Real Analysis Key Concepts: what you should know for the exam

References are to Davidson & Donsig, *Real analysis and applications*, Springer UTM. UoA link to PDF: `https://link-springer-com.ezproxy.auckland.ac.nz/book/10.1007/978-0-387-98098-0`.

1 Big ideas

- Limiting processes.
- Continuity.
- Approximation.

One of the main points of this course is to show the relationship between *sequential* and *analytic* definitions. E.g. there are three different definitions of continuity:

- (analytic) $f : \mathbb{R} \to \mathbb{R}$ is cts at x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.
- (sequential) $f : \mathbb{R} \to \mathbb{R}$ is cts at x_0 if, for every sequence (a_n) in $\mathbb{R}$ with $a_n \to x_0$, $f(a_n) \to f(x_0)$.
- (topological) $f : \mathbb{R} \to \mathbb{R}$ is cts at x_0 if, for every open set U containing x_0, $f^{-1}(U)$ is open.

You should be able to prove the equivalence of all three definitions.

In general everything in this course has at least 2 ways of looking at it (analytic and sequential) and many the third (topological).

2 Things you should know

What does 'know' mean? - be able to write it down immediately. - be able to justify why it is a necessary and/or important definition. - be able to give 3 examples and 1 counterexample.

- Suprema and infima. (two different definitions: both an $\varepsilon - \delta$ definition and an order-theoretic definition). The archimedian property/the LUB property. [DD SS2.1–2.3]
- Convergence of a sequence. [D SS2.4–2.5]
- Cauchy sequences and the relationship between 'Cauchy' and 'convergent'. [DD §2.8]
- Series and power series. (All 250 material. Main tool is the Cauchy criterion for series convergence.) [DD ch.3]
- Closed and open sets. (Sequential definitions? Analytic definitions?) [DD §4,1–4,3]
- Compactness. Again three definitions: K is compact if (1, analytic) it is a closed and bounded set; (2, sequential) every sequence in K has a subsequence which converges to a point of K; (3, topological, not so important in this course) for every collection $\{U_i\}_{i\in I}$ of open sets such that $K \subseteq \bigcup U_i$, there exist finitely many $i_1, ..., i_N \in I$ such that $K \subseteq \bigcup_{1\leq n\neq N} U_{i_n}$. [DD §4.4]

- Definition of continuity. [DD SS5.1–5.3]
- The big two continuity theorems: the intermediate value theorem and the extreme value theorem.[1] [DD SS5.4,5.6]
- Uniform continuity. Why is it important? Main theorem: functions on a compact set are uniformly continuous. (This is a hard theorem in general which shows the power of the sequential arguments.) Other examples: -linear functions [DD § 5.5]
- Lipschitz continuity. Examples of Lipschitz functions: -linear functions; -functions with bounded derivative (and conversely). Nonexample: non-linear polynomials on $\mathbb{R}$. Theorem: Lipschitz implies uniformly continuous. Example of uniformly cts but not Lipschitz: $f : [0,1] \to [0,1]$ defined by $f(x) = \sqrt{x}$. (Suppose it is K-Lipschitz, then $\left|\sqrt{0} - \sqrt{1/4K^2}\right| = 1/2K > 1/4K = K\left|0 - 1/4K^2\right|$.) [DD § 5.1]
- Differentiability (definition of). Should be able to prove the product rule, chain rule, etc. [DD§ 6.1]
- The mean value theorem. [DD § 6.2]
- Riemann-like integrals: two definitions, (1) via upper and lower sums and integrals, (2) $\varepsilon - P$ definition (a.k.a. Riemann's condition). Continuous functions on compact sets are Riemann integrable. Converse is not true (e.g. step functions or even worse the Topologist's sine function $x \mapsto \sin(1/x)$ which is integrable over $(0,1]$ by Riemann's condition argument). Example of non-integrable function: characteristic function of $\mathbb{Q}$; of the Cantor set; etc. [DD § 6.3]
- The fundamental theorem of calculus (both of them). Other results you should already know from college, like the change-of-variables rule and integration by parts. [DD § 6.4]
- Limits of functions. Pointwise convergence of continuous functions does not imply continuous limit. Uniform convergence of function sequences. [DD § 8.1–8.3]
- Stone-Weirstraß theorem for polynomials.[2] [DD ch.10]

3 Other useful books.

Books at the level of this course to look at:

- Two books on Canvas, including `https://www.jirka.org/ra/`
- Spivak's *Calculus*

More comprehensive undergraduate books:

- Rudin, *Principles of mathematical analysis*
- Loomis and Sternberg, *Advanced calculus* `http://people.math.harvard.edu/~shlomo/docs/Advanced_Calculus.pdf`
- Kolmogorov and Fomin, *Introductory real analysis*

[1] These theorems are reflections of the following properties: the IVT says that the image of a connected set is connected; the EVT says that the image of a compact set is compact. Proofs essentially elementary.

[2] Often you see the more general form, 'an algebra A of cts real-valued functions on a cpct metric space X that separates points and does not vanish at any point is dense in $C(X, \mathbb{R})$.