Toric Varieties

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Abstract

In this dissertation, we survey the basic theory of toric varieties as developed by Oda, Mumford, and others in the last 50 years. We give full proofs from first principles of the main theorems, modulo basic results from analysis and a first introduction to scheme theory. For this part we draw from [Oda78] and other historical sources. In the second part, we give detailed expositions of the two main classical applications of the theory: Ehrhart reciprocity and Stanley's resolution of McMullen's g-conjecture. For this part we give detailed proofs of all the toric results needed, and spend a significant amount of time motivating the theory and giving examples; we draw primarily from [CLS11].

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0 Introduction

The study of toric varieties (or, torus embeddings) is at the intersection of several different branches of mathematics: the theory of convex sets; algebraic geometry; complex geometry; and the theory of semigroups.

We begin by recalling the notion of a variety.

0.1 Definition. An **affine variety** is an affine scheme which is reduced, irreducible, and of finite type over an algebraically closed field. A **variety** is a reduced, irreducible, separated scheme with a finite open cover consisting of affine varieties.

A main purpose of the theory of combinatorial algebraic geometry is the study of the following questions:

To what extent is the global structure of a variety determined by the combinatorial properties of 'nice' finite affine open covers? To what extent are these combinatorial properties preserved by scheme morphisms?

Note that a scheme has two pieces of data: a topological space (which we have placed a condition on), and a sheaf of regular functions (each section of which is finitely generated). Hence any attempt to answer the questions must try to capture the combinatorial properties of both the topological space and the sheaf of regular functions.

In order to place a combinatorial structure on the sheaf of regular functions, we will study affine varieties whose ring of regular functions is generated by some additive structure embedded in a lattice (in particular, a convex semigroup). It will turn out that the simplest such structure — namely the semigroup consisting of the identity element of the lattice, which is clearly embedded in every other semigroup — corresponds to the ring of regular functions of $(\mathbb{C}^*)^n$ (the **torus** of rank n), where n is the rank of the lattice (Example 4.12). Further, this lattice substructure is embedded in the larger lattice in a very natural way; this leads us to guess that the copy of $(\mathbb{C}^*)^n$ is embedded in the variety in a natural way. We are therefore led to consider all the varieties which include $(\mathbb{C}^*)^n$ as open subvarieties, such that the torus is embedded in the most natural way possible: as a dense open subset of each of the finitely many open affine varieties in the cover, such that the natural multiplication action of $(\mathbb{C}^*)^n$ on itself extends naturally to its closure.

In order to make this work, we will eventually need a technical property that we might as well state now:

0.2 Definition. Let R be a ring, and let $K = \operatorname{Frac} R$. The **integral closure** of R in K is the set

$$\{\zeta \in K : f(\zeta) = 0 \text{ for some monic } f \in R[T]\}.$$

We say R is **integrally closed** if it is equal to its integral closure. A scheme is **normal** if all its local rings are integrally closed.

It is not immediately clear why this is at all related to the programme outlined above; the idea is basically that normal schemes are those where pathological gluing does not occur (see for example the intuition given at [MO109395]), and since we are studying varieties which are glued in a particularly nice manner this is a property we expect.

A feature of the theory which we will develop is a duality between normal varieties with an embedded torus ("toric varieties", see Definition 4.1) and certain polyhedra in Euclidean space. This duality allows us to apply powerful techniques from fields like convex geometry and linear optimisation to algebraic geometry, and vice versa.

0.1 Outline and novel elements

The dissertation is divided into two parts: Part I covers the basic theory of toric varieties, and Part II gives a brief overview of the cohomology theory and some classical applications.

In Part I, Sections 1, 2, and 3 give the required technical background from the theory of group varieties and convex geometry; section 4 develops the local theory of toric varieties; and Section 5 develops the global theory. The main novelty here is the manner in which the theory is developed: we follow the development in [CLS11] for most of the local theory, but in contrast to that book we immediately begin with a view to abstract varieties. In this way we are able to use the full power of scheme-theoretic language to give to give simplified or more enlightening statements in some cases. We also introduce new notation when it allows for a better exposition: the **E**, **H**, and **D** maps of Definition 4.4 and Definition 4.9 are novel and serve to make correspondences and identifications between points, semigroups, orbits, and distinguished open sets clearer than the notation of [Oda78] or [Ful93].

For the global theory we primarily follow the theoretical development pioneered by [Oda78], but with many added examples and with greater motivation: for example, the proof of the 'properness theorem' (Lemma 5.23) is given in terms of the combinatorial data rather than the pure valuation ring argument given in Oda (in this way, our proof is similar to that in [Ful93], but we have tried to simplify our argument compared to the proof there).

Another distinct feature of the development we pursue here is full acknowledgement of the combinatorial-commutative-algebraic side of the theory: as is well-known, the theory of toric varieties is intimately linked with the theory of monomial and binomial ideals and their combinatorics (see, for example, [MS05, Part II]). Adopting some of the language and methods of combinatorial commutative algebra allows us to choose at each point the language best suited for the task at hand.

In Part II, the goal of Section 6 is to develop the cohomology theory of toric varieties. We use this in section 7 to present a classification of quasi-projective toric varieties, and in section 8.1 a proof of Pick's formula using high-powered machinery: the divisor theory of toric varieties. This provides us with an opportunity to develop briefly the cohomology theory as well. In Section 8.2 we apply the theory of toric varieties to the study of the face lattice of polytopes, and briefly mention the recent solution of McMullen's g-conjecture for simplicial spheres, which uses modern techniques from this field.

0.2 Required background

The background assumed of the reader is, roughly speaking, undergraduate real analysis and a first course in scheme-theoretic algebraic geometry. More precisely, we will assume without proof anything found in [Rud13] for real analysis and in chapters I and II of [Har77] for geometry (along with the required algebra, which may generally be found in chapters I to VIII of [Alu09]). The reader may also find that some of the concepts we require from algebraic geometry may be covered in more detail in [Sha13, Chapter 2] (particularly the notion of normality, which Hartshorne relegates to exercises).

In addition, though our coverage of convex and affine geometry is mathematically self-contained, the reader less schooled in this area might find it a good idea to read an elementary treatment like that in chapters 12 and 13 of [Cox69] before trying one of the more relevant (but more abstract) treatments listed in the references section below.

0.3 References

The standard introductory references for this subject are [Ful93] and [CLS11] (the latter more comprehensive and covering newer topics in research using more abstract language). From a more combinatorial/convex-geometric viewpoint, the book [Ewa96] is comprehensive. One of the important historical figures in this subject is Tadao Oda; his book [Oda78] could also be viewed as an introductory text but assumes much more familiarity with algebraic geometry; it also has a different focus, the primary themes being the birational geometry of toric varieties and the classification of those of small dimension. Note also the existence of [Kem+73] which is one of the oldest references to the subject written by some of the other early modern practitioners.

For the necessary results in convexity theory, some good elementary references include [Bar02], [Ber09], and [Zie95]; an excellent historical reference is [Sch86, pp. 209–223]. The 'canonical' book on the subjects are [Grü03]. For the necessary results on algebraic groups, see [Hum75] or [Wat79].

Finally note that there are several books on the subject from the perspective of complex geometry (see the extensive monograph [BL04]) and tropical geometry (see the final chapter of [MS15]).

0.4 Terminology

The field K is assumed algebraically closed but may be of positive characteristic. If A is an algebra, the localisation of A by a multiplicative set S is denoted by $A[S^{-1}]$; the localisation by a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ is denoted by $A_{\mathfrak{p}}$ (so $A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$); and if $f \in A$ we denote by A_f the localisation $A[U^{-1}]$ where $U = \{f^n : n \in \mathbb{Z}\}$. When we write something like 'x is a point of Spec A' or '$x \in \operatorname{Spec} A$' we will usually mean that x is a *closed* point. If we mean to include arbitrary prime ideals of A we shall always say so, and will usually use an appropriate letter like $\mathfrak{p}$.

Occasionally we will write $\biguplus_{\alpha\in A} S_\alpha$ for the union $\bigcup_{\alpha\in A} S_\alpha$ if we wish to stress that the unioned sets are pairwise disjoint. We will write $\coprod_{\alpha\in A} S_\alpha$ for the *categorical* disjoint union.

If $p \in R[X]$ is a polynomial over a ring R in indeterminate X (we shall ordinarily use capital letters for indeterminates), we write ∂p for the degree of p, and $[X^i]p$ for the coefficient of X^i in p and extend this notation in the obvious way for multivariable polynomials.

We write $V^\vee$ for the dual space of a vector space V over K, and if $X \subseteq V$, we write $X^\perp$ for the set $\{a \in V^\vee : \forall_{x\in X} \langle x|a\rangle = 0\}$ where $\langle\cdot|\cdot\rangle : V \times V^\vee \to K$ is the duality pairing.)

We shall assume without comment the notation of [Har77]; we write $\mathbf{A}(Y)$ and $\mathbf{K}(Y)$ for, respectively, the affine coordinate ring and the function field of an affine variety Y; we write $D(f)$ for the principal open subvariety $\operatorname{Spec} A_f$ when $f \in A$.

We will occasionally use a Halmos ▮ at the end of the statement of a proposition; this means that the proof is (or should be) trivial.

0.5 Acknowledgements

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Part I
Basic theory

1 Algebraic tori and lattices

Fundamental to the study of toric varieties is the study of the canonical such variety: the algebraic torus itself. We are particularly interested here in the relationship between the algebraic torus and its lattices of characters and 1-psgs (Definition 1.3); in some sense the entire theory of toric varieties may be viewed as the study of embeddings of tori in varieties with the property that the characters of the torus are restrictions of regular functions of the ambient variety.

All the interesting theory occurs when we study closed points of varieties, and the extension to arbitrary points of a variety (i.e. actually using the scheme-theoretic definition) does not give us much more geometry. We will therefore spend most of our efforts on the closed case; for the general case, relevant introductory texts on algebraic groups include [Wat79] or [DG80]. For our purposes the relevant material is found entirely in [Hum75, Section 16].

1.1 Definition. A **group variety** is an affine variety X equipped with two morphisms of varieties $\mu : X \times X \to X$ and $\iota : X \to X$ such that μ and ι satisfy the axioms for a group multiplication and identity operation respectively. If X and Y are group varieties, a morphism $f : X \to Y$ is a morphism of varieties that is also a homomorphism of the groups.

1.2 Example. Let K be a field; then $K = \mathbb{A}^1_K$ is a group variety where $\mu(x, y) = x + y$ and $\iota(x) = -x$. Similarly K^* is a group variety where $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$; and $(K^*)^n$ is a group variety under componentwise multiplication. All these groups are abelian, and all their underlying varieties are (quasi)-affine.

1.3 Definition. Let G be a group variety; a **character** of G is a morphism of group varieties $\chi : G \to K^*$. The set of characters forms an abelian group, $X(G)$; note that $X(G)$ is a subgroup of $\mathbf{A}(G)$. If G is abelian, a **one-parameter subgroup** (**1-psg**) of G is a morphism $\lambda : K^* \to G$. The set of 1-psgs forms an abelian group, $Y(G)$.

1.4 Theorem (Torus characterisation)**.** *Let G be a group variety. Then the following are equivalent:*

1. *G is isomorphic to a closed subgroup of $(K^*)^n$ for some $n \in \mathbb{N}$;*
2. *$X(G)$ generates $\mathbf{A}(G)$;*
3. *$G \simeq (K^*)^m$ for some $m \in \mathbb{N}$.*

Such a group is called an ***algebraic torus****.*

Note also, by dimension counting we must have $m \leq n$ in the above theorem.

Proof. This is the combination of the results of [Hum75, Sections 16.1 and 16.2], noting that we assume varieties to be irreducible. ∎

We study the characters and 1-psgs of tori.

1.5 Definition. A **lattice** N is a free abelian group of finite rank; i.e. a group isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. If N is a lattice define the **dual lattice** $N^\vee \coloneqq \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ (i.e. the abelian group of $\mathbb{Z}$-linear maps $N \to \mathbb{Z}$).

By standard results similar to those for finite-dimensional vector spaces, if $N \simeq \mathbb{Z}^n$ then $N^\vee \simeq \mathbb{Z}^n$, there is a bijection between the set of isomorphisms $N \to N^\vee$ and the set of $\mathbb{Z}$-bilinear maps $N \times N^\vee \to \mathbb{Z}$ (we call such maps **duality pairings**), and there is a natural isomorphism $(N^\vee)^\vee \simeq \vee$.

1.6 Lemma. *Let T be a torus, with isomorphism $\tau : T \to (K^*)^k$. Then:*

1. *The characters of T are, after choosing coordinates, precisely the evaluations on T of monomials of the form $X_1^{m_1} \cdots X_k^{m_k}$ where $(m_1, ..., m_k) \in \mathbb{Z}^k$.*
2. *The 1-psgs of T are, after choosing coordinates, precisely the evaluations on $(K^*)^n$ of tuples of monomials of the form $(X^{n_1}, ..., X^{n_k})$ where $(n_1, ..., n_k) \in \mathbb{Z}^k$.*
3. *There is a canonical (coordinate-free) identification between $X(T)^\vee$ and $Y(T)$ as lattices of rank k.*

Proof. Every morphism of varieties $(K^*)^k \to K^*$ is the evaluation of a polynomial from the algebra $K[X_1^{\pm 1}, ..., X_k^{\pm 1}]$. Hence every character $m \in X(T)$ is of the form $f \circ \tau$ for such a polynomial f. Further, such a composition is a morphism of group varieties if and only if $f(X_1, ..., X_k) = X_1^{m_1} \cdots X_k^{m_k}$ for (possibly negative) integers m_i (otherwise, the polynomial map is not a homomorphism of groups). Thus the morphism of abelian groups $\phi : X(T) \to \mathbb{Z}^k$ sending $m \mapsto (m_1, ..., m_k)$ in this way is a bijection and hence an isomorphism. Similarly, all members of $Y((K^*)^k)$ are of the form $t \mapsto (t^{n_1}, ..., t^{n_k})$; so $\psi : Y(T) \to \mathbb{Z}^k$ mapping $n \mapsto (n_1, ..., n_k)$ is an isomorphism.

For all $m \in X(T)$ and $n \in Y(T)$, $m \circ n$ is a character of K^* and hence can be identified with an integer $[m, n] \in \mathbb{Z}$ by the previous paragraph. Note that unlike ϕ and ψ, $[\cdot, \cdot]$ does not depend on the choice of any isomorphism. Suppose $f \in X(T)^\vee$; let $(e_i)_{i=1}^k$ be a basis of $\mathbb{Z}^k$, and define $n_i \coloneqq f(\phi^{-1}(e_i)) \in \mathbb{Z}$ for each i. Set $n = \psi^{-1}(n_1, ..., n_k) \in Y(T)$.

Let $m \in X(T)$ be arbitrary, and set $(m_1, ..., m_k) = \phi(m)$. Now for all $t \in K^*$,

$$t^{[m,n]} = (m \circ n)(t) = (\phi(m) \circ \psi(n))(t) = \phi(m)(t^{n_1}, ..., t^{n_k}) = (t^{n_1})^{m_1} \cdots (t^{n_k})^{m_k} = t^{n_1 m_1 + \cdots + n_k m_k}$$

so $[m, n] = n_1 m_1 + \cdots + n_k m_k$, and

$$f(m) = f(\phi^{-1}(e_1)^{m_1} \cdots \phi^{-1}(e_k)^{m_k}) = m_1 f(\phi^{-1}(e_1)) + \cdots + m_k f(\phi^{-1}(e_k)) = m_1 n_1 + \cdots + m_k n_k$$

and so $[m, n] = f(m)$ for all $m \in X(T)$. Thus for each $f : X(T) \to \mathbb{Z}$ we obtain an $n \in Y(T)$ such that $f = [\cdot, n]$; and for each $n \in Y(T)$ it is obvious that $[\cdot, n]$ is a character. Further this correspondence is $\mathbb{Z}$-linear in f and n, and so we can identify $Y(T)$ and $X(T)^\vee$. Finally, note that this correspondence is coordinate-free as the pairing $[\cdot, \cdot]$ does not depend on coordinate choices; but upon choice of coordinates we have that $[m, n]$ is the usual dot product $\langle \phi(m) | \psi(n) \rangle$. ∎

We shall reserve the usual symbol $\langle \cdot | \cdot \rangle$ to denote the canonical inner product between the lattices of a torus, as defined in the proof of Lemma 1.6.

1.7 Example. As an example, consider the cusped cubic $k = \operatorname{Spec} K[X, Y]/(X^3 - Y^2)$ which we shall meet later on as a useful example of a singular affine toric variety (c.f. Example 4.17). The intersection $k \cap (K^*)^2$ is an algebraic torus under the isomorphism

$$\begin{aligned}\Phi : k \cap (K^*)^2 &\to K^* \\ (x, y) &\mapsto (y/x).\end{aligned}$$

(Note, this is an isomorphism as it is locally regular and has an inverse $\alpha \mapsto (\alpha^2, \alpha^3)$.) A character on $k \cap (K^*)^2$ is the pullback via Φ of a character of K^*, and a 1-psg on $k \cap (K^*)^2$ is a 1-psg into K^* followed by Φ^{-1}. Take the character $\chi X \mapsto X^3$ of K^*; then the induced character of the cusped cubic torus is $\chi' = \chi \circ \Phi : (X, Y) \mapsto (Y/X)^3$; further if we take the 1-psg of K^* defined by $\alpha : T \mapsto T^{-7}$ we end up with an induced 1-psg of the cusped cubic torus, namely $\alpha' = \Phi^{-1} \circ \alpha : T \mapsto ((T^{-7})^3, (T^{-7})^3) = (T^{-14}, T^{-21})$. The inner product $\langle \chi' | \alpha' \rangle$ is the exponent of T in $\chi' \circ \alpha'$; since $\chi' \circ \alpha' = (T^{-21}/T^{-14})^3 = T^{-21}$, we have $\langle \chi' | \alpha' \rangle = -21$. Note that the inner product $\langle \chi | \alpha \rangle$ is the exponent of T in $(T^{-7})^3$, which is -21 (as we would expect, since the inner products are coordinate independent).

We will give a name to the map ϕ from the proof of Lemma 1.6 as it will arise often.

1.8 Definition. Let $m = \lambda X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in K[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ be a monomial with $\lambda \neq 0$. We define the **exponent vector** of m to be the tuple $(\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$. Conversely, if $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$ we will write χ^α for the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in K[X_1^{\pm 1}, ..., X_n^{\pm 1}]$.

There is also an intrinsic embedding of an abstract lattice into K^n which satisfies similar compatibility requirements with respect to coordinate choices. We give a more general case first as we will meet several situations where a result of this form is useful.

1.9 Lemma (Base extension). *Let R be a ring and let M be an R-module. Let $f : R \to R'$ be a ring morphism. Then $M_{R'} := M \otimes_R R'$ is naturally an R'-module, in the sense that*

- *there is a morphism of abelian groups $\rho : M \to M_{R'}$ such that for all $a \in R$ and $x \in M$, $\rho(ax) = f(a)\rho(x)$;*
- *for every R'-module N and every morphism of abelian groups $\sigma : M \to N$ such that for all $a \in R$ and $x \in M$, $\sigma(ax) = f(a)\sigma(x)$, there is a unique R'-linear map $\tilde{\sigma} : M_{R'} \to N$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\sigma"'] & M_{R'} \arrow[d, dotted, "\tilde{\sigma}"] \\ & N \end{tikzcd}$$

- *if G is a set of generators for M over R, then $\rho(G)$ is a set of generators for $M_{R'}$ over R'.*

Proof. We first have to define the R'-module structure on $M_{R'}$. Note that R' is an R-module (in fact an R-algebra!) with scalar multiplication $(a, x) \mapsto f(a)x$; thus it makes sense to say that the map

$$\begin{aligned}\mu : M \times R' \times R' &\to M_{R'} \\ \mu(x, a, b) &:= x \otimes ab\end{aligned}$$

is R-trilinear. By the universal property of the tensor product we obtain an R-linear map $M \otimes_R R' \otimes_R R' \to M_{R'}$ and thus (composing with the map $M \times R \to M \otimes_R R'$) an R-bilinear map $\mu' : (M \otimes_R R') \times R' \to M_{R'}$, sending $(m \otimes a, b) \mapsto m \otimes ab$. But now note that this map μ' defines an R' multiplication on $M \otimes_R R'$, thus turning it into an R'-module. We equip $M_{R'}$ with the usual coprojection $\rho : M \ni x \mapsto x \otimes 1 \in M_{R'}$; this satisfies the linearity condition since, for $a \in R$ and $x \in M$, $\rho(ax) = ax \otimes 1 = f(a)(x \otimes 1) = f(a)\rho(x)$.

The universal property now follows directly from the usual universal property for a tensor product; the only thing needed is to verify that the induced pushforward $\tilde{\sigma}$ is indeed linear in the required manner, which is done in the same way it was showed for ρ.

Finally note that the statement about the generators follows directly from the construction, as $M \otimes_R R'$ is generated over R' by $m \otimes 1$ for $m \in M$. ∎

1.10 Corollary. *If N is a lattice of rank n and a particular isomorphism $\phi : N \to \mathbb{Z}^n$ is chosen, then:*

1. *$N_K = N \otimes_{\mathbb{Z}} K$ is canonically isomorphic to K^n under an extension of ϕ.*

2. *$T_N \coloneqq N \otimes_{\mathbb{Z}} K^*$ is canonically isomorphic to $(K^*)^n$ under an extension of ϕ. Further, $Y(T_N)$ is canonically isomorphic to N.*

3. *If T is a torus then T is canonically isomorphic to $T_{Y(T)}$.*

Proof. Part 1 and the first statement of part 2 follow readily from the lemma. We shall begin by proving the second statement of part 2. Note that the generators for the algebra T_N are just injective images of the generators of N; thus every element $\gamma = (g_1, ..., g_n) \in N$ corresponds bijectively to a 1-psg $(x_1, ..., x_n) \mapsto x_1^{g_1} \cdots x_n^{g_n}$ of T_n, and this bijection behaves naturally under addition of 1-psgs. Part 3 is just a restatement of this, noting that we have seen that there is a canonical identification $T = Y(T) \otimes_{\mathbb{Z}} K^*$. ∎

1.11 Example. Let $f_1 = e_1$ and $f_2 = e_1 + e_2$ be vectors in $\mathbb{R}^2$ (with (e_1, e_2) the standard basis). Consider the lattice $N \coloneqq \mathbb{Z}f_1 + \mathbb{Z}f_2$, together with the isomorphism $\phi : N \to \mathbb{Z}^2$ given by $f_1 \mapsto e_1$, $f_2 \mapsto e_2$. Consider now the tensor product $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$; this is the module over $\mathbb{R}$ generated by $f_1 \otimes 0$, $f_2 \otimes 0$ and is isomorphic to $\mathbb{R}^2$ via the isomorphism $f_1 \otimes 0 \to e_1$, $f_2 \otimes 0 \to e_2$ (which is an extension of ϕ given that there is an embedding of N into $N_{\mathbb{R}}$ by $n \mapsto n \otimes 0$).

We now look at the torus $T_N = N \otimes_{\mathbb{Z}} \mathbb{R}^*$; consider the canonical isomorphism $(\mathbb{R}^*)^2 \to T_N$ given by the extension of ϕ, that is $(x, y) \mapsto f_1 \otimes x + f_2 \otimes y$. Suppose $n = n_1 f_1 + n_2 f_2$ and $m = m_1 f_1 + m_2 f_2$, and let $x, y \in \mathbb{R}^*$. We may therefore compute that the product law induced by ϕ on T_N is

$$\begin{aligned}
(n \otimes x)(m \otimes y) &= ((n_1 f_1 + n_2 f_2) \otimes x)((m_1 f_1 + m_2 f_2) \otimes y) \\
&= (f_1 \otimes n_1 x + f_2 \otimes n_2 x)(f_1 \otimes m_1 y + f_2 \otimes m_2 y) \\
&= \phi(n_1 x, n_2 x)\phi(m_1 y, m_2 y) \\
&= \phi(n_1 m_1 xy, n_2 m_2 xy) \\
&= f_1 \otimes n_1 m_1 xy + f_2 \otimes n_2 m_2 xy = (n_1 m_1 f_1 + n_2 m_2 f_2) \otimes xy
\end{aligned}$$

(i.e. the coordinatewise products in each N-component and in the $\mathbb{R}$-component).

A significant part of the remainder of these notes will be spent studying the relationships between:

1. Varieties which include a torus as a dense open subset with an extended group action, or equivalently morphisms $\Phi : (K^*)^r \to K^n$ ('torus embeddings') and their closures;

2. Subsets of lattices which are closed under the group operation and contain 0 (**sub-semigroups**);

3. Convex hulls and positive hulls in $\mathbb{R}^n$.

We shall be particularly interested in how various gluing operations (forming varieties by gluing affine varieties along open subsets; forming sums of sub-semigroups; gluing convex sets in $\mathbb{R}^n$) map between these three viewpoints.

2 Convexity in $\mathbb{R}^n$

In this section $\mathbb{R}^n$ is endowed with the usual topology.

2.1 Types of convex set

2.1 Definition. Let $x_1, ..., x_r \in \mathbb{R}^n$, let $\lambda_1, ..., \lambda_r \in \mathbb{R}$, and let $x = \lambda_1 x_1 + \cdots + \lambda_r x_r$.

Consider the following independent conditions:

$$\lambda_1 + \cdots + \lambda_r = 1 \tag{Aff.}$$

$$\lambda_1 \geq 0, ..., \lambda_r \geq 0 \tag{Pos.}$$

If (Aff.) is satisfied, we say x is an **affine combination** of $x_1, ..., x_r$; if (Pos.) is satisfied, we say x is a **positive combination** of $x_1, ..., x_r$; and if both (Aff.) and (Pos.) are satisfied, we say x is a **convex combination** of $x_1, ..., x_r$. For brevity, let the acronym A/P/C stand for affine/positive/convex.

If a subset of $\mathbb{R}^n$ is closed under taking A/P/C combinations, we call it A/P/C. The smallest A/P/C subset of $\mathbb{R}^n$ containing a given set S is called the A/P/C **hull** of S, denoted by $\operatorname{aff} S$/$\operatorname{pos} S$/$\operatorname{conv} S$. By convention we set $\operatorname{aff} \emptyset = \operatorname{pos} \emptyset = \operatorname{conv} \emptyset = \{0\}$.

If S is finite, we call $\operatorname{conv} S$ the **convex polytope** (usually abbreviated to **polytope** in these notes) generated by S, and we call $\operatorname{pos} S$ the **polyhedral cone** (or just **cone**) of S.

2.2 Lemma. *If S and T are (affine/positive/convex) then $S \cap T$ is (affine/positive/convex).* ∎

2.3 Lemma. *Let $S \subseteq \mathbb{R}^n$.*

1. *S is convex iff for all $x, y \in S$, the segment $[x, y] = \{\lambda x + (1 - \lambda y) : \lambda \in [0, 1]\}$ lies in S.*
2. *S is positive iff for all $x \in S$, the ray $\mathbb{R}_{\geq 0} x$ lies in S.*
3. *S is affine iff for all $x, y \in S$, the line $\overline{x, y} = \mathbb{R}(x - y) + y$ lies in S.*

Proof. Suppose S is convex. Let $x, y \in S$; then if $z \in [x, y]$ there exists $\lambda \in [0, 1]$ such that $z = (1 - \lambda)x + \lambda y$: this is a convex combination of x, y; so $z \in S$.

Suppose that for all $x, y \in S$,the segment $[x, y]$ lies in S. Let $x = \lambda_1 x_1 + \cdots + \lambda_r x_r$ be a convex combination of $x_i \in S$; we proceed by induction on r. If $r = 1$, then $\lambda_1 = 1$ so $x = \lambda_1 x_1 \in S$ trivially. If $r > 1$, set $y = (\frac{\lambda_2}{1-\lambda_1} x_2 + \cdots \frac{\lambda_r}{1-\lambda_1} x_r)$ and consider $x = \lambda_1 x_1 + (1 - \lambda_1) y$; note that $\frac{1}{1-\lambda_1}(\lambda_2 + \cdots + \lambda_r) = 1$ since $\lambda_1 + \cdots + \lambda_r = 1$ by assumption. Further $\lambda_1 \leq 1$; so $\lambda_i/(1 - \lambda_1) \geq 0$ for each $i > 1$. Hence by induction, $y \in S$; thus $x \in [y, x_1]$, and $x_1, y \in S$, so $x \in S$.

Similar arguments work for the other two cases. ∎

2.4 Example. Every open or closed ball $B_\rho(0)$ is convex (easy to see via Lemma 2.3) but not a polytope. The set $X^2 + Y^2 = Z^2$ in $\mathbb{R}^3$ is a positive set but not a polyhedral cone. Every linear space is affine; every additive coset of a linear space is affine (in general every translation or linear transformation of a convex or affine set is convex or affine, since both these operations preserve linearity).

We formalise the last statement of that example as the following lemma and theorem.

2.5 Lemma. *Let $X \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$. If f is linear, then $f(X)$ is (affine/positive/convex) if X is (affine/positive/convex). If f is a translation, then $f(X)$ is (affine/convex) if X is (affine/convex).*

Proof. Suppose f is linear. If $P = \lambda_1 f(x_1) + \cdots + \lambda_r f(x_r)$ is an (affine/positive/convex) combination of elements of $f(X)$, then by linearity we have $P = f(\lambda_1 x_1 + \cdots + \lambda_r x_r) \in f(X)$ (since $\lambda_1 x_1 + \cdots + \lambda_r x_r \in X$ by the closure of X under the type of combination of interest).

Now suppose f is a translation, say $f : x \mapsto x + v$ for fixed $v \in \mathbb{R}^n$. If $P = \lambda_1 f(x_1) + \cdots + \lambda_r f(x_r)$ is a an (affine/convex) combination of elements of $f(X)$ then we may write $P = \lambda_1 x_1 + \cdots + \lambda_r x_r + (\lambda_1 + \cdots + \lambda_r)v$; now note that $\lambda_1 + \cdots + \lambda_r = 1$ and $\lambda_1 x_1 + \cdots + \lambda_r x_r \in X$, so $P = f(\lambda_1 x_1 + \cdots + \lambda_r x_r) \in f(X)$. ∎

2.6 Theorem. *Let $S \subseteq \mathbb{R}^n$ be affine. We say that a set $T \subseteq S$ is* ***affine dependent*** *if some $x \in T$ may be written as an affine combination of elements of $T \setminus \{x\}$, and that T is* ***affine independent*** *if it is not affine dependent. If $T \subseteq S$ is affine independent and $S = \operatorname{aff} T$ we call T an* ***affine frame*** *for S.*

1. *For every affine independent set $A \subseteq S$, there exists an affine independent set B such that $S = \operatorname{aff} B$ and $A \subseteq B$.*
2. *For every affine dependent set $C \subseteq S$, there exists an affine independent set B such that $S = \operatorname{aff} B$ and $B \subseteq C$.*
3. *Let B be an affine frame for S. Then $|B| \le n + 1$, and if C is another affine frame for S then $|B| = |C|$.*

Finally, suppose $\theta \in S$ is any point, let $B = (\theta, b_2, ..., b_k)$ be an affine frame for S containing θ. Define Λ to be the linear subspace of $\mathbb{R}^n$ spanned by $(b_2 - \theta, ..., b_k - \theta)$. Then Λ is independent of the choices of θ and B, $S = \Lambda + \theta$, and $\dim \Lambda = k$. ∎

By this theorem, every affine set S is an additive coset of some unique linear subspace; we denote by $\dim S$ the dimension of this subspace, and call it the **dimension** of S. Note also, every affine frame of S will have size $\dim S + 1$. If X is any subset of $\mathbb{R}^n$, define the dimension of X to be $\dim X := \dim \operatorname{aff} X$.

We will also, later on, need some concept of the interior of a convex set (we want the interior of a convex polygon embedded in $\mathbb{R}^3$, for example, to be the interior of that polygon if it were embedded in $\mathbb{R}^2$).

2.7 Definition. The relative interior of a set $S \subseteq \mathbb{R}^n$, $\operatorname{relint} S$, is the interior of S as a subset of $\operatorname{aff} S$ with the induced topology.

Note that by Theorem 2.6 it is obvious that this is just the interior of S when it is embedded in the smallest Euclidean space containing it.

We now study topological properties.

2.8 Lemma. *Let $S \subseteq \mathbb{R}^n$. Then:*

1. $\operatorname{aff} S$ *is always closed;*
2. $\operatorname{pos} S$ *is closed if S is compact and does not contain the origin; it is bounded iff $S = \emptyset$ or $S = \{0\}$.*
3. $\operatorname{conv} S$ *is bounded iff S is bounded;* $\operatorname{conv} S$ *is compact if S is compact.*

Proof.

Affine hull The set $\operatorname{aff} S$ is homeomorphic to a linear subspace (since translations are homeomorphisms), and every linear subspace is closed in $\mathbb{R}^n$ (since every linear subspace is the nullspace of a linear map, i.e. the inverse image of $\{0\}$ under a continuous function).

Positive hull Suppose S is compact and does not contain the origin. By Lemma 2.3, $S = \bigcup_{x \in S} \mathbb{R}_{\ge 0} x$. Suppose (x_i) is a sequence of points of $\operatorname{pos} S$ converging to a point $x \in \mathbb{R}^n$. Then each $x_i = \lambda_i y_i$ for $\lambda_i \in \mathbb{R}_{\ge 0}$, $y_i \in S$. By compactness of S, there is a convergent subsequence (y_{i_j}) converging to a nonzero point $y \in S$. Now the sequence $\lambda_{i_j} y_{i_j} / \left| y_{i_j} \right|$ converges to $x/|y|$; thus (taking absolute values) λ_{i_j} converges to $|x| / |y|$; therefore

$$x = \lim_{j\to\infty} x_{i_j} \lim_{j\to\infty} \left(\lambda_{i_j} y_{i_j} \right) = \lim_{j\to\infty} \lambda_{i_j} \lim_{j\to\infty} y_{i_j} = \frac{|x|}{|y|} y$$

and since $|x| / |y| \geq 0$ we have $x \in \operatorname{pos} S$ and $\operatorname{pos} S$ is closed.

Finally note that if $x \in S$ is non-zero then $\mathbb{R}_{\geq 0} x$ is unbounded, and $\mathbb{R}_{\geq 0} x \subseteq \operatorname{pos} S$. The converse is evident.

Convex hull Suppose S is bounded, and so there exists a ball $B_\rho(0)$ such that $S \subset B_\rho(0)$. By Example 2.4, $B_\rho(0)$ is an convex set containing S, hence contains the convex hull of S, hence $\operatorname{conv} S$ is bounded. The converse is trivial since $S \subseteq \operatorname{conv} S$.

Suppose S is compact; it only remains to show that $\operatorname{conv} S$ is closed. Let (x_i) be a sequence in $\operatorname{conv} S$, converging to some $x \in \mathbb{R}^n$. For each x_i, write $x_i = \lambda_i y_i + (1-\lambda_i) z_i$ for $y_i, z_i \in S$ (since by Lemma 2.3, the convex hull is characterised by closure with respect to segments). By compactness of S, there are convergent subsequences (y_{i_j}) and (z_{i_j}) converging to points y and z of S. The sequence (λ_{i_j}) in $[0,1]$ has a convergent subsequence $(\lambda_{i_{j_k}})$ converging to a point $\lambda \in [0,1]$. Now, $(x_{i_{j_k}})$ is a subsequence of (x_i), hence converges to x. In particular, the computation

$$\begin{aligned}
x &= \lim_{k\to\infty} x_{i_{j_k}} \\
&= \lim_{k\to\infty} \left(\lambda_{i_{j_k}} y_{i_{j_k}} + (1 - \lambda_{i_{j_k}}) z_{i_{j_k}} \right) \\
&= \lim_{k\to\infty} \lambda_{i_{j_k}} \lim_{k\to\infty} y_{i_{j_k}} + (1 - \lim_{k\to\infty} \lambda_{i_{j_k}}) \lim_{k\to\infty} z_{i_{j_k}} \\
&= \lambda y + (1 - \lambda) z
\end{aligned}$$

makes sense since all the limits exist; thus x lies on a segment joining points of S; thus $x \in \operatorname{conv} S$. Hence $\operatorname{conv} S$ is closed. ∎

2.9 Example. We cannot really do any better: it is possible for $\operatorname{conv} S$ to be compact but S be non-compact (e.g. take S to be the punctured closed disc). The convex hull of a closed set is not necessarily closed: take S to be the union of the $z = 1$ plane and the origin in $\mathbb{R}^3$, for example (then $\operatorname{conv} S$ is the set of points with $0 < z \leq 1$ together with the origin, which is not closed). The same S shows that $\operatorname{pos} S$ is not necessarily closed either if S is not compact. Further, if S is the closed unit ball centred at $(0,0,1)$, then S is compact but does contain the origin, and $\operatorname{pos} S$ is not closed as it is the union of the open half-space $z > 0$ with the origin.

The important result of Lemma 2.8 is that polyhedral cones are closed, and hence their intersection with compact sets is compact. This will be important in a few cases where we reduce results about cones to results about their intersection with polytopes.

2.10 Lemma. *The interior of a convex set is convex. The closure of a convex set is convex.*

Proof. Let K be convex, and let $x, y \in \operatorname{int} K$. Let $\varepsilon > 0$ be a real number such that $B_\varepsilon(x)$ and $B_\varepsilon(y)$ both lie inside K; then the 'open tube' $S = \operatorname{conv}(B_\varepsilon(x) \cup B_\varepsilon(y))$ has the property $B_\varepsilon(z) \subseteq S$ for all $z \in [x, y]$. Indeed, suppose $z = (1-\lambda)x + \lambda y$ for $\lambda \in [0,1]$ and suppose $p \in B_\varepsilon(z)$. Then take $p^x \coloneqq (p-z)+x \in B_\varepsilon(x)$ and $p^y \coloneqq (p-z)+y \in B_\varepsilon(y)$; and $z = (1-\lambda)p^x + \lambda p^y$ so lies in $[p^x, p^y]$ hence lies inside S since $p^x, p^y \in S$ and S is convex.

To see that $\overline{K}$ is convex, suppose $\lambda \in [0,1]$ and define $f_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $f(x, y) = (1-\lambda)x + \lambda y$. This map is bilinear, hence continuous. Now note that $f(K \times K) \subseteq K$, so $f(\overline{K \times K}) \subseteq \overline{K}$ (using the closure operator definition of continuity). ∎

2.2 Analytic-algebraic results

Much of the theory we will need to develop will be algebraic in nature; however, it will often be very useful to view convex sets in both analytic and algebraic settings (indeed, much of the theory above has an analytic or topological flavour). We have two basic tools to allow us to transform between the two worlds. The first tool is the Minkowski-Weil theorem, which we do not prove; it will have the consequence (Corollary 2.15) that polyhedral cones (which we defined algebraically) may be equivalently defined analytically as being intersections of finitely many halfspaces bounded by hyperplanes through the origin; this allows us to translate notions of faces and boundary of cones between the two worlds. The second tool is the Hahn-Banach theorem, and in particular the point separation lemma (Corollary 2.17) which will allow us to translate notions of inclusion between the analytic and algebraic worlds.

To fix notation we will repeat the standard definitions.

2.11 Definition. Let $x \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$. The set $\langle x|\cdot\rangle = \alpha$ is called a **hyperplane**. The sets $H^+ \coloneqq \langle x|\cdot\rangle \geq \alpha$ and $H^- \coloneqq \langle x|\cdot\rangle \leq \alpha$ are the **halfspaces** bounded by H. If A and B are subsets of $\mathbb{R}^n$, we say that A and B are **separated** by H if $A \subseteq H^+$ and $B \subseteq H^-$ (or vice versa). If, in addition, neither A nor B intersects H then the separation is called **strict**.

Remark. Note that the partition of $\mathbb{R}^n$ as $\mathbb{R}^n = H^+ \uplus H^-$ is independent of the choice of x and α; on the other hand, the choice of partity for each halfspace is dependent on these parameters. In the case that H is associated with the face of a closed convex set, there is a canonical choice of x which we shall introduce in Definition 2.23 below.

Note that the dimension of a hyperplane in $\mathbb{R}^n$ is $n - 1$.

The Minkowski-Weil theorem

2.12 Definition. A **polyhedron** is a set of the form $\{x \in \mathbb{R}^n : \forall_i \langle x|y_i\rangle \geq \lambda_i\}$ (where $y_1, ..., y_k \in \mathbb{R}^n$ and $\lambda_1, ..., \lambda_k \in \mathbb{R}$).

We will deduce the Minkowski-Weil theorem from the following result; we will in fact use this general characterisation of polyhedra later on, in Section 7.

2.13 Theorem (Motzkin). *Every polyhedron $P \subseteq M_{\mathbb{R}}$ may be written as a sum $P = Q + C$, for Q a polytope and C a cone.*

Proof. A proof may be found as [Zie95, Theorem 1.2] (who in fact deduces it from Corollary 2.15 below). ∎

2.14 Theorem (Minkowski-Weil). *Every bounded polyhedron is a polytope. Every polytope is a polyhedron.*

Historical remark. It is actually our Corollary 2.15 that historically comes first, and so Theorem 2.14 is often known as the *affine* Minkowski-Weil theorem to distinguish it from the conic case. Minkowski proved in the late 1800s that every intersection of half-spaces through the origin is a polyhedral cone [Min10, Section19], and later proved a similar result for polyhedra [Min97, p. 210, Lehrsatz II]; Weil proved the converse in the mid 1930s [Wey35; Wey50, Theorem 1]. According to [Sch86, p. 214], it is almost certain that Farkas was independently aware of a proof of both directions of this theorem as early as 1902, having explicitly proved Minkowski's theorem in a 1896 paper based on work by Gordan, and having essentially given a proof of the converse in a 1902 paper.

Proof. Let X be a bounded polyhedron. Then by Theorem 2.13 $X = Q + C$ for Q a polytope and C a cone. But the sum of a bounded and an unbounded set is clearly unbounded; thus C must be the trivial cone $\emptyset$ (by part 2 of Lemma 2.8), and so $X = Q$. The converse is evident. ∎

2.15 Corollary. *A positive set K is a polyhedral cone iff there exist $y_1, ..., y_k \in \mathbb{R}^n$ such that $K = \{x \in \mathbb{R}^n : \forall_i \langle x|y_i\rangle \geq 0\}$.*

Proof. Let $K = \operatorname{pos}\{k_1, ..., k_r\}$. The set $P = \operatorname{conv}\{k_1, ..., k_r\}$ is a polytope. Hence by the Minkowski-Weil theorem it is of the form $P = \{x \in \mathbb{R}^n : \forall_i \langle x|y_i\rangle \geq \lambda_i\}$ for some $y_1, ..., y_k \in \mathbb{R}^n, \lambda_1, ..., \lambda_k \in \mathbb{R}$. Let $k \in K$ (writing $k = \mu_1 k_1 + \cdots + \mu_r k_r$ for $\mu_1, ..., \mu_r \in \mathbb{R}_{\geq 0}$), and suppose that for some y_i we have $\langle k|y_i\rangle < 0$. Then for some sufficiently large $\Lambda \in \mathbb{R}_{\geq 0}$ we have $\langle \Lambda k|y_i\rangle < \lambda_i$. Further, by choosing Λ even larger if necessary we may ensure that $\Lambda > 1/(\mu_1 + \cdots + \mu_r)$. Thus

$$\lambda_i > \langle \Lambda k|y_i\rangle = \Lambda(\mu_1 \langle k_1|y_i\rangle + \cdots + \mu_r \langle k_r|y_i\rangle) \geq \Lambda(\mu_1\lambda_i + \cdots + \mu_r\lambda_i) = \Lambda(\mu_1 + \cdots + \mu_r)\lambda_i > \lambda_i$$

which is a contradiction. Hence $\langle k|y_i\rangle \geq 0$ for each y_i, and so $K \subseteq \{x \in \mathbb{R}^n : \forall_i \langle x|y_i\rangle \geq 0\}$. On the other hand, suppose $\langle x|y_i\rangle \geq 0$ for each i; then for some sufficiently large $\lambda \in \mathbb{R}_{\geq 0}$ we have $\langle \lambda x|y_i\rangle \geq \lambda_i$ for each i; hence $\lambda x \in P$; thus $x = (1/\lambda)\lambda x \in \operatorname{pos} P = K$.

Conversely, suppose there exist $y_1, ..., y_k \in \mathbb{R}^n$ such that $K = \{x \in \mathbb{R}^n : \forall_i \langle x|y_i\rangle \geq 0\}$. We must show that K is finitely generated. Let Π be a polytope surrounding 0; then Π is compact by Lemma 2.8. In particular since $0 \in K$ and K is closed (it is the intersection of inverse images of a closed subset of $\mathbb{R}$ by the linear, hence continuous, maps of the form $\langle \cdot|y_i\rangle$) the set $\Pi \cap K$ is non-empty and bounded. By the Minkowski-Weil theorem, $\Pi \cap K = \operatorname{conv}\{k_1, ..., k_r\}$. Let $k \in K$. Since K is a positive set (this is trivial to check), there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that $\lambda k \in \Pi \cap K$. Hence $\lambda k = \mu_1 k_1 + \cdots + \mu_r k_r$ (where $\mu_1 + \cdots + \mu_r = 1$ and $\mu_1, ..., \mu_r \in \mathbb{R}_{\geq 0}$). Thus $k = \frac{\mu_1}{\lambda} k_1 + \cdots + \frac{\mu_r}{\lambda} k_r$ and so $k \in \operatorname{pos}\{k_1, ..., k_r\}$. On the other hand if $x \in \operatorname{pos}\{k_1, ..., k_r\}$ we may find λ such that $\lambda x \in \operatorname{conv}\{k_1, ..., k_r\}$ (indeed, take $\lambda = 1/(\mu_1 + \cdots + \mu_r)$ if $x = \mu_1 k_1 + \cdots + \mu_r k_r$). Then $\lambda x \in K$; but K is positive; so $x \in K$. ∎

The Hahn-Banach theorem

2.16 Theorem (Hahn-Banach)**.** *Let $X \subseteq \mathbb{R}^n$ be an open non-empty convex set, and let L be an affine subset of $\mathbb{R}^n$ such that $X \cap L = \emptyset$. Then there exists a hyperplane containing L which does not intersect X.*

Historical remark. The classical Hahn-Banach theorem, as proved by Banach [Ban32; Ban87, Theorem II.1.1], states roughly that given a subspace M of a vector space X, and given a linear map $f : M \to \mathbb{R}$ and a map $p : X \to \mathbb{R}$ satisfying certain properties and bounding f above on M, there is a linear extension of f to the whole of X also bounded by p. This can then be used to prove the 'analytic' separation theorem, that is that if A and B are disjoint non-empty convex sets in a topological vector space X with A open then there is a linear functional positive on A and negative on B; noting that level sets of linear functionals on a vector space are precisely hyperplanes in that space, we obtain Theorem 2.16 below; this programme is carried out in detail in [Rud73, Chapter 3].

Proof. The proof is standard, and may be found in [Ber09, Section 11.4] or [Bar02, Theorem II.1.6]. ∎

The Hahn-Banach theorem has many useful corollaries. The one which will play the biggest role for us is the following.

2.17 Corollary (Point separation lemma)**.** *Let σ be a cone, and let $x \notin \sigma$. Then there exists a point $h \in \mathbb{R}^n$ such that $\langle h|x\rangle < 0$ and $\langle h|s\rangle \geq 0$ for all $s \in \sigma$.*

Proof. The proof is of the same flavour as that of Corollary 2.15. By [Ber09, Corollary 11.4.6] (a corollary to Theorem 2.16), since $\{x\}$ is a compact set and σ is non-empty, there exists a hyperplane strictly separating x and σ, say $\langle h|\cdot\rangle = \lambda$ with $\sigma \subseteq H^+$. Since $0 \in \sigma$, we have that $\lambda < 0$ and so $\langle h|x\rangle < \lambda < 0$. Now we claim that for every $s \in \sigma$, $\langle h|s\rangle \geq 0$. Indeed, suppose for some $s \in \sigma$ we have $\langle h|s\rangle < 0$. Then for sufficiently large $\mu \in \mathbb{R}_{\geq 0}$ we have $\langle h|\mu s\rangle < \lambda$, so $\mu s \in H^-$; but $\mu s \in \sigma$ since σ is closed under positive scaling. Thus h is the desired point and the proof is complete. ∎

2.3 Faces and vertices

An important object in the study of polytopes in general is the notion of the *face lattice* of the polytope: that is, the poset supported on the set of faces, with inclusion as the partial order. When we come to the study of toric varieties, the global structure of the variety will be determined by the relationships between face lattices of the local combinatorial objects.

2.18 Definition. Let $X \subseteq \mathbb{R}^n$ be closed convex. A hyperplane H is a **supporting hyperplane** of X if $X \cap H \neq \emptyset$ and $X \subseteq H^+$ or $X \subseteq H^-$; we respectively call H^+ or H^- a **supporting halfspace** of X bounded by H.

2.19 Lemma. *If H is a hyperplane then H, H^+, and H^- are closed sets.*

Proof. Each is the inverse image under the linear (hence continuous) map $\langle x|\cdot\rangle$ of a closed subset of $\mathbb{R}$. ∎

2.20 Definition. Let $X \subseteq \mathbb{R}^n$ be closed convex. A **face** of X is an intersection $X \cap H$ where H is a supporting halfplane of X. A face of dimension 0 is a **vertex**; a face of dimension 1 is an **edge**; a face of dimension $\dim X - 1$ is a **facet**. For convenience we also call X and $\emptyset$ faces of X. We denote the set of d-dimensional faces of X by $F_d(X)$ or $X(d)$, and the set of *all* faces of X, the **face complex** or **face lattice** of X, by $F_\bullet(X)$. The **boundary** of X is the union of the proper faces of X.

2.21 Lemma. *Let $X \subseteq \mathbb{R}^n$ be closed convex.*

1. *Every face of X is closed convex.*
2. *If F and G are faces of X and $G \subseteq F$ then G is a face of F.*
3. *If F and X are faces of X then $F \cap G$ is a face of X.*

Proof. Part 1 follows from Lemma 2.2 and Lemma 2.19. Now let F, G be faces of X; write $F = X \cap H_F$, $G = X \cap H_G$ for supporting halfplanes H_F, H_G of X. Suppose H_F has equation $\langle x_F|\cdot\rangle = \alpha_F$ and H_G has equation $\langle X_G|\cdot\rangle = \alpha_G$.

If $G \subseteq F$, then H_G is a supporting halfplane of F: indeed, it is a supporting half-plane for X and so $F \subseteq X$ lies in one of the half-spaces determined by H_G, and $G \cap H_G = G$ is non-empty and a subset of F. Hence G is a face of F.

Now consider $G \cap F$. We have that $F \cap G = X \cap (H_F \cap H_G)$, and $H_F \cap H_G$ is an affine subspace of $\mathbb{R}^n$. Further, we have $H_F \cap H_G \subseteq \partial X$; in particular $(H_F \cap H_G) \cap \operatorname{int} X = \emptyset$ and so by Theorem 2.16 there exists a hyperplane P containing $H_G \cap H_G$ disjoint from $\operatorname{int} X$. Clearly this is a supporting halfplane for X and has intersection $P \cap X = F \cap G$. ∎

2.22 Corollary. *The relation $\preceq$ defined by $\tau \preceq \sigma$ iff τ is a face of σ is a partial order, and so the faces $F_\bullet(X)$ of a closed convex set form a lattice.* ∎

The structure of polytope faces is quite elegant (see [Zie95, Chapter 2]); we primarily shall limit ourselves in Section 3.2 to studying cones. However, we will need the following general definition in Section 7, and we introduce it now so that we may state some simple corollaries of our results on cones in terms of this language.

2.23 Definition. Let H be a supporting hyperplane of $X \subseteq \mathbb{R}^n$ closed convex. The **root** of the face $F \coloneqq X \cap H$ with respect to H is the unique unit vector f such that

1. f is normal to H;
2. $\mathbb{R}_{>0} f \cap X = \emptyset$.

Thus H is described by $\langle f|\cdot\rangle = \alpha$ for some $\alpha \in \mathbb{R}$, and $X \subseteq H^-$ where the parity of the half-spaces is now chosen with respect to f. For the remainder of these notes, we shall always determine parity of supporting halfspaces with respect to a root.

If F is a face of X, we say that a vector y is an **outer normal** of F if there exists a supporting hyperplane H of X such that $F = X \cap H$ with the properties

1. y is normal to H; and
2. $y + h \in H^+$ for any (hence all) $h \in H$.

In this case we also say that $-y$ is an **inner normal** of F. The set of all inner normals of F is denoted by $N_X(F)$.

Note that our convention for the normal fan parity is opposite to [Ewa96] and [Zie95]: we take $N_X(F)$ to consist of *inner* rather than *outer* normals. The reason for this is Corollary 3.26.

3 Cones and lattice semigroups

In this section, we study particular cases of polyhedral cones: those whose generators lie in a lattice. Our main result, Corollary 3.33, is that if σ is a polyhedral cone whose generators are points of a lattice M then the intersection $S = M \cap \sigma$ can be viewed as the set of exponent vectors of the algebra of regular functions of a certain class of affine varieties. To obtain it we must study the behaviour of the intersection S and the nature of its dependancy on the way σ sits inside M.

For the remainder of the section, let M, N be lattices of rank n, dual under the pairing $\langle m|n\rangle$. Let $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$.

3.1 Semigroups related to cones

3.1 Definition. A set $\sigma \subseteq M \otimes_\mathbb{Z} \mathbb{R}$ is a **lattice cone** over M if $\sigma = \mathrm{pos}\{m_1, ..., m_k\}$ for $m_1, ..., m_k \in M$. A subset of M is said to be a **minimal generating set** for σ if it is a generating set minimal under inclusion.

There are various adjectives one might associate to a lattice cone; we state them all now, but most will be useful as they will give us nice properties of associated toric varieties.

3.2 Definition. A lattice cone over M is said to be **strongly convex** if it does not contain any nontrivial vector subspace of $M_\mathbb{R}$. A strongly convex lattice cone is **smooth** (or **regular** or **non-singular**) if its minimal generators form a basis for the lattice. It is **simplicial** if its minimal generators are linearly independent over $\mathbb{R}$.

3.3 Example. In $\mathbb{R}^3$ (with lattice $\mathbb{Z}^3$), in the plane $z = 1$ pick four points p, q, r, s; then $\sigma := \mathrm{pos}\{p, q, r, s\}$ is strongly convex. Further, if p, q, r, s are distinct then $\{p, q, r, s\}$ is a minimal generating set for σ. Suppose this is the case; if the four points are not collinear, then the cone is smooth. On the other hand, if p, q, r, s are distinct then σ cannot be simplicial.

Consider the halfplane $y \geq 0$ in $\mathbb{Z}^3$. This is a lattice (polyhedral!) cone with generators $\{\pm e_1, e_2\}$, and any set of generators must include one element each on the positive and negative ($y = 0$)-rays. Thus no minimal set of generators can be linearly independent over $\mathbb{Z}$ or $\mathbb{R}$; it satisfies none of the three properties of Definition 3.2.

Finally, note that the cone $\{0\}$ is (by convention) the positive hull of $\emptyset$; hence $\{0\}$ is smooth, as $\emptyset$ may be extended to a $\mathbb{Z}$-basis for any lattice.

3.4 Lemma. *Let σ be a lattice cone over M; then $\sigma \cap M$ is closed under addition and contains zero.* ∎

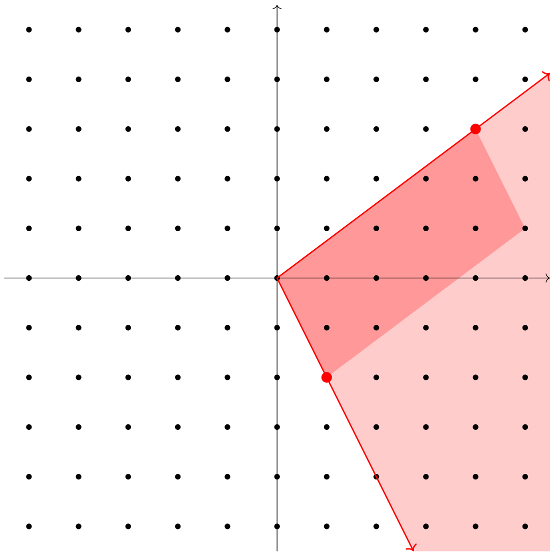


Figure 1: The construction of the finite generating set of a cone.

This provides an example of the following definition:

3.5 Definition. A **semigroup** is a pair $(S, +)$ such that $+$ is an associative, commutative binary operation on S with identity 0, and the cancellation rule $s + x = t + x \implies s = t$ holds for all $s, t, x \in S$. A semigroup S is **finitely generated** if there exist $s_1, ..., s_n \in S$ such that $S = \mathbb{Z}_{\geq 0}s_1 + \cdots + \mathbb{Z}_{\geq 0}s_n$. Such a generating set is **minimal** if none of the generators is a $\mathbb{Z}_{\geq 0}$-linear combination of the others.

3.6 Lemma (Gordan)**.** *If σ is a lattice cone over M, then $\sigma \cap M$ is a finitely generated semigroup.*

Historical remark. This result was first proved by Gordan [Gor73] (though not stated in this form — the statement due to Gordan is phrased in terms of Diophantine equations), and the uniqueness result Lemma 3.8 is due to van der Corput [Cor31].

Proof. Suppose $\sigma = \text{pos}\{x_1, ..., x_k\}$. Set

$$\Pi := \{\sum_{i=1}^{k} \alpha_i x_i \; : \; \alpha_i \in [0, 1]\}; \tag{1}$$

we will show that $\Pi \cap M$ is a generating set for $\sigma \cap M$ (Fig. 1); and this set is finite since it is the intersection of a compact set with a discrete set. The idea of the proof is that the parallellopiped Π can be used to tile σ.

Suppose $s \in \sigma \cap M$; then $s = \lambda_1 x_1 + \cdots + \lambda_k x_k$ for some $\lambda_i \in \mathbb{R}_{\geq 0}$. For each i, write $\lambda_i = \beta_i + \alpha_i$ where $\beta_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_i \in [0, 1)$. Then $s = \sum_i \beta_i x_i + \sum_i \alpha_i x_i$; since $\sum_i \beta_i x_i$ is an integral combination of the x_i, it is a lattice member. Hence $s - \sum_i \beta_i x_i = \sum_i \alpha_i x_i$ lies in the lattice; and also lies in σ since each α_i is non-negative. Further, it lies in Π by construction. ∎

This implies that there *is* a minimal generating set of $\sigma \cap M$:

3.7 Definition. A minimal generating set of a semigroup S is called an **Hilbert basis** for S.

Further, this generating set is unique under some mild conditions:

3.8 Lemma (van der Corput)**.** *If σ is strongly convex lattice cone, then $\sigma \cap M$ has a unique Hilbert basis.*

Proof. Let $\{x_1, ..., x_m\}$ and $\{y_1, ..., y_n\}$ be minimal generating sets. We will show that $x_1 \in \{y_1, ..., y_n\}$ and thus (since we can permute the x_i arbitrarily) $\{x_1, ..., x_m\} \subseteq \{y_1, ..., y_n\}$; then by minimality we must have equality.

We may write x_1 as the lattice sum of some subset of the y_i; relabelling if necessarily, suppose that

$$x_1 = \sum_{i=1}^{r} \lambda_i y_i \quad \text{and} \quad y_i = \sum_{j=1}^{m} \mu_{i,j} x_j \text{ for each } i;$$

with each $\lambda_i \in \mathbb{Z}_{>0}$ and each $\mu_{i,j} \in \mathbb{Z}_{\geq 0}$; hence

$$x_1 = \sum_{i=1}^{r} \left(\sum_{j=1}^{m} \mu_{i,j} x_j \right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{r} \lambda_i \mu_{i,j} \right) x_j.$$

Write ν_j for $\left(\sum_{i=1}^{r} \lambda_i \mu_{i,j} \right)$; so $x_1 = \sum_{j=1}^{m} \nu_j x_j$. Since ν_1 is a sum of non-negative integers and is non-zero (otherwise x_i could be written in terms of $x_2, ..., x_k$ contradicting minimality) it is at least 1. On the other hand, $\nu_1 \leq 1$: note that $x_1 - \nu_1 x_1 = \sum_{j=2}^{m} \nu_j x_j$ and the latter lies in σ, so $x_1 - \nu_1 x_1$ lies in σ and so $\mathbb{R}_{\geq 0} x_1 + \mathbb{R}_{\geq 0}(1 - \nu_1) x_1 \subseteq \sigma$; if $1 - \nu_i < 0$ then this set is $\mathbb{R} x_1$, contradicting strong convexity. Hence $\nu_1 = 1$. In particular, we have $0 = \sum_{j=2}^{m} \nu_j x_j$ and so each ν_j for $j > 1$, being a sum of non-negative integers, must be zero. Since each λ_i is non-zero and all the coefficients are non-negative, this implies that $\mu_{i,j} = 0$ for all i and all $j > 1$. In particular, we have

$$y_1 = \sum_{j=1}^{m} \mu_{1,j} x_j = \mu_{1,1} x_1$$

and since $1 = \nu_1 = \sum_{i=1}^{r} \lambda_i \mu_{i,1}$, the λ_i are all non-zero, and $\mu_{1,1} > 0$ since $y_1 \neq 0$, we must have $\mu_{1,1} = 1$; so $y_1 = x_1$ and $x_1 \in \{y_1, ..., y_n\}$ as desired. ∎

3.9 Lemma. *Let S be a semigroup, and let $R = \{x + y \in S : x \neq 0, y \neq 0\}$. Then $S \setminus R$, the set of **irreducible elements**, is a Hilbert basis for S.* ∎

3.10 Example. Again consider the half-plane $y \geq 0$; then every set of the form $\{e_1, e_2, -\lambda e_1\}$ where $\lambda \in \mathbb{N}$ is a distinct minimal generating set.

Finally, note that Gordan's lemma tells us that the intersection of a lattice cone $\sigma = \text{pos}\{s_1, ..., s_k\}$ and the lattice M is a finitely generated sub-semigroup of M. We have a similar property when it comes to the intersection of σ and the M-rational points of the ambient vector space; here we don't need to be clever to find a generating set.

The following definition will be useful in the proof and in similar situations.

3.11 Definition. Suppose $n \in N$; then $\sqrt{n} := \lambda^{-1} n$ where λ is the largest integer such that $\lambda^{-1} n \in N$. This quantity is well-defined as N is discrete. We call $\sqrt{n}$ the **radical** of n, and we call the integer λ the **lattice length** of n. A point is **primitive** if it has lattice length 1.

3.12 Example. In $\mathbb{Z}^2$, the point $e_1 + e_2$ is primitive; on the other hand, $4e_1 + 2e_2$ has lattice length 2 so is not primitive.

3.13 Lemma (Lattice rationality)**.** *If $\sigma = \text{pos}\{s_1, ..., s_k\}$ is a lattice cone over M, then*

$$\sigma \cap M_{\mathbb{Q}} = \mathbb{Q}_{\geq 0} s_1 + \cdots + \mathbb{Q}_{\geq 0} s_k.$$

Proof. The inclusion $\sigma \cap M_{\mathbb{Q}} \supseteq \sum_{i=1}^{k} \mathbb{Q}_{\geq 0} s_i$ is straightforward, so we only prove the converse. Suppose $x \in \sigma \cap N_{\mathbb{Q}}$; then there exists $A \in \mathbb{Z}_{\geq 0}$ such that $Ax \in \sigma \cap N_{\mathbb{Q}}$. By the proof of Gordan's lemma (Lemma 3.6), $\sigma \cap N$ is generated as a semigroup by the lattice points contained within the closed parallellopiped

$$\Pi := \{ \sum_{i=1}^{k} \alpha_i x_i \, : \, \alpha_i \in [0, 1]\}; \tag{2}$$

each such element is a sum $\sum_{i=1}^{k} \beta_i \sqrt{s_i}$, where $(\beta_1, ..., \beta_k) \in [0, 1]^k$.

We may therefore write $Ax = \sum_{i=1}^{l} \mu_i \pi_i$ for $\mu_1, ..., \mu_l \in \mathbb{Z}_{\geq 0}$, where each π_i is of the form $\pi_i = \sum_{j=1}^{k} \beta_{i,j} \sqrt{s_j}$ and $\pi_1, ..., \pi_l$ are precisely the set of lattice points of N in Π. Hence,

$$Ax = \sum_{i=1}^{l} \mu_i \left(\sum_{j=1}^{k} \beta_{i,j} \lambda_j^{-1} s_j \right)$$

where μ_i, $\beta_{i,j}$, and λ_j are all non-negative integers for all i, j. In particular, exchanging the summations, we may write $Ax = \sum_{i=1}^{k} \gamma_i s_i$ where each γ_i is a non-negative rational number; thus we have that $x = A^{-1} \sum_{i=1}^{k} \gamma_i s_i$ lies in $\sum_{i=1}^{k} \mathbb{Q}_{\geq 0} s_i$. ∎

3.2 Cones and the duality pairing

To every cone σ (not necessarily even a lattice cone) in $N_{\mathbb{R}}$, we may naturally assign a cone $\sigma^{\vee}$ in $M_{\mathbb{R}}$.

3.14 Lemma. *If σ is a cone in $N_{\mathbb{R}}$, let*

$$\sigma^{\vee} := \{m \in M_{\mathbb{R}} \, : \, \forall_{s \in \sigma} \langle m | s \rangle \geq 0\}.$$

*We call this subset of $M_{\mathbb{R}}$ the **dual cone** of σ. Indeed:*

1. *$\{0\}^{\vee} = M_{\mathbb{R}}$;*
2. *if $\sigma = \mathrm{pos}\{s_1, ..., s_r\}$, then $\sigma^{\vee} = \{m \in M_{\mathbb{R}} \, : \, \forall_i \langle m | s_i \rangle \geq 0\}$;*
3. *$\sigma^{\vee}$ is a cone;*
4. *$(\sigma^{\vee})^{\vee} = \sigma$;*
5. *if τ is a cone such that $\tau \subseteq \sigma$, then $\sigma^{\vee} \subseteq \tau^{\vee}$ (but in general the face relation is not reversed, see e.g. Fig. 3 on Page 30; there is a more complex face-relation-reversing correspondence between faces of σ and those of $\sigma^{\vee}$, see Corollary 3.21);*
6. *$(-\sigma)^{\vee} = -(\sigma^{\vee})$;*
7. *$\sigma^{\perp} = \sigma^{\vee} \cap (-\sigma^{\vee}) \subseteq \sigma^{\vee}$, with equality iff σ is a linear subspace;*
8. *for any pair σ_1, σ_2 of cones,*
   (a) $(\sigma_1 + \sigma_2)^{\vee} = \sigma_1^{\vee} \cap \sigma_2^{\vee}$ and
   (b) $(\sigma_1 \cap \sigma_2)^{\vee} = \sigma_1^{\vee} + \sigma_2^{\vee}$.

Proof.

1. Trivial.

2. Indeed, $y \in \sigma^\vee$ implies $\langle y|s_i\rangle \geq 0$ for all i; conversely, if $\langle y|s_i\rangle \geq 0$ for each i, then

$$\langle y|\lambda_1 s_1 + \cdots + \lambda_m s_m\rangle = \lambda_1 \langle y|s_1\rangle + \cdots + \lambda_m \langle y|s_m\rangle \geq 0$$

for all $\lambda_1, ..., \lambda_m \in \mathbb{R}_{\geq 0}$.

3. Follows from part 2 and Corollary 2.15.

4. Despite the claim of [Ewa96] this is actually nontrivial (c.f. [Ful93, Chap. 1, endnote 5]); it depends on one of the consequences of Theorem 2.16. For all $n \in \sigma$ and for all $m \in \sigma^\vee$, we have $\langle m|n\rangle \geq 0$. This implies that $n \in (\sigma^\vee)^\vee$. Conversely, we must show that $\langle m|n\rangle \geq 0$ for all $m \in \sigma^\vee$ *only if* $n \in \sigma$; but this follows directly from Corollary 2.17 since if $n \notin \sigma$ there exists $m \in M_\mathbb{R}$ such that $m \in \sigma^\vee$ and $\langle m|n\rangle < 0$.

5. Let $\tau \subseteq \sigma$; if $x \in \sigma^\vee$ then $\langle x|s\rangle \geq 0$ for all $s \in \sigma$; hence $\langle x|t\rangle \geq 0$ for all $t \in \tau$; hence $x \in \sigma^\vee$.

6. Trivial.

7. The statement $\sigma^\perp = \sigma^\vee \cap (-\sigma^\vee) \subseteq \sigma^\vee$ is trivial. Note that all positive sets are closed under vector sum (since sums of positive coefficients are positive). Thus it will suffice to show that equality holds iff $\sigma = -\sigma$. Suppose equality holds; then $\sigma^\vee = (-\sigma^\vee)$; hence $\sigma = (-\sigma^\vee)^\vee$; hence by parts 4 and 6 $\sigma = -(\sigma^\vee)^\vee = -\sigma$. Conversely, if $\sigma = -\sigma$ then $\sigma^\vee = -\sigma^\vee$ by part 6 and so equality holds.

8. (a) Let $m \in \sigma_1^\vee \cap \sigma_2^\vee$. Then $\langle m|s_1\rangle \geq 0$ and $\langle m|s_2\rangle \geq 0$ for all $s_1 \in \sigma_1$, $s_2 \in \sigma_2$. Hence $\langle m|s_1 + s_2\rangle = \langle m|s_1\rangle + \langle m|s_2\rangle \geq 0$ and $m \in (\sigma_1 + \sigma_2)^\vee$. Conversely, suppose $x \notin \sigma_1^\vee \cap \sigma_2^\vee$. Without loss of generality assume $x \notin \sigma_1^\vee$; then by Corollary 2.17 and part 4 there exists $n \in N_\mathbb{R}$ such that $n \in \sigma_1$ and $\langle x|n\rangle < 0$. But $n \in \sigma_1 + \sigma_2$; so $n \notin (\sigma_1 + \sigma_2)^\vee$.
   (b) By parts 4 and 8a we have $(\sigma_1 \cap \sigma_2)^\vee = (\sigma_1^\vee + \sigma_2^\vee)^{\vee\vee} = \sigma_1^\vee + \sigma_2^\vee$. ∎

3.15 Example. Let $e_1, ..., e_n$ be the standard basis of $\mathbb{Z}^n$, let $\mathbb{Z}^n$ be identified with its own dual lattice via taking transposes, and let $\sigma = \text{pos}\{e_1, ..., e_n\}$. Then $\sigma^\vee = \sigma$: indeed, suppose $m \in \sigma$; then for $s \in \sigma$, $\langle m|s\rangle$ is a sum of products of non-negative numbers, so $\langle m|s\rangle \geq 0$. Hence $\sigma \subseteq \sigma^\vee$. The same argument shows that $\sigma^\vee \subseteq \sigma$.

3.16 Example. If $v \in \mathbb{Z}^n$ is any point, then $\text{pos}\{v\}^\vee$ is the halfspace bounded by $v^\perp$ containing v. Indeed, $v^\perp$ is the hyperplane $\langle v|\cdot\rangle = 0$; and so $\text{pos}\{v\}^\vee = (v^\perp)^+$ which is the halfspace containing v.

3.17 Example. Let $f_1 = e_1$ and $f_2 = e_1 + e_2$ (e_1 and e_2 the standard basis of $\mathbb{R}^2$), and define $\sigma = \mathbb{Z}_{\geq 0} f_1 + \mathbb{Z}_{\geq 0} f_2$. Then $x = \lambda e_1 + \mu e_2 \in \sigma^\vee$ if and only iff $\langle x|s\rangle \geq 0$ for all $s \in S$. That is, for all $\alpha \geq 0$ and $\beta \geq 0$:

$$\begin{aligned}
0 \leq \langle x|s\rangle &= \langle \lambda e_1 + \mu e_2 | \alpha f_1 + \beta f_2\rangle \\
&= \langle \lambda e_1 + \mu e_2 | (\alpha + \beta) e_1 + \beta e_2\rangle \\
&= \lambda(\alpha + \beta)\langle e_1|e_1\rangle + (\mu(\alpha + \beta) + \beta\lambda)\langle e_1|e_2\rangle + \mu\beta\langle e_2|e_2\rangle \\
&= \lambda(\alpha + \beta) + \mu\beta.
\end{aligned}$$

Hence if $\lambda e_1 + \mu e_2 \in \sigma^\vee$ then $(\lambda + \mu)\beta \geq -\lambda\alpha$ for all $\alpha, \beta \in \mathbb{R}_{\geq 0}$; in particular for $\alpha = 0$ and $\beta = 1$ we obtain $(\lambda + \mu) \geq 0$. Similarly we obtain $\lambda \geq 0$. It is immediate that if these two inequalities hold then $\lambda e_1 + \mu e_2 \in \sigma^\vee$, and so the dual cone is the set of all point satisfying both inequalities simultaneously; this set is $\text{pos}\{e_2, e_1 - e_2\}$ (Fig. 2).

We will show that if σ is a lattice cone then $\sigma^\vee$ is a lattice cone. We will need some machinery to do so, primarily involving the relationship between the face lattice of a cone and the dual cone.

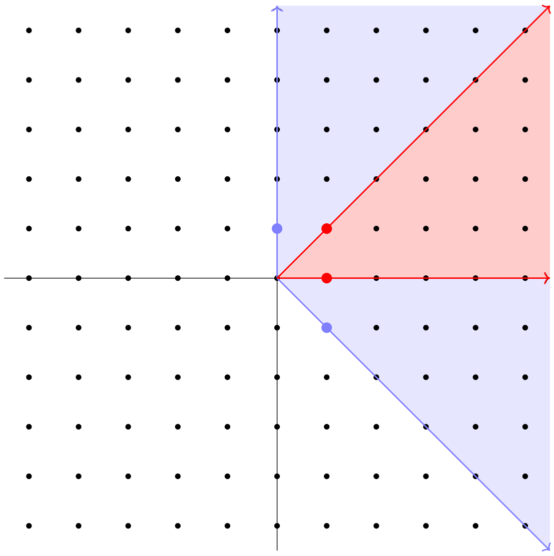


Figure 2: The cone $\sigma = \operatorname{pos}\{e_1, e_1 + e_2\}$ (red area) and its dual cone (combined blue and red area).

3.18 Definition. If σ is a cone and $m \in \sigma^\vee$, define the set

$$F_\sigma(m) := m^\perp \cap \sigma = \{s \in \sigma : \langle m|s\rangle = 0\}$$

called the **face of σ determined by** m.

3.19 Lemma. *A cone σ has only finitely many faces, each of which is a cone, and which inherit the property of being strongly convex if σ is strongly convex. Further, the following are equivalent:*

1. *τ is a face of σ;*
2. *$\tau = F_\sigma(m)$ for some $m \in \sigma^\vee$;*
3. *if $\sigma^\vee = \operatorname{pos} M$, then there is a finite subset $M' \subseteq M$ such that $\tau = \bigcap_{m \in M'} F_\sigma(m)$.*

Proof. Note that from part 3 we obtain that the cone has only finitely many faces, since a finite set has only finitely many subsets. Suppose $F_\sigma(m)$ is the face determined by $f_1, ..., f_\ell \in \sigma^\vee$. Then we may write

$$\begin{aligned} F_\sigma(m) &= \{s \in \sigma : \forall_{1\le i\le \ell} \langle f_i|s\rangle \le 0 \text{ and } \langle f_i|s\rangle \ge 0\} \\ &= \left\{ s \in \mathbb{R}^n : \left(\forall_{1\le i\le \ell} \langle f_i|s\rangle \le 0 \text{ and } \langle f_i|s\rangle \ge 0\right) \text{ and } \forall_{1\le j\le r} \langle k_i|s\rangle \ge 0 \right\} \end{aligned}$$

where the inequalities $\langle k_i|s\rangle \ge 0$ are those which define σ by one direction of Corollary 2.15; the other direction of the same result then tells us that $F_\sigma(m)$ is also a (finitely generated polyhedral) cone. Finally strong convexity of the faces is trivial if σ is strongly convex since if a face includes a line then σ includes it.

We now prove the equivalent definitions of a face.

1 $\Longrightarrow$ 2 Let H be a supporting halfplane for σ, say $\langle h|\cdot\rangle = \lambda$ where h is chosen so that $\sigma \subseteq H^+$. First note that $\lambda = 0$; for if $\lambda > 0$ then $\langle h|s\rangle > 0$ for all $s \in \sigma$, which is a contradiction since $0 \in \sigma$. Further note that this implies that $h \in \sigma^\vee$; so every face of σ is of the form $F_\sigma(h)$.

2 $\Longrightarrow$ 1 Conversely, if $m \in \sigma^\vee$ then for all $s \in \sigma$, $\langle m|s\rangle \geq 0$ and so $\sigma \subseteq (F_\sigma(m))^+$; since $0 \in \sigma \cap F_\sigma(m)$ the intersection is non-trivial and so $F_\sigma(m)$ is a face of σ.

2 $\Longleftrightarrow$ 3 Suppose $\sigma^\vee = \text{pos}\{m_1, ..., m_k\}$; then for $\lambda_1, ..., \lambda_k \in \mathbb{R}_{\geq 0}$ we have $F_\sigma(\lambda_1 m_1 + \cdots + \lambda_k m_k) = \{s \in \sigma : \sum_{i=1}^k \lambda_i \langle m_i|s\rangle = 0\}$; but $\langle m_i|s\rangle \geq 0$ and $\lambda_i \geq 0$ for all i. Let M' be the set of m_i such that $\lambda_i \neq 0$, so

$$F_\sigma(\lambda_1 m_1 + \cdots + \lambda_k m_k) = \{s \in \sigma : \forall_{1\leq i\leq k} \lambda_i \langle m_i|s\rangle = 0\} = \bigcap_{m\in M'} F_\sigma(m);$$

conversely every such intersection is a face since intersections of faces are faces. Hence every face is determined uniquely by the choice of some subset of the m_i (and setting $\lambda_i = 0$ for all other i). ∎

This previous lemma, and the following corollary, will often be used without reference as they simply tell us that our intuitive picture of the faces of a polyhedral cone is indeed correct.

3.20 Corollary. *Let $B = \{b_1, ..., b_k\}$. If $\sigma = \text{pos}\, B$ and $\tau \preceq \sigma$ then $\tau = \text{pos}\, B'$ for some (possibly empty) subset $B' \subseteq B$.*

Proof. If $\tau = 0$ the result is trivial, so suppose $\tau \neq 0$. Let $\tau = \text{pos}\{c_1, ..., c_m\}$ for $c_i \in \sigma \setminus 0$ be the face determined by the hyperplane H with equation $\langle m|\cdot\rangle \geq 0$. I claim that one of the b_i lives in H; indeed suppose not, so $\langle m|b_i\rangle > 0$ for each i. Let $t \in \tau$ be nonzero; so $t = \sum_{i=1}^k \lambda_i b_i$ where $\lambda_i \in \mathbb{R}_{\geq 0}$ for each i, and at least one of the $\lambda_i > 0$. In particular $\langle m|t\rangle = \lambda_i \sum_{i=1}^k \langle m|b_i\rangle > 0$ which is a contradiction since $t \in \tau \subseteq H$. Thus $b_i \in H$ for some i; let $B' = B \cap H$. Clearly then pos $B' \subseteq \tau$. Conversely, for each j we may write

$$c_j = \sum_{b\in B'} \lambda_b b + \sum_{b\in B\setminus B'} \lambda_b b.$$

From this we see that $\sum_{b\in B\setminus B'} \lambda_b b$ lies in H (since H is a hyperplane through 0, i.e. a vector subspace). But this implies

$$0 = \langle m| \sum_{b\in B\setminus B'} \lambda_b b\rangle = \sum_{b\in B\setminus B'} \lambda_b \langle m|b\rangle.$$

Since each $\langle m|b\rangle > 0$ for $b \in \sigma \setminus H$, we must have $\lambda_b = 0$ for each $b \in B \setminus B'$. In particular, each c_j is a positive combination of the members of B' and so $\tau = \text{pos}\{c_1, ..., c_m\} \subseteq \text{pos}\, B'$. ∎

Remark. The converse is not true unless B is minimal: take the cone $\text{pos}\{e_1, e_2, e_1 + e_2\} \subseteq \mathbb{R}^2$, then $\text{pos}\{e_1 + e_2\}$ is clearly not a face! Note also that $F_\sigma(m)$ for any $m \in \sigma^\perp$ is the trivial face σ.

3.21 Corollary. *Let $\sigma \subseteq N_\mathbb{R}$ be a lattice cone.*

1. *Let $s \in \sigma$. Then $s \in \text{relint}\,\sigma$ if and only if $\sigma^\vee \cap s^\perp = \sigma^\perp$.*
2. *The map $\tau \mapsto \sigma^\vee \cap \tau^\perp$ is an order reversing bijection between $F(\sigma)$ and $F(\sigma^\vee)$.*

Proof.

1. Suppose $s \in \text{relint}\,\sigma$. It is trivial that $\sigma^\perp \subseteq \sigma^\vee \cap s^\perp$; we prove the opposite inclusion. Since s does not lie in any nontrivial face of σ, by Lemma 3.19 $\langle m|s\rangle > 0$ for all $m \in \sigma^\vee \setminus \sigma^\perp$; therefore, if $\langle m|s\rangle = 0$ for some $m \in \sigma^\vee$, it must be the case that $m \in \sigma^\perp$. Suppose $m \in \sigma^\vee \cap s^\perp \setminus \sigma^\perp$; then $\langle m|s\rangle = 0$, so $m \in \sigma^\perp$.

   Conversely, suppose $\sigma^\vee \cap s^\perp = \sigma^\perp$; if $\langle m|s\rangle = 0$ for some $m \in \sigma^\vee$, then $m \in \sigma^\perp$. Thus if $s \in F_\sigma(m)$, $F_\sigma(m) = \sigma$; i.e. the only face s lies in is the trivial face σ.

2. Clearly $\sigma^\vee \cap \tau^\perp = \cap_{t\in\tau} F_{\sigma^\vee}(t)$, so the map defined does send faces to faces. Let $F_{\sigma^\vee}(t)$ be a face of $\sigma^\vee$; then $t \in \operatorname{relint} \tau$ for some $\tau \preceq \sigma$. By part (1), $\tau^\vee \cap t^\perp = \tau^\perp$. Thus

$$\sigma^\vee \cap \tau^\perp = \sigma^\vee \cap \tau^\vee \cap t^\perp = \sigma^\vee \cap t^\perp$$

so any map sending faces $F_\sigma(t) \in \sigma^\vee$ to some $\tau \preceq \sigma$ with $t \in \operatorname{relint} \tau$ is an inverse for the stated map; thus such a map is unique and we have a bijection.

∎

Some of the techniques in the proof of Lemma 3.19 may be used to form a more concrete view of the dual cone. More precisely, Proposition 3.22 will tell us how to compute the dual cone of a full-dimensional cone, and then Lemma 3.27 will tell us how to iteratively compute the dual cones of faces of larger cones.

3.22 Proposition. *Let $\sigma = \operatorname{pos}\{a_1, ..., a_k\}$ be an n-dimensional cone in $N_\mathbb{R}$; let $b_1, ..., b_r$ be the inner normals of the facets of σ. Then $\sigma^\vee$ is the positive hull of $b_1, ..., b_r$.*

Proof. For each i, let H_i be the hyperplane defined by $\langle \cdot | b_i \rangle = 0$; in particular, since H_i supports σ and b_i lies on the same side of H_i as σ we have $\langle s | b_i \rangle \geq 0$ for all $s \in \sigma$. Hence $b_i \in \sigma^\vee$ for all i and so $\operatorname{pos}\{b_1, ..., b_r\} \subseteq \sigma^\vee$ since $\sigma^\vee$ is a cone.

Conversely, suppose $x \in \sigma^\vee \setminus \operatorname{pos}\{b_1, ..., b_r\}$. Because σ is full-dimensional, $\{b_1, ..., b_r\}$ spans $\mathbb{R}^n$ and in particular we may write $x = \alpha_1 b_{i_1} + \cdots + \alpha_k b_{i_k}$ where $k \leq n$ and each α_j is nonzero. By assumption, one of the α_j must be negative; assume without loss of generality that $\alpha_1 < 0$. Since $\{a_1, ..., a_k\}$ spans $\mathbb{R}^n$ by assumption, there must be some a_ℓ lying in the orthogonal complement of the space spanned by $\{b_{i_2}, ..., b_{i_k}\}$ since the latter is at most $(n-1)$-dimensional. In particular $\langle a_\ell | b_{i_j} \rangle = 0$ for all $j \geq 2$, and so $\langle a_\ell | x \rangle = \alpha_1 \langle a_\ell | b_{i_1} \rangle$; by the first paragraph we have $\langle a_\ell | b_{i_1} \rangle > 0$ and so $\langle a_\ell | x \rangle < 0$ since $\alpha_1 < 0$. But this contradicts the assumption that $x \in \sigma^\vee$. ∎

We may immediately deduce the following useful corollary.

3.23 Corollary. *A cone σ is strongly convex if and only if $\sigma^\vee$ is full dimensional.*

Proof. Suppose first that σ is strongly convex. Note that $\sigma^\perp \subseteq \sigma^\vee$. It follows that we need only consider the case where σ is full dimensional in $N_\mathbb{R}$, as the complement always contributes fully to the dimension of the dual cone. If σ is full dimensional in $N_\mathbb{R}$, then by Proposition 3.22 we have that $\sigma^\vee$ is spanned by the set of all normal vectors to the facets of σ; but the facets of σ are spanned by a set which spans $N_\mathbb{R}$; hence the set of normal vectors is spanning in $M_\mathbb{R}$.

Suppose conversely that σ is not strongly convex. Then there exists a nonzero linear subspace $\Lambda \subseteq \sigma$. Now note that $\Lambda^\perp = \Lambda^\vee \supseteq \sigma^\vee$ (the first equality from part 7 of Lemma 3.14); but $\Lambda^\perp$ is a proper subspace of $M_\mathbb{R}$ so $\sigma^\vee$ cannot be full dimensional. ∎

3.24 Example. Note that the cone $\sigma \subseteq \mathbb{R}^2$ of Example 3.17 is strongly convex, and its dual is 2-dimensional. On the other hand, the cone consisting of the single ray $\operatorname{pos}\{v\}$ (for some $v \in \mathbb{R}^n$ for $n \geq 2$) of Example 3.16 is not full-dimensional and its dual cone is not strongly convex (it contains the space $\{v\}^\perp$ which is $(n-1)$-dimensional and hence is a nontrivial subspace).

As another application of Proposition 3.22, we may deduce:

3.25 Corollary. *If a full-dimensional lattice cone σ over M is smooth (resp. simplicial) then $\sigma^\vee$ is smooth (resp. simplicial).*

Proof. Let σ be simplicial, with $\mathbb{R}$-linearly independent generating set $\{x_1, ..., x_n\}$. Then there is an invertible linear transformation $A : M_{\mathbb{R}} \to \mathbb{R}^n$ sending $x_i \mapsto e_i$ for each i (the e_i being the usual basis for $\mathbb{R}^n$). We also consider the adjoint transformation $A' : N_{\mathbb{R}} \to \mathbb{R}^n$.

By Proposition 3.22, the dual cone $\sigma^\vee$ is generated by the inner normals of σ. Further, $(A\sigma)^\vee = A'\sigma^\vee$ by the properties of the adjoint. Now $A\sigma = \mathrm{pos}\{e_1, ..., e_n\}$ has dual $(A\sigma)^\vee = \mathrm{pos}\{e_1, ..., e_n\}$; the given generating set of the latter is linearly independent, so $\sigma^\vee = \mathrm{pos}\{(A')^{-1}e_1, ..., (A')^{-1}e_n\}$ also has a linearly independent generating set.

Clearly the above proof also goes through in the case where σ is smooth: the relevant observation needed is that, in this case, the matrix of A must have integer entries. ∎

Finally the following corollary will also be useful in Section 7:

3.26 Corollary. *If P is a full-dimensional polyhedron, and $x \in P$ is a vertex, the normal cone $N_P(x)$ is the dual cone of* $\mathrm{pos}(P - x)$*. (In particular, the normal cone is a cone.) We have an inclusion-reversing bijection between the faces F of P containing x and the faces of $N_P(x)$, given by $F \mapsto N_P(F)$.*

Proof. The statement $N_P(x) = \mathrm{pos}(P - x)^\vee$ follows directly from Proposition 3.22. The face correspondence follows from Corollary 3.21, noting that faces of P containing x correspond exactly to faces of $\mathrm{pos}(P - x)$ by the Minkowski-Weil theorem. ∎

3.27 Lemma. *If $\tau \preceq \sigma$, let $m \in \sigma^\vee$ such that $\tau = F_\sigma(m)$. Then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m)$.*

Proof. It suffices to show that $(\tau^\vee)^\vee = (\sigma^\vee + \mathbb{R}_{\geq 0}(-m))^\vee$; but $(\tau^\vee)^\vee = \tau$, and $(\sigma^\vee + \mathbb{R}_{\geq 0}(-m))^\vee = \sigma \cap (\mathbb{R}_{\geq 0}(-m))^\vee$. Now note that $n \in (\mathbb{R}_{\geq 0}(-m))^\vee$ iff $\langle n | -m\rangle \geq 0$, i.e. $\langle n|m\rangle \leq 0$; but $m \in \sigma^\vee$, so $\langle n|m\rangle \geq 0$ for all $n \in \sigma$; hence if $n \in \sigma \cap (\mathbb{R}_{\geq 0}(-m))^\vee$ iff $n \in m^\perp$, and thus $\sigma \cap (\mathbb{R}_{\geq 0}(-m))^\vee = \sigma \cap m^\perp = F_\sigma(m) = \tau$. ∎

3.28 Lemma (Farkas' theorem)**.** *If σ is a lattice cone over N, then $\sigma^\vee$ is a lattice cone over M.*

Proof. If σ is full dimensional, then by applying Proposition 3.22 we are done. Suppose then that $V \coloneqq \operatorname{span} \sigma$ is a proper subspace of $N_{\mathbb{R}}$. We may write $M_{\mathbb{R}} = V^\vee \oplus V^\perp$ ($V^\vee$ denoting the usual dual space); if $x \in M_{\mathbb{R}}$ we may therefore decompose x as $x_{V^\vee} + x_{V^\perp}$. Hence (noting that s is orthogonal to anything in $V^\perp$):

$$\begin{aligned} x \in \sigma^\vee &\iff \forall_{s\in\sigma}\, \langle s|x\rangle \geq 0 \\ &\iff \forall_{s\in\sigma}\, \langle s|x_{V^\vee}\rangle + \langle s|x_{V^\perp}\rangle \geq 0 \\ &\iff \forall_{s\in\sigma}\, \langle s|x_{V^\vee}\rangle \geq 0 \end{aligned}$$

i.e. $x_{V^\vee}$ lies in the dual of σ in $M_{\mathbb{R}}/V^\perp$; this reduces the problem to a full-dimensional problem, and we may take the facet normals of σ in $M_{\mathbb{R}}/V^\perp$ and lift those into $M_{\mathbb{R}}$, say as $\{b_1, ..., b_k\}$. Then the dual of σ is the cone generated by $\{b_1, ..., b_k\}$ and $\{\pm e_1, ..., \pm e_r\}$ where $\{e_1, ..., e_r\}$ is a basis for $V^\perp$. ∎

3.29 Definition. Let S be a semigroup. If K is a field, define the **semigroup algebra** $K[S]$ to be the algebra generated over K by the elements $\{X^\alpha : \alpha \in S\}$, modulo the relations $X^0 = 1$ and $X^\alpha X^\beta = X^{\alpha+\beta}$. If σ is a lattice cone over N, let $S_\sigma \coloneqq \sigma^\vee \cap M$ and let $A_\sigma = K[S_\sigma]$. If $\sigma^\vee$ is strongly convex, then $\sigma^\vee \cap M$ has a unique Hilbert basis (by Lemma 3.8); we denote this basis by H_σ.

In other words, we are interested in algebras whose generating sets form a semigroup which can be embedded into a lattice. This will correspond to varieties whose coordinate functions are generated by the extension of a lattice of functions to the entire variety. The following remarks follow directly from the definitions, but it will be convenient to mark them as a lemma so we may refer back to them later.

3.30 Lemma. *If S is a finitely generated semigroup, then $K[S]$ is finitely generated as an algebra over K. Conversely, if $K[S]$ is an algebra generated by a semigroup S and $K[S]$ is finitely generated as an algebra over K then S is finitely generated.*

If S is generated by $\{\beta_1, ..., \beta_k\}$, then $K[S]$ is the quotient of the free algebra over K with generators $\{X^{\beta_1}, ..., X^{\beta_k}\}$ by an ideal $\mathfrak{b}$ finitely generated by elements of the form $m - n$ for monomials m and n in the X^{β_i} (that is, $\mathfrak{b}$ is a ***binomial ideal****).* ∎

3.31 Example. $K[\mathbb{Z}^n] = K[\pm e_1, ..., \pm e_n]$. Hence $\mathrm{Spec}(K[\mathbb{Z}^n]) = (K^*)^n$.

3.32 Proposition. *If σ is a lattice cone over N, then A_σ is a finitely generated integral algebra over K.*

Proof. By Lemma 3.28, $\sigma^\vee$ is a lattice cone over M. By Lemma 3.6, $\sigma^\vee \cap M$ is finitely generared semigroup. Hence by Lemma 3.30 we are done. ∎

Our main result for this section is now immediate.

3.33 Corollary. *If S is a finitely generated semigroup, then* Spec $K[S]$ *is an affine variety cut out by binomials. Thus if σ is a lattice cone over N, then* Spec A_σ *is an affine variety cut out by binomials.* ∎

4 Affine toric varieties

Our strategy to prove general statements about toric varieties will be to perform as much work as possible locally (in affine toric varieties), and then 'glue' the pieces together in a manner which respects the toric structure. Thus, we must spend some time studying this local picture. This section is the most technically challenging of this dissertation; we shall use freely the results of earlier sections, as well as algebraic geometry (primarily the theories of normal and smooth varieties; all the results on these that we use, along with many further examples, may be found in [Sha13, Chapter 2]). We also make a convention for the remainder of our discussion that all cones associated with a vector space arising from a lattice will be implicitly lattice cones.

The primary results of this section are as follows:

- The closed points of a variety arising from a semigroup S are in natural correspondence with the group of semigroup morphisms $S \to K$ (Lemma 4.3). This correspondence also detects toric structure (Lemma 4.8).
- Lattice cones determine a toric variety (Corollary 4.11). The faces of the cone determine certain principal open subvarieties of this toric variety in a manner corresponding to the face lattice (Theorem 4.25), and in fact there is a bijection between toric principal open subvarieties and faces of the original cone (Corollary 4.45).
- We obtain a bijective correspondence (up to isomorphism) between *strongly convex* lattice cones, and *normal* affine toric varieties (Theorem 4.21). The tori of such varieties arise naturally from the lattice the cones are defined over (Proposition 4.20). These results may be viewed as a toric Nullstellensatz.
- Affine toric varieties are smooth if and only if they arise from a smooth cone (Theorem 4.30).
- The toric action on a normal affine toric variety has a unique fixed point (Theorem 4.31). The local properties of this fixed point determine combinatorial properties of the related cone; in particular, the dimension of the tangent space is precisely the size of a Hilbert basis for the semigroup of the cone (Lemma 4.33).

- As a generalisation of the fixed point result, *all* orbits of the torus on an affine toric variety may be classified and associated with a face of the associated cone (Theorem 4.38); the closures of these orbits are also classified (Corollary 4.44).

Let M, N be lattices of rank n, dual under the pairing $\langle m|n\rangle$. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let K be an algebraically closed field.

We begin by giving our main definition.

4.1 Definition. A **toric variety** is an inclusion of varieties $\Phi : T \to X$, where T is an algebraic torus and $\Phi(T)$ is open in X, such that there is a continuous group action $\Phi(T) \times X \ni (t, x) \mapsto t \cdot x \in X$ which is an extension of the induced multiplication within $\Phi(T)$; i.e. $\Phi(t) \cdot \Phi(s) = \Phi(ts)$ for all $t, s \in T$. We will often refer to $\Phi(T)$ as 'the' torus of X, and will often refer to the variety X as the toric variety with the torus embedding itself carried implicitly.

If X and Y are toric varieties with respective tori T_X and T_Y, then a morphism $\phi : X \to Y$ is a **toric morphism** if $\phi(T_X) \subseteq T_Y$ and $\phi(t \cdot x) = \phi(t) \cdot \phi(x)$ (i.e. ϕ is a homomorphism of the tori).

4.2 Example. K^k and $(K^*)^k$ are toric varieties; $\mathbb{P}^k$ is a toric variety whose torus is the canonical projection of $(K^*)^{k+1}$.

In this section we will be interested only in *affine* toric varieties; then in Section 5.1 we shall study the extent to which general toric varieties may be obtained by gluing affine toric varieties together in the usual way.

There are broadly speaking two ways of developing the local theory of toric varieties: via the characters, and via the 1-psgs. The character method, studied by Oda, Miyake, and others (c.f. [Oda78]), views affine toric varieties and their structure through the way their toric characters extend, and is coloured by the lens of monomial algebra. This viewpoint may be profitably generalised, and indeed is still a fundamental part of modern combinatorial commutative algebra (c.f. [Sta96], [MS05], or [HH11]). The 1-psg method also dates back to the 1970s (c.f. [Kem+73]), and studies toric varieties via the extension (through limits) of curves in the torus to curves on the variety (this is the primary viewpoint of [CLS11]). For the remainder of these notes we shall prefer the character method over the 1-psg method as this is the method that will best enable us to develop the applications of toric varieties given in later sections (which are primarily commutative-algebraic in flavour), though it may be conceptually more challenging at first.

4.1 Fundamentals of affine toric varieties

In this subsection, we will develop various fundamental algebraic notions underpinning the study of toric varieties. Chief among these are the correspondences between semigroup homomorphisms and points or subsets of an affine toric variety.

4.3 Lemma. *Let $S = \mathbb{Z}_{\geq 0}\beta_1 + \cdots + \mathbb{Z}_{\geq 0}\beta_r$ be a finitely generated semigroup embedded as a sub-semigroup of M; then closed points of* $\operatorname{Spec} K[S]$ *are in bijection with semigroup morphisms $S \to K$ (where K is a semigroup under multiplication).*

Proof. Closed points of $\operatorname{Spec} K[S]$ are in bijective correspondence with morphisms from $\operatorname{Spec} K$ to $\operatorname{Spec} K[S]$, which are in turn in bijective correspondence with algebra morphisms $K[S] \to K$. If $f : K[S] \to K$ is an algebra morphism, then it induces a semigroup homomorphism $\tilde{f} : S \to K$ defined on the generators by

$$\tilde{f}(\beta_i) := \lambda_i \text{ if } f(X^{\beta_i}) = \lambda_i;$$

this construction obviously reverses, so if $f : S \to K$ is a semigroup homomorphism we obtain an algebra homomorphism. ∎

To fix notation, we shall make some definitions.

4.4 Definition. We denote by γ_x the morphism $S \to K$ corresponding to a closed point $x \in \operatorname{Spec} K[S]$.

If $H \subseteq \operatorname{Hom}_{\mathsf{SemiGrp}}(S, K)$, we define the **evaluation set** of H to be the set of points that H gives evaluations of; that is,

$$\mathbf{E}(H) := \{x \in \operatorname{Spec} K[S] \,:\, \gamma_x \in H\}.$$

If $X \subseteq \operatorname{Spec} K[S]$, we define the **homomorphism set** of X to be the set of evaluation homomorphisms of X; that is,

$$\mathbf{H}(X) := \{\gamma_x \in \operatorname{Hom}_{\mathsf{SemiGrp}}(S, K) \,:\, x \in X\}.$$

4.5 Lemma. *The pair of maps*

$$\mathbf{E} \,:\, \mathcal{P}(\operatorname{Hom}_{\mathsf{SemiGrp}}(S, K)) \to \mathcal{P}(\operatorname{Spec} K[S]),$$
$$\mathbf{H} \,:\, \mathcal{P}(\operatorname{Spec} K[S]) \to \mathcal{P}(\operatorname{Hom}_{\mathsf{SemiGrp}}(S, K))$$

are inclusion-preserving and are inverses.

Proof. This is an immediate consequence of Lemma 4.3. ∎

The next examples should illustrate the ease with which we may actually compute the semigroup morphisms from points and vice versa.

4.6 Example. Consider the cusped cubic $k = \operatorname{Spec} K[X, Y]/(X^3 - Y^2)$ (where $S = \mathbb{Z}_{\geq 0} f_1 + \mathbb{Z}_{\geq 0} f_2 \subseteq \mathbb{Z}$ for $f_1 = 2$ and $f_2 = 3$); the point $(0, 0)$ corresponds to the evaluation morphism $\operatorname{eval}_{(0,0)} \,:\, K[X, Y]/(X^3 - Y^2) \to K$ which acts on the characters as $\chi^m \mapsto 0$ unless $m = 0$, and so induces a semigroup morphism $\gamma_{(0,0)} \,:\, S \to K$ defined by

$$m \mapsto \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To take the general case, consider $(x, y) \in K$; this corresponds to the algebra morphism $\operatorname{eval}_{(x,y)} \,:\, K[X, Y]/(X^3 - Y^2) \to K$, which behaves on the characters as $\operatorname{eval}_{(x,y)}(X) = x$ and $\operatorname{eval}_{(x,y)}(Y) = y$; i.e. it sends $f_1 = 2 \mapsto x$ and $f_2 = 3 \mapsto y$, so the semigroup morphism acts on $S = \{0\} \cup \mathbb{Z}_{\geq 2}$ as

$$s = 3a + 2b \mapsto x^a y^b.$$

4.7 Example. Let $\sigma = \operatorname{pos}\{e_1 + 2e_2\} \subseteq \mathbb{R}^2$ for $e_1, e_2 \in \mathbb{Z}^2$ the usual basis vectors. Then $S = \sigma \cap \mathbb{Z}^3$ is generated by the three elements $\alpha_1 = e_1$, $\alpha_2 = e_1 + e_2$, and $\alpha_3 = e_1 + 2e_2$, with the relation $2e_2 = e_1 + e_3$; so $\operatorname{Spec} K[S] = \operatorname{Spec} \frac{K[X,Y,Z]}{Y^2 - XZ}$. We may construct a semigroup morphism $f \,:\, S \to K$ by

$$\alpha_1 \mapsto 7, \quad \alpha_2 \mapsto 4, \quad \alpha_3 \mapsto 1.$$

This corresponds to the algebra homomorphism defined on the monomials by

$$X \mapsto 7, \quad Y \mapsto 4, \quad Z \mapsto 1;$$

thus the point corresponding to f is the point $(7, 4, 1) \in \operatorname{Spec} \frac{K[X,Y,Z]}{Y^2 - XZ}$.

4.8 Lemma. *If $T \simeq (K^*)^r$ is a torus, then $\mathbf{H}(T) = \operatorname{Hom}_{\mathsf{Grp}}(X(T), K^*)$. Further, if $x, y \in T$ then $\gamma_{x \cdot y}$ is the map $m \mapsto \gamma_x(m)\gamma_y(m)$.*

Proof. To simplify notation, we will identify T with $(K^*)^r$ using the given isomorphism. Under this identification, $X(T) = \mathbb{Z}^r$. By Lemma 4.3 and Example 3.31 there is a bijection between closed points $x = (x_1, ..., x_r) \in (K^*)^r$ and semigroup homomorphisms $\gamma_x \,:\, \mathbb{Z}^r \to K$. Note by the construction in Lemma 4.3 that such a homomorphism is a map sending $(m_1, ..., m_r) \mapsto x_1^{m_1} \cdots x_r^{m_r}$; in particular, since $x \in (K^*)^r$ no x_i is zero and so the image of γ_x is nonzero: i.e. $\gamma_x(T) \subseteq K^*$ and so the bijection is between closed points x of T and semigroup homomorphisms $\gamma_x \,:\, \mathbb{Z}^r \to K^*$. Further, if $y = (y_1, ..., y_r)$ is a second point we have $\gamma_{x \cdot y}(n) = (x_1 y_1)^{m_1} \cdots (x_r y_r)^{m_r} = (x_1^{m_1} \cdots x_r^{m_r})(y_1^{m_1} \cdots y_r^{m_r}) = \gamma_x(m)\gamma_y(m)$. ∎

Lemma 4.8 is a special case of a more general phenomenon. Let R be a subsemigroup of a semigroup S embedded in a lattice M. Then $\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}R, K^*)$ may be viewed as the set of semigroup homomorphisms which do not vanish on monomials with exponents from $\mathbb{Z}R$; that is, $\mathbf{E}(\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}R, K^*))$ is the set

$$\{x \in \operatorname{Spec} K[S] : \forall_{r \in \mathbb{Z}R} \gamma_x(r) \neq 0\} = \{x \in \operatorname{Spec} K[S] : \forall_{r \in \mathbb{Z}R} \chi^r(x) \neq 0\} = \operatorname{Spec} K[S] \setminus \mathbf{Z}(\chi^{\mathbb{Z}R})$$

(where we use $\chi^{\mathbb{Z}R}$ to denote the set of monomials $\{\chi^r : r \in \mathbb{Z}R\}$). We therefore make the following definition:

4.9 Definition. If $R \subseteq S$ is a subsemigroup of a semigroup S embedded in M, we set

$$\mathbf{D}(R) := \mathbf{E}(\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}R, K^*)) = \operatorname{Spec} K[S] \setminus \mathbf{Z}(\chi^{\mathbb{Z}R}).$$

This notion will become very useful in Section 4.4, where we will see that particular choices for the subsemigroup R correspond to toric orbits and subvarieties; the content of the following theorem is, in part, that $\mathbf{D}(S)$ is the torus of $\operatorname{Spec} K[S]$.

4.10 Theorem. *If S is a finitely generated semigroup embedded as a sub-semigroup of M, then* $\operatorname{Spec} K[S]$ *is a toric variety. If T is the torus of* $\operatorname{Spec} K[S]$*, then $X(T) = \mathbb{Z}S$. In particular if S contains a basis for M then $X(T) = M$ and $Y(T) = N$.*

Proof. Suppose S is generated by $\{\beta_1, ..., \beta_k\}$; we shall use $K[X^{\beta_1}, ..., X^{\beta_k}]$ to denote the free algebra over K in the symbols X^{β_i}, and we will denote by $\mathfrak{b}$ the binomial ideal of relations for $K[S]$ (i.e. $K[S] = K[X^{\beta_1}, ..., X^{\beta_k}]/\beta$, c.f. Lemma 3.30). Note that the scheme $\operatorname{Spec} K[X^{\pm\beta_1}, ..., X^{\pm\beta_k}]/\mathfrak{b}$ is a closed subscheme of $(K^*)^k$. Suppose $\mathfrak{b}$ is generated by binomial relations $f_1, ..., f_n$ of the form

$$f_i = \prod_{p \in P_i} X^{\beta_p} - \prod_{q \in Q_i} X^{\beta_q}$$

for finite subsets $P_i, Q_i \subseteq \{1, ..., k\}$. Then if $x = (x_1, ..., x_k)$ and $y = (y_1, ..., y_k)$ are closed points of the quotient $\operatorname{Spec} K[X^{\pm\beta_1}, ..., X^{\pm\beta_k}]/\mathfrak{b}$ (note, this is *not* $K[S]$) which are viewed by canonical inclusion as members of the k-torus $\operatorname{Spec} K[X^{\pm\beta_1}, ..., X^{\pm\beta_k}]$, we have $xy = (x_1y_1, ..., x_ky_k)$ and for each i,

$$\begin{aligned}
f_i(xy) &= \prod_{p \in P_i} (x_p y_p)^{\beta_p} - \prod_{q \in Q_i} (x_q y_q)^{\beta_q} \\
&= \prod_{p \in P_i} x_p^{\beta_p} \prod_{p \in P_i} y_p^{\beta_p} - \prod_{q \in Q_i} x_q^{\beta_q} \prod_{q \in Q_i} y_q^{\beta_q} \\
&= \prod_{p \in P_i} x_p^{\beta_p} \prod_{p \in P_i} y_p^{\beta_p} - \prod_{q \in Q_i} x_q^{\beta_q} \prod_{q \in Q_i} y_q^{\beta_q} + \prod_{p \in P_i} x_p^{\beta_p} \prod_{q \in Q_i} y_q^{\beta_q} - \prod_{p \in P_i} x_p^{\beta_p} \prod_{q \in Q_i} y_q^{\beta_q} \\
&= \prod_{p \in P_i} x_p^{\beta_p} \left(\prod_{p \in P_i} y_p^{\beta_p} - \prod_{q \in Q_i} y_q^{\beta_q} \right) + \prod_{q \in Q_i} y_q^{\beta_q} \left(\prod_{p \in P_i} x_p^{\beta_p} - \prod_{q \in Q_i} x_q^{\beta_q} \right) \\
&= 0 - 0
\end{aligned}$$

(where the final equality comes since $f_i(x) = f_i(y) = 0$ for each i). In particular, $\operatorname{Spec} K[X^{\pm\beta_1}, ..., X^{\pm\beta_k}]/\mathfrak{b}$ forms a subgroup of $\operatorname{Spec} K[X^{\pm\beta_1}, ..., X^{\pm\beta_k}]$; thus by Theorem 1.4, $T := \operatorname{Spec} K[X^{\pm\beta_1}, ..., X^{\pm\beta_k}]/\mathfrak{b}$ is a torus of rank at most k. Further, T is obtained from $\operatorname{Spec} K[S]$ by localising away from the multiplicatively closed set $\{(X_i^{\beta})^j : 1 \leq i \leq k, j \in \mathbb{Z}_{\geq 0}\}$, hence it is open in $\operatorname{Spec} K[S]$.

We next must show that the action of T on itself extends to the whole variety. We have that the action is given by multiplying semigroup morphisms in the manner of Lemma 4.8; if $x \in \operatorname{Spec} K[S]$ and $t \in T$ define $\delta_{x,t} : S \to K$ by

$$\delta_{t,x}(s) = \gamma_t(s)\gamma_x(s).$$

Clearly, if $\delta_{t,x} = \gamma_y$ for some $y \in \operatorname{Spec} K[S]$ then setting $x \cdot t = y$ extends the group operation of T. By Lemma 4.3 it just suffices to check that $\delta_{t,x} : S \to K$ is a morphism of semigroups. Suppose $s, s' \in S$; then

$$\delta_{t,x}(s+s') = \gamma_t(s+s')\gamma_x(s+s') = \gamma_t(s)\gamma_t(s')\gamma_x(s)\gamma_x(s') = \gamma_t(s)\gamma_x(s)\gamma_t(s')\gamma_x(s') = \delta_{t,x}(s)\delta_{t,x}(s')$$

which is what we wanted.

Now consider the character group $X(T)$. By Lemma 1.6, we have that the characters are precisely the maps given on T by monomials in the generators; that is, $f \in X(T)$ iff f is given by

$$f(x_{\beta_1}, ..., x_{\beta_k}) = x_{\beta_1}^{m_1} \cdots x_{\beta_k}^{m_k}$$

and so $X(T)$ is naturally isomorphic to the free abelian group over the set $\{\beta_1, ..., \beta_k\}$ modulo the relations of S, that is $X(T) = \mathbb{Z}S$. The remaining claim, that if S contains a basis for M then $X(T) = M$ and $Y(T) = N$, follows from Lemma 1.6 directly. ∎

In particular, if S is full-dimensional, then $\operatorname{rank} T = \operatorname{rank} N$. Thus by Corollary 1.10 we can write $T = N \otimes_{\mathbb{Z}} K^*$ and so we have a very natural relationship between T, N, and M.

4.11 Corollary. *If σ is a lattice cone over N, then $U_\sigma := \operatorname{Spec} A_\sigma$ is an affine toric variety. If T is the torus of U_σ, the product $x \cdot t$ for $x \in U_\sigma$ and $t \in T$ is given by the semigroup morphism $\gamma_{x \cdot t} : S_\sigma \to K$ that is the pointwise product of γ_x and γ_t.* ∎

4.12 Example. Affine space K^n is a toric variety arising from a cone. Indeed, $\mathbb{A}^n = \operatorname{Spec} K[X^{\alpha_1}, ..., X^{\alpha_n}]$, where $\alpha_1, ..., \alpha_n$ are independent members of some semigroup; we may take this semigroup to be the positive quadrant of $\mathbb{Z}^n$: $\sigma^\vee = \operatorname{pos}\{e_1, ..., e_n\}$. But $\operatorname{pos}\{e_1, ..., e_n\}$ is its own dual cone (Example 3.15), so $\mathbb{A}^n = U_\sigma$.

We also have that the n-torus, $(K^*)^n$, arises from a cone. Indeed, $(K^*)^n = \operatorname{Spec} K[X^{\pm\alpha_1}, ..., X^{\pm\alpha_n}]$; hence we have a semigroup generated by n independent elements, together with their inverses; thus the semigroup is $\mathbb{Z}^n$; and the dual cone of $\mathbb{Z}^n$ is $\{0\}$. Hence $(K^*)^n = U_{\{0\}}$.

4.13 Lemma. *A morphism $\phi : \operatorname{Spec} K[S_1] \to \operatorname{Spec} K[S_2]$ is a toric morphism if and only if the corresponding map $\phi^* : K[S_2] \to K[S_1]$ of coordinate rings restricts to a semigroup homomorphism $\tilde{\phi}^* : S_2 \to S_1$.*

Proof. To fix notation, let T_1 be the torus of $\operatorname{Spec} K[S_1]$ and let T_2 be the torus of $\operatorname{Spec} K[S_2]$; so $X(T_1) = \mathbb{Z}S_1$ and $X(T_2) = \mathbb{Z}S_2$ (by Theorem 4.10) and thus $\mathbf{A}(T_1) = K[\mathbb{Z}S_1]$ and $\mathbf{A}(T_2) = K[\mathbb{Z}S_2]$.

Suppose $\phi : \operatorname{Spec} K[S_1] \to \operatorname{Spec} K[S_2]$ is toric; then $\phi(T_1) \subseteq T_2$ and so we obtain a commutative diagram

$$\begin{tikzcd} K[S_2] \arrow[r, "\phi^*"] \arrow[d] & K[S_1] \arrow[d] \\ K[\mathbb{Z}S_2] \arrow[r, "\hat{\phi}^*"] & K[\mathbb{Z}S_1]. \end{tikzcd} \tag{3}$$

We must show that the map ϕ^* induces a map $S_2 \to S_1$, by sending $s \in S_2$ to the exponent vector of $\phi^*(\chi_s) \in S_1$; *a priori*, the problem is that it might be that $\phi^*(\chi_s)$ is not monomial and so the exponent vector is not defined. Let $m \in \mathbb{Z}S_2$; observe that the image of χ^m under the induced map $\hat{\phi}^*$ is a regular map $\hat{\phi}^*(\chi^m) : T_1 \to K$ with the property that

$$\begin{aligned}[\hat{\phi}^*(\chi^m)](a \cdot b) = \chi^m(\phi(a \cdot b)) = \chi^m(\phi(a) \cdot \phi(b)) \\ = (\chi^m \circ \phi)(a) \cdot (\chi^m \cdot \phi)(b) = [\hat{\phi}^*(\chi^m)](a) \cdot [\hat{\phi}^*(\chi^m)](b);\end{aligned}$$

i.e. $\hat{\phi}^*(\chi^m)$ is a character of T_1. Thus $\hat{\phi}^*$ sends monomials in $K[\mathbb{Z}S_2]$ to monomials in $K[\mathbb{Z}S_1]$ and so induces a well-defined homomorphism of semigroups $\tilde{\phi^*} : \mathbb{Z}S_2 \to \mathbb{Z}S_1$. Since $\hat{\phi}^*$ restricts to the map ϕ^*, $\tilde{(}\phi^*(S_2)) \subseteq S_1$ and so we have obtained the homomorphism desired.

Conversely, suppose $\phi : \operatorname{Spec} K[S_1] \to \operatorname{Spec} K[S_2]$ has the property that the induced map $\phi^* : K[S_2] \to K[S_1]$ restricts to a semigroup homomorphism $\tilde{\phi^*} : S_2 \to S_1$. Since S_2 spans $X(T_2)$ (by the preliminary paragraph above), this induces a semigroup homomorphism $\tilde{\phi^*} : \mathbb{Z}S_2 \to \mathbb{Z}S_1$ and this produces the diagram of Eq. (3) above. The image of the diagram under Spec is

$$\begin{tikzcd} \operatorname{Spec} K[S_2] & \operatorname{Spec} K[S_1] \arrow[l, "\phi"'] \\ \operatorname{Spec} K[\mathbb{Z}S_2] \arrow[u, hook] & \operatorname{Spec} K[\mathbb{Z}S_1]. \arrow[l, "\phi\restriction_{T_1}"'] \arrow[u, hook] \end{tikzcd}$$

In particular, $\phi(T_1) \subseteq T_2$. We finally must show that the map $\phi\restriction_{T_1}$ is a homomorphism of groups. Indeed, suppose $a, b \in T_1$; by Lemma 4.8, we may canonically identify a and b with a pair of maps $\gamma_a, \gamma_b \in \mathrm{Hom}_{\mathsf{SemiGrp}}(X(T_1), K^*)$. For notational convenience, for all $x \in T_1$ let

$$\delta_x := \gamma_x \circ \tilde{\phi^*} \in \mathrm{Hom}_{\mathsf{SemiGrp}}(X(T_2), K^*);$$

since δ_x is the semigroup homomorphism representing $\phi(x) \in T_2$, we wish to prove that $\delta_{ab} = \delta_a \delta b$ (since then $\phi(ab) = \phi(a)\phi(b)$). But this is immediate by direct computation, recalling that $\gamma_{ab} = \gamma_a \gamma_b$. ∎

4.14 Example. Let $\{x_1, ..., x_k\} \in N$ (for some $k \leq n$) be a set of lattice points that may be extended to a $\mathbb{Z}$-basis $(x_1, ..., x_n)$ of N, and consider $\sigma = \operatorname{pos}\{x_1, ..., x_k\}$. We may choose an isomorphism of lattices $N \simeq \mathbb{Z}^n$ such that the image of $x_1, ..., x_n$ is the standard basis $(e_1, ..., e_n)$; let $\tilde{\sigma}$ be the image of σ under this isomorphism. Then:

$$\begin{aligned} \tilde{\sigma}^\vee &= \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}e_k + \mathbb{R}e_{k+1} + \cdots + \mathbb{R}e_n, \text{ so} \\ S_{\tilde{\sigma}} &= \mathbb{Z}_{\geq 0}e_1 + \cdots + \mathbb{Z}_{\geq 0}e_k + \mathbb{Z}e_{k+1} + \cdots + \mathbb{Z}e_n \text{ and thus} \\ A_{\tilde{\sigma}} &= K[X_1, ..., X_k, \pm X_{k+1}, ..., \pm X_n] = K[X_1, ..., X_k] \otimes_K K[\pm X_{k+1}, ..., \pm X_n]. \end{aligned}$$

It follows immediately that $U_{\tilde{\sigma}} = K^k \times (K^*)^{n-k}$; and by Lemma 4.13 we have $U_\sigma \simeq K^k \times (K^*)^{n-k}$.

4.15 Example. Consider the lattice cone σ generated over $\mathbb{Z}^2$ by $(1, 0)$ and $(-1, r)$ $(r \in \mathbb{Z})$. Then $\sigma^\vee$ is the cone generated by $(0, 1)$ and $(r, 1)$; by the construction in the proof of Lemma 3.6, S_σ is generated by the $(r + 1)$ points $(0, 1), ..., (r, 1)$; call these $\alpha_0, ..., \alpha_r$. We have relations $\alpha_i + \alpha_j = \alpha_k + \alpha_\ell$ if $i + j = k + \ell$.

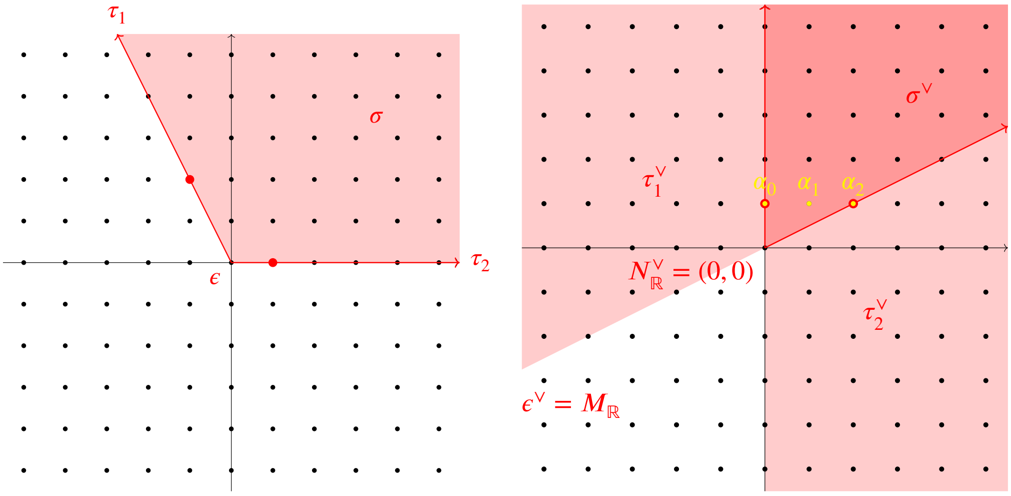
For the sake of simplicity, suppose $r = 2$ (Fig. 3); then the three generators have the relations $\alpha_0 + \alpha_2 = 2\alpha_1$, and so

$$A_\sigma = \frac{K[X^{\alpha_0}, X^{\alpha_1}, X^{\alpha_2}]}{\Big((X^{\alpha_1})^2 - X^{\alpha_0}X^{\alpha_2}\Big)};$$

so the toric variety is the cone $U_\sigma = \mathbf{Z}(Y^2 - XZ) \subseteq K^3$ (Fig. 4). The torus of the variety is $(\mathbb{C}^*)^2 \simeq U_\sigma \setminus (\mathbf{Z}(X) \cup \mathbf{Z}(Z))$, with embedding

$$(\mathbb{C}^*)^2 \ni (\lambda, \mu) \mapsto (\lambda, \mu, \mu^2/\lambda) \in U_\sigma \setminus (\mathbf{Z}(X) \cup \mathbf{Z}(Z))$$

(indeed, this is clearly an isomorphism as it is invertible; and the group action of $\mathbb{C}^*$ extends in the obvious way, since if $(x, y, z) \in U_\sigma$ and $(\lambda, \mu, \mu^2/\lambda)$ is in the embedded torus then the product $(\lambda x, \mu y, \mu^2/\lambda z)$ still lies in U_σ).



(a) The cone $\sigma = \mathrm{pos}\{(-1,2),(1,0)\}$ and its faces τ_1, τ_2 and ϵ.

(b) The dual cones of σ, τ_1, τ_2 and ϵ. The generators of S_σ are highlighted in yellow.

Figure 3: A strongly convex cone σ and its dual cone.

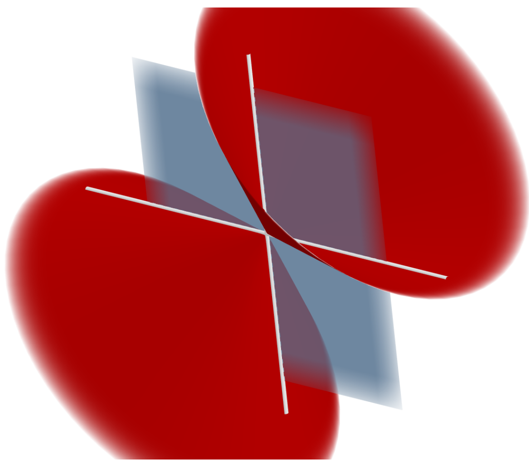


Figure 4: The toric variety of Example 4.15, with the embedded torus cut out by the white lines.

The converse of Theorem 4.10 is also true: every affine toric variety arises from a semigroup (but the reader should beware Example 4.17 and the remark following it). The proof indicates a key heuristic about toric varieties:

Toric varieties are precisely those varieties embedding a torus such that regular maps out of the torus extend to maps on the whole variety.

4.16 Theorem. *If Y is an affine toric variety then there exists a finitely generated semigroup S embedded in a lattice M such that $Y = \operatorname{Spec} K[S]$; further, $\mathbb{Z}S = M$.*

Historical remark. Theorem 4.16 appears as early as 1973, as proposition 1 of [Kem+73, p. 5]. An even earlier paper of 1972 includes a result which is a germ of this duality between semigroups and affine toric varieties [Hoc72, Lemma 3].

Proof. Suppose $Y \subseteq \mathbb{A}^k$, and let $M = \mathbb{Z}^k$. Consider the characters which extend to regular functions on the whole variety Y; that is, regular maps $f : T \to K^*$ of the form $X_1^{m_1} \cdots X_k^{m_k}$ such that $m_i < 0$ only if X_i^{-1} is regular on Y. Let $S \subseteq M$ be the set of exponent vectors of such characters. Then clearly S is a sub-semigroup of M (that is, it is closed under multiplication, has identity, and satisfies a cancellation law); further, it is finitely generated by $\Gamma = \{\pm e_i\} \cap S$. Finally note that $\operatorname{Spec} K[S] = Y$ by the proof of Theorem 4.10. Observe also that for all i, X_i is a character of T extending to a regular function on Y; hence S contains e_i for each i, and so $\mathbb{Z}S = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k = M$. ∎

We now give a word of warning, in the way of an example.

4.17 Example. Consider the following two toric varieties.

1. Affine 1-space, $\mathbb{A}^1_K$, which arises from the sub-semigroup of $\mathbb{Z}$ generated by $\{1\}$ (the torus is K^*);
2. The cusped cubic, $k = \operatorname{Spec} K[X, Y]/(X^3 - Y^2)$, which arises from the sub-semigroup generated by $\{2, 3\}$ (the torus is $(K^*)^2 \cap k$).

Clearly $\mathbb{A}^1_K \ncong k$ (indeed, exactly one of these varieties is smooth). On the other hand, the cones $\operatorname{pos}\{1\}$ and $\operatorname{pos}\{2, 3\}$ are equal (both being $\mathbb{Z}_{\geq 0}$).

In particular, we cannot hope for a bijection between arbitrary affine toric varieties and cones. Note though that the problem is essentially that the semigroup of k is 'missing some elements': this is why the variety has none of the nice properties we would expect: it clearly isn't smooth, and it is also not normal: if $A = K[X, Y]/(X^3 - Y^2)$, the polynomial $T^2 - X \in A[T]$ has two roots in $\operatorname{Frac} A$, neither of which are in A itself (namely, $\pm Y/X$ since $Y^2/X^2 = X^3/X^2$).

Remark. We shall prove an equivalence in Section 4.2 between a certain class of cones and a certain class of varieties (i.e. we restrict both the category of cones and the category of varieties). If we forget the idea of trying to use cones as our dual category and simply consider semigroups, we may obtain a duality of categories between *arbitrary* toric varieties and finitely generated sub-semigroups of a lattice which contain a generating set for that lattice. We have essentially proved this above (note that the problem with Example 4.17 was at the level of cones, not the level of semigroups); however, to obtain results about *general* toric varieties we will need to work on the level of cones (as it will be the cone structure that determines the gluing properties of the variety), and so we will develop the local theory accordingly. In any case, the more general situation is 'difficult to describe' [Oda78, p. 15].

4.2 A toric Nullstellensatz

We shall now study how combinatorial properties of a cone σ over N and the geometric properties of the toric variety U_σ interact. Most of these properties will generalise fairly readily to the more general situations we study in the next section.

We shall begin by making precise the problems with the naïve correspondence which was ruled out by Example 4.17, in order to actually give the 'correct' bijection. The first issue is, in some sense, the same that which occurs when trying to set up a correspondence between ideals in a polynomial ring and varieties cut out by polynomials in that ring: if the algebraic objects (ideals in the variety case, semigroups here) do not include the nth radicals of the objects they contain then the geometric objects (varieties/toric varieties) are pathological. The relevant definition is as follows.

4.18 Definition. Generalising Definition 3.11, let $S \subseteq N$ be a subsemigroup. We define for $s \in S$ the **radical** of s with respect to S to be $\sqrt[S]{s} := \lambda^{-1}s$ where λ is the largest integer such that $\lambda^{-1}s \in S$.

The semigroup S is called **saturated** if $\sqrt{s} = \sqrt[S]{s}$ for all $s \in S$ (equivalently, $\sqrt[S]{s}$ is primitive for all $s \in S$).

4.19 Example. Compare Example 4.17: $\mathbb{Z}_{\geq 0}$ is saturated, but $S := \mathbb{Z}_{\geq 0}\{2, 3\}$ is not since it does not include $1 = \sqrt[S]{2 \cdot 1}$ despite including $2 \cdot 1$.

The second issue that occurs is that, given our cone σ, the cone $\sigma^\vee$ might not be full-dimensional and so the characters of the torus of U_σ do not span the vector space of characters of the 'ambient torus' (and consequently the toric variety will not be of the right dimension). This is an aesthetic consideration more than a practical one, but it will simplify the theory if we take it seriously (it will enable us to characterise the torus of a toric variety U_σ as being simply the torus associated with the lattice N containing σ).

These issues are clarified and resolved in Proposition 4.20 and Theorem 4.21 below.

4.20 Proposition (Natural torus theorem)**.** *Let U_σ be the toric variety associated with a lattice cone σ over N (*rank $N = n$*). Then the following are equivalent:*

1. *The torus of U_σ is naturally isomorphic to $T_N = N \otimes_{\mathbb{Z}} K^*$, the natural torus of N;*
2. rank $\mathbb{Z}S_\sigma = n$*;*
3. $\dim U_\sigma = n$*;*
4. *$\sigma^\vee$ is full-dimensional in $M_\mathbb{R}$; and*
5. *σ is strongly convex in N.*

Proof. By Theorem 4.10, the torus T of U_σ has character lattice $\mathbb{Z}S_\sigma$. Thus $T \simeq T_N$ iff $M = X(T_N) = \mathbb{Z}S_\sigma$, i.e iff rank $\mathbb{Z}S_\sigma = n$. This shows the equivalence of (1) and (2).

Note that $\dim U_\sigma = \dim T$, since T is a dense open subset of U_σ; and $\dim T = \operatorname{rank} X(T) = \operatorname{rank} \mathbb{Z}S_\sigma$. This shows the equivalence of (2) and (3).

Next, observe that $\operatorname{rank} \mathbb{Z}S_\sigma = \dim \operatorname{span} S_\sigma$ (if B is a free generating set for $\mathbb{Z}S_\sigma$ then clearly it spans $\operatorname{span} S_\sigma$ so $\dim \operatorname{span} S_\sigma \leq \operatorname{rank} \mathbb{Z}S_\sigma$; on the other hand, an $\mathbb{R}$-basis for S_σ may be found that consists entirely of elements of S_σ which gives the opposite inequality) and $\dim \operatorname{span} S_\sigma = \dim \sigma^\vee$ since $\operatorname{span} S_\sigma$ is the smallest affine subspace containing $\sigma^\vee$. This shows that $\operatorname{rank} \mathbb{Z}S_\sigma = n$ iff $\dim \sigma^\vee = n$, giving the equivalence of (2) and (4).

Finally, the equivalence of (4) and (5) was proved as Corollary 3.23. ∎

The natural torus theorem tells us that in order to have an affine toric variety U_σ whose torus is natural with respect to the lattice embedding σ (i.e. in order to resolve the second issue above), we must require σ

to be strongly convex. It turns out that this, in fact, resolves the first issue as well and we finally obtain a correspondence between some class of semigroup, some class of cone, and some class of affine toric variety.

4.21 Theorem (Global correspondence theorem, affine case)**.** *Let Y be an affine toric variety with torus T. The following are equivalent:*

1. *Y is normal (Definition 0.2);*
2. *$Y = \operatorname{Spec} K[S]$ for S a saturated finitely generated semigroup such that $\mathbb{Z}S = M$;*
3. *$Y = U_\sigma$ where $\sigma \subseteq N_\mathbb{R}$ is a strongly convex lattice cone over N.*

Historical remark. Theorem 4.16 appears in 1973 as theorem $1'$ of [Kem+73, p. 8] and independently as proposition 1 of [OM75].

Before giving the proof of Theorem 4.21, we pause for a few technical lemmata.

4.22 Lemma. *Let $S_1, ..., S_r$ be sub-semigroups of N; let $S = \bigcap_{i=1}^r S_i$. In order for $K[S]$ to be integrally closed, it suffices for each $K[S_i]$ to be integrally closed.*

Proof. Note first that $K[S]$ may be canonically identified with a subalgebra of $K[S_i]$ for each i; so $\operatorname{Frac} K[S] \subseteq \operatorname{Frac} K[S_i]$ for each i. Let f be a monic polynomial over $K[S]$ with a root $\alpha \in \operatorname{Frac} K[S]$; then $\alpha \in \operatorname{Frac} K[S_i]$ for each i, so $\alpha \in K[S_i]$ for each i and thus $\alpha \in K[S]$. ∎

4.23 Lemma. *Any unique factorisation domain A is integrally closed.*

Proof. Let $K = \operatorname{Frac} A$. Let $f = \sum_{i=0}^r f_i T^i \in A[T]$ be monic, so we may assume $f_r = 1$; suppose that $\alpha \in K$ is a root of f. We may write $\alpha = p/q$ for $p, q \in A$ and we may assume that p and q have no common irreducible factors. Then

$$0 = q^r f(\alpha) = q^r f_0 + q^{r-1} f_1 p + \cdots + p^r;$$

in particular $p^r = -(q^r f_0 + q^{r-1} f_1 p + \cdots + q f_{r-1} p^{r-1})$ so $q \mid p^r$. Since q and p have no common irreducible factors, it must be the case that q is a unit in A and thus $\alpha \in A$. ∎

4.24 Lemma. *Let S be a sub-semigroup of N, and let U be a multiplicatively closed subset of $K[S]$. Then $(K[S])[U^{-1}]$ is integrally closed if $K[S]$ is integrally closed. In particular, an affine variety Y is normal if $\mathbf{A}(Y)$ is integrally closed; and a general variety X is normal if it has an affine cover consisting of normal affine varieties.*

Proof. Note that $\operatorname{Frac}(K[S])[U^{-1}] = \operatorname{Frac} K[S]$; hence if f is a monic polynomial over $(K[S])[U^{-1}]$ which has a root in $\operatorname{Frac}(K[S])[U^{-1}]$ then that root also lies in $\operatorname{Frac} K[S]$ and hence in $K[S]$, thus in $(K[S])[U^{-1}]$. ∎

Proof of Theorem 4.21. We first show that (2) implies (3). Suppose S is a saturated semigroup of the form $\mathbb{Z}_{\geq 0} s_1 + \cdots \mathbb{Z}_{\geq 0} s_k$ ($s_1, ..., s_k \in M$). Note first that $S \subseteq \operatorname{pos} S \cap \mathbb{Z}S = \operatorname{pos} S \cap M$ trivially; we proceed to show the opposite inclusion. Suppose that $x \in \operatorname{pos} S \cap M$. Note that $\operatorname{pos} S \cap M \subseteq \operatorname{pos} S \cap M_\mathbb{Q}$, and by Lemma 3.13 we have that $\operatorname{pos} S \cap M_\mathbb{Q} = \mathbb{Q}_{\geq 0} s_1 + \cdots + \mathbb{Q}_{\geq 0} s_k$; thus we may write $x = \frac{\lambda_1}{\mu_1} s_1 + \cdots + \frac{\lambda_k}{\mu_k} s_k$ and so $(\mu_1 \cdots \mu_k)x$ is a $\mathbb{Z}_{\geq 0}$-combination of the generators $s_1, ..., s_k$; by saturatedness of S, $(\mu_1 \cdots \mu_k)x \in S$ implies that $x \in S$, so $S = \operatorname{pos} S \cap \mathbb{Z}S$. Since $\operatorname{pos} S$ is full-dimensional in $M_\mathbb{R}$, we have by Corollary 3.23 that $\sigma := (\operatorname{pos} S)^\vee$ is strongly convex over N with the property that $S = \sigma^\vee \cap M$: i.e. $\operatorname{Spec} K[S] = U_\sigma$.

The converse implication, that (3) implies (2), is almost by definition; the only thing we must check is that $S = \sigma^\vee \cap M$ is in fact saturated. If $\lambda x \in S$ for some $x \in M$ and $\lambda \in \mathbb{Z}_{>0}$ then $x = \frac{1}{\lambda}(\lambda x)$ lies in $\sigma^\vee$ by positivity of $\sigma^\vee$ and lies in M by definition, so lies in S.

We finally must show that (1) and (2) are equivalent. Suppose $Y \subseteq \mathbb{A}^n$ is a normal affine toric variety. Then by Theorem 4.16 we may write $Y = \operatorname{Spec} K[S]$ for some finitely generated sub-semigroup $S \subseteq M = \mathbb{Z}^k$ where $\mathbb{Z}S = M$. Let $s \in M$ and $\lambda \in \mathbb{Z}_{\geq 0}$ be such that $\lambda s \in S$. Let χ^s be the character of T with exponent vector s (recalling Definition 1.8); since $\lambda s \in S$, $(\chi^s)^\lambda$ extends to a regular function on the entirety of Y, so $(\chi^s)^\lambda \in \mathbf{A}(Y)$. Consider the polynomial $p = T^\lambda - (\chi^s)^\lambda \in \mathbf{A}(Y)[T]$. Since p is monic and $\chi^s \in \mathbf{K}(Y)$, we may conclude that $\chi^s \in \mathbf{A}(Y)$ and so the exponent vector s of χ^s lies in S; this shows that (1) implies (2).

Conversely, suppose S is a saturated finitely generated semigroup such that $\mathbb{Z}S = M$. By Corollary 2.15, S is the set of lattice points of an intersection of finitely many lattice halfspaces $H_1, ..., H_r$ (i.e. halfspaces of the form $\operatorname{pos}\{\pm x_1, ..., \pm x_{n-1}, x_n\}$ for some $x_1, ..., x_n \in N$). Thus by Lemma 4.22 it will suffice to show that $K[S]$ is integrally closed in the special case where S is the lattice points of such a halfspace $\operatorname{pos}\{\pm e_1, ..., \pm e_{n-1}, e_n\}$. Because S is saturated, we may assume that each e_i is primitive and hence $(e_1, e_2, ..., e_n)$ is a $\mathbb{Z}$-basis of N. By a similar argument to that given in Example 4.14 it is therefore the case that $K[S] \simeq K[X_1^{\pm 1}, ..., X_{n-1}^{\pm 1}, X_n]$. But this latter set is a localisation of $K[X_1, ..., X_n]$; since the latter is integrally closed (Lemma 4.23) we may conclude that $K[X_1^{\pm 1}, ..., X_{n-1}^{\pm 1}, X_n]$ is integrally closed and hence $K[S]$ is integrally closed by Lemma 4.24; thus $Y = \operatorname{Spec} K[S]$ is normal (since by Lemma 4.24 again all its local rings, which are of the form $K[S]_{\mathfrak{p}}$ for primes $\mathfrak{p}$, are integrally closed). This proves that (2) implies (1). ∎

The following proposition will give us (at the very minimum) a method to compute manually the embedded torus of a toric variety U_σ based only on the original cone σ: one computes the toric variety corresponding to the minimal face ϵ of the cone, and uses the natural inclusion $S_\sigma \subseteq S_\epsilon$ of the semigroups to construct an inclusion of varieties.

4.25 Theorem. *If $\tau \preceq \sigma$ are lattice cones over N, then U_τ may be canonically identified with a principal open subvariety of U_σ. Further, this map $\tau \mapsto U_\tau$ sets up a bijection between faces $\tau \preceq \sigma$ and affine open subvarieties of U_σ that are invariant under the action of the torus, in such a way that the torus of U_σ corresponds to the minimal face of σ.*

Remark. Corollary 4.45 will furnish us with a converse to this statement.

Proof. If $\tau \preceq \sigma$ then by Lemma 3.19 we have $\tau = F_\sigma(m)$ for some $m \in \sigma^\perp$; hence by Lemma 3.27, $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m)$. In particular (taking an intersection with M), $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m)$ from whence it follows immediately that $K[S_\tau] = K[S_\sigma]_{\chi^m}$ and so U_τ is a principal open subvariety of U_σ; in fact, $U_\tau = \mathbf{D}(m)$. It is also clear that if $\rho \preceq \tau \preceq \sigma$ then $U_\rho \subseteq U_\tau \subseteq U_\sigma$.

Let T be the torus of U_σ, and note that $\mathbf{H}(U_\tau)\mathbf{H}(T) \subseteq \mathbf{H}(U_\tau)$; thus $T \cdot U_\tau \subseteq U_\tau$ and U_τ is invariant under products. ∎

4.26 Example. Recall (Example 4.12) that affine space K^n is a toric variety arising from the cone $\operatorname{pos}\{e_1, e_2\}$. This cone has two non-trivial faces; namely, $\operatorname{pos}\{e_1\}$ and $\operatorname{pos}\{e_2\}$. The dual cones of these faces are respectively $\operatorname{pos}\{e_1, \pm e_2\}$ and $\operatorname{pos}\{e_2, \pm e_1\}$. Hence the principal open subvarieties of K^n corresponding to the cone faces are $\operatorname{Spec} K[X,Y]_Y = K^n \setminus \mathbf{Z}(Y) = \mathbf{D}(e_2)$ and $\operatorname{Spec} K[X,Y]_X = K^n \setminus \mathbf{Z}(X) = \mathbf{D}(e_1)$ respectively.

4.27 Example. Consider the cone of Example 4.15 with $r = 2$; label the faces as in Fig. 3. Then we have the following varieties associated to the faces:

- $U_\sigma = \operatorname{Spec} \frac{K[X^{\alpha_0}, X^{\alpha_1}, X^{\alpha_2}]}{\left((X^{\alpha_1})^2 - X^{\alpha_0}X^{\alpha_2}\right)} = \mathbf{Z}((X^{\alpha_1})^2 - X^{\alpha_0}X^{\alpha_2})$
- $U_{\tau_1} = \operatorname{Spec} K[X^{\alpha_0}, X^{\alpha_1}, X^{\alpha_2}, X^{-\alpha_2}] = K[X^{\alpha_0}, X^{\alpha_1}, X^{\alpha_2}][X^{-\alpha_2}] = \mathbf{Z}((X^{\alpha_1})^2 - X^{\alpha_0}X^{\alpha_2}) \setminus \mathbf{Z}(X^{\alpha_2})$

- $U_{\tau_2} = \operatorname{Spec} K[X^{\alpha_0}, X^{\alpha_1}, X^{\alpha_2}, X^{-\alpha_0}] = K[X^{\alpha_0}, X^{\alpha_1}, X^{\alpha_2}][X^{-\alpha_0}] = \mathbf{Z}((X^{\alpha_1})^2 - X^{\alpha_0}X^{\alpha_2}) \setminus \mathbf{Z}(X^{\alpha_0})$
- $U_\epsilon = \operatorname{Spec} K[X^{\alpha_0}, X^{\alpha_2}, X^{-\alpha_0}, X^{-\alpha_2}] = \mathbf{Z}((X^{\alpha_1})^2 - X^{\alpha_0}X^{\alpha_2}) \setminus \big(\mathbf{Z}(X^{\alpha_0}) \cup \mathbf{Z}(X^{\alpha_2})\big)$

Note that we do not always use Hilbert bases for the semigroups — we need to be able to glue and localise with respect to the generators of S_σ. In particular, with the embedding in $\mathbb{C}^3$ shown in Fig. 4, we have that the two principal open subvarieties corresponding to τ_1 and τ_2 are obtained by cutting out the Z and X axes respectively.

4.3 Dimension, fixed points, and smoothness

We continue to study the pathological properties of Example 4.17. Note that the cusped cubic k was neither normal nor smooth.

4.28 Lemma. *Let Y be a scheme; if Y is smooth then Y is normal.*

Proof. Let R be a local ring of Y; so R is regular. By the Auslander-Buchsbaum theorem [Eis95, Theorem 19.19], R is a UFD. Hence by Lemma 4.23, R is integrally closed. ∎

On the other hand, normality does not imply smoothness:

4.29 Example. Consider the affine toric variety of Example 4.15 in the case $r = 2$; that is, let $\sigma = \operatorname{pos}\{e_1, -e_1 + 2e_2\}$ be a cone over $\mathbb{Z}^2$ so $U_\sigma = \mathbf{Z}(Y^2 - XZ) \subseteq \mathbb{A}^3$. Then $\mathbf{A}(U_\sigma)$ is not a UFD because $Y^2 = XZ$, and in fact the local ring $\mathbf{A}(U_\sigma)_{(X,Y,Z)}$ has the same factorisation (one must check that X, Y, and Z remain irreducible). On the other hand, we show that U_σ is normal: elements of the field $\mathbf{K}(U_\sigma)$ are of the form $u + vY$ for $u, v \in K(X, Z)$ (since $Y^2 = XZ$ in $\mathbf{K}(U_\sigma)$); similarly, elements of $\mathbf{A}(U_\sigma)$ are of the form $u + vY$ for $u, v \in K[X, Z]$. In particular, $\mathbf{A}(U_\sigma)$ is a finitely generated $K[X, Y]$-module and is thus integral[1] over $K[X, Y]$; so any element $\alpha \in \mathbf{K}(U_\sigma)$ which is integral over $\mathbf{A}(U_\sigma)$ is in particular integral over $K[X, Z]$. Suppose $\alpha = u + vY$ ($u, v \in K(X, Z)$) is such an element; then the minimal polynomial for α over $K(X, Z)$ is

$$T^2 - 2uT + (u^2 - v^2XZ);$$

it is a standard fact that the coefficients of the minimal polynomial of an elemement integral over a ring lie in that ring, so $u \in K[X, Z]$ and $v^2XZ \in K[X, Z]$. But XZ is the product of two coprime irreducibles, and if the denominator of $v \in K(X, Y)$ were nontrivial then v^2 would have four irreducibles in the denominator and cancellation could not occur. Thus $v \in K[X, Z]$, and hence $u + vY \in \mathbf{A}(U_\sigma)$. Thus $\mathbf{A}(U_\sigma)$ is integrally closed and U_σ is normal.

It follows that smoothness is a stronger condition than normality. Further:

4.30 Theorem. *An affine toric variety Y is smooth if and only if $Y = U_\sigma$ for σ a smooth cone (Definition 3.2) over N.*

We prove one direction immediately:

Proof of the easy direction (U_σ is smooth if σ is smooth). By Example 4.14, if σ is a smooth cone then $U_\sigma \simeq \prod_{i=1}^k K^* \times \prod_{i=k+1}^n K$ (for some $k \leq n$). ∎

[1] Recall, an element $\alpha \in \operatorname{Frac} R$ is **integral over** R if α lies in the integral closure of R; i.e. if α is a root of a monic polynomial over R.

The proof of the converse will require the development of some further theory that will, at the outset, seem completely unrelated. Thus we present a brief 'roadmap' of the next few propositions. The idea of the proof is to find a distinguished point $p_\sigma \in Y_\sigma$ that will encapsulate the combinatorial properties of σ; more precisely, in Theorem 4.31 we will show that the properties of U_σ imply the existence of a unique fixed point for the torus action on the variety. This point will be useful in other circumstances as well, as the semigroup homomorphism it is represented by is, in some sense, the *minimal* homomorphism for S_σ. We shall use this 'minimality' property to show that the dimension of the variety (which, by smoothness, is just the dimension at this distinguished point) is precisely the size of the Hilbert basis H_σ (this equality being the content of Lemma 4.33). Finally, we may use the combinatorial properties of H_σ to show that σ is smooth (Lemma 4.34).

4.31 Theorem. *Let σ be a strongly convex cone over N. The torus action of U_σ has a fixed point if and only if σ is full dimensional in $N_\mathbb{R}$. In this case, the fixed point is unique; it is the point p_σ corresponding (c.f. Lemma 4.3) to the semigroup morphism $\gamma_{p_\sigma} : S \to K$ given by*

$$\gamma_{p_\sigma}(s) = \begin{cases} 1 & s = 0 \\ 0 & \textit{otherwise} \end{cases}$$

(i.e. the 'most efficient extension' of the identity morphism $1 : 0 \to K$ to the entirety of M) and the maximal ideal of $K[S_\sigma]$ generated by the set

$$\{\chi^s : s \in S_\sigma \setminus 0\}.$$

Proof. Note first that γ_{p_σ} as defined above is a semigroup homomorphism if and only if $\sigma^\vee$ is strongly convex: if $\sigma^\vee$ is not strongly convex then there exist nonzero $x, y \in S_\sigma$ such that $x + y = 0$; but $\gamma_{p_\sigma}(x) + \gamma_{p_\sigma}(y) = 0 + 0 \neq 1 = \gamma_{p_\sigma}(0) = \gamma_{p_\sigma}(x + y)$. Conversely, if γ_{p_σ} is not a semigroup homomorphism then there exist nonzero $x, y \in S_\sigma$ such that $\gamma_{p_\sigma}(x+y) \neq \gamma_{p_\sigma}(x)+\gamma_{p_\sigma}(y)$; this can only occur if $\gamma_{p_\sigma}(x+y) = 1$ and then $x + y = 0$; but $x, y \neq 0$ and so the cone $\sigma^\vee$ is not strongly convex.

Define p_σ as above; then p_σ is a fixed point iff for all $\gamma_t \in \mathbf{H}(T_N)$, $\gamma_{p_\sigma} = \gamma_{p_\sigma}\gamma_t$; i.e.

$$\gamma_{p_\sigma}(s) = \gamma_{p_\sigma}(s)\chi^s(t) \tag{4}$$

for all $s \in S_\sigma$, and all $t \in T_N$. Since $\gamma_{p_\sigma}(0) = 1$, Eq. (4) is satisfied for $s = 0$. For $s \neq 0$, note both sides of Eq. (4) are zero. Hence p_σ is a fixed point. Further, suppose x is any fixed point; then replacing p_σ with x in Eq. (4) and substituting $s = 0$, we see that $\gamma_x(0) = 1$. For $s \neq 0$, we may find some t such that $\chi^s(t) \neq 1$; then the equality holds only if $\gamma_x(s) = 0$. Thus $\gamma_x = \gamma_{p_\sigma}$, and hence the fixed point is unique.

Finally note that γ_{p_σ} corresponds to the homomorphism $K[S_\sigma] \to K$ sending every nontrivial monomial to 0; therefore (since the monomials generate $K[S_\sigma]$ over K) the point p_σ corresponds to the maximal ideal stated above. ∎

4.32 Example. We computed the semigroup morphisms of the fixed point $(0, 0)$ of the cusped cubic of Example 4.17 above, as Example 4.6.

4.33 Lemma. *Let σ be a strongly convex cone over N, full dimensional in $N_\mathbb{R}$. Let $T_{p_\sigma}(U_\sigma)$ be the Zariski tangent space to U_σ at p_σ. Then $\dim T_{p_\sigma}(U_\sigma) = |H_\sigma|$.*

Proof. By Theorem 4.31, $\dim T_{p_\sigma} = \dim \mathfrak{m}/\mathfrak{m}^2$ for $\mathfrak{m}$ the ideal of $K[S_\sigma]$ generated by $\{\chi^s : s \in S_\sigma \setminus 0\}$. Note that $\mathfrak{m} = \bigoplus_{s \in S_\sigma \setminus \{0\}} K\chi^s$; we may split this into two sums, a sum over s irreducible and a sum over s reducible. Consider the sum over *reducible* $s \in S_\sigma \setminus \{0\}$; such an element must be of the form $s = s_1 + s_2$ for nonzero $s_1, s_2 \in S_\sigma$, and so $\chi^s = \chi^{s_1}\chi^{s_2} \in \mathfrak{m}^2$. Since clearly every monomial in $\mathfrak{m}^2$ has a reducible exponent vector in S, we have that $\mathfrak{m} = \bigoplus_{\substack{s \in S_\sigma \setminus \{0\} \\ s \text{ reducible}}} K\chi^s + \mathfrak{m}^2$ so $\mathfrak{m}/\mathfrak{m}^2 \simeq \bigoplus_{\substack{s \in S_\sigma \setminus \{0\} \\ s \text{ reducible}}} K\chi^s$; hence the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is the cardinality of the set of irreducible elements of S_σ, which is $|H_\sigma|$ by Lemma 3.9. ∎

Recall that the easy implication of Theorem 4.30 was proved immediately following the theorem statement, and the difficult implication was postponed pending further theoretical development. The following lemma completes the proof by providing the difficult implication.

4.34 Lemma. *If an affine toric variety Y is smooth then $Y = U_\sigma$ for σ a smooth cone (Definition 3.2) over N.*

Proof. By Lemma 4.28, if Y is smooth then Y is normal. Thus if Y is toric, by Theorem 4.21 it is of the form U_σ for σ a strongly convex cone over N. It remains to show that σ is smooth.

We have two cases: $\dim \sigma = n$, and $\dim \sigma < n$.

Case I ($\dim \sigma = n$)**.** Since U_σ is normal, by Proposition 4.20 we have that $\dim U_\sigma = n$; thus since U_σ is smooth, in particular we have $\dim T_{p_\sigma}(U_\sigma) = n$. Hence, by Lemma 4.33, $|H_\sigma| = n$. Observe that $|H_\sigma| \geq |F_1(\sigma^\vee)|$: this is because $\sigma^\vee = \operatorname{pos} H_\sigma$ (by definition), so each one-dimensional face of $\sigma^\vee$ is generated by a subset of H_σ (by Corollary 3.20), hence is generated by exactly one such element (since H_σ is a minimal generating set and S_σ is saturated). On the other hand, $|F_1(\sigma^\vee)| \geq n$ since $\sigma^\vee$ has dimension n: a choice of direction vectors for the edge rays of $\sigma^\vee$ span the vector space spanned by $\sigma^\vee$. Thus we have $n = |H_\sigma| \geq |F_1(\sigma^\vee)| \geq n$ and so $|F_1(\sigma^\vee)| = n$. Thus H_σ consists exactly of n lattice points generating the edges of $\sigma^\vee$. Because $M = \mathbb{Z}S_\sigma$, it follows that H_σ is a $\mathbb{Z}$-basis for M and so $\sigma^\vee$ is smooth; by Corollary 3.25, σ is smooth as well.

Case II ($\dim \sigma < n$)**.** Let N_1 be the smallest saturated sublattice of N containing the generators of σ. By the classification of finitely generated modules over a PID, there exists a complement sublattice N_2 such that $N = N_1 \oplus N_2$; observe that $\operatorname{rank} N_1 = \dim \sigma$ and $\operatorname{rank} N_2 = n - \dim \sigma$, and that we obtain a dual decomposition $M = M_1 \oplus M_2$ with $M_1 = N_1^\vee$.

Since σ is a lattice cone over both N and N_1, we obtain two toric varieties $U_\sigma = U_{\sigma,N}$ and U_{σ,N_1} with $S_{\sigma,N} \subseteq M$ and $S_{\sigma,N_1} \subseteq M_1$. We show next that $S_{\sigma,N} = S_{\sigma,N_1} \oplus M_2$; indeed, $S_{\sigma,N} = (\sigma^\vee \cap M) = (\sigma^\vee \cap M_1) \oplus (\sigma^\vee \cap M_2) = S_{\sigma,M_1} \oplus M_2$ (noting that $\sigma^\vee \cap M_2 = M_2$ as $M_2 \subseteq \sigma^\perp$). This induces an isomorphism $K[S_{\sigma,N}] \simeq K[S_{\sigma,N_1}] \otimes_K K[M_2]$; hence

$$U_\sigma = \operatorname{Spec} K[S_{\sigma,N}] \simeq \operatorname{Spec} K[S_{\sigma,N_1}] \otimes_K K[M_2] = U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-\dim\sigma}.$$

Since U_σ is smooth, U_{σ,N_1} is smooth (as the tangent space of a product variety at a point is the product of the tangent spaces of the components). It follows by case I that σ is smooth in N_1, hence is smooth in N (as any basis of N_1 may be extended to a basis of N). ∎

4.35 Example. Recall that $(\mathbb{C}^*)^n = U_\epsilon$ for $\epsilon = \{0\}$ (Example 3.31); i.e. $\epsilon = \operatorname{pos}\emptyset$, and $\emptyset$ can be extended to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

As a final note, we consider the case that the cone σ is simplicial, not just smooth.

4.36 Example. Let σ be a simplicial cone over N, with linearly independent generating set $\{s_1, ..., s_k\}$. Then $N' := \bigoplus_{i=1}^k \mathbb{Z}s_i$ is a full-dimensional sublattice of N. Set $G := N/N'$. From the inclusion $N' \hookrightarrow N$, we obtain a toric morphism $\phi : U_{\sigma,N'} \to U_{\sigma,N}$. One can show (c.f. [CLS11, Proposition 1.3.18]) that the morphism ϕ is in fact constant on G-orbits and so we obtain a bijection $U_{\sigma,N'}/G \simeq U_{\sigma,N}$.

Since σ is a smooth cone over N', $U_{\sigma,N'} \simeq K^n$ (Example 4.14); thus we may write $U_\sigma = U_{\sigma,N}$ as the group quotient $U_{\sigma,N'}/G$. Since G is finite, we say that U_σ is an **orbifold**.

4.4 Affine torus orbits

Our final family of local results will be finer versions of Theorem 4.31 and its corollaries. We generalise the notation of that theorem:

4.37 Definition. If R is a subsemigroup of S and $\gamma \in \mathrm{Hom}_{\mathsf{SemiGrp}}(R, K^*)$ we define the **skyscraper extension** of γ to S, denoted $\gamma^{R \nearrow S} : S \to K$, by

$$\gamma^{R \nearrow S}(s) = \begin{cases} \gamma(s) & s \in R \\ 0 & \text{otherwise.} \end{cases}$$

We let $\mathrm{Hom}_{\mathsf{SemiGrp}}(R \nearrow S, K)$ denote the set of all such extensions.

If τ is a lattice cone over N, define p_τ to be the closed point of U_τ corresponding to the semigroup morphism $\gamma_{p_\tau}(s) : S_\tau \to K^*$ given by the skyscraper extension to S_τ of the trivial morphism $1 : \tau^\perp \cap M \to K^*$.

Note that, by direct application of the definition, γ_{p_τ} has the property that the product $\mathrm{Hom}_{\mathsf{Grp}}(M, K^*)\gamma_{p_\tau}$ is precisely

$$\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap M \nearrow S_\tau, K)$$

and so the orbit of p_τ under the torus, $T_N \cdot p_\tau$, is just

$$\mathbf{E}(\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap M \nearrow S_\tau, K)).$$

If $\tau \preceq \sigma$ then it is clear that

$$\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap M \nearrow S_\tau, K) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap M \nearrow S_\sigma, K)$$

are identified under the inclusion $U_\tau \subseteq U_\sigma$.

4.38 Theorem. *Let σ be a strongly convex cone over N. Then there is a bijection between the set of faces of σ and the orbits of the torus action on U_σ; this bijection is given by the map* orb *defined for $\tau \preceq \sigma$ by*

$$\tau \mapsto \operatorname{orb} \tau := T_N \cdot p_\tau = \mathbf{E}(\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap M \nearrow S_\sigma, K)).$$

A different proof and description of orb dates back to at least [Kem+73], and may be found in [CLS11, Section 3.2]; it uses limits of 1-psgs to compute the torus orbits. Our proof, which follows [Oda78, p. 16] but is (according to Oda) due originally to Ramanan, relies primarily on the following technical lemma.

4.39 Lemma. *Let σ be a lattice cone over N. Call a decomposition of S_σ as a disjoint union $S_\sigma = A \cup B$* permissible *if it satisfies the following two axioms:*

- *P-1. B is a subsemigroup of S_σ;*
- *P-2. $S_\sigma + A \subseteq A$.*

A decomposition $A \cup B$ is permissible if and only if there exists a face $\tau \preceq \sigma^\vee$ such that $B = \tau \cap M$ and $A = S_\sigma \setminus B$.

Proof. Suppose first that $\tau \preceq \sigma^\vee$, and let $B = \tau \cap M$ and $A = S_\sigma \setminus B$. We show that $S_\sigma = A \cup B$ is permissible. Indeed, clearly B is a subsemigroup of S_σ; we need only show P-2. By By Lemma 3.19, there exists $n \in \sigma$ such that $\tau = n^\perp \cap \sigma^\vee$. We can therefore write

$$A = S_\sigma \setminus B = S_\sigma \setminus (\tau \cap M) = S_\sigma \setminus \tau = S_\sigma \setminus (n^\perp \cap \sigma^\vee) = S_\sigma \setminus n^\perp.$$

Let $s \in S_\sigma$, $a \in A$; then $\langle s|n \rangle \geq 0$ by definition of $S^\vee$, and $\langle a|n \rangle > 0$ by Section 4.4. Hence $\langle s + a|n \rangle = \langle s|n \rangle + \langle a|n \rangle > 0$; i.e. $s + a \notin n^\perp$ so $s + a \in S_\sigma \setminus n^\perp = A$; this shows that P-2 holds.

Conversely, suppose $S_\sigma = A \cup B$ is permissible. Let τ be the minimal face of $\sigma^\vee$ containing B; we show that $B = \tau \cap M$. Indeed, let $x \in \tau \cap M$; it will suffice to show that $(S_\sigma + x) \cap B$ is nonempty (since by P-2, $x \in S_\sigma$ lies in B if $(S_\sigma + x) \cap B \neq \emptyset$). Pick $b \in \operatorname{relint} \tau \cap M \subseteq B$; if we show that $(S_\sigma + x) \cap \mathbb{Z}_{\geq 0} b \neq \emptyset$ then we are done. Indeed, since $b \in \operatorname{relint} \tau$ (i.e. b does not lie in a face of τ), we must have $\langle b|a_i \rangle > 0$ for $a_1, ..., a_k$ a minimal generating set for $\tau^\vee$ (again by Lemma 3.19). Note that every element $s \in S_\sigma + x$ has the property $\langle s|n_i \rangle \geq \langle x|n_i \rangle$; conversely, if $s \in M$ has the property that $\langle s|n_i \rangle \geq \langle x|n_i \rangle$ for each i then $\langle s - x|n_i \rangle \geq 0$ for each i and so $s \in S_\sigma + x$. Hence

$$S + x = \{ s \in M \, : \, \forall_i \, \langle s|n_i \rangle \geq \langle x|n_i \rangle \}.$$

Since $\langle b|n_i \rangle > 0$ for all i, there exists $\lambda \in \mathbb{Z}_{>0}$ such that $\lambda \langle b|n_i \rangle > \langle x|n_i \rangle$ for each i. Thus $\lambda b \in S + x$, so $(S_\sigma + x) \cap \mathbb{Z}_{\geq 0} b \neq \emptyset$. ∎

The idea is that a decomposition is 'permissible' whenever it is a decomposition into a 'small' torus fixed set (corresponding to B), and 'the rest' (corresponding to A).

4.40 Example. Let $\sigma = \operatorname{pos}\{e_1, e_2\} = \sigma^\vee$; then a permissible decomposition of S_σ is $S_\sigma = A \cup B$ for $B = \operatorname{pos}\{e_1\}$, $A = S_\sigma \setminus B$.

The main purpose of Lemma 4.39 is to prove the following result on partitioning subsets of U_σ into pieces corresponding to the faces of σ.

4.41 Lemma. *Let σ be a lattice cone over N, and let $\epsilon \preceq \sigma$. Then*

$$\operatorname{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \epsilon^\perp, K) = \biguplus_{\epsilon \preceq \tau \preceq \sigma} \operatorname{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap S_\sigma \not\to S_\sigma, K).$$

Proof. Let $\gamma \in \operatorname{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \epsilon^\perp, K)$. Then $S_\sigma \cap \epsilon^\perp$ is the disjoint union of $A := \gamma^{-1}(0)$ and $B := \gamma^{-1}(K^*)$. This disjoint union $S_\sigma \cap \epsilon^\perp = A \cup B$ satisfies the properties P-1 and P-2 of Lemma 4.39: $\gamma^{-1}(K^*)$ is a semigroup since if $s, t \in \gamma^{-1}(K^*)$ we have $\gamma(s + t) = \gamma(s)\gamma(t) \neq 0$, and if $s \in S_\sigma \cap \epsilon^\perp$ and $t \in \gamma^{-1}(0)$ then $\gamma(s + t) = \gamma(s)0 = 0$, so $S_\sigma \cap \epsilon^\perp + \gamma^{-1}(0) \subseteq \gamma^{-1}(0)$. Thus by that lemma, $B = \tau_\gamma \cap M$ for some $\tau_\gamma \preceq \sigma^\vee \cap \epsilon^\perp$.

By part 2 of Corollary 3.21, these τ_γ are in correspondence with faces ρ_γ of σ that *contain* ϵ. Hence we may partition:

$$\operatorname{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \epsilon^\perp, K) = \biguplus_{\epsilon \preceq \tau \preceq \sigma} \operatorname{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap S_\sigma \not\to S_\sigma, K),$$

since every $\gamma \in \operatorname{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \epsilon^\perp, K)$ belongs to a unique set in the partition (namely, that corresponding to $\tau = \rho_\gamma$). ∎

Proof of Theorem 4.38. By Lemma 4.41 (with $\epsilon = \emptyset$) we may conclude that

$$\operatorname{Hom}_{\mathsf{SemiGrp}}(S_\sigma, K) \simeq \biguplus_{\tau \preceq \sigma} \operatorname{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap S_\sigma \not\to S_\sigma, K^*).$$

Hence (recalling Definition 4.37) we have the following chain of natural bijections:

$$\begin{aligned} \mathbf{H}(U_\sigma) = \operatorname{Hom}_{\mathsf{SemiGrp}}(S_\sigma, K) &= \biguplus_{\tau \preceq \sigma} \operatorname{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap S_\sigma \not\to S_\sigma, K^*) \\ &\simeq \biguplus_{\tau \preceq \sigma} \operatorname{Hom}_{\mathsf{SemiGrp}}(\tau^\perp \cap S_\sigma \not\to S_\tau, K^*) = \biguplus_{\tau \preceq \sigma} T_n \cdot p_\tau. \end{aligned} \tag{5}$$

Since the orbits of T_N partition U_σ we must obtain all of them in this disjoint union, and so the defined map orb is indeed a bijection. ∎

From the proof of Theorem 4.38, we obtain the following equivalent characterisation of $\operatorname{orb}\tau$:

4.42 Corollary. $\operatorname{orb}\tau = \mathbf{D}(\tau^{\perp} \cap S_\sigma) \cap \mathbf{Z}(\chi^{S_\sigma \setminus \tau^{\perp}})$

Proof. $\operatorname{orb}\tau = \mathbf{E}(\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^{\perp} \cap S_\sigma \rightharpoondown S_\sigma, K))$; this is the subset of $\mathbf{D}(\tau^{\perp} \cap S_\sigma)$ which kills $S_\sigma \setminus \tau^{\perp}$, and so we obtain the desired result. ∎

Using the same techniques as Theorem 4.38, we may characterise the Zariski closures of the orbits of the torus. To phrase it simply we will introduce a standard definition from polyhedral geometry.

4.43 Definition. Let σ be a cone; for each face $\epsilon \preceq \sigma$ the **star of σ at ϵ**, $\star_{F(\sigma)}(\epsilon)$, is the subset

$$\star_{F(\sigma)}(\epsilon) \coloneqq \{\tau \in F(\sigma) \,:\, \epsilon \preceq \tau\}.$$

4.44 Corollary. *Let $\tau \preceq \sigma$ for σ a strongly convex lattice cone over N. Then*

$$\overline{\operatorname{orb}\tau} = \biguplus_{\rho \in \star_{F(\sigma)}(\tau)} \operatorname{orb}\rho = \mathbf{E}(\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^{\perp} \cap S_\sigma, K)).$$

In particular, $\overline{\operatorname{orb}\sigma} = \operatorname{orb}\sigma = \{p_\sigma\}$ *is the unique closed orbit of* U_σ*; and* $\overline{\operatorname{orb}\emptyset} = U_\sigma$.

Proof. Note that for all τ, $\operatorname{orb}\tau = T_N p_\tau = \mathbf{E}(\mathrm{Hom}_{\mathsf{SemiGrp}}(\tau^{\perp} \cap S_\sigma, K^*)) = \mathbf{D}(\tau^{\perp} \cap S_\sigma)$, and hence the closure of $\operatorname{orb}\tau$ corresponds to the points corresponding to the vanishing ideal of these points: i.e. $\operatorname{orb}\tau$ corresponds to the ideal generated by the monomials with exponent vectors in $S_\sigma \setminus \tau^{\perp}$. To this end, for each $\tau \preceq \sigma$ let $\mathfrak{p}_\tau$ be the subset of A_σ defined by

$$\mathfrak{p}_\tau \coloneqq \langle \chi^{\sqrt{f}} \,:\, f \in S_\sigma \setminus \tau^{\perp} \rangle \tag{6}$$

The set $\mathfrak{p}_\tau$ is a prime ideal of A_σ, by the properties of $\tau^{\perp}$. Further, $A_\sigma/\mathfrak{p}_\tau$ is canonically isomorphic to $\langle \chi^{\sqrt{f}} \,:\, f \in S_\sigma \cap \tau^{\perp} \rangle$; thus the closed subscheme of U_σ corresponding to $\mathfrak{p}_\tau$ has closed points $\mathbf{E}(\mathrm{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \tau^{\perp}, K))$ and by Lemma 4.41 with $\epsilon = \tau$ we have that

$$\mathrm{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \tau^{\perp}, K) = \biguplus_{\tau \preceq \rho \preceq \sigma} \mathrm{Hom}_{\mathsf{SemiGrp}}(S_\sigma \cap \rho^{\perp}, K^*) = \biguplus_{\tau \preceq \rho \preceq \sigma} \operatorname{orb}\rho.$$ ∎

We now state and prove the promised converse to Theorem 4.25.

4.45 Corollary. *Let σ be a strongly convex lattice cone over N; then the mapping $\tau \mapsto U_\tau$ is a bijection between the faces of σ and the affine open subsets of U_σ closed under the torus action; moreover,*

$$U_\tau = \biguplus_{\rho \preceq \tau} \operatorname{orb}\rho$$

and so the correspondence preserves inclusion.

Proof. We showed that the map $\tau \mapsto U_\tau$ does indeed give torus-fixed affine open subsets of U_σ, as Theorem 4.25.

Conversely, let $\operatorname{Spec} B$ be an affine open subset of U_σ fixed under the torus action. By Theorem 4.21, since the torus of $\operatorname{Spec} B$ is T_N, we may find a cone $\tau \subseteq \sigma$ such that $\operatorname{Spec} B = U_\tau$. It remains to show that $\tau \preceq \sigma$. Let σ_0 be the smallest face of σ containing τ; then there exists $n \in \operatorname{relint}\sigma_0 \cap \tau \cap N$. Then $F_{\sigma_0}(n) = \tau^{\perp} \cap n^{\perp}$ is a face of $\sigma_0{}^{\vee}$, and by Corollary 3.21 $\sigma_0^{\vee} \cap n^{\perp} = \sigma_0^{\perp}$.

In particular, the prime ideal $\mathfrak{p}B$ where $\mathfrak{p} = \langle \chi^{\sqrt{f}} \,:\, f \in S_\sigma \setminus \sigma_0^{\perp} \rangle$ is the ideal of U_{σ_0} (c.f. Eq. (6)) in A is a proper ideal of B: $\mathfrak{p}B \subseteq \mathfrak{p} \cap B$, and $\mathfrak{p} \cap B$ is generated by monomials with exponents in

$$(S_\sigma \setminus \sigma_0^{\perp}) \cap \tau^{\perp} = (S_\sigma \setminus n^{\perp}) \cap \tau^{\vee};$$

noting that $n \in \tau$ so $n^\perp \supseteq \tau^\perp$, $\mathfrak{p} \cap B$ has exponents drawn from $S_\sigma \cap (\tau^\vee \setminus \tau^\perp)$ which is clearly a proper subset of $\tau^\vee \cap M$, the exponent vector set of B.

This shows that $\overline{\operatorname{orb}\sigma_0} = \operatorname{Spec} A_\sigma/\mathfrak{p}$ (the equality from Corollary 4.44) is contained in Spec B. Since by Corollary 4.44 the closure of any T_N-orbit of U_{σ_0} contains $\overline{\operatorname{orb}\sigma_0}$, every T_N-orbit of U_{σ_0} is contained in Spec B; thus since U_{σ_0} is the union of such orbits we have $U_{\sigma_0} \subseteq \operatorname{Spec} B$; but we know $\operatorname{Spec} B \subseteq U_{\sigma_0}$ by hypothesis, so $\operatorname{Spec} B = U_{\sigma_0}$ and thus by Theorem 4.21 we must have $\tau = \sigma_0$.

The final disjoint union decomposition then follows directly from Theorem 4.38. ∎

5 The global theory of toric varieties

The purpose of this section is to obtain global results about toric varieties. In Section 5.1 we shall prove the global analogues of the main results for affine varieties that were proved in the previous section, and will classify all normal toric varieties. In Section 7 we will present some examples of projective toric varieties that arise from lattice polytopes.

5.1 General structure theorems

In order to state a global version of the correspondence theorem for affine toric varieties (Theorem 4.21), we will need a notion of a global combinatorial object which is locally a strongly convex cone.

5.1 Definition. A **strongly convex lattice fan** over N (we will normally simply abbreviate this to **fan**) is a finite[2] set Σ of strongly convex lattice cones over N such that:

1. If $\sigma \in \Sigma$ and $\tau \preceq \sigma$ then $\tau \in \Sigma$;
2. If $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of τ and σ.

We turn Σ into a poset by inclusion. Since inclusion in Σ generalises the face relation $\preceq$ on an individual cone, we use the same symbol to denote the partial order on Σ.

The **support** of a fan Σ is the set $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$.

If Σ and Σ' are fans of lattice cones over N and N' respectively, a **morphism** $\phi : \Sigma \to \Sigma'$ is a map $N_\mathbb{R} \to N'_\mathbb{R}$ which restricts to a morphism of lattices $N \to N'$ and has the property that for each $\sigma \in \Sigma$, there exists $\sigma' \in \Sigma'$ such that $\phi(\sigma) \subseteq \sigma'$.

We now state the global correspondence theorem.

5.2 Theorem (First structure theorem (global correspondence))**.** *There is an (covariant) equivalence of categories between the category of normal separated toric varieties with toric morphisms and the category of fans with fan morphisms.*

Historical remark. Theorem 4.16 appears in 1973 as theorem 6(i) and theorem 7 of of [Kem+73, p. 8] and independently as theorem 6 of [OM75].

We will require the following theorem due to Sumihiro, and which is rather deeper than results we have needed so far.

5.3 Theorem (Sumihiro)**.** *Let $\Phi : T \to X$ be a normal toric variety. Then for every point[3] $x \in X$ there is an affine neighbourhood of x which is stable under the action of T.*

[2] We may delete the finiteness restriction if we no longer require our varieties to be of finite type. Compare with [Oda78], where the ‘finite type’ hypotheses (and separatedness hypotheses) for these theorems are stated in full.

[3] The theorem holds for arbitrary points, not just closed points.

References to proof. The proof is lengthy and technical and we shall only need to use it once (in the proof of Theorem 5.2). The original proof (proceeding via a representation theory approach) appears as [Sum74, Corollary 2], generalised to group schemes as [Sum75, Corollary 3.11]. Alternative proofs appear in [Fin93] (which also provides an alternative proof of the part of Theorem 5.2 for which we use Sumihiro's theorem), and in [Kno+89]. All of the known proofs appear to use the line bundle structure of X. ∎

Proof of Theorem 5.2. We first show that with every fan we may associate a normal toric variety. Let Σ be a fan over N; consider $\sigma, \tau \in \Sigma$. By Corollary 4.45, $U_{\sigma\cap\tau}$ is canonically an open subvariety of both U_σ and U_τ, and so we may glue U_σ and U_τ along $U_{\sigma\cap\tau}$ (for the details of the gluing process, see [Har77, example II.2.3.5]). Since normality is a local property, the scheme X_Σ which results from gluing all the cones of Σ in this way is normal; this scheme is actually a variety (it is of finite type since Σ is finite, and is separated[4] since $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$ is affine open and $K[S_{\sigma\cap\tau}]$ is generated by S_σ and S_τ as $(\sigma \cap \tau)^\vee = \sigma^\vee + \tau^\vee$). Further, the action of T_N on each maximal cone of Σ clearly is compatible with the gluing (e.g. by Theorem 4.38).

We next consider the relationship between morphisms of fans and toric morphisms. Suppose Σ and Σ' are fans over N and N' respectively, and $\phi : \Sigma \to \Sigma'$ is a morphism of fans. Then for each $\sigma \in \Sigma$, $\phi\restriction_\sigma$ is a morphism of cones by Lemma 4.13; since these restrictions are (trivially) compatible with respect to the face relation we have an induced morphism of toric varieties $X_\Sigma \to X_{\Sigma'}$.

Conversely, suppose $f : X_\Sigma \to X_{\Sigma'}$ is a morphism of normal toric varieties, where the varieties in question arise from fans Σ over N and Σ' over N' (note that we have not yet proved that *all* normal toric varieties arise in this way). Since the morphism maps the torus, it induces a map $\phi : N \to N'$ of the tori by successive restrictions $X_\Sigma \hookleftarrow T_N = N \otimes_{\mathbb{Z}} K^* \hookleftarrow N$. Further, by Theorem 4.38 for each $\sigma \in \Sigma$ the unique T_N-orbit $\operatorname{orb}\sigma$ is mapped into some $T_{N'}$-orbit, say $\operatorname{orb}\tau$ for some $\tau \in \Sigma'$. Thus $f(U_\sigma) \subseteq U_\tau$, and so $f\restriction_{U_\sigma}$ is a toric morphism $U_\sigma \to U_\tau$, and so the base extension of ϕ to $N_{\mathbb{R}}$ restricts to a morphism of cones $\sigma \to \tau$.

Finally we describe how an arbitrary normal toric variety $\Phi : T \to X$ induces a fan. Let N be the character group of T, and consider the collection of affine open subvarieties of X that are stable under the torus action. By Theorem 4.21 there exists a collection Σ of cones corresponding to these affine open subvarieties. By Theorem 5.3, the set $\{U_\sigma : \sigma \in \Sigma\}$ covers X. It remains to see that Σ is indeed a fan.

If $\sigma \in \Sigma$ and $\tau \preceq \sigma$, by Corollary 4.45 $U_\tau \subseteq U_\sigma$ and thus $\tau \in \Sigma$. Now note that for $\sigma, \tau \in \Sigma$ the set $U_\sigma \cap U_\tau$ is stable under the torus action; since it is affine by separatedness, it equals U_ρ for some $\rho \in \Sigma$ (by Theorem 4.21). Further by separatedness, $K[S_\rho]$ is generated by S_σ and S_τ; i.e. $S_\rho = S_\sigma + S_\tau$ and so $\rho = \sigma \cap \tau$. Since U_ρ is affine open in U_σ and U_τ, by Corollary 4.45 it must be that ρ is a face of both σ and τ. Finally note that Σ is finite as each U_σ has only a finite number of affine open subvarieties stable under the torus action. ∎

Having proved this, given a fan Σ we may always construct a normal toric variety X_Σ. Conversely, when we write 'a toric variety X_Σ' we will always implicitly mean a normal variety (whose associated fan is called Σ).

5.4 Example. Consider the fan Σ in $\mathbb{R}^2$ with maximal cones $\sigma_1 = \operatorname{pos}\{e_1, e_2\}$ and $\sigma_2 = \{e_1, -e_2\}$ (depicted on the left of Fig. 7 on Page 47). Then $U_{\sigma_1} = K[X_1, X_2]$ and $U_{\sigma_2} = K[X_1, X_2^{-1}]$; they are glued along the open subvariety $U_{\sigma_1\cap\sigma_2} = K[X_1, X_2^{\pm 1}]$, and so the variety X_Σ is the gluing of two copies of $\mathbb{A}^2$ but one 'inverted' on the other to form a cylinder (Fig. 5).

5.5 Example. It is easy to see that projective space $\mathbb{P}^n$ is a toric variety, with torus

$$\{[x_0 : \ldots : x_n] \in \mathbb{P}^n : \forall_i x_i \neq 0\} \simeq (K^*)^n$$

[4] Here we use the condition given as [Sha13, Proposition 5.4]

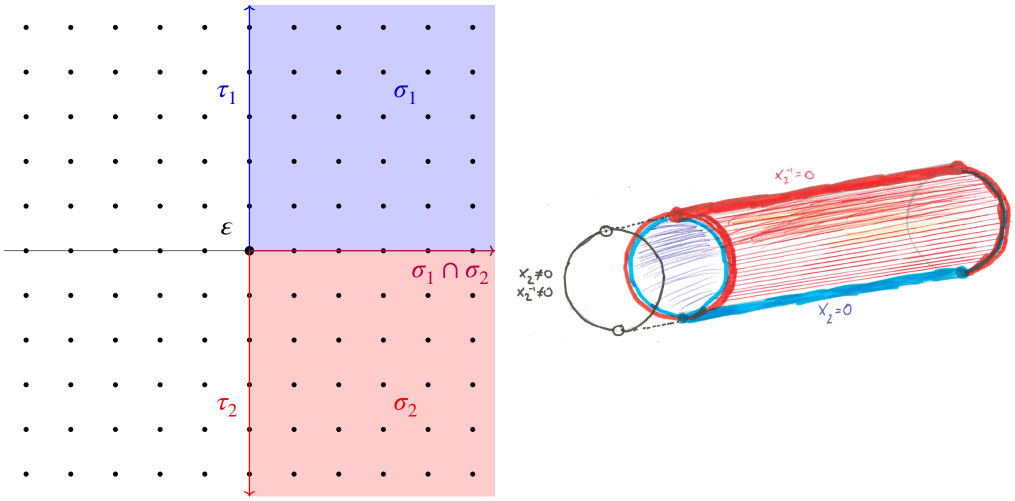


Figure 5: A fan Σ, and the variety X_Σ it corresponds to.

and the obvious action. More precisely, projective space may be obtained as a toric variety by applying the projectivisation functor $\mathbb{P}$ to affine space: $\mathbb{P}(\mathbb{A}^{n+1}) = \mathbb{P}^n$, with the torus of $\mathbb{P}^n$ exactly the image of the torus of $\mathbb{A}^{n+1}$ under $\mathbb{P}$. (One may formalise this notion by considering the subcategory of toric varieties with toric morphisms; then this result shows that projectivisation is a well-behaved operation in this category as it preserves the toric structure.)

One can also construct projective space via a fan. Indeed, let $e_1, ..., e_n$ be the usual basis vectors of $\mathbb{R}^n$ and set $e_0 \coloneqq e_{n+1} \eqqcolon -e_1 - \cdots - e_n$; then let Σ be the fan with facets $\sigma_i \coloneqq \operatorname{pos}\{e_0, ..., \hat{e}_i, ..., e_n\}$ for every $0 \leq i \leq n$ (see Fig. 6 for the case $n = 2$). Note that $\sigma_0^\vee = \operatorname{pos}\{e_1, ..., e_n\}$, and a lengthy but simple calculation shows that $\sigma_i^\vee = \operatorname{pos}\{-e_i, e_1 - e_i, ..., e_n - e_i\}$ for each $i > 0$. Then X_Σ is covered by the affine sets $\operatorname{Spec} K[X_1, ..., X_n]$ and $\operatorname{Spec} K[X_i^{-1}, X_1 X_i^{-1}, ..., X_n X_i^{-1}]$ $(0 < i \leq n)$ glued in the obvious fashion; it is clear that these correspond to the standard cover of $\mathbb{P}^n$ by $n + 1$ affine spaces.

5.6 Theorem (Second structure theorem (orbit correspondence))**.** *Let X_Σ be a normal toric variety with embedded torus T_N.*

1. *Let $\operatorname{orb}(X_\Sigma)$ denote the set of orbits of the action of T_N, and define a partial order $\preceq$ on $\operatorname{orb}(X_\Sigma)$ by $Y \preceq Y'$ if $Y' \subseteq \overline{Y}$. Then, by the map*

$$\operatorname{orb} : \Sigma \ni \sigma \mapsto \mathbf{E}(\operatorname{Hom}(\sigma^\perp \cap M, K^*)) \simeq (\sigma^\perp \cap M) \otimes_{\mathbb{Z}} K^* \in \operatorname{orb}(X_\sigma)$$

(where we interpret $\mathbf{E}$ as giving the set of points in $U_{\hat{\sigma}}$, $\hat{\sigma}$ being the maximal cone of Σ containing σ), the poset $(\Sigma, \preceq)$ is isomorphic to the poset $(\operatorname{orb}(X_\sigma), \preceq)$, and $\operatorname{orb}(\sigma)$ is a closed subtorus of T_N for each $\sigma \in \Sigma$.

2. *Let $U(X_\Sigma)$ denote the set of affine open subvarieties of X_Σ stable under the action of T_N. Then, by the map*

$$U : \Sigma \ni \sigma \mapsto U_\sigma \coloneqq \biguplus_{\tau \preceq \sigma} \operatorname{orb} \tau = \mathbf{E}(\operatorname{Hom}(\sigma^\vee \cap M, K^*)) \in U(X_\sigma),$$

the poset $(\Sigma, \preceq)$ is isomorphic to the poset $(U(X_\sigma), \subseteq)$.

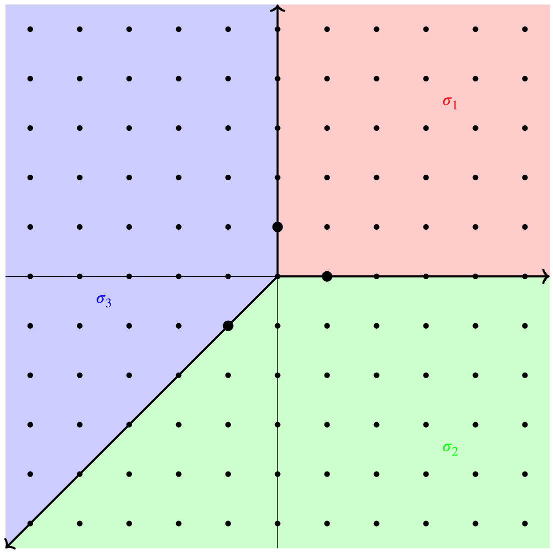


Figure 6: The fan for $\mathbb{P}^2$.

Proof.

1. For $\sigma \in \Sigma$, $\operatorname{orb}\sigma$ is the unique closed orbit of U_σ (Corollary 4.44). On the other hand, each orbit $\operatorname{orb}\tau$ for $\tau \in X_\sigma$ must lie in some affine open subvariety stable under the torus action, and so (since it lies in U_τ) we have the desired bijection.

2. By the construction in Theorem 5.2, X_Σ is covered by affine open subvarieties of the form U_σ stable under the toric action, with $U_{\sigma\cap\tau} = U_\sigma \cap U_\tau$. Thus the map $U : \Sigma \to U(X_\Sigma)$ is injective, by Corollary 4.45.

   Conversely, let U be an affine open subvariety of X_Σ closed under the torus action. By Theorem 4.21, there exists a unique cone σ over N with $U = U_\sigma$. Let $\tau \in \Sigma$, so $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$ is an affine open subvariety of U_σ and U_τ. Since it is an affine open subvariety of U_τ, by Corollary 4.45 it may be written as a disjoint union $U_{\sigma\cap\tau} = \biguplus_{\rho\preceq(\sigma\cap\tau)} U_\rho$. On the other hand, since it is an affine open subvariety of U_σ this union must (since the decomposition of Corollary 4.45 is unique) be taken over the faces of U_σ. Hence for all $\tau \in \Sigma$, $\sigma \cap \tau \in \Sigma$ and every face of σ is of this form. Moreover, $U_\sigma = \biguplus_{\rho\preceq\sigma} U_\rho$. Hence $\operatorname{orb}\sigma \subseteq U_\rho$ for some $\rho \preceq \sigma$; but of course in this case $\sigma \preceq \rho$ since U_ρ is the disjoint union of $\operatorname{orb}\tau$ for $\tau \preceq \rho$ (Corollary 4.45). Thus $\sigma = \rho \in \Sigma$, so $U = U_\sigma$ for $\sigma \in \Sigma$. ∎

5.7 Corollary. *X_Σ is affine iff Σ is the fan consisting of the faces of a single cone.*

Proof. X_Σ is affine $\iff$ $U(X_\Sigma)$ has a unique maximal element $\iff$ $(\Sigma, \preceq)$ has a unique maximal element $\iff$ Σ is the faces of a single cone. ∎

The most useful result for computing orbits is usually Corollary 4.42:

5.8 Example. Consider again the toric cylinder of Example 5.4. The fan decomposes to give the orbits as follows:

- We first compute $\operatorname{orb}\sigma_1$. We have

$$\operatorname{orb}\sigma_1 = \mathbf{D}\left(\sigma_1^\perp \cap S_{\sigma_1}\right) \cap \mathbf{Z}\left(\chi^{S_{\sigma_1}\setminus\sigma_1^\perp}\right);$$

now $\mathbf{D}\left(\sigma_1^{\perp} \cap S_{\sigma_1}\right) = \mathbf{D}(0) = U_{\sigma_1}$, and

$$\mathbf{Z}\left(\chi^{S_{\sigma_1} \setminus \sigma_1^{\perp}}\right) = \bigcap_{\substack{\alpha_1 \geq 0 \\ \alpha_2 \geq 0 \\ (\alpha_1, \alpha_2) \neq (0,0)}} \mathbf{Z}\left(X_1^{\alpha_1} X_2^{\alpha_2}\right) = \mathbf{Z}(X_1) \cap \mathbf{Z}(X_2);$$

therefore $\operatorname{orb} \sigma_1$ is the subset of U_{σ_1} with $(x_1, s_2) = (0, 0)$.

- By essentially the same calculation, $\operatorname{orb} \sigma_2$ is the subset of U_{σ_2} with $\left(x_1, x_2^{-1}\right) = (0, 0)$.
- To compute $\operatorname{orb} \tau_1$, note that

$$\operatorname{orb} \tau_1 = \mathbf{D}\left(\tau_1^{\perp} \cap S_{\sigma_1}\right) \cap \mathbf{Z}\left(\chi^{S_{\sigma_1} \setminus \tau_1^{\perp}}\right) = \mathbf{D}(\mathbb{Z}_{\geq 0}) \cap \mathbf{Z}\left\{X_1^{\alpha_1} X_2^{\alpha_2} : \alpha_1 \geq 0, \alpha_2 > 0\right\};$$

noting that $\left\{X_1^{\alpha_1} X_2^{\alpha_2} : \alpha_1 \geq 0, \alpha_2 > 0\right\}$ has greatest common divisor X_2, we find

$$\operatorname{orb} \tau_1 = \left(U_{\sigma_1} \setminus \mathbf{Z}(X_1)\right) \cap \mathbf{Z}(X_2);$$

i.e. $\operatorname{orb} \tau_1$ is the subset of U_{σ_1} given by $\left\{(x_1, x_2) : x_1 \neq 0, x_2 = 0\right\}$.

- By essentially the same calculation, $\operatorname{orb} \tau_2$ is the subset of U_{σ_2} given by

$$\left\{\left(x_1, x_2^{-1}\right) : x_1 \neq 0, x_2^{-1} = 0\right\}.$$

- We may compute $\operatorname{orb}(\sigma_1 \cap \sigma_2)$ either with respect to U_{σ_1} or U_{σ_2}; in either case it is very similar to the computation for $\tau_\bullet$. With respect to σ_1, we have

$$\mathbf{D}\left((\sigma_1 \cap \sigma_2)^{\perp} \cap S_{\sigma_1}\right) \cap \mathbf{Z}\left(\chi^{\left(S_{\sigma_1} \setminus (\sigma_1 \cap \sigma_2)^{\perp}\right)}\right) = \left(U_{\sigma_1} \setminus \mathbf{Z}(X_2)\right) \cap \mathbf{Z}(X_1),$$

so

$$\operatorname{orb}(\sigma_1 \cap \sigma_2) = \left\{(x_1, x_2) \in U_{\sigma_1} : x_1 = 0, x_2 \neq 0\right\}.$$

Similarly we have (with respect to σ_2) that

$$\operatorname{orb}(\sigma_1 \cap \sigma_2) = \left\{\left(x_1, x_2^{-1}\right) \in U_{\sigma_2} : x_1 = 0, x_2^{-1} \neq 0\right\}.$$

As these subsets are identified by the gluing on the cylinder, we have no inconsistency.

- Finally, since ε is the only cone remaining we may use the fact that the orbits partition X_Σ to see that

$$\varepsilon = \left\{(x_1, x_2) \in U_{\sigma_1} : x_1 \neq 0, x_2 \neq 0\right\} = \left\{\left(x_1, x_2^{-1}\right) \in U_{\sigma_2} : x_1 \neq 0, x_2^{-1} \neq 0\right\}.$$

We now move to 'topological'-type properties.

5.9 Definition. Let Σ be a fan over N. Then Σ is called **smooth** if every cone in Σ is smooth, and **simplicial** if every cone in Σ is simplicial (see Definition 3.2). The fan is **complete** if $|\Sigma| = N_{\mathbb{R}}$.

5.10 Theorem (Third structure theorem (local and topological properties))**.**

1. *A morphism $f : X_\Sigma \to X_\Pi$ of normal toric varieties with associated morphism of fans $\tilde{f} : \Sigma \to \Pi$ is proper iff for each $\pi \in \Pi$, $\tilde{f}^{-1}(\pi) = \bigcup\{\sigma \in \Sigma : \tilde{f}(\sigma) \subseteq \pi\}$.*

2. *A toric variety X_Σ is smooth iff Σ is smooth. If X_Σ is defined over $\mathbb{C}$, it is an **orbifold** (has an affine cover such that each open subvariety in the cover is the quotient of affine space by a finite abelian group) iff Σ is simplicial.*

Remark. Recall that a morphism of fans is in actuality a map of the underlying $\mathbb{R}$-vector spaces; thus $\tilde{f}^{-1}(\pi)$ denotes the inverse image of π under this map of vector spaces, not simply the inverse umage under the restriction to $|\Sigma|$. The theorem says that f is proper iff these inverse images do in fact coincide.

The proof of Theorem 5.10 requires different techniques to those we have used so far: part 1 will be proved using techniques related to discrete valuation rings, and so we postpone the proof of that part until Lemma 5.23. The result on orbifolds is relatively recent and requires cohomology theory, so we only give a reference for the proof.

Sketch of proof of part 2 of Theorem 5.10. The equivalence "X_Σ smooth if and only if Σ is smooth" follows directly from Theorem 4.30, as smoothness is a local property. We will not prove the orbifold statement in detail; one direction was sketched (locally) in Example 4.36. For the other direction, the idea is to consider the singularity properties of X_Σ and indeed one can show (this is done as [CLS11, example 11.4.4 and theorem 11.4.8] essentially using results of Brion [Bri99]) that X_Σ is an orbifold iff it is **rationally smooth** (we do not give the definition here as it is essentially cohomological), and this result along with a dimension/lattice counting argument similar in flavour to Lemma 4.34 and its preceeding lemmata gives the desired conclusion (that U_σ an orbifold implies σ is simplicial). ∎

It is easy to see that the following follows directly from part 1 of Theorem 5.10:

5.11 Corollary. *A morphism $f : X_\Sigma \to X_\Pi$ of normal toric varieties with associated morphism of fans $\tilde{f} : \Sigma \to \Pi$ is proper iff $\tilde{f}^{-1}(|\Pi|) = |\Sigma|$.* ∎

We also have the following special case:

5.12 Corollary. *A toric variety X_Σ is complete iff Σ is complete. If X_Σ is defined over $\mathbb{C}$, X_Σ is compact in the usual topology iff Σ is complete.*

Proof. Recall that X_Σ is complete if X_Σ is proper over Spec K; let f denote the canonical map $X_\Sigma \to$ Spec K and suppose Σ is defined over the lattice N. Note that Spec $K = U_\epsilon$ for $\epsilon = 0$, i.e. the trivial cone in the zero lattice. Hence f is proper iff $\tilde{f} : \Sigma \to$ Spec K has $|\Sigma| = \tilde{f}^{-1}(0)$ (by Corollary 5.11); but $\tilde{f}$ sends every element of $N_{\mathbb{R}}$ to 0. ∎

We give one classical application of Theorem 5.10.

5.13 Example (Toric blowups)**.** We remind the reader briefly about the blowup construction for varieties; see [Har77, pp. 28–30]. Let $X \subseteq \mathbb{A}^n$ be an affine variety, with $x \in X$. Consider the subvariety $U \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} = \operatorname{Spec} K[X_1, ..., X_n] \times \operatorname{Proj} K[Y_1, ..., Y_n]$ given by $U = \mathbb{Z}(\{X_i Y_j - X_j Y_i : 1 \le i \le n, 1 \le j \le n\})$. Let $\phi : U \to \mathbb{A}^n$ be the projection of U onto the affine factor. Then the **blowup** of X at x is the closure of $\varphi^{-1}(X - x)$, where $X - x$ denotes pointwise vector subtraction. One can show that (up to isomorphism) the blowup of X at x is independent of the embedding in to affine space; further, the map φ is an isomorphism

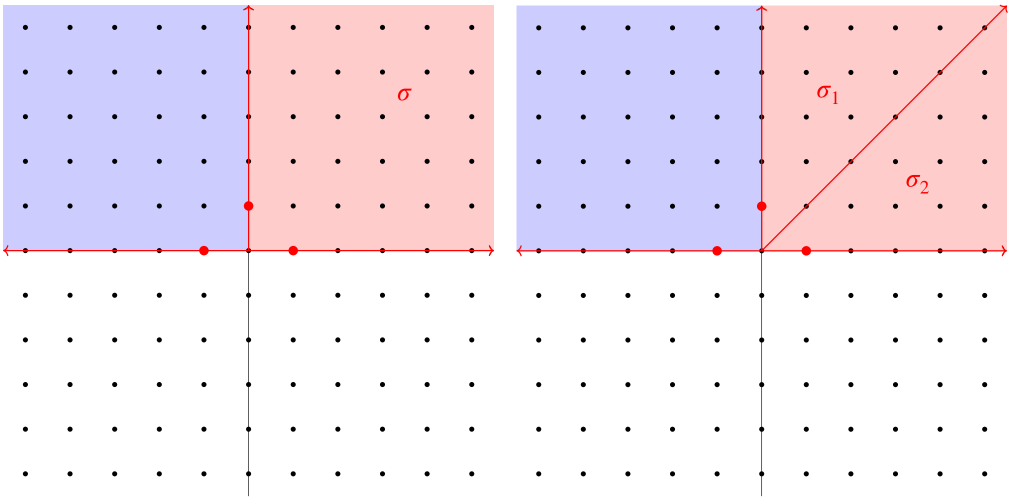


Figure 7: A fan Σ, and a refinement Σ' corresponding to a blowup of X_Σ.

on $\mathbb{A}^n \setminus 0$ and so we may define the blowup of an arbitrary variety X at a point $x \in X$ by choosing an affine cover $\{U_\alpha\}$ of X such that x lies in a unique U_α and then blowing up U_α.

Now suppose X_Σ is a normal toric variety (Σ over N), and let $\sigma \in \Sigma$ be a smooth cone generated by some basis $\{e_1, ..., e_n\}$ for N. Set $e_0 = e_1 + \cdots + e_n$, and let Σ' be the refinement of Σ obtained by replacing σ with the the set of all cones of the form pos S, where S is a subset of $\{e_0, e_1, ..., e_n\}$ not containing $\{e_1, ..., e_n\}$ (see Fig. 7; here $\sigma = \text{pos}\{e_1, e_2\}$ so the refinement cones are $\text{pos}\{e_1 + e_2\}$, $\text{pos}\{e_1 + e_2, e_1\}$, and $\text{pos}\{e_1 + e_2, e_2\}$). It is clear by part 1 of Theorem 5.10 that the natural map $\phi : \Sigma' \to \Sigma$ induces a proper morphism $X_{\Sigma'} \to X_\Sigma$; we check that $X_{\Sigma'}$ is a blowup of X_Σ.

Since blowing up is local, we need to consider only the relationship between $U_\sigma = \text{Spec}\, K[X_1, ..., X_n] \simeq \mathbb{A}^n$ and the variety Y coming from the fan refining σ; further since everything is defined up to isomorphism we may assume $N = \mathbb{Z}^n$. Then Y is the gluing of the varieties U_{σ_i} where $\sigma_i = \text{pos}(\{e_0, ..., e_n\} \setminus \{e_i\})$, so $S_{\sigma_i} = \mathbb{Z}_{\geq 0} e_i + \mathbb{Z}_{\geq 0}(e_1 - e_i) + \cdots + \mathbb{Z}_{\geq 0}(e_n - e_i)$ and so

$$U_{\sigma_i} = \text{Spec}\, K[X_i, X_1 X_i^{-1}, ..., X_n X_i^{-1}].$$

On the other hand, the blowup B of $\mathbb{A}^n$ is cut out of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ by the polynomials $X_i Y_j - X_j Y_i$ for $i, j \in \{1, ..., n\}$. For each i consider the principal open subvariety $D_i = B \setminus \mathbb{Z}(Y_i)$; then, on D_i, points have coordinates of the form

$$(x_1, ..., x_n, \frac{y_1}{y_i}, ..., \frac{y_n}{y_i}) = (x_i \frac{y_1}{y_i}, ..., x_i, ..., x_i \frac{y_n}{y_i}, \frac{y_1}{y_i}, ..., \frac{y_n}{y_i})$$

since $X_j = X_i Y_j Y_i^{-1}$ for each j; thus,

$$D_i = \text{Spec}\, K[X_i, X_i(Y_1 Y_i^{-1})^{-1}, ..., X_i(Y_n Y_i^{-1})^{-1}] \simeq \text{Spec}\, K[X_i, X_i X_1^{-1}, ..., X_i X_n^{-1}] = U_{\sigma_i}$$

for each i; thus $B \simeq Y$ and we have indeed constructed the blowup.

5.2 Discrete valuation rings

Many of the applications which we shall study will require the notion of a valuation on a field. The motivation for this concept is as follows:

5.14 Example. If $U \subseteq \mathbb{C}$ is open, $z_0 \in U$, and $f : U \to \mathbb{C} \cup \{\infty\}$ is meromorphic and non-zero then we may expand f as a Laurent series about z_0, say

$$f(z) = \sum_{n=\omega(f)}^{\infty} a_n (z - z_0)^n$$

where $\omega(f) \in \mathbb{Z}$ and $a_\omega(f) \neq 0$. The quantity $\omega(f)$ is an integer which tells us the behaviour of f at z_0: if $\omega(f) < 0$ then f has a pole of order $-\omega(f)$ at z_0, if $\omega(f) > 0$ then f vanishes to order $\omega(f)$ at z_0, and if $\omega(f) = 0$ then $f(z_0) = a_\omega(f)$. Let $M(U)$ denote the field of meromorphic functions on U; then the map

$$f \mapsto \begin{cases} \omega(f) & f \neq 0 \\ \infty & f = 0 \end{cases}$$

is a map $\omega : M(U) \to \mathbb{Z} \cup \{\infty\}$. Note that the set $\{f \in M(U) : \omega(f) \geq 0\}$ is a ring (namely, the ring of functions holomorphic on a neighbourhood of z_0), and this ring is **local**: it has a unique maximal ideal, namely the set of elements of K with positive valuation. (An ad-hoc proof is easy, but we shall prove a more general statement as Lemma 5.17.)

Axiomatising the properties of ω, we obtain the following definition.

5.15 Definition. Let K be a field. A map $v : K \to \mathbb{R} \cup \{\infty\}$ is a **valuation** if, for all $f, g \in K$,

1. $v(f) = \infty$ iff $f = 0$;
2. $v(fg) = v(f) + v(g)$;
3. $v(f + g) \geq \min\{v(f), v(g)\}$.

Because every valuation must send 0 to ∞, we often identify valuations with their restriction to K^*. Note also that every field has at least one valuation: the **trivial valuation** mapping every nonzero element to $0 \in \mathbb{R}$. We pause to recall some standard elementary results.

5.16 Lemma. *Let $f, g \in K$.*

1. $v(1) = 0$.
2. $v(-f) = v(f)$.
3. *If $v(f) \neq f(g)$ then $v(f + g) = \min\{v(f), v(g)\}$.* ∎

5.17 Lemma. *The image $v(K^*)$ is an additive subgroup of $\mathbb{R}$, known as the **value group** of v; this is sometimes denoted Γ_v. The set $R := \{f \in K : v(f) \geq 0\}$ is a local ring with unique maximal ideal $\mathfrak{m}_K := \{f \in K : v(f) > 0\}$, which we call the **local ring** of v and which is precisely the set of non-units of R. Then $R/\mathfrak{m}_K$ is a field, the **residue field** of K under v.* ∎

5.18 Example.

1. The map ω of Example 5.14 is a valuation on the field of functions meromorphic on $U \subseteq \mathbb{C}$ open. In fact, note that ω depends on the choice of an element $z_0 \in U$ and we obtain a different valuation for each such choice.
2. Let p be a prime; then every $\rho \in \mathbb{Q}^*$ may uniquely be written in the form $p^{v(\rho)}a/b$ where p divides neither a nor b; then the map $\rho \mapsto v(\rho)$ gives a valuation on $\mathbb{Q}$, the **p-adic valuation**. The value group of ρ is $\mathbb{Z}$. The local ring is the localisation of $\mathbb{Q}$ at the ideal (p); the maximal ideal of the valuation ring is $p\mathbb{Q}$; hence the residue field is $\mathbb{Q}/p\mathbb{Q} = \mathbb{Z}/p\mathbb{Z}$.

We may view both parts of Example 5.18 as facets of the same general principle, Proposition 5.19. The general theory of these objects is expounded in detail in [Ser79, Chapter I].

5.19 Proposition. *Let A be a ring. The following are equivalent:*

1. *A is a local principal ideal domain (PID).*
2. *There exists a field K and a* surjective *valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ such that A is the local ring of K.*
3. *A is a Noetherian local ring with maximal ideal generated by a non-nilpotent element.*
4. *A is a Noetherian integral domain which is integrally closed and has a unique non-zero prime ideal.*

If any (hence all) of these conditions are satisfied, we call A a ***discrete valuation ring****.* ∎

There is a difference, though, between the two situations of Example 5.18: given an open set $U \subseteq \mathbb{C}$, it is possible for a meromorphic function to have infinitely many zeroes or poles on U. For example, $z \mapsto \sin z$ has infinitely many zeros on $\mathbb{C}$. This means that it is possible, having fixed a function $f \in M(U)$, for $\omega_{z_0}(f)$ to be non-zero for infinitely many $z_0 \in U$ (where here ω_{z_0} denotes the valuation at z_0). On the other hand, given a number $\rho \in \mathbb{Q}$, it is clearly the case that $v_p(\rho)$ is non-zero for only finitely many primes p (this is a consequence of the fundamental theorem of arithmetic, of course). The problem is that rings of holomorphic functions are not Noetherian: Proposition 5.20 guarantees that if our ring which we are localising in is Noetherian then we obtain a result analogous to a fundamental theorem of arithmetic: namely, that only finitely many primes divide each function in the ring.

5.20 Proposition. *Let A be an integrally closed Noetherian integral domain with field of fractions K. Let D be the set of non-zero ideals of A of height 1. Then for every $\mathfrak{p} \in D$ the localisation $A_\mathfrak{p}$ is a discrete valuation ring with valuation $v_\mathfrak{p}$. Further, for fixed $f \in K^*$, the set $\{\mathfrak{p} \in D : v_\mathfrak{p}(f) \neq 0\}$ is finite.*

Proof. Every localisation of a Noetherian integrally closed ring is Noetherian and integrally closed. Note that the maximal ideal $\mathfrak{m}$ of $A_\mathfrak{p}$ is precisely the set of elements of $A_\mathfrak{p}$ with numerator lying in $\mathfrak{p}$. Since $\mathfrak{p}$ has height 1, $\mathfrak{m}$ contains no nontrivial prime ideals and so $A_\mathfrak{p}$ satisfies part 4 of Proposition 5.19: $A_\mathfrak{p}$ is a discrete valuation ring.

We next show that if $f \in K^*$ is non-zero then only finitely many height 1 prime ideals of A contain f. If, for all $\mathfrak{p} \in D$ we have $v_\mathfrak{p}(f) = 0$, then we are done since $0 < \infty$. Let

$$\mathfrak{D} := \bigcap \{\mathfrak{p} \in D : v_\mathfrak{p}(f) \neq 0\}.$$

Since $\mathfrak{D}$ is an ideal in a Noetherian ring we may write $\mathfrak{D} = \cap_{\mathfrak{p} \in D'} D'$ for some finite subset $D' \subseteq D$. Let D'' be the set of height 1 ideals of D'. Now suppose $\mathfrak{p}$ is an arbitrary prime of A of height 1; if $v_\mathfrak{p}(f) \neq 0$ then $\mathfrak{p} \cap \mathfrak{D}$ is non-empty, so $\mathfrak{p}$ (being prime) is contained within one of the ideals in D'. Further, since D is of height 1 it must be equal to one of the ideals in D''; thus there are only finitely choices for $\mathfrak{p}$. ∎

5.21 Example. Consider as a final example a normal affine variety Spec A; then A is a Noetherian integral domain by definition. Consider a closed irreducible subvariety of codimension 1 (a **prime divisor** of Spec A); such a subvariety is given by a prime ideal $\mathfrak{p} \subseteq A$ which is of height 1. It follows that the local ring $A_\mathfrak{p}$ is a discrete valuation ring. The valuation on this ring, analogously to Example 5.14, counts the degree of vanishing of rational functions in Frac A on the prime divisor.

We shall now prove part 1 of Theorem 5.10. The link with valuation rings comes from the following standard result (a slight generalisation of [Har77, Exercise II.4.11] and [EGA II, Théorème 7.3.8]; compare [Ful93, note 10 to chapter 2]):

5.22 Theorem ([Stacks, Tag 0CM1])**.** *Let $f : X \to Y$ and $h : U \to X$ be morphisms of schemes, with Y locally Noetherian, f and h of finite type, and $h(U)$ dense in X. Then f is proper if and only if for each discrete valuation ring R with fraction field K, and every pair of morphisms* $\operatorname{Spec} K \to U$ *and* $\operatorname{Spec} R \to Y$ *making the solid square in the following diagram commute:*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "i"] & U \arrow[r, "h"] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[rr] \arrow[urr, dotted, "\exists!"] & & Y
\end{tikzcd}$$

(i being the morphism induced by $R \subseteq K$), there exists a unique morphism $\operatorname{Spec} R \to X$ *making the entire diagram commute.* ∎

Remark. For our purposes, since all schemes are varieties (more precisely, they are separated), we need not check uniqueness of $\dashrightarrow$ in Theorem 5.22: the uniqueness follows from the valuative criterion of separatedness [Har77, Theorem II.4.3].

5.23 Lemma (Part 1 of Theorem 5.10)**.** *A morphism $f : X_\Sigma \to X_\Pi$ of normal separated toric varieties with associated morphism of fans $\tilde{f} : \Sigma \to \Pi$ is proper iff*

$$\textit{for each } \pi \in \Pi,\ \tilde{f}^{-1}(\pi) = \bigcup \{\sigma \in \Sigma : \tilde{f}(\sigma) \subseteq \pi\}. \tag{F}$$

The reader should recall the remark following Theorem 5.10: namely, the morphism $\tilde{f} : \Sigma \to \Pi$ is supported on a function $N_{\mathbb{R}} \to N'_{\mathbb{R}}$ that restricts to a homomorphism $N \to N'$.

Proof. To fix notation, assume Σ is a fan over N, Π is a fan over N', and N and N' have respective dual lattices M and M',

Suppose f satisfies the property (F). We apply Theorem 5.22 with $X = X_\Sigma$, $Y = X_\Pi$, and $U = T_N$; that is, for some discrete valuation ring (R, ν) we wish to find a filler $\operatorname{Spec} R \to X_\Sigma$ for the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "i"] & T_N \arrow[r, hook] & X_\Sigma \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[rr] & & X_\Pi .
\end{tikzcd}$$

We may assume the closed point of $\operatorname{Spec} R$ maps into some affine open subset $U_\pi \subseteq X_\Pi$. Reversing arrows, our goal is to find some $\sigma \in \Sigma$ such that $\tilde{f}(\sigma) \subseteq \pi$ and providing a filler for the following diagram:

$$\begin{tikzcd}
K & K[M] \arrow[l, "\phi"'] & K[S_\sigma] \arrow[l, hook'] \arrow[dll, dashed] \\
R \arrow[u, hook] & & K[S_\pi] \arrow[ll] \arrow[u, "f^*"]
\end{tikzcd}$$

The existence of the map $K[S_\sigma] \to R$ making the diagram commute is equivalent to the inclusion $(\phi \circ f^*)(K[S_\pi]) \subseteq R$ or equivalently $(\nu \circ \phi \circ f^*)(K[S_\pi]) \subseteq \mathbb{Z}_{\geq 0}$. Note that $\nu \circ \phi$ restricts to a group morphism $M \to \mathbb{Z}$, so is identified with a member of the dual lattice N. Similarly, $\nu \circ \phi \circ f^* \in N'$. The condition $(\nu \circ \phi \circ f^*)(\pi^\vee \cap M') \subseteq \mathbb{Z}_{\geq 0}$ is now equivalent (recalling that the duality pairing of M' and N' is given by composition) to the condition $\nu \circ \phi \circ f^* \in (\pi^\vee)^\vee = \pi$; and $(\nu \circ \phi)(\sigma^\vee \cap M') \subseteq \mathbb{Z}_{\geq 0}$ implies $\nu \circ \phi \in \sigma$.

Now, note that $\tilde{f}$ gives a morphism of lattices $N \to N'$ with the property that $\tilde{f}(\nu \circ \phi) = \nu \circ \phi \circ f^*$. In particular, the existence of $\dashrightarrow$ in the diagram is equivalent to the existence of a cone σ over N with the property that $\tilde{f}(\sigma) \subseteq \pi$ and $\nu \circ \phi \in \sigma$.

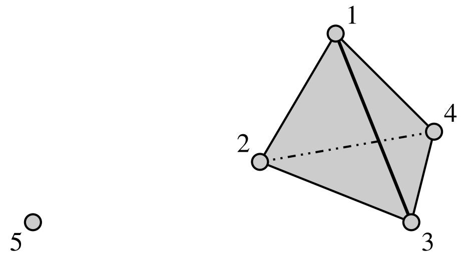


Figure 8: The simplicial complex with facets $\{1, 2, 3, 4\}$ and $\{5\}$.

Since $\tilde{f}(\nu \circ \phi) \in \pi$, and $\nu \circ \phi \in N$, it follows that $\tilde{f}^{-1}(\pi) \neq \emptyset$. Hence, since $\tilde{f}^{-1}(\pi) = \bigcup \{\sigma \in \Sigma : \tilde{f}(\sigma) \subseteq \pi\}$ and the latter is non-empty, we may find σ such that $\tilde{f}(\sigma) \subseteq \pi$, so $\dashrightarrow$ exists (and is unique by the remark following Theorem 5.22); in particular, σ exists if and only if f is proper. ∎

Part II
Cohomology theory and applications

6 Cohomology of sheaves and divisors

In this section, we present basic results on sheaf cohomology and the theory of divisors. The treatment is essentially elementary; the reader might also refer to [Osb00] and [Eis95, Appendix A] for homological algebra, [Har77, Chapter III] for cohomology of sheaves and Čech cohomology, and [Har77, Section II.6] for the theory of divisors.

6.1 Review of homological algebra

Let R be a ring.

6.1 Definition. Recall that a **complex** of R-modules is a sequence of R-modules and homomorphisms

$$A_\bullet := \cdots \longrightarrow A_{i+1} \xrightarrow{\partial_{i+1}} A_i \xrightarrow{\partial_i} A_{i-1} \longrightarrow \cdots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for each i (i.e. $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$). We call the module A_i the **term of degree** i of $A_\bullet$, and the map ∂_i the **ith boundary operator**. The elements of $\operatorname{im} \partial_i$ are called **boundaries**, and the elements of $\ker \partial_i$ are called **cycles**. The ith **homology module** of $A_\bullet$ is $H_i A_\bullet := \ker \partial_i / \operatorname{im} \partial_{i+1}$.

Intuitively, a complex is a chain of maps, each of which lowers the dimension by 1, which kill higher-dimensional boundaries at each step.

6.2 Example (Simplicial complexes)**.** A **simplicial complex** Δ on the **vertex set** $\{1, ..., n\}$ is a collection of subsets called **simplices** closed under taking subsets (i.e. if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$). The **dimension** of a face $\sigma \in \Delta$ is $|\sigma| - 1$ (note that $\dim \emptyset = -1$). The **dimension** of Δ is $\max_{\sigma \in \Delta} \dim \sigma$, or is defined to be $-\infty$ if $\Delta = \emptyset$. A **facet** of Δ is a maximal face.

For example, given the vertex set $\{1, 2, 3, 4, 5\}$ take the facets of Δ_5 to be $\{1, 2, 3, 4\}, \{5\}$. Then the complex has the face structure depicted in Fig. 8, where the tetrahedron has one three-dimensional face, four two-dimensional faces, and so forth.

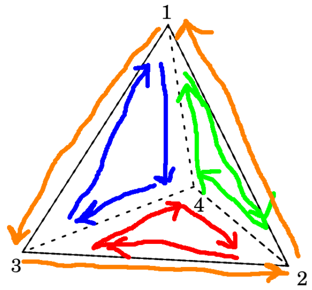


Figure 9: Boundaries on the simplicial complex T.

For each integer i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ. Let $k\{F_i(\Delta)\}$ be a k-vector space with (free) basis $\{e_\sigma : \sigma \in F_i(\Delta)\}$. Then the **reduced chain complex** of Δ (where Δ is supported on $\{1, ..., n\}$) is the complex

$$\tilde{C}_\bullet(\Delta, k) \coloneqq 0 \longrightarrow k\{F_{n-1}(\Delta)\} \xrightarrow{\partial_{n-1}} k\{F_{n-2}(\Delta)\} \cdots\cdots \longrightarrow k\{F_i(\Delta)\} \xrightarrow{\partial_i} k\{F_{i-1}(\Delta)\} \cdots\cdots \longrightarrow k\{F_0(\Delta)\} \xrightarrow{\partial_0} k\{F_{-1}(\Delta)\} \longrightarrow 0$$

where we define $\partial_i : k\{F_i(\Delta)\} \to k\{F_{i-1}(\Delta)\}$ by setting $\operatorname{sign}(j, \sigma)$ to be $(-1)^{r-1}$ if j is the rth element of the set $\sigma \subseteq \{1, ..., n\}$ written in ascending order and then defining

$$\partial_i(e_\sigma) = \sum_{j\in\sigma} \operatorname{sign}(j, \sigma) e_{\sigma\setminus j}.$$

One can formally show that $\partial_i \partial_{i+1} = 0$ for each i by direct computation but it is tedious.

In order to explain what this means consider the tetrahedron T, that is the complex on $\{1, 2, 3, 4\}$ with $\{1, 2, 3, 4\}$ as the only facet. Then:

$$\partial_3\{1, 2, 3, 4\} = \{2, 3, 4\} - \{1, 3, 4\} + \{1, 2, 4\} - \{1, 2, 3\}$$

and:

$$\partial_2\{2, 3, 4\} = \{3, 4\} - \{2, 4\} + \{2, 3\}.$$

We draw the image of $\partial_3\{1, 2, 3, 4\}$ as Fig. 9. Intuitively, the boundary map takes the oriented face (the entire tetrahedron with orientation $(1, 2, 3, 4)$) and cuts it up into oriented faces such that the sum of the orientations of those faces is zero. The composition $\partial_2\partial_3$ is zero since the cutting up of the tetrahedron into oriented faces by ∂_3 gives (triangular) faces whose orientations mutually cancel at the edges, so when the triangular faces are cut up again each edge is covered twice in opposing directions. (This explains the terminology of Definition 6.1; recall that in the integration theory of manifolds a cycle is a union of closed paths such that the formal sum of all the paths is zero; i.e. it is the image under a homeomorphism of a simplex with an orientation killed by the boundary operator defined here.)

Now for each i, the ith homology of $\tilde{C}_\bullet(\Delta, k)$ is the k-vector space

$$\tilde{H}_i(\Delta, k) = \ker \partial_i / \operatorname{im} \partial_{i+1}.$$

We shall compute the homologies of the tetrahedron T. We have that

$$\begin{aligned}
F_3(T) &= \{\{1,2,3,4\}\} \\
F_2(T) &= \{\{2,3,4\},\{1,3,4\},\{1,2,4\},\{1,2,3\}\} \\
F_1(T) &= \{\{3,4\},\{2,4\},\{2,3\},\{1,4\},\{1,3\},\{1,2\}\} \\
F_0(T) &= \{\{4\},\{3\},\{2\},\{1\}\} \\
F_{-1}(T) &= \{\emptyset\}
\end{aligned}$$

and so, taking these as our basis orderings we have the following boundary maps:

$$[\partial_3] = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$[\partial_2] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[\partial_1] = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$[\partial_0] = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

We find after a short computation that these matrices have ranks 1, 3, 3, 1 respectively; so $\dim \operatorname{im} \partial_3 = 1$ and $\dim \ker \partial_2 = 4 - 3 = 1$, allowing us to conclude that $\tilde{H}_3(T,k) \simeq k^0$. Similarly $\dim \operatorname{im} \partial_2 = 3$ and $\dim \ker \partial_1 = 3$ so $\tilde{H}_2(T,k) \simeq k^0$; and $\dim \operatorname{im} \partial_1 = 3$, $\dim \ker \partial_0 = 3$ so $\tilde{H}_1(T,k) = k^0$. Thus all the homologies are trivial.

On the other hand, the homomlogies are not trivial precisely when some chain is killed by a boundary operator that is not obtained by cutting higher dimensional faces; for example, in Δ_5 above the operator ∂_0 kills $\{5\} - \{1\}$, but the only possible face with this as boundary would be $\{1,5\}$, which is not a face of Δ_5 since $\{5\}$ is disconnected from the rest of the simplex. Thus $\operatorname{im} \partial_1$ is a strict subset of $\ker \partial_0$, and so $\tilde{H}_0(\Delta,k) = \ker \partial_0 / \operatorname{im} \partial_1$ detects connectedness. More generally, the ith homology $\tilde{H}_i(\Delta,k)$ detects existence of i-dimensional 'holes': $\dim \tilde{H}_i(\Delta,k)$ is the *number* of such holes.

Often in algebraic geometry it is the dual notion which appears; consider now a sequence of R-modules and homomorphisms

$$A^\bullet := \cdots \longleftarrow A^{i+1} \xleftarrow[d^{i+1}]{} A^i \xleftarrow[d^i]{} A^{i-1} \longleftarrow \cdots$$

such that $d^{i+1} \circ d^i = 0$ for each i (i.e. $\ker d^{i+1} \supseteq \operatorname{im} d^i$). We call the module A^i the term of degree i of $A^\bullet$, and the map d^i the ith coboundary operator. The elements of $\operatorname{im} d^i$ are called coboundaries, and the elements of $\ker d^i$ are called cocycles. The ith **cohomology** of $A_\bullet$ is $H^i A^\bullet := \ker d^i / \operatorname{im} d^{i+1}$.

6.2 Sheaf cohomology

Let X be a scheme. Recall that the functor $\Gamma(X, \cdot)$ from sheaves on X to abelian groups is left exact [Har77, Exercise II.1.8]; that is, if

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is a short exact sequence, then

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H}) \tag{7}$$

is exact. The goal of sheaf cohomology is to study the manner in which the induced morphism $\Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{H})$ is *not* surjective; this will allow us in many useful cases to compute properties of one sheaf using known properties of the other two.

In order to do cohomology we need to extend Eq. (7) to the right somehow. We will do this in a way analogous to constructing a free resolution of modules (recall that if A is an R-module that is a quotient of some free module F_1 by a relation module M, then we obtain an exact sequence $0 \to M \to F_1 \to A \to 0$; if M is not free, we may write $M = F_2/M'$ and extend the exact sequence to the left as $0 \to M' \to F_2 \to F_2 \to A \to 0$; and so forth). Instead of every element in our resolution being free, we want every element to make Hom right-exact.

The following theorem is standard [Osb00, Section 2.4]:

6.3 Theorem. *For $A \in \operatorname{Ob} \mathsf{Mod}(R)$ (more generally in any abelian category) the following are equivalent:*

1. $\operatorname{Hom}(\cdot, A)$ *is an exact functor from* $\mathsf{Mod}(R)$ *to* Ab.
2. *For every morphism $\alpha : B \to A$ and every monomorphism $\iota : B \to C$ there is a $\theta : C \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} & A & \\ 0 \arrow[r] & B \arrow[u, "\alpha"] \arrow[r, hook, "\iota"] & C \arrow[ul, "\theta"'] \end{tikzcd}$$

We call an object satisfying either (hence both) of these conditions ***injective****.* ∎

6.4 Definition. Given any sheaf $\mathscr{F}$ on X, a **injective resolution** of $\mathscr{F}$ is an exact sequence

$$\mathscr{F}^\bullet := 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \mathscr{F}^2 \longrightarrow \cdots$$

such that each $\mathscr{F}^i$ is injective.

6.5 Theorem (Enough injectives)**.** *We say that an abelian category* C *has* ***enough injectives*** *if every object of* C *is isomorphic to a subobject of an injective object.* Ab *has enough injectives.* $\mathsf{Mod}(R)$ *has enough injectives. If $(X, \mathscr{O}_X)$ is a ringed space, $\mathfrak{Mod}(X)$ and $\mathfrak{Ab}(X)$ (the categories of sheaves of $\mathscr{O}_X$-modules and sheaves of abelian groups on X respectively) have enough injectives.*

Proof. The case of Ab is due to Baer (1940) [Eis95, Corollary A3.7]. The case of $\mathsf{Mod}(R)$ is ue to Eckmann and Schöpf (1953) [Eis95, Corollary A3.9]. The cases of $\mathfrak{Mod}(X)$, $\mathfrak{Ab}(X)$ may be found as [Har77, proposition III.2.2 and corollary III.2.3]. ∎

6.6 Corollary. *Every sheaf $\mathscr{F}$ on X has an injective resolution.*

Proof. Embed $\mathscr{F}$ in an injective module $\mathscr{F}^0$; embed $\mathscr{F}^0/\mathscr{F}$ in an injective module $\mathscr{F}^1$; embed $\frac{\mathscr{F}^1}{\mathscr{F}^0/\mathscr{F}}$ in an injective module $\mathscr{F}^2$; and continue inductively in the obvious fashion. ∎

Injective resolutions are important because (since $\Gamma(X, \bullet)$ is left exact) they induce complexes:

$$\begin{gathered}
\mathscr{F}^\bullet = 0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \mathscr{F}^2 \longrightarrow \cdots \\
\Gamma(X, \mathscr{F}^\bullet) = 0 \xrightarrow{d^{-1} := 0} \Gamma(X, \mathscr{F}^0) \xrightarrow{d^0} \Gamma(X, \mathscr{F}^1) \xrightarrow{d^1} \Gamma(X, \mathscr{F}^2) \longrightarrow \cdots
\end{gathered} \tag{8}$$

The pth **sheaf cohomology group** is then $H^p(X, \mathscr{F}) := H^p(X, \mathscr{F}^\bullet) = \ker d^p / \operatorname{im} d^{p-1}$.[5] Every choice of injective resolution will give the same cohomology groups; this result is not trivial, and may be found as [Eis95, cor. A3.14].

Note that we start our complex at 0, *not* at $\Gamma(X, \mathscr{F})$. We do not lose any information, as part 1 of the following straightforward result tells us.

6.7 Proposition. *With the same notation as above,*

1. $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$*;*
2. *for all* $i \geq 0$, $\mathscr{F} \mapsto H^i(X, \mathscr{F})$ *is a functor from* $\mathfrak{Ab}(X)$ *to* Ab*. This is refered to as the ith* ***right derived functor*** *of* Γ.[6] ∎

Part 2 of the above proposition tells us that morphisms of sheaves induce individual morphisms of homology groups. When given a short exact sequence, this can be strengthened to a chain of morphisms of *all* the homology groups.

6.8 Theorem (The long exact sequence)**.** *Let*

$$0 \longrightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \longrightarrow 0 \tag{9}$$

be a short exact sequence of complexes in an abelian category (that is, each induced $0 \to A^i \to B^i \to C^i \to 0$ *is exact). Then for each i there is a natural map*

$$\partial_i : H^i(C^\bullet) \to H^{i+1}(A^\bullet)$$

such that the following ***long exact sequence in cohomology*** *of Eq.* (9) *is exact:*

$$\begin{aligned}
0 \longrightarrow H^0(A^\bullet) \xrightarrow{\alpha} H^0(B^\bullet) \xrightarrow{\beta} H^0(C^\bullet) &\xrightarrow{\partial_0} \\
H^1(A^\bullet) \longrightarrow \cdots \longrightarrow H^{i-1}(C^\bullet) &\xrightarrow{\partial_{i-1}} \\
H^i(A^\bullet) \xrightarrow{\alpha} H^i(B^\bullet) \xrightarrow{\beta} H^i(C^\bullet) &\xrightarrow{\partial_i} \\
\cdots\cdots\cdots\cdots\cdots\cdots &
\end{aligned}$$

References to proof. [Eis95, Secs. A3.7 and A3.8] or [Osb00, Theorem 3.3]. ∎

[5] [Har77] uses h^i not H^i.

[6] If F is a covariant left exact functor then [Har77] uses $R^i F$ to denote the ith right derived functor of F.

6.9 Corollary. *A short exact sequence of sheaves*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{aligned}
0 \longrightarrow H^0(X,\mathscr{F}) &\xrightarrow{\alpha} H^0(X,\mathscr{G}) \xrightarrow{\beta} H^0(X,\mathscr{H}) \xrightarrow{\partial_0} \\
H^1(X,\mathscr{F}) &\longrightarrow \cdots \longrightarrow H^{i-1}(X,\mathscr{H}) \xrightarrow{\partial_{i-1}} \\
H^i(X,\mathscr{F}) &\xrightarrow{\alpha} H^i(X,\mathscr{G}) \xrightarrow{\beta} H^i(X,\mathscr{H}) \xrightarrow{\partial_i} \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Proof. A short exact sequence of sheaves lifts to a short exact sequence of injective resolutions. Taking global sections of this short exact sequence gives a short exact sequence

$$0 \to \Gamma(X,\mathscr{F}^\bullet) \to \Gamma(X,\mathscr{G}^\bullet) \to \Gamma(X,\mathscr{H}^\bullet) \to 0$$

and then applying Theorem 6.8 finishes the proof. ∎

Toric varieties have nice vanishing properties for their cohomologies, so the long exact sequence becomes finite and we may compute properties of the sections using standard results like the rank-nullity theorem and its generalisations. We shall give some exemplary results later; see Theorem 7.12 and Theorem 7.13.

Sheaf cohomology turns out to be theoretically useful but difficult to compute with. The standard solution to this (for Noetherian separated schemes, at least) is to turn to a different cohomology theory which is easier to compute with and that gives the same cohomologies as sheaf cohomology. In fact, this cohomology will be very similar in style to the homology theory of the reduced chain complex of a simplex, which we discussed in Example 6.2.

6.10 Definition. Let X be a topological space, and pick an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ where A is endowed with some (fixed) well-ordering. For each finite subset $\{i_0, ..., i_p\} \subseteq A$, we write $U_{i_0,...,i_p} \coloneqq U_{i_0} \cap \cdots \cap U_{i_p}$. Let $\mathcal{F} \in AAb(X)$, and define a complex $\check{C}^\bullet(\mathcal{U},\mathscr{F})$ as follows:

- For all $p \geq 0$, set
$$\check{C}^p(\mathcal{U},\mathscr{F}) \coloneqq \prod_{i_0<\cdots<i_p} \mathscr{F}(U_{i_0,...,i_p}).$$

- Define $d : \check{C}^p(\mathcal{U},\mathscr{F}) \to \check{C}^{p+1}(\mathcal{U},\mathscr{F})$ by setting, for $\alpha \in \check{C}^p(\mathcal{U},\mathscr{F})$,
$$(d\alpha)_{i_0,...,i_p} = \prod_{j=0}^{p+1}(-1)^j \operatorname{res}^{U_{i_0,...,\hat{i}_j,...,i_{p+1}}}_{U_{i_0,...,i_{p+1}}}(\alpha_{i_0,...,\hat{i}_j,...,i_{p+1}}).$$

We define the pth **Čech cohomology group** of $\mathscr{F}$, *with respect to* $\mathcal{U}$, to be

$$\check{H}^p(\mathcal{U},\mathscr{F}) \coloneqq H^p\check{C}^\bullet(\mathcal{U},\mathscr{F}).$$

The important theorem is the following [Har77, Theorem III.4.5]:

6.11 Theorem. *Let X be a Noetherian separated scheme, $\mathcal{U}$ an open affine cover of X, and $\mathcal{F}$ a quasi-coherent sheaf on X. Then for all $p \geq 0$, there is a natural isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F}).$$

∎

6.3 Divisors in general

The subject of divisors on schemes is discussed in [Har77, Section II.6]; we pause to state the definitions and results we will need. Throughout this section, X is a normal variety.

6.12 Definition. A **prime divisor** D on X is an irreducible subvariety of codimension 1. The **local ring** of D is the ring

$$\mathscr{O}_{X,D} := \{f \in \mathbf{K}(X) : f \text{ is regular on some open} U \text{ with } D \cap U \neq \emptyset\}$$

We may restate Example 5.21 as the following:

6.13 Proposition. *The ring $\mathscr{O}_{X,D}$ is a discrete valuation ring.* ∎

Motivated by the discussion surrounding that example and Example 5.14, if $f \in \mathbf{K}(X)^*$ has $v_D(f) = n < 0$ then we say f has a **pole of order $-n$** on D; if $v_D(f) = n > 0$ we say it **vanishes to order** n on D. By Proposition 5.20, if $f \in \mathbf{K}(X)^*$ then $v_D(f) = 0$ for all but finitely many prime divisors D.

6.14 Definition. The **Weil divisor group** of X, denoted $\operatorname{Div}(X)$, is the free abelian group generated by the prime divisors of X. A member of $\operatorname{Div}(X)$ is a **Weil divisor**. If every coefficient of $D \in \operatorname{Div}(X)$ is nonnegative then we say D is **effective** and write $D \geq 0$.

If X is normal and $f \in \mathbf{K}(X)^*$ we define the **divisor** of f to be $\operatorname{div} f := \sum v_D(f) D$ where the sum is over all prime divisors D of X; by Proposition 5.20 this sum is finite. Any divisor of this form is called a **principal divisor**.

6.15 Theorem. *If D is a Weil divisor on X then we may define a coherent sheaf $\mathscr{O}_X(D)$ of $\mathscr{O}_X$-modules as follows:*

$$\Gamma(U, \mathscr{O}_X(D)) := \{f \in \mathbf{K}(X) : \operatorname{div}(f) + D \geq 0 \text{ on } U\} \cup \{0\}$$

(i.e. the set of functions which are regular on U modulo the poles measured by D).

Proof. For coherence see [CLS11, proposition 4.0.27]; the property of being a sheaf of $\mathscr{O}_X$-modules is left there to the reader and so for completeness we verify it here. Write D in the form $D = \sum_{P \in \operatorname{Div} X} a_P P$ where all but finitely many of the a_P are 0. First we show that each $\Gamma(U, \mathscr{O}_X(D))$ is an $\mathscr{O}_X(U)$-module: if $f, g \in \Gamma(U, \mathscr{O}_X(D))$ and $\lambda, \mu \in \mathscr{O}_X(U)$ then we wish to show that $v_P(\lambda f + \mu g) + a_P \geq 0$ for each $P \in \operatorname{Div} X$. Now $v_P(\lambda f + \mu g) \geq \min\{v_P(\lambda) v_P(f), v_P(\mu) v_P(g)\}$, and by assumption $v_P(\lambda), v_P(\mu) \geq 0$ (since λ and μ are regular on U) and $v_P(f), v_P(f) \geq -a_P$ (by assumption) so $\min\{v_P(\lambda) v_P(f), v_P(\mu) v_P(g)\} \geq -a_P$ and we are done.

It remains to show that $\mathscr{O}_X(D)$ is a sheaf; this follows easily since $\mathbf{K}(X)$ is a sheaf. ∎

Finally, we give a name to the Weil divisors which are locally of the form $\operatorname{div} f$ for some $f \in \mathbf{K}(X)^*$.

6.16 Definition. A Weil divisor $D = \sum_{P \in \operatorname{Div} X} a_P P$ on X is **Cartier** if there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that the divisor $\sum_{U_\alpha \cap P \neq 0} a_P (U_\alpha \cap P)$ is principal in U_α for each $\alpha \in A$.

Remark. We may extend the definition of Cartier divisor to an arbitrary scheme, see [Har77, p. 141].

7 Quasi-projective toric varieties

It is *not* true that a toric variety is always quasi-projective. In fact, one can characterise all quasi-projective toric varieties; we will do this as Theorem 7.18 and Theorem 7.25.

7.1 Divisors on toric varieties

Our goal is to state two vanishing theories for the cohomology groups of a toric variety: Theorem 7.12 and Theorem 7.13.

We shall first study the properties of torus-invariant prime Weil divisors of X_Σ.

7.1 Lemma. *The map* $\Sigma(1) \to \operatorname{Div}(X_\Sigma)$ *defined by* $\rho \mapsto D_\rho := \overline{\operatorname{orb}\rho}$ *is a bijection between the rays of* Σ *and the torus-invariant prime Weil divisors of* X_Σ.

Proof. This follows immediately from Theorem 5.6 and Corollary 4.44: any $\rho \in \Sigma(1)$ corresponds to an $(n-1)$-dimensional orbit $\operatorname{orb}\rho$ such that $\overline{\operatorname{orb}\rho}$ is an irreducible closed subvariety of codimension 1 invariant under the torus action. ∎

We first give a preliminary computational result.

7.2 Proposition. *If* $\rho \in \Sigma(1)$ *has minimal generator* u_ρ, *and* $m \in M$ *(where* M *is the character lattice of* X_Σ*), then:*

$$v_\rho(\chi^m) = \langle m|u_\rho\rangle\text{ ; and}$$
$$\operatorname{div}\chi^m = \sum_{\rho\in\Sigma(1)} \langle m|u_\rho\rangle\, D_\rho.$$

Proof. Extend u_ρ to a basis of M. Without loss of generality, we may assume $M = \mathbb{Z}^n$ and $u_\rho = e_1$, so $\rho = \mathbb{R}_{\geq 0}e_1$. Let (f_i) be the dual basis of N according to the pairing of characters and 1-psgs. Then $U_\rho = K \times (K^*)^{n-1}$ by Example 4.14; and one can easily show with the formalism developed above that $\overline{\operatorname{orb}\rho} = \{0\} \times (K^*)^{n-1} = \mathbf{Z}(X^{f_1})$, and

$$\mathbf{D}(\rho) \cap U_\rho = \{0\} \times (K^*)^{n-1} \cap (K \times (K^*)^{n-1}) = \{0\} \times (K^*)^{n-1}.$$

Hence the relevant local ring is $K[X^{f_1}, X^{\pm f_2}, ..., X^{\pm f_n}]_{(X^{f_1})}$ and the valuation v_ρ is defined by $v_\rho(f) = k$ for $f = (X^{f_1})^k(g/h)$ $(g, h \in K[X^{f_1}, X^{\pm f_2}, ..., X^{\pm f_n}]_{(X^{f_1})} \setminus (X^{f_1}))$. Now finally note that

$$\chi^m = (x^{f_1})^{m_1} \cdots (x^{f_n})^{m_n} = (x^{f_1})^{\langle m|e_1\rangle} \cdots (x^{f_n})^{\langle m|e_n\rangle}$$

so $v_D(\chi^m) = \langle m|e_1\rangle = \langle m|u_\rho\rangle$.

For the second part, note that the $\mathbf{D}(\rho)$ are the irreducible components of $X_\Sigma \setminus T_N$; since χ^m is non-zero on T, $\operatorname{div}\chi^m$ is supported on $X_\Sigma \setminus T = \bigcup_{\rho\in\Sigma(1)} D(\rho)$. Hence $\operatorname{div}\chi^m = \sum_{\rho\in\Sigma(1)} v_{D_\rho}(\chi^m)D(\rho)$. The result then follows from the first part. ∎

7.3 Definition. Given a Weil divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$, we may define a related polyhedron by

$$P_D = \{m \in M_\mathbb{R} : \forall_{\rho\in\Sigma(1)} \langle m|u_\rho\rangle \geq -a_\rho\} \tag{10}$$

where u_ρ denotes the minimal generator of ρ.

In this language, we see that the global sections of $\mathcal{O}_{X_\Sigma}(D)$ are determined by the characters given by the lattice points of P_D.

7.4 Theorem. *If D is a torus-invariant Weil divisor of X_Σ, then*

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} K\chi^m = \bigoplus_{m\in P_D\cap M} K\chi^m. \tag{11}$$

Proof. Let $f \in \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$. Let $\rho \in \Sigma(1)$; note that $\overline{\operatorname{orb}\rho} \cap T_N = \emptyset$, since $\overline{\operatorname{orb}\rho}$ is the union of orbits of cones containing ρ, and $0 \not\succeq \rho$. In particular, each prime divisor of X_Σ restricts to the zero divisor on T_N and so $D\restriction_{T_N} = 0$. By definition of $\mathscr{O}_{X_\Sigma}(D)$, we have $\operatorname{div} f + D \geq 0$; restricting to T_N, we therefore obtain $(\operatorname{div} f)_{T_n} \geq 0$, so f is regular on T_N and $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ is a subring of $\mathbf{A}(T_N) = K[M]$. Note that $K[M]$ is generated as a vector space over K by the characters on T_N; hence $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ is generated by the set of characters of T_N which lie inside it, i.e.

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\chi^m\in\Gamma(X_\Sigma),\mathscr{O}_{X_\Sigma}(D)} K\chi^m = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} K\chi^m.$$

It is easy to see that the second equality in Eq. (11) holds: $\operatorname{div}(\chi^m) + D \geq 0$ iff $v_\rho(\chi^m) + a_\rho \geq 0$ for all $\rho \in \Sigma(1)$, and by Proposition 7.2 we have $v_\rho(\chi^m) = \langle m|u_\rho\rangle$. The result is now immediate by comparison with Eq. (10). ∎

7.5 Definition. A variety V is a **vector bundle** of rank r over a variety X if there is a morphism $\pi : V \to X$ and an open cover $\{U_i\}_{i\in I}$ of X such that (i) for every i, there is an isomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that ϕ_i followed by projection onto U_i is just π restricted to $\pi^{-1}(U_i)$; and (ii) for every $i, j \in I$ there is $g_{i,j} \in \mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathscr{O}_X))$[7] such that the following diagram commutes:

$$\begin{tikzcd} & \pi^{-1}(U_i \cap U_j) \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\ U_i \cap U_j \times \mathbb{C}^r \arrow[rr, "g_{i,j}"] & & U_i \cap U_j \times \mathbb{C}^r \end{tikzcd}$$

A **section** of V over an open $U \subseteq X$ is a morphism $s : U \to V$ such that $\pi s = 1$.

Recall that for a scheme X, a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is **locally free** if there is an open cover $\{U_i\}_{i\in I}$ of X such that for each U_i, $\Gamma(\mathscr{F}, U_i)$ is a free $\mathscr{O}_{U_i}$-module.

The following result is standard [Har77, exercise II.5.18]:

7.6 Proposition. *With the notation of Definition 7.5, for each open subset U of X let $\mathscr{F}(U)$ be the set of all sections of V over U. Then $\mathscr{F}$ with the obvious restriction maps is a locally free sheaf of $\mathscr{O}_X$-modules.* ∎

A vector bundle of rank 1 is a **line bundle**. We have that ([Har77, propositions II.6.13 and II.6.15]):

7.7 Proposition. *If $\mathscr{L}$ is an locally free sheaf of rank 1 on a normal variety X, then there is a Cartier divisor D on X such that $\mathscr{L} \simeq \mathscr{O}_X(D)$ and there is a line bundle $V_\mathscr{L} \to X$ whose sheaf of sections is $\mathscr{L}$. Further, this line bundle is unique up to isomorphism.* ∎

7.8 Definition. Let $\mathscr{F}$ be a sheaf, and let $\mathscr{L}$ be a locally free sheaf of rank 1 (both on a normal variety X).

1. We say that $\mathscr{F}$ is **generated by its global sections** if there is a set $S \subseteq \Gamma(X, \mathscr{F})$ such that at any $x \in X$ the images of S under the canonical map generate the stalk $\mathscr{F}_x$.

2. We say that a subspace $W \subseteq \Gamma X, \mathscr{L}$ is **basepoint free** if for every $x \in X$ there exists $s \in W$ with $s(x) \neq 0$. Then $\mathscr{L}$ is **basepoint free** if $W = \Gamma X, \mathscr{L}$ is basepoint free.

[7] i.e. invertible linear maps in r variables over $\Gamma(U_i \cap U_j, \mathscr{O}_X)$

The primary motivation for the definition of *basepoint free* is the following trivial remark (compare [Har77, Remark II.7.8.1]).

7.9 Lemma. *If $\mathscr{L}$ is a basepoint-free sheaf on a normal variety X, and $\Gamma(X, \mathscr{L})$ has a finite basis $(e_0, ..., e_n)$, then the map*

$$X \ni x \to (a_0(x), ..., a_n(x)) \in K^{n+1}$$

induces a well-defined function $X \to \mathbb{P}^n_K$. ∎

7.10 Definition. Let D be a basepoint free torus-invariant divisor on X_Σ. By Theorem 7.4, $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is finite-dimensional. We denote by ϕ_D the map of Lemma 7.9.

We say that D and $\mathcal{O}_{X_\Sigma}(D)$ are **very ample** if ϕ_D is a closed embedding. We say that D and $\mathcal{O}_{X_\Sigma}(D)$ are **ample** if kD is very ample for some $k \in \mathbb{Z}_{>0}$.

It is a standard result that (when $\mathscr{F} = \mathscr{L}$) the two concepts of Definition 7.8 are equivalent. More formally,

7.11 Lemma ([Har77, Lemma II.7.8])**.** *Let $\mathscr{L}$ be a locally free sheaf of rank 1 on a normal variety X. Then $\mathscr{L}$ is generated by its global sections iff it is basepoint free.* ∎

We now state the promised vanishing theorems. The proofs are not too involved, but are fairly lengthy and so we omit them; the idea is to use the combinatorics of the divisors of X_Σ to compute Čech cohomologies via the enumeration of lattice points.

7.12 Theorem (Demazure vanishing [CLS11, Theorem 9.2.3])**.** *Let D be a divisor on X_Σ such that nD is Cartier for some $n \in \mathbb{N}$. If $|\Sigma|$ is convex as a subset of $\mathbb{R}^n$ and if $\mathcal{O}_{X_\Sigma}(nD)$ is generated by its global sections, then*

$$i > 0 \implies H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0.$$ ∎

and

$$i = 0 \implies H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} K\chi^m.$$

(Note that the second part of this follows immediately from Theorem 7.4.)

The second vanishing theorem will state that the cohomology of a negated divisor vanishes everywhere except in a single degree, which may be determined combinatorially.

7.13 Theorem (Batyrev-Borisov vanishing [CLS11, Theorem 9.2.7])**.** *Let D be a divisor on X_Σ such that nD is Cartier for some $n \in \mathbb{N}$. If X_Σ is complete and if $\mathcal{O}_{X_\Sigma}(nD)$ is generated by its global sections, then*

$$i \neq \dim P_D \implies H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(-D)) = 0.$$

and

$$i = \dim P_D \implies H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(-D)) = \bigoplus_{m \in \mathrm{relint}(P_D) \cap M} K\chi^{-m}.$$ ∎

The similarity between Theorem 7.12 and Theorem 7.13 is the first sign we see of the Ehrhart duality which appears in Theorem 8.8.

7.2 A cohomology computation

We will now compute some Čech cohomology groups for projective space, filling in some of the details of [CLS11, Example 9.1.1]. In fact, we will prove Theorem 7.12 and Theorem 7.13 for the special case $X_\Sigma = \mathbb{P}^2$ and a particularly natural choice of D.

Recall that a Čech cohomology complex depends on a choice of ordered open cover; our discussion of divisors above, as well as our classification theorems, provide us with a canonical open cover for a toric variety X_Σ (where $\dim \Sigma = n$), namely

$$X_\Sigma = \bigcup_{\sigma \in \Sigma(n)} U_\sigma.$$

We shall fix some arbitrary ordering for this cover, and write $\mathcal{U} = \{U_1, ..., U_k\}$ for it.

Let D be a torus-invariant Weil divisor on X_Σ. By definition, we may write the pth Čech complex module as

$$\check{C}^p(\mathcal{U}, \mathcal{O}_X(D)) = \prod_{i_0 < \cdots < i_p} \Gamma(U_{i_0,\ldots,i_p}, \mathcal{O}_X(D)) = \prod_{i_0 < \cdots < i_p} H^0(U_{i_0,\ldots,i_p}, \mathcal{O}_X(D));$$

using Theorem 7.4, we have

$$H^0(U_{i_0,\ldots,i_p}, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D(U_{i_0,\ldots,i_p}) \cap M} K\chi^m$$

(where we write $P_D(V)$ for the polyhedron corresponding to the restriction of D to an open subvariety V) and so we may set up a grading on $\check{C}^p(\mathcal{U}, \mathcal{O}_X(D))$ by writing

$$\check{C}^p(\mathcal{U}, \mathcal{O}_X(D)) = \bigoplus_{i_0 < \cdots < i_p} \left(\bigoplus_{m \in P_D(U_{i_0,\ldots,i_p}) \cap M} K\chi^m \right) = \bigoplus_{i_0 < \cdots < i_p} \left(\bigoplus_{m \in M} A(U_{i_1,\ldots,i_p})_m \right)$$

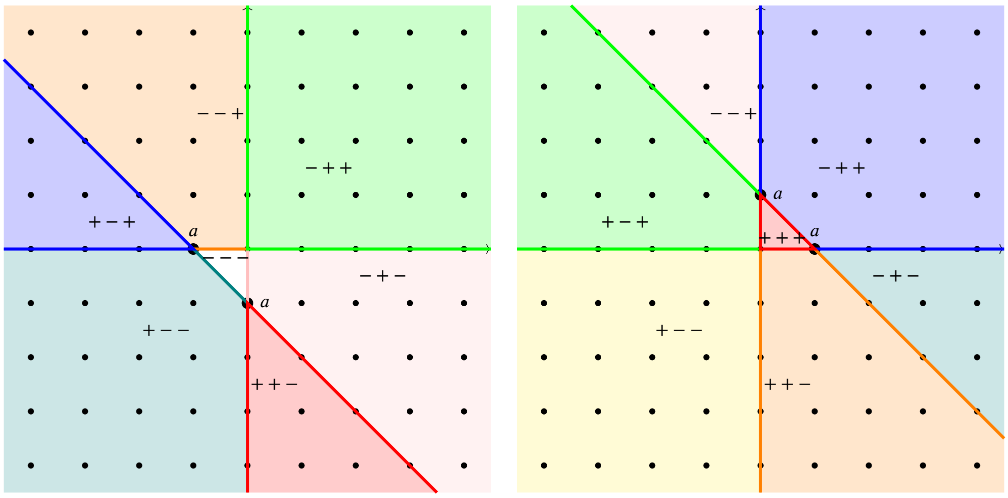
where $A(U_{i_1,\ldots,i_p})_m$ is defined to be $K\chi^m$ for $m \in P_D(U_{i_0,\ldots,i_p}) \cap M$ and 0 otherwise. Note that now the dummy variable of the inner sum is independent of the outer sum and so we may swap them; that is, we may grade each $\check{C}^p(\mathcal{U}, \mathcal{O}_X(D))$ by M and hence obtain a grading of each Čech cohomology module.

The cohomology of $\mathbb{P}^2$. Specialising to the fan $\Sigma \subseteq \mathbb{R}^2$ with facets $U_0 = \operatorname{pos}\{e_1, e_2\}$, $U_1 = \{e_0, e_2\}$, and $U_2 = \{e_0, e_1\}$ (where $e_0 = -e_1 - e_2$) we obtain (with $M = \mathbb{Z}^2$) the toric variety $X \coloneqq X_\Sigma = \mathbb{P}^2$ (compare Example 5.5 and Fig. 6 on Page 44). Set $aD_0 \coloneqq aD_{e_0}$, for some $a \in \mathbb{Z}$.

The case $a < 0$. By our discussion above, $\check{C}^0(X, \mathcal{O}_X(aD_0)) = \Gamma(U_0, \mathcal{O}_X(aD_0)) \oplus \Gamma(U_1, \mathcal{O}_X(aD_0)) \oplus \Gamma(U_2, \mathcal{O}_X(aD_0))$. Now note that $P_D(U_0) \cap \mathbb{Z}^2 = \{m \in \mathbb{Z}^2 : \langle m|e_1 \rangle \geq 0, \langle m|e_2 \rangle \geq 0\}$ and so by Theorem 7.4 we have $\Gamma(U_0, \mathcal{O}_X(aD_0)) = \bigoplus_{m \in \mathbb{Z}^2_{\geq 0}} K\chi^m$.

Similarly, $\Gamma(U_1, \mathcal{O}_X(D_1))$ is generated by the characters of $\{m \in \mathbb{Z}^2 : \langle m|e_0 \rangle \geq -a, \langle m|e_2 \rangle \geq 0\}$, and this set is precisely the set of lattice points $\mathbb{Z}^2 \cap \{(x, y) : x + y \leq a, y \geq 0\}$; i.e. $\Gamma(U_1, \mathcal{O}_X(D_1)) = \bigoplus_{m \in \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0}} K\chi^{ae_1}\chi^m$. Finally, $\Gamma(U_2, \mathcal{O}_X(D_2)) = \bigoplus_{m \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0}} K\chi^{-ae_2}\chi^m$.

Motivated by these computations, and to simplify notation, we set up the chamber complex C displayed in Fig. 10a. Here, the cell C_{---} is the set of points $m \in \mathbb{Z}^2$ with $\langle m|e_0 \rangle < a$, $\langle m|e_1 \rangle < 0$, and $\langle m|e_2 \rangle < 0$; C_{+--} is the set of points m with $\langle m|e_0 \rangle \geq a$, $\langle m|e_1 \rangle < 0$, and $\langle m|e_2 \rangle < 0$; and in general a $+/-$ in the ith place ($i \in \{0, 1, 2\}$) denotes a respective $\geq/<$ in the e_ith inequality. Then the computations above show that $A(U_0)_m \neq 0 \iff m \in C_{-++}$; $A(U_1)_m \neq 0 \iff m \in C_{+-+}$; and $A(U_2)_m \neq 0 \iff m \in C_{++-}$.



(a) The case $a < 0$ (reproduced from [CLS11, p. 400]).

(b) The case $a > 0$.

Figure 10: The cohomology chamber complex of $\mathbb{P}^2$

This figure simplifies the higher order computations in the following way. Consider $U_{0,1} = U_0 \cap U_1$; this is the subvariety $U_{\mathrm{pos}\, e_1}$, so we have the inequality $\langle m|e_1\rangle \geq 0$: thus $A(U_{0,1})_m \neq 0$ when m lies in the union of $C_{\dots}$ with a $+$ in the second position, i.e. $A(U_{0,1})_m \neq 0 \iff m \in C_{++-} \cup C_{-+-} \cup C_{-++}$. Similarly, $A(U_{0,2})_m \neq 0 \iff m \in C_{-++} \cup C_{--+} \cup C_{+-+}$ and $A(U_{1,2})_m \neq 0 \iff m \in C_{+-+} \cup C_{+--} \cup C_{++-}$.

Finally for the single third order intersection, we have $U_{0,1,2} = U_{\mathrm{pos}\, 0}$ and so we have no inequalities: $A(U_{0,1,2})_m \neq 0$ for all $m \in \mathbb{Z}^2$.

By definition of the gradings on the Čech complex components, we have the following:

$$
\begin{aligned}
\check{C}^0(X, \mathcal{O}_X(aD_0)) &= \bigoplus_{m \in M} A(U_0)_m \oplus \bigoplus_{m \in M} A(U_2)_m \oplus \bigoplus_{m \in M} A(U_2)_m \\
&= \bigoplus_{m \in C_{-++}} K\chi^m \oplus \bigoplus_{m \in C_{+-+}} K\chi^m \oplus \bigoplus_{m \in C_{++-}} K\chi^m \\
\check{C}^1(X, \mathcal{O}_X(aD_0)) &= \bigoplus_{m \in M} A(U_{0,1})_m \oplus \bigoplus_{m \in M} A(U_{0,2})_m \oplus \bigoplus_{m \in M} A(U_{1,2})_m \\
&= \bigoplus_{m \in C_{++-} \cup C_{-+-} \cup C_{-++}} K\chi^m \oplus \bigoplus_{m \in C_{-++} \cup C_{--+} \cup C_{+-+}} K\chi^m \oplus \bigoplus_{m \in C_{+-+} \cup C_{+--} \cup C_{++-}} K\chi^m \\
\check{C}^2(X, \mathcal{O}_X(aD_0)) &= \bigoplus_{m \in M} A(U_{0,1,2})_m = \bigoplus_{m \in M} K\chi^m.
\end{aligned}
$$

Let us now fix some $m \in \mathbb{Z}^2$; from the equations just above we can read off the dimension of the mth graded part of each $\check{C}^p(X, \mathcal{O}_X(aD_0))$, and the results are given in Table 1.

Table 1: Dimension of $\check{C}^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))$ for $a < 0$.

$m \in ...$	$\dim \check{C}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))$	$\dim \check{C}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))$	$\dim \check{C}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))$
$C_{-++} \cup C_{+-+} \cup C_{++-}$	1	2	1
$C_{--+} \cup C_{-+-} \cup C_{+--}$	0	1	1
C_{---}	0	0	1

The graded Čech complex, for $m \in \mathbb{Z}^2$, is given by the following diagram (c.f. Example 6.2):

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \check{C}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \arrow[r] \arrow[d, equal] & \check{C}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \arrow[r] \arrow[d, equal] & \check{C}^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^2 \Gamma(U_i, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \arrow[r, swap, "\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}"] & \bigoplus_{0 \le i < j \le 2} \Gamma(U_{i,j}, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \arrow[r, swap, "{[1\ -1\ 1]}"] & \Gamma(U_{0,1,2}, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \arrow[r] & 0
\end{tikzcd}$$

Suppose for the sake of argument that $m \in C_{-++}$. Then, specialising the diagram above, we have

$$0 \longrightarrow K\chi^m \oplus 0 \oplus 0 \xrightarrow[d^1]{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}} K\chi^m \oplus K\chi^m \oplus 0 \xrightarrow[d^2]{[1\ -1\ 1]} K\chi^m \longrightarrow 0$$

We have $d_1(x,0,0) = (x,x,0)$ and so $\operatorname{im} d_1 = \{(x,y,0) : x = y\}$, $\ker d_1 = 0$. Note $d_2(x,y,0) = (x-y)$, so $\ker d_2 = \{(x,y,0) : x = y\}$ and $\operatorname{im} d_2 = K\chi^m$; this exhibits the exactness of the sequence in this case, and by almost the exact same computation the sequence is exact for every other m lying in some cell listed in the first column of Table 1.

In particular, since taking cohomology is compatible with the grading, we have $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m = 0$ if $m \notin \operatorname{int} C_{---} = \operatorname{int} \operatorname{conv}\{(0,0),(a,0),(0,a)\}$. Conversely, if $m \in \operatorname{int} C_{---}$ then $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \neq 0$ (clearly the sequence is exact everywhere that $p \neq 2$ by rank considerations).

That is (setting $\Delta_2 := \operatorname{conv}\{0, e_1, e_2\}$, so $C_{---} = a\Delta_2$),

$$\dim H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0)) = \begin{cases} 0 & p \neq 2 \\ \left|\operatorname{int} a\Delta_2 \cap \mathbb{Z}^2\right| & p = 2. \end{cases}$$

The case $a > 0$. When $a > 0$, much the same argument as the previous case will work. We now obtain the chamber complex of Fig. 10b; and using the combinatorial methods above, we see that $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m = 0$ if $m \notin C_{+++} = \operatorname{conv}\{(0,0),(a,0),(0,a)\}$. Conversely, if $m \in C_{+++}$ then $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0))_m \neq 0$ iff $p = 0$. In particular,

$$\dim H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0)) = \begin{cases} 0 & p \neq 0 \\ \left|a\Delta_2 \cap \mathbb{Z}^2\right| & p = 0. \end{cases}$$

The main theorem. We have a simple combinatorial identity for counting the lattice points inside the n-simplex:

7.14 Lemma. *Let $\Delta_n = \mathrm{conv}\{0, e_1, \ldots, e_n\}$. Then, for $a \in \mathbb{Z}$:*

1. *$\left|a\Delta_n \cap \mathbb{Z}^n\right| = \binom{|a|+n}{n}$; and*
2. *$\left|\mathrm{int}\, a\Delta_n \cap \mathbb{Z}^n\right| = \binom{|a|-1}{n}$.* ∎

Using this, and the results above, we obtain the following theorem.

7.15 Theorem. *Let D_0 be the divisor of $\mathbb{P}^2$ associated to $\mathrm{pos}\{-e_1 - e_2\}$, and let $a \in \mathbb{Z}$. Then, if $a < 0$ we have*

$$\dim H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0)) = \begin{cases} 0 & p \neq 2 \\ \left|\mathrm{int}\, a\Delta_2 \cap \mathbb{Z}^2\right| = \binom{-a-1}{2} & p = 2; \end{cases}$$

and if $a > 0$ then we have

$$\dim H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(aD_0)) = \begin{cases} 0 & p \neq 0 \\ \left|a\Delta_2 \cap \mathbb{Z}^2\right| = \binom{a+2}{2} & p = 0. \end{cases}$$ ∎

We shall prove a more general version of this statement as Theorem 8.4.

7.3 The classification theorems

7.16 Definition. Let $P \subseteq M_{\mathbb{R}}$ be a polyhedron. We say that the cone C of Theorem 2.13 is the **recession cone** of P. The polyhedron P is a **lattice polyhedron** over M if $F_0(P) \subseteq M$ and C is a strongly convex lattice cone over M.

Note that if P is a lattice polyhedron over M, we may refine Corollary 3.26 by replacing $\mathrm{pos}(P - x)$ by $\mathrm{pos}(P \cap M - x)$ as each vertex $x \in F_0(P)$ is the intersection of defining hyperspaces and the pos is computed by intersecting some subset of translates of these hyperplanes by a lattice point.

7.17 Lemma. *If P is a full-dimensional lattice polyhedron over M with recession cone C, and F is a face of P, then $N_P(F) \subseteq N_{\mathbb{R}}$ is a lattice cone over N. Let $\Sigma(P) := \{N_P(F) : F \preceq P\}$; then $\Sigma(P)$ is a fan (the **normal fan** of P) with $\left|\Sigma(P)\right| = C^\vee$. In particular, if P is a polytope, $\Sigma(P)$ is complete.*

Proof. By Corollary 3.26 and Lemma 3.28, $N_P(F)$ is a lattice cone over N. It is explicit in the construction that if $\sigma \in \Sigma(P)$ and if $\tau \preceq \sigma$ then $\tau \in \Sigma(P)$. Finally, let $N_P(F), N_P(G) \in \Sigma(P)$. Consider the intersection $I := N_P(F) \cap N_P(G)$; for each $y \in I$ there exists a pair of hyperplanes H, L with $F = P \cap H$, $G = P \cap L$, and:

1. $y \in H^\perp \cap L^\perp$;
2. $y + h \in H^-$, $y + l \in L^-$ for each $h \in H, l \in L$.

Suppose $H = L$. Then clearly $I = N_P(H \cap P)$, which is a face of both $N_P(F)$ and $N_P(G)$. If $H \neq L$, we have two cases: if H and L are not parallel, $H^\perp \cap L^\perp = 0$ and so $y = 0$; i.e. I is the zero face. On the other hand, if H and L are parallel but not equal then let x be a vector perpendicular to H such that $L = H + x$; note that $P \subseteq H^- \cap L^-$. If $y \neq 0$, then $\mathbb{R}_{\geq 0} y$ is unbounded and perpendicular to both H and L (i.e. it lies in the direction of x) so cannot lie entirely in $H^- \cap L^-$; but $y \in I \subseteq H^- \cap L^-$, which is a contradiction and so $y = 0$; thus $I = 0$. In each of the three cases, I is a face of a cone in $\Sigma(P)$ and lies in $\Sigma(P)$.

For each $v \in F_0(P)$, let σ_v be the maximal cone of $N_P(v)$. To prove that $|\Sigma(P)| = C^\vee$, it will suffice to see that $\cup_{v \in F_0(P)} \sigma_v = C^\vee$. By Corollary 3.26, we have $\sigma_v = \mathrm{pos}(P-v)^\vee$. If $c \in C$, then $(v+c)-v \in C-v$ so $C \subseteq \mathrm{pos}(P-v)$ and thus $\sigma_v = \mathrm{pos}(P-v)^\vee \subseteq C^\vee$. Conversely, suppose $n \in \sigma_v$ and pick $v \in F_0(P)$ minimising $\langle \cdot | n \rangle : F_0(P) \to \mathbb{R}$ (this exists since $F_0(P)$ is finite). We claim that $n \in \sigma_v$. Indeed, by the remark following Definition 7.16 it suffices to show that $\langle m - v | n \rangle \geq 0$ for all $m \in P \cap M$. Note that if $m \in P \cap M$ we may write it (since P is convex and generated by its vertices) in the form $m = \sum_{w \in V} \lambda_w w + c$ for $\lambda_w \in \mathbb{R}_{\geq 0}$, $\sum_{w \in W} \lambda_1 = 1$, and $c \in C$. Then:

$$\langle m | n \rangle = \langle \sum_{w \in V} \lambda_w w + c | n \rangle = \sum_{w \in V} \lambda_w \langle w | n \rangle + \langle c | n \rangle \geq \sum_{w \in V} \lambda_w \langle v | n \rangle = \langle v | n \rangle$$

and by subtraction we have $\langle m - v | n \rangle \geq 0$.

The final claim follows since, if P is a polytope (hence bounded), $C = 0$. ∎

We wish to show that the variety $X_{\Sigma(P)}$ is quasi-projective, and that every quasi-projective variety arises in this way. More precisely, we will prove one direction of the following:

7.18 Theorem (Classification of quasi-projective toric varieties, I)**.** *Let P be a full-dimensional lattice polyhedron over M. Then $X_{\Sigma(P)}$ is quasi-projective. Conversely, if X_Σ is quasi-projective where $|\Sigma|$ is full-dimensional and convex then $\Sigma = \Sigma(P)$ for some full-dimensional lattice polyhedron P over M.*

The implication 'P full dimensional lattice polyhedron $\implies X_{\Sigma(P)}$ is quasiprojective' is proved below as Lemma 7.23. The converse is more involved, and may be found as [CLS11, Proposition 7.2.9].

We shall develop the relevant theory first, following [CLS11, Sections 6.1, 7.1].

7.19 Definition. Let P be a full-dimensional lattice polyhedron over M, written as $P = \cap_{i=1}^k (\langle \cdot | f_i \rangle \geq \alpha_i)$ for $f_1, ..., f_k \in N_\mathbb{R}$ and $\alpha_1, ..., \alpha_k \in \mathbb{R}$. Then P is **very ample** if for every $v \in F_0(P)$, the semigroup $\mathbb{Z}_{\geq 0}(P \cap M - v)$ is saturated.

The following result in convex geometry is non-trivial but essentially elementary (we neglect the proof as it requires Carathéodory's theorem, which we do not need at any other point: the idea is to do the proof for simplices with empty lattice interior first, and then to use an argument based on Carathéodory's theorem to decompose kP for some $k$into such polytopes).

7.20 Theorem ([BGT97, Theorem 1.3.1],[CLS11, Proposition 7.1.9])**.** *Suppose* $\dim M \geq 2$*, and let P be a full-dimensional lattice polyhedron over M. Then kP is very ample for all $k \geq n-1$.* ∎

We may now define a 'natural' divisor on $X_{\Sigma(P)}$. Note that by Corollary 3.26 the rays of $\Sigma(P)$ correspond to the *facets* of P. Thus, comparing Lemma 7.1, torus-invariant prime divisors of $X_{\Sigma(P)}$ are given by $D_F := \overline{\mathrm{orb}\, N_P(F)}$.

7.21 Definition. Let P be a full-dimensional lattice polyhedron over M, written in intersection form as $P = \cap_{F \in F_{n-1}(P)} (\langle \cdot | f_F \rangle \geq \alpha_F)$ where $\alpha_F \in \mathbb{R}$ for all F. Define D_P, the **divisor of** P, by

$$D_P := \sum_{\sigma \in F_{n-1}(P)} \alpha_F D_F$$

We now prove some properties of the divisor of a polyhedron; part 2 of the following proposition shows that D_P is locally determined by the vertices of P, and part 3 and the proof of part 4 of the following proposition indicate how the properties of D_P relate to possible projective embeddings of $X_{\Sigma(P)}$.

7.22 Proposition. *Let P be a full-dimensional lattice polyhedron over M. Set $X := X_{\Sigma(P)}$. Then:*

1. $P_{D_P} = P$.

2. *D_P is Cartier; more precisely, if $\sigma \in F_n(\Sigma(P))$ then $D_P \restriction_{U_\sigma} = \operatorname{div} \chi^{m_\sigma}$ for some $m_\sigma \in F_0(P)$.*

3. *$\mathcal{O}_X(D_P)$ is basepoint free.*

4. *D_P is very ample iff P is very ample.*

5. *D_P is ample.*

Proof.

1. This follows immediately from the definitions, Definition 7.3 and Definition 7.21.

2. We show that D_P is principal on each open subset U_σ ($\sigma \in \Sigma$ maximal). Fix such a σ. There exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$ (where u_ρ is the minimal generator of ρ): since σ is maximal, it corresponds to a vertex $v \in F_0(P)$ and a ray ρ lies in $\sigma(1)$ iff v is a face of the facet F of P corresponding to ρ; i.e. $\langle v | u_\rho \rangle = -a_\rho$; and $v \in M$ since P is a lattice polytope. We may now apply Proposition 7.2 to see that $D_P = \operatorname{div} \chi^{m_\sigma}$ on U_σ.

3. Let $\sigma \in \Sigma(n)$. In particular, Theorem 7.4 applies and thus we may view χ^{m_σ} as identified with an element of $\Gamma(X, \mathcal{O}_X(P_D))$. But on U_σ, χ^{m_σ} is non-zero (in fact, $U_\sigma = \mathbf{D}(m_\sigma)$) (again by Proposition 7.2) and so we may choose $s = \chi^{m_\sigma}$ in Definition 7.8.

4. Observe that by part 3 applied to Lemma 7.9 we have a well-defined map $\phi_D : X \to \mathbb{P}^k_K$ induced by an affine morphism $\tilde{\phi}_D x \mapsto (\chi^{m_0}(x), ..., \chi^{m_k}(x))$ where $m_0, ..., m_k \in P \cap M$ and $(\chi^{m_0}, ..., \chi^{m_k})$ is a K-basis for $\Gamma(X, \mathcal{O}_X(D_P))$; and D_P is very ample iff this map is a closed embedding. Moreover, note that $\phi_D(X)$ is a projective toric variety: the image of the map $\tilde{\phi}_D$ is an affine toric variety (i.e. a closed subset of $\mathbb{A}^{k+1}_K$) with torus $\tilde{T} = ((\chi^{m_0}(t), ..., \chi^{m_k}(t)) : t \in T_N)$, and the quotient is again a toric variety as the image of an affine toric variety under $\mathbb{P}$ is a projective toric variety (c.f. Example 5.5). In particular, $\phi_D(X)$ is closed in $\mathbb{P}^k$, and thus it suffices to show that ϕ_D is an isomorphism onto its image (the half of 'closed embedding' that is not always true) iff P is very ample.

   Let $\mathcal{V} \subseteq \{0, ..., k\}$ be the set of indices such that $F_0(P) = \{m_v : v \in \mathcal{V}\}$, and let $U_{\sigma_v} = \mathbf{D}(m_v)$ be the toric open subvariety of X given by the cone of m_v. Note that $\phi_D(U_{\sigma_v}) \subseteq U_v := \{[x_0 : ... : x_k] \in \mathbb{P}^k : x_v \neq 0\}$. Now note that $\{U_{\sigma_v}\}_{v \in \mathcal{V}}$ is an affine open cover of X and $\{U_v\}_{v \in \mathcal{V}}$ is an affine open cover of $\mathbb{P}^k$, so it will suffice to show that the affine restrictions $\phi^v_D := \phi_D \restriction_{U_{\sigma_v}} : U_{\sigma_v} \to \phi_D \cap U_v$ are all isomorphisms.

   We may now apply our theory of affine toric varieties. Fix some $v \in \mathcal{V}$. Then ϕ^v_D is an isomorphism iff the dual semigroup map $\sigma_v^\perp \cap M \to \mathbb{Z}_{\geq 0}(P \cap M - m_v)$ is an isomorphism; but recall that $\sigma_v^\perp = N_P(v)^\perp = \operatorname{pos}(P \cap M - m_v)$ (by Corollary 3.26), and so taking an intersection with M equality between $\sigma_v^\perp \cap M$ and $\mathbb{Z}_{\geq 0}(P \cap M - m_v)$ holds if and only if $\mathbb{Z}_{\geq 0}(P \cap M - v)$ is saturated.

5. By the previous part, it suffices to show that kP is very ample for some $k \geq 0$ (indeed, then $D_{kP} = kD_P$ is very ample, so D_P is ample); by Theorem 7.20, this is true if $\dim M \geq 2$, and if $\dim M = 1$ then $\dim P$ is a closed segment or a ray and so P is obviously very ample. ∎

The proof of the direction of Theorem 7.18 which we do prove uses very similar ideas to those developed in the proof of Proposition 7.22""

7.23 Lemma. *If P is a full-dimensional lattice polyhedron, then $X_{\Sigma(P)}$ is quasiprojective.*

Proof. Suppose that P is very ample, and let $\phi_{D_P} : X_{\Sigma(P)} \to \mathbb{P}^k$ be the induced morphism of D_P. As in the proof of part 3 of Proposition 7.22, we have:

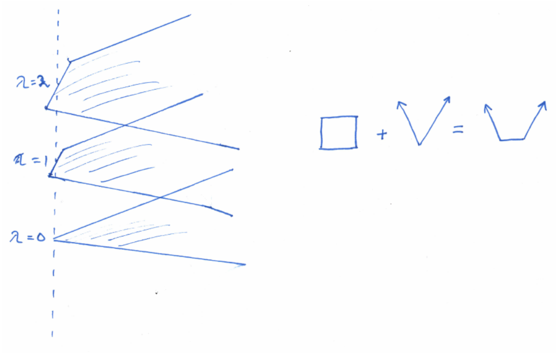


Figure 11: Sections of $C(P)$ for P the Minkowski sum of a square and a cone.

- A subset $\mathcal{V} \subseteq \{0, ..., k\}$;
- An affine open cover $\{U_v\}_{v \in \mathcal{V}}$; and
- An affine open cover $\{U_{\sigma_v}\}_{v \in \mathcal{V}}$

such that

$$\phi_{D_P}\restriction_{U_{\sigma_v}} : U_{\sigma_v} \to U_i \cap \phi_{D_P}(X_{\Sigma_P})$$

is an isomorphism onto its image. Further, we showed that $\phi_{D_P}(X_{\Sigma_P})$ is projective. Hence we may write ϕ_{D_P} as

$$\phi_{D_P} : X_{\Sigma(P)} = \bigcup U_{\sigma_v} \xrightarrow{\sim} \bigcup U_i \cap \phi_{D_P}(X_{\Sigma_P}) \subseteq \mathbb{P}^k$$

which exhibits an isomorphism of $X_{\Sigma(P)}$ with an open subset of a projective variety: $X_{\Sigma(P)}$ is quasiprojective. ∎

Given a full-dimensional lattice polyhedron P, we wish to describe a semigroup S_P such that $X_{\Sigma(P)} \simeq \operatorname{Proj} K[S_P]$. Recall that we will need a grading on $K[S_P]$ for this to make sense.

7.24 Definition. Let P be a full-dimensional lattice polyhedron over M. Let $C(P)$ be the convex set

$$C(P) := \left\{ (m, \lambda) \in M_{\mathbb{R}} \times \mathbb{R}_{\geq 0} : m \in \begin{cases} \lambda P & \lambda > 0 \\ C & \lambda = 0 \end{cases} \right\}$$

(see Fig. 11) where C is the recession cone of P; note that if $P = Q + C$ then $\lambda P = \lambda Q + C$ for $\lambda > 0$ and so as $\lambda \to 0$ we have continuity in the definition. With this defined, we set $S_P := C(P) \cap (M \times \mathbb{Z})$, and define the **graded semigroup algebra** $K[S_P]$ to be the semigroup algebra $K[S_P]$ with grading $\deg \chi^m t^k := k$ (t the algebra generator corresponding to the $\mathbb{Z}$-component).

7.25 Theorem (Classification of quasi-projective toric varieties, II)**.** *Let P be a full-dimensional lattice polyhedron over M. There is a natural isomorphism $X_{\Sigma(P)} \simeq$* Proj *$K[S_P]$.*

Proof. Set $A := K[S_P]$. The strategy is to show that Proj A has affine open cover

$$\left\{ \operatorname{Spec} A_{\chi^v t} : v \in F_0(P) \right\},$$

and that this corresponds to the usual affine open cover of $X_{\Sigma(P)}$ with the same gluing data.

It is a standard result (e.g. [Har77, Proposition II.2.5]) that Proj A is covered by

$$\mathcal{U} = \left\{ \operatorname{Spec} A_f \,:\, f \in A_+ \right\};$$

it is easy to show that (since Proj A is a variety) if $A_+ = \operatorname{rad} \mathfrak{a}$ for an ideal $\mathfrak{a} \subseteq A$ then we may throw away all elements of A_+ which do not lie in $\mathfrak{a}$. In particular, if $\mathfrak{a} = (\chi^v t \,:\, v \in F_0(P))$ then it suffices to show that $A_+ = \operatorname{rad} \mathfrak{a}$ to see that Proj A has the claimed open cover. To do this, it suffices ([Har77, Exercise I.2.2]) to show that $\operatorname{rad} \mathfrak{a} \neq A$ (which is obvious: $1 \notin \operatorname{rad} \mathfrak{a}$) and $\mathfrak{a} \supseteq A_d$ for some $d > 0$ (again obvious: $\mathfrak{a} \supseteq A_1$).

Now note that $A_{\chi^v t} = K[\sigma_v^\vee \cap M]$ where σ_v is the cone of $\Sigma(P)$ corresponding to v: this follows from the fact that the degree 1 component of A is P combined with the usual theory of normal toric varieties. Further, it is easy to see that for higher-dimensional faces of P we obtain open sets of Proj A corresponding in the usual way to the intersections of cones of $\Sigma(P)$, such that the gluing data between the $K[\sigma_v^\vee \cap M]$ in $X_{\Sigma(P)}$ corresponds exactly to the gluing of the elements of $\mathcal{U}$. ∎

7.26 Corollary. *If P is a polytope, $X_{\Sigma(P)}$ is projective over K.*

Proof. Recall that Proj $K[S_P]$ is projective over $K[S_P]_0$ (c.f. [Har77, Example II.4.8.1]); in addition, $K[S_P]_0 = \{\chi^m t^k \in K[S_P] \,:\, k = 0\} = K[C]$; but since P is a polytope, C is trivial; i.e. Proj $K[S_P]$ is projective over K. ∎

8 Classical applications of the theory

In this section we present two well-known applications of the cohomology theory of toric varieties: the proof of Pick's formula (Corollary 8.14), and Stanley's proof of the necessity portion of McMullen's g-theorem (Theorem 8.23).

All varieties in this section are taken to be over $\mathbb{C}$.

8.1 The Ehrhart polynomial of a polytope

We shall now show that the cohomology of toric varieties may be applied to geometry. Essentially the kinds of cohomology theorems that will be useful here are vanishing theorems like Theorem 7.12 and Theorem 7.13 above.

Recall that a sheaf $\mathcal{F}$ on X is **quasicoherent** if X has an affine open cover $\{U_\alpha = \operatorname{Spec} R_\alpha\}$ such that for each α, $\Gamma(U_\alpha, \mathcal{F}) \simeq \widetilde{M_\alpha}$ for some R_α module M_α (recall that the squiggled module is a sheaf of $\mathcal{O}_X$-modules which restricts at principal open subsets to localisations of the module being hit). If each M_α may be chosen to be finitely generated then $\mathcal{F}$ is **coherent**.

8.1 Definition. If $\mathcal{F}$ is a coherent sheaf on a complete variety X, define the **Euler characteristic** $\chi(\mathcal{F})$ by

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}).$$

By results of Grothendieck and Serre [Har77, thm. III.2.7 and thm. III.5.2] this is well-defined (i.e. a finite integer).

8.2 Example (Euler characteristic is analogue of simplicial definition)**.** Let Δ be a simplicial complex. We shall write $f_d(\Delta)$ for $|F_d(\Delta)|$. The standard definition of Euler characteristic of a complex is

$$\chi(\Delta) = \sum_{i \geq 0} (-1)^i f_i(\Delta).$$

While this does look similar to the definition for coherent sheaves, it is not true that $\dim \tilde{H}_i(\Delta, k) = f_i(\Delta)$: e.g. for the tetrahedron each $\tilde{H}_i$ is trivial but it has faces! But note that by the rank-nullity theorem $\dim \tilde{H}_i(\Delta, k) = f_i(\Delta) - \operatorname{rank} \partial_i - \operatorname{rank} \partial_{i+1}$ and so (since $\operatorname{rank} \partial_0 = 0$, and $\operatorname{rank} \partial k = 0$ for all $k > \dim \Delta$) we have

$$\begin{aligned}
\sum_{i \geq 0} (-1)^i \dim \tilde{H}_i(\Delta, k) = (-1)^i \sum_{i \geq 0} & f_i(\Delta) - \operatorname{rank} \partial_i - \operatorname{rank} \partial_{i+1} \\
&= (f_0 - 0 - \operatorname{rank} \partial_1) - (f_1 - \operatorname{rank} \partial_1 - \operatorname{rank} \partial_2) + \cdots \\
&\quad \cdots + \left((-1)^{\dim \Delta} f_{\dim \Delta} - \operatorname{rank} \partial_{\dim \Delta} - 0\right) + (0 - 0 - 0) + \cdots \\
&= \sum_{i \geq 0} (-1)^i f_i(\Delta).
\end{aligned}$$

8.3 Theorem (Snapper)**.** *The Euler characteristic is intimately related to line bundles on the variety. Let X be a complete variety over $\mathbb{C}$, let $\mathscr{F}$ be coherent. Recall that for integers n we define the nth **twist** of $\mathscr{F}$ by a Cartier divisor D by*

$$\mathscr{F}(nD) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(nD).$$

Then there is some polynomial $P_h(z) \in \mathbb{Q}[z]$ such that for all $n \in \mathbb{Z}$,

$$P_h(n) = \chi(\mathscr{F}(nD)) = \chi(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(nD)) = \chi(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)^{\otimes n});$$

*this is called the **Hilbert polynomial** of $\mathscr{F}$ with respect to D.*

References to proof. This theorem was proved for projective varieties as [Sna59, Theorem 9.1], and Snapper conjectured that it should extend to arbitrary varieties; for a general proof for schemes of finite type over a field, see [Kle66, Section 1]. Compare [Har77, Exercise III.5.2] which is a special case (where the divisor is the divisor associated with $\mathscr{O}_X(1)$, in which case the Hilbert polynomial obtained is the classical one). ∎

In the special case of toric varieties, computing the Hilbert polynomial turns into an exercise in the combinatorics of the lattice. This is because computing with divisors can be done by using the vanishing theorems, Theorem 7.12 and Theorem 7.13, to convert dimension computations into lattice point enumerations of polytopes.

We first give a useful generalisation of Theorem 7.15.

8.4 Theorem. *Let D_0 be the divisor of $\mathbb{P}^n$ associated to $\operatorname{pos}\{-e_1 - \cdots - e_n\}$, and let $a \in \mathbb{Z}$. Then, if $a < 0$ we have*

$$\dim H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(aD_0)) = \begin{cases} 0 & p \neq n \\ \left|\operatorname{int} a\Delta_n \cap \mathbb{Z}^n\right| = \binom{-a-1}{n} & p = n; \end{cases}$$

and if $a > 0$ then we have

$$\dim H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(aD_0)) = \begin{cases} 0 & p \neq 0 \\ \left|a\Delta_n \cap \mathbb{Z}^n\right| = \binom{a+n}{n} & p = 0. \end{cases}$$

Proof. Suppose $a < 0$. Note that $|a|\, D_0$ is basepoint free with $P_{|a|D_0} = a\Delta_n$ (by Proposition 7.22) and so $\mathscr{O}_{\mathbb{P}^n}(aD_0)$ is generated by its global sections (by Lemma 7.11); hence we may apply Theorem 7.13 to obtain (since $n = \dim a\Delta_n$)

$$\dim H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-|a|\, D_0)) = \begin{cases} 0 & p \neq n \\ \dim \bigoplus_{m \in \operatorname{relint}(a\Delta_n) \cap \mathbb{Z}^n} K\chi^{-m} = \left|a\Delta_n \cap \mathbb{Z}^n\right| & p = n \end{cases}$$

which is one of the desired results.

If $a > 0$, by similar arguments to the previous case we may apply Theorem 7.12 to conclude

$$\dim H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(aD_0)) = \begin{cases} 0 & p \neq 0 \\ \dim \bigoplus_{m \in a\Delta_n \cap \mathbb{Z}^n} K\chi^m = \left|a\Delta_n \cap \mathbb{Z}^n\right| & p = 0, \end{cases}$$

the second desired result. ∎

8.5 Example. We shall now compute $\chi(\mathscr{O}_{P^n}(aD_0))$. For $a > 0$, we have by Theorem 8.4 that

$$\chi(\mathscr{O}_{\mathbb{P}^n}(aD_0)) = \dim H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(aD_0)) = \left|a\Delta_n \cap \mathbb{Z}^n\right| = \binom{a+n}{n};$$

and for $a < 0$ we have

$$\chi(\mathscr{O}_{\mathbb{P}^n}(aD_0)) = (-1)^n \dim H^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(aD_0)) = (-1)^n \left|\operatorname{int} a\Delta_n \cap \mathbb{Z}^n\right| = (-1)^n \binom{-a-1}{n}.$$

Finally note that if $p(x) = \frac{(x+n)(x+n-1)\cdots(x+1)n!}{n!}$ then $p(a)$ agrees with these two values for all $a \in \mathbb{Z}$: for $a > 0$ we have $p(a) = \left|a\Delta_n \cap \mathbb{Z}^n\right|$ and $p(-a) = (-1)^n \left|\operatorname{int} a\Delta_n \cap \mathbb{Z}^n\right|$.

With this as motivation, we make the following definition.

8.6 Definition. Let $P \subseteq \mathbb{R}^r$ be an full-dimensional lattice polytope. The function taking $m \in \mathbb{Z}_{\geq 0}$ to the number $\mathscr{E}_P(m) = |mP \cap \mathbb{Z}^r|$ of lattice points of mP is the **Ehrhart polynomial** of P.

8.7 Example. Consider the cube $C = [0,1]^3 \subseteq \mathbb{Z}^n$. If $m \in \mathbb{Z}_{\geq 0}$ then the number of lattice points in mC is $(m+1)^3$; that is, $\mathscr{E}_C(m) = m^3 + 3m^2 + 3m + 1$.

8.8 Theorem (Ehrhart reciprocity)**.** *The function $\mathscr{E}_P(m) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ agrees with a polynomial function of degree r, and so can be extended to a polynomial function on $\mathbb{R}$ (which we will denote by $\mathscr{E}_P$ without any loss of clarity). Further, we have the following* ***Ehrhart reciprocity****:*

- *For all $m \in \mathbb{Z}_{\geq 0}$, $\mathscr{E}_P(m) = |mP \cap \mathbb{Z}^r|$.*
- *For all $m \in \mathbb{Z}_{\geq 0}$, $\mathscr{E}_P(-m) = (-1)^r \left|\operatorname{int} mP \cap \mathbb{Z}^r\right|$.*

Historical remark. Ehrhart polynomials were introduced by Ehrhart in the 1960s; see [Ehr67] as well as the papers cited at [MS05, p. 246]. The reciprocity result was independently given by Macdonald, see [Mac71].

Proof. The proof is exactly the argument of Theorem 8.4 and Example 8.5. We will first show that $\mathscr{E}_P(m) = \chi(\mathscr{O}_{X_P}(mD_P))$ for all $m \geq 0$ (and thus $\mathscr{E}_P(m)$ is polynomial by Theorem 8.3). Indeed, for $m \geq 0$, mD_P is basepoint free by Proposition 7.22 and so by Theorem 7.12,

$$\chi(\mathscr{O}_X(D_P)) = H^0(X_{\Sigma(P)}, \mathscr{O}_{X_{\Sigma(P)}}(D_P)) = \left|mD_P \cap \mathbb{Z}^r\right| = \mathscr{E}_P(m).$$

By the theorem, we define $\mathscr{E}_P(-m)$ for $m \geq 0$ to be the value of the polynomial that the positive arguments satisfy; thus we may apply Theorem 7.13 to $\chi(\mathscr{O}_X(D_P))$ to see that $\mathscr{E}_P(-m) = (-1)^r \left|\operatorname{int} mP \cap \mathbb{Z}^r\right|$. ∎

Remark. 1. In fact, one may show that if P is very ample then the Ehrhart polynomial of P is the *usual* Hilbert polynomial) of $X_{\Sigma(P)}$ under the projective embedding induced as in Lemma 7.9. The proof uses the famous 'Serre vanishing' result, and is given as [CLS11, Proposition 9.4.3].

2. An alternative proof of Ehrhart reciprocity proceeds via **Brion's formula**:

$$\sum_{p\in mP\cap\mathbb{Z}^r} t^p = \sum_{v\in F_0(P)} \left(t^{mv} \frac{K_v(t)}{D_v(t)} \right)$$

where we use multinomial notation for t; here, K_v and D_v are polynomials depending on the Hilbert basis of $\mathbb{Z}_{\geq 0}(P-v)$. This approach is detailed in [MS05, Section 12.3].

We will use the Ehrhart polynomial of a polytope P to compute some geometric information. We shall need the notion of a *lattice volume*.

8.9 Definition. Let M be a lattice with basis $(e_1, ..., e_n)$; then the **covolume** of M is $d(M) \coloneqq \det(e_1, ..., e_n)$. The **normalised volume measure** on $M_\mathbb{R}$ is the usual Lebesgue measure divided by $d(M)$. We write $\operatorname{Vol} E$ for the measure of $E \subseteq M_\mathbb{R}$.

8.10 Lemma. *If $X \subseteq \mathbb{R}^d$ is full-dimensional, then*

$$\operatorname{Vol}(X) = \lim_{n\to\infty} \frac{1}{n^d} \left| nX \cap \mathbb{Z}^d \right|.$$

Proof. We have, approximating X by boxes of size n^{-1} (hence volume n^{-d}); these boxes span the lattice $n^{-1}\mathbb{Z})^d$, so

$$\operatorname{Vol}(X) = \int_X 1 = \lim_{n\to\infty} n^{-d} \cdot \left| X \cap n^{-1}\mathbb{Z})^d \right| = \lim_{n\to\infty} n^{-d} \cdot \left| nX \cap \mathbb{Z}^d \right|. \qquad \blacksquare$$

The following general remark will also be useful. The proof is easy.

8.11 Lemma. *Let $p \in \mathbb{Q}[z]$ be a polynomial so that $L \coloneqq \lim_{z\to\infty} z^{-n} p(z)$ exists and is nonzero for some $n \in \mathbb{Z}$. Then $\partial p = n$ and $L = [z^n]p$.* $\blacksquare$

8.12 Theorem. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional lattice polygon, and for notational convenience let $f \in K[X]$ be the Ehrhart polynomial of P. Then:*

1. $\partial f = n$;
2. $[X^n]f = \operatorname{Vol}(P)$;
3. $[X^{n-1}]f = \frac{1}{2} \sum_{F\in F_{n-1}(P)} \operatorname{Vol}(F)$ *(where* $\operatorname{Vol}(F)$ *is normalised with respect to* $(\operatorname{aff} F) \cap M$*);*
4. $[X^0]f = 1$.

Proof. Parts 1 and 2 follow easily from Lemma 8.10 and Lemma 8.11. Part 4 is trivial: $f(0) = \left|0P \cap \mathbb{Z}^r\right| = \left|\{0\}\right| = 1$. Finally we show part 3, which is more difficult. Note that $|P \cap \mathbb{Z}^n| - \left|\operatorname{int} P \cap \mathbb{Z}^n\right|$ is the number of lattice points on ∂P. By the other parts of this theorem, we may write

$$f(x) = \operatorname{Vol}(P)x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + 1$$

where each α_i is rational. By Theorem 8.8, we have

$$\begin{aligned}
\left|xP \cap \mathbb{Z}^n\right| - \left|\operatorname{int} xP \cap \mathbb{Z}^n\right| &= f(x) - (-1)^n f(-x) \\
&= \operatorname{Vol}(P)x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + 1 \\
&\quad - \operatorname{Vol}(P)x^n + \alpha_{n-1}x^{n-1} - \cdots - (-1)^n \alpha_{n-i} x^{n-i} - \cdots - (-1)^n \\
&= 2\alpha_{n-1}x^{n-1} + 2\alpha_{n-3}x^{n-1} + \cdots + 2\alpha_{n-\lfloor n/2\rfloor} x^{n-\lfloor n/2 \rfloor}.
\end{aligned}$$

On the other hand, we may compute the number of boundary points by direct enumeration:

$$\left|xP \cap \mathbb{Z}^n\right| - \left|\mathrm{int}\, xP \cap \mathbb{Z}^n\right| = \sum_{j=0}^{d-1} \sum_{F \in F_j(P)} \left|\mathrm{int}\, xF \cap \mathbb{Z}^n\right| = \sum_{j=0}^{d-1} \sum_{F \in F_j(P)} (-1)^j \mathscr{E}_F(-x).$$

Hence we obtain the equality

$$\alpha_{n-1} x^{n-1} + \alpha_{n-3} x^{n-3} + \cdots + \alpha_{n-\lfloor n/2 \rfloor} x^{n-\lfloor n/2 \rfloor} = \frac{1}{2} \sum_{j=0}^{d-1} (-1)^j \sum_{F \in F_j(P)} \mathscr{E}_F(-x); \tag{12}$$

observe that (using part 1)

$$[x^{d-1}] \left(\frac{1}{2} \sum_{j=0}^{d-1} (-1)^j \sum_{F \in F_j(P)} \mathscr{E}_F(-x) \right) = \frac{1}{2} (-1)^{d-1} \sum_{F \in F_{d-1}(P)} (-1)^{d-1} [x^{d-1}] \mathscr{E}_P(x) = \frac{1}{2} \sum_{F \in F_{d-1}(P)} \mathrm{Vol}(F)$$

and comparing coefficients in Eq. (12) the claim is proved. ∎

8.13 Example. Take $P = [0,1]^2$; then $\mathscr{E}_P(m) = (m+1)^2 = m^2 + 2m + 1$ and it is easy to see that the coefficients are as claimed.

Remark. For some results on the other coefficients of $\mathscr{E}_P$, which are in general less well-understood, see the elementary reference [BR15, Chapter 5] or the papers [Bec+04] and [RS17] and the references therein.

8.14 Corollary (Pick's formula)**.** *Let $P \subseteq \mathbb{R}^2$ be a lattice polygon. Then* $\mathrm{Vol}(P) = \left|\mathrm{int}\, P \cap \mathbb{Z}^2\right| + \frac{1}{2}\left|\partial P \cap \mathbb{Z}^2\right| - 1$.

Historical remark. This theorem was first proved in [Pic99]; an induction based on a triangulation of P suffices to prove the theorem (the argument is an easy exercise, see [BR15, Theorem 2.8] for example).

Proof. In Theorem 8.12, take $n = 2$; then $\mathscr{E}_P(1) = \mathrm{Vol}(P) + \frac{1}{2}\left|\partial P \cap \mathbb{Z}^2\right| + 1$ and so $\mathrm{Vol}(P) = \left|P \cap \mathbb{Z}^2\right| - \frac{1}{2}\left|\partial P \cap \mathbb{Z}^2\right| - 1$; using Ehrhart reciprocity this is equal to the claimed formula. ∎

Remark. There is a generalisation of the Ehrhart polynomials which simultaneously generalises the Dehn-Sommerville relations (i.e. the symmetry relations in the face complex of a polytope which are in turn a generalisation of Euler's identity $f_0 - f_1 + f_2 = 2$). The idea is that instead of counting the lattice points of a polytope we count the lattice points of i-dimensional faces and relate them to sheaves of p-forms, obtaining the so-called ***p*-Ehrhart polynomials**. The theory relies on cohomology (in particular Serre duality, which is not surprising as Serre duality gives us a symmetry result reminiscent of the Dehn-Sommerville relations).

A second direction for generalisation is to extend the class of convex objects we enumerate within: we may extend Ehrhart reciprocity to rational polytopes (i.e. polytopes with vertices in $M_{\mathbb{Q}}$), though the Ehrhart function ceases to be polynomial (in general it will be only piecewise polynomial); see for example [CLS11, Section 9.4 (exercises)], or [MS05, Chapter 12], or [BR15, Chapter 4].[Ehrhart reciprocity!rational]

8.2 McMullen's g-theorem

In this section we present the proof of the necessity of McMullen's conditions for the f-vector of a simplicial polytope: that is, the half of McMullen's theorem which uses the theory of toric varieties for the proof. We note that, in addition to the references included below, [Sta96, Section II.2] may be of interest.

We say that a polytope P of dimension d is **simplicial** if each facet of P is a simplex. The f**-vector** of P is the vector $(f_0(P), ..., f_{d-1}(P))$. It is usual also to define the h**-vector**, by setting

$$h_i(P) := \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{i-1}(P); \tag{13}$$

here (as is the convention) we set $f_{-1}(P) = 1$ as there is a single face of dimension -1, namely $\emptyset$. The most elementary relationship between the f_i is then the following relation most conveniently stated in terms of the h-vector:

8.15 Theorem (Dehn-Sommerville relations). *If P is a simplicial d-polytope, we have for each i $(0 \leq i \leq d)$ the relation*

$$h_i = h_{d-i}.$$

Historical remark. According to [Grü03, Section 9.8], the relations for $d \in \{4, 5\}$ were found by Dehn in 1905, and the general case was proved in 1927 by Sommerville. The Dehn-Sommerville relations are essentially a duality result and are a version of **Poincaré duality** for simplicial complexes.

Proof. The proof is standard, see for example [Ewa96, pp. 82–83] or [Zie95, p. 252]. ∎

We naturally ask ourselves, what other relations hold between the face numbers? Further, we ask for the 'maximal' such relation: that is,

which possible vectors $f \in \mathbb{N}^d$ have the property $f = f(P)$ for some simplial polytope P?

This is answered by McMullen's famous g-theorem for simplicial polytopes (conjectured in [McM71]), which was proved in the 1980s: sufficiency was proved by Billera and Lee in [BL80], and we shall prove the necessity result (following Stanley) as Theorem 8.23 below.

8.16 Theorem. *A vector $h = (h_0, ..., h_d) \in \mathbb{N}^{d+1}$ is the h-vector of some simplicial d-polytope if and only if:*

1. $h_0 = 1$*;*
2. $h_i = h_{d-i}$ *for all i (that is, h satisfies the Dehn-Sommerville relations); and*
3. $(h_0, h_1 - h_0, h_2 - h_1, ..., h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ *is an M-vector, in the sense of Definition 8.17 below.*

8.17 Definition. A vector $(k_0, ..., k_d) \in \mathbb{Z}^{d+1}$ is an **M-vector**[8] if there exists a graded algebra $R = R_0 \oplus \cdots \oplus R_d$, generated as an algebra over $K = R_0$ by R_1, such that the Hilbert function $H(R, n) := \dim_K R_n$ is given by $H(R, n) = k_n$.

Remark.
1. The terminology 'g-theorem' comes from the original paper of McMullen, who wrote g_i for $h_{i+1} - h_i$.
2. Classically, following [Kru63], one would define M-vectors in terms of so-called r**-canonical forms**, which have a more obvious combinatorial meaning and are more easily computable. It is difficult, however, to prove that the two definitions are equivalent: the original proof of equivalence may be found in [Mac27]; according to that author, "the proof of [equivalence]... is too long and complicated to provide any but the most tedious reading" (see also the discussion surrounding Theorem II.2.2 of [Sta96]). In addition, the definition of the r-canonical form is incredibly technical and so we do not include the details of that approach. We are therefore essentially giving the version of the theorem discussed in [Oda91].

[8]Following [Sta80] the M is for F. S. Macaulay.

We shall need some properties of the singular cohomology of X_Σ to write the proof of necessity.

8.18 Definition. Let X be a topological space, and let R be a ring. If $S \subseteq \{1, ..., n\}$, we may identify S with $\Delta^S := \operatorname{conv}\{s \in S : e_s\}$, a copy of the 'usual' $|S|$-simplex embedded in $\mathbb{R}^n$. We may therefore set up a chain complex $C_\bullet(X, R)$ as follows: we define $C_p(X, R)$ to be the free R-module defined on the set of continuous maps $\Delta^p \to X$, with boundary maps $\partial_p : C_p(X, R) \to C_{p-1}(X, R)$ defined in the same way as Example 6.2 under the identification. More precisely, given some $S \subseteq \{1, ..., n\}$ we define a basis element of $C_{|S|}(X, R)$ to be a composition $S \to \Delta^S \to X$ (the second map continuous and the first the standard identification), and we define boundary maps to be precisely the boundary maps ∂_p of Example 6.2.

For each each $C_p(X, R)$ we define $C^p(X, R) := C_P^*(X, R) = \operatorname{Hom}_R(C_P(X, R), R)$, and define $d^p : C^p(X, R) \to C^{p+1}(X, R)$. The **singular cochain complex** of X is then $(C^\bullet(X, R), d^\bullet(X, R))$, and the associated cohomology $H^p(X, R) = \ker d^i / \operatorname{im} d^{i-1}$ is the **singular cohomology** of X. As is usual, we set $h^p(X, R) := \dim_R H^p(X, R)$.

8.19 Theorem. *If X_Σ (defined over $\mathbb{C}$) is complete and an orbifold, then:*

$$h^p(X_\Sigma, \mathbb{Q}) = \begin{cases} \sum_{i=k}^n (-1)^{i-k}\binom{i}{k}\Sigma(n-i) & p = 2k, \\ 0 & p = 2k+1. \end{cases}$$

Proof. The proof uses spectral sequences and so we omit it. A modern exposition may be found as [CLS11, Theorem 12.3.11]; alternative and historical perspectives may be found in [Dan78, Section 10] and as [Oda91, Theorem 4.1]. ∎

The graded singular cohomology module $C^\bullet(X, R)$ may be turned into a ring by the following standard construction (see [Hat01, Section 3.2]).

8.20 Definition. We define the **cohomology ring** of X to be the ring supported on

$$C^\bullet(X, R) = \bigoplus_{p=0}^{\infty} H^p(X, R)$$

with addition induced by the addition in each free module $H^p(X, R)$, and multiplication the **cup product** defined by

$$C^k(X, R) \times C^l(X, R) \ni (f, g) \to f \smile g \in C^{k+l}(X, R)$$

where $f \smile g \in C_{k+l}^*(X, R)$ is the coelement

$$\operatorname{Hom}_{\mathsf{Set}}(\Delta^{k+l}, X) = C_{k+l}(X, R) \ni \sigma \mapsto f(\sigma\restriction_{\{0,...,k\}})g(\sigma\restriction_{\{k,...,k+l\}}) \in R.$$

We will need to see that a particular cohomology ring is commutative; the standard result on commutativity of the cup product is the following.

8.21 Lemma ([Hat01, Theorem 3.11]). *The identity $f \smile g = (-1)^{kl} g \smile f$ holds for $f \in H^k(X, R)$ and $g \in H^l(X, R)$.* ∎

Finally, we will need a version of the famous **hard Lefschetz theorem**.

8.22 Theorem (Théorème de Lefschetz vache [Ste76, Theorem 1.13]). *If $L \in H^2(X, \mathbb{Q})$ is the class of an ample divisor on a normal orbifold projective toric variety X of dimension n, then for all $q \in \mathbb{N}$ the map $\omega \mapsto L^q \smile \omega$ induces an isomorphism between $H^{n-q}(X, \mathbb{C})$ and $H^{n+q}(X, \mathbb{C})$.* ∎

With these preliminaries out of the way, we prove the necessity of the conditions of Theorem 8.16.

8.23 Theorem (Necessity of McMullen's conditions [Sta80]). *The conditions of Theorem 8.16 are necessary for a vector $h = (h_0, ..., h_d) \in \mathbb{N}^{d+1}$ to be the h-vector of some d-polytope.*

Proof. A simple computation shows that $h_0(P) = 1$ and by Theorem 8.15 the Dehn-Sommerville relations hold. Hence it suffices to check only the third condition.

Let $P \subseteq N_{\mathbb{R}}$ be a d-polytope. By small pertubations of the vertices, which do not change the combinatorial structure of P, we may assume that $F_0(P) \subseteq N_{\mathbb{Q}}$; by refining the lattice N, we may in fact assume that $F_0(P) \subseteq N$. In addition we may assume that $0 \in \operatorname{int} P$.

Let $\Sigma \coloneqq \Sigma(P)$. By Lemma 7.17 and Theorem 5.10, $X_{\Sigma(P)}$ (defined over $\mathbb{C}$) is complete and orbifold. Since the odd singular cohomologies of $X_{\Sigma(P)}$ vanish by Theorem 8.19, an application of Lemma 8.21 shows that the cohomology ring $C^\bullet(X_{\Sigma(P)}, \mathbb{Q})$ is commutative. We may then regrade it to form a ring $A^\bullet \coloneqq \bigoplus_{k=0}^{\infty} \bigoplus_{p=0}^{\infty} H^{2p}(X, R)$, and write $a^p \coloneqq h^{2p}(X, R)$ for each p. Observe, comparing Theorem 8.19 and Eq. (13), that $a_p = h_p$ for each p.

By Corollary 7.26, $X_{\Sigma(P)}$ is projective; and since $\Sigma(P)$ is simplicial, $X_{\Sigma(P)}$ is orbifold. Thus by Theorem 8.22 and Proposition 7.22 we may conclude that there exists some $L \in A^1$ such that for $0 \leq i \leq \lfloor d/2 \rfloor$, the map $g_i : A^i \to A^{d-i}$ given by $\omega \mapsto L^{d-2i} \smile \omega$ is a bijection. Define for every j ($0 \leq j \leq \lfloor d/2 \rfloor$) a map $f_j : A^j \to A^{j+1}$ given by $\omega \mapsto L \smile \omega$; then $g_i = f_{d-i-1} \circ \cdots \circ f_{i+1} \circ f_i$, and since g_i is bijective each f_i in this composition is injective. Let $\mathfrak{a} \subseteq A^\bullet$ be the ideal generated by L and $A_{\lfloor d/2 \rfloor + 1}$. We claim that

$$H(A^\bullet/\mathfrak{a}, n) = \begin{cases} h_n - h_{n-1} & 1 \leq n \leq \lfloor d/r \rfloor \\ 0 & n > \lfloor d/2 \rfloor, \end{cases}$$

and this claim will prove the theorem.

Indeed, note that $H(A^\bullet/\mathfrak{a}, n) = \dim_K A^n/\mathfrak{a} \cap A^n$; for $n \leq \lfloor d/2 \rfloor$, $\mathfrak{a} \cap A^n = LA^{n-1}$ since each f_i is injective, and for $n > \lfloor d/2 \rfloor$, $\mathfrak{a} \cap A^n = A^n$ since $A^n = L^{n-\lfloor d/2 \rfloor - 1} A^{\lfloor d/2 \rfloor + 1} = A^{\lfloor d/2 \rfloor + 1} \subseteq \mathfrak{a}$. Thus for $n > A^{\lfloor d/2 \rfloor + 1}$, $\dim_K A^n/\mathfrak{a} \cap A^n = 0$; and for $n \leq \dim_K A^n/\mathfrak{a} \cap A^n$, $\dim_K A^n/\mathfrak{a} \cap A^n = \dim A^n - \dim A^{n-1} = h_n - h_{n-1}$. ∎

Remark. In the papers [Sta80; McM71], a generalisation of Theorem 8.16 where the d-polytope of P is replaced with an arbitrary triangulation of the $(d-1)$-sphere is given as an open problem. The fully general version of this conjecture was proved only recently [Adi19]; the main ingredients for the proof are similar to those of Stanley's proof above, including a more general version of the hard Lefschetz theorem for toric varieties and a more general version of the Dehn-Sommerville relations (via Poincaré duality). A new ingredient needed is the introduction of the so-called *Hall-Laman relations*. Some idea of the general direction of the proof and the related concepts and conjectures is given in a blog post by Gil Kalai, [Kal18].

References

[Adi19] Karim Adiprasito. *Combinatorial Lefschetz theorems beyond positivity*. 2019. arXiv: `1812.10454 [math.CO]`. URL: `https://arxiv.org/abs/1812.10454` (cit. on p. 75).

[Alu09] Paolo Aluffi. *Algebra: Chapter 0*. Graduate studies in mathematics 104. American Mathematical Society, 2009 (cit. on p. 4).

[Ban32] Stephan Banach. *Théorie des Opérations linéaires*. Polish. Monografie matematyczne 1. Warszawa: Fundusz Kultury Narodowej, 1932 (cit. on p. 13).

[Ban87] Stephan Banach. *Theory of linear operations*. Trans. by F. Jellett. Elsevier Science, 1987 (cit. on p. 13).

[Bar02] Alexander Barvinok. *A course in convexity*. Graduate studies in mathematics 54. 516.08 B29. American Mathematical Society, 2002 (cit. on pp. 4, 13).

[Bec+04] M. Beck et al. *Coefficients and roots of Ehrhart polynomials*. 2004. arXiv: `math/0402148 [math.CO]` (cit. on p. 72).

[Ber09] Marcel Berger. *Geometry I & II*. Universitext. Springer, 2009 (cit. on pp. 4, 13).

[BGT97] Winfried Bruns, Joseph Gubeladze, and Ngô Viêt Trung. "Normal polytopes, triangulations, and Koszul algevras". In: *Journal für die riene und angewandte Mathematik* 485 (1997), pp. 123–160 (cit. on p. 65).

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*. 2nd ed. Grundlehren der mathematischen Wissenschaften 302. 516.353 L27 2004. Springer, 2004 (cit. on p. 4).

[BL80] Louis J. Billera and Carl W. Lee. "Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes". In: *Bulletin of the Americal Mathematical Society* 2.1 (1980), pp. 181–185 (cit. on p. 73).

[BR15] Matthias Beck and Sinai Robins. *Computing the continuous discretely*. 2nd ed. Undergraduate Texts in Mathematics. Springer, 2015 (cit. on p. 72).

[Bri99] Michel Brion. "Rational smoothness and fixed points of torus actions". In: *Transformation Groups* 4.2-3 (1999), pp. 127–156 (cit. on p. 46).

[CLS11] David Cox, John Little, and Hal Schenck. *Toric varieties*. Graduate studies in mathematics 124. 516.35 C87t. American Mathematical Society, 2011 (cit. on pp. A, 3, 4, 25, 37, 38, 46, 57, 60–62, 65, 70, 72, 74).

[Cor31] J. G. van der Corput. "Konstruktion der Minimalbasis für spezielle Diophantische Systeme von linear-homogenen Gleichungen und Ungleichungen". Dutch. In: *Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam* 34 (1931), pp. 515–523 (cit. on p. 16).

[Cox69] H. S. M. Coxeter. *Introduction to geometry*. 2nd ed. John Wiley & Sons, Inc., 1969 (cit. on p. 4).

[Dan78] V. I. Danilov. "The geometry of toric varieties". Trans. by Miles Reid. In: *Russian Mathematical Surveys* 33 (1978), pp. 97–154 (cit. on p. 74).

[DG80] Michel Demazure and Peter Gabriel. *Introduction to algebraic geometry and algebraic groups*. North-Holland mathematics studies 39. Elsevier North-Holland, 1980 (cit. on p. 5).

[EGA II] Alexander Grothendieck. *Éléments de géométrie algébrique*. Vol. II: *Étude globale élémentairede quelques classes de morphismes*. French. Publications mathéatiques de l'I.H.É.S. 8. 1961 (cit. on p. 49).

[Ehr67] E. Ehrhart. "Sur un problème de géometrié diophantienne linéaire I". French. In: *Journal für die reine und angewandte Mathematik* 226 (1967), pp. 1–29 (cit. on p. 70).

[Eis95] David Eisenbud. *Commutative algebra with a view toward algebraic geometry*. Graduate texts in mathematics 150. Springer-Verlag, 1995 (cit. on pp. 35, 51, 54, 55).

[Ewa96] Günter Ewald. *Combinatorial convexity and algebraic geometry*. Graduate texts in mathematics 168. 516.08 E94. Springer, 1996 (cit. on pp. 4, 15, 19, 73).

[Fin93] Jonathan Fine. "Another proof of Sumihiro's theorem on torus embeddings". In: *Duke Mathematical Journal* 69.1 (1993), pp. 243–245 (cit. on p. 42).

[Ful93] William Fulton. *Introduction to toric varieties*. Annals of mathematics studies 131. 516.353 F97. Princeton University Press, 1993 (cit. on pp. 3, 4, 19, 49).

[Gor73] Paul Gordan. "Ueber die Auflösung linearer Gleichungen mit reellen Coefficienten". German. In: *Mathematische Annalen* 6 (1873), pp. 23–28 (cit. on p. 16).

[Grü03] Branko Grünbaum. *Convex polytopes*. Ed. by Volker Kaibel, Victor Klee, and Günter M. Ziegler. 2nd ed. Graduate texts in mathematics 221. Springer, 2003 (cit. on pp. 4, 73).

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate texts in mathematics 52. 512.33 H33. Springer, 1977 (cit. on pp. 4, 5, 42, 46, 49–51, 54, 55, 57, 59, 60, 68, 69).

[Hat01] Allen Hatcher. *Algebraic topology*. 2001. URL: `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf` (visited on 2020) (cit. on p. 74).

[HH11] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*. Graduate texts in mathematics 260. Springer, 2011 (cit. on p. 25).

[Hoc72] M. Hochster. "Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes". In: *Annals of Mathematics* 96.2 (July 1972), pp. 318–337 (cit. on p. 31).

[Hum75] James E. Humphreys. *Linear algebraic groups*. Graduate texts in mathematics 21. 512.2 H92 1988. Springer-Verlag, 1975 (cit. on pp. 4–6).

[Kal18] Gil Kalai. *Amazing: Karim Adiprasito proved the g-conjecture for spheres!* Blog post. Dec. 25, 2018. URL: `https://gilkalai.wordpress.com/2018/12/25/amazing-karim-adiprasito-proved-the-g-conjecture-for-spheres/` (cit. on p. 75).

[Kem+73] G. Kempf et al. *Toroidal embeddings I*. Lecture Notes in Mathematics 339. Springer, 1973 (cit. on pp. 4, 25, 31, 33, 38, 41).

[Kle66] Steven L. Kleiman. "Towards a numerical theory of ampleness". In: *Annals of Mathematics* 84.3 (1966), pp. 293–344 (cit. on p. 69).

[Kno+89] F. Knop et al. "Local properties of algebraic transformation groups". In: *Algebraische Transformationsgruppen und Invariantentheorie*. Ed. by H. Kraft, P. Slodowy, and T. A. Springer. Deutsche Mathematiker-Vereinigung Seminar 13. Birkhäuser, 1989, pp. 63–75 (cit. on p. 42).

[Kru63] Joseph B. Kruskal. "The number of simplices in a complex". In: *Mathematical optimisation techniques*. Ed. by Richard Bellman. University of California Press, 1963. Chap. 12, pp. 251–278 (cit. on p. 73).

[Mac27] F. S. Macaulay. "Some properties of enumeration in the theory of modular systems". In: *Proceedings of the London Mathematical Society* 26.1 (1927), pp. 531–555 (cit. on p. 73).

[Mac71] I. G. Macdonald. "Polynomials associated with finite cell-complexes". In: *Journal of the London Mathematical Society* s2-4 (1 1971). Erroneously catalogued as 'Polynomials Associated with Finite Gell-Complexes', pp. 181–192. DOI: `https://doi.org/10.1112/jlms/s2-4.1.181` (cit. on p. 70).

[McM71] P. McMullen. "The numbers of faces of simplicial polytopes". In: *Israel Journal of Mathematics* 9 (1971), pp. 559–570 (cit. on pp. 73, 75).

[Min10] Hermann Minkowski. *Geometrie der Zahlen*. German. Zeipzig: Teubner, 1910. URL: `https://archive.org/details/geometriederzahl00minkrich` (cit. on p. 12).

[Min97] Hermann Minkowski. "Allgemeine Lehrsätze über die convexen Polyeder". German. In: *Gesellschaft der Wissenschaften zu Gottingen* (1897), pp. 198–219 (cit. on p. 12).

[MO109395] aglearner (https://mathoverflow.net/users/13441/aglearner). *Is there a "geometric" intuition underlying the notion of normal varieties?* MathOverflow. Oct. 12, 2012. URL: `https://mathoverflow.net/q/109395` (cit. on p. 3).

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*. Graduate texts in mathematics 227. Springer, 2005 (cit. on pp. 3, 25, 70–72).

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*. Graduate studies in mathematics 161. 516.35 M16. American Mathematical Society, 2015 (cit. on p. 4).

[Oda78] Tadao Oda. *Lectures on torus embeddings and applications*. 510.4 T21 57. Bombay: Tata Institute of Fundamental Research, 1978 (cit. on pp. A, 3, 4, 25, 31, 38, 41).

[Oda91] Tadau Oda. "Simple convex polytopes and the strong Lefschetz theorem". In: *Journal of Pure and Applied Algebra* 71 (1991), pp. 265–286 (cit. on pp. 73, 74).

[OM75] Tadao Oda and Katsuya Miyake. "Almost homogeneous algebraic varieties under algebraic torus action". In: *Manifolds Tokyo 1973*. International Conference on Manifolds and Related Topics in Topology, 1973. Ed. by Akio Hattori. University of Tokyo Press, 1975, pp. 373–381 (cit. on pp. 33, 41).

[Osb00] M. Scott Osborne. *Basic homological algebra*. Graduate texts in mathematics 196. Springer, 2000 (cit. on pp. 51, 54, 55).

[Pic99] Georg Pick. "Geometrisches zur Zahlenlehre". In: *Sitzungsberichte des deutschen naturwissenschaftlich–medicinischen Vereines für Böhmen "Lotos" in Prag*. Ed. by Johannes Gad. Neue Folge 19. 1899, pp. 311–319. URL: `https://www.biodiversitylibrary.org/item/50207#page/327` (cit. on p. 72).

[RS17] Maren H. Ring and Achill Schürmann. *Local Formulas for Ehrhart Coefficients from Lattice Tiles*. 2017. arXiv: `1709.10390 [math.MG]` (cit. on p. 72).

[Rud13] Walter Rudin. *Principles of mathematical analysis*. McGraw Hill Education (India) edition. International series in pure and applied mathematics. 2013 (cit. on p. 4).

[Rud73] Walter Rudin. *Functional analysis*. McGraw Hill. McGraw-Hill series in higher mathematics. 1973 (cit. on p. 13).

[Sch86] Alexander Schrijver. *Theory of linear and integer programming*. corrected 1987 reprinting. Wiley-Interscience series in discrete mathematics and optimization. Wiley-Interscience, 1986 (cit. on pp. 4, 12).

[Ser79] Jean-Pierre Serre. *Local fields*. Graduate texts in mathematics 67. 512.74 S48cy. Springer-Verlag, 1979 (cit. on p. 49).

[Sha13] Igor R. Shafarevich. *Basic algebraic geometry 1 & 2*. 3rd ed. Springer, 2013 (cit. on pp. 4, 24, 42).

[Sna59] Ernst Snapper. "Multiples of divisors". In: *Journal of Mathematics and Mechanics* 8.6 (1959), pp. 967–992 (cit. on p. 69).

[Sta80] Richard P. Stanley. "The number of faces of a simplicial convex polytope". In: *Advances in Mathematics* 35 (1980), pp. 236–238 (cit. on pp. 73, 75).

[Sta96] Richard P. Stanley. *Combinatorics and commutative algebra*. 2nd ed. Progress in mathematics 41. Birkhäuser, 1996 (cit. on pp. 25, 72, 73).

[Stacks] The Stacks project authors. *The Stacks project*. 2020. URL: `https://stacks.math.columbia.edu` (cit. on p. 50).

[Ste76] J. H. M. Steenbrink. "Mixed Hodge structure on the vanishing cohomology". In: *Real and complex singularities, Oslo 1976*. Ed. by P. Holm. Sijthoff & Noordhoff International Publishers, 1976, pp. 525–563 (cit. on p. 74).

[Sum74] Hideyasu Sumihiro. "Equivariant completion". In: *Journal of Mathematics of Kyoto University* 14.1 (1974), pp. 1–28 (cit. on p. 42).

[Sum75] Hideyasu Sumihiro. "Equivariant completion II". In: *Journal of Mathematics of Kyoto University* 15.3 (1975), pp. 573–605 (cit. on p. 42).

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*. Graduate texts in mathematics 66. 512.2 W32. Springer-Verlag, 1979 (cit. on pp. 4, 5).

[Wey35] Hermann Weyl. “Elementare Theorie der konvexen Polyeder”. German. In: *Commentarii Mathematici Helvetici* 7 (1935), pp. 290–306 (cit. on p. 12).

[Wey50] Hermann Weyl. “The elementary theory of convex polyhedra”. Trans. by H. W. Kuhn. In: Annals of Mathematics Studies 1.24 (1950). Ed. by H. W. Kuhn and A. W. Tucker, pp. 3–18 (cit. on p. 12).

[Zie95] Günter M. Ziegler. *Lectures on polytopes*. Graduate texts in mathematics 152. 516.35 Z66. Springer-Verlag, 1995 (cit. on pp. 4, 12, 14, 15, 73).

Index of symbols

Index of terms