Two-bridge links and Heegaard splittings

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§1. Maximum in interior

(1.1) Proposition. *Let L be a 2-bridge link with at least two twist regions. Then the volume map V : $A(T)$ cannot have a maximum on $\partial A(T)$ for special choice of T as given earlier talk.*

Proof is technical and relies on following argument. First restrict tetrahedra shapes that can occur using certain algebraic restrictions. On remainder show that there exist paths from boundary into the interior along which V is strictly increasing.

§2. Unknotting tunnels

In this section we survey work of Kobayashi [Kob99], Morimoto and Sakuma [MS91], Adams and Reid [AR96], and Kuhn [Kuh96], following in part the nice survey of Sakuma [Sak98].

Let k be a link in $\mathbb{S}^3$, and let M be its complement (topological) manifold. The **tunnel number** $t(k)$ is the smallest number of properly embedded arcs in M (i.e. endpoints on k) such that the complement of a tubular neighbourhood of the arcs in M is a handlebody. We are particularly interested in the case $t(k) = 1$. A properly embedded arc τ in M such that $M \setminus N(\tau)$ is a handlebody is called an **unknotting tunnel**. Since the complement of a handlebody in $\mathbb{S}^3$ is also a handlebody, we see that if k admits an unknotting tunnel then it is either a knot or a two-component link with the components joined by τ.

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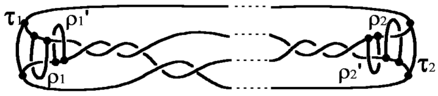
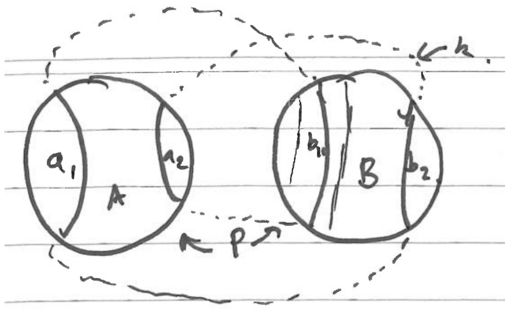
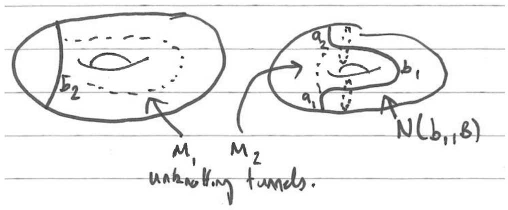


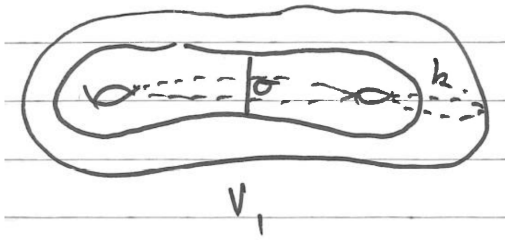
Figure 1: The six unknotting tunnels for a two-bridge knot. For a two-bridge link only τ_1 and τ_2 (the **upper and lower tunnels**) are unknotting tunnels. Figure 1.1 of [Kob99].



(a) Genus 0 2-bridge presentation.



(b) Genus 1 1-bridge presentation.



(c) Genus 2 0-bridge presentation.

Figure 2: Heegaard splittings of $\mathbb{S}^3 \setminus k$ occuring in the proof of Kobayashi's theorem

(2.1) Theorem (Kobayashi, 1999). *Every unknotting tunnel for a non-trivial two-bridge knot is isotopic (in the knot complement) to one of the six shown in figure 1.*

Sketch of proof. Let $k \subset \mathbb{S}^3$ be 2-bridge and fix a sphere P in $\mathbb{S}^3$ which cuts the knot complement M into two Conway balls $A = \mathbb{B}^3 \setminus a_1 \cup a_2$ and $B = \mathbb{B}^3 \setminus b_1 \cup b_2$, figure 2a. Consider $A \cup N(b_1, B)$: this is a solid torus with an arc drilled out and its complement in M is also a solid torus with an arc (b_2) drilled out, figure 2b. This is called a genus 1 one-bridge presentation of k.

Given such a decomposition there are two obvious unknotting tunnels μ_1 and μ_2. In addition it is easy to see from the construction that one of the visible unknotting tunnels is τ_1 or τ_2 and the other is one of $\rho_1, \rho_1', \rho_2, \rho_2'$.

A result of kobayashi and Saeki [KS00] is that every genus 1 one-bridge presentation of a two-bridge knot is obtained from a rational splitting as above. Hence the result is shown if we can show that every unknotting tunnel arises as the unknotting tunnel of a genus 1 one-bridge splitting, for then by the kobayashi–Saeki theorem it is one of the six known tunnels.

Suppose k admits an unknotting tunnel σ and the induced genus two Heegaard splitting of $\mathbb{S}^3$ is $V_1 = N(k \cup \sigma)$, $V_2 = \mathbb{S}^3 \setminus V_1$, c.f. figure 2c. We say the splitting is **weak** if there exist k-compressing discs D_1 and D_2 respectively properly embedded in V_1 and V_2 with disjoint boundaries ∂D_1 and ∂D_2.

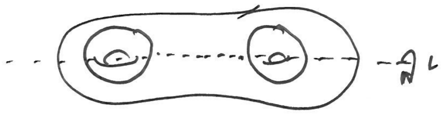


Figure 3: The hyperelliptic involution.

If there are no such discs then the splitting is **strong**.

If the splitting is weak, it can be shown that the discs D_1 and D_2 can be choosen to be non-separating in their respective handlebodies[1] There are two possibilities now: either D_1 intersects k or it doesn't (c.f. the two dotted discs in figure 2c).

- If D_1 does not intersect k, then cut the handlebody along D_1 to give a solid torus T. Since D_2 is non-separating in V_2, ∂D_2 cannot bound a disc in T for then the union of this disc with D_2 would be a non-separating 2-sphere in $\mathbb{S}^3$. We see that ∂D_2 must be a latitude of T, because if it was more twisted then we would be able to construct a lens space inside $\mathbb{S}^2$ from a piece of V_2 and T. Hence ∂D_2 is isotopic in M to the knot k, in particular k is the boundary of an embedded disk, and k is trivial.
- If D_1 intersects k, then let N be a regular neighbourhood of D_1 in V_1. Set $T_1 = \overline{V_1 \setminus N}$, $T_2 = V_2 \cup N$. Then T_2 is a solid torus with $N \cap k$ a trivial embedded arc[2] (i.e. it is of the form the left image of figure 2b with $b_2 = N \cap k$), and $T_1 \cup T_2$ is a genus 1 one-bridge presentation for k with σ the unknotting tunnel associated with T_1 in the one-bridge presentation.

If the splitting is strong, then a detailed study of the embedded discs in V_1 (carried out in §4 of [Kob99]) shows that if σ is not isotopic to τ_1 or τ_2, then there is an essential annulus in the manifold M. Roughly speaking, the idea of this analysis is to consider the interaction between the sphere P inducing the rational decomposition and the genus two surface Q that induces the genus two Heegaard splitting arising from $k \cup \sigma$: the simple closed curves of $P \cap Q$ bound a number of discs which intersect k in various configurations and the proof proceeds by (i) induction on the number of discs and (ii) cases on the different combinatorics of the discs.

Once it is known that there is an essential annulus then the knot is a torus knot, for which the result is known due to Boileau, Rost, and Zieschang [BRZ88] (and in fact the six unknotting tunnels reduce to only three up to isotopy). Since these are not hyperbolic we ignore this case.

(2.2) Theorem (Adams and Reid, 1996; Kuhn, 1996)**.** *Every unknotting tunnel for a non-trivial two-bridge two-component link is isotopic (in the knot complement) to either τ_1 or τ_2 shown in figure 1.*

Sketch of proof. We follow the proof in [AR96] which uses hyperbolic geometry. Suppose first that the link k is not a closed 2-braid (i.e. it has at least two twist regions). Then the complement is hyperbolic. Let ι be the hyperelliptic involution of the handlebody containing $k \cup \sigma$, σ being the unknotting tunnel, figure 3. Lifting to the universal cover $\mathbb{H}^3$, the involution becomes a $\pi_1(k)$-invariant set of elliptic involutions of order 2 and the unknotting tunnel lifts to some subset of the axes of these involutions. This shows that every unknotting tunnel is isotopic to a geodesic which ends on the cusps.

The link k always has orientation preserving symmetry group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, figure 4. Only one of the involutions preserves the two components. Hence the only possible unknotting tunnels are the four arcs in the knot complement which make up the axis of this unique involution. Two of these are the upper and lower tunnels. Let α and β be the other arcs, and suppose for a contradiction that

[1] Claim 1 in the proof of Proposition 2.15 of [Kob99].

[2] Claim 2 in the proof of Proposition 2.15 of [Kob99].

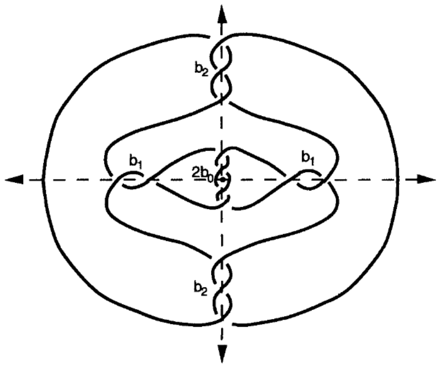


Figure 4: The symmetry group of $[2b_0, 2b_1, 2b_2]$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The hyperelliptic involution is the rotation by π around the horizontal line. Figure 1 of [AR96].

α is an unknotting tunnel. Let M be the link complement. The hyperelliptic involution ι extends to the whole of $\mathbb{S}^3$; let $p : \mathbb{S}^3 \to \mathbb{S}^3$ be the quotient. Under this quotient, the image $p(\beta)$ is unknotted (since it lies on the axis of the involution). This means that $p(\alpha \cup k \cup \beta)$ is unknotted (i.e. a trivially embedded genus three trivalent graph). But consideration of the diagram figure 4 shows that the quotient is actually $p(\alpha \cup k \cup \beta) = [b_0, 4b_1, b_2, 4b_3, ...]$ which is knotted.

§3. Kleinian groups generated by two parabolics

We now proceed to study the family of Kleinian groups $\langle X, Y \rangle$ where X and Y are parabolic with distinct fixed points. Suitable normalisation allows us to assume that our group is

$$\Gamma_\rho = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle.$$

The fundamental questions are:

1. For what ρ are these groups discrete? and,
2. When Γ_ρ is discrete, what is the isometry type of $\mathbb{H}^3/\Gamma_\rho$?

These questions have a long history (the earliest papers which I am aware of are by Sanov in 1947 [San47] and Brenner in 1955 [Bre55], see [EMS23]), but the modern point of view was initiated by Riley [Ril72; Ril75a; Ril75b; Ril13; BJS13; Ril79] who found these groups from his study of the hyperbolisation of two-bridge links. In the following, $\mathfrak{b}(q, p)$ denotes the q/p 2-bridge link.

(3.1) Proposition ([Ril72, Proposition 1]). *Fix a 2-bridge link $\mathfrak{b}(q, p)$. For any $s \neq q$ write $\bar{s}$ for the reprentative of s mod $2q$ in the interval $(-q, q)$. For each i set $\varepsilon_i = -\operatorname{sgn}(\overline{ip})$. Define a word $R_{p/q}$ in the symbols X and Y by*

$$R_{p/q} = X^{\varepsilon_1} Y^{\varepsilon_2} \cdots (X \text{ or } Y \text{ depending on } q)^{\varepsilon_{q-1}},$$

so $R_{p/q}$ is a word of length $q-1$. Then, if q is odd (so the link is a knot) we have

$$\pi_1(\mathfrak{b}(q, p)) \simeq \langle X, Y : R_{p/q} X = Y R_{p/q} \rangle;$$

if q is even (so the link has two components) then

$$\pi_1(\mathfrak{b}(q, p)) \simeq \langle X, Y : R_{p/q} Y = Y R_{p/q} \rangle.$$

The essence of the proof of proposition (3.1) is to compute a Wirtinger representation for the link such that two of the generators are the bridge arcs, and then eliminate all other generators. *The single relator which remains is exactly the word which represents a loop around the upper unknotting tunnel of the knot.* We denote this formal word in X and Y by $W_{p/q}$: under the act0427ual representation $F(X,Y) \to \Gamma_\rho$, the word $W_{p/q}$ is sent to the identity.

Thus we have a large family of discrete groups Γ_ρ: for each 2-bridge link $\mathfrak{b}(q,p)$ there is some ρ (exactly four choices which give either the same knot or highly related knots) such that $W_{p/q} = I_2$, in which case the group is not only discrete but $\mathbb{H}^3/\Gamma_\rho$ is the 2-bridge link complement.[3] This family of groups all have representations

$$\langle X, Y : X^\infty = Y^\infty = W_{p/q} = 1 \rangle$$

where the relation G^∞ is to be read as "G is parabolic".

Riley studied the representations of the strongly related groups

$$\langle X, Y : X^\infty = Y^\infty = W_{p/q}^n = 1 \rangle,$$

where $n \in \mathbb{Z}$. Geometrically these correspond to 2-bridge knot complements where the unknotting tunnel is replaced by a cone arc (with endpoints on the knot) with cone angle $2\pi/n$: the element $W_{p/q}$ is an elliptic of order n. He called these groups **Heckoid groups** [Ril92].

As $n \to \infty$ the cone angle decreases to zero, and in the limit we obtain free groups

$$\langle X, Y : X^\infty = Y^\infty = W_{p/q}^\infty = 1 \rangle$$

where $W_{p/q}$ has gone parabolic. When these groups are discrete they correspond to **cusp groups**, where the unknotting tunnel is deleted and replaced by a rank one cusp (the vertices where it meets the knot are thrice-punctured spheres). When ρ is increased further, $W_{p/q}$ becomes loxodromic and the manifold $\mathbb{H}^3/\Gamma_\rho$ is homeomorphic to a 3-ball with two drilled arcs.

The geometric procedure is shown in figure 5.

In 2002 Agol [Ago02] sketched an incomplete proof of the following theorem; two complete proofs were given by Aimi, Lee, Sakai, and Sakuma [Aim+20], and Akiyoshi, Ohshika, Parker, Sakuma, and Yoshida [Aki+21].

(3.2) Theorem. *A non-free non-Fuchsian Kleinian group G is generated by two non-commuting parabolic elements if and only if one of the following holds:*

1. *G is conjugate to some hyperbolic 2-bridge link group; or*
2. *G is conjugate to the Heckoid group $\langle X, Y : W_{p/q}^n = 1 \rangle$ for some $p/q \in \mathbb{Q}$ and some $n \in \mathbb{Z}_{>1}$; or*
3. *G is conjugate to the orbifold holonomy of a quotient of a Heckoid manifold by order two involutions of the p/q-knot.*

If G is a hyperbolic 2-bridge link group then it has exactly two parabolic generating pairs, up to conjugacy. If G is a Heckoid group then it has a unique parabolic generating pair up to conjugacy.

Remark. The Fuchsian groups generated by two non-commuting parabolics are fully classified by Knapp [Kna68]. The case that X and Y are allowed to be finite order is qualitatively very similar, and has been fully studied by Chesebro, Martin, and Schillewaert [CMS24].

[3] It is a theorem of Riley which can be found in [Ril72; Ril75b] and [KAG86, Problem 86] that so long as a knot group representation is faithful and has parabolics in the correct places then it is actually giving the correct action on $\mathbb{H}^3$, for some examples and applications see [KAG86, Examples 59, 60].

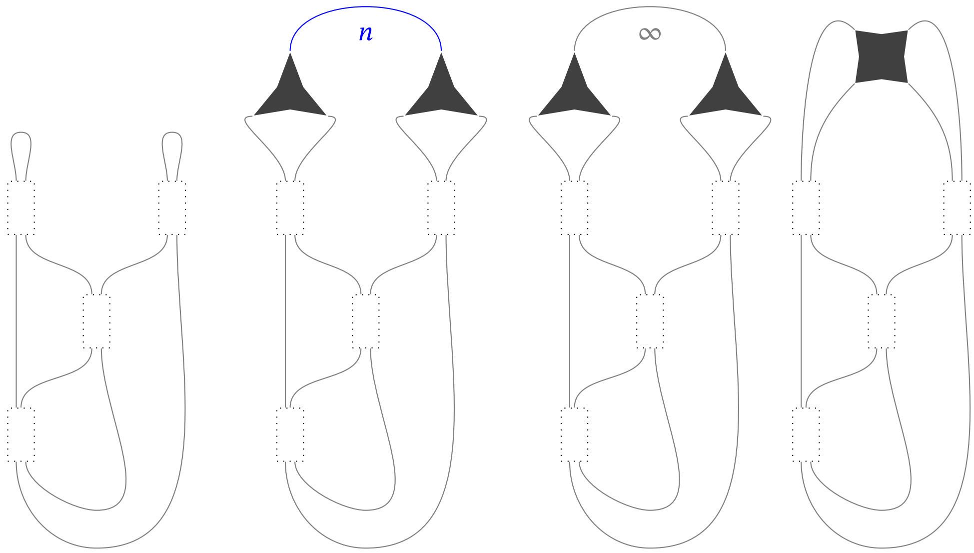


Figure 5: The four kinds of orbifolds found along an extended pleating ray. Left to right: a 2-bridge link or knot complement; a Heckoid orbifold; a cusp group; and a Riley group. In the elliptic case, all can occur and the two cone orders a and b correspond to the two arcs of the link separated by the unknotting tunnel. If $a \neq b$ and the ray corresponds to a knot then the procedure must stop at the Heckoid group with unknotting tunnel a cone arc of order 2. From [EMS24].

The Kleinian groups that are *freely* generated by parabolic elements are called **Riley groups** and have been studied in detail by a wide range of people as they are the easiest example of quasi-Fuchsian groups of the second kind. The primary sources in this direction are the work of Keen and Series [KS94], Komori and Series [KS98], and Ohshika and Miyachi [OM10]. Additional work is surveyed in [EMS23].

The study of the Riley groups is still essentially derived from the 2-bridge link structures: although an unknotting tunnel has been deleted, the data associated to the knot (encoded in the family of words $W_{p/q}$) is still visible in the conformal structure of the four-times punctured sphere on the conformal boundary. One point of view is that the conformal structure comes from the way that a braid group acts on the Conway ball to produce a rational tangle: this action extends to the bounding sphere $S_{0,4}$ and hence induces a twisting of the conformal structure. Details may be found in [Elz23, §3.1].

A full picture of the entire deformation space of ρ is seen in figure 6. It should be noted that this is only heuristic and in fact the boundary is highly non-circular (one can show it is not even a quasicircle).

§3.1. Optional background in character theory

For later use we relate this discussion to representation theory. We recall that a **representation** of a group G over a vector space V is a homomorphism $\rho : G \to \mathrm{Aut}(V)$. We will restrict to the setting $V = \mathbb{C}^2$ and ask for images $\rho(G)$ to lie in $\mathrm{SL}(2,\mathbb{C})$. Suppose G is finitely presented on the generators $g_1, ..., g_n$ and relators $R_1 = \cdots = R_m = 1$. Then a map $\check{\rho} : \{g_1, ..., g_n\} \to \mathrm{SL}(2,\mathbb{C})$ extends to a representation if and only if for every relator $R_j = \prod_{i=1}^{\ell_j} g_{k_j}$ the equation $\prod_{i=1}^{\ell_j} \check{\rho}(g_{k_j}) = I_2$ for all j. This sets up three polynomial equations (one entry is determined by the determinant condition) on the entries of the $\check{\rho}(g_k)$ for all relators, giving a total of $3m$ equations in a space of dimension $3n$.

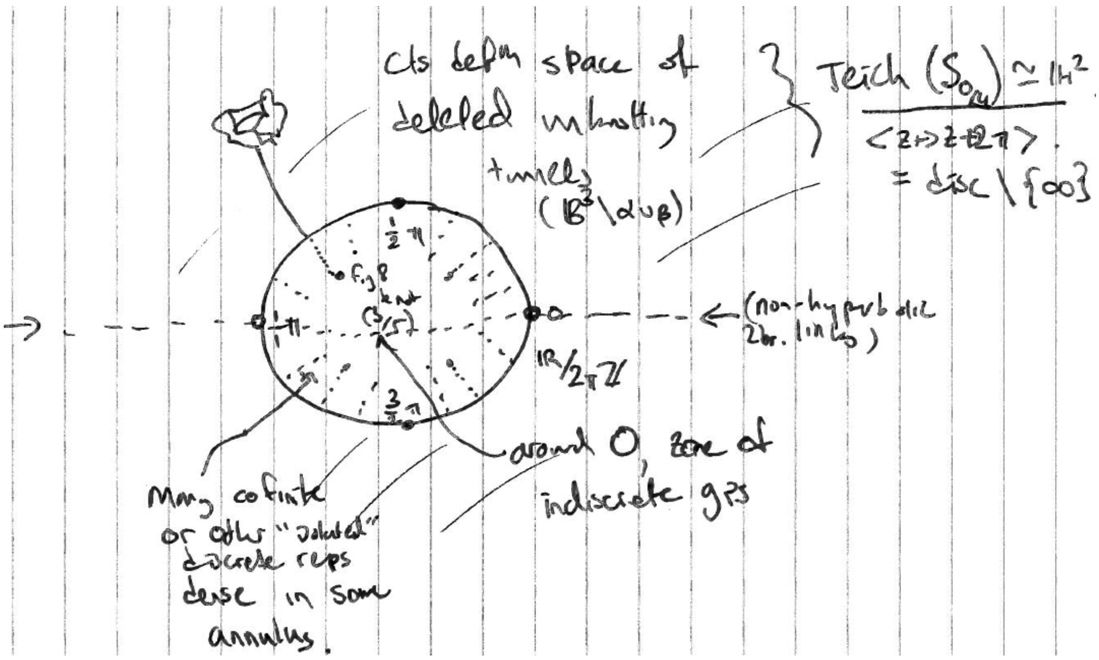


Figure 6: Rough map of the Kleinian groups generated by a pair of parabolics.

The set of all representations is therefore generically dimension $3(n-m)$. For example take a surface group which has $2g$ generators and one relation and look at representations in $\mathsf{SL}(2,\mathbb{R})$; we end up with final dimension $3(2g-1) = 6g-3$. This is three too many since we recall from Teichmüller theory that the correct number of real dimensions should be $6g-6$. The remaining three are lost if we mod out conjugacy in $\mathsf{PSL}(2,\mathbb{R})$.

Given a representation $\rho : G \to \mathsf{SL}(2,\mathbb{C})$ the **character** is the map $\chi_\rho : g \mapsto \operatorname{tr}\rho(g)$. If we know the value of sufficiently many characters, this gives us a number of polynomial equations in the parameters of the representation ρ. It is intuitive then that the character map should determine the representation fully. This is not always the case[4] but it is true in this setting. Again the group presentation gives us a number of polynomial equations in the entries of the images of the generators (that this is true is a consequence of the famous theorem that all traces in a finitely generated subgroup of $\mathsf{SL}(2,\mathbb{C})$ are integer polynomials in the traces of words in the generators of some bounded length, c.f. [MR03, §3.5]). The subvariety of $\mathbb{C}^{3n}$ cut out by these equations, (GIT) quotient by the conjugation action of $\mathsf{SL}(2,\mathbb{C})$, is called the **character variety** $X(G)$. The seminal work on these varieties is the work of Culler and Shalen [CS83].

It is easy to see [GMM98] that exactly three traces determine the representations of a group on two generators $\langle X, Y\rangle$: $\operatorname{tr} X$, $\operatorname{tr} Y$, and $\operatorname{tr}[X,Y]$. (We have two generators and no relations and indeed $3(2-0) = 6$ minus 3 for conjugacy gives that we should only need three traces). In the case of the knot groups above, $\operatorname{tr} X = \operatorname{tr} Y = 2$ and so we have exactly one complex dimension of freedom, $\operatorname{tr}[X,Y] = \rho^2 + 2$. (This reflects the fact that the groups Γ_ρ and $\Gamma_{-\rho}$ are indistinguishable, differing only in choice of generators).

Warning. As the audience of this note is not composed solely of algebraic geometers, we assume all things that we see are varieties even when they are blatently not (e.g. they might be non-reduced, certainly we have already taken a GIT quotient without asking theological questions). In the cases

[4]for a counterexample with a linear group see `https://en.wikipedia.org/wiki/Character_variety#Variants`

of interest to us, everything will turn out to be a variety (i.e. reduced over $\mathbb{C}$, though maybe not irreducible).

§4. The enumeration of unknotting tunnels

Another unifying point of view, this time from 2-bridge knots to arbitrary knots with tunnel number 1, may be found in work of Cho and McCollough [CM09]. Roughly speaking their point of view is similar to the proof of theorem (2.1) outlined above: knot tunnels are detected by using complexes of embedded discs in the genus two Heegard splitting. This point of view is philosophically isomorphic (isosophic?) to an extension of the study of upper/lower unknotting tunnels of two-bridge knots as carried out by Sakuma et. al., indeed one can view the enumeration of two-generated Kleinian groups as an enumeration of non-separating discs in genus two surface if two compressing discs have already been chosen (dual to the bridges). In [CM09] this setting is the setting of 'simple tunnels'. Just like how the isolated groups (the non-free ones, i.e. two-bridge link groups, Heckoid groups, and certain quotients) in the space of discrete representations are indexed by vertices of the Farey triangulation and permuted around semi-transitively (transitive on triangles not vertices) by $\mathsf{SL}(2,\mathbb{Z}) \approx \mathrm{Mod}(S_{0,4})$ where the indeterminacy comes from some global symmetry (the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ group which is the symmetry group of the knot complement), in general there is a large complex of discs called the **disc complex** $\mathcal{D}(H)$ and some large group $\mathcal{G}$ acting on $\mathcal{D}(H)$ such that $\mathcal{D}(H)/\mathcal{G}$ is a complex indexing all the possible tunnels. There is even a natural spanning tree akin to the Stern-Brocot tree. One can extend this picture to define combinatorial structures on which to hang trace polynomials for general Schottky groups and general function groups.