Is $\mathrm{PSL}(2,\mathbb{Z})$ discrete?

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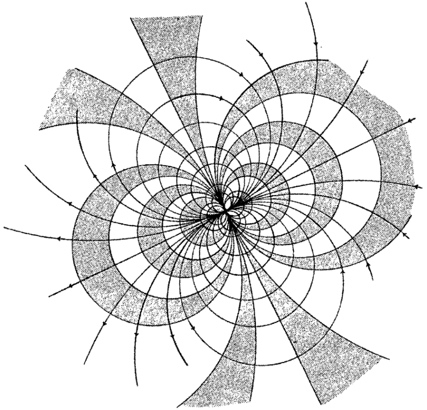
§I. Introduction

A nontrivial Möbius transformation acts in one of three ways:

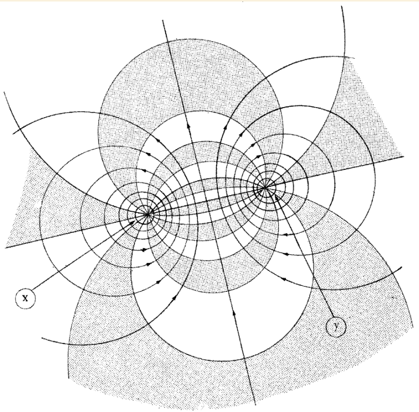
- fixing a point, and permuting a pencil of mutually tangent circles through that point (*parabolic*);



M. Berger. *Geometry I*. Springer (1987), p. 313.

A nontrivial Möbius transformation acts in one of three ways:

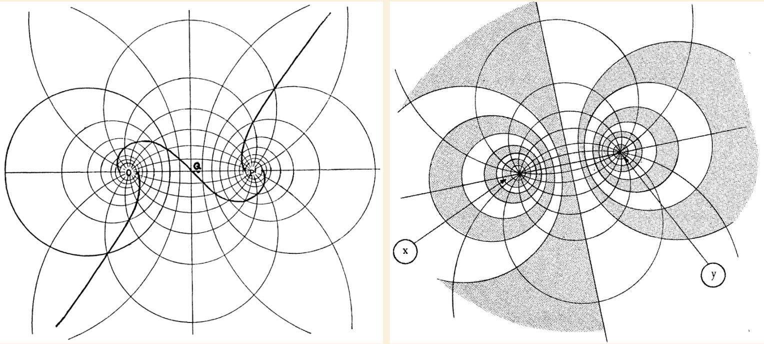
- fixing two points, and permuting the pencil of circles passing through both those points (*elliptic*);



M. Berger. *Geometry I*. Springer (1987), p. 314.

A nontrivial Möbius transformation acts in one of three ways:

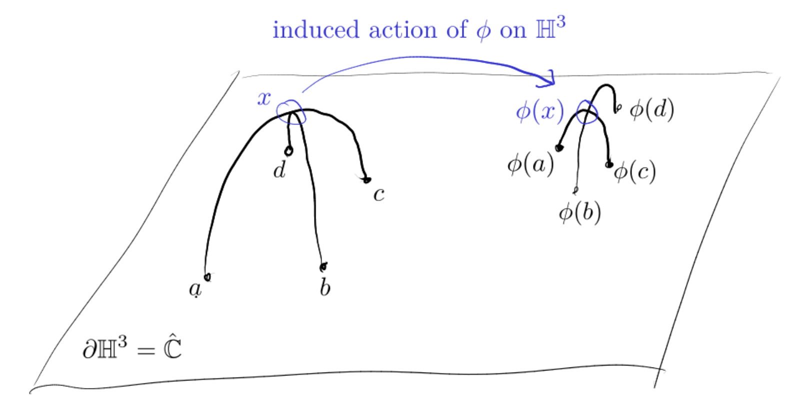
- fixing two points, and pushing points along the leaves of a pencil of loxodromes limiting to the fixed points (*loxodromic*).
  - *Special case:* fixing two points, and pushing points along the leaves of the pencil of circles through the fixed points (*hyperbolic*).



Left: J. Wilker, repro. in H.S.M. Coxeter, "Mid-circles and loxodromes". The Mathematical Gazette **52** (1968), pp. 1–8.
Right: M. Berger. *Geometry I*. Springer (1987), p. 314.

Theorem (Rigidity)

Isometries of $\mathbb{H}^3$ determine, and are determined uniquely by, a conformal map on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$.



Definition

A **Kleinian group** is one of the following equivalent things:

1. a holonomy group of some hyperbolic 3-orbifold.
2. a discrete subgroup of the isometry group of $\mathbb{H}^3$.
3. a discrete group of fractional linear transformations.
4. a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.
5. a discrete group of conformal maps of the sphere.

§II. The modular group

- The modular group is $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$.
- The action on $\hat{\mathbb{C}}$ preserves $\mathbb{R}$ and the upper half-plane.
- It is well-known that Γ admits the presentation

$$\langle R, S \,:\, R^2 = S^3 = 1 \rangle$$

where

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; S = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$\Gamma = \left\langle R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\rangle$$

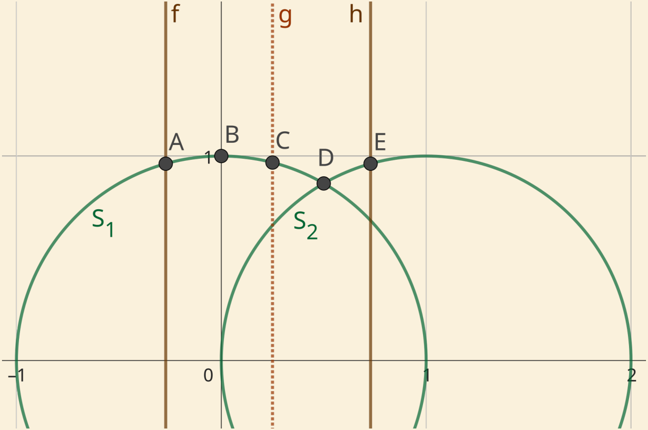
- R has fixed points $\pm i$, preserves the circle centred at 0 through these points, and flips its inside and outside.
- S has fixed points $(1 \pm i\sqrt{3})/2$, and moves the circle through these points centred at 0 to the circle through these points centred at 1. These circles intersect with angle $2\pi/3$.

These are the *isometric circles* of R and S: for every Möbius transformation which does not fix ∞ there is a unique circle which is sent onto a circle of the same radius.

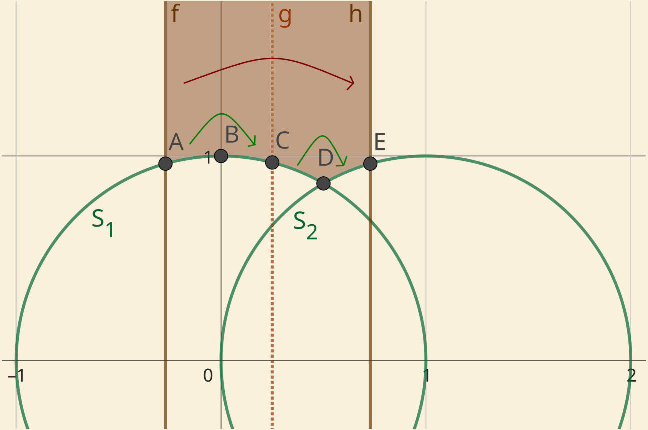
- We consider also a third element,

$$(RS)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

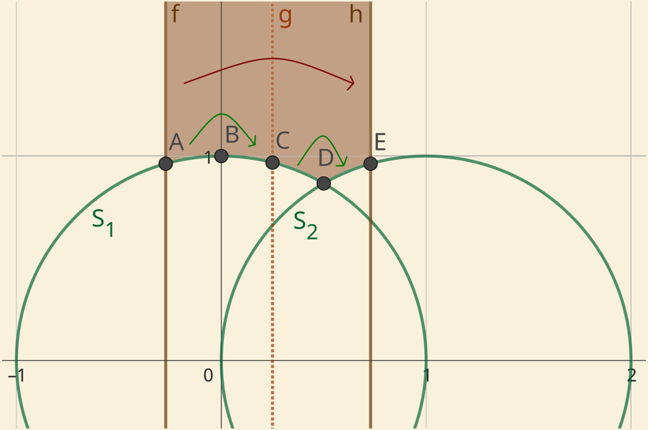
which acts as the translation $z \mapsto z + 1$.



The lines f and h are arbitrary vertical lines with distance 1. The vertical line g is the unique vertical line such that $|AB| = |BC|$ and $|CD| = |DE|$.



$(RS)^{-1}$ sends f to h.
R sends $[A, B]$ to $[C, B]$.
S sends $[C, D]$ to $[E, D]$.



$\mathbb{H}^2/\Gamma$ is a sphere with a puncture, a cone point with angle $2\pi/2$, and a cone point with angle $2\pi/3$. The marked points are the projections of the fixed points of $(RS)^{-1}$, R, and S.

By similar arguments:

- The quotient of the *lower* half-plane by Γ is an identical surface.
- The quotient $\mathbb{H}^3/\Gamma$ is homeomorphic to $\mathbb{H}^2/\Gamma \times (-1, 1)$.

One of the main conclusions of this proof is actually that Γ is discrete: existence of a set in $\hat{\mathbb{C}}$ which is moved entirely off itself and which tiles the plane is a certificate of discreteness. This was known to Poincaré.*

*H. Poincaré. "Theorie des groupes Fuchsiennes". Acta Mathematica **1** (1882), pp. 1–62. Trans. by J. Stillwell in *Papers on Fuchsian functions*. Springer-Verlag (1985).

The tessellation shown is induced by the $(2, 3, \infty)$ triangle group (which is $\mathrm{PSL}(2, \mathbb{Z})$), normalised to act on the radius $1/2$ disc around $i/2$, and extended by adding an additional parabolic $z \mapsto z + 1$.

R. Fricke and F. Klein. *Vorlesungen über die Theorie der automorphen Functionen* **1**. B.G. Teubner, Leipzig (1897), p. 432.

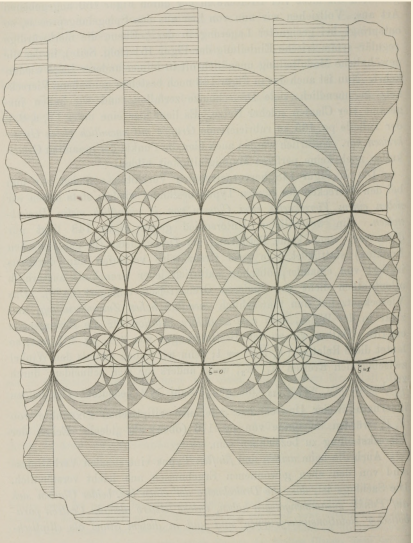
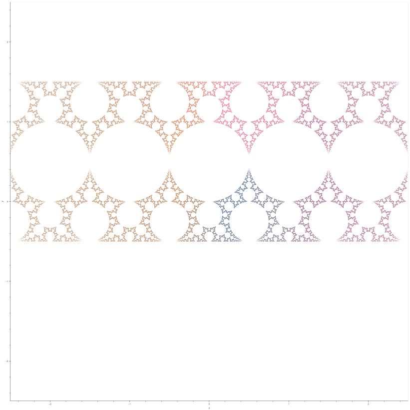
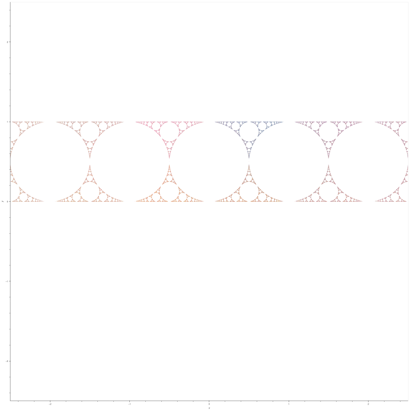


Fig. 151.

free limit set



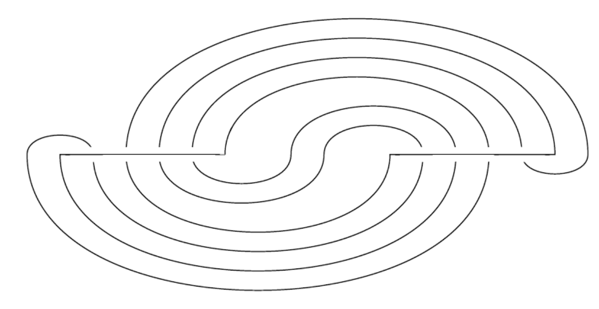


§III. Two-bridge link groups

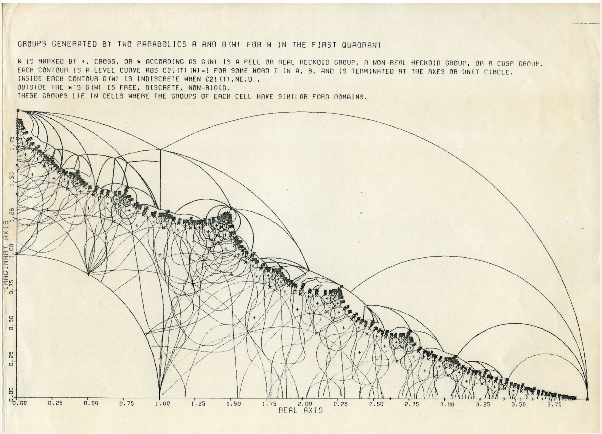
In the early 1970s R. Riley studied discrete representations of $\pi_1(k)$, where k is a two-bridge link, of the form

$$\pi_1(k) \to \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$

where the generators X and Y are the images of the elements in the fundamental group representing loops around the bridges.



$$\left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$



Many people have considered the problem of bounding this fractal-like space, including Sanov (1947), Brenner (1955), Chang, Jennings, and Ree (1958), and Lyndon and Ullman (1969).

We consider more generally groups of the form

$$\Gamma_\rho = \left\langle X = \begin{bmatrix} e^{\pi i/p} & 1 \\ 0 & e^{-\pi i/p} \end{bmatrix}, \ Y = \begin{bmatrix} e^{\pi i/q} & 0 \\ \rho & e^{-\pi i/q} \end{bmatrix} \right\rangle$$

so X and Y are finite order or parabolic (if p or q is ∞). If $\Gamma_\rho = \langle X \rangle * \langle Y \rangle$ then the group is called *simple monotunnel*. Set $\alpha = e^{\pi i/p}$ and $\beta = e^{\pi i/q}$.

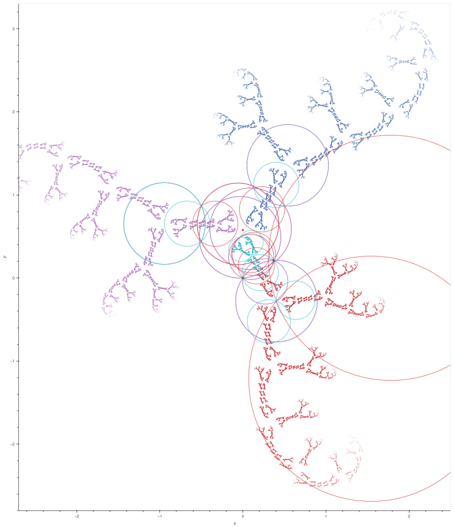
Example

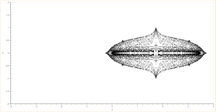
$\mathrm{PSL}(2, \mathbb{Z})$ is such a group: when $p = 3$ and $q = 2$ and $\rho = -2 + \sqrt{3}$, the group $\langle X, Y \rangle$ is conjugate to the standard embedding of $\mathrm{PSL}(2, \mathbb{Z})$ with $X \to S$ and $Y \to R$.

Example

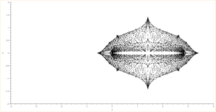
The law specifies that we must give an example of a limit set.

$$\left\langle X = \begin{bmatrix} e^{i\pi/3} & 1 \\ 0 & e^{-i\pi/3} \end{bmatrix}, \quad Y = \begin{bmatrix} e^{i\pi/7} & 0 \\ 1.01 + 0.77i & e^{-i\pi/7} \end{bmatrix} \right\rangle$$

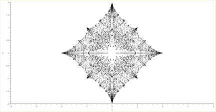




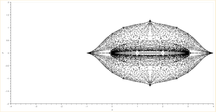
(a) (2, 3)



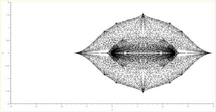
(b) (2, 4)



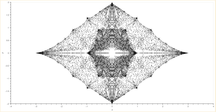
(c) $(2, \infty)$



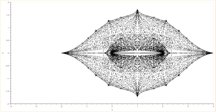
(d) (3, 3)



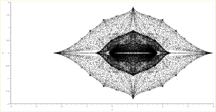
(e) (3, 4)



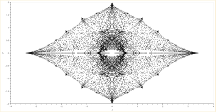
(f) $(3, \infty)$



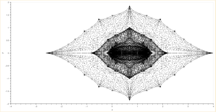
(g) (4, 4)



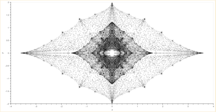
(h) (4, 5)



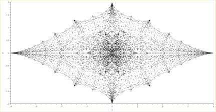
(i) $(4, \infty)$



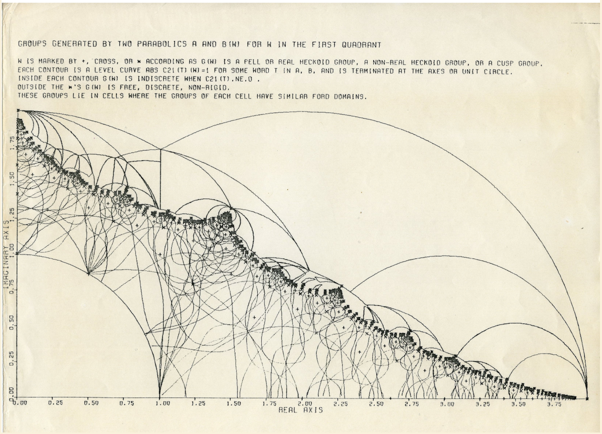
(j) (5, 5)



(k) $(5, \infty)$



(l) (∞, ∞)



- Many elements marked in Riley's figure are lattices (i.e. finite covolume).
- A finite number of these are arithmetic.

The strategy to enumerate all ρ such that Γ_ρ is arithmetic:

1. F.W. Gehring, C. Maclachlan, G.J. Martin, and A.W. Reid (1997) gave a necessary criterion for Γ_ρ to be arithmetic. Let the set of all ρ satisfying their condition be $\mathcal{A}$.

2. Classify the orbifolds $\mathbb{H}^3/\Gamma_\rho$ topologically, which gives an enumeration of all possible relators for discrete Γ_ρ.

3. For each $\rho \in \mathcal{A}$, either certify freeness or certify nonfreeness (by finding a nontrivial relator).

4. Derive *a priori* bounds on free representations in order to reduce the amount of ad-hoc computational work.

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   - We focus on this problem here.

§IV. Chang–Jennings–Ree bounds

Let a group G act on a space X. We would like to give conditions for the group to be discrete.

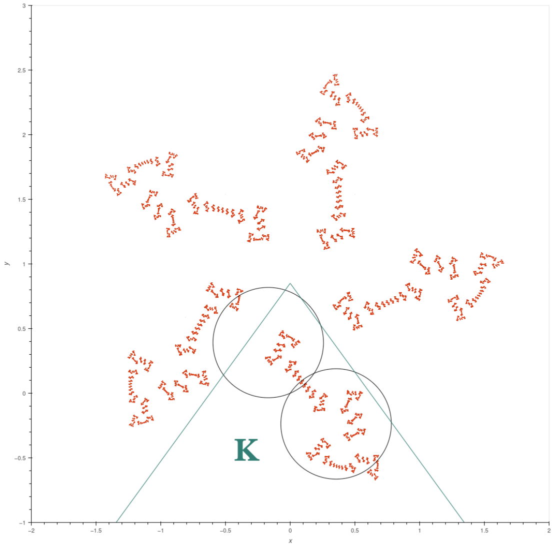
Theorem (Klein combination)

Suppose that U_1 and U_2 are disjoint subsets of X. Suppose that $G_1, G_2 < G$ are subgroups and that $G = \langle G_1, G_2 \rangle$. If:

1. *$g_1(U_1) \subseteq U_2$ for all $g_1 \in G_1 \setminus 1$;*
2. *$g_2(U_2) \subseteq U_1$ for all $g_2 \in G_2 \setminus 1$; and*
3. *$G_2(U_1) \nsubseteq U_1$ and $G_1(U_2) \nsubseteq U_2$;*

then we can conclude that G is discrete and $G = G_1 * G_2$.

We will apply this to the isometric discs of X and Y: we obtain conditions on ρ such that the intersection of the isometric discs of X and its complement form an interactive pair and conclude that $\langle X, Y \rangle$ is discrete and non-cocompact.



X sends $\mathbb{K}$ off itself regardless of ρ

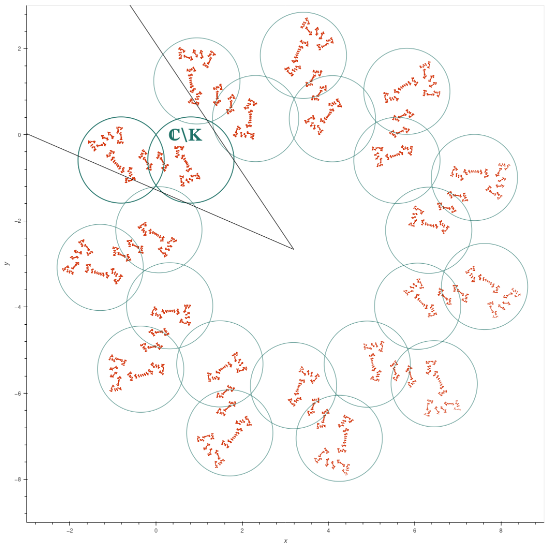


Image under $J(z) = 1/z$. If JYJ sends $J\mathbb{K}^{\mathsf{C}}$ off itself then the group is discrete.

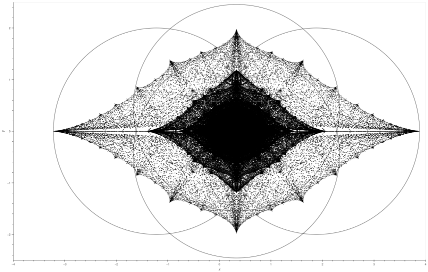
Recall $X(z) = \alpha^2 z + \alpha$ and $Y(z) = \beta^2/(\beta\rho z + 1)$ where $\alpha = \exp(\pi i/p)$ and $\beta = \exp(\pi i/q)$.

Lemma

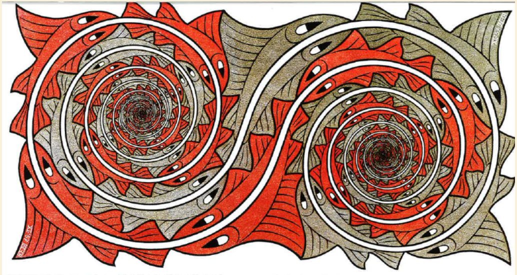
Sufficient conditions for $\langle X, Y\rangle$ to be discrete and non-cocompact are:

$$|\alpha\left(\beta - \overline{\beta}\right) + \rho| > 2, \quad |\beta\overline{\alpha} + \alpha\overline{\beta} - \rho| > 2,$$
$$|\overline{\alpha}\left(\overline{\beta} - \beta\right) + \rho| > 2, \quad |\overline{\alpha}\overline{\beta} + \alpha\beta + \rho| > 2.$$

These bounds are displayed when $p = 5$ and $q = 11$:



It is possible to extend this argument to the case that one of the generators is loxodromic:



M.C. Escher (1957)

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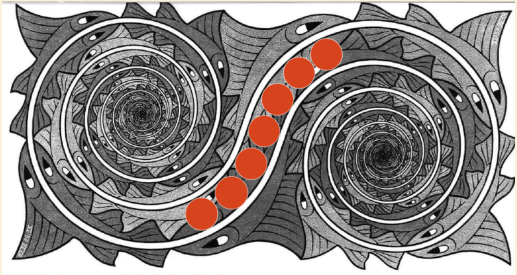


M.C. Escher (1957)

Pick two loxodromic orbits of your generator Y; choose a disc which is tangent to each and whose $Y^{\pm 1}$-images are disjoint; now choose X so that its isometric circles lie within this disc.

§V. Lyndon–Ullman bounds

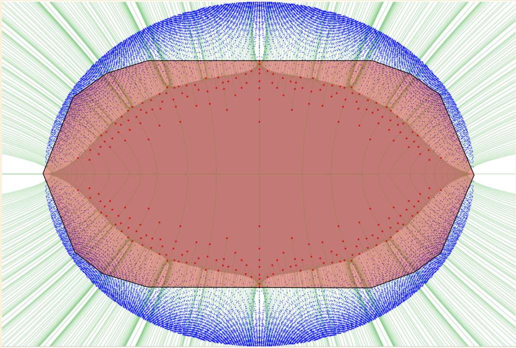
By more detailed (but still very classical) geometric analysis, we can significantly improve these bounds.

Theorem (E.–Gong–Martin–Schillewart, 2024.)

When ρ lies outside a certain (Euclidean) dodecagon with edges depending on α and β, then the corresponding group $\langle X, Y\rangle$ is discrete and non-cocompact.

The bound in general is far away from $\partial\mathcal{R}$, but is sharp at exactly four cusp points.

The bounds for $p = q = 3$:



Red points are arithmetic lattices or non-cocompact generalised triangle groups, there are 129 of them, reduced from 15,909 possibilities given by the Gehring–Maclachlan–Martin–Reid criterion.

We perform two kinds of improvements.

1. Isometric circles are not conjugacy invariants. Therefore we may choose better generators (in this case more symmetric generators) in order to obtain better bounds.
2. Pioneered by Lyndon and Ullman when both generators are parabolic: given an interactive pair for a group with parameter ρ, it can be modified by a Möbius transformation Φ in order to produce an interactive pair for a group with parameter $\Phi(\rho)$. We conduct a study of the simplest choices of Φ which continue to produce discrete groups, and these allow us to slice off certain half-planes.

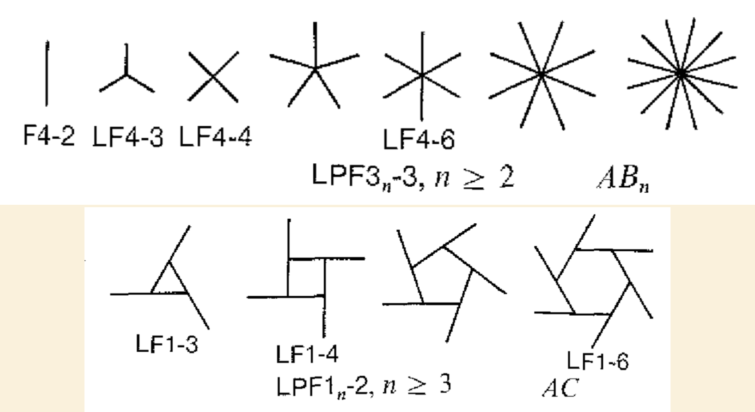
Lemma

Let $u \in \mathbb{C}^*$ be arbitrary. Let $\Phi(z) = (1/u)z$.

1. $\Phi Y(\rho)\Phi^{-1} = Y(u\rho)$.
2. *If $Y(\rho)$ maps $\hat{\mathbb{C}} \setminus \mathbf{K}$ off itself, then $Y(u\rho)$ maps $\hat{\mathbb{C}} \setminus \Phi(\mathbf{K})$ off itself.*

Qn. For what choices of $(1/u)$ does X map $(1/u)\mathbf{K}$ off itself?

How does $\langle X \rangle$ act on the cone $\Phi(\mathbb{K})$?



B. Grünbaum and G.C. Shepard. *Tilings and patterns.* Freeman and co. (1987), Fig. 7.4.2.

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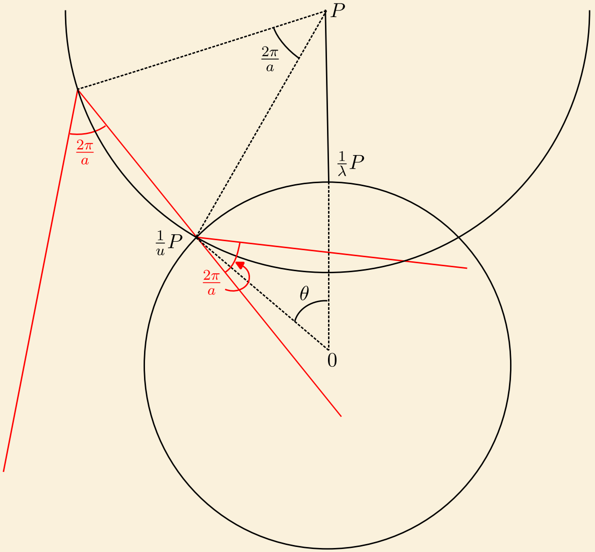


Key piece of proof comes from analysis of the diagram:

Lemma

If ρ works, then $u\rho$ works whenever $\operatorname{Im} u \geq 1$: equality is the 'boundary' where all the images $X^n(1/u)\mathbb{K}$ form a pinwheel pattern.

Optimal choices for ρ then allow us to cut off 12 lines from the Chang–Jennings–Ree bounds giving us a dodecagon.

§VI. Faithfulness of B_3-reps

Motivated by the study of q-rationals, S. Morier-Genoud, V. Ovsienko, and A. P. Veselov (2023) studied the locus of $\mu \in \mathbb{C}$ such that the family of representations

$$\tilde{\rho}_\mu : B_3 \xrightarrow{\rho} \mathrm{SL}(2, \mathbb{Z}[t^{\pm 1}]) \xrightarrow{t \mapsto \mu} \mathrm{SL}(2, \mathbb{C})$$

is faithful, where B_3 is the 3-strand braid group and ρ is the reduced Burau representation

$$\rho(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho(\sigma_2) = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}.$$

(σ_1 and σ_2 are the standard Artin generators).

Theorem (Morier-Genoud–Ovsienko–Veselov, 2023)

The specialised Burau representations are faithful for μ outside the annulus $3 - 2\sqrt{2} \leq |\mu| \leq 3 + 2\sqrt{2}$

Conjecture (M-G–O–V, 2023)

The theorem bound may be improved to the exterior of the annulus

$$\frac{3-\sqrt{5}}{2} \leq |\mu| \leq \frac{3+\sqrt{5}}{2}.$$

Let $\rho_\mu : B_3 \to \mathrm{PSL}(2, \mathbb{C})$ be specialisation. When $\mu = -1$, the image is $\mathrm{PSL}(2, \mathbb{Z})$. In general the images are conjugate to groups $\langle X, Y \rangle$ with $p = 3$ and $q = 2$.
The change of variables between μ and ρ is:

$$\rho = i\sqrt{\mu} + \sqrt{3} - \frac{i}{\sqrt{\mu}}.$$

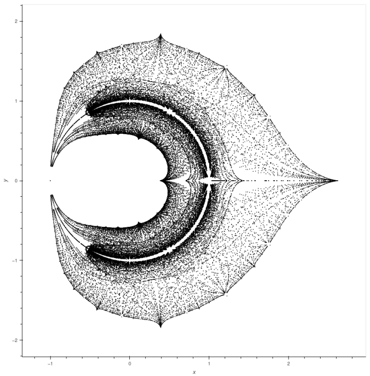
Pulling a version of the Chang–Jennings–Ree bounds with more symmetric isometric circles back to the Burau matrices:

Lemma

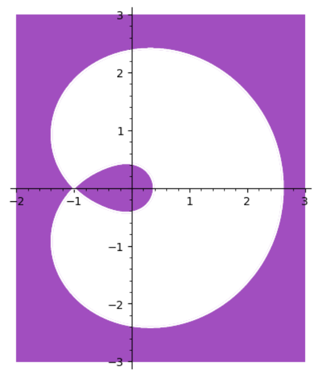
For $\mu \in \mathbb{C}$, let $z = \sqrt{\mu} - 1/\sqrt{\mu}$. If

$$3 \leq |z \pm \sqrt{z^2 + 3}|$$

then $B_3/Z(B_3) \to \mathrm{PSL}(2, \mathbb{C})$ is injective.



2
1
0
−1
−2
−1
0
1
2
x
y



3
2
1
−1
−2
−3
−2
−1
1
2
3

A fairly straightforward argument in $\mathrm{PSL}(2, \mathbb{C})$ (using the fact that we know $Z(B_3)$ explicitly from the theory of braid groups) shows that for these representations, injectivity of $B_3/Z(B_3) \to \mathrm{PSL}(2, \mathbb{C})$ implies injectivity of the specialised Burau representation $B_3 \to \mathrm{SL}(2, \mathbb{C})$:

Theorem (E.–Gong–Martin–Schillewart, 2024.)

The specialised Burau representation $B_3 \to \mathrm{SL}(2, \mathbb{C})$ is faithful within a closed semialgebraic region strictly containing the region conjectured by Morier-Genoud, Ovsienko, and Veselov, except at one point where both the conjectured bound and our bound are tight.

Bedtime reading

- A.J.E., Jianhua Gong, Gaven Martin, and Jeroen Schillewaert. “Bounding deformation spaces of 2-generator Kleinian groups”, 2024. arXiv:2405.15970 [math.CV].
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- R.C. Lyndon and J.L. Ullman, “Groups generated by two parabolic fractional linear transformations”. In: *Canadian Journal of Mathematics* **21** (1969), pp. 1388–1403.
- S. Morier-Genoud, V. Ovsienko, and A.P. Veselov, “Burau representation of braid groups and *q*-rationals”. In: *International Mathematics Research Notices* (2024), pp. 1–10.