MTH 1020 Week 6 tutorial

1. Some reminders on geometry of complex numbers
2. Fractals

Definition of complex numbers

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A **complex number** is a sum $x + yi$ where $x, y \in \mathbb{R}$ and where i is a symbol satisfying $i^2 = -1$.

Example

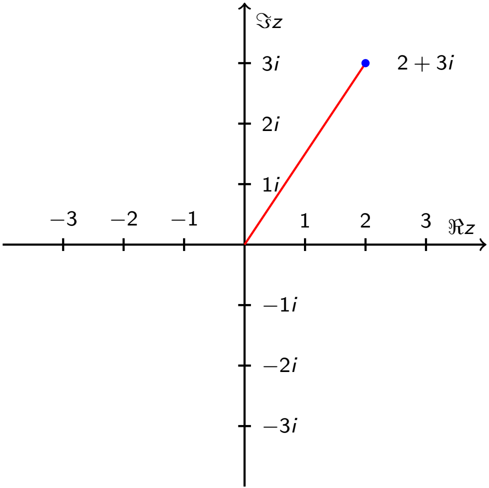
If $z = 2 + 3i$ then

$$\begin{aligned} z^2 + z + 1 &= (2+3i)^2 + (2+2i) + 1 \\ &= (4 + 12i + 9i^2) + (2+2i) + 1 \\ &= (4 + 12i - 9) + (2+2i) + 1 \\ &= -2 + 14i. \end{aligned}$$

Hurwitz (1923) showed that there are only 3 possible ways to extend the real numbers by adding new symbols to get a system of numbers which still has an absolute value: the complex numbers (2D), the **quaternions** (4D), and the **octonions** (8D).

Argand diagram

- Since $i^4 = (i^2)^2 = 1$, it is natural to think about multiplication by i as some kind of rotation by an angle $2\pi/4 = 90^\circ$.
- So $4i$ should be the number obtained by rotating 4 by 90°.



Geometric meaning of complex numbers

- Complex numbers *add* like vectors, easiest in rectangular form.
- Complex numbers *multiply* by rotating and scaling, easiest in polar form.

All the geometry that you can do is shifts, and rotating and scaling.

While we are thinking geometrically...

Here are two functions:

$$f(z) = z + 1 \text{ and } g(z) = \frac{z}{2i \cdot z + 1}.$$

We will make a list of numbers $z_1, z_2, \ldots$.

1. Take $z_1 = 0$.
2. To produce point z_n:
   2.1 Randomly choose f, f^{-1}, g, or g^{-1}.
   2.2 Apply the function you just chose to z_{n-1}; so you randomly have

$$z_n = f(z_{n+1}) \text{ or } z_n = g(z_{n+1}) \text{ or}$$
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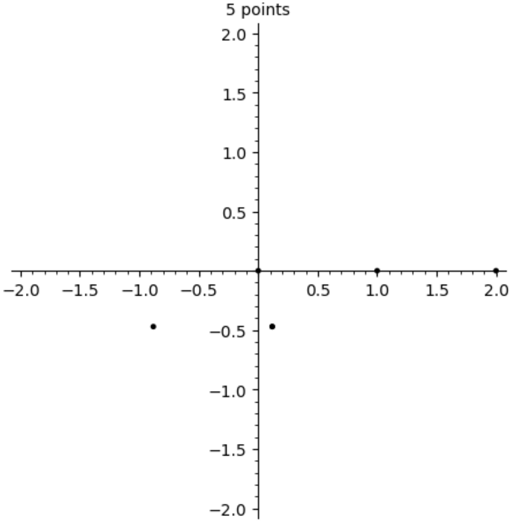

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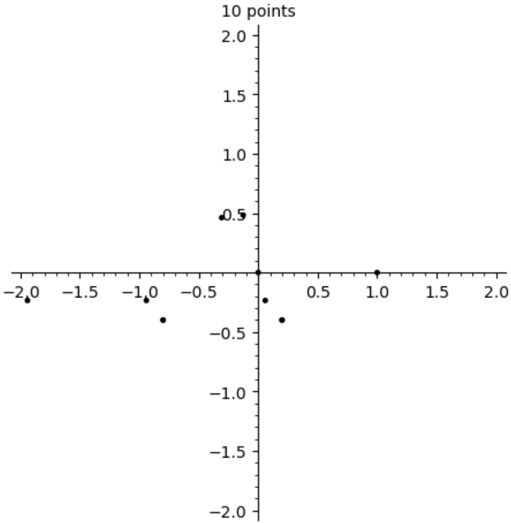

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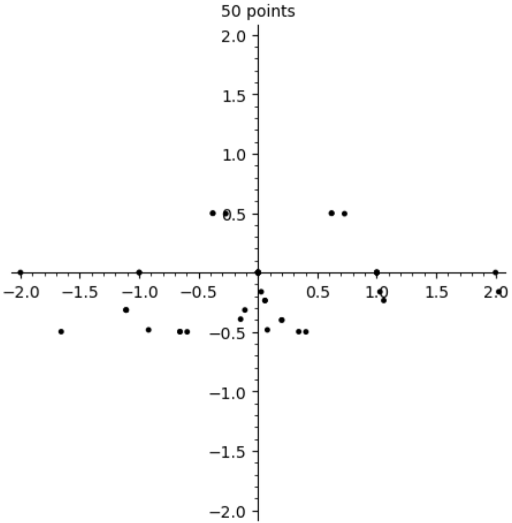

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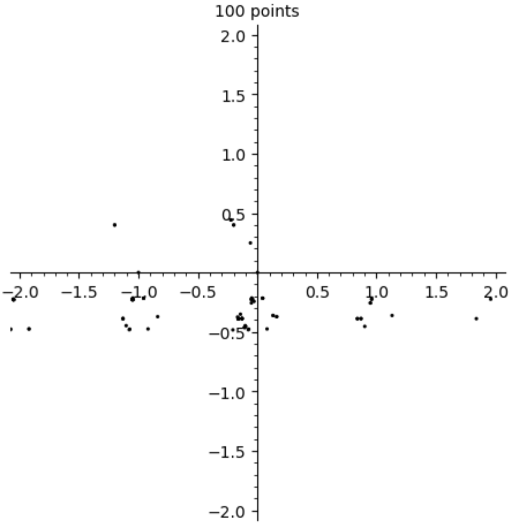

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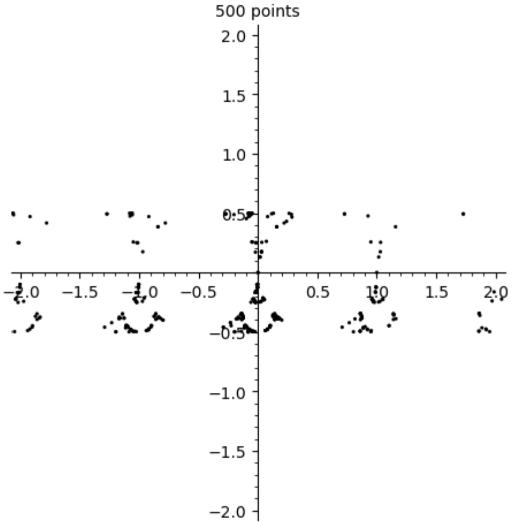

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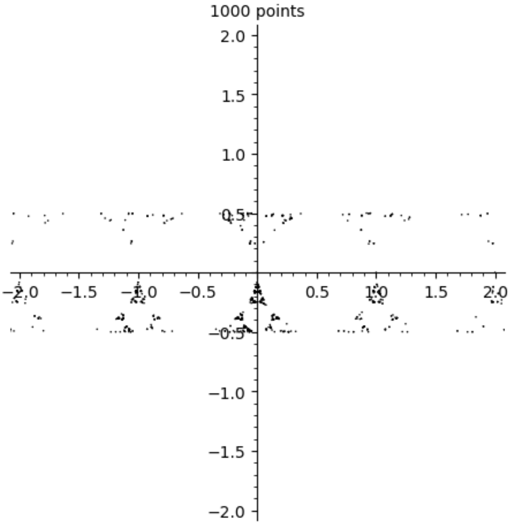

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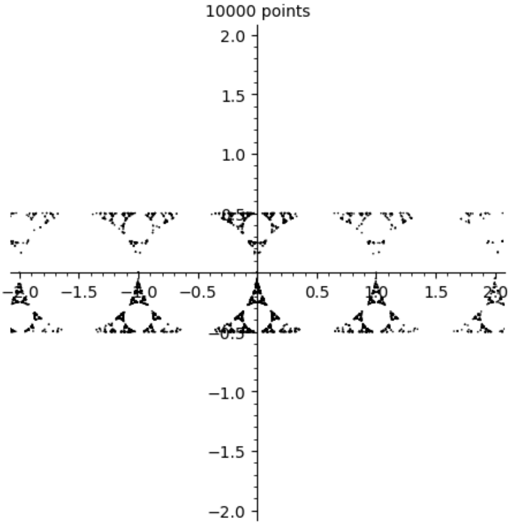

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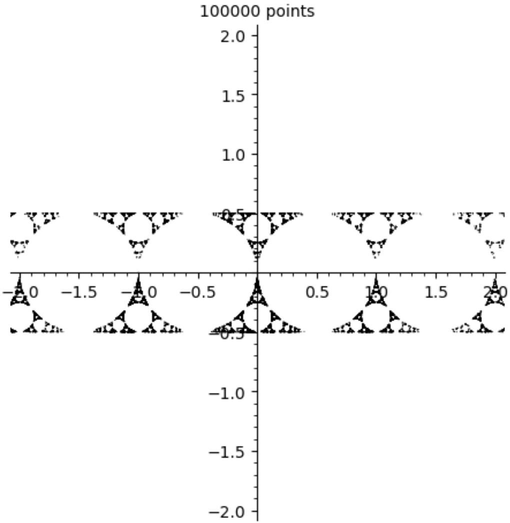

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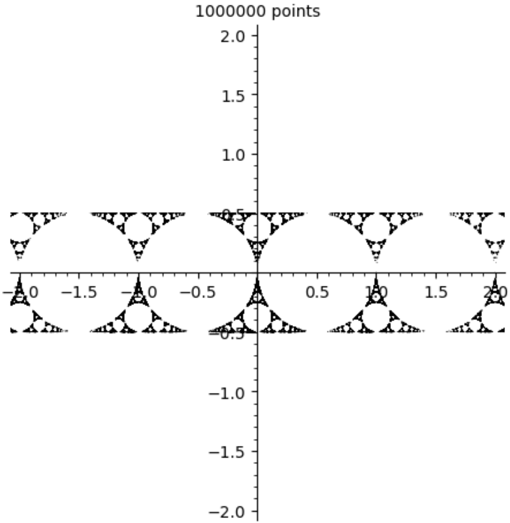
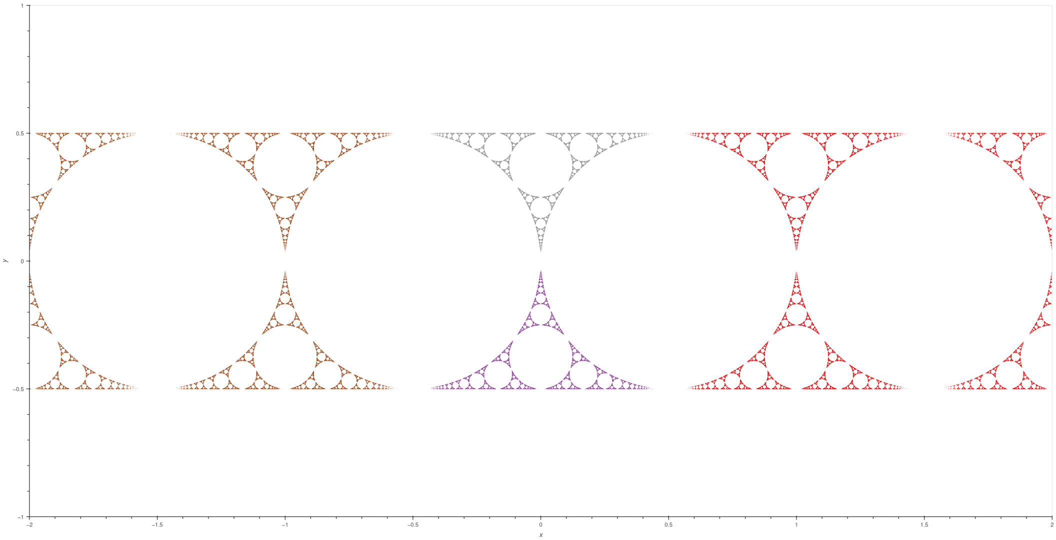
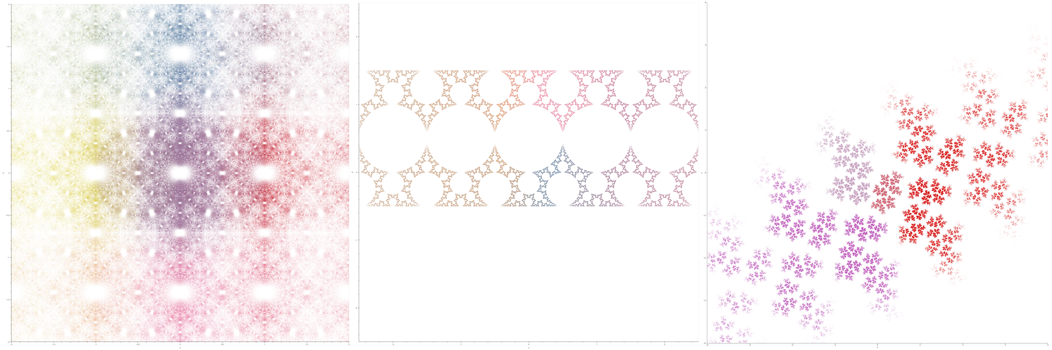




This is a fractal called the **Apollonian gasket**. We just make it by writing down a list of complex numbers one after another, following a simple arithmetic rule at each step.



Three other fractals you can make in the same way (just different randomly chosen functions. These are called **Kleinian fractals**. Other kinds of fractals you can get from the complex numbers are **Julia sets** and the **Mandelbrot set**. There are nice animations on YouTube. It is all just arithmetic of complex numbers.

MTH 1020 Week 5 tutorial

1. Get into groups of 3-4 people who all prepared a different question in advance.
2. Write your **preferred name** and **ID number** on the whiteboards so I can take attendance
3. Present your prepared question to each other as I come around, you should only take about 5min each for this.
4. Then get started on the other questions **in your groups**.
5. **At the end:** please erase the boards and return any markers etc that you used (you do not need to return the handouts)