

PICTURES OF HYPERBOLIC SPACES

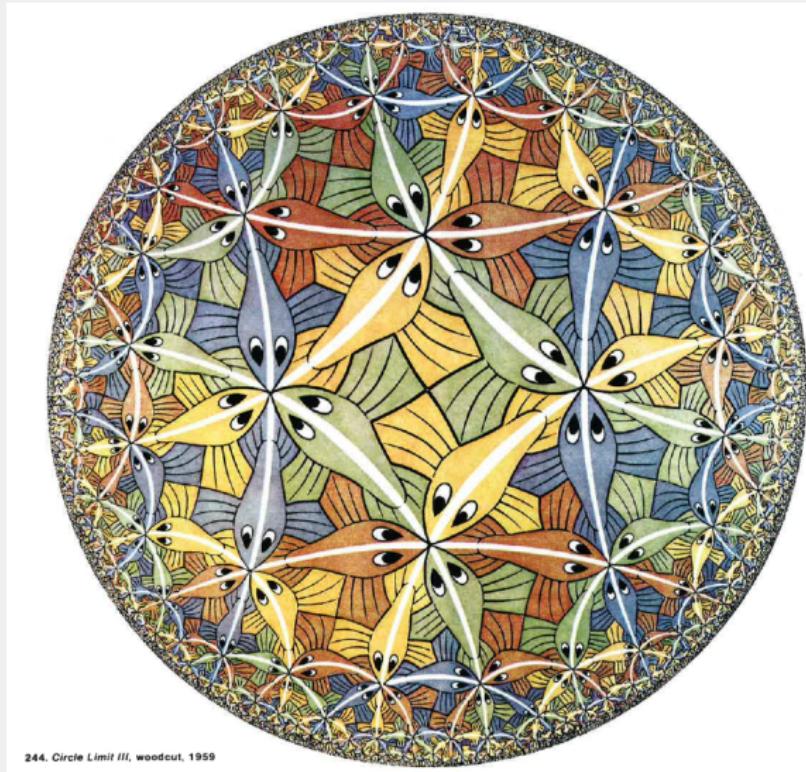
ALEX ELZENAAR

(MPI-MIS)

MAY 3, 2022



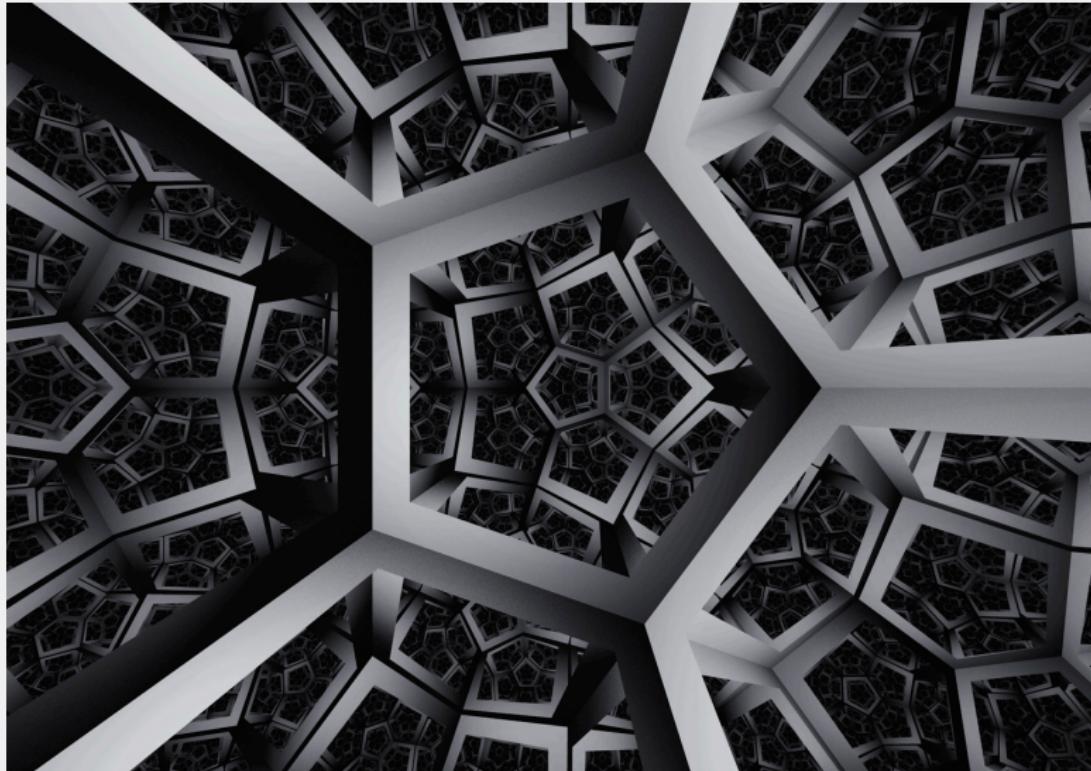
HYPERBOLIC 2-SPACE, \mathbb{H}^2



244. *Circle Limit III*, woodcut, 1959

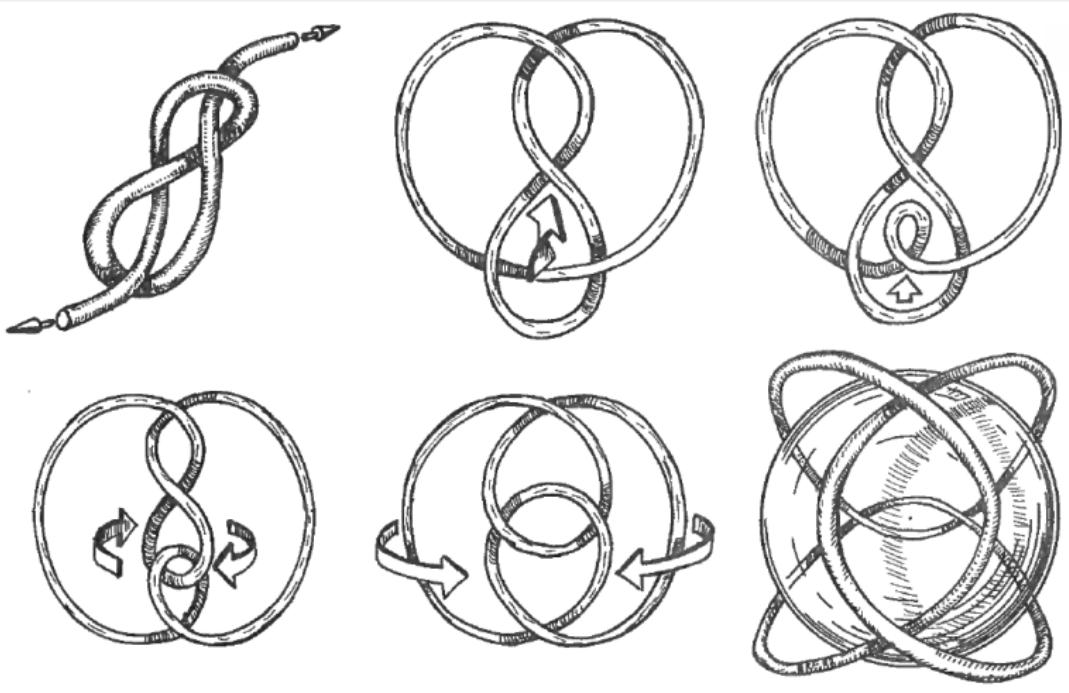
[M.C. Escher (1959)]

HYPERBOLIC 3-SPACE, \mathbb{H}^3 : TILING BY DODECAHEDRA



[Pierre Berger <https://www.espaces-imaginaires.fr/works/ExpoEspacesImaginaires2.html>]

THE FIGURE 8 KNOT $k(5/3)$



[Francis, p.150]

DAWN OF 3-DIMENSIONAL GEOMETRY AND TOPOLOGY

Theorem (Robert Riley (c.1974); William P. Thurston (c.1975))

The complement of the figure 8 knot,

$$S^3 \setminus k(5/3),$$

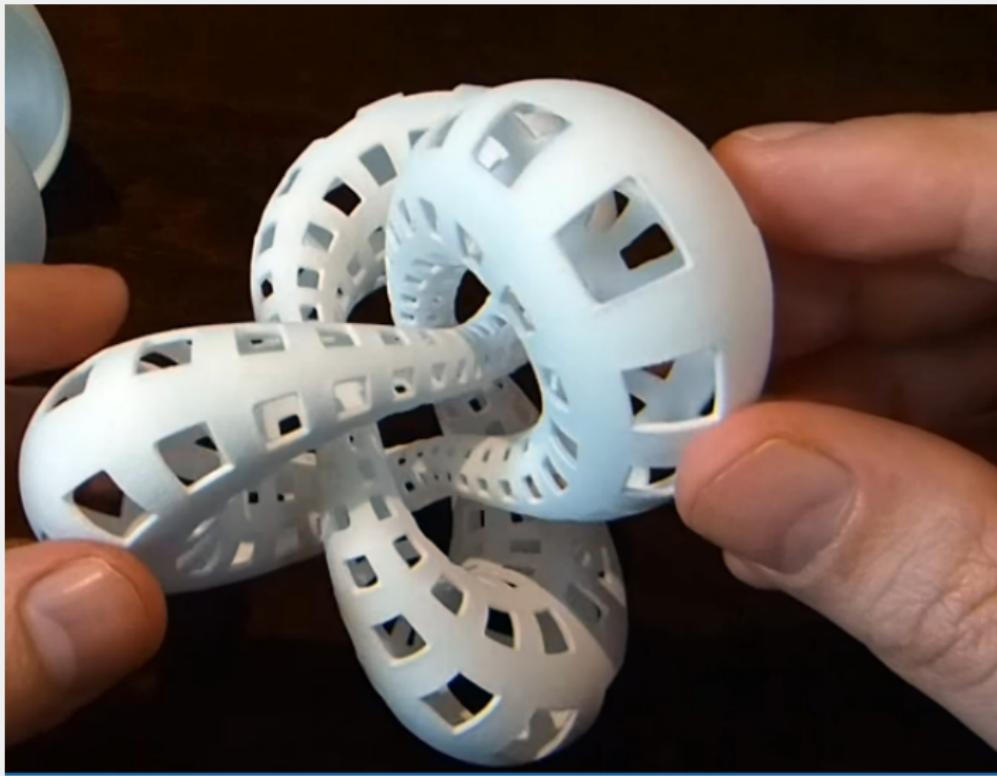
admits a hyperbolic geometry.

Theorem (Thurston (c.1979))

Almost every knot complement¹ admits a hyperbolic geometry.

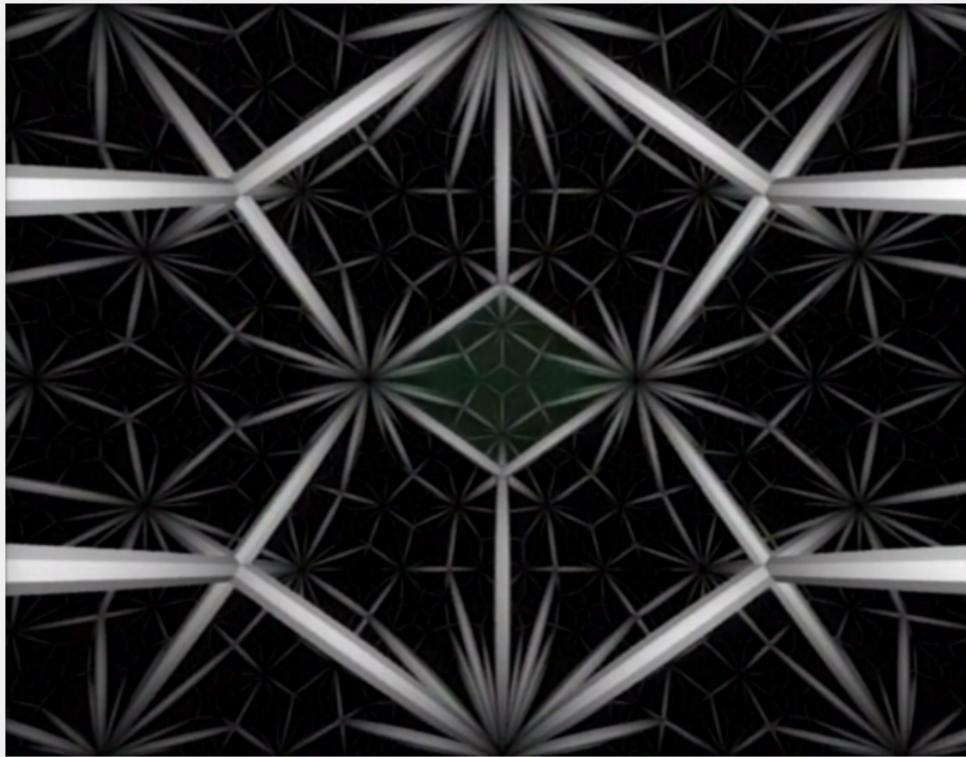
¹All but a small family of tabulated exceptions

THE HYPERBOLIC STRUCTURE



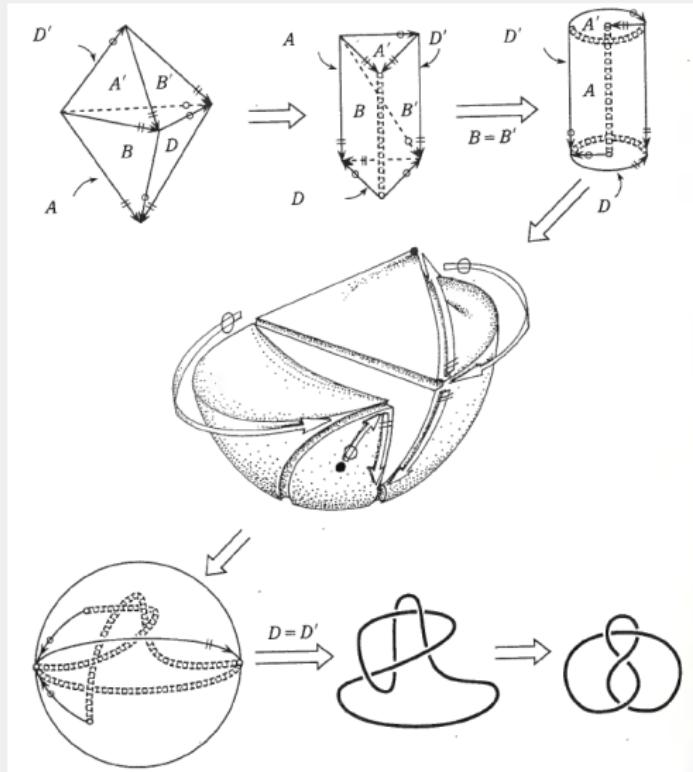
[Guéritaud/Segerman/Schleimer, https://youtu.be/xGf5jY_v5GE]

THE BORROMEEAN RINGS COMPLEMENT



[Gunn/Maxwell, Not Knot]

IDEA: VISUALISE KNOT COMPLEMENTS BY TILING

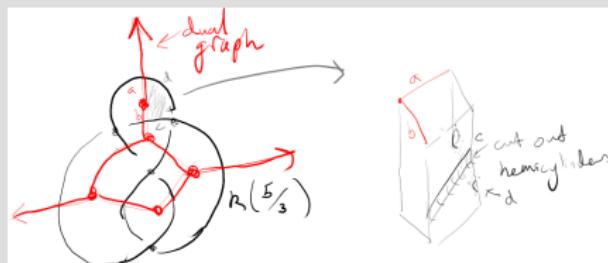


[Matsuzaki/Taniguchi, p.34]

WEEKS' ALGORITHM

SnapPea Algorithm (Jeff Weeks, c.1985)

1. Embed the knot in $S^2 \times [-1, 1]$, 'flatly' around $S^2 \times 0$.
2. Cut straight down along the dual graph & the knot graph.



3. Collapse these quadrilateral tubes to tetrahedra (pinch the upper & lower faces to points, and the two hollowed tubes).
4. Glue four cusps onto these vertices to get spherical tetrahedra.
5. Do a bit of fiddling to get the hyperbolic geometry back.

THIS GIVES ALL HYPERBOLIC MANIFOLDS

Theorem (Thurston, c.1979)

Every hyperbolic 3-manifold can be obtained by ‘Dehn surgery’ along some hyperbolic link.

Thus Weeks’ algorithm triangulates every hyperbolic 3-manifold.

KLEINIAN GROUPS

Theorem (Hyperbolic uniformisation)

*Let M be a hyperbolic orbifold. Then M is isometric to a orbifold of the form \mathbb{H}^3/G for some discrete group G of hyperbolic isometries (called the **holonomy group** of M). Conversely, given any discrete group $G \leq \text{Isom}^+(\mathbb{H}^3)$, \mathbb{H}^3/G is a hyperbolic orbifold.*

Definition

A discrete group $G \leq \text{Isom}^+(2, \mathbb{C})$ is called a **Kleinian group**.

ACTION AT INFINITY

Theorem (Poincaré extension)

There is a natural isomorphism between the group $\text{Isom}^+(\mathbb{H}^3)$ of orientation-preserving hyperbolic isometries and the group $\text{PSL}(2, \mathbb{C})$ of conformal maps on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$.

Example (Robert Riley, c.1972)

The holonomy group of the figure 8 knot complement is

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\exp(2\pi i/3) & 1 \end{bmatrix} \right\rangle.$$



[M.C. Escher (1957)]

THE LIMIT SET

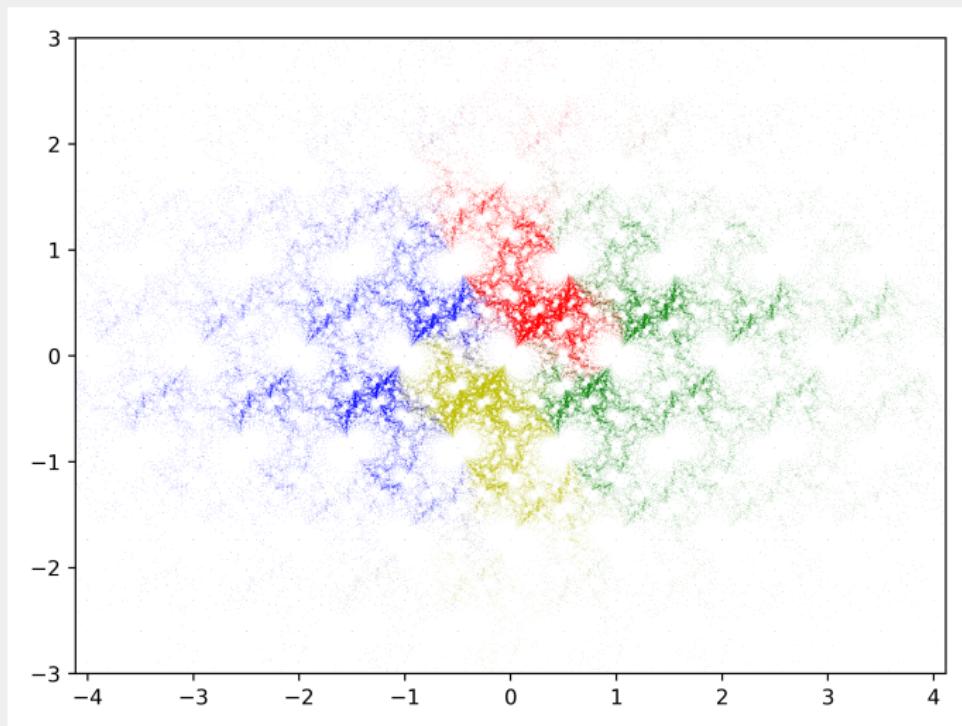
The dynamics of the action of a Kleinian group G on $\hat{\mathbb{C}}$ are complicated. There is a partition $\hat{\mathbb{C}} = \Omega(G) \cup \Lambda(G)$ similar to the partition between the Fatou and Julia sets of a holomorphic dynamical system.

Definition

If G is non-elementary, then the **limit set** of G is the closure of the set of fixed points of elements of G .

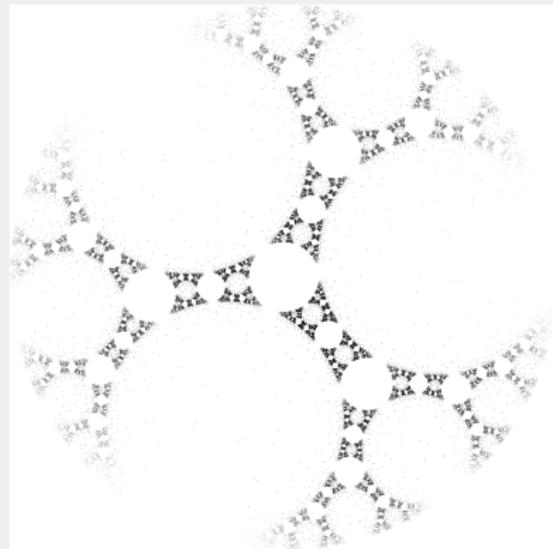
EXAMPLES

Figure 8 knot group (dense in $\hat{\mathbb{C}}$)



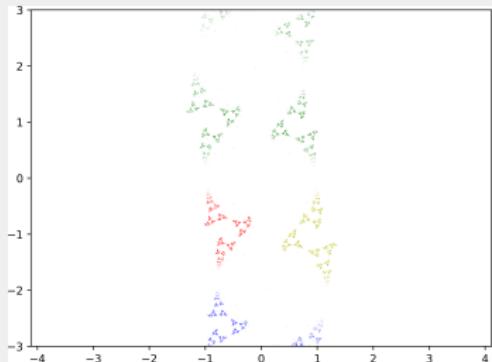
EXAMPLES

Elliptic Riley groups



EXAMPLES

Left: parabolic Riley group. Right: Indra's Necklace group.



REMARK: WHY IS THIS IMPORTANT?

Theorem (Thurston (c.1979); The ending lamination theorem (Epstein/Marden/Minsky))

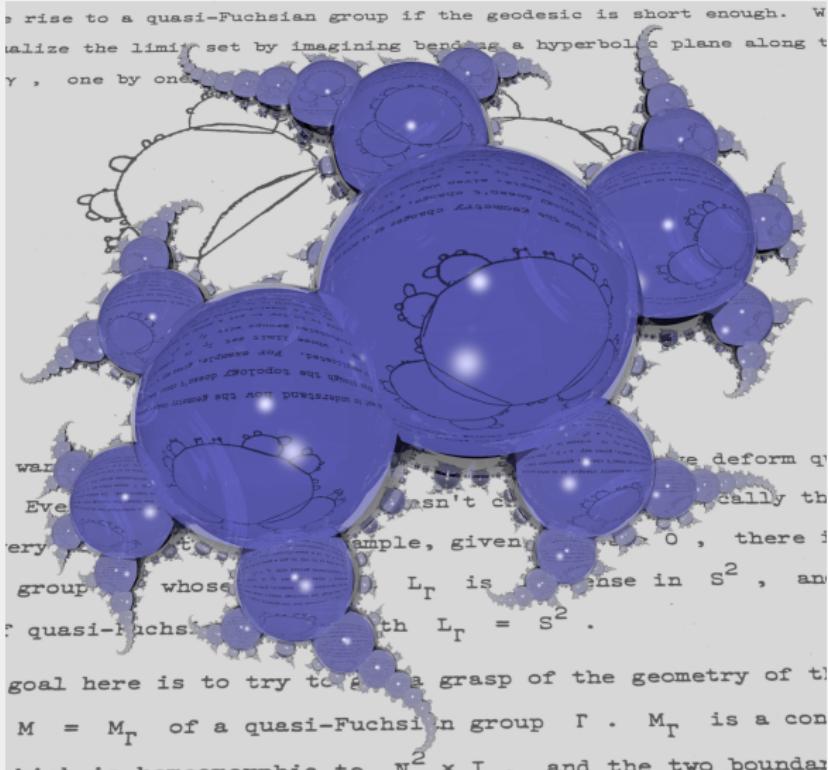
If G is non-degenerate² then there is a strong deformation retract

$$\frac{\mathbb{H}^3 \cup \Omega(G)}{G} \twoheadrightarrow \frac{\text{h.conv } \Lambda(G)}{G}$$

and the ‘folding structure’ on the convex hull determines the hyperbolic geometry entirely.

²non-Fuchsian and non-elementary

BUG ON NOTES OF THURSTON



[Jeffrey Brock and David Dumas, <https://www.dumas.io/poster/>]

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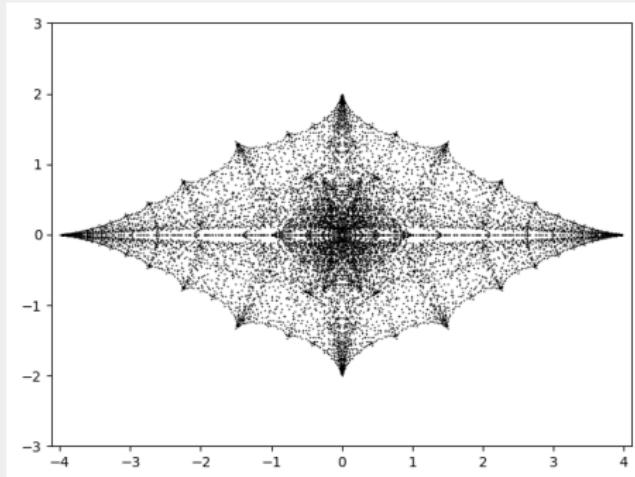
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- So the problem is reduced to (1) enumerating ‘lots of words’, and (2) doing matrix products quickly.
- These are standard problems in computational combinatorial group theory.

A DEFINITION

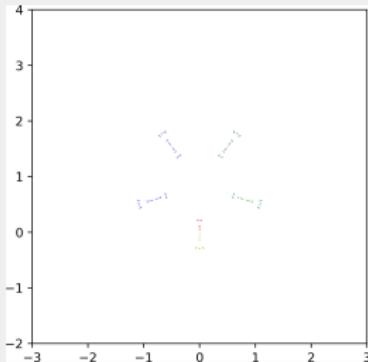
The **(a, b) -Riley slice**, $\mathcal{R}^{a,b}$, is the set of $\rho \in \mathbb{C}$ such that the limit set of the matrix group

$$\Gamma_\rho := \left\langle \begin{bmatrix} e^{\pi i/a} & 1 \\ 0 & e^{-\pi i/a} \end{bmatrix}, \begin{bmatrix} e^{\pi i/b} & 0 \\ \rho & e^{-\pi i/b} \end{bmatrix} \right\rangle$$

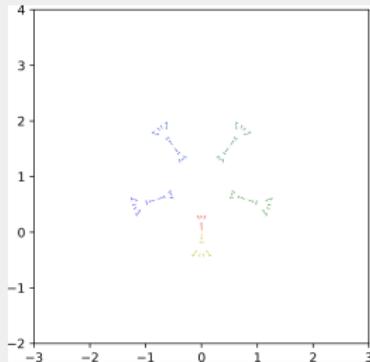
is *neither* dense in a circle packing, *nor* dense in $\hat{\mathbb{C}}$.



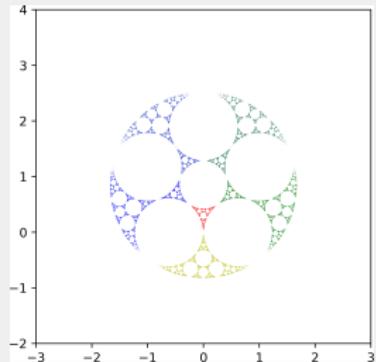
LIMIT SET DEFORMATIONS IN $\mathcal{R}^{5,\infty}$



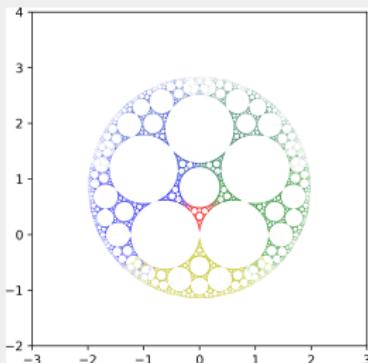
(a) $\rho = 4i$



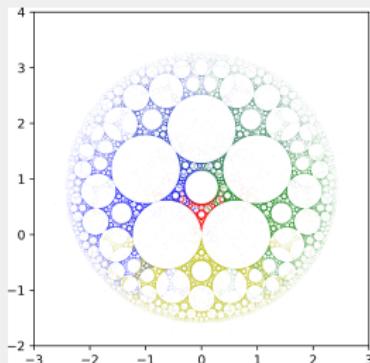
(b) $\rho = 3i$



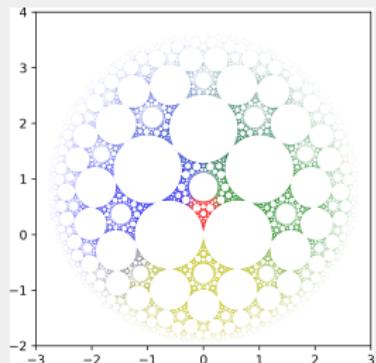
(c) $\rho = 2i$ (cusp group)



(d) $\rho = 1.7i$



(e) $\rho = 1.5i$



(f) $\rho = \sqrt{2}i$

SMALL PRINT

- The definition I gave is hiding the term ‘quasiconformal deformation space’ behind the geometry of limit sets.
- Equivalent definition: $\mathcal{R}^{a,b}$ is the moduli space of hyperbolic orbifolds (...together with conformal boundary...) homeomorphic to a 3-ball with two cone arcs, one of order a and one of order b (whose boundary is a sphere with two a -cone points and two b -cone points).
- The closure $\overline{\mathcal{R}^{a,b}}$ is the moduli space of discrete groups, free on two elliptic generators (where we view parabolic elements as limiting cases of elliptic elements).
- Discrete groups in the exterior $\mathbb{C} \setminus \overline{\mathcal{R}^{a,b}}$ parameterise the 2-bridge link groups and their ‘untwistings’ (Heckoid groups).

THE KEEN-SERIES RATIONAL LAMINATION

Theorem (Linda Keen & Caroline Series (1994); Yohei Komori & Series (1998); Elzenaar, Martin, Schillewaert (2022))

There exists a dense foliation of $\mathcal{R}^{a,b}$ by smooth analytic curves, indexed by $p/q \in \mathbb{Q}$, such that

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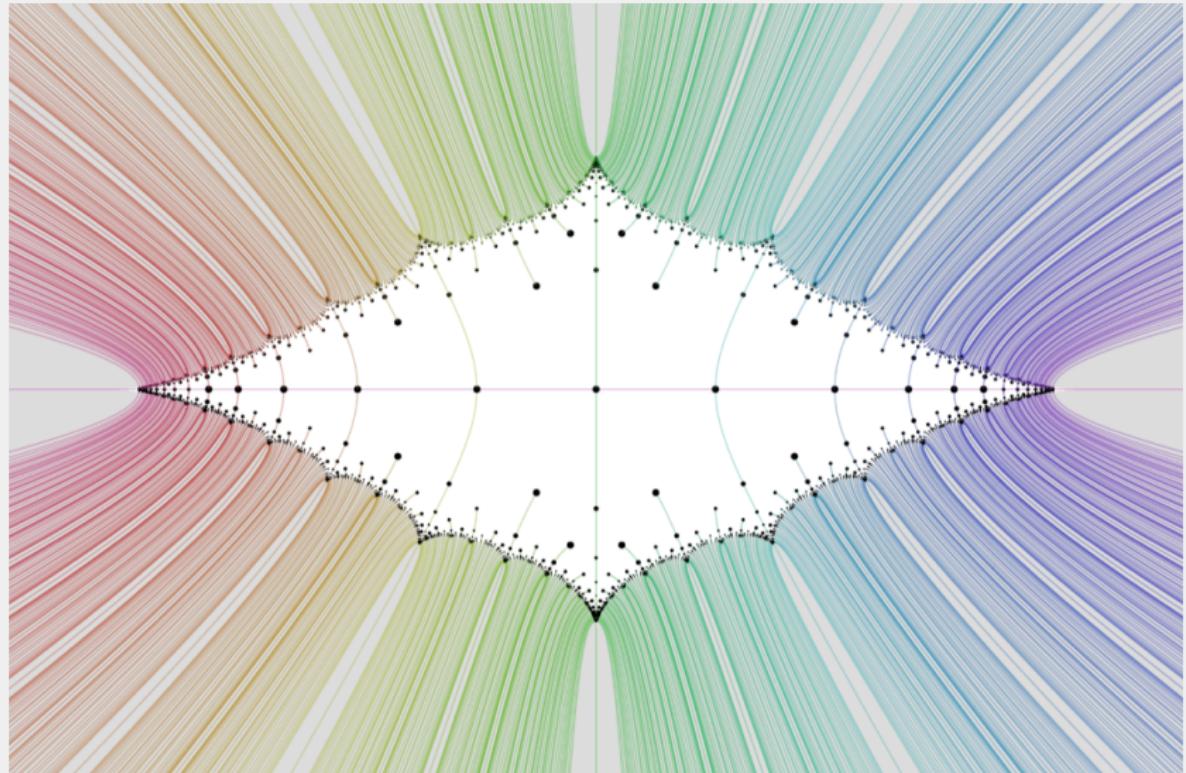
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3. *The inverse images of -2 which lie at the ends of the curves (called **cusp points**) are dense in the boundary of $\mathcal{R}^{a,b}$ [Curtis McMullen, 1991]; they have circle packing limit sets and the points corresponding to different curves are distinct [Keen/Maskit/Series, 1991].*

THE KEEN-SERIES RATIONAL LAMINATION OF $\mathcal{R}^{\infty, \infty}$



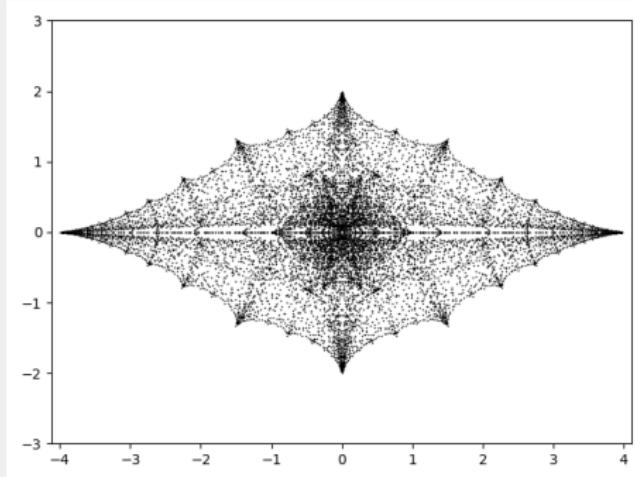
[Yasushi Yamashita, c.2007]

A METHOD FOR DRAWING THE RILEY SLICE BOUNDARY AND EXTERIOR

1. Compute all of the Farey polynomials. (*A priori*, each such computation requires $2q$ matrix multiplications. We have a recurrence relation that computes them all just with polynomial arithmetic [EMS22].)
2. Compute all of the inverse images $\Phi_{p/q}^{-1}(-2)$. (This is the computationally hard step. Geometric arguments show that even the inverse images which are not cusps lie in the exterior of the slice)

Conjecture

The points $\Phi_{p/q}^{-1}(-2)$ ($p/q \in \mathbb{Q}$) are dense in $\mathcal{R}^{a,b}$.



Some work has been done by Jane Gilman (2008) to explain the patterns of density that are visible (the ‘parabolic dust’).

BEDTIME READING

- George K. Francis, *A topological picturebook* (Springer, 1987)
- William P. Thurston, *Geometry and topology of 3-manifolds* (unpublished lecture notes, c.1979)
- William P. Thurston, *Three-dimensional geometry and topology, Vol. 1* (Princeton, 1997)
- Jeff Weeks, “Computation of hyperbolic structures in knot theory”. In: *Handb. of Knot Theory* (Elsevier, 2005)
- David Mumford, Caroline Series, David Wright, *Indra's pearls* (Cambridge, 2002)
- Albert Marden, *Hyperbolic manifolds* (Cambridge, 2015)
- Jessica Purcell, *Hyperbolic knot theory* (AMS, 2021)