Astronomy 792
Spring 2024
Homework #1 solution

In all cases "log" refers to "log_{10}" and "ln" refers to the natural logarithm.

1. a. **What is the central surface brightness of a galaxy having 15 mag arcsec^{-2} in the R-band when expressed in $L_\odot$ pc^{-2}.**

All units will be written in [<unit>]. We must first convert our observed surface brightness μ_I[mag/arcsec^2] to Σ_θ[L$_\odot$/arcsec^2], or the number of solar luminosities per square arcsecond. We can do this using the definition $M = m - 5\ \log\left(\frac{d}{10\text{pc}}\right)$ and applying it to the surface brightness $\mu_I = 15$mag/arcsec^2. The absolute magnitudes per arcsec^2 is

$$\mu_{abs} \equiv \mu\ [\text{mag asec}^{-2}] - 5\ \log\left(\frac{d}{10\text{pc}}\right), \tag{1}$$

Where d is the distance to the galaxy. We can convert this to the surface brightness in solar luminosities Σ_θ using

$$\begin{aligned}
\Sigma_\theta &= 10^{-0.4(\mu_{abs} - M_{I,\odot})}\ [L_\odot\ \text{arcsec}^{-2}] && (2)\\
&= 10^{-0.4(\mu - 5\ \log\left(\frac{d}{10\text{pc}}\right) - M_{I,\odot})}\ [L_\odot\ \text{arcsec}^{-2}] && (3)\\
&= 10^{-0.4(\mu - M_{I,\odot} - 2.5\ \log\left(\frac{d}{10\text{pc}}\right)^2)}\ [L_\odot\ \text{arcsec}^{-2}] && (4)\\
&= 10^{-0.4(\mu - M_{I,\odot})} 10^{-0.4(-2.5\ \log\left(\frac{d}{10\text{pc}}\right)^2)}\ [L_\odot\ \text{arcsec}^{-2}] && (5)
\end{aligned}$$

$$\Sigma_\theta = 10^{-0.4(\mu - M_{I,\odot})}\left(\frac{d}{10\text{pc}}\right)^2\ [L_\odot\ \text{arcsec}^{-2}]. \tag{6}$$

To now convert this to [L$_\odot$ pc^{-2}] we need to convert an arcsecond to a pc. We use the small angle theorem for the angular size θ of an object of physical size r, or $\theta = \frac{r}{d}$, where theta is in radians and there are 206265 arcsec in a radian. So,

$$\frac{1\text{asec}}{206265\text{asec}} = \frac{r}{d} \tag{7}$$

and assuming $r = 1pc$,

$$\frac{1\text{asec}}{1\text{pc}} = \frac{206265\text{asec}}{d[\text{pc}]} \tag{8}$$

Which we can use to modify Eq. 6 to read

$$\begin{aligned}
\Sigma &= \Sigma_\theta \; [L_\odot \text{ arcsec}^{-2}] \left(\frac{206265\text{arcsec}}{d[\text{pc}]}\right)^2 && (9)\\
&= 10^{-0.4(\mu - M_{I,\odot})} \left(\frac{d}{10\text{pc}}\right)^2 \; [L_\odot \text{ pc}^{-2}] \left(\frac{206265\text{pc}}{d}\right)^2 && (10)\\
&= 10^{-0.4(\mu - M_{I,\odot})} \left(\frac{206265}{10}\right)^2 \; [L_\odot \text{ pc}^{-2}]. && (11)
\end{aligned}$$

Plugging in $\mu = 15$ and $M_{I,\odot} = 4.11$ (From internet or other sources like `https://mips.as.arizona.edu/~cnaw/sun.html`)

$$\underline{\Sigma = 10^{-0.4(15-4.11)} \left(\frac{206265}{10\text{pc}}\right)^2 \; [L_\odot \text{ pc}^{-2}] = 18,232 \; [L_\odot \text{ pc}^{-2}].} \tag{12}$$

b. **Explain why the surface brightness of a galaxy does not depend on the galaxy's distance for nearby galaxies (where space-time is locally flat).**

To understand qualitatively why there is no dependence of surface brightness on distance, as an object gets farther away its light is dimmed by a factor proportional to d^{-2}. At the same time, the physical scale that corresponds to 1 arcsecond increases as d and the area of 1 arcsec^2 increases as d^2, so the distances cancel out. In other words, the stars in 1 arcsec^2 get fainter with distance, but the number of stars in your area increases in just the right amount to compensate.

2. a. **If a galaxy has absolute magnitude M, derive an expression for the apparent magnitude m in terms of the redshift $z = V_r/c$, the absolute magnitude M, and a constant C, which is the same for all objects.**

Starting with the formula for apparent magnitude and absolute magnitude.

$$m - M = -2.5 \log(F) + 2.5 \log(F_{10\text{pc}}), \tag{13}$$

where F_{10pc} is the flux of an object at 10pc, which is the distance at which the absolute magnitude is computed. Then if you use eq. 1.1 from S&G in place of F and $F_{10\text{pc}}$ you get

$$m - M = -2.5 \log\frac{L}{4\pi d^2} + 2.5 \log\frac{L}{4\pi(10 \text{ pc})^2}. \tag{14}$$

We expand out all the terms in the logarithms, including the exponent of the distance and get

$$m - M = -2.5 \log L + 2.5 \log 4\pi + 5 \log d + 2.5 \log L - 2.5 \log 4\pi - 5 \log 10 \text{ pc} \tag{15}$$

after cancelling all terms and consolidating the logarithms we get the familiar distance modulus equation:

$$m - M = 5\ \log\frac{d}{10\ \mathrm{pc}}. \tag{16}$$

Given the equation that relates recession velocity, the Hubble Constant, and the distance

$$d = V_r/(h\ 100\ \mathrm{km\ s^{-1}\ Mpc^{-1}}) \tag{17}$$

We can substitute $zc = V_r$ to get

$$d = h^{-1}[zc(\mathrm{km\ s^{-1}})/100]\ \mathrm{Mpc}. \tag{18}$$

Putting this into the distance modulus formula from above we get

$$m - M = 5\log\left(\frac{h^{-1}[zc(\mathrm{km\ s^{-1}})/100]\ \mathrm{Mpc}}{10^{-5}\ \mathrm{Mpc}}\right) = 5\ \log z + 5\ \log\left(\frac{h^{-1}[c(\mathrm{km\ s^{-1}})/100]\ \mathrm{Mpc}}{10^{-5}\ \mathrm{Mpc}}\right). \tag{19}$$

The last term consists only of constants and so we can consolidate it into a term C and rewrite the equation as the answer to the first part of the problem

$$m = M + 5\ \log z + C \tag{20}$$

b. **Draw an approximate straight line through the points in the figure below, which shows the apparent magnitude for the brightest galaxy in a galaxy cluster and calculate its slope. Compare its slope to what you would expect if the brightest galaxy in a rich cluster always has the same luminosity.**

The second part of the problem asks us to draw a straight line through the data and compare its slope to what we would get if all the plotted objects are "standard candles", that is if they all have the same luminosity. The slope I get is approx 8.5 mag for every 1.7 in $\log(z)$, or a slope of 5. To compare this to what I would expect I set up the equations from the previous part for the points at the lower left (m_1) and upper right (m_2)

$$M = m_1 - 5\ \log z_1 + C \tag{21}$$

$$M = m_2 - 5\ \log z_2 + C \tag{22}$$

If the absolute magnitudes of both objects are the same then we can set these equations equal to each other and rerrange the terms

$$m_1 - m_2 = 5\ \log z_1 - 5\ \log z_2 \tag{23}$$

and solving for the slope from these two points

$$\frac{m_1 - m_2}{\log z_1 - \log z_2} = 5 \tag{24}$$

which is exactly what I measured from this plot, implying that the brightest cluster galaxies have the same luminosity.

3. **The ALMA radio telescope can observe over a simultaneous frequency range of 8 GHz. Assume that the frequency of the ALMA instrument (called a "correlator") can be centered at any observed frequency, e.g. that of a galaxy at rest in a cluster.**

   The CO(2-1) spectral line has a rest-frame frequency of 230.538 GHz. You would like to observe this line coming from galaxies in a cluster at $z = 1.60$. How fast can a galaxy be moving along the line of sight with respect to the center of the cluster such that you will still be able to detect the CO line? Express your answer in km/s.

   We will need to define two redshifts, frequencies, and velocities corresponding to that for an observed galaxy moving with respect to the cluster (z_{obs}, ν_{obs}, and v_{obs}) which has the values $z_{sys} = 1.60, \nu_{sys}$. We will solve for the redshift that corresponds to a certain shift in frequency from the systemic frequency. Then we will determine what velocity that corresponds to.

   The generic relation between the observed and rest-frame frequency is

$$\nu_{obs} = \frac{\nu_{rest}}{1 + z_{obs}}. \tag{25}$$

   We also know that the peculiar velocity of a galaxy v_{obs} with redshift z_{obs} moving with respect to some systemic redshift z_{sys} is given by

$$v_{obs} = c\frac{(z_{obs} - z_{sys})}{1 + z_{sys}}. \tag{26}$$

   We can then rearrange Eq. 25 to get

$$z_{obs} = \frac{\nu_{rest} - \nu_{obs}}{\nu_{obs}} \tag{27}$$

   and

$$z_{sys} = \frac{\nu_{rest} - \nu_{sys}}{\nu_{sys}} \tag{28}$$

   For a galaxy that is moving so fast that its CO(2-1) line just moves out of the spectral bandpass, we know that the maximum frequency is $\Delta\nu = 4$ GHz above the center frequency, remembering that the whole bandpass is 8 GHz. Stated otherwise, $\nu_{obs,lim} = \nu_{sys} + \Delta\nu$.

We can then plug this into Eq. 27 to get

$$z_{obs,lim} = \frac{\nu_{rest} - \nu_{obs,lim}}{\nu_{obs,lim}} = \frac{\nu_{rest} - (\nu_{sys} \pm \Delta\nu)}{\nu_{sys} \pm \Delta\nu}, \tag{29}$$

where we use the $\pm$ to denote that we have $\Delta\nu$ on either side of our systemic frequency. We could solve the problem by first finding the redshifts and then plugging those numbers into Eq. 26 but I would rather plug everything symbolically and only evaluate at the end.

Therefore, we can then plug Eq. 29 into Eq. 26 to get the velocity corresponding to the edge of our frequency band

$$v_{obs,lim} = c\frac{(z_{obs,lim} - z_{sys})}{1 + z_{sys}} = c\frac{\frac{\nu_{rest} - (\nu_{sys} \pm \Delta\nu)}{\nu_{sys} \pm \Delta\nu} - z_{sys}}{1 + z_{sys}}. \tag{30}$$

If we now plug in our numbers we get $\mathbf{v_{obs,lim}} = \mathbf{-12949}$ km/s and **14172** km/s. These correspond to redshifts of 1.488 and 1.723. These velocities are far in excess of the typical cluster velocity dispersion of ~ 1000 km/s and so we are ensured of being able to observe all cluster members.

4. **Compute the comoving volume in the two intervals $0.05 < z < 1$ and $1 < z < 2$ that is contained within a field whose solid angle is 0.25 deg^2. Assume $\Omega_m = 0.3, \Omega_\Lambda = 0.7$, and $h = 0.7$. I want you to write a program to solve this problem by numerically integrating $E(z)$. Don't use the astropy.cosmology package nor should you use Ned Wright's cosmology calculator. You may want to refer to "Distance Measures in Cosmology" by Hogg for an alternate formulation of the Volume element.**

   You must include your code with this assignment, which you can upload as a jupyter notebook or python code.

   At the following link is my solution to this, including a check using the astropy.cosmology package. `https://canvas.ku.edu/files/9540248/download?download_frd=1`