

# Ephemera and apocrypha

## 2. Sociology

1. Let  $T = \mathbb{R}^2/\mathbb{Z}^2$  be the affine torus.
  - (a) Show that every geodesic on  $T$  is either (i) dense in  $T$  or (ii) the projection of a line of rational slope. Call the latter  $\gamma(p/q)$  where  $(p, q) = 1$  or  $p/q \in \{1/0, 0/1\}$ .
  - (b) Show that every geodesic lamination on  $T$  consists of a single geodesic.
  - (c) Show that there is a natural  $\mathrm{PSL}(2, \mathbb{Z})$  action on non-dense geodesic laminations and this action preserves the simplicial structure with cells  $(p/q, r/s, (p+r)/(q+s))$  where  $|ps - qr| = 1$ .
2. (a) Find conditions on  $\lambda, \mu \in \mathbb{R}$  such that the group  $G$  generated by the two elements  $\begin{bmatrix} \lambda & -\lambda \\ \lambda(1-\lambda^2)/2 & \end{bmatrix}$  and  $\begin{bmatrix} \mu & -\mu \\ -\mu(1+\mu^2)/2 & \end{bmatrix}$  is discrete and has quotient a disjoint union of two punctured tori.  
 The group should be Fuchsian of the first kind and hence the hyperbolic metrics on  $\mathbb{H}^2$  and  $-\mathbb{H}^2$  descend. Let  $T^*$  be the quotient of  $\mathbb{H}^2$ .
  - (b) Show that every *closed* geodesic on  $T^*$  is either (i) dense in  $T^*$  minus some small open neighbourhood of the cusp or (ii) has homology class  $p\gamma_0 + q\gamma_\infty$  for  $(p, q) = 1$  or  $p/q \in \{1/0, 0/1\}$  and with some fixed homology basis  $(\gamma_0, \gamma_\infty)$ . Even better, there is a bijection between points visible from the origin in  $H_1(T^*) \simeq \mathbb{Z}^2$  and non-dense closed geodesics on  $T^*$ .
  - (c) Show that every geodesic lamination on  $T^*$  which does not meet the cusp end ( $\iff$  has compact support) consists of a single geodesic.
  - (d) Show that there is a natural  $\mathrm{PSL}(2, \mathbb{Z})$  action on non-dense geodesic laminations with compact support and that this action preserves the simplicial structure with cells  $(p/q, r/s, (p+r)/(q+s))$  where  $|ps - qr| = 1$ .
3. (We saw the Maskit and Riley slices in the lecture. This problem contains both of them.) Let  $G = G(\lambda, \mu, \rho)$  be the group generated by

$$X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix}, Y = \begin{bmatrix} \mu & 0 \\ \rho & \mu^{-1} \end{bmatrix};$$

the generic situation is that  $G$  is free and  $\Omega(G)/G$  (it is a genus two Schottky group). Suppose  $\lambda = 2$ , then  $X$  represents the transformation  $z \mapsto 4z + 2$  which sends the vertical line  $\mathrm{Re} z = -1$  to the vertical line  $\mathrm{Re} z = 1$ . If  $\mu = 2$  then the isometric circles of  $Y$  are the circles of radius  $1/|\rho|$  centred around  $-1/2\rho$  and  $2/\rho$ .

- (a) Prove that these circles are contained strictly within the vertical strip  $-1 < \mathrm{Re} z < 1$  iff the two following conditions hold (where  $\rho = re^{i\theta}$ ):

$$r > \frac{1}{2}(\cos \theta + 2) < 1 \text{ and } r > (2 \cos \theta + 1).$$

That is,  $\rho$  lies in the mutual exterior of the two cardioids depicted in fig. 1. This gives a rough bound on the family  $\mathrm{Fam}(G(\lambda = 2, \mu = 2, \rho), |\rho| \gg 0)$ , which is a one-dimensional slice through genus two Schottky space. (Compare this with the Riley slice, which is the slice  $\lambda = \mu = 1$  through the boundary of this space.) Which points of the cardioids actually lie on the actual boundary of the deformation space?

- (b) Anyway, carry out an analysis similar to the torus questions for a genus two surface  $S$ . Your theory should include such things as:
  - i. Maximal geodesic laminations all have three geodesics, and there is a natural partition of the space of laminations given via density in non-zero measure subsets of the surface;
  - ii. The geodesics can be viewed in some way as (isotopic to) projections of combinatorially defined curves with respect to the fundamental domain given in (a);

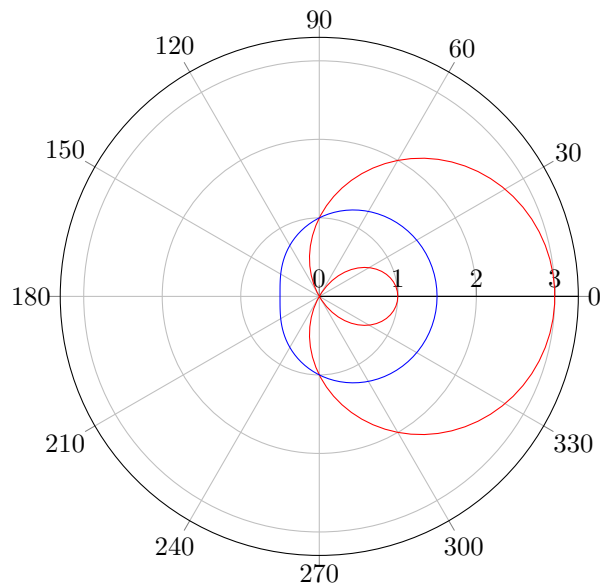


Figure 1: Bounds on the one-dimensional slice through genus two Schottky space.

- iii. There is a bijection between the space of non-dense maximal geodesics and some natural subset of  $H_1(S)$ ;
  - iv. There is an action of some group like  $\mathrm{PSL}(3, \mathbb{Z})$  on this subset.
- (c) If you take a maximal geodesic lamination with one lamination separating the surface into two tori-minus-discs (each with one of the other laminations) then this lamination can be pinched to a cusp. This reduces (topologically) to question 2 above. In particular, this explains why we ignore geodesics which hit the cusp. Check that your theory contains the theory of question 2.
4. (Open) Give a strictly algebraic method for determining the canonical lamination on a Kleinian group.