3Y_1Z_2 + X_4Y_1Z_2 + X_1Y_3Z_2 - X_4Y_3Z_2 - X_1Y_4Z_2 + X_3Y_4Z_2 \\ + X_2Y_1Z_3 - X_4Y_1Z_3 - X_1Y_2Z_3 + X_4Y_2Z_3 + X_1Y_4Z_3 - X_2Y_4Z_3 \\ - X_2Y_1Z_4 + X_3Y_1Z_4 + X_1Y_2Z_4 - X_3Y_2Z_4 - X_1Y_3Z_4 + X_2Y_3Z_4 \end{aligned}$$

V_o is the volume of the tetrahedron at the reference configuration.

Now, for the **F** tensor we simply dot product the gradient vector by the relevant current position vector (data vectors, as opposed to physical vectors):

$$F_{xx} = \frac{\partial x}{\partial Y} = [G_X] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \quad F_{xy} = \frac{\partial x}{\partial Y} = [G_Y] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \quad F_{xz} = \frac{\partial x}{\partial Z} = [G_Z] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$F_{yx} = \frac{\partial y}{\partial X} = [G_X] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \quad F_{yy} = \frac{\partial y}{\partial Y} = [G_Y] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \quad F_{yz} = \frac{\partial y}{\partial Z} = [G_Z] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$$F_{zx} = \frac{\partial z}{\partial X} = [G_X] \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \quad F_{zy} = \frac{\partial z}{\partial Y} = [G_Y] \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \quad F_{zz} = \frac{\partial z}{\partial Z} = [G_Z] \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix}$$