APANASOV'S WILD KNOT

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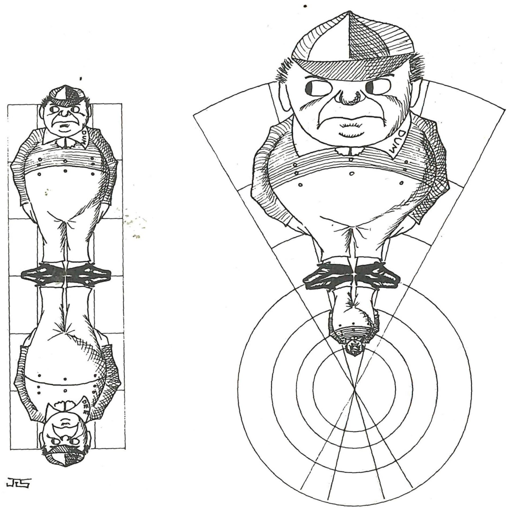
ABSTRACT. We fill in some missing details from a theorem of Apanasov.

1. CONFORMAL MAPS

An (orientation-preserving) *Möbius transformation* is a map $\mathbb{S}^n \to \mathbb{S}^n$ which is a finite product of an even number of inversions in $(n-1)$-spheres. The group of Möbius transformations on $\mathbb{S}^n$ will be denoted $\mathbb{M}(n)$.



Drawing by John Stillwell from [Sve91, p. 30].

We are familiar with the result that all holomorphic functions are conformal wherever they have non-zero derivative [Ahl79, §2.3]. A short argument shows that every conformal automorphism from $\mathbb{S}^2$ to itself is a Möbius transformation.[1] On the other hand, there are many conformal maps $\mathbb{S}^2 \to \mathbb{S}^2$ which are not bijective and not Möbius: the Riemann mapping theorem gives a vast supply of such maps!

It is therefore very remarkable that, for $n > 2$, *all C^1 conformal maps are Möbius transformations*; this is an 1850 result of Liouville in the case of C^3 maps, see the discussion in [GMP17, §3.8] (who give a proof for C^4 maps due to Nevanlinna). To put it another way, unlike the case in two dimensions where every open simply-connected subset of $\mathbb{C}$ can be mapped conformally onto the round 2-ball, in n dimensions the only thing conformally equivalent to the n-ball is itself!

A subgroup $\Gamma \leq \mathbb{M}(n)$ is called *Kleinian*[2] if it acts discontinuously on a non-empty open subset of $\mathbb{S}^n$; this domain will be denoted $\Omega(\Gamma)$. It is called *m-Fuchsian*

[1] Suppose that $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a conformal bijection, in particular f is meromorphic. We may assume by composition with an appropriate Möbius transformation that it fixes ∞. Hence $f(1/z)$ has a pole at 0. Expand $f(1/z) = z^{-k} \sum_{n=0}^{\infty} a_n z^n$, so $f(z) = z^k \sum_{n=0}^{\infty} a_n z^{-n}$. Since f does not have a pole at 0, $a_n = 0$ when $n > k$. Thus f is polynomial. Since it is injective it must be degree 1, so $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

[2] Despite Klein [Gra13, pp. 233–234]!

if there is some round $\mathbb{B}^m \subset \mathbb{S}^n$ which it preserves. It is called *m-quasi-Fuchsian* if there is a quasiconformal map $\phi : \mathbb{S}^n \to \mathbb{S}^n$ such that $\phi\Gamma\phi^{-1}$ is m-Fuchsian.

How to visualise a quasi-Fuchsian group: in n-space there is a crinkled embedded sphere, left invariant by some global group action.
Anatoliĭ Fomenko, *Homotopy groups of spheres* (1971), [Fom90, pp. 34–35].

We will primarily be interested in $m = 3$ and $n = 4$. So the setup is, we consider groups of conformal maps of $\mathbb{S}^3$; and these are all Möbius transformations so extend naturally to act conformally on $\mathbb{B}^4$ as hyperbolic isometries. This is really the meaning of the word 'Fuchsian': a n-Fuchsian group is just a Kleinian group acting on $\mathbb{S}^n$ that is embedded conformally into $\mathbb{S}^m$ for $m > n$.

2. Apanasov's example

We describe an example due to Apanasov, in §7 of [Apa80; Apa81]. Let $L \subset \mathbb{S}^3$ be a polygonal knot. Choose a family of spheres $S_1, ..., S_k$ covering L so that each sphere meets exactly two others, and such that all intersection points are points of tangency and lie at vertices of L. Define $G < \mathbb{M}(3)$ by taking the orientation-preserving half of the group generated by reflections in these spheres. This group is Kleinian by the Poincaré polyhedron theorem: a fundamental domain is the set

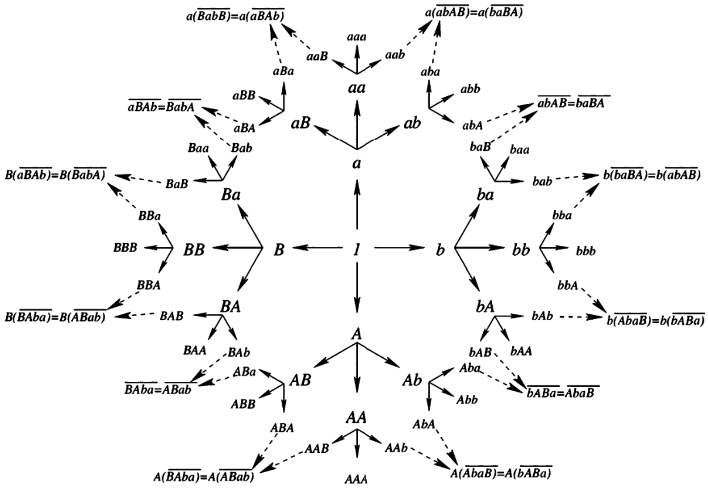
$$(*) \qquad P(G) = \left(\bigcap_{i=1}^{k} \operatorname{ext} S_i \right) \cup f_k \left(\bigcap_{i=1}^{k} \operatorname{ext} S_i \right)$$

where f_k is inversion in the single sphere S_k. The limit set of such a group has been constructed by Othoniel.

Jean-Michel Othoniel, *Lacan's Knot*; from [Día+19].

Theorem 1. *The limit set $\Lambda(G)$ is a wild knot in $\mathbb{S}^3$.*

Proof. In this proof we let $\oplus$ denote connect sum of knots. Finite index subgroups of discrete groups have the same limit set, and so we can work in the slightly easier to describe setting of the double cover $\tilde{G}$ of G generated by reflections in the S_i. First, observe that $\Lambda(\tilde{G})$ is homeomorphic to $\mathbb{S}^1$: this is proved by giving a very explicit homeomorphism and we neglect the details [Mas87, §VIII.F.2].



The construction of the homeomorphism to $\mathbb{S}^2$ is a standard Gromov-type argument about identifying $\partial\tilde{G}$ (i.e. the boundary of its Cayley graph) with the boundary of $\mathbb{H}^2$. This pictorial proof is taken from [MSW02, Fig. 6.13].

It remains to show that $\Lambda(\tilde{G})$ has infinitely many crossings. A fundamental domain of $\tilde{G}$ is the common exterior of the spheres S_i. The vertices of this domain are the vertices of the polygonal link $L = L_0$ and clearly $\Lambda(\tilde{G})$ is a simple closed curve passing through the vertices of L_0. Let f_k be reflection in S_k; then $f_k(S_1), ..., f_k(S_{k-1})$ are a family of tangent circles inside S_k, and passing through the tangency points is a copy of the portion of the link L_0 lying outside S_k; again $\Lambda(G)$ passes through these limit points, and so it passes through all vertices of the polygonal link L_1 which is equivalent to the connect sum $L_0 \oplus L_0$. Pick one of the $k-1$ spheres inside S_k and continue inductively to show that $\Lambda(G)$ has infinitely many crossings. $\square$

Theorem 2. *The group G is quasi-Fuchsian; $\Lambda(G)$ is a quasicircle.*

Proof. Cover $\mathbb{S}^1 \subset \mathbb{S}^2$ by k discs, forming a tangency chain. Embed this conformally in $\mathbb{S}^3$. The orientation-preserving half Γ of the group generated by reflections in these spheres clearly has $\Lambda(\Gamma) = \mathbb{S}^1$ and a fundamental domain $P(\Gamma)$ defined using ($*$). Embed $\mathbb{S}^3$ into $\mathbb{S}^4$ as the boundary of the upper half-space $\mathbb{H}^4$; let $\tilde{G}$ and $\tilde{\Gamma}$ denote the extensions of G and Γ respectively to $\mathbb{S}^4$, and define $P(\tilde{\Gamma})$ and $P(\tilde{G})$ by direct extension. We can now construct a PL-homeomorphism $f : P(\tilde{\Gamma}) \to P(\tilde{G})$, defined by choosing basepoints in $\mathbb{H}^4$, coning from them to $\partial P(\tilde{G})$ and $\partial P(\tilde{\Gamma})$, definining homeomorphisms on the boundaries, and extending pos-linearly to the remainder of the cone. Since the cones are compact, they admit a finite simplicial decomposition so that f is linear on each simplex (apply [RS82, Theorem 2.2] to the graph of f, as in the proof of Corollary 2.3, *op. cit.*); again by compactness each face of this decomposition has finite 3-measure in $\mathbb{S}^4$ so we can apply [Väi71, Theorem 35.1] to see that f is actually quasiconformal. Extend f to a quasiconformal homeomorphism $\Omega(\tilde{G}) \to \Omega(\tilde{\Gamma})$ by the group action. By a modification of the argument in [Mas87, §VIII.F.2], f can be extended to a homeomorphism between the limit sets. Since the limit sets are contained in $\mathbb{S}^3$, they are nil-measure in $\mathbb{S}^4$ and so $f : \mathbb{S}^4 \to \mathbb{S}^4$ is a quasiconformal homeomorphism such that $\tilde{\Gamma} = f\tilde{G}f^{-1}$. $\square$

We remark that G is quasi-Fuchsian because it preserves a quasidisc, bounded by the wild knot, that is bent through $\mathbb{R}^4$. So it is a 2-quasi-Fuchsian and 4-Kleinian group. Of course it is also 3-Kleinian, but the quasidisc is not visible in its action on $\mathbb{S}^3$.

An amusing but involved extension of these ideas allows one to construct groups in $\mathbb{M}(3)$ whose limit sets are wild knotted 2-spheres in $\mathbb{R}^3$ [Apa91, Example 7.19].

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