A Combinatorial Approach to Fibonacci Identities

with an Introduction to Fibonomial Coefficients

Bennett Alex Myers, Maggie Gourlay

Metropolitan State University of Denver

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Table of Contents

1. The Art of Combinatorial Proof
2. The Fibonacci Numbers
3. Fibonacci Identities
4. Fibonomial Coefficients

Binomial Coefficients and Combinatorial Arguments

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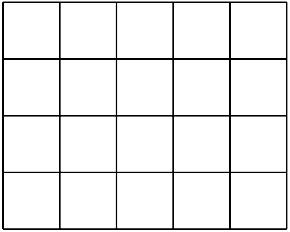
- Equivalently, how many ways are there to exclude $n-k$ objects from n objects?

Another Interpretation of the Binomial Coefficients

- Consider a $m \times n$ grid. A (northeast) *lattice path* is a path from $(0, 0)$ to (n, m) along the grid lines where steps to the north and to the east are permitted.



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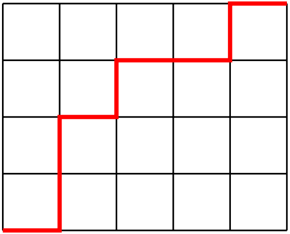

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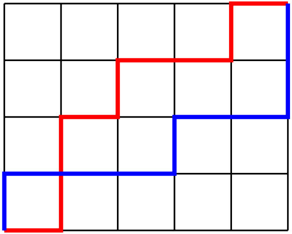

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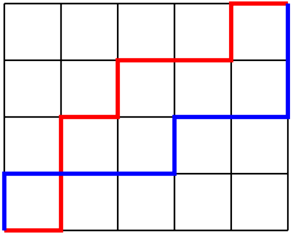

- The binomial coefficient $\binom{m+n}{n}$ counts the number of lattice paths from $(0,0)$ to (n,m).

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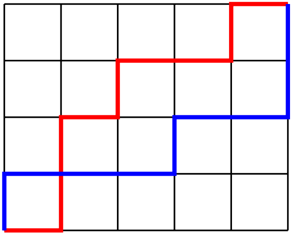

- $m + n$ steps are necessary to travel from $(0, 0)$ to (n, m), of which n steps must be east and m steps must be north.

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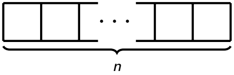
n	1	2	3	4	5	6	7	8	9	10	$\cdots$
f_n	1	2	3	5	8	13	21	34	55	89	$\cdots$

Tiling n-Boards

- Define an n-board to be a $1 \times n$ grid 

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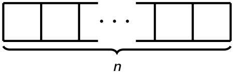
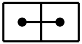
- Define an n-board to be a $1 \times n$ grid 
- How many ways can we tile an n-board with squares  and dominoes  ?

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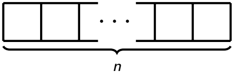
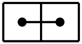
- Define an n-board to be a $1 \times n$ grid 

- How many ways can we tile an n-board with squares  and dominoes  ?

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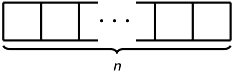
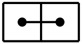
- Define an n-board to be a $1 \times n$ grid 

- How many ways can we tile an n-board with squares  and dominoes  ?

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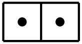
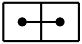
$n = 2$: , 

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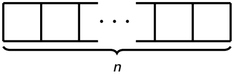
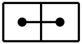
- Define an n-board to be a $1 \times n$ grid 

- How many ways can we tile an n-board with squares  and dominoes  ?

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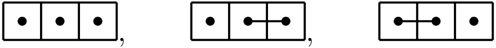
$n = 3$: 

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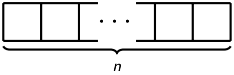
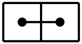
- Define an n-board to be a $1 \times n$ grid 

- How many ways can we tile an n-board with squares  and dominoes  ?

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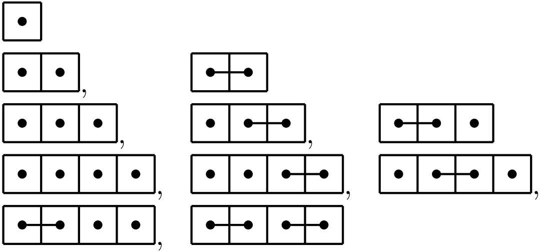

A Combinatorial Interpretation

Proposition

Let f_n count the ways to tile an n-board with squares and dominoes, define $f_{-1} = 0$, and let $f_0 = 1$ count the empty tiling of the 0-board. Then $f_n = f_{n-1} + f_{n-2}$ and $f_n = F_{n+1}$.

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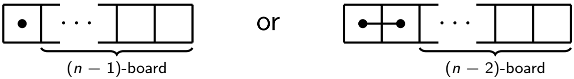

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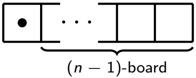

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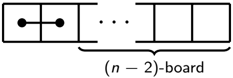

- Thus: $f_n = f_{n-1} + f_{n-2}$.

Proving Fibonacci Identities: Double-Counting

Identity 1

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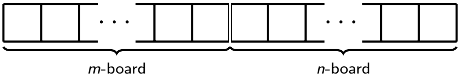

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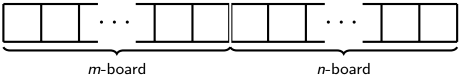

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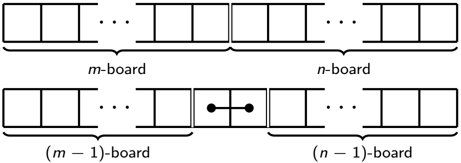

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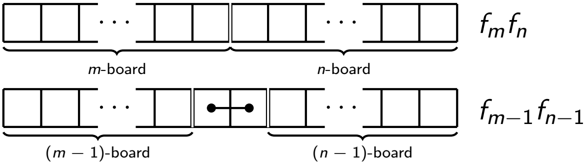

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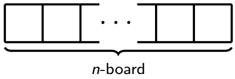

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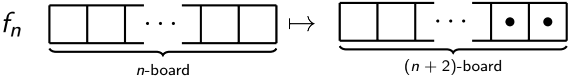

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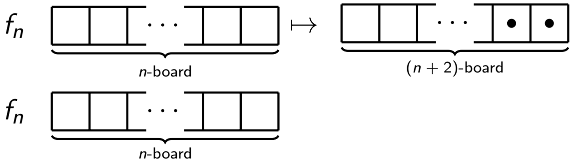

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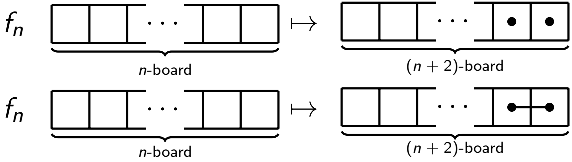

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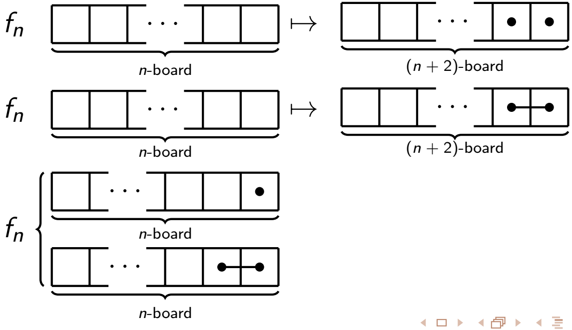

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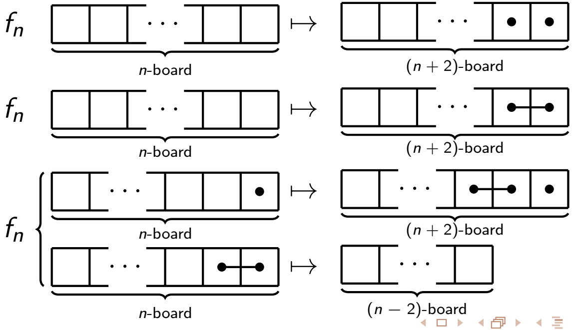


Another Example: Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Another Example: Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & \cancel{1} & 1 & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Another Example: Pascal's Triangle

1

1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Another Example: Pascal's Triangle

1

1

1

1 1 2

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Another Example: Pascal's Triangle

1

1

1

1 1

2

3

1 2 1

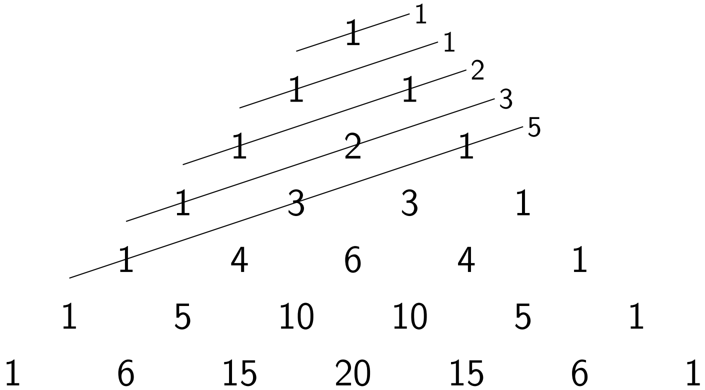
1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Another Example: Pascal's Triangle



Another Example: Pascal's Triangle

1

1

1

2

1 1

3

5

1 2 1

8

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Another Example: Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

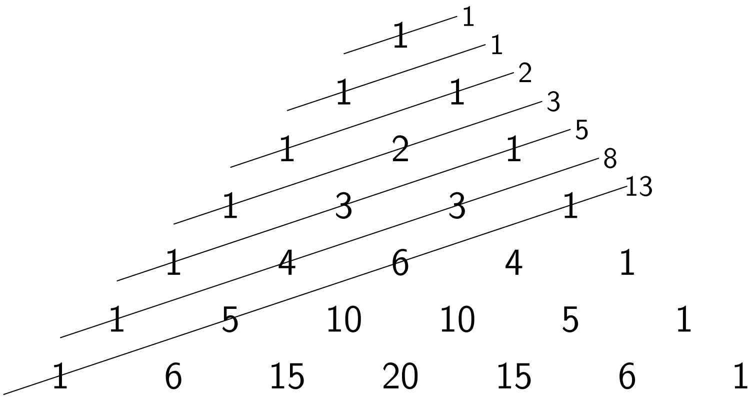
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1
1
2
3
5
8
13

Another Example: Pascal's Triangle



- Summing across diagonals in Pascal's Triangle generates the Fibonacci numbers.

Another Example: Pascal's Triangle

- Recall that the numbers in Pascal's Triangle correspond to the binomial coefficients.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{6}{2} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{4}{3} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

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- Recall that the numbers in Pascal's Triangle correspond to the binomial coefficients.

$$
\begin{array}{c}
\binom{0}{0} \qquad F_1 = f_0 \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{6}{2} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
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\end{array}
$$

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$F_1 = f_0$

$F_2 = f_1$

$$
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 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{6}{2} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
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 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & F_3 = f_2 \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{6}{2} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
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$$

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- Recall that the numbers in Pascal's Triangle correspond to the binomial coefficients.

$F_1 = f_0$
$F_2 = f_1$
$F_3 = f_2$
$F_4 = f_3$

$$
\begin{array}{ccccccccccccc}
&&&&&& \binom{0}{0} &&&&&& \\
&&&&& \binom{1}{0} && \binom{1}{1} &&&&& \\
&&&& \binom{2}{0} && \binom{2}{1} && \binom{2}{2} &&&& \\
&&& \binom{3}{0} && \binom{3}{1} && \binom{3}{2} && \binom{3}{3} &&& \\
&& \binom{4}{0} && \binom{4}{1} && \binom{4}{2} && \binom{6}{2} && \binom{4}{4} && \\
& \binom{5}{0} && \binom{5}{1} && \binom{5}{2} && \binom{5}{3} && \binom{5}{4} && \binom{5}{5} & \\
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\end{array}
$$

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$$F_1 = f_0 \qquad F_2 = f_1 \qquad F_3 = f_2 \qquad F_4 = f_3 \qquad F_5 = f_4$$

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{6}{2} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{4}{3} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

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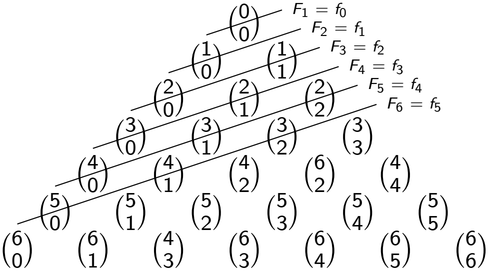

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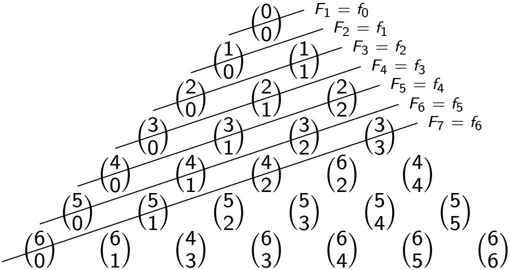

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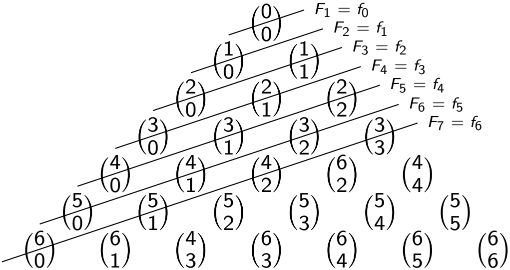

- Thus it seems reasonable to conjecture that

$$F_{n+1} = f_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

Another Example: Pascal's Triangle

Identity 3

For $n \geq 0$,

$$f_n = \sum_{k \geq 0} \binom{n-k}{k} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots .$$

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- How many n-tilings use exactly k dominoes? Note that $0 \leq k \leq n/2$.
- If we use k dominoes, we use $n - 2k$ squares, which means we use $k + (n - 2k) = n - k$ tiles.

Fibotorials and Fibonomials

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The *nth Fibotorial* is $F_n^! = \prod_{i=1}^{n} F_i$.

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Definition

For $0 < k \leq n$, the *Fibonomial coefficients* are given by

$$\binom{n}{k}_F = \frac{F_n^!}{F_k^! F_{n-k}^!} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1}.$$

Are Fibonomial Coefficients Always Integers?

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Using Identity 1 from earlier, we have

$$\begin{aligned} F_n = F_{k+(n-k)} &= f_{(k-1)+(n-k)} \\ &= f_{k-1}f_{n-k} + f_{k-2}f_{n-k-1} \\ &= F_k F_{n-k+1} + F_{k-1}F_{n-k}. \end{aligned}$$

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Replacing F_n by this expression in the numerator of $\binom{n}{k}_F$ results in

$$\begin{aligned} \binom{n}{k}_F &= \frac{(F_k F_{n-k+1} + F_{k-1}F_{n-k})F_{n-1}\cdots F_{n-k+1}}{F_k F_{k-1}\cdots F_1} \\ &= F_{n-k+1}\frac{F_{n-1}F_{n-2}\cdots F_{n-k+1}}{F_{k-1}F_{k-2}\cdots F_1} + F_{k-1}\frac{F_{n-1}F_{n-2}\cdots F_{n-k}}{F_k F_{k-1}\cdots F_1} \\ &= F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n}{k-1}_F. \end{aligned}$$

Are Fibonomial Coefficients Always Integers?

Identity 4

For $n \geq 2$ and $0 < k \leq n$,

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n}{k-1}_F.$$

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- Using this recursion, an induction argument can be used to prove that the Fibonomial coefficients always takes on integer values.

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- Furthermore, this forms the basis of Sagan and Savage's combinatorial interpretation of the Fibonomial coefficients!

Partitions and Lattice Paths

Definition

An *integer partition* of a positive integer n is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $0 \leq \lambda_i \leq n$ such that $\sum \lambda_i = n$. Each λ_i is called a *part* of our partition.

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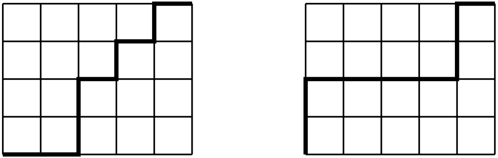

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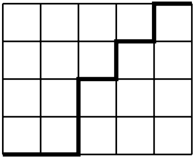

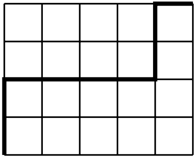

- Equivalently, this also counts the number of integer partitions with m parts, none of which are greater than n. $(\lambda \subseteq m \times n)$

A Combinatorial Interpretation: Partition Tilings

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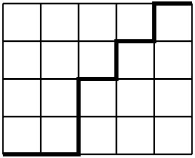
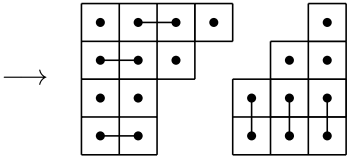


$\lambda \subseteq 4 \times 5$ with $\lambda = (4, 3, 2, 2)$.
Note that $\lambda^* = (4, 3, 2, 0, 0) \subseteq 5 \times 4$.



A tiling of λ in L_λ and a tiling of λ^* in L'_{λ^*}

Partition Tilings, Continued

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$$\binom{m+n}{m}_F = \sum_{\lambda \subseteq m \times n} \sum_{T \in L_\lambda \times L'_{\lambda^*}} 1.$$

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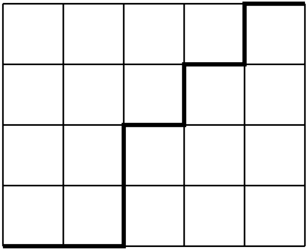

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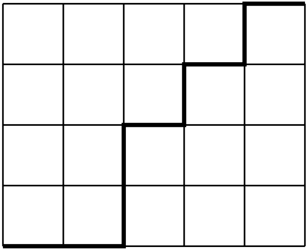
$f_4 = 5$



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$f_4 = 5$

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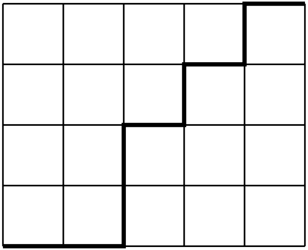

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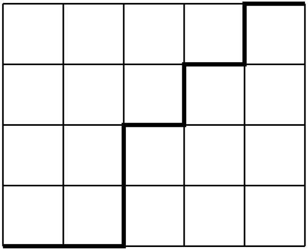
$f_4 = 5$

$f_3 = 3$

$f_2 = 2$



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$f_2 = 2$

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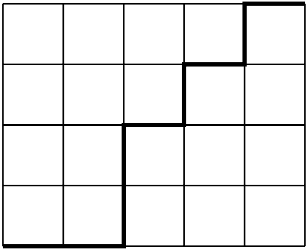

Partition Tilings, Continued

A *partition tiling* for $\lambda \subseteq m \times n$ is defined as a tiling $T \in L_\lambda \times L'_{\lambda^*}$.

Theorem (Sagan and Savage)

$$\binom{m+n}{m}_F = \sum_{\lambda \subseteq m \times n} \sum_{T \in L_\lambda \times L'_{\lambda^*}} 1.$$

$f_4 = 5$

$f_3 = 3$

$f_2 = 2$

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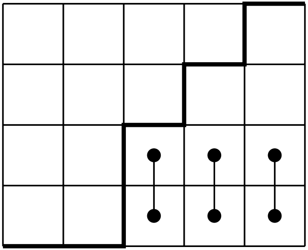

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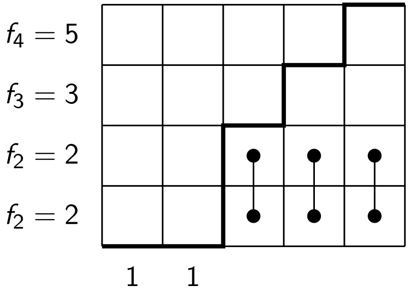

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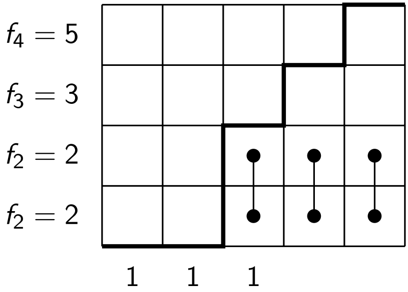

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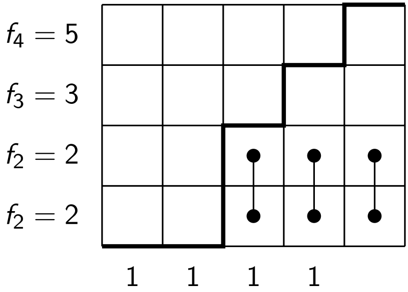

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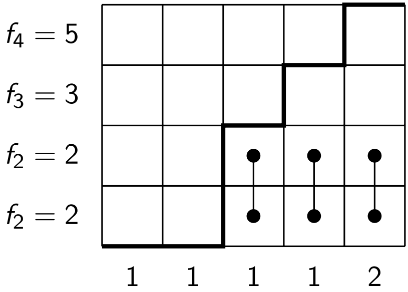

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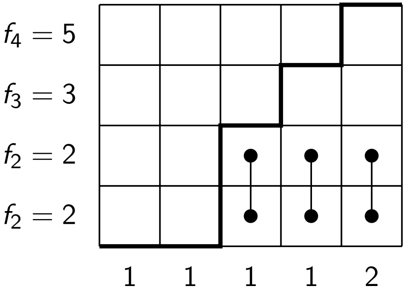
$$\begin{array}{r} 5 \cdot 3 \cdot 2 \cdot 2 = 60 \\ 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = \underline{\quad 2} \\ 120 \end{array}$$



Using the Combinatorial Interpretation

Identity 4, revisited

For $m, n \geq 1$,

$$\binom{m+n}{m}_F = F_{n+1}\binom{m+n-1}{m-1} + F_{m-1}\binom{m+n-1}{n-1}.$$

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- If $\lambda_1 < n$, then $\lambda_1^* = m$, so there are $f_{m-2} = F_{m-1}$ ways to tile that row and $\binom{m+n-1}{n-1}$ ways to tile the remaining $m \times n-1$ grid.

Future Work

- Bruce Sagan posed a number of open problems regarding the combinatorial interpretations of analogues to the Fibonomial numbers such as the FiboCatalan numbers and LucaCatalan numbers.
- These were outlined in his talk "Open problems for Catalan number analogues" given at JMM in January 2015.

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- These were outlined in his talk "Open problems for Catalan number analogues" given at JMM in January 2015.
- Maggie and I did an independent study in Spring 2015 on these sorts of combinatorial proofs. Much of the content of this presentation was derived from the exercises we did in Art Benjamin and Jennifer Quinn's *Proofs that Really Count: The Art of Combinatorial Proof*.

Thank You!

References

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