Serie 04

1. $\begin{cases} x^3 y^2 z^6 = 1 \\ x^4 y^5 z^{12} = 2 \\ x^2 y^2 z^5 = 3 \end{cases} \iff \begin{cases} 3\ln(x) + 2\ln(y) + 6\ln(z) = 0 \\ 4\ln(x) + 5\ln(y) + 12\ln(z) = \ln(2) \\ 2\ln(x) + 2\ln(y) + 5\ln(z) = \ln(3) \end{cases}$

posons: $\begin{cases} X = \ln(x) \\ Y = \ln(y) \\ Z = \ln(z) \end{cases} \iff (S): \begin{cases} 3X + 2Y + 6Z = 0 \\ 4X + 5Y + 12Z = \ln(2) \\ 2X + 2Y + 5Z = \ln(3) \end{cases}$

$$(S) \iff \begin{pmatrix} 3 & 2 & 6 \\ 4 & 5 & 12 \\ 2 & 2 & 5 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ \ln(2) \\ \ln(3) \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -4/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 6 \\ 4 & 5 & 12 \\ 2 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 0 & 7/3 & 4 \\ 0 & 2/3 & 1 \end{pmatrix}$$

$-\frac{2}{\cancel{3}} \times \frac{\cancel{3}}{7}$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/7 & 1 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/7 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 6 \\ 0 & 7/3 & 4 \\ 0 & 2/3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 0 & 7/3 & 4 \\ 0 & 0 & -1/7 \end{pmatrix}$$

$-\frac{2}{\cancel{3}} \times \frac{\cancel{3}}{7}$

$-\frac{8}{7} + 1$

d'où

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ 2/3 & 2/7 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 3 & 2 & 6 \\ 0 & 7/3 & 4 \\ 0 & 0 & -1/7 \end{pmatrix}}_{U}$$

$AX = b \iff \begin{cases} UX = Y \\ LY = b \end{cases}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ 2/3 & 2/7 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \ln(2) \\ \ln(3) \end{pmatrix} \iff \begin{cases} y_1 = 0 \\ \frac{4}{3}y_1 + y_2 = \ln(2) \\ \frac{2}{3}y_1 + \frac{2}{7}y_2 + y_3 = \ln(3) \end{cases} \Rightarrow \begin{cases} y_1 = 0 \\ y_2 = \ln(2) \\ y_3 = \ln(3) - \frac{2}{7}\ln(2) \end{cases}$$

$$AX = Y \iff \begin{pmatrix} 3 & 2 & 6 \\ 0 & 7/3 & 4 \\ 0 & 0 & -1/7 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ \ln(2) \\ \ln(3) - \frac{2}{7}\ln(2) \end{pmatrix}$$

$$\iff \begin{cases} 3X + 2Y + 6Z = 0 \\ \frac{7}{3}Y + 4Z = \ln(2) \\ -\frac{1}{7}Z = \ln(3) - \frac{2}{7}\ln(2) \end{cases}$$

$$\iff \begin{cases} 3X + 2Y + 6Z = 0 \\ \frac{7}{3}Y = \ln(2) - 8\ln(2) + 28\ln(3) = -7\ln(2) + 28\ln(3) \\ Z = 2\ln(2) - 7\ln(3) \end{cases}$$

$$\cancel{\iff \begin{cases} 3X + 2Y + 6Z = 0 \\ Y = 12\ln(3) - \frac{3}{7}\ln(2) \\ Z = 2\ln(2) - 7\ln(3) \end{cases}}$$

$$\cancel{\iff \begin{cases} 3X = -12\ln(2) + 42\ln(3) - 24\ln(3) + \frac{6}{7}\ln(2) \\ \quad = 18\ln(3) - \frac{78}{7}\ln(2) \\ X = 54\ln(3) - \frac{234}{7}\ln(2) \end{cases}}$$

$$\iff \begin{cases} 3X + 2Y + 6Z = 0 \\ Y = -3\ln(2) + 12\ln(3) = \ln\left(\frac{3^{12}}{2^3}\right) \\ Z = 2\ln(2) - 7\ln(3) = \ln\left(\frac{2^2}{3^7}\right) \end{cases}$$

$$\iff \begin{cases} 3X = 6\ln(2) - 24\ln(3) - 12\ln(2) + 42\ln(3) \\ \Rightarrow 3X = -6\ln(2) + 18\ln(3) \\ \Rightarrow X = -2\ln(2) + 6\ln(3) = \ln\left(\frac{3^6}{2^2}\right) \end{cases}$$

$$\begin{cases} X = \ln\left(\frac{3^6}{2^2}\right) \\ Y = \ln\left(\frac{3^{12}}{2^3}\right) \\ Z = \ln\left(\frac{2^2}{3^7}\right) \end{cases} \iff \begin{cases} \ln(x) = \ln\left(\frac{3^6}{2^2}\right) \\ \ln(y) = \ln\left(\frac{3^{12}}{2^3}\right) \\ \ln(z) = \ln\left(\frac{2^2}{3^7}\right) \end{cases} \iff \begin{cases} x = \frac{3^6}{2^2} \\ y = \frac{3^{12}}{2^3} \\ z = \frac{2^2}{3^7} \end{cases}$$

qst 2 : $s(t) = (at^3 + bt^2 + ct + d) \cdot e^{2t}$

$s'(t) = (3at^2 + 2bt + c) \cdot e^{2t} + 2(at^3 + bt^2 + ct + d) \cdot e^{2t}$

$= (2at^3 + (3a+2b)t^2 + (2b+2c)t + c + 2d) \cdot e^{2t}$

$\left[-3s'(t) = (-6at^3 + (-9a-6b)t^2 + (-6b-6c)t - 3c - 6d) \cdot e^{2t} \right]$

$s''(t) = (6at^2 + (6a+4b)t + 2b + 2c) \cdot e^{2t} + (4at^3 + (6a+4b)t^2 + (4b+4c)t + 2c + 4d) \cdot e^{2t}$

$\left[s''(t) = (4at^3 + (12a+4b)t^2 + (6a+8b+4c)t + 2b + 4c + 4d) \cdot e^{2t} \right]$

$\left[2s(t) = (2at^3 + 2bt^2 + 2ct + 2d) \cdot e^{2t} \right]$

→ $s''(t) - 3s'(t) = (-2at^3 + (3a-2b)t^2 + (6a+2b-2c)t + 2b + c - 2d) \cdot e^{2t}$

→ $s''(t) - 3s'(t) + 2s(t) = (0 + 3at^2 + (6a+2b)t + 2b + c + 0) \cdot e^{2t}$

on a $s(t)$ est une solution de l'équation $y'' - 3y' + 2y = (t^2 + t - 1) \cdot e^{2t}$

alors $s''(t) - 3s'(t) + 2s(t) = (t^2 + t - 1) \cdot e^{2t}$

$\Leftrightarrow (3at^2 + (6a+2b) \cdot t + 2b + c) \cdot e^{2t} = (t^2 + t - 1) \cdot e^{2t}$

$\Leftrightarrow 3at^2 + (6a+2b)t + 2b + c = t^2 + t - 1$

$\Leftrightarrow \begin{cases} 3a = 1 \\ 6a + 2b = 1 \\ 2b + c = -1 \end{cases} \Leftrightarrow \begin{cases} a = 1/3 \\ b = -1/2 \\ c = 0 \\ d = d \end{cases}$

$s(t) = \left(\frac{1}{3} t^3 - \frac{1}{2} t^2 + d \right) e^{2t}$ est une solution de $y'' - 3y' + 2y = (t^2 + t - 1) e^{2t}$

Exercice 02 :

$$(S)\quad \begin{cases} 2x_1+3x_2+x_3=1 \\ 4x_1+4x_2-3x_3=4 \\ -2x_1+3x_2-4x_3=-2 \end{cases} \qquad (S) \Leftrightarrow \underbrace{\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

- $\det(2) = 2 \neq 0$
- $\det\begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix} = -4 \neq 0$

$\det(A) = 2(-16+9) - 4(-12+3) - 2(-9+4) = -14 + 36 + 10 = 32 \neq 0$

donc A est décomposable en $L.U$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{(2)} = L_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 6 & -5 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$A^{(3)} = L_2 A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -8 \end{pmatrix}$$

donc $A = L^{-1} A^{(3)}$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -8 \end{pmatrix}}_{U}$$

on a $AX = b$

$\Leftrightarrow LUX = b$ poson $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = UX$.

$$(S) \Leftrightarrow \begin{cases} LY = b \\ Y = UX \end{cases}$$

on a ~~$UX = Y$~~ $LY = b$ ~~$Y = UX$~~

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \iff \begin{cases} y_1 = 1 \\ 2y_1 + y_2 = 4 \\ -y_1 - 3y_2 + y_3 = -2 \end{cases} \iff \begin{cases} y_1 = 1 \\ y_2 = 2 \\ y_3 = 5 \end{cases}$$

donc $UX = Y$

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{cases} 2x_1 + 3x_2 + x_3 = 1 \\ -2x_2 - x_3 = 2 \\ -8x_3 = 5 \end{cases} \iff \begin{cases} \cancel{x_1 = 39/32} \\ \cancel{x_2 = -3/2} \\ \cancel{x_3 = -5/8} \end{cases}$$

$x_1 = 39/32$

$x_2 = -11/16$

$x_3 = \frac{-5}{8}$

ene

qst 2/

$$(S): \begin{cases} x_1 + x_2 + 3x_4 = 4 \\ 2x_1 + x_2 - x_3 + x_4 = 1 \\ 3x_1 - x_2 - x_3 + 2x_4 = -3 \\ -x_1 + 2x_2 + 3x_3 - x_4 = 4 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$

$(S) \iff AX = b$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{(2)} = L_1 A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$$

!

$$A^{(3)} = L_2 A^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{(4)} = L_3 A^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}$$

d'où $A = L^{-1} A^{(4)}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}$$

$A = LU.$

on a $(S) \iff AX = b$

$\iff LUX = b$

posons $Y = UX$ $\iff \begin{cases} LY = b \\ Y = UX \end{cases}$ avec $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

$$LY = b \iff \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\iff \begin{cases} y_1 = 4 \\ 2y_1 + y_2 = 1 \\ 3y_1 + 4y_2 + y_3 = -3 \\ -y_1 - 3y_2 + y_4 = 4 \end{cases} \iff \begin{cases} y_1 = 4 \\ y_2 = -7 \\ y_3 = 13 \\ y_4 = -13. \end{cases}$$

$$UX = Y \iff \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 13 \\ -13 \end{pmatrix}$$

$$\iff \begin{cases} x_1 + x_2 + 3x_4 = 4 \\ -x_2 - x_3 - 5x_4 = -7 \\ 3x_3 + 13x_4 = 13 \\ -13x_4 = -13 \end{cases} \iff \begin{cases} x_1 = -1 \\ x_2 = 2 \\ x_3 = 0 \\ x_4 = 1 \end{cases}$$

Exercice 03 :

1 . $(S) : \begin{cases} 2x_1 + 3x_2 = 4 \\ 3x_1 + 6x_2 + x_3 = 14 \\ x_2 + 2x_3 = 12 \end{cases}$

posons $A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 14 \\ 12 \end{pmatrix}$

$(S) \Leftrightarrow AX = b$

$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A^{(2)} = L_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$

$$A^{(3)} = L_2 A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

d'où $$A = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix} = LU$$

on a $(S) \Leftrightarrow AX = b \Leftrightarrow LUX = b$

poson $Y = UX$ avec $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$(S) \Leftrightarrow \begin{cases} LY = b \\ UX = Y \end{cases}$$

$$LY = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 12 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} y_1 = 4 \\ \frac{3}{2} y_1 + y_2 = 14 \\ \frac{2}{3} y_2 + y_3 = 12 \end{cases} \Leftrightarrow \begin{cases} y_1 = 4 \\ y_2 = 8 \\ y_3 = \frac{20}{3} \end{cases}$$

$$UX = Y \Leftrightarrow \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 20/3 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 2x_1 + 3x_2 = 4 \\ 3/2\, x_2 + x_3 = 8 \\ \frac{4}{3} x_3 = \frac{20}{3} \end{cases} \Leftrightarrow \begin{cases} x_1 = -1 \\ x_2 = 2 \\ x_3 = 5 \end{cases}$$

2/ La méthode de Cholesky ?

Montrons que $A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ est symetrique, définie positive :

- $A^T = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A \Rightarrow A$ est symetrique.
- $\det(\Delta_1) = \det(2) = 2 > 0$
- $\det(\Delta_2) = \det \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} = 12 - 9 = 3 > 0$

- $\det(\Delta_3) = \det(A) = 2(12-1) - 3(6-0) = 22 - 18 = 4 > 0$

donc A est une matrice symetrique, définie positive alors elle peut être résolu par la méthode de Cholesky.

3/ Résounement par la méthode de cholesky :

on a A est une matrice symetrique, définie positive, alors il existe une matrice $L = (l_{ij})_{1 \leq i,j \leq 3}$ triangulaire inferieure tq :

$$A = L \cdot L^T.$$

d'après qst 1)

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix} \quad (L.U)$$

$x_3 =$

alors

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3}/\sqrt{2} & 0 \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{3}{2}\sqrt{2} & \sqrt{3}/\sqrt{2} & 0 \\ 0 & \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$

donc $A = L . L^T$

$(S) \Leftrightarrow AX = b \Leftrightarrow L L^T X = b$, posons $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = L^T . X$

$$(S) \Leftrightarrow \begin{cases} LY = b \\ L^T . X = Y \end{cases}$$

$$LY = b \Leftrightarrow \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{3}{2}\sqrt{2} & \frac{\sqrt{3}}{\sqrt{2}} & 0 \\ 0 & \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 12 \end{pmatrix} \Leftrightarrow \begin{cases} \sqrt{2}\, y_1 = 4 \\ \frac{3}{2}\sqrt{2}\, y_1 + \frac{\sqrt{3}}{\sqrt{2}} y_2 = 14 \\ \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} y_2 + \frac{2}{\sqrt{3}} y_3 = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1 = 2\sqrt{2} \\ y_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(14 - \frac{3}{2}\sqrt{2} . 2\sqrt{2}\right) = 8\frac{\sqrt{2}}{\sqrt{3}} \\ y_3 = \frac{\sqrt{3}}{2}\left(12 - \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} . 8\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{10}{3}\sqrt{3} \end{cases}$$

$$L^T X = Y \iff \begin{pmatrix} \sqrt{2} & \frac{3}{2}\sqrt{2} & 0 \\ 0 & \sqrt{3}/\sqrt{2} & \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 8\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{10}{3}\sqrt{3} \end{pmatrix}$$

$$\iff \begin{cases} \sqrt{2}\, x_1 + \frac{3}{2}\sqrt{2}\, x_2 = 2\sqrt{2} \\ \frac{\sqrt{3}}{\sqrt{2}} x_2 + \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} x_3 = 8\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} x_3 = \frac{10}{3}\sqrt{3} \end{cases}$$

$$\iff \begin{cases} x_1 = \frac{1}{\sqrt{2}}\left(2\sqrt{2} - \frac{3}{\cancel{2}}\sqrt{2} \cdot \cancel{2}\right) = 2 - 3 = -1 \\ x_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(8\frac{\sqrt{2}}{\sqrt{3}} - \frac{2}{3}\frac{\sqrt{3}}{\sqrt{2}} \cdot 5\right) = \frac{8 \cdot 2}{3} - \frac{10}{3} = \frac{6}{3} = 2 \\ x_3 = \frac{10}{3}\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 5 \end{cases}$$

d'où

$$\begin{cases} x_1 = -1 \\ x_2 = 2 \\ x_3 = 5 \end{cases}$$

Exercice 04

<u>qst 1</u>

$$A = \begin{pmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha & 0 \\ \alpha^2 & 0 & \alpha^2 \end{pmatrix}$$

- $A^T = \begin{pmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha & 0 \\ \alpha^2 & 0 & \alpha^2 \end{pmatrix} = A \Rightarrow A$ est symetrique

- $\det(\Delta_1) = \det(1) = 1$

  $\det(\Delta_1) = \det\begin{pmatrix} 1 & \alpha \\ \alpha & \alpha \end{pmatrix} = \alpha - \alpha^2$

il faut que $\alpha - \alpha^2 > 0 \iff \alpha > \alpha^2 \iff \alpha < 1$ (1)

$$\det(A) = \begin{vmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{vmatrix} - \alpha \begin{vmatrix} \alpha & \alpha^2 \\ 0 & \alpha^2 \end{vmatrix} + \alpha^2 \begin{vmatrix} \alpha & \alpha^2 \\ \alpha & 0 \end{vmatrix}$$

$$= \alpha^3 - \alpha^4 + \alpha^2 \times (-\alpha^3)$$

$$= -\alpha^5 - \alpha^4 + \alpha^3 + \alpha^2$$

il faut que :

$$-\alpha^5 - \alpha^4 + \alpha^3 + \alpha^2 > 0$$

$$\iff -\alpha^2(\alpha^3 + \alpha^2 - \alpha - 1) > 0$$

$$\iff -\alpha^2(\alpha^2(\alpha+1) - \alpha - 1) > 0$$

$$\iff -\alpha^2(\alpha^2(\alpha+1) - (\alpha+1)) > 0$$

$$\iff -\alpha^2(\alpha+1)(\alpha^2 - 1) > 0$$

$$\iff \alpha^2(\alpha+1)(\alpha^2-1) < 0$$

α^2 et $(\alpha+1)(\alpha^2-1)$ ont deux signes différents

$$\iff \begin{cases} \alpha^2 < 0 \quad \text{(absurde)} \\ (\alpha+1)(\alpha^2-1) > 0 \end{cases} \quad \underline{\text{ou}} \quad \begin{cases} \alpha^2 > 0 \\ (\alpha+1)(\alpha^2-1) < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \emptyset \\ (\alpha+1)(\alpha^2-1) > 0 \end{cases} \quad \text{ou} \quad \begin{cases} \alpha^2 > 0 \\ (\alpha+1)(\alpha^2-1) < 0 \end{cases}$$

ont m̂ signe

$$\Leftrightarrow \begin{cases} \emptyset \\ \begin{cases} \alpha+1 > 0 \\ \alpha^2-1 > 0 \end{cases} \\ \text{ou} \\ \begin{cases} \alpha+1 < 0 \\ \alpha^2-1 < 0 \end{cases} \end{cases} \quad \underline{\text{ou}} \quad \begin{cases} \alpha^2 > 0 \\ \begin{cases} \alpha+1 > 0 \\ \alpha^2-1 < 0 \end{cases} \\ \underline{\text{ou}} \\ \begin{cases} \alpha+1 < 0 \\ \alpha^2-1 > 0 \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \emptyset \\ \begin{cases} \alpha > -1 \\ \alpha^2 > 1 \end{cases} \\ \text{ou} \\ \begin{cases} \alpha < -1 \\ \alpha^2 < 1 \end{cases} \end{cases} \quad \text{ou} \quad \begin{cases} \alpha^2 > 0 \\ \begin{cases} \alpha > -1 \\ \alpha^2 < 1 \end{cases} \\ \text{ou} \\ \begin{cases} \alpha < -1 \\ \alpha^2 > 1 \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \emptyset \\ \begin{cases} \alpha > -1 \\ \alpha \in]-\infty,-1[\cup]1,+\infty[\end{cases} \\ \underline{\text{ou}} \\ \begin{cases} \alpha < -1 \\ \alpha \in]-1,1[\end{cases} \end{cases} \quad \underline{\text{ou}} \quad \begin{cases} \alpha \in \mathbb{R} \setminus \{0\} \\ \begin{cases} \alpha > -1 \\ \alpha \in]-1,1[\end{cases} \\ \underline{\text{ou}} \\ \begin{cases} \alpha < -1 \\ \alpha \in]-\infty,-1[\cup]1,+\infty \end{cases} \end{cases}$$

l'intersection donne :

$$\Leftrightarrow \begin{cases} \emptyset \\ \text{et} \\ \alpha \in]1,+\infty[\end{cases} \quad \underline{\text{ou}} \quad \begin{cases} \alpha \neq 0 \\ \begin{cases} \alpha \in]-1,1[\\ \text{ou} \\ \alpha \in]-\infty,-1[\end{cases} \end{cases}$$

$$\Leftrightarrow \left\{ \emptyset \right. \quad \underline{\text{ou}} \quad \left\{ \alpha \in]-\infty,-1[\cup]-1,0[\cup]0,1 \right.$$

d'où $\alpha \in D =]-\infty,-1[\cup]-1,0[\cup]0,1[$

dc ① : $\alpha < 1$

et ② : $\alpha \in D$

alors pour que A soit symétrique définie positive

il faut que $\alpha \in]-\infty, 1[\setminus \{0, -1\}$

$2 - \alpha = 1/2 \Rightarrow A = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \end{pmatrix} , b = \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix}$

$(\alpha \in D) \Rightarrow D.P$

$A = L \cdot L^T$

$$\begin{pmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ y & z & 0 \\ e & f & h \end{pmatrix} \cdot \begin{pmatrix} x & y & e \\ 0 & z & f \\ 0 & 0 & h \end{pmatrix} = \begin{pmatrix} x^2 & xy & xe \\ xy & y^2+z^2 & ye+zf \\ ex & ey+zf & e^2+f^2+h^2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x^2 = 1 & \Rightarrow x = \pm 1 \text{ , prenons } x = 1 \\ xy = \frac{1}{2} & \Rightarrow y = \frac{1}{2} \\ xe = \frac{1}{4} & \Rightarrow e = \frac{1}{4} \\ y^2 + z^2 = \frac{1}{2} & \Rightarrow z = \frac{1}{2} \\ ye + zf = 0 & \Rightarrow f = \frac{-1}{4} \\ e^2 + f^2 + h^2 = \frac{1}{4} & \Rightarrow h = \frac{\sqrt{2}}{4} \end{cases}$$

$$\Leftrightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{-1}{4} & \frac{\sqrt{2}}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1/2 & \frac{1}{4} \\ 0 & 1/2 & -\frac{1}{4} \\ 0 & 0 & \frac{\sqrt{2}}{4} \end{pmatrix} = LL^T$$

$$AX = b \Leftrightarrow LL^T X = b \quad , \text{ posons } Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = L^T X$$

$$\Leftrightarrow \begin{cases} LY = b \\ L^T X = Y \end{cases}$$

$$LY = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & \frac{1}{2} & 0 \\ 1/4 & -\frac{1}{4} & \frac{\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \Leftrightarrow \begin{cases} y_1 = 1 \\ \frac{1}{2} y_1 + \frac{1}{2} y_2 = \frac{1}{2} \\ \frac{1}{4} y_1 - \frac{1}{4} y_2 + \frac{\sqrt{2}}{4} y_3 = \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1 = 1 \\ y_2 = 0 \\ y_3 = \frac{\sqrt{2}}{2} \end{cases}$$

$$L^T X = Y \Leftrightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{-1}{4} \\ 0 & 0 & \frac{\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \Leftrightarrow \begin{cases} x_1 + \frac{1}{2} x_2 + \frac{1}{4} x_3 = 1 \\ \frac{1}{2} x_2 - \frac{1}{4} x_3 = 0 \\ \frac{\sqrt{2}}{4} x_3 = \frac{\sqrt{2}}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 1 - \frac{1}{2} - \frac{1}{2} = 0 \\ x_2 = 1 \\ x_3 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$$