Deformation Spaces of Kleinian Groups

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Abstract

It has been known since at least the time of Poincaré that isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$ can be represented by 2×2 matrices over the complex numbers: the matrices represent fractional linear transformations on the sphere at infinity, and hyperbolic space is rigid enough that every hyperbolic motion is determined by such an action at infinity. A discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is called a Kleinian group; the quotient of $\mathbb{H}^3$ by the action of such a group is an orbifold, and its boundary at infinity is a (possibly empty or disconnected) Riemann surface.

The *Riley slice* is the moduli space of Kleinian groups generated by a pair of parabolic elements which are free on those generators and whose corresponding surface is supported on a 4-punctured sphere; Robert Riley introduced this object in the 1970s while studying two-bridge knot groups. The Riley slice is naturally embedded in $\mathbb{C}$ and so is particularly amenable to study since one can draw pictures of it. Linda Keen and Caroline Series studied this embedding in the early 1990s via a family of polynomials which gave a foliation (local product decomposition) of the slice. We will discuss the Keen–Series theory and extend it to allow torsion elements as generators. We also discuss some new results of a combinatorial flavour and some applications. We aim for the exposition to be accessible to beginning graduate students, despite the high bar for entry to this subject in terms of prerequisite material.

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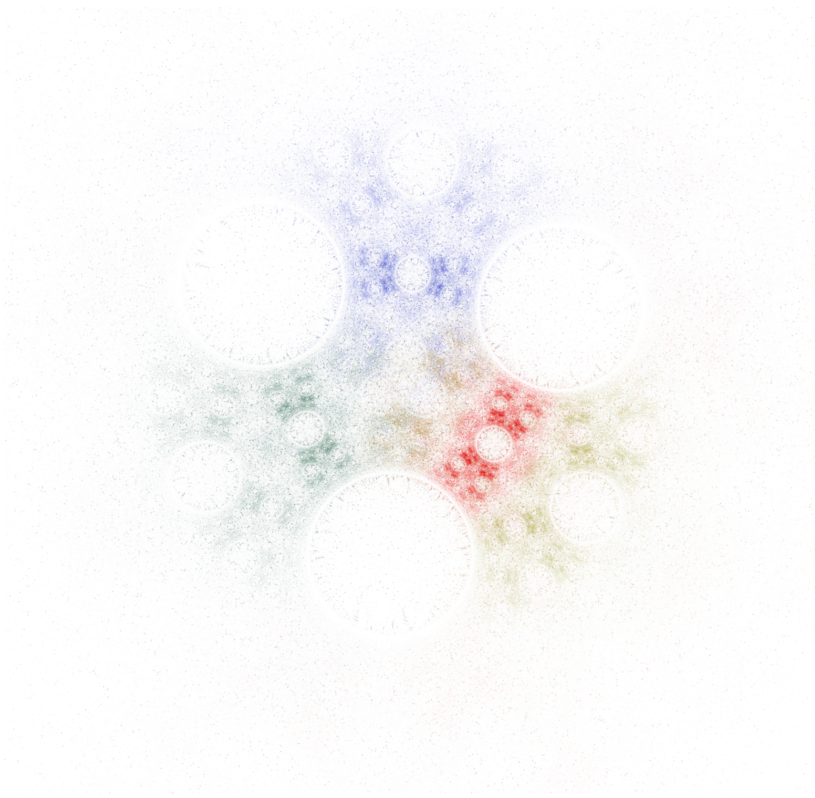

Contents

List of Figures **vii**

List of Tables **ix**

1 Introduction **1**
- 1.1 Non-technical précis . . . 1
- 1.2 Technical introduction . . . 4
- 1.3 Outline of the thesis . . . 6
- 1.4 Assumed background . . . 6
- 1.5 Miscellaneous conventions . . . 8
- 1.6 Papers based on this thesis . . . 8
- 1.7 Acknowledgements . . . 8

2 Geometric spaces **9**
- 2.1 Hyperbolic space . . . 9
- 2.2 Geometric manifolds and orbifolds . . . 11
- 2.3 The Poincaré polyhedron theorem . . . 13
- 2.4 The topology of 3-manifolds . . . 16

3 The geometry of Kleinian groups **19**
- 3.1 Kleinian groups . . . 19
- 3.2 Marked Riemann surfaces . . . 24
- 3.3 Geometry of the 3-manifold near infinity . . . 26
- 3.4 The convex core of a hyperbolic 3-manifold and measured laminations . . . 28
- 3.5 Schottky groups and deformations . . . 30
- 3.6 Quasiconformal deformation spaces . . . 34
  - 3.6A Quasiconformal mappings . . . 35
  - 3.6B Classical theory of Teichmüller space . . . 35
  - 3.6C Deformation spaces of Kleinian groups . . . 36
  - 3.6D Holomorphic motions . . . 39
  - 3.6E Measured laminations and quasiconformal deformation spaces . . . 40

4 The Riley slice: motivation and definition **43**
- 4.1 Two-bridge links . . . 43
- 4.2 The features of Riley's plot . . . 47
- 4.3 Elliptic Riley slices . . . 51

5 Analytical and topological properties **55**
5.1 The quasiconformal deformation space . . . 55
5.2 Topology of the boundary . . . 57

6 Curves on the 4-marked sphere **61**
6.1 Geodesic coding on the 4-marked sphere . . . 61
6.2 The geodesic representation in terms of Dehn twists . . . 70
6.3 The Farey triangulation and spines . . . 71

7 The foliation theory of Keen and Series **75**
7.1 Motivation . . . 75
7.2 F-peripheral subgroups . . . 80
7.3 Measured laminations and rational pleating rays . . . 85
7.3A The Fuchsian case . . . 87
7.3B The non-Fuchsian case . . . 90
7.3C Completing the proof of Theorem 7.3.4 . . . 91
7.4 Irrational pleating rays . . . 92
7.4A Irrational properties analogous to the rational properties . . . 92
7.4B The family of normalised complex length functions . . . 95
7.4C The pleating ray coordinate system for $\mathcal{R}$ . . . 100

8 Neighbourhoods of pleating rays **103**
8.1 The main result and a sketch of the proof . . . 103
8.2 An aside: Lyndon and Ullman's results . . . 105
8.3 Proof of the main theorem . . . 107
8.3A Products of parabolics . . . 107
8.3B Rotation angles and isometric discs . . . 108
8.3C Canonical peripheral quasidiscs . . . 111
8.3D Completing the proof . . . 113

9 The combinatorics of the Farey polynomials **117**
9.1 A recursion formula to generate Farey polynomials . . . 117
9.2 Some properties of the recurrence . . . 123
9.2A A Fibonacci-like subsequence of the homogeneous Farey polynomials . . . 124
9.2B Solving the homogeneous recursion relation in general . . . 126
9.3 Approximating irrational pleating rays . . . 130

10 Conjectures and open problems **133**
10.1 General problems . . . 133
10.2 The structure of the Riley slice exterior . . . 134
10.3 Some computational problems . . . 136
10.4 Factorisation properties of the Farey polynomials . . . 137

A Some remarks on computation **141**
A.1 Farey words and polynomials . . . 141
A.2 Limit sets . . . 143

Bibliography **147**

Index **157**

List of Figures

1/3 cusp for the $(2\pi/3, 2\pi/3)$ Riley group iii

1.1 Keeping track of untying a 2-tangle corresponding to the figure 8 knot. 2
1.2 The coordinate system for the Riley slice. 4

2.1 The shapes of the orbits of the different types of fractional linear transformation. . . . 11
2.2 A surface at infinity of a 3-manifold that bounds an embedded disc. 17
2.3 Two examples of incompressible surfaces in 3-manifolds. 17

3.1 Various limit sets of Kleinian groups. 23
3.2 A doubly cusped region for a Kleinian group 28
3.3 A pleated surface. 29
3.4 The surface for a loxodromic Schottky group. 32
3.5 The surface for a parabolic Schottky group. 33
3.6 The 3-manifold for a parabolic Schottky group. 33
3.7 The surface for an elliptic Schottky group. 34
3.8 The 3-manifold for an elliptic Schottky group. 34

4.1 Two two-bridge knots. 44
4.2 Pushing disjoint arcs on the sphere into the interior to form a 2-tangle. 44
4.3 The two ‘generating’ Dehn twists for 2-tangles. 45
4.4 The figure 8 knot. 45
4.5 Riley’s plot of the Riley slice 47
4.6 Upper and lower unknotting tunnels for a two-bridge knot with four braid regions. . . 48
4.7 The four kinds of orbifolds found along an extended pleating ray. 48
4.8 Riley slice exterior 50
4.9 A branched cover with branching curve the figure 8 knot in S^3 51
4.10 Riley slices for various cone point orders (a, b). 53

5.1 The action of an elementary parabolic group on $\mathbb{H}^2$. 57

6.1 A labelled ideal quadrilateral on the 4-marked sphere. 62
6.2 Fundamental domain for a Riley group with two parabolic generators. 62
6.3 Fundamental domain for a Riley group with two elliptic generators, of orders p and q. 63
6.4 Fundamental domain for a Riley group with one elliptic and one parabolic generator. 63
6.5 Topological cover of the 4-marked sphere induced by an ideal quadrilateral. 65
6.6 Three rational curves on the sphere. 66
6.7 Three rational ideal arcs on the sphere. 67
6.8 An ideal triangulation corresponding to a Farey triangle. 73

6.9 Collapsing the spine corresponding to a triangle to that of one of its sides. 74

7.1 A full limit set and two F-peripheral subgroup limit sets. 76
7.2 The hyperbolic convex hull, h.conv $\Lambda(\Gamma_{4i}^{\infty,\infty})$. 77
7.3 Limit sets of $\Gamma_{\rho}^{\infty,\infty}$ for various $\rho \in \mathbb{R}i$. 78
7.4 Limit sets of $\Gamma_{\rho}^{5,\infty}$ for various $\rho \in \mathbb{R}i$. 79
7.5 Two F-peripheral subgroups of $\Gamma_{1.7i}^{5,\infty}$. 80
7.6 The canonical polygon of a F-peripheral group F in $\Gamma_{3i}^{\infty,\infty}$ 81
7.7 The pleated surface of a group on $\mathcal{P}_{1/2}$. 84
7.8 The hyperbolic locus of slope 1/3 for $\mathcal{R}^{\infty,\infty}$. 86
7.9 The union of the hyperbolic locii of small slope for a representative elliptic Riley slice. 87
7.10 The pleated surface of a Fuchsian group. 88
7.11 Modification of Beardon's Figure 11.4.5 in the case that both generators are known to be elliptic. 89
7.12 Ramifications of a singularity on a pleating ray. 91
7.13 Figure for the proof of Proposition 7.4.4. 93
7.14 The universal cover of $\mathcal{R}^{a,b}$. 95
7.15 Pleating rays have uncountably many points. 98
7.16 The 'cut sets' $\mathbf{L}_{p/q}$ and $\mathbf{R}_{p/q}$ in $\mathbb{H}^+$, from the proof of Theorem 7.4.14. 98
7.17 Irrational pleating rays lie on exactly one side of every rational ray. 99
7.18 The quotient surface from part C. of the proof of Theorem 7.4.14. 100
7.19 Objects in the proof of Theorem 7.4.15. 101

8.1 The Riley slice with neighbourhoods for our pleating ray values illustrated. 104
8.2 The convex hull of $B \cup \{\pm 4\}$ contains $\mathbb{C} \setminus \mathcal{R}$. 106
8.3 Preimages of the sector $\{\arg(z) = -\frac{\pi}{6}\} \cup \{\arg(z) = \frac{\pi}{6}\}$. 107
8.4 Geometric objects in the limit sets of two cusp groups. 108
8.5 The isometric circles of Word(3/4). 110
8.6 A log-spiral connecting the fixed points of $W_{3/4}(\tilde{\rho})$. 113

9.1 Another view of the cutting sequence for Word(1/2). 118
9.2 The Farey addition graph. 118
9.3 Farey addition versus word multiplication for $W_{p/q}$ (red) and $W_{r/s}$ (green). 120
9.4 The induced colouring of the Stern-Brocot tree. 122
9.5 An embedding of the Farey graph into $\mathbb{H}^2$. (By Hyacinth—Own work, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=63343697`.) 127

10.1 A portion of the Riley slice exterior. 134

A.1 The figure 8 knot lies on the real locii of two Farey polynomials. 142

List of Tables

6.1 Farey words Word(p/q) for small q. See Example A.1.2. 68
6.2 Farey polynomials $\Phi^{a,b}_{p/q}(z)$ for small q. Here, $\alpha = \exp(\pi i/a)$ and $\beta = \exp(\pi i)/b$. 69

9.1 Selected homogeneous Farey polynomials Φ^h of slope p/q for small q, with the initial values as given. 124

10.1 Coefficients of the r/s Farey polynomial. 138

Chapter 1

Introduction

Section 1.1 gives a non-technical précis sketch of the problem which we tackle in this thesis, hopefully accessible to the layperson. We then proceed in Section 1.2 to give a technical introduction for an expert mathematical audience. An outline of the thesis, listing the main results, appears in Section 1.3; and in Section 1.4 we give references to some introductory texts covering the fields of mathematics that are required to read this thesis in full detail.

1.1 Non-technical précis

Broadly speaking, in this thesis we study the following problem:

1.1.1 Problem. Classify tangles made up of two pieces of string, with the four endpoints glued onto a table.

In fact, to be more precise, we consider also the *orientation* of the table: that is, if we rotate the table by 90° we consider the resulting tangle to be different than the one we started with. The more correct problem is then the following:

1.1.2 Problem. Attach four pegs to a table in a square shape, such that the pegs are paired up by two red lines on the table. What essentially different ways are there to tie two tangled-up pieces of blue string to the pegs, where we count two such arrangments as being 'different' if you cannot physically deform one arrangement to the other while keeping all the string above the table and without untying the strings from the pegs?

One such tangle is shown in the top-left diagram of Figure 1.1 (subfigure (a)); the curve made up of the two pieces of blue string together with the red lines drawn on the table is in fact the **figure 8 knot** (see Figure 4.4 below).

Let us make a further modification to the problem. To formalise the idea that we should not be able to 'cheat' by moving a big loop of blue string entirely around the table (looping under all the table legs and back up around), we replace the table with a sphere—we attach the pegs to the inside of the sphere and we allow ourselves only to move the string around within the bounds of the sphere (see (b) of Figure 1.1). Observe that the two red arcs now lie on the outside of the sphere, continuing to keep track of the orientation of the entire system.

We would like to replace the complicated situation of this tangle sitting inside the sphere with a pair of coordinates. To do this, draw a third line on the sphere (in green) which is parallel to the two red arcs but goes the whole way around. This is drawn in (c) of Figure 1.1, where we have also

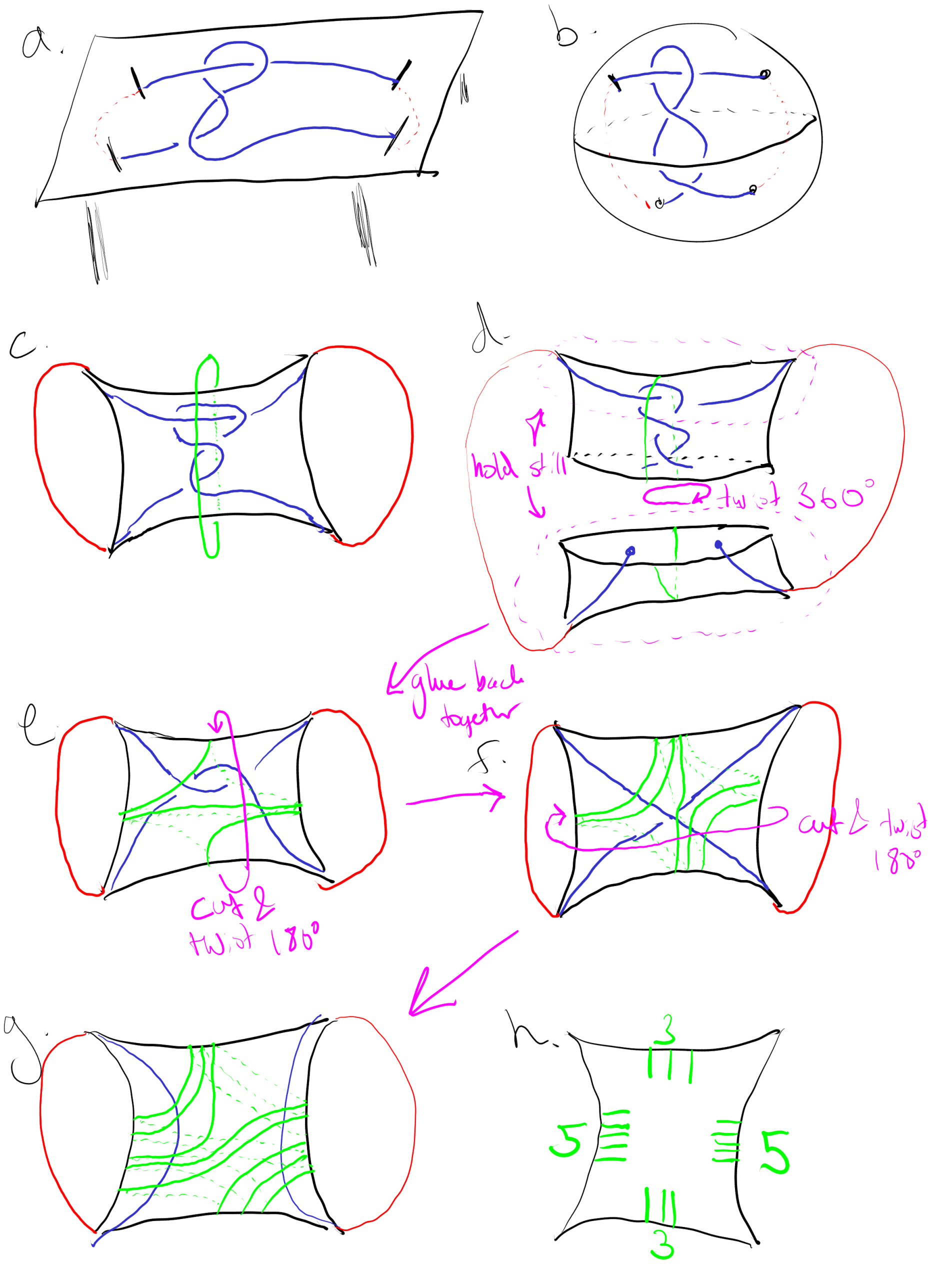

Figure 1.1: Keeping track of untying a 2-tangle corresponding to the figure 8 knot.

deformed the sphere a bit to make the structure of the tangles clearer (we have pushed and pulled the sphere surface so that it is in a 'pillowcase' shape, with the pegged ends of the blue tangle at the corners; we also pulled the red arcs off the sphere and into the surrounding space). We now proceed to untangle the tangle by cutting and twisting the sphere and its interior, while keeping track of what twists we have made using this green line.

Consider now diagram (d) of Figure 1.1. We cut along the horizontal diameter of the sphere, hold the lower half steady and the upper two corners steady, and twist the bottom part of the top hemisphere by 360°. This untwists the lower twisted part of the tangle, and when we glue it back together we have only one twist in the tangle left (that seen in the lower-left diagram of Figure 1.1). Observe that when we cut, twist, and glue, the green curve gets spiralled around the equator of the sphere: it is by looking at this that we keep track of the way that the tangle was tangled: as we untwist the tangle, we twist up this green line, so when we are done the totality of the twists in the green line represents all of the information that was present in the original tangle. The benefit of this is that we have reduced all of the three-dimensional information down to a piece of two-dimensional information, namely the way a single curve is draped on a sphere. (What we are doing here is a *Dehn twist* along the horizontal equator—this is the curve we will later in Chapter 4 call $\gamma(0/1)$.)

In order to untwist the final piece of the tangle, we do another two cuts and twists in this way. We do not show the cut-open views this time, but we cut along a horozontal equator and twist 180° to unwind diagram (e) of Figure 1.1 to diagram (f); we then cut along a vertical equator for a second time and again twist 180° to finally obtain the untwisted pair of arcs in diagram (g). Of course, these twists make the green curve even more complicated.

It turns out that the green curve, in the end, twists 5 times horizontally and 3 times vertically (see diagram (h) of Figure 1.1): for the reader who is coming back to this after having read Chapters 4 and 6 this corresponds to the fact that the figure 8 knot has *Schubert normal form* of 5/3 and the green line has become $\gamma(3/5)$—c.f. Example A.1.1).

The reader must now take on faith the following fact (Theorem 6.1.5): for every possible green curve that you can obtain by untangling a tangle in this way, you can obtain a fraction p/q; and for every fraction p/q there exists a tangle which will give the p/q curve back after being untangled. It is this number which is the first coordinate we associate to the tangle. It turns out that this fraction naturally represents an *angle*: the tangle which is associated to p/q should appear $p/q \cdot 360°$ around, and eventually you get back to where you started.

The second coordinate which we associate is the length of the green curve: different lengths correspond to the same tangle, but scaled appropriately. To be honest, the *real* statement of the problem which we are solving is the following:

1.1.3 Problem. Classify the possible *geometries* of tangles made up of two strands attached at four points to a sphere, with those four points arranged in pairs like above.

With this formulation, it is clear that both the *size* and the *shape* of the tangle determine the geometry: the shape gives an angle around a circle, and the size gives a distance away from the centre of the circle, so the set of possible geometries is an annulus (ring-shape). This set is called the **Riley slice**, and is the portion of Figure 1.2 which lies *outside* the shaded-in area in the middle; we have indicated the two 'axes', namely length of the green curve (moving outwards, blue) and the position of the green curve (moving around, red). In that figure we have called the green curve the **pleating locus** of the geometry; this is because it turns out that the geometry will have a natural 'pleat' on the sphere following the green curve. The radial coordinate lines corresponding to different lengths of the same curve are called **rational pleating rays**. The precise definition of this coordinate system (which is slightly more complicated than we have described here, for technical reasons: we need to 'fill in the gaps' between the rational pleating rays, and doing this with the definitions we outlined above gives curves of infinite length, so we need to normalise our lengths to make them finite in this

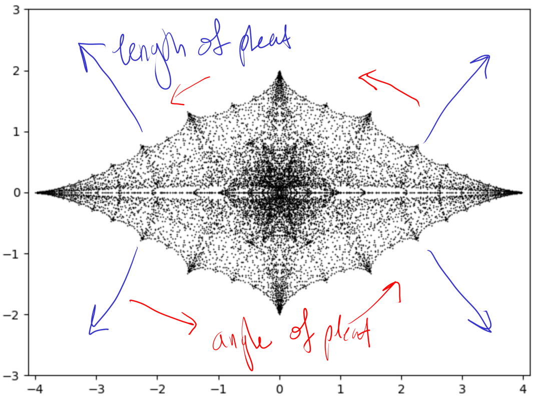

Figure 1.2: The coordinate system for the Riley slice.

case as well) is found as Theorem 7.4.15.

The reader should take some time to consider the (really quite complicated) geometry of the Riley slice: one sees from Figure 1.2 that the boundary curve separating the slice interior (the white unshaded region) from the exterior (the central region shaded with black dots) is very intricate; it is, in fact, fractal. The patterns of dot density which are visible in the shaded exterior are also very subtle and exhibit complicated behaviour, which we still do not fully understand.

Briefly, we explain the word 'geometry' which we have used in relation to our final formulation of the problem. It turns out that if you take a 3-dimensional ball and drill out two arcs, the resulting solid object has a natural way of measuring distance within itself so that the spherical surface and the two deleted arcs are 'infinitely far away' from all the interior points. This geometry is *non-Euclidean*: parallel lines eventually get further and further apart from each other, rather than staying parallel. More precisely, the geometry is *hyperbolic*. There is a way of associating to every such geometry a 'system of symmetries', called a **Kleinian group** (formally, it is the 'holonomy group' of the geometry: the geometry can be flattened out onto the largest-possible hyperbolic space and the group measures the way in which this flattening must take place). It is these Kleinian groups which we parameterise by the Riley slice; and the Riley slice is a so-called *moduli space* for Kleinian groups. For more information on hyperbolic geometry and its relationship with knots, the non-technical reader is directed to the excellent video *Not Knot* [53].

1.2 Technical introduction

A Kleinian group may be equivalently defined as (a) a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, or (b) a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. The relationship between these two definitions comes from the fact that isometries of hyperbolic 3-space are uniquely characterised by their actions on the sphere at infinity: namely, there is a natural bijection between $\mathrm{Isom}^+(\mathbb{H}^3)$ and the group of conformal automorphisms of S^2. After identifying S^2 with the Riemann sphere $\hat{\mathbb{C}}$, we may characterise the conformal automorphisms as none other than the Möbius transformations, those maps of the form $z \mapsto \frac{az+b}{cz+d}$ ($a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$). Performing one final identification, of $\hat{\mathbb{C}}$ with $\mathbb{PC}^1$, we see that the Möbius transformations are in natural correspondence with $\mathrm{PSL}(2,\mathbb{C})$ via the identification

$$\left(z \mapsto \frac{az+b}{cz+d}\right) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The Riley slice $\mathcal{R}$ (the *exterior* of the filled-in fractal 'eye' of Figure 1.2) is a moduli space parameterising the hyperbolic structures on the 3-manifold with conformal boundary consisting of a four-times punctured sphere $S_{0,4}$ with the punctures joined by arcs at infinity. More precisely, define a family $(\Gamma_\rho)_{\rho\in\mathbb{C}\setminus\{0\}}$ of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ by

$$\Gamma_\rho := \left\langle X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, Y_\rho = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} \right\rangle;$$

The assumption $\rho \neq 0$ implies that Γ_ρ is not elementary. The group Γ_ρ acts on the Riemann sphere $\hat{\mathbb{C}}$ and there is a largest open (possibly empty) set $\Omega(\Gamma_\rho) \subset \mathbb{C}$ on which this group acts discontinuously (the **ordinary set**); the complement of this set in $\hat{\mathbb{C}}$ is the **limit set** $\Lambda(\Gamma_\rho)$ and is the closure of the set of fixed points of elements of Γ_ρ (so, since ∞ is fixed by X, $\infty \in \Lambda(\Gamma_\rho)$).[1]

The quotient $\Omega(\Gamma_\rho)/\Gamma_\rho$ is a Riemann surface. When Γ_ρ is free and discrete, the Riemann surfaces so obtained are supported on one of three homeomorphism classes of topological space: the empty set; a disjoint union of two three-times punctured spheres; and a four-times punctured sphere. It happens that the first two types of space may be viewed as geometric deformations of four-times punctured spheres, and so it is natural to consider the set of all ρ such that Γ_ρ is free and discrete and such that the quotient Riemann surface is supported on a four-times punctured sphere; the other two kinds of space then form the boundary of this set (though this observation is highly non-trivial: we study it in Chapter 5).

Thus, the Riley slice is defined by

$$\mathcal{R} = \{\rho \in \mathbb{C} : \Omega(\Gamma_\rho)/\Gamma_\rho \text{ is topologically a four-times punctured sphere}\}.$$

This set has been studied since the mid-1900s; for a non-exhaustive list of literature, see the paragraph following Definition 4.2.2.

The theory of Keen and Series [63] (with corrections by Komori and Series [66]) endows the Riley slice with a foliation structure that measures the geometry of the surface and its underlying hyperbolic 3-manifold. The structure consists of a set of curves parameterised by $\mathbb{Q}$ which radiate out from the boundary of the slice and which are dense in the slice (the so-called **rational pleating rays**) together with a natural completion (in the sense that we may add curves parameterised by $\mathbb{R}\setminus\mathbb{Q}$ in order to fill out the entire slice). These curves are arcs in the deformation space corresponding to pinching a particular simple closed geodesic represented by a loxodromic element (which we call the p/q-**Farey word**[2]) down to a parabolic element, splitting the surface into two pieces joined by a new pair of cusps.

The goal of this thesis is to describe in detail this foliation theory along with a generalisation to the case of the 4-marked sphere (allowing cone points as well as punctures). We will also give some extensions of the theory, for instance by constructing open neighbourhoods of cusp points in the slice; as well as giving some additional combinatorial results.

[1] One may also define the ordinary set in the following way, if Γ_ρ is discrete and non-elementary (true for every group in $\mathcal{R}$): it is the largest domain in $\mathbb{C}$ on which the transformations of Γ_ρ are equicontinuous (Definition 7.4.9). In this way the ordinary set is analogous to the **Fatou set** of a dynamical system.

[2] We name these words after John Farey Sr. as they are closely related to the so-called Farey sequences of rational numbers which we will discuss briefly in Section 6.3 and Chapter 9; with regard to this attribution, we quote from the historical notes to Chapter III of Hardy and Wright [54, pp. 36–37]: "The history of 'Farey series' is very curious... [their properties] seem to have been stated and proved first by Haros in 1802... Farey did not publish anything on the subject until 1816. [...] Mathematicians generally have followed Cauchy's example in attributing the results to Farey, and the series will no doubt continue to bear his name. Farey has a notice of twenty lines in the *Dictionary of national biography* where he is described as a geologist. As a geologist he is forgotten, and his biographer does not mention the one thing in his life which survives."

1.3 Outline of the thesis

This thesis is split, broadly, into three main parts.

The first part consists of Chapters 2 and 3, and is a brief review of specific results which we will use—the vast majority of the definitions and results contained in this part are expected to be known by the reader, and so it is recommended that these chapters be skimmed over and returned to only to check notation and theorem statements. Below in Section 1.4 we list some books which give detailed introductions to all of the mathematics which we describe in these chapters. *These chapters are intended only to fix notation, and the reader is strongly advised to consult the references below for examples and motivation—for instance, the purpose of Section 3.1 is to define a Kleinian group and the related notions as efficiently as posible, at the expense of pedagogical value for readers unfamiliar with the concepts.*

The second part consists of Chapters 4 to 7, and develops the theory of the parabolic and elliptic Riley slices. Chapter 4 may be viewed as a secondary introduction to this part and the majority of the results here are simple extensions of results found in the literature. Chapter 5 gives some analytical results, primarily that the Riley slice, as a subset of $\mathbb{C}$, is biholomorphic to the quasiconformal deformation space of any particular group in the Riley slice; then we give some results on the topology of the Riley slice boundary, though since most of these results are very deep we only give references to proofs. Chapter 6 studies the enumeration of simple closed non-boundary-parallel geodesics, needed for the Riley slice theory. Chapter 7 extends the Keen–Series theory of the parabolic Riley slice developed in [63, 66] to allow the two group generators to be elliptic rather than parabolic. Throughout this part of the thesis, we emphasise intuitive understanding of the results as well as mathematical correctness.

The final part consists of Chapters 8 to 10; these chapters are more significant extensions to the Riley slice theory and the majority of these are due to the author and his two supervisors. Chapter 8 gives a detailed description of some open neighbourhoods of cusp points in the parabolic Riley slice, following our preprint [39]. Chapter 9 describes some properties of the so-called Farey polynomials (the trace polynomials of the Farey words), including a recursion formula in both elliptic and parabolic situations, and some closed-form formulae in special cases. Finally, in Chapter 10 we list some conjectures and open problems.

1.4 Assumed background

The reader is assumed to have prior knowledge of the theory of Kleinian groups and hyperbolic geometry. Unfortunately the amount of background we require is voluminous and so in this section we give a concrete list of references (primarily textbooks) which are sufficient to read this thesis in full detail. It is our hope that this thesis will be of some use to other students who are attempting to understand the theory of deformations of Kleinian groups (as much of this theory is found scattered among a diverse range of different papers and books); we will give copious references and will give some preference to references which are geared towards students rather than researchers for basic results.

Three modern books which will be very helpful to the beginning graduate student are those by Thurston [124], Marden [79], and Purcell [99]; these three books do not always contain sufficient detail for every subject, but give an idea of the landscape. The reader may also be interested in our survey of the background material and the history of the Riley slice written for beginning graduate students and only assuming basic topology and complex analysis [40].

Classical theory of Kleinian groups. We will make significant use of the material in Chapters I to VII of Maskit's textbook [83] (though we will often pause to refresh ourselves on definitions). An alternative elementary reference is the book by Beardon [13] (the early chapters do discuss the 3-dimensional case, though this book deals almost exclusively with the geometry of Fuchsian groups).

Complex analysis and Riemann surfaces. We also assume some basic knowledge of the theory of Riemann surfaces: to give a concrete reference, the book by Farkas and Kra [47] is most relevant in flavour. Note that the proof techniques used for theorems like the Riemann mapping theorem/uniformisation are not so relevant for applying these results, which is what we are primarily interested in. The reader must have a passing understanding of mapping class groups and Teichmüller theory; two good references are the textbooks by Farb and Margalit [46] (more geometric) and by Imayoshi and Taniguchi [56] (more analytic and explicitly dealing with the relationships to Kleinian groups).

We will also need some of the quasiconformal deformation theory of Kleinian groups; this is very seldom found in textbooks, but one self-contained introduction is the monograph by Matsuzaki and Taniguchi [88] (particularly Sections 4.3 and 5.3, and Chapter 7). Since this subject is less well-known than the standard mapping class theory, we will spend some time in this thesis motivating and discussing it. We would also like to make the reader aware of the triplet of papers [91, 120, 121] which actually provide much of the motivation for the study of deformations of Kleinian groups in this way.

Geometric manifolds. We will also need to make use of the modern (that is, post-Thurston) theory of geometric manifolds. The reader should be comfortable with the basic theory of (X, G)-manifolds; these are motivated and then studied in Chapter 3 of the book by Thurston [126], though the reader will find it helpful to have a deeper knowledge of the relationship with Kleinian groups as discussed in Chapters 3, 4, and 8 of Thurston's famous lecture notes [124]. An alternative reference here is the book by Ratcliffe [100] which develops Thurston's theory in much greater depth than the lecture notes and in greater generality, though this book is perhaps not as useful as an initial introduction. Three more books which discuss the theory developed by Thurston and which the author found particularly useful are the textbook by Benedetti and Petronio [14] (which comes at the theory from a Riemannian viewpoint), the monograph by Kapovich [57] (which comes at the theory from geometric group theory), and the monograph [88]. A very nice modern introductory book from the geometric point of view is [79].

Because we are generalising the Keen-Series theory from cusps to cone points, we will also need the theory of orbifolds; in particular, we will need to be able to compute with orbifold fundamental groups and coverings. For the reader unfamiliar with orbifold theory we recommend starting with Chapter 13 of Thurston's notes [124] and Chapter 13 of Ratcliffe [100]: the former studies the fundamental group from the viewpoint of deck transformations, and the latter studies them from the viewpoint of loops. Also useful is Chapter 6 of Kapovich [57], in particular the 'glossary' on pp.148–149. Also useful for us is Chapter III.$\mathcal{G}$ of [24] which covers almost exactly what we need (but unfortunately in a slightly more general setting, that of étale groupoids, which may be easier to grasp if the reader first studies Thurston's treatment of geometric manifolds via pseudogroups in [126]).

Finally, we will need some of the theory of measured laminations and foliations. The classical references are Chapter 8 of Thurston's notes [124] and Thurston's *FLP* [45], as well as the pair of expository papers [28, 43]. See also Section 3.9 of [79].

1.5 Miscellaneous conventions

We will occasionally use a Halmos ∎ at the end of the statement of a proposition without giving an explicit proof. This means one of two things: either the proof should be immediately obvious to the reader (in which case we will always state this), or the proof is to be found in a reference (which will usually be given either immediately prior to or immediately subsequent to the theorem statement).

Often, if $X \subseteq \mathbb{C}$, we write $\hat{X}$ for the set $X \cup \{\infty\} \subseteq \hat{\mathbb{C}} \coloneqq \mathbb{C} \cup \{\infty\}$. In the rational case, when we write '$p/q \in \hat{\mathbb{Q}}$' we implicitly assume that $(p, q) = 1$ (unless of course $p = 0$ or $q = 0$, in which case we require the non-zero integer to be 1).

We write $B(x, \varepsilon)$ for the open ball of radius ε about the point x. It will be clear from context which space this ball is taken with respect to (usually, $\mathbb{C}$ with the Euclidean metric). Similarly, $S(x, \varepsilon)$ is the sphere $\partial B(x, \varepsilon)$. The unit ball in $\mathbb{R}^n$ about 0 is denoted by $\mathbb{B}^n$ (and $\mathbb{B}^2$ is also taken as a subset of $\mathbb{C}$).

For clarity of typography, we sometimes swap between subscript notation and function application notation: that is, sometimes we will write $\gamma_{p/q}$ and sometimes $\gamma(p/q)$, where both typographical entities represent the same mathematical object. Similarly, if we hold some parameters in an object fixed in a section, we often drop these parameters from the notation: if a, b are fixed, then we write $\mathcal{R}$ for $\mathcal{R}^{a,b}$ *et cetera*.

1.6 Papers based on this thesis

Much of the content in this thesis has appeared (or will soon appear) in preprint form, jointly authored with Gaven Martin and Jeroen Schillewaert: chapter 8 has appeared as [39]; parts of chapters 6 and 9, together with some extensions and major updates, have appeared as [41]; and chapters 4 to 7 will soon appear as [42] together with their application to the study of arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{C})$. We have also prepared an elementary introduction to the theory of Kleinian groups and their deformation spaces which gives a historical overview of many of the subjects which we touch upon in this thesis [40].

1.7 Acknowledgements

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Chapter 2

Geometric spaces

The theme of the subject which we are studying is the interplay between group theory and geometry. The geometry will come from manifolds with hyperbolic or complex structure, and in this chapter we formalise what this means and give a quick overview of some of the major theorems of a geometric flavour. Most of the material in this chapter may be found in [100, 126] and the reader is expected to be familiar with it already; we discuss it here only for context and to fix notation and terminology.

2.1 Hyperbolic space

We primarily make use of two models of hyperbolic space: the ball model and the half-space model. The material we discuss in this section is very standard, and may be found (for example) in chapter IV of [83] or the first few chapters of [100].

2.1.1 Definition (The ball model)**.** The **ball model** of hyperbolic space is the open set

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$$

equipped with the Riemann metric

$$ds^2 = \frac{2|x|^2}{(1-|x|^2)^2}.$$

The boundary of the ball model is the sphere S^{n-1}; we refer to it as the **sphere at infinity** S^{n-1}_∞. The union $S^{n-1}_\infty \cup \mathbb{B}^n$ is the **closure** of the ball model, denoted $\overline{\mathbb{B}^n}$.

2.1.2 Proposition. *For arbitrary $x, y \in \mathbb{B}^n$ there exists a unique geodesic joining x to y; this geodesic is a (Euclidean) circle arc or line segment such that the extension of the arc or segment is orthogonal to S^{n-1}.* ∎

2.1.3 Definition (The half-space model)**.** The **half-space model** of hyperbolic space is the open set

$$\mathbb{H}^n = \{(x_1, ..., x_{n-1}, t) \in \mathbb{R}^n : t > 0\}$$

equipped with the Riemann metric

$$ds^2 = \frac{2|x|^2}{(1-|x|^2)^2}.$$

The boundary of the ball model is the space $\hat{\mathbb{R}}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$, where $\mathbb{R}^{n-1}$ is injected into $\mathbb{R}^n$ as the set $\{(x,t) \in \mathbb{R}^n : t = 0\}$. As with the ball model, we refer to it as the **sphere at infinity** S^{n-1}_∞. The union $\hat{\mathbb{R}}^{n-1} \cup \mathbb{H}^n$ is the **closure** of the half-space model, denoted $\overline{\mathbb{H}^n}$.

2.1.4 Proposition. *For arbitrary $x, y \in \mathbb{H}^n$ there exists a unique geodesic joining x to y; this geodesic is a (Euclidean) circle arc or line segment such that the extension of the arc or segment is orthogonal to $\mathbb{R}^{n-1}$.*

The isometry group of $\mathbb{H}^n$ is the group generated by motions of $\mathbb{R}^n$ of the following forms (which clearly all preserve $\mathbb{H}^n$):

1. *Translations: $(z, t) \mapsto (z + a, t)$ for $a \in \mathbb{R}^{n-1}$;*
2. *Rotations: $(z, t) \mapsto (zx, t)$ for $r \in O(n-1)$;*
3. *Dilations: $z \mapsto \lambda z$ for $\lambda \in \mathbb{R}$;*
4. *Inversion in the unit sphere: $z \mapsto z/|z|^2$.*

Isometries of $\mathbb{H}^n$ have natural continuous extensions to the sphere $\hat{\mathbb{R}}^{n-1}$, on which they act as conformal maps. Conversely, every conformal map of $\hat{\mathbb{R}}^{n-1}$ extends uniquely to a hyperbolic isometry on $\mathbb{H}^n$ (the ***Poincaré extension*** *of the conformal map).* ∎

The two models $\mathbb{B}^n$ and $\mathbb{H}^n$ of hyperbolic space are naturally isometric via a sphere inversion: see [83, IV.B.1]. We will not use the hyperboloid model of hyperbolic space in this thesis.

We shall usually be interested in the case that $n = 3$. In this case, the sphere at infinity of the half-space model is $\hat{\mathbb{R}}^2$, which may be identified with the Riemann sphere, $\hat{\mathbb{C}}$. In this case, a short argument gives the following:

2.1.5 Proposition. *Every conformal bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is of the form*

$$z \mapsto \frac{az + b}{cz + d} \tag{2.1.6}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. ∎

Such maps are called **fractional linear transformations**; they preserve the property of "being a circle" (viewing a Euclidean line as a 'circle through ∞'), and the group of such maps, denoted $\mathbb{M}$ (for Möbius), is transitive on triples of points of $\hat{\mathbb{C}}$. One readily observes that there is a natural isomorphism between this group and the group $\mathrm{PSL}(2, \mathbb{C})$ acting on $\hat{\mathbb{C}} = \mathbb{P}\mathbb{C}$, which sends the map of Equation (2.1.6) to the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix};$$

we will often use this isomorphism without comment. We place a norm structure and the associated topology on $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathbb{M}$ by defining

$$\left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

(where the representative matrix is chosen to have determinant 1)—so $\mathrm{PSL}(2, \mathbb{C})$ has the induced topology as a subset of $\mathbb{C}^4$.

2.1.7 Example. Let $f \in \mathbb{M}$ be defined by $f(z) = z + 1$; then f has matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ so $\|f\| = \sqrt{3}$ and $\|f^n\| = \sqrt{2 + n^2}$. This shows that $\mathrm{PSL}(2, \mathbb{C})$ is not compact (it is not bounded in $\mathbb{C}^4$).

The isometries of $\mathbb{H}^3$ may be classified into the following types:

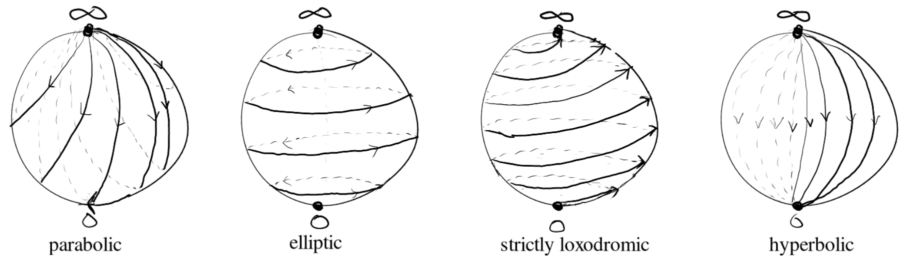


Figure 2.1: The shapes of the orbits of the different types of fractional linear transformation.

1. **loxodromic transformations**: these transformations act as a translation along a geodesic axis together with a possible twist about the axis. If the transformation acts solely as a translation with no rotation, then the transformation is known as a **hyperbolic transformation**; a loxodromic transformation which is not hyperbolic is known as **strictly loxodromic**. The action of a loxodromic transformation on $\hat{\mathbb{C}}$ has two fixed points, and every such transformation is conjugate to one of the form
$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \text{ or in function notation } z \mapsto \lambda^2 z$$
where $\lambda \in \mathbb{C}$ is nonzero and $|\lambda| \neq 1$. A transformation $f \in \mathbb{M}$ is loxodromic iff $\operatorname{tr}^2 f \notin [0,4]$, and is hyperbolic iff $\operatorname{tr}^2 f \in (4, \infty)$.

2. **parabolic transformations**: these transformations act as a translation on $\hat{\mathbb{C}}$ with a single fixed point, and no fixed points in $\mathbb{H}^3$. Every such transformation is conjugate to the transformation
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ or in function notation } z \mapsto z + 1$$
A transformation $f \in \mathbb{M}$ is parabolic iff $\operatorname{tr}^2 f = 4$.

3. **elliptic transformations**: these transformations act as a rotation along a geodesic axis in $\mathbb{H}^3$, and as a rotation in $\hat{\mathbb{C}}$ with two fixed points. Every such transformation is conjugate to one of the form
$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \text{ or in function notation } z \mapsto \lambda^2 z$$
where $\lambda \in \mathbb{C}$ is of unit norm and $\lambda \neq 1$. A transformation $f \in \mathbb{M}$ is elliptic iff $\operatorname{tr}^2 f \in [0, 4)$.

The typical shapes of the orbits of each type of element on $\hat{\mathbb{C}}$ are depicted in Figure 2.1.

2.2 Geometric manifolds and orbifolds

In this section we briefly recall the definitions of geometric manifolds and orbifolds in order to refresh the memory of the reader, following [57, 100, 124, 126].

2.2.1 Definition. Let X be a metric space; a **geodesic segment** in X is a distance-preserving map $\gamma : [a, b] \to X$ (for some reals $a < b$). As usual we often identify γ with its image $\gamma([0, 1])$ and say that γ joins $\gamma(0)$ to $\gamma(1)$. A **geodesic line** in X is a continuous map $\lambda : \mathbb{R} \to X$ which is locally distance-preserving. (Thus restrictions of geodesic lines to segments in $\mathbb{R}$ are not necessarily geodesic segments unless they are sufficiently short.)

The space X is a **geometric space** of dimension n if it satisfies suitable generalisations of the Euclidean axioms. The following definition states Euclid's axioms (which we take verbatim from [33, section 1.2]) in italics together with the appropriate generalisation (described, for instance, in chapter 8 of [100]).

1. *A straight line may be drawn from any point to any other point.* For any pair $x, y \in X$ there exists a geodesic segment joining x to y.

2. *A finite straight line may be extended continuously in a straight line.* Given a geodesic segment $\gamma : [0,1] \to X$ there exists a unique geodesic line $\hat{\gamma} : \mathbb{R} \to X$ with $\hat{\gamma}\restriction_{[0,1]} = \gamma$.

3. *A circle may be described with any centre and any radius.* There exists a continuous function $\sigma : \mathbb{R}^n \to X$ and a real number $\varepsilon > 0$ such that σ is a homeomorphism of $B(0,\varepsilon)$ onto $B(\sigma(0),\varepsilon)$, such that for any $x \in S^{n-1}$ the map $\gamma : \mathbb{R} \to X$ defined by $\gamma(t) = \sigma(tx)$ is a geodesic line such that $\gamma\restriction_{[-\varepsilon,\varepsilon]}$ is a geodesic segment.

4. *All right angles are equal to each other.* X is homogeneous (i.e. for all $x, y \in X$ there exist isometric neighbourhoods U of x and V of y).

A bijection $\phi : X \to Y$ between two metric spaces is a **similarity** of scale factor $\kappa \neq 0$ if, for all $x_1, x_2 \in X$, $d_Y(\phi(x_1), \phi(x_2)) = \kappa d_X(x_1, x_2)$.

2.2.2 Definition. Let G be a group of similarities of a geometric space X, and let M be a topological manifold. An (X,G)-**atlas** on M is an open cover $\{U_\alpha\}_{\alpha\in A}$ of M together with a family of topological embeddings $(\phi_\alpha : U_\alpha \to X)_{\alpha\in A}$ such that, for every $\alpha, \beta \in A$, the composition $\phi_\alpha\phi_\beta^{-1}$ (defined on $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta)$) is a restriction of some element of G; if M admits such an atlas then it is called an (X,G)-**manifold**, or a **geometric manifold modelled on** X.

The morphisms of this category are the (X,G)-**maps**: if M, N are (X,G)-manifolds, such a map is a map $f : M \to N$ such that for every pair of charts ϕ on M and ψ on N, the composition $\psi f \phi^{-1}$ is a restriction of an element of G on its domain of definition. We say that M and N are (X,G)-**equivalent** if there is an invertible (X,G)-map $f : M \to N$ such that f^{-1} is also an (X,G)-map.

We will also be interested in allowing quotient singularities on manifolds. The correct notion was defined by Satake in 1957 [111] and was reintroduced by Thurston in the 1970s (see Chapter 13 of [124]). We follow the treatment of Chapter III.$\mathcal{G}$ of [24].

2.2.3 Definition. Let G be a group of similarities of some geometric space X. An (X,G)-**orbifold** O is a Hausdorff topological space $|O|$ together with the following data (known as an **orbifold atlas**):

1. An open cover $\{U_\alpha\}_{\alpha\in A}$ of $|O|$; and

2. For each $\alpha \in A$: a connected and simply connected (X,G)-manifold X_α, a finite group Γ_α of diffeomorphisms of X_α, and a continuous map $p_\alpha : X_\alpha \to U_\alpha$ inducing a homeomorphism $X_\alpha/\Gamma_\alpha \to U_\alpha$

satisfing the condition that, if $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$ have the property that $p_\alpha(x_\alpha) = p_\beta(x_\beta)$, then there are connected open neighbourhoods V_α of x_α and V_β of x_β together with a (X,G)-equivalence $h : V_\alpha \to V_\beta$ such that $p_\beta h = p_\alpha\restriction_{V_\alpha}$.

A point x of O is said to be **singular** if there exists some chart p_α containing X such that the stabiliser $\mathrm{Stab}_{\Gamma_\alpha}(x)$ is nontrivial. The set of singular points is the **singular locus**, denoted $\operatorname{sing} O$.

Many clear geometric examples may be found in [124, Chapter 13], [100, Chapter 13], and [57, Chapter 6].

We will need to know that the Euler characteristic and the Gauss-Bonnet theorem hold for orbifolds. This is studied in Proposition 13.3.4 of [124] and the surrounding discussion. We recall the main points.

2.2.4 Definition. The **Euler characteristic** of a 2-orbifold O is defined by

$$\chi(O) := \chi(|O|) - \frac{1}{2}\sum(1 - 1/n_i) - \sum(1 - 1/m_i)$$

where the n_i are the orders of the corner reflectors of O and the m_i are the orders of the cone points.

2.2.5 Proposition (Covering formula for orbifold Euler characteristic, [124, Proposition 13.3.4])**.** *If $\tilde{O} \to O$ is an orbifold covering map of degree k (that is, the preimage of a nonsingular point is of cardinality k) then $\chi(\tilde{O}) = k\chi(O)$.* ∎

2.2.6 Theorem (Gauss-Bonnet for orbifolds, [124, Section 13.20])**.** *If an orbifold O is equipped with a metric coming from invariant Riemann metrics on each chart X_α then*

$$\int_O K dA = 2\pi\chi(O)$$

where K is the curvature. ∎

As an easy corollary of the Gauss-Bonnet theorem, for a hyperbolic orbifold we obtain the following area computation:

2.2.7 Corollary (Siegel area formula)**.** *Let S be a hyperbolic Riemann surface of genus g with n marked points of order $a_1, \ldots, a_n$ (punctures being marked with order ∞). Then the hyperbolic area of S is*

$$\mathrm{Area}(S) = 2\pi\left(2g - 2 + \sum_{i=1}^{n}\left(1 - \frac{1}{a_i}\right)\right).$$ ∎

2.3 The Poincaré polyhedron theorem

In this section we introduce one of our primary tools which relates the geometry and topology of geometric manifolds to combinatorial group theory: this tool is the Poincaré polyhedron theorem (Theorem 2.3.5 below), which is a result guaranteeing that sufficiently regular face gluing structures on polyhedra do induce hyperbolic quotient structures.

Notation. In this section, we restrict the geometric spaces of interest to be $\mathbb{R}^n$ and $\mathbb{H}^n$. Let X denote a fixed geometric space which is either one of these two.

2.3.1 Definition. A subset $K \subseteq X$ is said to be **convex** if, for every pair $x, y \in K$, the geodesic arc $[x, y]$ lies in K.

The theory of convex sets from $\mathbb{R}^n$ (for instance, as described in [44]) carries over almost without change to $\mathbb{H}^n$. We describe the relevant structures in the ball model.

A **hyperbolic hyperplane** in $\mathbb{B}^n$ is the intersection with $\mathbb{B}^n$ of a sphere in $\mathbb{R}^n$ orthogonal to S^{n-1}_∞. A **hyperbolic halfspace** is a component of the complement of a hyperbolic hyperplane in $\mathbb{B}^n$ (we take the convention that $\varnothing$ and $\mathbb{B}^n$ are hyperplanes); a **hyperbolic polyhedron** is a nonempty intersection of a countable family of hyperbolic halfspaces. We will usually drop the qualifier 'hyperbolic' when it is clear from context that we consider hyperbolic objects rather than classical Euclidean objects.

Let P be a polyhedron; a **supporting hyperplane** of P is a hyperplane H such that $\overline{P} \cap H$ is nonempty and such that P lies entirely in one of the halfspaces determined by P. A **face** of P is an interesection $\overline{P} \cap H$ where H is a supporting hyperplane; each face f has a well-defined dimension, namely the largest $k \in \mathbb{Z}_{0 \le n-1}$ such that there exists a sphere S of dimension k orthogonal to S^{n-1}_∞ with the property that f is exactly $\overline{P} \cap S$. The **relative interior** of the face f, denoted $\operatorname{relint} f$, is the interior of f as a subset of the sphere S. By convention we take the empty set to be a face of every polyhedron (of dimension -1) and the closure $\overline{P}$ itself to be a face (of dimension n). A face of dimension $n-1$ is a **facet** or a **side**; a face of dimension $n-2$ is a **ridge**; a face of dimension 1 is an **edge**; a face of dimension 0 is a **vertex**. We write $P(k)$ for the set of k-dimensional faces of P.

Given a polyhedron P, it is possible for the defining hyperplanes to intersect on the sphere S^{n-1}_∞; faces which are subsets of the sphere at infinity are called **ideal faces**. We also allow ourselves to take convex hulls of points on the boundary, and other such things. (As a simple example, take two diametrically opposite points on $\partial\mathbb{B}^2$; then the convex hull of these points is the diameter joining them.)

2.3.2 Definition. Let $\mathcal{P}$ be a family of finitely many disjoint polyhedra in X. For convenience, we say 'a facet of $\mathcal{P}$' to mean 'a facet of some polyhedron in $\mathcal{P}$'. (Of course this can easily be made precise: take the disjoint union of the relevant face complexes, this itself is a face complex and so everything is well-defined.) A **facet-pairing structure** Φ on $\mathcal{P}$ consists of the following data:

- A map $(\cdot)' : \mathcal{P}(n-1) \to \mathcal{P}(n-1)$; and
- For each facet F a similarity $\phi_F : X \to X$ (called a **facet-pairing transformation**)

such that for every facet $F \in \mathcal{P}$, (i) $(F')' = F$, (ii) $\phi_F(F) = F'$, and (iii) $\phi_F^{-1} = \phi_{F'}$.

We say that two points $x, y \in \overline{\mathcal{P}}$ are **tiled adjacently** by Φ if there exists a facet F of $\mathcal{P}$ such that $x \in F$, $y \in F'$, and $y = \phi_F(x)$; in this case, we write $x \simeq y$. Observe that $\simeq$ is a symmetric relation. We extend it to an equivalence relation in the following way: if $x, y \in \overline{\mathcal{P}}$, we say that x and y are **tiled** by Φ and write $x \sim y$ if either $x = y$ or there is a finite sequence $x_1, ..., x_m$ of points of $\overline{\mathcal{P}}$ such that

$$x = x_1 \simeq \cdots \simeq x_m = y.$$

An equivalence class of related points is called a **cycle** of Φ; the cycle containing $x \in \mathcal{P}$ is denoted $[x]$.

In order to understand the meaning of these definitions geometrically, consider the group Γ generated by the face-pairing transformations of Φ, and suppose that the images of $\mathcal{P}$ under Γ tessellate X. Two points of the boundary $\partial\mathcal{P}$ are tiled by Φ if they are glued onto each other at a tile boundary of the tessellation, and they are *adjacently* tiled if, at the place where they are glued together, the two copies of the tile are glued there facet-to-facet.

Remark. The adjectives 'tiled' and 'tiled adjacently' used here are not standard terminology; Maskit does not introduce specific names for these relations (see paragraph IV.F.5 of [83]), and Ratcliffe uses **paired** for 'tiled adjacently' and **related** for 'tiled' (see section 6.8 of [100]).

Suppose $x \in \operatorname{relint} e$ for some ridge e of $\mathcal{P}$; then every point of $[x]$ lies in the relative interior of some ridge of $\mathcal{P}$, and we call $[x]$ a **ridge cycle** of Φ. Let $[x] = \{x_1, ..., x_m\}$ be a *finite* ridge cycle of Φ. For each i, the element x_i is paired to at most two other elements of $[x]$ by Φ (since each ridge is a subset of exactly two facets of $\mathcal{P}$) and so we can reindex $[x]$ such that $x_1 \simeq x_2 \simeq \cdots \simeq x_m$. Such a cycle is said to be **dihedral** if there is a facet F of $\mathcal{P}$ containing x_1 such that $F = F'$ and $\phi_F(x_1) = x_1$.

A ridge cycle which is not dihedral is called **cyclic**. In either case, we may define the **dihedral angle sum** of $[x]$ to be

$$\theta[x] \coloneqq \theta(x_1) + \cdots + \theta(x_m)$$

where, for each i, $\theta(x_i)$ is the dihedral angle between the two facets of $\mathcal{P}$ which meet along the ridge x_i.

There are some easy-to-see local (at ridges) necessity conditions for facet-pairings to produce tilings:

2.3.3 Definition. A facet-pairing transformation Φ for an polyhedron $\mathcal{P}$ in X is said to be **subproper** if

- each cycle of Φ is finite,
- each dihedral edge cycle of Φ has dihedral angle sum a submultiple of π, and
- each cyclic edge cycle of Φ has dihedral angle sum a submultiple of 2π.

It turns out that these conditions are also sufficient. For a proof of the following theorem, see [100, Theorem 13.4.2].

2.3.4 Theorem. *Let X be $\mathbb{R}^n$ or $\mathbb{H}^n$, let G be a discrete group of similarities of X, and let M be the space obtained by gluing a family of X-polyhedra $\mathcal{P}$ according to a subproper facet-pairing structure Φ: that is, M is the space $\mathcal{P}/\sim$ of cycles endowed with the quotient topology. Then M is an (X, G)-orbifold such that the natural injection $\mathcal{P} \hookrightarrow M$ is an (X, G)-map.* ∎

The content of the Poincaré polyhedron theorem is that, if one is given a family of polyhedra $\mathcal{P}$ together with a facet pairing structure Φ which has sufficient regularity properties, then the group $\Gamma = \langle\Phi\rangle$ has a presentation determined exactly by the combinatorial properties of Φ and $\Gamma\mathcal{P}$ tiles the quotient space X/Γ. Before giving the statement of the theorem, we explicitly state the relations which are sufficient to determine the group:

- For every facet F, the **side-pairing relation** for F is $\Phi_F\Phi_{F'} = 1$;
- For every facet F, and every face $G \preceq F$, define sequences $(F_i)_{i\in\mathbb{N}}$ and $(G_i)_{i\in\mathbb{N}}$ of faces (with $G_i \preceq F_i$ for each i) and inductively by setting $F_1 \coloneqq F$ and $G_1 \coloneqq G$ and then defining F_{i+1} to be the face of $\mathcal{P}$ adjacent to F_i' such that $\phi_{F_i}(F_i' \cap F_{i+1}) = G_i$, and G_{i+1} to be the side of F_{i+1} given by $F_i' \cap F_{i+1}$. The sequences are periodic (Theorem 6.8.7 of [100]), say (F_i) is of period k; then the **cycle relation** for $G \preceq F$ is $g_{F_1} \cdots g_{F_k} = 1$.

2.3.5 Theorem (Poincaré (1883)). *Let Φ be a subproper facet pairing for a polyhedron $\mathcal{P}$ in X, such that the glued orbifold M is complete.*[1] *Then:*

1. $\Gamma \coloneqq \langle\Phi\rangle$ *is a discrete group of X-similarities with $M = X/\Gamma$;*
2. $\Gamma\mathcal{P}$ *tiles the quotient space X/Γ, in the sense that $\mathcal{P}$ satisfies the following:*

   FP1. For every nontrivial $\gamma \in \Gamma$, $\gamma\mathcal{P} \cap \mathcal{P} = \emptyset$;

   FP2. For every $x \in X$, there exists some $\gamma \in \Gamma$ with $\gamma(x) \in \overline{\mathcal{P}}$;

   FP3. Any compact subset of X meets only finitely many translates of X.
3. $\mathcal{P}$ *is exact, that is for each facet $S \in \mathcal{P}(n-1)$ there exists some $\gamma \in \Gamma$ such that $S = \mathcal{P} \cap \gamma\mathcal{P}$;*

[1] See Theorem 13.3.7 of [100]: in all cases of interest to us, the orbifolds will have a hyperbolic metric and so we may take ‘complete’ to mean ‘complete as a metric space’.

4. *If R is the set of words in the symbols $\mathcal{P}(n-1)$ corresponding to all of the side-pairing and cycle relations of Φ, then $\langle \mathcal{P}(n-1) : R \rangle$ is a presentation for Γ under the isomorphism $\mathcal{P}(n-1) \ni S \mapsto f_S \in \Phi$.* ∎

2.3.6 Definition. If Γ is a discrete group of isometries of X, then an X-polyhedron $\mathcal{P}$ is said to be a **fundamental polyhedron** for Γ if it satisfies FP1 – FP3 above, together with

FP4. $\mathcal{P}$ admits a facet-pairing structure such that the facet-pairing transformations are elements of the group Γ.

If Γ admits such a fundamental polyhedron with finitely many sides, then Γ is called **geometrically finite**.

In order to apply this theorem, we need a criterion for completeness of hyperbolic orbifolds. Let $\mathcal{P} \subseteq \mathbb{B}^3$ be a polyhedron; a **cusp point** of $\mathcal{P}$ is a point $c \in \overline{\mathcal{P}} \cap S^2_\infty$ which has a neighbourhood U in $\mathbb{R}^3$ such that the intersection of the closures in $\overline{B^3}$ of all the facets of $\mathcal{P}$ which meet U is $\{c\}$. (Compare the discussion below in Section 3.3.)

Suppose c is such a cusp point, and let $b \in [c]$. The **link** of b is the Euclidean polygon $L(b)$ obtained by intersecting $\mathcal{P}$ with a horosphere Σ_b based at b which meets only the sides of $\mathcal{P}$ incident with b. It is easy to see that we may choose the horospheres Σ_b to be sufficiently small that the $L(b)$ are mutually disjoint (suppose not; then there must be a sequence (b_n) of points of $[c]$ such that $b_n \to c$; in particular, some subsequence of the b_n must lie on an edge incident with c; and the two facets of $\mathcal{P}$ incident with that edge intersect at infinitely many points in any neighbourhood in the sense above of c). We now show that if Φ is a facet-pairing for $\mathcal{P}$, then Φ induces a set Ψ of Euclidean similarities which acts as a side-pairing for the disjoint union of the set of polygons $\{L(b) : b \in [c]\}$ after they have been embedded into $\mathbb{R}^2$. Suppose e is an edge of $L(b)$; we define the side-pairing transformation g_e. The edge e lies in some facet S of $\mathcal{P}$; now take $f_S(e)$, this lies on some facet $S' = f_S(S)$ incident with $b' = f_S(b) \in [c]$; and take g_e to be the Euclidean similarity in $\mathbb{R}^2$ which sends e to the edge corresponding to the radial projection of $f_S(e)$ onto the horosphere $\Sigma_{b'}$. Define $L[c]$ to be the space obtained by taking the quotient of $\{L(b) : b \in [c]\}$ according to Ψ; this space is called the **link space of the cusp point** $[c]$. By Theorem 2.3.4, the link $L[c]$ is a connected $(\mathbb{R}^2, S(\mathbb{R}^2))$-orbifold.

The following theorem is proved as Theorem 13.4.7 of Ratcliffe [100].

2.3.7 Theorem. *With the above notation, the link $L[c]$ for a cusp point $[c]$ of $\mathcal{P}$ is complete iff each $L(b)$ for $b \in [c]$ can be chosen such that Φ restricts to a side-pairing for $\{L(b) : b \in [c]\}$ (i.e. if the radial projections in the definition are trivial). The orbifold M obtained by gluing $\mathcal{P}$ is complete iff $L[c]$ is complete for each cusp point $[c]$ of $\mathcal{P}$.* ∎

2.4 The topology of 3-manifolds

As well as the geometric theory above, we need some topological theory of 3-manifolds. This theory, which predates Thurston's study of 3-manifolds via the geometric structures which they accept, is conceptually quite similar to the classical topological theory of surfaces.

The main theorem which we need is the 'loop theorem', Theorem 4.2 of [55]:

2.4.1 Theorem (The loop theorem). *Let M be a 3-manifold and let $S \subseteq \partial M$ be a connected 2-manifold. If $N \lhd \pi_1(S)$ and if $\ker(\pi_1(S) \to \pi_1(M))$ is not contained wholly within N, then there is a proper embedding $g : (B, \partial B) \to (M, S)$ (B the usual unit disc) such that the image of ∂B under g (which is a closed curve in S so may be identified with an element of $\pi_1(S)$) does not lie in N.* ∎

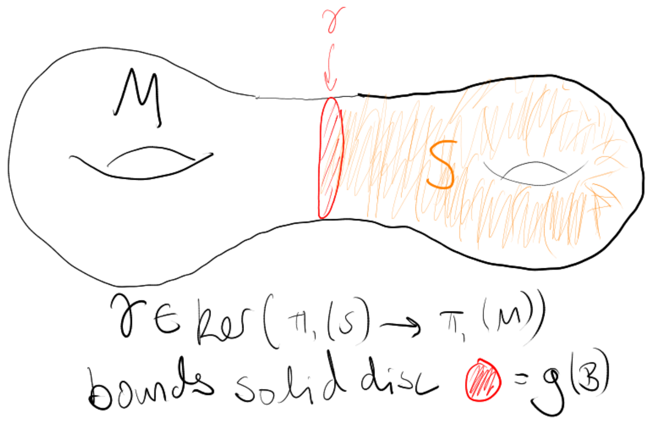


Figure 2.2: A surface S on the boundary of a 3-manifold which bounds an embedded disc, as in Theorem 2.4.1.

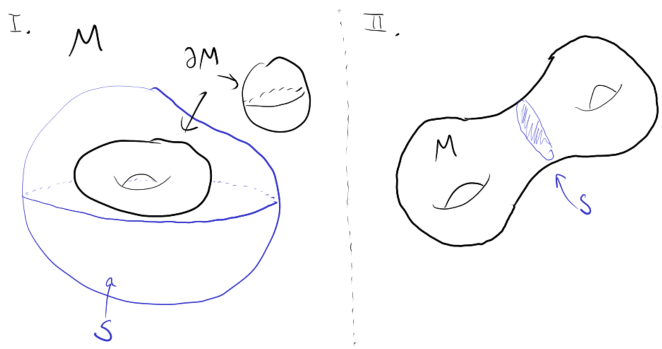


Figure 2.3: Two examples of incompressible surfaces in 3-manifolds.

In Figure 2.2 we see an example of the theorem where $N = 1$.

It is useful to rephrase this in terms of so-called *compressible surfaces*; for more context, see for example p.63ff. of [88].

2.4.2 Definition. Let M be a 3-manifold and let S be a surface such that either $S \subseteq \partial M$ or $S \cap \partial M = \partial S$. If S satisfies any of the following conditions, then it is **incompressible**; otherwise, we say it is **compressible**.

1. S is a smooth topological sphere which bounds no balls (in the sense that $M \setminus S$ has two components, neither of which is a topological ball);

2. S is a topological disc with boundary ∂S a homotopically nontrivial simple closed curve in ∂M;

3. S is a surface other than a sphere or a disc such that $\ker(\pi_1(S) \to \pi_1(M)) = 1$.

(See Figure 2.3 for examples of (1) and (2). The surface S of Figure 2.2 is compressible, because of the loop γ.)

2.4.3 Theorem (Dehn's lemma)**.** *Let M be a 3-manifold and let S be a compressible surface with nontrivial fundamental group such that either $S \subseteq \partial M$ or S is properly embedded and 2-sided in M. Then there exists an embedded disc B in M such that $B \cap S = \partial B$ and ∂B is a homotopically non-trivial closed curve in S. Moreover, if a simple closed curve $\gamma \subseteq S$ is trivial in M, then D may be chosen such that $\partial D = \gamma$.*

Historical remark. Dehn gave a faulty proof of this lemma in 1910 [36]; the first correct proof was due to Papakyriakopoulos [97], who also provided the first proof of the loop theorem [98]. For further references, see the discussion in Chapter 4 of [55].

Proof of Theorem 2.4.3. We will apply Theorem 2.4.1. Suppose first that $S \subseteq \partial M$ and let $N = 1$. We must show that $\ker(\pi_1(S) \to \pi_1(M))$ is nontrivial. Suppose that the kernel is trivial; then since option (3) of Definition 2.4.2 does not hold, S is either a disc or a sphere. If S is a topological sphere, then it must bound a ball in M. Thus $\ker(\pi_1(S) \to \pi_1(M)) = \pi_1(S)$ (since everything contracts through the interior of the ball), and by assumption $\pi_1(S)$ is nontrivial; this contradicts that the kernel is trivial. Similarly, if S is a disc then its boundary must be homotopically trivial in ∂M; but if the boundary of the disc contracts to a point then any curve on the interior also contracts to a point and so again we have $\ker(\pi_1(S) \to \pi_1(M)) = \pi_1(S)$ which is a contradiction.

For the case that S is properly embedded, first use that S is two-sided to cut the manifold along S (see Chapter 2 of [55]); now S remains compressible in one of the two halves and is contained in the boundary of that half which reduces the problem to the first case. ∎

Similar definitions and results hold for orbifolds. Following the glossary given in Section 6.4 of [57], we modify Definition 2.4.2 as follows:

2.4.4 Definition. Let O be a 3-orbifold and let S be a 2-orbifold such that either $S \subseteq \partial O$ or $S \cap \partial O = \partial S$. If S satisfies any of the following conditions, then it is **incompressible**; otherwise, we say it is **compressible**.

1. S is diffeomorphic to a 2-orbifold covered by S^2 such that $O \setminus S$ has two components, neither of which is of the form $\mathbb{B}^3/\Gamma$ for some finite group $\Gamma \leq O(3)$;
2. S is diffeomorphic to $\mathbb{B}^2/\Gamma$ (where Γ is a finite subgroup of $O(2)$) with boundary ∂S a homotopically nontrivial simple closed curve in ∂O;
3. S is a surface other than case (1) or (2) such that $\ker(\pi_1(S) \to \pi_1(O)) = 1$.

Chapter 3

The geometry of Kleinian groups

In this chapter we recall the definition of a Kleinian group and then recall various standard results which we refer to in the main body of the thesis. Most of the material in this chapter may be found in [83, 100] and the reader is expected to be familiar with it with the exception of the final two sections on the Teichmüller theory of Kleinian groups, as this material is a little less elementary.

3.1 Kleinian groups

A central idea in algebraic topology is the notion of a covering space: instead of studying a space X, one studies a simpler space $\hat{X}$ together with a projection map $\hat{X} \to X$ such that inverse images of objects in X behave in some predictable way in $\hat{X}$. In studying hyperbolic manifolds, we are interested primarily in covering spaces which are also hyperbolic. It will turn out that we can always find such a covering, and that this covering in fact exhibits X as a quotient of $\mathbb{H}^3$ by some group of hyperbolic isometries.

Let X be a topological space, and let $p : \hat{X} \to X$ be a covering of X (we make the standing assumption that, whenever we have a covering, the upper space is connected and locally path connected). Recall that a **deck transformation** of p is a homeomorphism $f : \hat{X} \to \hat{X}$ such that $pf = p$; the set of deck transformations forms a group under composition, which we denote $\operatorname{Aut} p$. We say that p is **regular** if the action of $\operatorname{Aut} p$ on each fibre of p is transitive. This is equivalent to asking that the group $p_*\pi_1(\hat{X}, \hat{x})$ is normal in $\pi_1(X, p(\hat{x}))$ for each $\hat{x} \in \hat{X}$. The reader should now remember that the following holds [22, Corollary III.6.9]:

3.1.1 Proposition. *If $p : \hat{X} \to X$ is a regular covering map, with $\hat{x} \in \hat{X}$ and $x = p(\hat{x})$, then* $\operatorname{Aut}(p) \simeq \pi_1(X, x)/p_*\pi_1(\hat{X}, \hat{x})$. ∎

Suppose now that our covering comes from a group action. The notion of interest turns out to be the following:

3.1.2 Definition. Let X be a topological space, and let G be a group with an action as a group of homeomorphisms on X. The group action is said to be **discontinuous** at a point $x \in X$ if there exists a neighbourhood U of x such that $gU \cap U \neq \emptyset$ for only finitely many $g \in G$. If a neighbourhood of some x can be chosen such that the only $g \in G$ with $gU \cap U \neq \emptyset$ is the identity transformation, then the action is said to be **freely discontinuous** at x.

Notation. If G acts on X as in the previous definition, then the set of all $x \in X$ at which G acts discontinuously is called the **regular set** of the action and is denoted by $\Omega(G)$. The set of all $x \in X$

at which G acts freely discontinuously is called the **free regular set** of the action and is denoted by $^\circ\Omega(G)$. If we which to emphasise the space X, we will write $\Omega(G,X)$ or $^\circ\Omega(G,X)$.

3.1.3 Proposition. *If a group G acts freely discontinuously on a path connected and locally path connected Hausdorff space X, then $p : X \to X/G$ is a regular covering map such that* $\mathrm{Aut}(p) = G$. ∎

We now state precisely the existence of a hyperbolic covering space for hyperbolic manifolds (a slightly more general statement is found as theorem 8.5.9 of [100]).

3.1.4 Proposition. *Let Γ be a group of isometries of $\mathbb{H}^n$, and let M be a complete connected $(\mathbb{H}^n,\Gamma)$-manifold. Then M is $(\mathbb{H}^n,\Gamma)$-equivalent to a manifold of the form $\mathbb{H}^n/G$, for G a discrete group of hyperbolic isometries.* ∎

Let us move to the case of orbifolds; really, the point is to replace 'freely discontinuously' with 'discontinuously' in Proposition 3.1.3. The reader will recall that, given an orbifold O, we may define an **orbifold fundamental group** in two equivalent ways: first, as the group of loops on O modulo homotopies compatible with the local group quotients (for a precise definition, see Section 13.3 of [100] or Section III.$\mathcal{G}$.3 of [24]); and second, as the group of deck transformations of the universal orbifold cover (see Section 13.2 of [124]).

Notation. If M is a manifold, then M is naturally an orbifold (with all the groups Γ_α of Definition 2.2.3 trivial) and the orbifold fundamental group of M is equal to the classical fundamental group. In the remainder of this thesis, every topological space which appears will be a manifold or an orbifold, and we make the convention that $\pi_1(X)$ always denotes the orbifold fundamental group of the space X.

In any case, we have the following pair of results:

3.1.5 Proposition (Analogue of Proposition 3.1.1)**.** *If $p : \hat{O} \to O$ is a regular covering map, with $\hat{o} \in \hat{O}$ and $o = p(\hat{o})$, then* $\mathrm{Aut}(p) \simeq \pi_1(O,o)/p_*\pi_1(\hat{O},\hat{o})$. ∎

3.1.6 Proposition (Analogue of Proposition 3.1.3)**.** *If a group G acts discontinuously on a path connected and locally path connected Hausdorff space X, then $p : X \to X/G$ is a regular covering map* of orbifolds *such that* $\mathrm{Aut}(p) = G$. ∎

These two propositions are proved exactly analogously to the standard topological ones: all the necessary machinery is developed on pp.611–612 of [24] for the adaptation of the proofs of Propositions 3.1.1 and 3.1.3 in [22] cited above. The combination of the two,

$$G \simeq \frac{\pi_1(X/\Gamma, p(x_0))}{p_*\pi_1(X,x_0)},$$

may also be found in [100] as Exercise 13.3.2. We also get a uniformisation result, Theorem 13.3.10 of [100]:

3.1.7 Proposition (Analogue of Proposition 3.1.4)**.** *Let Γ be a group of isometries of $\mathbb{H}^n$, and let O be a complete connected $(\mathbb{H}^n,\Gamma)$-orbifold. Then O is $(\mathbb{H}^n,\Gamma)$-equivalent to a manifold of the form $\mathbb{H}^n/G$, for G a discrete group of hyperbolic isometries (in fact, G can be taken to be naturally isomorphic to $\pi_1(O)$).* ∎

In the three-dimensional case, we make the following definition:

3.1.8 Definition. A **Kleinian group** is a discrete group of isometries of $\mathbb{H}^3$ (equivalently, a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$; or a discrete group of conformal automorphisms of $\hat{\mathbb{C}}$).

Manu of the results on Kleinian groups which we will need have the hypothesis that the groups are not **elementary** i.e. that they do not contain a finite-index abelian subgroup. Elementary Kleinian groups are completely classified and have very simple geometry (see chapter 5 of [13], chapter V of [83], or section 5.5 of [100]).

A useful inequality for proving that a group is not discrete is that of Jørgensen:

3.1.9 Theorem (Jørgensen). *If $A, B \in \mathrm{PSL}(2,\mathbb{C})$ generate a non-elementary discrete group, then*

$$\left|\operatorname{tr}^2 A - 4\right| + |\operatorname{tr}[A,B] - 2| \geq 1 \tag{3.1.10}$$

(where here and elsewhere the notation $[A,B]$ denotes the commutator $ABA^{-1}B^{-1}$). ∎

For a proof of this inequality, see [13, Theorem 5.4.1]. As a consequence, we obtain the following in the parabolic case:

3.1.11 Corollary (Shimizu-Leutbecher lemma). *Suppose $A, B \in \mathrm{PSL}(2,\mathbb{C})$ are of the form*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $\langle A, B\rangle$ is discrete, then either $c = 0$ or $|c| \geq 1$.

Proof. One can check that $\langle A, B\rangle$ is elementary iff $c = 0$. Explicit computation with Equation (3.1.10) shows that, if $c \neq 0$ and $\langle A, B\rangle$ is discrete, then $|c| \geq 1$. ∎

Remark. A hands-on proof without using the high-powered machinery of Jørgensen's inequality may be found as Proposition II.C.5 of [83].

If G is a Lie group acting transitively on a manifold M with compact stabilisers, then any discrete $\Gamma \leq G$ acts discontinuously on M (see, e.g. corollary 3.5.11 of [126]). One can show (theorem 4.2.2 of [13]) that if $\mathrm{PSL}(2,\mathbb{C})$ acts as the group of hyperbolic isometries of $\mathbb{H}^3$ then each point stabiliser is a conjugate of $\mathrm{SU}(2,\mathbb{C})$; and $\mathrm{SU}(2,\mathbb{C})$ is compact in $\mathrm{PSL}(2,\mathbb{C})$. Thus, in the cases of interest, we always have that the group Γ of Proposition 3.1.4 acts discontinuously on $\mathbb{H}^3$; in addition, by the classification of hyperbolic isometries, the set $\Omega(\mathbb{H}^3,\Gamma)\setminus{}^\circ\Omega(\mathbb{H}^3,\Gamma)$ consists only of elliptic fixed points, so Proposition 3.1.3 holds in this case if Γ does not contain any elliptic elements: the quotient map $\mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ is regular and has deck transformation group Γ. On the other hand, if Γ includes rotations around some axis in $\mathbb{H}^3$ then the quotient $\mathbb{H}^3/\Gamma$ is an orbifold with quotient singularities.

Consider next a Kleinian group G acting on $\hat{\mathbb{C}}$. The stabilisers of this group action are not necessarily compact; we give a simple example.

3.1.12 Example. Consider the subgroup of $\mathbb{M}$ generated by the single element $f : z \mapsto z + 1$. This is clearly discrete. On the other hand, it is not compact (by Example 2.1.7).

Motivated by this, we recall some of the most fundamental results on the dynamics of Kleinian groups. These results may be found in various guises in chapter 12 of [100], chapter 6 of [13], and chapter II of [83].

3.1.13 Definition. The **limit set** of G, $\Lambda(G)$, is the set of accumulation points of the orbits of G on $\overline{\mathbb{H}^3}$; that is, $\Lambda(G)$ is the set of all $x \in \overline{\mathbb{H}^3}$ such that there exists a point $x_0 \in \overline{\mathbb{H}^3}$ and a sequence (γ_n) of distinct elements of G such that $x = \lim_{n\to\infty} \gamma_n x_0$.

3.1.14 Example. In Figure 3.1, we show four limit sets with pleasant appearance. The groups from which they are generated are:

- Figure 3.1a: $\left\langle \begin{bmatrix} \exp(2\pi i/3) & 1 \\ 0 & \exp(-2\pi i/3) \end{bmatrix}, \begin{bmatrix} \exp(2\pi i/7) & 0 \\ 5 & \exp(-2\pi i/7) \end{bmatrix} \right\rangle$
- Figure 3.1b: $\left\langle \begin{bmatrix} 1 & 0 \\ -2i & 1 \end{bmatrix}, \begin{bmatrix} 1-i & 1 \\ 1 & 1+i \end{bmatrix} \right\rangle$
- Figure 3.1c: $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1.5+1.3i & 1 \end{bmatrix} \right\rangle$
- Figure 3.1d: $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+2i & 1 \end{bmatrix} \right\rangle$

Our algorithm for drawing these is essentially a very simple depth-first walk of the tree of words in the two generators of each group, based on that described in the book *Indra's Pearls* [94] (in which can be found many other nice computer-generated pictures related to this subject). Observe that most of the pictures we have given consist of circle packings of various shapes, or at least contain obvious 'chains' of circles; the theory of Keen and Series which we shall study in a later chapter is essentially based on the observation that we can continuously deform the coefficients of the generators of these groups such that the bounding circles of the circle packing remain bounding circles.

3.1.15 Theorem (Dynamics of a Kleinian group)**.** *Let G be a Kleinian group acting on $\overline{\mathbb{H}^3}$.*

1. *The limit set $\Lambda(G)$ is the set of accumulation points of the orbits of G on $\hat{\mathbb{C}}$.*
2. $\Omega(G, \hat{\mathbb{C}}) = \hat{\mathbb{C}} \setminus \Lambda(G)$. ∎

This theorem exhibits every element of $\hat{\mathbb{C}} \setminus \Omega(G)$ as a point at which the quotient $\hat{\mathbb{C}}/G$ fails to be Hausdorff. On the other hand, the quotient $\Omega(G, \hat{\mathbb{C}})/G$ is a **possibly disconnected Riemann surface**—that is a union of at most countably many (connected) Riemann surfaces — with possible quotient singularities (at projections of elliptic fixed points which are not limit points); and ${}^\circ\Omega(G, \hat{\mathbb{C}})/G$ is a possibly disconnected Riemann surface with these singularities deleted.

From this point, when a Kleinian group G is given, the notation $\Omega(G)$ (without topological space indicated) refers to the action of G on $\hat{\mathbb{C}}$.

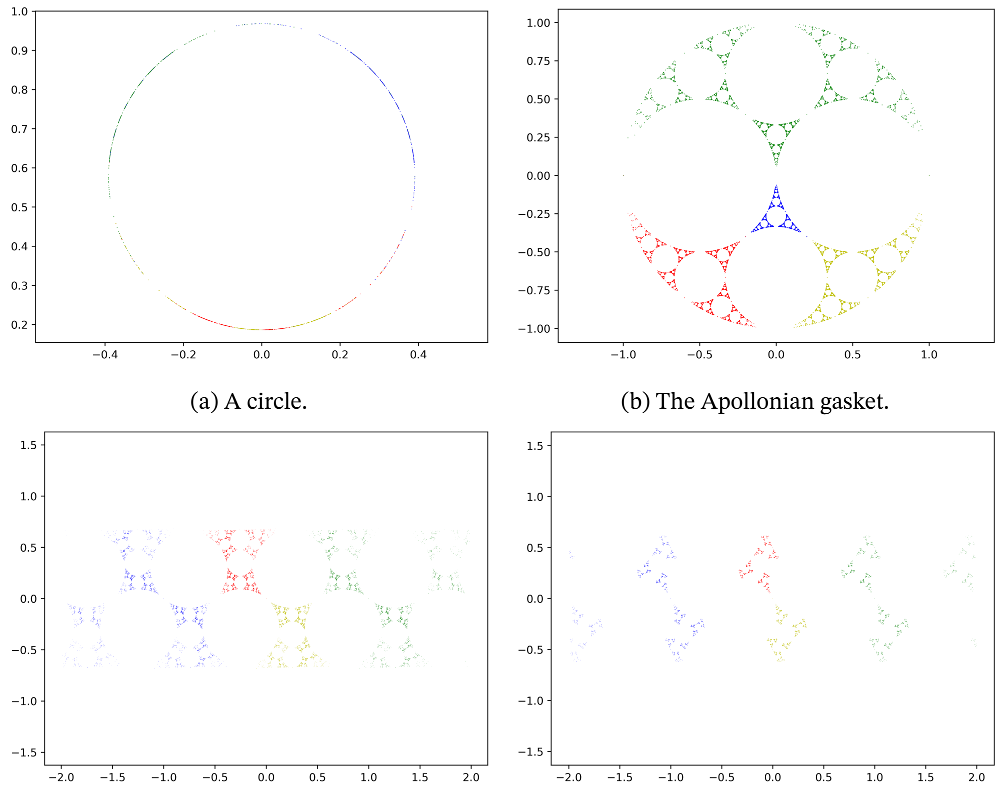
Remark. Note that it is not always the case that $\Omega(G) \setminus {}^\circ\Omega(G)$ contains all the elliptic fixed points of G; consider a group generated by an elliptic element with fixed points at 0 and ∞, and a loxodromic element with fixed points at 0 and 1.

3.1.16 Definition. Let G be a Kleinian group. We have three quotient spaces of interest:

- The **Riemann surface of** G, the possibly disconnected Riemann surface
$$\mathcal{S}(G) := \Omega(G)/G;$$
- The **hyperbolic orbifold of** G, the hyperbolic 3-orbifold
$$\mathcal{M}(G) := \mathbb{H}^3/G;$$
- The **Kleinian orbifold of** G, the hyperbolic 3-orbifold-with-boundary
$$\mathcal{K}(G) := (\mathbb{H}^3 \cup \Omega(G))/G.$$

Occasionally, these objects will be manifolds (not just orbifolds).

Remark. Thurston refers to $\mathcal{M}(G)$ and $\mathcal{K}(G)$ as N_G and O_G respectively [124, Definition 8.3.5].



(a) A circle.

(b) The Apollonian gasket.

(c) A 'cusp group'.

(d) A 4-punctured sphere group.

Figure 3.1: Various limit sets of Kleinian groups. Points are coloured according to the first letter in the word (in the two generators of the group) which moves the generating point to the limit point. (See Example A.2.1 for the computer code to draw these.)

3.2 Marked Riemann surfaces

Let G be Kleinian. We will usually consider the possibly disconnected Riemann surface $\mathcal{S}(G)$ to be *marked*. In order to explain what we mean by this, we must formally define the notion of a puncture on a possibly disconnected Riemann surface S. Recall that a map $f : R \to S$ of Riemann surfaces is said to be **conformal** if, for every pair of charts ϕ for R and ψ for S, the composition $\psi f \phi^{-1}$ is conformal. If $D \subseteq S$ is an open subset that is conformally equivalent via some map ϕ to the punctured disc $\Delta_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}$, then we say that D **bounds a puncture** and we may complete S by extending ϕ^{-1} to be defined at 0 and adjoining the image of 0 to S. We often abuse language and notation and move back and forth between thinking of punctures as either deleted points on a surface or points which are on the surface but have been painted in some way to indicate that they are special.

Define on $\mathcal{S}(G)$ a map $\mu : \mathcal{S}(G) \to \mathbb{N} \cup \{\infty\}$ such that:

1. If x is a puncture, then $\mu(x) = \infty$;
2. If the covering $\Omega(G) \to G$ is ramified at x with degree $\delta \in \mathbb{N}$ (i.e. is locally of the form $z \mapsto z^\delta$), then $\mu(x) = \delta$;
3. Otherwise, $\mu(x) = 1$.

Then the map μ is called a **marking** on $\mathcal{S}(G)$, and points $x \in G$ with $\mu(x) \geq 2$ are called **marked points**—they are either punctures, or **cone points** (in which case they have neighbourhoods which are isometric to cones with cone angle $2\pi/\mu(x)$ with x at the cone vertex).

An *abstract* Riemann surface S is said to be **analytically finite** if it has finitely many boundary components, all of which are punctures (that is, there is a compact Riemann surface S' such that $S' \setminus S$ consists of finitely many points); this is equivalent to the quotient being of finite area (this can be proved, for example, using the Gauss-Bonnet theorem Theorem 2.2.6). We say that a Kleinian group G is **of finite type** if $\mathcal{S}(G)$ is a finite union of analytically finite Riemann surfaces and the covering $\Omega(G) \to \mathcal{S}(G)$ is ramified at only finitely many points (equivalently, $\mathcal{S}(G)$ has only finitely many marked points and there are no deleted discs on the boundary). The famous finiteness theorem of Ahlfors states that this happens often enough to be useful:

3.2.1 Theorem (Ahlfors' finiteness theorem). *If G is a non-elementary finitely generated Kleinian group, then G is of finite type.* ∎

Historical remark. This theorem was originally stated by Ahlfors in [3] with corrections in [2], generalising similar results of Bers in the two-dimensional case; the proof uses Beltrami differentials and quasiconformal techniques. A modern account of Ahlfors' proof together with copious references to other proofs may be found in Section 8.14 of [57].

We now consider the converse problem: given a possibly disconnected Riemann surface S, does there exist a Kleinian group G such that $\mathcal{S}(G) = S$? The answer is almost always *yes*, but we can be slightly more precise. The resulting theorem is the *uniformisation theorem* for Riemann surfaces, which we state in two parts (Theorems 3.2.2 and 3.2.3).

Historical remark. The theorem was originally conjectured by Klein and Poincaré in the early 1880s; Poincaré gave an incomplete proof, and a complete proof was asked for as part of Hilbert's 22nd problem. The result was eventually proved by 1907, independently by Poincaré and Koebe. A very nice history with full references may be found in Section 8.3 of Bottazzini and Gray's history of complex analysis [19].

First, we classify the universal coverings, in a generalisation of the Riemann mapping theorem:

3.2.2 Theorem (Uniformisation I). *If D is a connected and simply connected Riemann surface, then D is conformally equivalent to precisely one of 1. $\hat{\mathbb{C}}$; 2. $\mathbb{C}$; 3. $\mathbb{B}^2$.* ∎

(For a proof, see the theorem of Paragraph IV.6.1 of [47].)

Now, we may classify every marked Riemann surface.

3.2.3 Theorem (Uniformisation II). *Let S be a Riemann surface and let $\mu : S \to \mathbb{N} \cup \{\infty\}$ be a map such that the set $\mathcal{M}$ of points $x \in S$ for which $\mu(x) \neq 1$ is discrete and such that if $S = \hat{\mathbb{C}}$, neither (a) $\mathcal{M} = \{x\}$ with $\mu(x) = \infty$, or (b) $\mathcal{M} = \{x, y\}$ and $\mu(x) \neq \mu(y)$.*

Let $S' = S \setminus \mu^{-1}(\infty)$, $S'' = S' \setminus \mu^{-1}(\mathbb{Z}_{\geq 2})$. There exists a simply connected Riemann surface $\tilde{S}$ (which is assumed to be embedded in $\hat{\mathbb{C}}$ by Theorem 3.2.2) and a Kleinian group G leaving $\tilde{S}$ invariant such that

1. *$\tilde{S}/G \simeq_{conf.} S'$ and $\tilde{S}^*/G \simeq_{conf.} S''$ (where $\tilde{S}^*$ denotes $\tilde{S}$ with the elliptic fixed points of G deleted); and*

2. *The induced projection $\tilde{S} \to S'$ is unramified except over the points of $\mu^{-1}(\mathbb{Z}_{\geq 2})$; if $2 \leq \mu(x) < \infty$, then the cover has ramification degree $\mu(x)$ over x.*

Further, G is uniquely defined up to conjugation in the full group of conformal maps leaving $\tilde{S}$ invariant. ∎

(For a proof, see Theorem IV.9.12 of [47]. The theorem extends to possibly disconnected Riemann surfaces, as the so-called **simultaneous uniformisation theorem**, [83, Section VIII.B].)

If S is covered by $\tilde{S} = \mathbb{B}^2$ and uniformised by the Kleinian group G leaving the disc invariant (using the notation of Theorem 3.2.3), then the hyperbolic metric on $\mathbb{B}^2$ descends via the projection map to S. Such groups G are called **Fuchsian groups**; they will be important later on.

Warning. Suppose $\mathcal{S}(\Gamma)$ admits a Fuchsian uniformisation in this way, so $\mathcal{S}(\Gamma) \simeq \mathbb{B}^2/G$ for some group G—there is usually no relationship at all between Γ and G, and the Euclidean metric on the surface (coming from the Riemann surface structure) is different to the hyperbolic metric.

A general Kleinian group F is Fuchsian if one of the following equivalent properties holds:

1. There is an open Euclidean disc $\Delta \subseteq \Omega(F)$ left invariant by F;

2. F is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

In either case, the limit set $\Lambda(F)$ is a subset of a Euclidean circle, and the two discs bounded by this circle are left invariant by F. Let Δ be either of these invariant discs; then there is a conformal map ϕ sending Δ to $\mathbb{H}^2$, and $\phi F \phi^{-1}$ acts on $\mathbb{H}^2$ as a discrete group of isometries of the hyperbolic plane. Standard references for the theory of Fuchsian groups which may be useful are [13] and [58]. Primarily we will be interested in certain polygons related to Fuchsian groups.

3.2.4 Definition. Let F be a non-elementary Fuchsian group acting on a disc Δ such that $\Lambda(F) \neq \partial\Delta$. Then $\partial\Delta$ is the disjoint union of $\Lambda(\Delta)$ with a countable set of open arcs σ_i (the **intervals of discontinuity**). For each σ_j let S_j be the hyperbolic line which meets the circle at infinity $\partial\Delta$ at the endpoints of σ_j and let H_j be the hyperbolic halfplane bounded by L_j away from σ_j. Then the intersection

$$N(F) := \bigcap_j H_j$$

is called the **Nielsen region** for F.

The Nielsen region is the minimal non-empty open convex F-invariant subset of Δ [13, Theorem 8.5.2], it is precisely the hyperbolic convex hull of its limit set taken within the disc it acts on as a group of hyperbolic plane isometries (so it is the 2-analogue of the convex core for Kleinian groups defined in Section 3.4), and in the situation we will study in Chapter 7 it will be a fundamental polygon (see Lemma 7.2.9).

We say that $f \in F$ is a **boundary hyperbolic element** if it leaves invariant one of the intervals σ_i. These are studied in Sections 10.3 and 10.4 of [13]; we recall the main results here. For the sake of language, if S is a hyperbolic surface then a **cylinder** on S is a boundary component corresponding to a deleted disc.

3.2.5 Proposition. *A finitely generated Fuchsian group F has finitely many conjugacy classes of maximal hyperbolic boundary elements*[1]*; and these conjugacy classes are in bijective correspondence with the cylinders of Δ/F.* ∎

Another polygon of interest is the **canonical Fricke polygon**. We will not need too much (we will use it exactly once, in the proof of Lemma 7.2.2), just the following:

3.2.6 Lemma ([59, Theorem 6]). *Let F be a finitely generated Fuchsian group with $\Lambda(F) \neq \Delta$. There exists a finite-sided fundamental polygon R for the action of F as a group of hyperbolic isometries on Δ such that $R \cap \partial\Delta$ consists only of parabolic fixed points of F and subintervals of the intervals of discontinuity of F.* ∎

3.3 Geometry of the 3-manifold near infinity

In the previous section, we studied how the structure of a Kleinian group induces covering data on the Riemann surface. In this section we give more precise geometric information about the behaviour of the Riemann surface, and how this is reflected in the geometry of the interior hyperbolic 3-manifold. The discussion is based on Chapter VI of [83].

3.3.1 Lemma. *Let G be a Kleinian group. If $j \in G$ is parabolic with fixed point $w \in \hat{\mathbb{C}}$, then there is a horoball based at w left precisely invariant*[2] *by a maximal parabolic group (namely,* $\operatorname{Stab}_G w$*).*

Proof. Note that the stabiliser of w in G is a subgroup of G with a global fixed point (w) and which contains a parabolic element. Therefore by the classification of elementary Kleinian groups, $\operatorname{Stab}_G w$ is a parabolic group and must be a maximal such group (if a parabolic group strictly contains $\operatorname{Stab}_G w$ then it contains a parabolic k not fixing w and so $[j,k]$ is loxodromic).

Without loss of generality, the parabolic element may be chosen to be $j : z \mapsto z + 1$ and so $w = \infty$. Let $J = \operatorname{Stab}_G(\infty)$; this group contains no loxodromic elements (suppose it contained such an element f, then f would share a fixed point with j; we can therefore conjugate f to $z \mapsto \lambda z$; if $|\lambda| > 1$ then $f^{-k} j f^k = \lambda^{-k}(\lambda^k z + 1) = z + \lambda^{-k} \to z$, and similarly if $|\lambda| < 1$ then $f^k j f^{-k} \to 1$, either way contradicting discreteness). Thus every element of J is an elliptic or parabolic element with fixed point at infinity, so is a Euclidean transformation on $\mathbb{C}$ and leaves every horoball based at ∞ fixed.

Suppose now that $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ is arbitrary. By the Shimizu-Leutbecher lemma (Corollary 3.1.11), either $c = 0$ (in which case $g \in J$) or $|c| > 1$; in this latter case the radius of the isometric circle of g is strictly less than 1, and so the action of g on $\mathbb{H}^3$ maps the horoball $H = \{(z,t) \in \mathbb{H}^3 : t > 1\}$ below a dome of radius 1 centred on $\hat{\mathbb{C}}$. In any case, H is mapped strictly off itself by g. ∎

[1] In a group G an element g is **maximal** if, whenever there is some $h \in G$ and $n \in \mathbb{Z}$ with $h^n = g$, the element h actually *is* g.

[2] A subset $T \subseteq X$ is left **precisely invariant** by $H \leq G$ if $HT \subseteq T$ and $(G \setminus H)T \subseteq X \setminus T$.

We may generalise Lemma 3.3.1 to the case that we have finitely many punctures. We give a proof following Proposition A.11 of [83] (which gives the proof for a pair of punctures):

3.3.2 Lemma. *Let G be a Kleinian group and let $J_1, ..., J_k$ be pairwise non-conjugate maximal parabolic subgroups of G, such that the fixed point of J_m is x_m for each m. Then there are horoballs $T_1, ..., T_k$, based at $x_1, ..., x_k$ respectively, such that each T_m is precisely invariant under J_m and such that if $n \neq m$ and $g \in G$ is arbitrary then $gT_n \cap T_m = \varnothing$. (In particular, the T_i are mutually disjoint.)*

Proof. The proof is by induction on k; the base case is Lemma 3.3.1 (see the discussion immediately following the statement of that lemma). Suppose that we have found horoballs $T_1, ..., T_{k-1}$ based respectively at $x_1, ..., x_{k-1}$, satisfying the precise invariance conditions of the lemma statement.

Let T_k be a horoball disjoint from $T_1, ..., T_{k-1}$ which is precisely invariant under J_m. We can construct such a horoball in the following way: let H be the horoball constructed for J_k in Lemma 3.3.1; let T_k be a horoball based at x_k which is contained in H and which is disjoint from $T_1, ..., T_{k-1}$. Clearly every element of $G \setminus J_k$ moves T_k off itself (indeed, every such element moves T_k out of H entirely); we also see easily that every element of J_k preserves the smaller horoball T_k (*a priori*, J_k might move elements of T_k into $H \setminus T_k$) by conjugating x_k to ∞; then the group J_k becomes a group of Euclidean transformations preserving each horizontal plane above $\mathbb{C}$ in $\mathbb{H}^3$ (as in the proof of Lemma 3.3.1) and thus preserves every horoball based at x_k.

We now show that the horoball T_k satisfies the precise invariance condition of the lemma statement; by the inductive hypothesis we need only check that whenever $m \neq k$, $T_m \cap gT_k = \varnothing$ for all $g \in G$. We will do this by replacing T_k, if necessary, with a smaller horoball which does satisfy this condition (which we need to do at most finitely many times). To this end, fix some $m \neq k$; we may normalise so that $x_m = \infty$ and $x_k = 0$, and hence $T_m = \{(z,t) \in \mathbb{H}^3 \,:\, t > r\}$ for some $r > 0$. To save ink, we write $T_{m,s}$ for the set $\{(z,t) \in \mathbb{H}^3 \,:\, t > s\}$ $(s > 0)$.

Suppose, for contradiction, that there is no $s \geq r$ with the property that for all $g \in G$, $gT_{m,s} \cap T_k = \varnothing$. Then there exists, for each $n \in \mathbb{N}$, a group element $g_n \in G$ with $T_{m,n} \cap g_n T_k \neq \varnothing$. For each $n \in \mathbb{N}$ let (ρ_n, z_n) be the Euclidean radius and centre of $g_n(T_k)$ in $\mathbb{H}^3$, and define $a_n \in \mathbb{M}$ to be a transformation of the form $a_n(z) = \lambda_n^2 z$ with λ_n^2 chosen such that $a_n(T_k)$ is a horosphere of Euclidean radius ρ_n. Let b_n be the transformation $b_n(z) = z + z_n$; then $b_n a_n(T_k)$ and $g_n(T_k)$ are horospheres with the same radius and centre. Since both $b_n a_n$ and g_n map the Euclidean sphere T_k to the same Euclidean sphere, by Euclidean geometry the two transformations differ only by a rotation about the axis (z_n, ∞). More precisely, there exists an elliptic element c_n with fixed point set $\{z_n, \infty\}$ such that $g_n = c_n b_n a_n$. Observe now that $\lambda_n^2 \to \infty$ (since the $\rho_n \to \infty$ in order for the spheres $g_n T_k$ to continue to hit the ceilings $T_{m,n}$); if $j_k \in J_k$ then j_k is represented by a matrix of the form $\begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix}$, so for each n

$$a_n j_k a_n^{-1} = \begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} \lambda_n^{-1} & 0 \\ 0 & \lambda_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda_n^{-2} w & 1 \end{bmatrix}$$

and the lower-left entry of this matrix tends to 0 as $n \to \infty$; thus the isometric circle radius of $a_n j_k a_n^{-1}$ goes to infinity. We have seen that g_n and a_n differ only by Euclidean motion factors, and Euclidean motion factors do not change the isometric circle radius. Hence the lower-left entry of g_n also must tend to 0 as $n \to \infty$; in particular for sufficiently large n, this entry is strictly less than 1 and is non-zero since a_n and hence g_n have finite-radii isometric circles for each n. But this contradicts the Shimizu-Leutbecher lemma (Corollary 3.1.11). ∎

Lemma 3.3.1 and Lemma 3.3.2 give neighbourhoods of points on the boundary corresponding to punctures. We can get a similar result in the surface case:

3.3.3 Lemma. *Let F be a Fuchsian group acting on $\mathbb{H}^2$ and containing a primitive parabolic element j. Then there is a horoball $H \subseteq \mathbb{H}^2$ based at the fixed point of j such that*

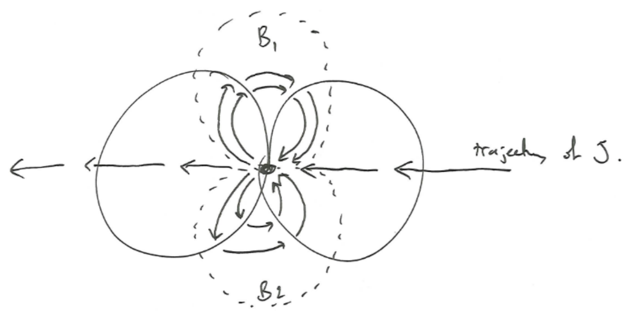


Figure 3.2: A doubly cusped region for a Kleinian group; observe that the solid regions are mapped onto each other by the indicated action and so form a pair of 'funnels' which wrap up to form two cusps.

1. $\mathrm{Stab}_F\, H = \langle j \rangle$;
2. *H is precisely invariant in F under $\langle j \rangle$; and*
3. *the image of H under the canonical projection $\mathbb{H}^2 \to \mathbb{H}^2/F$ is a punctured disc conformally embedded in $\mathbb{H}^2/F$ such that under the natural homomorphism $\pi_1(\mathbb{H}^2 \cap {}^\circ\Omega(F)/F) \to F$ [3] of Propositions 3.1.1 and 3.1.3, j corresponds to a small loop about the puncture.* ∎

The proof is essentially the same argument as in the 3-dimensional case; see Propositions VI.A.6 and VI.A.7 of [83] for the details.

Let G be a Kleinian group, and let J be a rank 1 parabolic subgroup of G. If B_1, B_2 are two disjoint open discs such that $B = B_1 \cup B_2$ is precisely invariant under J in G, then B is called a **doubly cusped region**. In the case of a geometrically finite group, all cusps are doubly cusped (see [83, Proposition VI.A.10] or [80]):

3.3.4 Lemma. *If G is a geometrically finite Kleinian group with finite-sided fundamental polyhedron D, $x \in \overline{D} \cap \hat{\mathbb{C}}$, and J is a rank 1 parabolic subgroup of G with fixed point x, then J is doubly cusped.* ∎

3.4 The convex core of a hyperbolic 3-manifold and measured laminations

It is well-known that the basic theory of convexity in $\mathbb{R}^3$ carries over almost without change to $\mathbb{H}^3$ and hyperbolic manifolds in general (c.f. [124, Section 8.3], [28], and [43]; more specifically geared towards the topic of this thesis is [61]). The hyperbolic theory is made richer by the existence of a geometric compactification $\overline{\mathbb{H}^3}$, in that we may take convex hulls of sets at infinity. If C is a circle on $\hat{\mathbb{C}} = \partial\mathbb{H}^3$, then the hyperbolic convex hull $\mathrm{h.conv}\, C$ is the hyperbolic plane spanned by C. We often will say that $\mathrm{h.conv}\, C$ is obtained by 'erecting a dome' above C. Of particular interest to us are hyperbolic convex hulls of limit sets of Kleinian groups; various people have made excellent visualisations of these objects, for instance the image *Bug on Notes of Thurston* by Jeffrey Brock and David Dumas [26].

The *convex core* of a hyperbolic 3-manifold is a convex subset which captures all of the geometric information about M while being combinatorial.

[3]The intersection with ${}^\circ\Omega(F)$ is taken so as only to deal with the nonsingular part of the surface.

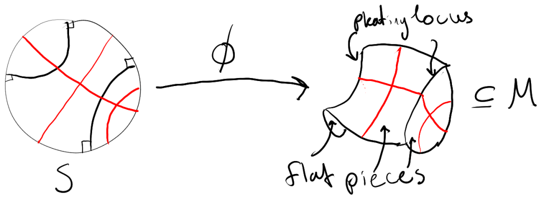


Figure 3.3: A pleated surface: the map ϕ embeds S into M, possibly with some bends along S-geodesics.

3.4.1 Definition. Let G be a Kleinian group; the **convex core** of G (which we denote by $\mathcal{C}(G)$ in harmony with the notation of Definition 3.1.16) is the quotient space $(\mathrm{h.conv}\, G \cap \mathbb{H}^n)/G$.

3.4.2 Lemma. *The convex core $\mathcal{C}(G)$ is a strong deformation retract of the hyperbolic manifold $\mathcal{M}(G)$ and is homeomorphic to the Kleinian manifold $\mathcal{K}(G)$.* ∎

For a proof of this lemma, see Proposition 3.1 of [88]. The proof given there extends with minimal change to the orbifold case, where G is allowed to contain elliptics.

The boundary of $\mathcal{C}(G)$ is a *pleated surface* with some additional data.

3.4.3 Definition. A **pleated surface** in a hyperbolic 3-manifold M is a complete (abstract) hyperbolic surface S together with a smooth embedding $\phi : S \to M$ such that every $s \in S$ is contained in the interior of some geodesic arc which is mapped to a geodesic arc in $\phi(S)$ (where $\phi(S)$ is given the induced metric from M). The **pleating locus** of the pleated surface is the set of points contained in precisely *one* such arc. A connected component of the complement of the pleating locus in $\phi(M)$ is called a **flat piece**, and a complete arc in the pleating locus is called a **bending line**.

3.4.4 Example. An example of a pleated surface is shown in Figure 3.3: the two geodesics of S drawn in black become the two pleats of $\phi(S) \subset M$, while their complements become smooth hyperbolic surfaces in M. Two of the three geodesics in S drawn in red remain so in M; the other, which is transverse to the pleating locus, is only piecewise geodesic in the image surface.

A **geodesic lamination** on a hyperbolic surface S is a union of disjoint complete geodesic arcs, called **leaves**. In this thesis, we will shorten 'geodesic lamination' to simply 'lamination'. The pleating locus of a pleated surface is a geodesic lamination [28, Lemma 5.1.4].

We draw heavily from [62, §6.2], [61], and [57, Chapter 11] for the theory of laminations that now follows.

Let L be a lamination on a surface S; a **transverse measure** on L is a regular Borel measure[4] ν defined on the set of embedded intervals in S which are transverse to every bending line that they meet. The pair (L, ν) is said to be a **measured lamination** (and by usual abuses of notation we refer to the single object L or ν as the measured lamination; often we write $|\nu|)$ for the set of leaves of the measured lamination).

The point is that given a pleated surface S, there is a natural transverse measure β, the **bending measure**, on the pleating locus which measures the bending angle across a particular pleat. The technical parts of such a definition are worked out in [43] (see for instance Definition 1.11.2 for the formal definition of the measure); roughly speaking, a **roof** above some portion of S is a finite hyperbolic polyhedral approximation to S; on a polyhedron, bending angles (that is, dihedral angles between planes in $\mathbb{H}^3$) may be define since all the bends are 'far apart', so if I is an open subset of an

[4]We are not interested in the technicalities of analysis; one may refer to Definition 2.15 of [108] and the surrounding discussion.

interval in S then we may define $\beta(I)$ to be $\inf_R\{$sum of bending angles of I along $R\}$ where R ranges over all the roofs above S around I.

Denote by $\mathcal{ML}(S)$ the set of measured laminations on S, and let $\mathcal{ML}_0(S)$ denote the subset of $\mathcal{ML}(S)$ consisting of those laminations which do not have leaves which tend asymptotically to marked points (i.e. $\nu \in \mathcal{ML}_0(S)$ iff $|\nu|$ is compact and lies in the nonsingular set of S).[5] There is a natural topology on $\mathcal{ML}(S)$ given by the weak topology on measures [108, Exercise 18 of Chapter 11]: declare a sequence (ν_n) to converge to some $\nu_\infty \in \mathcal{ML}(S)$ if

$$\int_I f d\nu_n \to \int_I f d\nu$$

for every open interval I transverse to all the $|\nu_k|$ ($k \in \hat{\mathbb{N}}$) and for all $f : S \to \mathbb{R}$ compactly supported on S.

We now define two functionals on $\mathcal{ML}(S)$, one which will measure 'length' and one which will measure 'transversality'.

3.4.5 Definition. For $\nu \in \mathcal{ML}(S)$, define

- the **lamination length**, $l(\nu)$, to be the total mass of the measure on S that is locally the product of ν on intervals transverse to $|\nu|$ and the usual hyperbolic length measure on intervals parallel to $|\nu|$; and
- for any simple closed geodesic γ on S, the **intersection number** of ν with γ, $i(\nu, \gamma)$, to be

$$\inf\left\{ \int_{\gamma'} d\nu \,:\, \gamma' \text{ a curve isotopic to } \gamma \right\}.$$

If ν is the measured lamination with the single leaf η for some simple closed geodesic η and with transverse measure given by the Dirac measure on transversals to η, then $l(\nu)$ is the usual hyperbolic length of η and $i(\nu, \gamma)$ is the usual intersection number of η with γ.

The following continuity result is actually a special case of a pair of uniform continuity results for measured laminations in quasiconformal deformation spaces, Theorems 3.6.14 and 3.6.16 below.

3.4.6 Lemma. *Both $l : \mathcal{ML}_0(S) \to \mathbb{R}$ and $i(\cdot, \gamma) : \mathcal{ML}_0(S) \to \mathbb{R}$ are continuous with respect to the weak topology.* ∎

3.5 Schottky groups and deformations

The group-theoretic definitions of the previous sections are hard to visualise in general; in this section, we discuss a special class of groups for which the geometry is easy to see and which will be critical in what follows.

3.5.1 Definition. A **classical Schottky group** of rank n is defined by the following data:

- $2n$ pairwise disjoint circles in $\mathbb{C}$, labelled $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_n$, which bound a common region D; and

[5] Kapovich [57] uses the term 'measured laminations' to denote only the laminations in $\mathcal{ML}_0(S)$, and terms the more general laminations in $\mathcal{ML}(S)$ 'measured *quasi*laminations'.

- n elements of $\mathbb{M}$, $g_1, ..., g_n$, such that for all i, $g_i(A_i) = A'_i$ and $g_i(D) \cap D = \emptyset$.

The Schottky group is then the group $G = \langle g_1, ..., g_n \rangle$.

Let $\mathcal{A}$ be the polyhedron in $\mathbb{H}^3$ with faces consisting of the hyperbolic planes erected over the $2n$ circles $A_1, ..., A_n$ and $A'_1, ..., A'_n$, and consider the system of side-pairings of $\mathcal{A}$ induced by the maps $g_1, ..., g_n$. Observe that every point of $S^2 \cap \overline{\mathcal{A}}$ is contained in exactly one facet, and so the intersection of the facets containing a given boundary point is an entire circle: thus there are no cusp points and the completeness condition of Theorem 2.3.7 is vacuous. We may therefore apply Theorem 2.3.5 to glue the sides of $\mathcal{A}$; considering the boundary at infinity of the resulting manifold, we conclude that:

3.5.2 Proposition. *With G defined by means of a side-pairing as just described:*

1. *G is a discrete group, free on the generators $g_1, ..., g_n$ (since the family of relations in the presentation for G given by Theorem 2.3.5 are all trivial);*
2. *D is a fundamental domain for the action of G on $\Omega(G)$;*
3. *$\mathcal{S}(G)$ is obtained by taking the surface D (a sphere with n deleted discs) and gluing the boundaries together according to the g_i, so $\mathcal{S}(G)$ is topologically a genus n handlebody.*
4. *Every element of G is loxodromic.*

Proof. Conclusions (1) to (3) follow from the preceeding discussion. Conclusion (4) requires a small amount of work, and our argument is adapted from paragraph VII.C.5 of [83]. The idea is to show that every non-identity element of G has an attractive fixed point. We proceed by induction on n; the case $n = 1$ is trivial, so assume that $n > 1$.

Suppose $g \in G \setminus \{1\}$ is arbitrary, and write g as a word in the generators g_i. Suppose without loss of generality that the rightmost letter in this word is g_1, and write $g = h_n \cdots h_1$ where every h_i with i odd is a power of g_1 and every h_i with i even is a product of some string of the g_j with $j \neq 1$. If $h_n = g_1^k$ ($k \neq 0$) then replace g with the conjugate $g_1^{-k} g g_1^k$ (and it suffices to check that this is loxodromic), and relabel $g_1, ..., g_n$ such that the rightmost letter of this new word is g_1; repeat this process (at most finitely many times) until either (a) the rightmost letter in the word in the h's for g differs from the leftmost (in which case n is necessarily even), or (b) the length of the word — in terms of the number of h's—becomes 1. Clearly in case (b) the word g is either a power of g_1 so is loxodromic, or lies in G' so is loxodromic by the inductive hypothesis; so it remains only to resolve case (a).

Now g_1 (resp. g_1^{-1}) maps A_1 into A'_1 (resp. A'_1 into A_1) and so by continuity must move the two components of $\hat{\mathbb{C}} \setminus A_1$ into the two components of $\hat{\mathbb{C}} \setminus A'_1$; since $g_1 D \cap D = \emptyset$, g_1 must move the exterior of A_1 (the component containing D) into the interior; this implies that the attractive fixed point of g_1 lies in B_1 and so every positive power of g_1 has the same property. Let γ be a Jordan curve which separates $A_1 \cup A'_1$ from $\bigcup_{i>1}(A_i \cup A'_i)$ and label the *closures* of the two components of $\hat{\mathbb{C}} \setminus \gamma$ by B_1 (this is the closure of the component containing A_1) and B_2 (the closure of the other component). Since B_2 lies in the exterior of A_1, $g_1^k B_2 \subseteq B_1$ ($k \geq 1$). By a similar argument, $g_i^k B_1 \subseteq B_2$ for all $i > 1$. Let $G' := \langle g_2, ..., g_n \rangle$; if $h \in G'$ then we even have that $hB_1 \subseteq B_2$, since h may be written as a product of g_i^k where each g_i moves the circles corresponding to the other g_j into A'_i in the same fashion as described.

Since n is even, this ping-pong game between B_1 and B_2 necessarily leads to $g(B_2) \subset$ in B_2 (the inclusion is proper since the ping-pong game is played more than once, and so some g_i with $i \neq 1$ is involved which maps B_1 into the interior of one of its circles). In particular, the map on B_2 induced by g is contractive, and so g has an attractive fixed point in B_2; thus it is loxodromic. ∎

Remark. All of the statements of Proposition 3.5.2 hold if the disjoint circles in the Schottky data are replaced with pairwise disjoint *topological* circles; see for instance exercise VII.F.8 of [83].

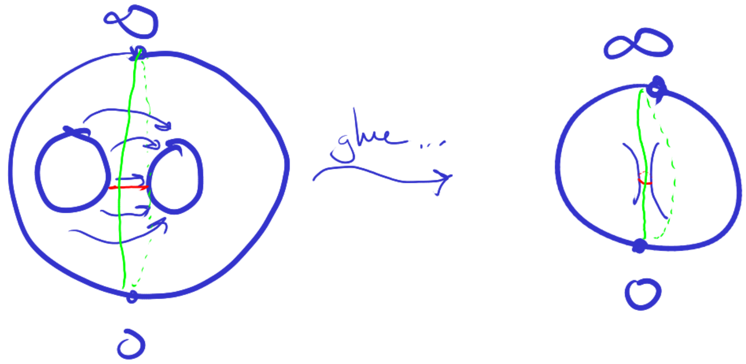


Figure 3.4: The portion of a Riemann surface due to two circles paired by a loxodromic element in a Schottky group.

We may see the action visually in Figure 3.4.

Let us now allow the circles to meet pairwise. The following proposition follows immediately from the proof of Proposition 3.5.2: observe that to prove part 4 of that proposition, we showed that every non-loxodromic element must be conjugate to a generator.

3.5.3 Proposition. *Let $A_1, ..., A_n$ and $A_1', ..., A_n'$ be circles bounding a common region D such that if A_i and A_j' meet then $i = j$, and let $g_1, ..., g_n \in \mathbb{M}$ be such that for all i, $g_i(A_i) = A_i'$ and $g_i(D) \cap D = \emptyset$. Let $\mathcal{A}$ be the family of polyhedrons in $\hat{\mathbb{C}}$ made up of the 1-gons $A_1, ..., A_n$ and $A_1', ..., A_n'$, and consider the system of side-pairings of $\mathcal{A}$ induced by the maps $g_1, ..., g_n$. Define $G \coloneqq \langle g_1, ..., g_n \rangle$.*

1. *G is a discrete group, and is the free product $*_{i=1}^n \langle g_i \rangle$;*
2. *D is a fundamental domain for the action of G on $\Omega(G)$;*
3. *$\mathcal{S}(G)$ is obtained by taking the surface D (a sphere with n deleted discs) and gluing the boundaries together according to the g_i, so $\mathcal{S}(G)$ is topologically a genus n handlebody with $2n$ punctures identified in pairs.*
4. *If an element $h \in G$ is not loxodromic, then $\langle h \rangle$ is conjugate in G to one of the groups $\langle g_i \rangle$, and h is elliptic (resp. parabolic) if A_i intersects A_i' transversely (resp. tangentially).* ∎

We call groups generated via the data of Proposition 3.5.3 **generalised Schottky groups**; usually we will just shorten this to *Schottky group*. We now study the qualitative geometry of the resulting quotients.

Suppose first that we have a pair of tangent circles paired by one of the generators; necessarily, it is parabolic. Applying the Poincaré polyhedron theorem (Theorem 2.3.5), the Riemann surface locally looks like a pair of cusps coming together at a deleted point (Figure 3.5). It is more interesting (but harder to visualise) if we consider the interior of the 3-manifold. The best way to do this is to erect the domes above $\hat{\mathbb{C}}$ in the half-plane model (the left of Figure 3.6), and then perform the edge-gluing by folding $\hat{\mathbb{C}}$ "down"; thus the two dome surfaces combine to give a single disc in the quotient which we may pass through (this is the disc shaded to the right of Figure 3.6: the reader should note that this is the perspective from *inside* the 3-manifold, and the cusps that can be seen on the surface are the insides of those visible in Figure 3.5). Observe that there is a nontrivial loop (in blue, passing through the shaded disc and then around the deleted point) which exhibits that there is a deleted

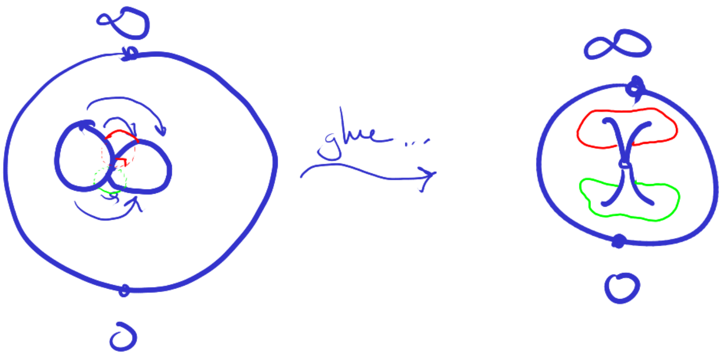


Figure 3.5: The portion of a Riemann surface due to two circles paired by a parabolic element in a Schottky group.

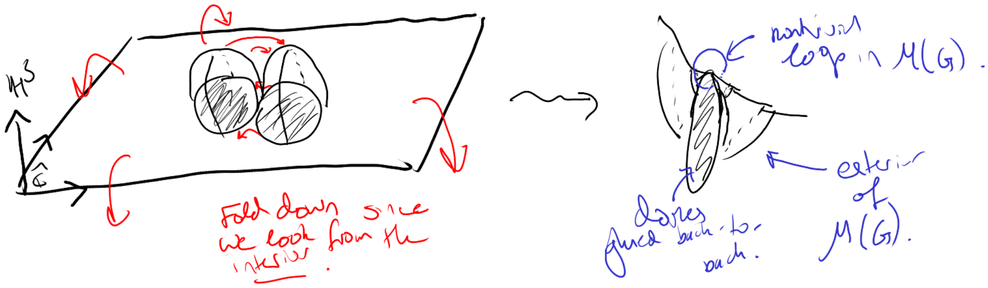


Figure 3.6: The portion of 3-manifold interior due to two circles paired by a parabolic element in a Schottky group.

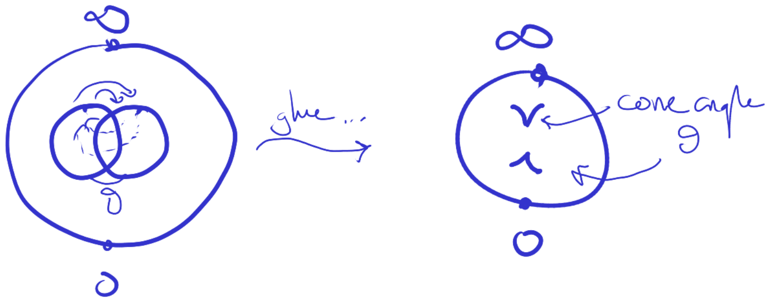


Figure 3.7: The portion of a Riemann surface due to two circles paired by an elliptic element in a Schottky group.

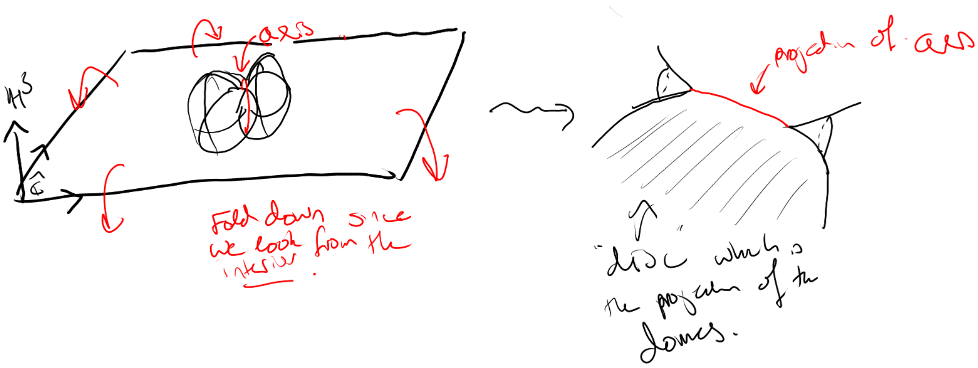


Figure 3.8: The portion of 3-manifold interior due to two circles paired by an elliptic element in a Schottky group.

arc in the interior of the 3-manifold (though in some sense this deleted arc is "infinitely short" in the picture). Since the manifold is given as a quotient of a simply connected space $\mathbb{H}^3$ by a group G acting freely discontinuously on that space, we may apply standard covering space theory to see that $\pi_1(\mathcal{M}(G)) = G$; so for instance if we are considering a Schottky group on one parabolic generator then we see that there is a single nontrivial loop in the space, and it is precisely this one. We can extend this to multiple generators using the various combination theorems of Chapter VII of [83], in which case we get exactly one nontrivial loop per generator in the purely parabolic case.

In the case of elliptic pairings, the pictures (Figure 3.7 for the surface and Figure 3.8 for the interior) are similar to the parabolic case, except that instead of two cusp neighbourhoods meeting at a deleted point we have a pair of cone points joined by a cone arc in the interior of the manifold; this arc is exactly the projection of the axis of the elliptic element representing the cone points.

3.6 Quasiconformal deformation spaces

The Riley slice has more structure than just a set parameterising group representations: it is also a moduli space. In this section we recall first the some basic terminology from the Teichmüller theory of Riemann surfaces and the theory of quasiconformal deformation spaces, following a mixture of

[46, 56, 72]; the reader is expected to have met this subject already and so our discussion is mainly to fix notation. We then study more carefully the Teichmüller theory of Kleinian groups, which is not expected to be as familiar to the reader; this is discussed at an elementary level in Chapter 8 of [57], and in more detail in [88]. We will emphasise the parallels with the classical case.

3.6A Quasiconformal mappings

Define the differential operators $\partial_z := (1/2)(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} := (1/2)(\partial_x + i\partial_y)$, where $x = \Re z$ and $y = \Im z$. Let $f : D \to D'$, where D and D' are domains in $\mathbb{C}$, be a homeomorphism. If there exists some Lebesgue measurable function $\mu_f : D \to \mathbb{C}$ and some $k \in \mathbb{R}_{>0}$ with $\sup_{z\in D} |\mu(z)| \leq k < 1$ such that f satisfies the **Beltrami equation**

$$\partial_z f = \mu_f(z)\, \partial_{\overline{z}} f,$$

then f is said to be **quasiconformal with dilatation at most** K, where $K = \frac{1+k}{1-k}$; often this is shortened to 'K-quasiconformal'. The function μ_f is called the **Beltrami coefficient** of f, and intuitively it measures the failure of f to be conformal at a given point.

A homeomorphism $f : S \to S'$ between Riemann surfaces is said to be K-quasiconformal if, whenever ρ and σ are complex charts on S and S' respectively, the composition $\sigma f \rho^{-1}$ is K-quasiconformal. If there is some $K < \infty$ such that f is K-quasiconformal, then f is simply called **quasiconformal**. The supremum of all K such that f is K-quasiconformal is called the **maximal dilatation** of f.

Let $\pi : D \to S$ be the universal cover of S; computing Beltrami coefficients for $\sigma f \rho^{-1}$ where each ρ is chosen to be a suitable restriction of π and gluing the results together, we obtain a function $\mu_f : D \to D$ such that whenever $\gamma \in \operatorname{Aut} \pi$ and $z \in D$,

$$\mu(z) = \mu(\gamma(z))\, \frac{\overline{\gamma'(z)}}{\gamma'(z)}$$

(that is, μ is a $(-1, 1)$-automorphic form for $\operatorname{Aut} \pi$). For the details, see Section V.1 of [72].

The reason that we want to study these mappings is that they will arise naturally when studying deformation spaces of Kleinian groups. It is a consequence of the λ-lemma (stated below as Theorem 3.6.8) that if the entries of the matrices of a Kleinian group vary holomorphically, then the ordinary set (and therefore the group action) varies quasiconformally in $\hat{\mathbb{C}}$.

The next theorem, variously called the **Alhfors-Bers Riemann mapping theorem** or the **measurable Riemann mapping theorem**, guarantees a sufficient supply of quasiconformal maps.

3.6.1 Theorem. *Let μ be a measurable function, compactly supported on $\mathbb{C}$, with $\|\mu\|_\infty < 1$. Then there is a unique solution $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to the Beltrami equation*

$$\partial_{\overline{z}} f = \mu(z)\, \partial_z f, \quad \textit{for almost all } z \in \mathbb{C}$$

satisfying the normalisation condition that f fixes $\{0, 1, \infty\}$ pointwise. ∎

For a proof, see Section 5.3 of [9].

3.6B Classical theory of Teichmüller space

Let S be a hyperbolic[6] Riemann surface.

[6] Of course one can do Teichmüller theory on any Riemann surface, but we are only interested in hyperbolic manifolds in this thesis.

3.6.2 Definition. A **hyperbolic structure** on S is a pair (R, f) where R is a Riemann surface and $f : S \to R$ is a quasiconformal homeomorphism. We say that two such pairs (R, f) and (R', f') are equivalent if $f'f^{-1} : R \to R'$ is homotopic to a conformal map. The **Teichmüller space** of S is the set of equivalence classes of hyperbolic structures on S with respect to this equivalence; we denote it by Teich(S).

We may place a natural metric on Teich(S) via the following formula:

$$\rho(A, B) := \inf \left\{ \log K(f'f^{-1}) : (R, f) \in A, (R', f') \in B \right\} \tag{3.6.3}$$

where $K(f'f^{-1}) = \frac{1+\|\mu_f\|_\infty}{1-\|\mu_f\|_\infty}$ is the quasiconformal deformation coefficient of $f'f^{-1}$.

Because of the duality between quasiconformal maps and Beltrami coefficients, we may equivalently define the Teichmüller space of S in the following way: uniformise S as $\mathbb{H}^2/G$ for some Fuchsian group G, then Teich(S) is the space of all bounded measurable functions μ on $\mathbb{H}^2$ (the universal cover of any hyperbolic Riemann surface) such that $\|\mu\|_\infty < 1$ and

$$\mu(z) = \mu(g(z)) \frac{\overline{g'(z)}}{g'(z)}$$

for almost all $z \in \mathbb{H}^2$ and for all $g \in G$, modulo the relation $\mu \sim \nu$ if $f_\mu^{-1} f_\nu$ is homotopic to the identity on $\mathbb{H}^2$ (f_μ and f_ν being solutions to the Beltrami equations for μ and ν respectively, on $\mathbb{H}^2$).

The **mapping class group** of the surface S, Mod(S), is the group of isotopy classes of orientation-preserving homeomorphisms $S \to S$, where both the isotopies and the homeomorphisms are required to fix the boundary of S pointwise. It has a natural action on Teich(S): if $[\Sigma, \phi] \in$ Teich(S) is represented by (Σ, ϕ), and $[f] \in$ Mod(S) is represented by $f : S \to S$, then define $[\Sigma, \phi] \cdot [f]$ to be represented by $(\Sigma, \phi f^{-1})$ (it requires some thought to see that this is well-defined). The **Riemann moduli space** of S is defined to be the quotient space

$$\mathcal{M}(S) := \mathrm{Teich}(S)/\,\mathrm{Mod}(S).$$

The action of Mod(S) on Teich(S) is discontinuous, and so $\mathcal{M}(S)$ admits a compatible hyperbolic geometry. If $S = \bigcup S_\alpha$ is a possibly disconnected Riemann surface, then we define Teich(S) $= \prod_\alpha$ Teich(S_α) and similar results hold.

Finally recall that, given a closed curve ω on a marked Riemann surface S (that is, $\omega \in \pi_1(S)$; by our standing assumption that $\pi_1(S)$ is the orbifold fundamental group, ω misses all the marked points of S), we may define a homeomorphism $\tau_\omega : S \to S$ by cutting S along ω, twisting one side of the cut through a rotation of 2π, and then regluing. Similarly, we may define a half-twist σ_ω along ω by cutting along ω, twisting one side by π, and then regluing. The map τ_ω is called a **Dehn twist**, and σ_ω a **Dehn half-twist**. The mapping class group of $S_{g,n}$ is generated by finitely many Dehn twists and half-twists [46, Corollary 4.16].

3.6C Deformation spaces of Kleinian groups

A Kleinian group represents not just a Riemann surface, but also an interior hyperbolic 3-manifold. The deformation theory of Kleinian groups therefore must take into account the interior geometry. In practice, this will mean for us the following: suppose the 3-manifold $M = \mathbb{H}^3/\Gamma$ is a braid manifold (that is, ∂S has $2n$ punctures paired by n deleted tubes through the interior); then another manifold is equivalent to M if it has the same braid structure (i.e. it can be obtained via boundary isotopies which fix the positions of the punctures or which permute punctures in such a way as to keep the braid group the same). In terms of the objects of interest, we will obtain a diagram of the following form:

$$\begin{tikzcd} \mathrm{Teich}(\Gamma) \arrow[r, "\simeq"] \arrow[d, two heads, "\widehat{\mathrm{Mod}}(\Gamma)"] & \mathrm{Teich}(S) \arrow[d, two heads, "\mathrm{Mod}(S)"] \\ \mathrm{QH}(\Gamma) \arrow[r, two heads] & \mathcal{M}(S) \end{tikzcd}$$

The theory of Teichmüller spaces for Kleinian groups is a special case of the concept of the Teichmüller space of a dynamical system introduced by McMullen and Sullivan [120],[121], and [91, c.f. Section 4], but was originally studied first by Bers [15, 16, 17], Kra [68, 69], Marden [78], and Maskit [84, 86]; the specific theorems which we will need in the Kleinian group situation may be found in [88, Section 5.3] (though this reference restricts itself to the torsion-free case) and Chapter 5 of [79].

3.6.4 Definition (C.f. Definition 3.6.2)**.** A **quasiconformal conjugate** of Γ is a pair (G, f) where G is a Kleinian group, and f is a quasiconformal automorphism of $\hat{\mathbb{C}}$ such that the map $\gamma \mapsto f\gamma f^{-1}$ is an isomorphism $\theta_f : \Gamma \to G$. We say that two quasiconformal conjugates $(G_1, f_1), (G_2, f_2)$ of Γ are equivalent if there exists a Möbius transformation $t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ isotopic to $f_2 f_1^{-1}$, via a family of quasiconformal automorphisms of $\hat{\mathbb{C}}$ with uniformly bounded Beltrami coefficients and which all induce the isomorphism $\theta_{f_2}\theta_{f_1}^{-1} : G_1 \to G_2$ via conjugation. The **Teichmüller space** of Γ is the set of equivalence classes of quasiconformal conjugates of Γ with respect to equivalence; we denote it by $\mathrm{Teich}(\Gamma)$.

We place a metric space structure on $\mathrm{Teich}(\Gamma)$ by the same formula as Equation (3.6.3) (with the obvious small modifications).

The Teichmüller space of a Kleinian group can also be defined as a space of particular automorphic forms, but now with respect to the Kleinian group rather than to some Fuchsian uniformisation group: $\mathrm{Teich}(\Gamma)$ is the space of all measurable functions μ on $\Omega(\Gamma)$[7] such that $\|\mu\|_\infty < 1$, and

$$\mu(z) = \mu(\gamma(z))\,\frac{\overline{\gamma'(z)}}{\gamma'(z)}$$

for almost all $z \in \Omega(\Gamma)$ and for all $\gamma \in G$, modulo the relation $\mu \sim \nu$ if $f_\mu^{-1} f_\nu$ is homotopic to the identity on $\Omega(\Gamma)$ (where f_μ and f_ν solve the Beltrami equations for μ and ν respectively on $\Omega(\Gamma)$).

We now define the analogue of the moduli space. Unfortunately we cannot simply quotient by a group action, since it is not immediately clear what the ‘correct’ definition for the Kleinian mapping class group is. Instead we will define what we want the moduli space of a Kleinian group to be, and then we will show that it is the quotient of Teichmüller space by a subgroup of the surface mapping class group.

The **quasiconformal representation space** of Γ, denoted $\mathrm{QHom}(\Gamma)$, is the space of representations $\theta : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ such that

1. θ is faithful and $\theta\Gamma$ is discrete;

2. θ is type-preserving, that is if $\gamma \in \Gamma$ is parabolic (resp. elliptic of order n) then $\theta\gamma$ is parabolic (resp. elliptic of order n); and

3. the groups $\theta\Gamma$ are all quasiconformally conjugate.

[7] One often sees ‘...functions on $\hat{\mathbb{C}}$’ (for instance, it is this definition which appears in [87]); however by a result of Sullivan [119] it suffices to look only at $\Omega(\Gamma)$.

The subset of $\mathrm{Hom}(\Gamma)$ consisting of representations satisfying conditions (1) and (2) is closed, and hence the closure of $\mathrm{QHom}(\Gamma)$ in $\mathrm{Hom}(\Gamma)$ consists of discrete groups [88, Proposition 4.18].

The **quasiconformal deformation space** of Γ is the space $\mathrm{QH}(\Gamma) = \mathrm{QHom}(\Gamma)/\approx$, where $H \approx K$ if there is some $g \in \mathrm{PSL}(2,\mathbb{C})$ with $H = gKg^{-1}$. Since this is the space of different ways on which to place the Kleinian group structure on Γ up to conjugacy, it is analogous to the moduli space in the classical theory.

There remain two problems: (1) to clarify the relationship between $\mathrm{Teich}(\mathcal{S}(\Gamma))$ and $\mathrm{Teich}(\Gamma)$; and (2) to show that there is in fact a natural surjection $\mathrm{Teich}(\Gamma) \to \mathrm{QH}(\Gamma)$ (and to give an explicit description of this, analogous to the surface case where the surjection is the projection induced by the mapping class group).

It turns out that the answer to (1) is straightforward to state: the two spaces are naturally homeomorphic if Γ is finitely generated. By Theorem 3.2.1 we may write $\mathcal{S}(\Gamma) \coloneqq S_1 \cup \cdots \cup S_k$ where each S_i is an analytically finite Riemann surface; for each S_i, view $\mathrm{Teich}(S_i)$ as a space of equivalence classes of Beltrami coefficients, and $\mathrm{Teich}(\mathcal{S}(\Gamma))$ as the Cartesian product of these spaces.

3.6.5 Theorem. *Let Γ be a finitely generated Kleinian group with $\Omega(\Gamma) \neq \emptyset$. Given some*

$$\mu = ([\mu_1], \ldots, [\mu_k]) \in \mathrm{Teich}(\mathcal{S}(\Gamma)),$$

there exist lifts $(\hat{\mu}_1, \ldots, \hat{\mu}_k)$ of $(\mu_1, \ldots, \mu_k)$ (respectively) to $\Omega(\Gamma)$ and quasiconformal automorphisms $f_\mu : \Omega(\Gamma) \to \Omega(\Gamma)$ with Beltrami coefficient $\hat{\mu}$. Suppose that Γ_μ is the group which is quasiconformally conjugate to Γ via f_μ. Then the map $\iota : \mathrm{Teich}(\mathcal{S}(\Gamma)) \to \mathrm{Teich}(\Gamma)$ defined by

$$\iota([\mu]) \coloneqq [(\Gamma_\mu, f_\mu)]$$

is a well-defined function, and is a homeomorphism. ∎

The torsion-free version of this theorem is proved in [88] as Theorem 5.26, and the proof goes through in the torsion case as well with minimal changes (merely replace the classical Teichmüller space with the version detecting cone singularities).

We now answer the second question posed: that of the relationship between $\mathrm{Teich}(\Gamma)$ and $\mathrm{QH}(\Gamma)$.

3.6.6 Theorem. *Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ be a finitely generated non-elementary Kleinian group with $\Omega(\Gamma) \neq \emptyset$. For some Beltrami coefficient μ, the map $p : \mathrm{Teich}(\Gamma) \to \mathrm{QH}(\Gamma)$ defined by*

$$p([\mu]) \coloneqq \theta_\mu$$

is a well-defined holomorphic surjection. Further, there is a discrete subgroup $\widehat{\mathrm{Mod}}(\mathcal{S}(\Gamma)) \leq \mathrm{Mod}(\mathcal{S}(\Gamma))$ and a natural bijection $\mathrm{QH}(\Gamma) \approx \mathrm{Teich}(\mathcal{S}(\Gamma))/\widehat{\mathrm{Mod}}(\mathcal{S}(\Gamma))$ such that the two projection maps agree. ∎

In the case that Γ is geometrically finite, the group $\widehat{\mathrm{Mod}}(\mathcal{S}(\Gamma))$ is the subgroup generated by Dehn twists along simple closed curves which bound compression discs. The torsion-free version of this theorem appears in [88] as Theorem 5.27 and the following discussion. The proof given there actually works for the torsion case as well, as long as we are careful to replace all manifold theorems and definitions with the corresponding 'orbi-theorems' and 'orbi-definitions': for example, the usual definition of compression disc should be replaced with Definition 2.4.4. Another version of this theorem is found as Theorem 5.1.3 of [79] (where the theorem is also stated without any conditions on torsion; the proof is sketched as Exercise 5-35, p.367ff).

Historical remark. The two theorems Theorem 3.6.5 and Theorem 3.6.6 are originally attibuted to Bers and Greenberg [17] and Marden [78] (see [16, §2.4]).

3.6D Holomorphic motions

An important tool in dealing with deformations of Kleinian groups is the theory of holomorphic motions. A pertubation in the matrix entries of a Kleinian group will induce some kind of motion of the limit and ordinary sets, and vice versa; the analytical properties of this motion turn out to be shared by motions of Julia sets in deformation spaces of dynamical systems of rational maps (e.g. the Mandelbrot set), and so deserve to be axiomatised and studied abstractly. The definition of a holomorphic motion which we now state was given first by Mañé, Sad, and Sullivan [77], who also proved the so-called 'λ-lemma' (which we will state a more general version of below, Theorem 3.6.8). A good modern introduction to the theory may be found in Chapter 12 of [9].

3.6.7 Definition. Let $A \subseteq \hat{\mathbb{C}}$. A **holomorphic motion** of A is a map $\Phi : \mathbb{B}^2 \times A \to \hat{\mathbb{C}}$ such that

1. For each $a \in A$, the map $\mathbb{B}^2 \ni \lambda \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is holomorphic;
2. For each $\lambda \in \mathbb{B}^2$, the map $A \ni a \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is injective;
3. The mapping $A \ni a \mapsto \Phi(0, a) \in \hat{\mathbb{C}}$ is the identity on A.

By abuse of the English language, we say that ρ **moves holomorphically** in a set D (with no further context) if there is some holomorphic motion of a subset of A containing ρ which it is not important to give explicitly.

The following result, due to Slodkowski [116, 117], shows that holomorphic motions are 'rigid': they are determined everywhere even when defined on 'small' sets.

3.6.8 Theorem (Extended λ-lemma). *If $\Phi : \mathbb{B}^2 \times A \to \hat{\mathbb{C}}$ is a holomorphic motion of $A \subseteq \mathbb{C}$, then Φ has an extension to $\tilde{\Phi} : \mathbb{B}^2 \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

1. *$\tilde{\Phi}$ is a holomorphic motion of $\hat{\mathbb{C}}$;*
2. *For each $\lambda \in \mathbb{B}^2$, the map $\tilde{\Phi}_\lambda$ defined by $\hat{\mathbb{C}} \ni a \mapsto \tilde{\Phi}(\lambda, a) \in \hat{\mathbb{C}}$ is a K-quasiconformal homeomorphism with*
$$K \leq \frac{1 + |\lambda|}{1 - |\lambda|};$$
3. *$\tilde{\Phi}$ is jointly continuous in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$; and*
4. *For all $\lambda_1, \lambda_2 \in \mathbb{B}^2$, $\tilde{\Phi}_{\lambda_1} \tilde{\Phi}_{\lambda_2}^{-1}$ is K-quasiconformal with $\log K \leq \rho(\lambda_1, \lambda_2)$ (where ρ is the hyperbolic metric on $\mathbb{B}^2$).* ∎

There is also an equivarant version due to Earle, Kra, and Krushkal', appearing as Theorem 1 of their paper [37]:

3.6.9 Theorem (Equivariant λ-lemma). *Let $A \subseteq \hat{\mathbb{C}}$ have at least three points, and let Γ be a group of conformal motions preserving A. Let $\Phi : \mathbb{B}^2 \times A \to \hat{\mathbb{C}}$ be a holomorphic motion on A, and suppose that for each $\gamma \in \Gamma$ and each $\lambda \in \mathbb{B}^2$ there is a conformal map $\theta_\lambda(\gamma)$ such that*

$$\Phi(\lambda, \gamma(z)) = \theta_\lambda(\gamma)(\Phi(\lambda, z)) \tag{3.6.10}$$

for all $z \in A$. Then Φ can be extended to a holomorphic motion on $\hat{\mathbb{C}}$ such that Equation (3.6.10) *holds for all $z \in \hat{\mathbb{C}}$.* ∎

Theorem 3.6.9 is important to us because it allows us to write proofs following the schema we now outline. If $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ is a representation with image $\tilde{\Gamma}$, then an element $\tilde{\gamma} \in \tilde{\Gamma}$ is said to be an **accidental parabolic** if $f^{-1}\tilde{\gamma}f$ is *not* parabolic in Γ.

3.6.11 Proof Schema.

1. Assume that a non-elementary Kleinian group Γ_ρ moves holomorphically with $\rho \in U \subseteq \mathbb{C}$.
2. Then the fixed points of Γ_ρ move holomorphically with ρ since they are solutions to polynomial equations in the matrix coefficients, and do not collide as long as Γ_ρ remains non-elementary and free of accidental parabolics.
3. But the fixed points of a Kleinian group are dense in the limit set, so the limit set moves holomorphically.
4. Thus by Theorem 3.6.9 the Kleinian groups Γ_ρ are all quasiconformally conjugate (and in fact the quasiconformal maps are quasisymmetric on the limit sets, see [9, Section 3.2]).

We will see in Figures 7.3 and 7.4 some snapshots of the holomorphic flows of limit sets of certain Kleinian groups as their matrix coefficients are varied.

3.6E Measured laminations and quasiconformal deformation spaces

Finally we recall some results on the behaviour of $\mathcal{ML}(S)$ as S moves through a quasiconformal deformation space, referencing primarily [61]; an alternative source for the continuity results here is Chapter 6 of [88]. All of this theory is intimitely related to the Fenchel-Nielsen coordinates [46, Part 2, 56, Chapter 3] and Thurston's work on foliations [45]: the main importance of $\mathcal{ML}_0(S)$[8] is that it compactifies the Teichmüller space Teich(S) (this is the **Thurston compactification** [88, §6.1.2])—and we will see concretely that this is exactly what happens on the boundary of the Riley slice (see the remark following the proof of Theorem 7.4.15).

Fix a quasiconformal family of Kleinian groups Γ_ρ with the parameter ρ moving holomorphically through a connected open set $D \subseteq \mathbb{C}$, and let $\Omega^*(\Gamma_\rho)$ be a connected component of $\Omega(\Gamma_\rho)$ (such that as ρ moves, the induced homeomorphisms of $\hat{\mathbb{C}}$ move the $\Omega^*(\Gamma_\rho)$ onto each other).

3.6.12 Lemma (Proposition 3.1 of [61])**.** *All of the surfaces*

$$\mathcal{C}(\Gamma_\rho) = \frac{\partial\, \mathrm{h.conv}\, \Lambda(\Gamma_\rho)}{\Gamma_\rho} \quad \text{and} \quad \frac{\Omega^*(\Gamma_\rho)}{\mathrm{Stab}_{\Gamma_\rho} \Omega^*(\Gamma_\rho)}$$

are homeomorphic for all $\rho \in D$. ∎

3.6.13 Lemma (Section 3.7 of [61])**.** *The space of measured laminations on $\partial\mathcal{C}(\Gamma_\rho)$ is independent of ρ.* ∎

Given Lemmata 3.6.12 and 3.6.13, we denote by S the homeomorphism class of the surfaces $\mathcal{C}(\Gamma_\rho)$ and write $\mathcal{ML}(S)$ for the space of measured laminations on any of the surfaces of that type.

3.6.14 Theorem (Continuity of lamination length: Theorem 4.5 of [61])**.** *For $\rho \in D$, let l_ρ denote the length function of Definition 3.4.5 on $\mathcal{C}(\Gamma_\rho)$. The map*

$$D \times \mathcal{ML}(S) \to \mathbb{R}_{>0}$$

defined by

$$(\rho, \nu) \mapsto l_\rho(\nu)$$

is jointly continuous in both arguments. ∎

[8] Properly this should be $\mathcal{PML}_0(S)$, the quotient of $\mathcal{ML}_0(S)$ by the relation $\sim$ given by $\nu \sim \mu$ iff there exists some nonzero $c \in \mathbb{R}$ such that $\nu = c\mu$.

Finally,

3.6.15 Theorem (Continuity of bending measure: Theorem 4.6 of [61]). *The map*

$$\mathrm{pl} : D \to \mathcal{ML}(S)$$

defined by

$$\rho \mapsto \beta_\rho$$

is continuous, where β_ρ denotes the usual hyperbolic bending measure on $\mathcal{C}(\rho)$. ∎

Though not proved in [61], a very similar argument to the proofs of Theorems 3.6.14 and 3.6.15 gives the next theorem:

3.6.16 Theorem (Continuity of intersection number). *For $\rho \in D$, let i_ρ denote the intersection number function of Definition 3.4.5 on $\mathcal{C}(\Gamma_\rho)$. Fix a simple closed geodesic γ on S, and write γ_ρ for the simple closed geodesic on $\mathcal{C}(\Gamma_\rho)$ which is isotopic to γ. The map*

$$D \times \mathcal{ML}(S) \to \mathbb{R}_{>0}$$

defined by

$$(\rho, \nu) \mapsto i_\rho(\nu, \gamma_\rho)$$

is jointly continuous in both arguments. ∎

Chapter 4

The Riley slice: motivation and definition

In this chapter, we discuss the historical motivation for the study of the Riley slice, and define both the parabolic Riley slice and its elliptic cousins. Our discussion here is informal, and its purpose is to orient the reader before we give an account of the topological and geometric properties of these moduli spaces in the coming chapters.

4.1 Two-bridge links

Recall that a **link** L of m components is a subset of S^3 that consists of m disjoint simple closed curves, each finitely piecewise-linear. A **knot** is a link with only one component. Some nice introductory and reference books for knot theory which we find particularly helpful for the material which we need are those by Cromwell [34], by Crowell and Fox [35], by Lickorish [73], and particularly the recent book by Purcell [99].

The most fundamental topological invariant of a link is its associated complement manifold.

4.1.1 Definition. Let L be a link. A **tubular neighbourhood** of a component k of L is the image U of an embedding of a solid torus into S^3 such that k lies in the interior of the embedding; a tubular neighbourhood N of L is a union of tubular neighbourhoods of each of its components, with the component neighbourhoods chosen to be mutually disjoint. The **link complement manifold** of L is the manifold $S^3 \setminus N$. The **fundamental group** of the link L is the fundamental group $\pi_1(S^3 \setminus N)$. We will often abuse notation and denote this group by $\pi_1(L)$.

Recall now that a **two-bridge link** or a **rational link** ([34, Section 4.10], [99, Chapter 10]) is a nontrivial link which can be embedded (without crossings) in a plane $\Sigma \subseteq \mathbb{R}^3$ apart from two arcs (the 'bridges') whose projections onto the plane consist of two disjoint straight segments; after adding a point at ∞, this is the same as saying that there exists a (topological) 2-sphere $\Sigma \subseteq S^3$ such that

1. L intersects Σ transversely at precisely four points, and
2. if U is either of the 3-balls bounded by Σ, $L \cap U$ projects radially onto Σ as a pair of disjoint arcs (we call the system $(U, U \cap L)$ a **tangle**).

An equivalent definition of two-bridge knots is the following: if D is a diagram of a knot k (viewed as a 4-valent graph together with crossing information) then a **braid region** in D is a sequence of

Figure 4.1: Two two-bridge knots; the braid regions on the left are labelled a_1, a_2, a_3. Diagram from [73, p. 9].

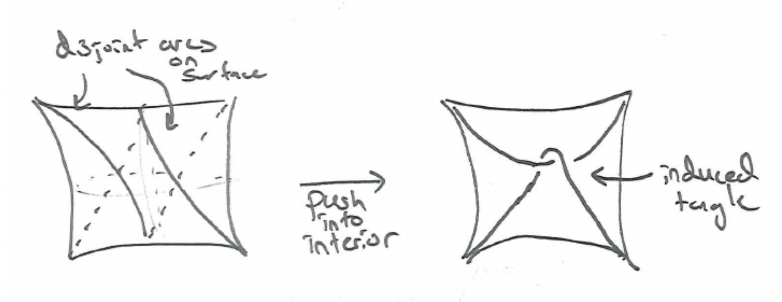


Figure 4.2: Pushing disjoint arcs on the sphere into the interior to form a 2-tangle.

regions $(b_1, ..., b_n)$ of the graph complement such that (1) each b_i is bounded by two edges and two vertices, (2) b_i and b_j are incident iff $|i - j| = 1$ and b_i and b_j meet at a single vertex, (3) overcrossings and undercrossings alternate, and (4) the sequence is of maximal length; a two-bridge link is then a link which admits a diagram D such that (a) every crossing is contained in a braid region, and (b) D admits an embedding into $\mathbb{R}^2$ such that the x coordinate has exactly two local minima. (This description is much clearer when the diagrams are drawn out, as in Figure 4.1.)

There is an important procedure due to Schubert [112] and Conway [32], described in [99, Chapter 10], which gives a surjection φ from $\hat{\mathbb{Q}}$ to the set of isotopy classes of rational links, with the property $\varphi(p/q) \simeq \varphi(r/s)$ iff $p = r$ and either $q \equiv s \pmod{p}$ or $q \equiv s^{-1} \pmod{p}$.

We quickly indicate the computation of the rational number corresponding to a rational link (though we do not prove well-definedness or bijectivity of this representation); this number is known as the **Schubert normal form** or simply **normal form**:

4.1.2 Algorithm. Let L be a two-bridge link, and pick a representation of L in space such that it is embedded on a plane except for two arcs. Complete this plane to a sphere S and push the planar pieces of the curve into the interior slightly to form a ball with four marked points on the boundary and two twisted arcs on the interior (Figure 4.2). Delete the arcs to form a 4-punctured sphere with two deleted interior arcs. By performing Dehn half-twists along the oriented curves $\gamma(0/1)$ and $\gamma(1/0)$ indicated in Figure 4.3,[1], where we have been careful to draw points on the surface paired by the original pair of bridge arcs, *not the interior arcs forming the tangle*, as the two horizontal pairs (top and bottom), we may 'unwind' the knot—starting from the left, unwind each of the braid regions in the knot one at a time, corresponding to alternating the applications of the 'horizontal' or 'vertical' Dehn twist (c.f. Figure 10.1 of [99], but we perform the process in reverse). Let $(a_n, ..., a_1)$ be the sequence of twisting numbers alternating in this way, with the sign of each a_i denoting whether you apply the twist or its inverse. Then the Schubert normal form is the rational number p/q given by

[1]This labelling is compatible with our labelling of geodesics that we will develop in Chapter 6; see Section 6.2.

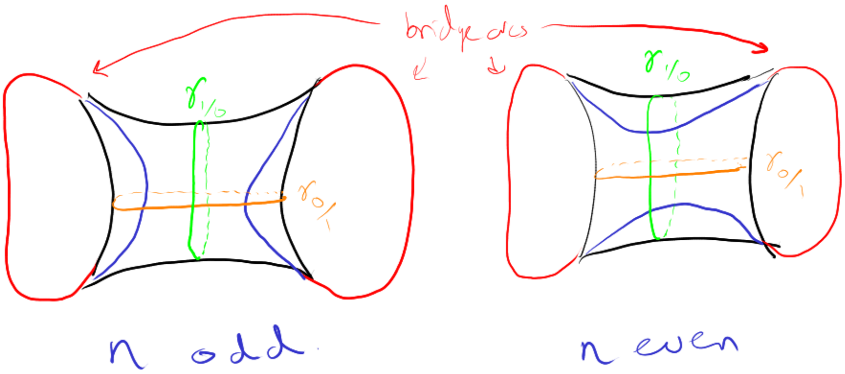


Figure 4.3: The two 'generating' Dehn twists for 2-tangles. Observe that $\gamma(0/1)$ and $\gamma(1/0)$ orient according to the bridge arcs, not the tangle arcs.

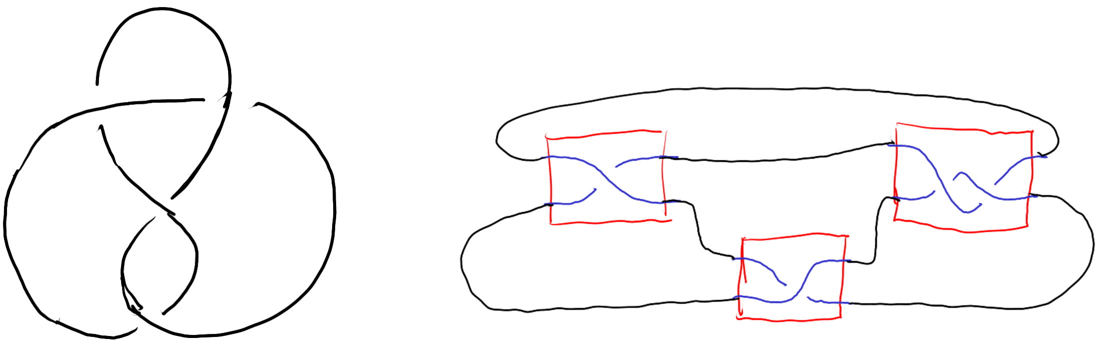


Figure 4.4: The figure 8 knot (left) and a diagram exhibiting that it is two-bridge with Schubert normal form $1 + \frac{1}{1+\frac{1}{2}} = 5/3$ (right).

the continued fraction

$$p/q := [a_n, \ldots, a_1] = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{a_1}}}.$$

The unknotted form of the link after performing this procedure is either an unknot or a trivial link of two components, depending on the parity of n: these are illustrated in Figure 4.3.

For convenience, we define the **slope** of a two-bridge knot to be the reciprocal of the Schubert normal form.

Remark. A useful list of rational knots and links up to 13 crossings was compiled by Herman Gruber; unfortunately it is no longer available on his website, but it may be accessed through the *Internet Archive* [52].

We note that a two-bridge link has either one or two components, according to whether its slope has odd or even numerator respectively [34, Corollary 8.7.8].

One of the simplest examples of a two-bridge link is the figure 8 knot (Figure 4.4). It is important historically as it was the first known example of a link whose complement manifold M admits a hyperbolic structure. In fact, William Thurston (informed in part by evidence presented by Robert

Riley) in the early 1970s proved that "most" knot complements (in a certain sense) admit a hyperbolic structure. The history surrounding this discovery is very interesting but out of scope for this thesis; various accounts of interest are [102] and the accompanying commentary [25], Thurston's account [125], and the additional references given in the historical notes to Section 10.3 of [100] (p.504).

Thurston's proof of the hyperbolicity of M is sketched in [124, Section 3.1], and a detailed description may be found in Section 10.3 of [100]. The manifold that is constructed is very closely related to one first constructed in 1912 by Hugo Gieseking in his PhD thesis [50]; the idea is simply to apply the Poincaré polyhedron theorem to a pair of ideal tetrahedra with some carefully chosen facet-pairings. In addition to the sources just given, we direct the reader to the very nice picture given as Figure 1.9 of [88].

Riley's proof, on the other hand, is more interesting to us at the moment; it is described in [103] (see also the conference proceeding [105] for an overview). It is a consequence of Riley's algebraic theory of parabolic representations of two-bridge link groups (developed in [106, 107]). More precisely, Riley computes as Proposition 1 of [106] an over-presentation (in the sense of Chapter VI of [35]) for the fundamental group G of the figure 8 knot complement of the form

$$G = \langle x, y : wx = yw \rangle \tag{4.1.3}$$

where

$$w = x^{-1}yxy^{-1}.$$

(The cited proposition is applied directly, with the note that the figure 8 knot has Schubert normal form 5/3. One can also compute this presentation from the Wirtinger presentation: this method is detailed in Example (4.3) of Chapter VI of [35]. The advantage of using an over-presentation rather than the Wirtinger presentation is that the over-presentation has only two generators.)

Riley then defined a representation $\theta : G \to \mathrm{PSL}(2, \mathbb{C})$ by

$$\theta x := A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \theta y := B = \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix} \tag{4.1.4}$$

where ω is a primitive cube root of unity. Then Riley's main result is the following:

4.1.5 Theorem (Theorem 1 of [103]). *If $\Gamma := \langle \theta x, \theta y \rangle$ then Γ has presentation*

$$\langle A, B : WAW^{-1} = B \rangle$$

where $W = A^{-1}BAB^{-1}$, and so θ gives an isomorphism $G \simeq \Gamma$. ∎

Riley used Theorem 4.1.5 to prove the hyperbolicity of the manifold M (Corollary on p.284 of [103]), appealing to some results on 3-manifold topology. In general his theory shows that every two-bridge link has a fundamental group with an over-presentation on two generators like Equation (4.1.3) for the figure 8 knot, and that every such group admits a discrete non-abelian representation with the images of the two generators being parabolic (Theorem 2 of [106]). These representations are not necessarily faithful: not every two-bridge link is hyperbolic, but the only exceptions are torus knots [105, Corollary to Theorem 1] (for example, the trefoil knot is two-bridge but its complement has $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$-geometry since it is given by $\widetilde{\mathrm{PSL}(2, \mathbb{R})}/\widetilde{\mathrm{PSL}(2, \mathbb{Z})}$). The Fuchsian groups generated by two parabolic generators were classified by Knapp [64]. In 2002 Agol [1] conjectured the following theorems (Theorems 4.1.6 and 4.1.7) and sketched an incomplete proof with some errors; complete proofs were given by Aimi, Lee, Sakai, and Sakuma [5], and Akiyoshi, Ohshika, Parker, Sakuma, and Yoshida [6].

4.1.6 Theorem. *A non-free Kleinian group G is generated by two non-commuting parabolic elements iff one of the following holds:*

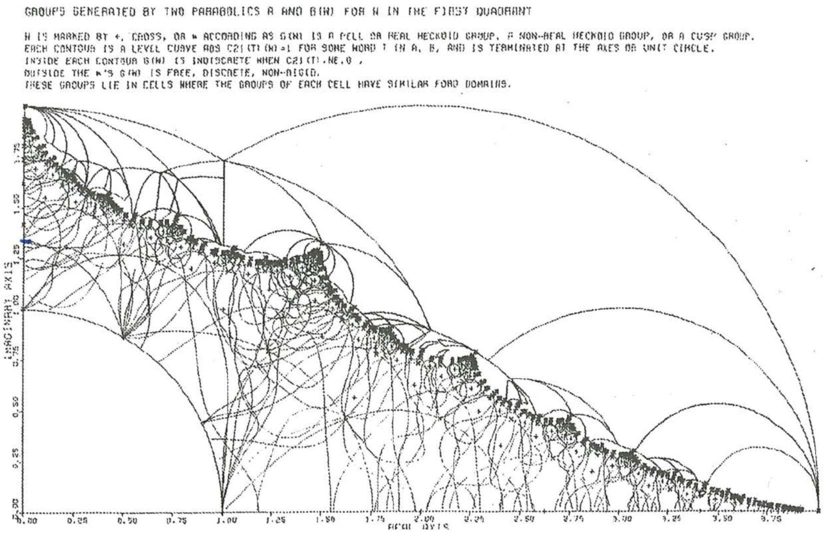


Figure 4.5: Riley's plot of two-bridge link groups in the $(+,+)$-quadrant of $\mathbb{C}$, reproduced from the monograph [7, p. VIII].

1. *G is conjugate to some hyperbolic two-bridge link group; or*
2. *G is conjugate to the Heckoid group*[2] $\mathrm{Heck}(p/q, n)$ *for some* $p/q \in \mathbb{Q}$ *and some* $n \in \frac{1}{2}\mathbb{Z}_{\geq 3}$. ∎

4.1.7 Theorem. *If G is a hyperbolic two-bridge link group then it has exactly two parabolic generating pairs, up to conjugacy. If G is a Heckoid group then it has a unique parabolic generating pair up to conjugacy.* ∎

If X and Y are two parabolics which do not share fixed points (which must be the case, otherwise the group would be elementary) then, up to conjugacy, they are represented by matrices

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y_\rho = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix}$$

where $\rho \in \mathbb{C}$. This normalisation associates a complex number ρ with each two-bridge link group representation. Riley devoted much effort to finding the set of $\rho \in \mathbb{C}$ such that

$$\Gamma_\rho := \langle X, Y_\rho \rangle \tag{4.1.8}$$

is a two-bridge link group, producing the famous image Figure 4.5; we will now spend some time discussing the features of this plot in order to motivate the rest of this thesis.

4.2 The features of Riley's plot

The most obvious feature of Figure 4.5 is the fractal curve running diagonally across the centre of the frame. This is the set of so-called **cusp groups**. In order to explain what these groups are, we must consider the two 'halves' which the curve separates the plot into. The half of the plot on the left (closer to 0) contains (marked with $+$ or $\times$) various **Heckoid groups** (introduced by Riley in [104] and formalised by Lee and Sakuma in [71]). Observe that these points appear to lie on curves which lead to large peaks on the boundary curve. We now explain the construction of these groups.

[2] We will define Heckoid groups in the next subsection.

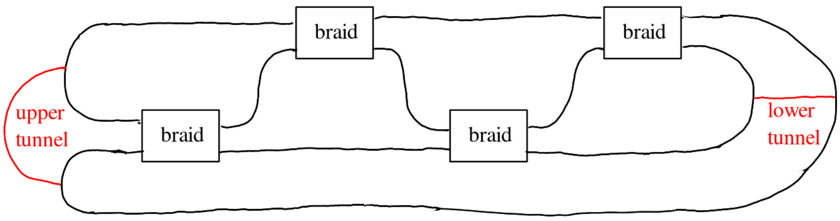


Figure 4.6: Upper and lower unknotting tunnels for a two-bridge knot with four braid regions.

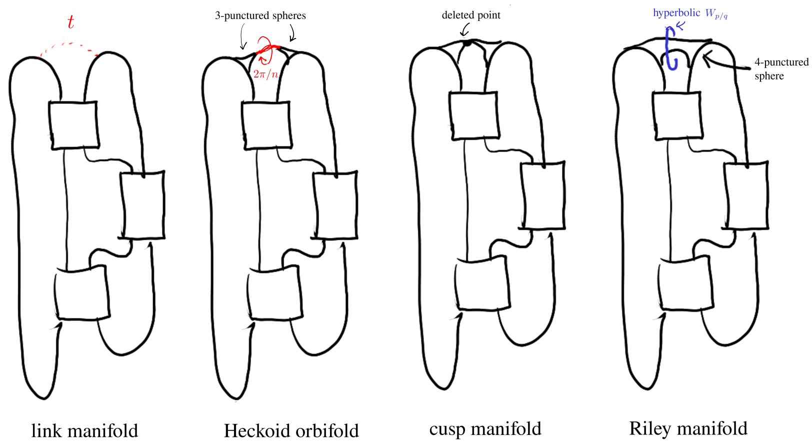


Figure 4.7: The four kinds of orbifolds found along an extended pleating ray.

The idea is very simple: recall that a **unknotting tunnel** for a link L is a properly embedded arc t in the complement manifold M for L such that if N is a tubular neighbourhood of t then $M \setminus N$ is a handlebody; a two-bridge link has two natural unknotting tunnels, depicted in Figure 4.6, called the **upper** and **lower tunnels** [109]. Let L be the two-bridge knot of slope p/q; let $M(p/q, 0)$ be the knot complement manifold for L and $\mathrm{Heck}(p/q, 0)$ be the fundamental group of $M(p/q, 0)$. Now we informally[3] define the **Heckoid orbifold** of of slope p/q and index $n \in \frac{1}{2}\hat{\mathbb{N}}_{\geq 3}$, denoted $M(p/q, n)$, to be the orbifold obtained by replacing the arc $t \subseteq M(p/q)$ with a cone arc of angle $2\pi/n$ (or with a deleted arc if $n = \infty$); the **Heckoid group** of slope p/q and index n, denoted $\mathrm{Heck}(p/q, n)$, is then the orbifold fundamental group $\pi_1(p/q, n)$.

Let $\Gamma_\rho = \langle X, Y_\rho : WX = Y_\rho W\rangle$ be a faithful representation of the two-bridge link group of slope p/q (where W is a word in X and Y_ρ depending on the fraction p/q; compare Equation (4.1.3) and [106, Proposition 1]). The element in Γ_ρ corresponding to a loop around the upper tunnel t is represented by the word $WXY_\rho^{-1}W^{-1}$, which is necessarily trivial (since the tunnel arc is not deleted); the Heckoid group $\mathrm{Heck}(p/q, n)$ has the property that $WXY_\rho^{-1}W^{-1}$ is an elliptic element of order $2\pi/n$, corresponding to replacing the normal points on the tunnel arc with singular points of order n [6, Proposition 3.3]. The representatives for $\mathrm{Heck}(p/q, n)$ in $\mathbb{C}$ therefore lie on the curve $\mathrm{tr}^2(WXY_\rho^{-1}W^{-1}) \in (0, 4]$, where the 'limit' as the parameter goes to 0 is the two-bridge knot group itself, and where the 'limit' as the parameter goes to 4 is a manifold where the cone arc is replaced with a deleted point encircled by a parabolic tunnel word (the corresponding manifold is called a **cusp**

[3]The purpose of this discussion is only to motivate the definition of the Riley slice, and we will not be dealing with Heckoid groups in subsequent chapters; the formal definition, which may be found as Definition 3.2 of [6], involves quite a bit of technical machinery and since we will not need it we omit it.

manifold). We can push the parameter past 4 as well, past the point where the tunnel word turns parabolic: the word turns hyperbolic, and now gives a curve of finite length on the Riemann surface at infinity: the boundary becomes a 4-times punctured sphere with two arcs joining the punctures (and we call the corresponding manifold a **Riley manifold**). This limiting procedure is depicted in Figure 4.7 (following Figure 0.1 of [7]).

We now give some definitions to fix notation for the subsequent discussion. We will redefine all of these in later chapters, and the later definitions agree with the ones given here; the difference is the point of view (namely, in this chapter we are starting in the space of two-bridge link groups and then pushing outwards through the Heckoid orbifolds to reach the moduli space of 4-punctured spheres; in later chapters, we will be studying the latter moduli space intrinsically and we will replace these definitions with ones coming from the 4-punctured sphere geometry).

4.2.1 Definition. If $p/q \in \mathbb{Q}$, then:

1. The **admissable loop** of slope p/q, denoted $\gamma_{p/q}$, is the (homotopy class of a) loop surrounding the upper tunnel of the two-bridge link of slope p/q in its complement manifold. If $\rho \in \mathbb{C}$ is such that Γ_ρ is a Heckoid group of slope p/q and index n (for some n), then $\gamma_{p/q}$ is the (orbifold homotopy class of a) loop around the cone arc of the corresponding orbifold. Finally, if Γ_ρ has the property that $\mathcal{S}(\Gamma_\rho)$ is a 4-times punctured sphere (perhaps not even obtained by pushing out from a two-bridge link group of slope p/q) then define $\gamma_{p/q}$ in the following way: take the admissible loop of slope p/q in the Heckoid orbifolds, this naturally corresponds to a homotopy class of loops in the corresponding 3-manifold with boundary a 4-times punctured sphere; these loops can be pushed onto the boundary and so we obtain a homotopy class of simple closed curves on the 4-times punctured sphere which separate the punctures into pairs; now for each possible hyperbolic structure on the 4-punctured sphere there is a unique geodesic in this homotopy class; and we define $\gamma_{p/q}(\rho)$ to be this unique geodesic on the 4-times punctured sphere $\mathcal{S}(\Gamma_\rho)$. (We will give a much easier and more useful definition in Chapter 6.)

2. The **Farey word** of slope p/q is the word $\mathrm{Word}(p/q) := WXY_\rho^{-1}W^{-1}$ (where W is the word in X and Y_ρ coming from the two-bridge knot representation). Observe that $\mathrm{Word}(p/q)$ depends on the value of ρ; if we wish to emphasise this dependence we will write $\mathrm{Word}(p/q) = \mathrm{Word}(p/q)(\rho)$.

3. The **Farey polynomial** of slope p/q is $\Phi_{p/q} := \operatorname{tr} \mathrm{Word}(p/q)(\rho)$, which is a polynomial of degree q in the variable ρ.

4. The **extended rational pleating ray** of slope p/q, $\mathcal{EP}(p/q)$ is the connected component of $\Phi_{p/q}^{-1}(\mathbb{R})$ which contains the Heckoid groups of slope p/q.

5. The **rational pleating ray** of slope p/q, $\mathcal{P}(p/q)$ is the subset of $\mathcal{EP}(p/q)$ corresponding to $\Phi_{p/q}^{-1}((-\infty, -2))$.

6. The **cusp group** of slope p/q is the group Γ_ρ on the extended pleating ray of slope p/q with $\Phi_{p/q}(\rho) = -2$.

Looking back at Riley's plot, Figure 4.5, the fractal boundary is the closure of the set of cusp groups, and the empty space in the upper right half of the plot is the moduli space of the four-times punctured spheres.

4.2.2 Definition. The **parabolic Riley slice**, which we will often shorten to **Riley slice** and denote by $\mathcal{R}$, is the set of $\rho \in \mathbb{C}$ such that the group Γ_ρ of Equation (4.1.8) is free, discrete, and $\mathcal{S}(\Gamma_\rho)$ is a 4-times punctured sphere. We denote by $\overline{\mathcal{R}}$ the closure of $\mathcal{R}$, $\mathcal{R} \cup \partial\mathcal{R}$.

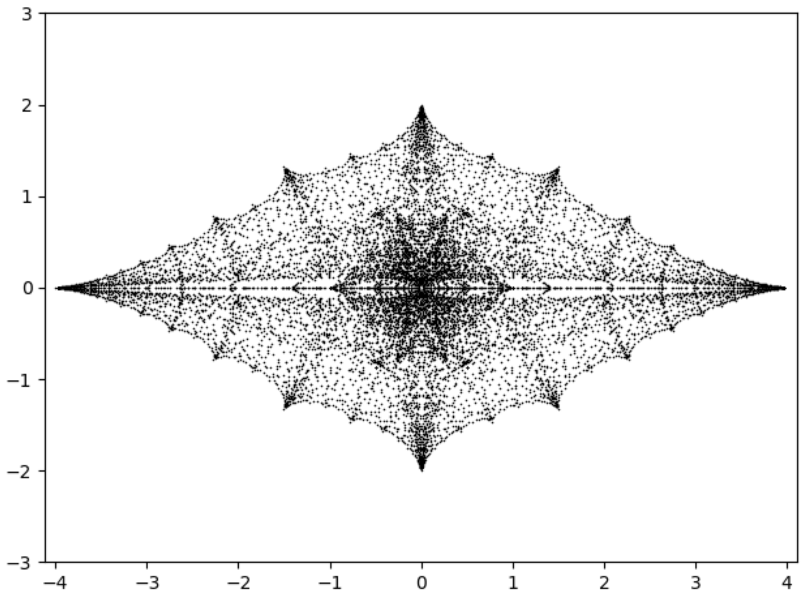


Figure 4.8: Shaded is the Riley slice *exterior* (that is, the area of $\mathbb{C}$ containing the Heckoid groups); the portion of the boundary in the northeasterly quadrant of this figure is the curve of Figure 4.5.

This set and the group Γ_ρ were also studied prior to the 1990s by Sanov [110], Brenner [23], Chang, Jennings, and Ree [29], Ree [101], Lyndon and Ullman [74], Lyubich and Suvorov [75] (who termed the discreteness problem for Kleinian groups on two parabolic generators the **eye problem** due to the shape of $\mathbb{C} \setminus \mathcal{R}$), and Maskit and Swarup [87]. Work subsequent to that of the papers by Keen, Komori, and Series includes papers and talks by Beardon [11, 12], Gehring, Maclachlan, and Martin [48], Bamberg [10], Agol [1], Gilman [51], Ohshika and Miyachi [96], Aimi, Lee, Sakai, and Sakuma [5], Akiyoshi, Ohshika, Parker, Sakuma, and Yoshida [6], Martin [81, 82], and Elzenaar, Martin, and Schillewaert [39] (see Chapter 8).

The Riley slice $\mathcal{R}$ consists of the area *outside* the shaded region of Figure 4.8. For the sake of saying something concrete (and giving at least some credibility to Figure 4.8), we will prove the following bound:

4.2.3 Lemma. *If* $|\rho| < 1$, *then* $\rho \notin \mathcal{R}$.

Proof. Suppose $\rho \in \mathcal{R}$, so Γ_ρ is discrete. The Shimizu–Leutbecher inequality (Corollary 3.1.11) applied to X and Y_ρ is exactly $|\rho| \geq 1$. ∎

Of course, this bound is quite bad. A slightly better one is given in Theorem 8.2.1, and some bounds for sets that are essentially the same as the elliptic Riley slice exterior can be found in [82] (see in particular Figure 1).

We also note that Γ_ρ is elementary iff $\rho = 0$ (since this is the only situation where the fixed points of the generators collide). In the sequel we will apply results to the groups Γ_ρ which require the groups to be elementary; the reader should exclude $\rho = 0$ in all of these situations. (Of course, 0 does not lie in $\mathcal{R}$ by Lemma 4.2.3 so this is hardly ever something one needs to think about.)

In the next chapter we will study the topological and analytical properties of the Riley slice: for instance, we will show that it is an open set and that it is actually the quasiconformal deformation

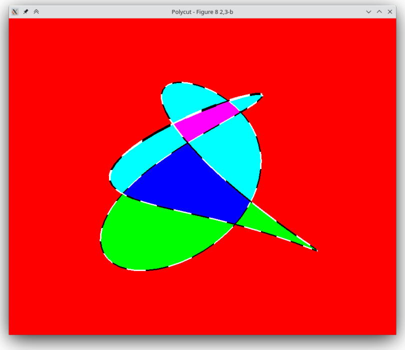


Figure 4.9: A branched cover with branching curve the figure 8 knot in S^3, generated by the Polycut software [21]; different colours represent different sheets of the cover.

space of any of the groups in its interior. Then, in Chapter 7 we will study the more modern theory of the slice due to Keen and Series [63] which uses the rational pleating rays to describe the geometry of the moduli space.

4.3 Elliptic Riley slices

In this thesis, one of the primary goals is to generalise the theory of the Riley slice to the case that X and Y_ρ are elliptic. This corresponds to the following geometric situation: instead of considering the complement manifold M of a link L, we take an orbifold O supported on S^3 with L as the singular locus. More precisely, we take an orbifold induced by viewing L as a branching curve, and gluing different S^3-sheets together through the 'portals' of the knot. This idea is due to Thurston, in particular his 1992 talk *Knots to Narnia* [123]. Various attempts have been made to visualise this geometric phenomenon, including the software *Polycut* [21] (see Figure 4.9) and a virtual reality model by Sümmerman [122].

To make this precise algebraically, for $\rho \in \mathbb{C}$ and $a, b \in \mathbb{N}$ we define $\Gamma_\rho^{a,b}$ to be the group generated by

$$X = \begin{bmatrix} \exp(\pi i/a) & 1 \\ 0 & \exp(-\pi i/a) \end{bmatrix}, Y_\rho = \begin{bmatrix} \exp(\pi i/b) & 0 \\ \rho & \exp(-\pi i/b) \end{bmatrix};$$

note that, with the convention $\pi i/\infty \coloneqq 0$ we may view the parabolic groups Γ_ρ of the previous section as limiting cases.

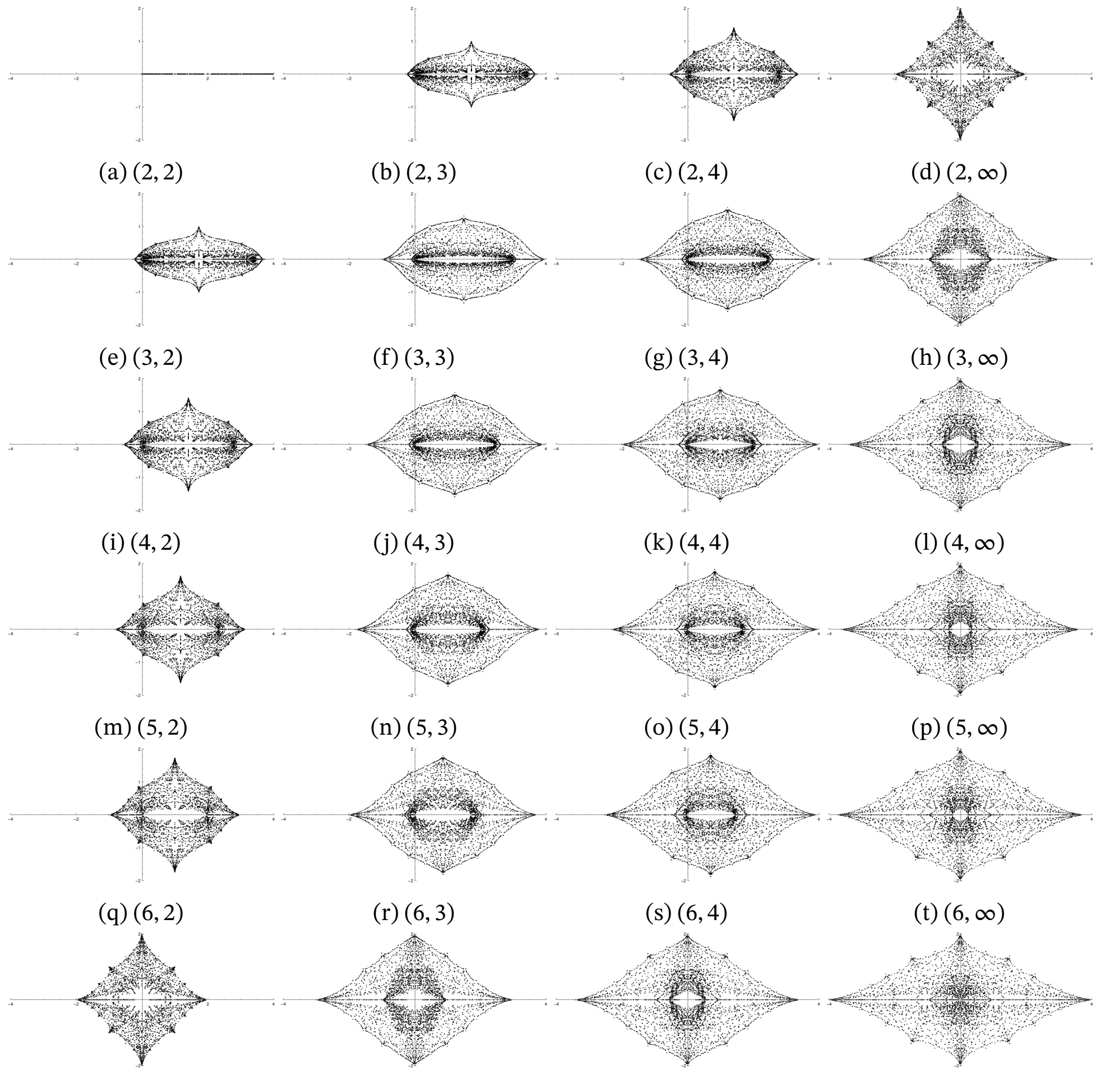
Warning. The matrices X and Y have respective orders a and b. This is because $X^a = Y^b = -I$ and the ambient group which $\langle X, Y_\rho \rangle$ is taken within is $\mathrm{PSL}(2, \mathbb{C})$, *not* $\mathrm{SL}(2, \mathbb{C})$.

The idea now is that for certain values of ρ, the orbifold $\mathbb{H}^3/\Gamma_\rho^{a,b}$ has singular locus a two-bridge link, with one of the two bridges corresponding to a branching of order a and the other a branching of order b (the generators X and Y_ρ here are analogous to A and B in [123], where Thurston takes all the branching orders equal to 2). We may then define elliptic Heckoid groups as in the previous section, giving a sequence of orbifolds precisely analogous to those of Figure 4.7 but with the cusp orbifolds bounded by a pair of 3-marked spheres with one pair of marked points (the pair surrounded by the tunnel loop) being punctures and the other marked points being the endpoints of two cone arcs in the orbifold of respective cone angle $2\pi/a$ and $2\pi/b$; the Riley orbifolds are then bounded by

4-times marked spheres with one pair of marked points being cone points of angle $2\pi/a$ joined by an order a cone arc and the other pair being of angle $2\pi/b$ and joined by an order b cone arc. We then make the analogous definitions to Definition 4.2.1 and Definition 4.2.2; in particular,

4.3.1 Definition. The **elliptic Riley slice** for cone orders a and b, which we denote by $\mathcal{R}^{a,b}$, is the set of $\rho \in \mathbb{C}$ such that $\Gamma_\rho^{a,b}$ is discrete, isomorphic to $\mathbb{Z}/(a) * \mathbb{Z}/(b)$ and $\mathcal{S}(\Gamma_\rho)$ is supported on a sphere with four cone points, two of order a and two of order b. We denote by $\overline{\mathcal{R}^{a,b}}$ the closure of $\mathcal{R}^{a,b}$, $\mathcal{R}^{a,b} \cup \partial\mathcal{R}^{a,b}$. We also identify $\mathcal{R}$ with $\mathcal{R}^{\infty,\infty}$, and we allow any combination of finite and infinite values for a, b.

Some illustrations of the elliptic Riley slice exteriors may be found as Figure 4.10 (the generating computer code may be found in Example A.1.3). Intitial work towards understanding the elliptic Riley slices was carried out in [128].



(a) $(2, 2)$ (b) $(2, 3)$ (c) $(2, 4)$ (d) $(2, \infty)$

(e) $(3, 2)$ (f) $(3, 3)$ (g) $(3, 4)$ (h) $(3, \infty)$

(i) $(4, 2)$ (j) $(4, 3)$ (k) $(4, 4)$ (l) $(4, \infty)$

(m) $(5, 2)$ (n) $(5, 3)$ (o) $(5, 4)$ (p) $(5, \infty)$

(q) $(6, 2)$ (r) $(6, 3)$ (s) $(6, 4)$ (t) $(6, \infty)$

(u) $(\infty, 2)$ (v) $(\infty, 3)$ (w) $(\infty, 4)$ (x) (∞, ∞)

Figure 4.10: Riley slices for various cone point orders (a, b).

Chapter 5

Analytical and topological properties

In this chapter, we begin with the definition given in the previous chapter (Definition 4.3.1) and deduce topological and analytic properties of the Riley slices $\mathcal{R}^{a,b}$. We do not rely on any of the motivating discussion of the previous chapter in our proofs: while we motivated the Riley slice via deforming knot groups (moving 'from the outside in'), it is far easier to work 'from the inside out' with the purely Kleinian-group-theoretic definition.

Our main results are the following:

- The Riley slices as defined in Definition 4.3.1 are equal to the quasiconformal deformation spaces of any group in the interior (Lemma 5.1.7);
- $\mathcal{R}^{a,b}$ is a connected set, homeomorphic to an open annulus (Corollary 5.1.6);
- The boundary $\partial\mathcal{R}^{a,b}$ (as a subset of $\hat{\mathbb{C}}$) consists of precisely two varieties of group: a group Γ on the boundary is either a Kleinian group of the first kind (i.e. $\Omega(\Gamma) = \emptyset$), or represents a surface consisting of a pair of thrice-marked spheres, with at least one marking on each sphere being parabolic (the *cusp groups* of Definition 4.2.1); this is proved in Theorem 5.2.4.

5.1 The quasiconformal deformation space

Throughout this section, fix $\rho \in \mathcal{R}^{a,b}$ and set $\Gamma = \Gamma_\rho^{a,b}$ (where $a, b \in \mathbb{N}_{\geq 2}$ and $\max\{a, b\} \geq 3$).

5.1.1 Proposition. *Let* $[\tilde{\Gamma}, f] \in \mathrm{QH}(\Gamma)$ *(where* $[\tilde{\Gamma}, f]$ *represents the equivalence class of* $(\tilde{\Gamma}, f)$*); then we may choose the representative* $(\tilde{\Gamma}, f)$ *such that* f *fixes the points* $\{0, X(0), \infty\}$*. With this choice, there exists some* $\tilde{\rho} \in \mathbb{C}$ *such that*

$$\tilde{\Gamma} = \langle X, Y_{\tilde{\rho}} \rangle.$$

The map $i : \mathrm{QH}(\Gamma) \to \hat{\mathbb{C}}$ *defined by* $f \mapsto \tilde{\rho}$ *where* f *is the unique representative defined above is a well-defined injective map.*

Proof. Let $(w_1, w_2, w_3) \coloneqq (f(0), f(1), f(\infty))$; since Möbius transformations are triply transitive we may define a function t by defining $(t(w_1), t(w_2), t(w_3)) \coloneqq (0, 1, \infty)$; now set $f' \coloneqq tf$. If $\tilde{\Gamma}'$ is the conjugate group induced by f', then the map $t = f'f^{-1}$ induces an equivalence between $(\tilde{\Gamma}, f)$ and

$(\tilde{\Gamma}', f')$ (with a trivial homotopy that satisfies the quasiconformality conditions by virtue of everything being 0-quasiconformal), and f' fixes $(0, 1, \infty)$. We may then replace $(\tilde{\Gamma}, f)$ with $(\tilde{\Gamma}', f')$.

Assuming now that f fixes $(0, 1, \infty)$, $f^{-1}Xf$ is a Möbius transformation which fixes ∞ (so is upper triangular), has the same order and type as X, and sends $0 \mapsto X(0)$; this is enough to force $f^{-1}Xf = X$. We also see that $f^{-1}Y_\rho f$ is of the same order and type as Y_ρ and fixes 0 so is of the form $Y_{\tilde{\rho}}$ for some $\tilde{\rho}$. This shows that the map $i : \mathrm{QH}(\Gamma) \to \hat{\mathbb{C}}$ is well-defined; and it is clearly injective. ∎

5.1.2 Proposition. *If $\tilde{\rho} \in \mathcal{R}^{a,b}$, then there exists a path $\gamma : [0,1] \to \mathcal{R}^{a,b}$ such that*

1. *$\gamma(0) = \rho$ and $\gamma(1) = \tilde{\rho}$*
2. *For all $t \in [0,1]$ there exists a quasiconformal homeomorphism $w(t) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $(\Gamma_{\gamma(t)}, w(t))$ is a quasiconformal conjugate of Γ.*

In particular, $\mathcal{R}^{a,b}$ is a path connected quasiconformal deformation space.

Remark. For this proposition, see also [121, Theorem 2] (observe that torsion-free is not assumed) and [87].

Proof. Let $\tilde{\rho} \in \mathcal{R}^{a,b}$. Then $\mathcal{S}(\Gamma_{\tilde{\rho}})$ is a 4-times punctured sphere and so (considering the Teichmüller space of such surfaces) there exists a quasiconformal mapping $f : \mathcal{S}(\Gamma_{\tilde{\rho}}) \to \mathcal{S}(\Gamma_\rho^{a,b})$; lift f to $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and let μ be the Beltrami coefficient of this mapping; by the measurable Riemann mapping theorem (Theorem 3.6.1) we may solve the Beltrami equation

$$\partial_{\bar{z}} f_t = t\mu(z)\partial_z f_t$$

for each $t \in [0,1]$; in particular, f_0 is a conformal map, and $f_1 = \tilde{f}$ (by the uniqueness part of the measurable Riemann mapping theorem). This gives a path in $\mathrm{Teich}(\mathcal{S}(\Gamma_\rho^{a,b}))$ which projects to the desired path in $\mathrm{QH}(\Gamma_\rho^{a,b})$. ∎

5.1.3 Corollary. *For any $\rho \in \mathcal{R}^{a,b}$, $\mathcal{R}^{a,b} = i(\mathrm{QH}(\Gamma_\rho^{a,b}))$.*

Proof. By Proposition 5.1.2, $\mathcal{R}^{a,b} \subseteq \mathrm{QH}(\Gamma_\rho^{a,b})$. On the other hand, if $\tilde{\rho} \in \mathrm{QH}(\Gamma_\rho^{a,b})$ then $\mathcal{S}(\Gamma_{\tilde{\rho}})$ is a Kleinian group with $\mathcal{S}(\Gamma_{\tilde{\rho}}) = \mathcal{S}(\Gamma_\rho^{a,b})$; since $\Gamma_{\tilde{\rho}}$ is isomorphic to $\Gamma_\rho^{a,b}$ it is isomorphic to $\mathbb{Z}/(a) * \mathbb{Z}/(b)$. Combining these observations, we see $\tilde{\rho} \in \mathcal{R}^{a,b}$ which proves the opposite inclusion. ∎

5.1.4 Corollary. *No group in $\mathcal{R}^{a,b}$ contains accidental parabolics.*

Proof. First note that the groups are quasi-Fuchsian (e.g. $\Gamma_{57}^{a,b}$ is Fuchsian for any a, b); then apply [83, p. IX.D.17]. ∎

5.1.5 Corollary. $\dim \mathrm{QH}(\Gamma_\rho^{a,b}) = 1$.

Proof. By Theorem 3.6.5 and Theorem 3.6.6 $\mathrm{QH}(\Gamma_\rho^{a,b})$ is a quotient of $\mathrm{Teich}(\mathcal{S}(\Gamma_\rho^{a,b}))$ by a discontinuous group action and so $\dim \mathrm{QH}(\Gamma_\rho^{a,b}) = \dim \mathrm{Teich}(\mathcal{S}(\Gamma_\rho^{a,b})) = 1$. ∎

In fact, $\mathrm{QH}(\Gamma_\rho^{a,b}) = \mathrm{Teich}(\mathcal{S}(\Gamma_\rho^{a,b}))/\langle\omega\rangle$ where ω is the Dehn twist along a geodesic loop bounding a compression disc in $\mathcal{S}(\Gamma_\rho^{a,b})$. The element ω is a parabolic element of the mapping class group (see [46, p. 383]), so the quotient is a sphere with one puncture corresponding to ω and one deleted disc (see Figure 5.1, where the indicated horoball is that of Lemma 3.3.3); in particular,

5.1.6 Corollary. $\mathrm{QH}(\Gamma_\rho^{a,b})$ *is topologically an annulus.* ∎

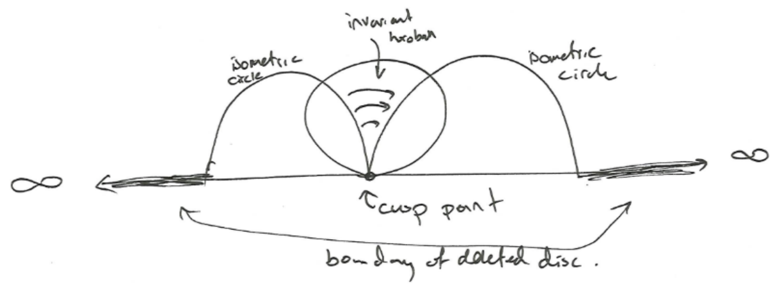


Figure 5.1: The action of an elementary parabolic group on $\mathbb{H}^2$.

Corollary 5.1.6 is also an easy consequence of Theorem 7.4.15 which we will prove using the Keen–Series theory.

Fix some $\rho \in \mathcal{R}^{a,b}$, and let X and Y_ρ denote the generators of Γ_ρ as above. We have seen that there is a natural bijection (namely, i) between $\mathcal{R}^{a,b}$ and $\mathrm{QH}(\Gamma_\rho^{a,b})$.

5.1.7 Lemma. *The map i is a holomorphic embedding of* $\mathrm{QH}(\Gamma_\rho^{a,b})$ *into $\mathbb{C}$ for all $\rho \in \mathcal{R}^{a,b}$.*

Proof. We will show that i^{-1} is holomorphic following Proof Schema 3.6.11. Recall that, by Theorem 3.6.8, if ρ is allowed to move holomorphically in $\mathcal{R}^{a,b}$ then the fixed points of Γ_ρ move holomorphically (since they are given by polynomial equations in the matrix coefficients) and therefore (since the fixed points are dense in the limit set) the limit set moves holomorphically; by the λ-lemma, this holomorphic motion extends to a holomorphic motion of $\hat{\mathbb{C}}$ and therefore assigns quasiconformal deformations of Γ_ρ to points near to ρ in $\mathbb{C}$ in a holomorphic map; by uniqueness of the representation we chose in the definition of i, this assignment must be i^{-1}.

Of course for this proof to work we need to check that for $\rho \in \mathcal{R}^{a,b}$ then Γ_ρ is (a) non-elementary and (b) has no accidental parabolics: property (a) is true since the two generators X, Y_ρ have nontrivial commutator as long as $\rho \neq 0$, and (b) is true since every surface in the slice has exactly four marked points. ∎

The guiding question which was studied by Keen and Series [63] in the parabolic case, and which we will study for the remainder of this thesis, is the following:

5.1.8 Question. What is the *geometry* of $\mathcal{R}^{a,b}$ induced by the embedding i?

Remark. In the sequel, we will implicitly identify $\mathcal{R}^{a,b}$ and $\mathrm{QH}(\Gamma_\rho^{a,b})$ for some fixed $\rho \in \mathcal{R}^{a,b}$ under the map i.

5.2 Topology of the boundary

In this section, we discuss some results about the structure of the boundary of the Riley slice. We state these results for the general situation of $\mathcal{R}^{a,b}$, but the proofs we cite only mention the case $a = b = \infty$. The majority of the proofs transfer without significant incident to the torsion case, and we discuss those where this might not be true.

5.2.1 Lemma. *Every group $\Gamma = \Gamma_\rho^{a,b}$ for $\rho \in \partial\mathcal{R}^{a,b}$ is discrete and isomorphic to the groups in the interior of $\mathcal{R}^{a,b}$.*

Proof. This is immediate from the observation that convergence in the quasiconformal deformation space is exactly algebraic convergence. ∎

For the next set of results we apply the theory of Bers and Maskit [15, 84] on deformation spaces of quasi-Fuchsian groups (that is, groups which are quasiconformally conjugate to Fuchsian groups—or, equivalently, have limit set lying on a quasicircle); since there are Fuchsian groups in each Riley slice (e.g. $\Gamma_{72}^{a,b}$ lies in every slice) the Riley slices are quasi-Fuchsian deformation spaces.

Remark. Since the time of the classical papers on boundaries of deformation spaces by Bers, Maskit, and Swarup (among others) [15, 84, 87], much work has been done to develop powerful hammers that can deal with the degenerate groups which might lie on the boundaries. In particular, the **density conjecture** (that every group on the boundary is an algebraic limit of groups on the interior) and the **ending lamination conjecture** (that hyperbolic manifolds with finitely generated fundamental group are determined by their conformal boundary and their ending laminations) are germane to our study of deformation spaces. A very nice discussion of these modern results (and others) may be found in Section 5.4 and subsequent sections of [79], as well as the final sections of Chapter 4 of [88].

Given a quasi-Fuchsian deformation space $D = \mathrm{QH}(F)$ (F Fuchsian), a **cusp group** is a group on the boundary ∂D which has an accidental parabolic.

We first give a result which comes from the set of results of the type 'geometrically finite groups on the boundary appear only by pinching nontrivial loops down'; it appears in [84] (essentially it is a result of the discussion culminating in Theorems 5 and 6 of that paper).

5.2.2 Theorem. *If $\rho \in \partial\mathcal{R}^{a,b}$ contains at least three conjugacy classes of non-loxodromic elements, then Γ_ρ is a cusp group.* ∎

We now recall the famous **Ahlfors measure zero theorem**, which was conjectured by Ahlfors in 1966 and proved by him when the group G is geometrically finite [4]; Canary [27] proved the theorem for the case that G is 'geometrically tame', and so the full theorem follows from Marden's tameness conjecture (for which see the discussion starting on p.292 of [79]).

5.2.3 Theorem (Ahlfors measure zero theorem)**.** *The limit set of a finitely generated Kleinian group G has either the entire Riemann sphere, or is of measure 0.* ∎

For a modern, readable proof in the geometrically finite case see [88, §3.2.2].

The proof of the following theorem makes up the main part of [87], which also shows that every group in $\overline{\mathcal{R}}$ with $\Omega(G) \neq \varnothing$ is geometrically finite. One should note that, at the time of this paper, the Ahlfors measure zero conjecture was not yet proven and so the authors must rely on some heavy machinery of Sullivan [119] to see that the quasiconformal deformations in the space are not supported on the limit set. Since everything in $\overline{\mathcal{R}^{a,b}}$ is finitely generated we may apply Theorem 5.2.3 to skip this difficulty when reading the paper. In fact, as a consequence of Bowditch's discussion of Maskit's planarity theorem [20], every point in an elliptic Riley slice corresponds to a geometrically finite group, and so we only need the 'easy' version of the measure zero theorem. Bowditch also shows that every complex structure on the 4-marked sphere has a realisation in one of the Riley slices.

5.2.4 Theorem. *Every group $\Gamma = \Gamma_\rho^{a,b}$ for $\rho \in \partial\mathcal{R}^{a,b}$ is either 'of the first kind' (i.e. $\Lambda(\Gamma) = \hat{\mathbb{C}}$), or $\mathcal{S}(\Gamma)$ is a disjoint union of a pair of spheres, each with three marked points.* ∎

The following theorem is due to McMullen [90] and resolved a classical conjecture of Bers [15] (who proved, at least, that cusps exist and are of null measure in the boundary):

5.2.5 Theorem. *Cusp groups are dense in $\partial\mathcal{R}^{a,b}$.* ∎

We are sure that an even stronger result holds, if cusp groups are defined in terms of the Keen–Series pleating laminations; this appears as Conjecture 10.2.2.

The next theorem is proved by Ohshika and Miyachi as Theorem 1.2 of [96] as a consequence of the ending lamination theorem. We believe that a similar result for the torsion case might be obtained by modifying the proof of the special case of the ending lamination theorem given in that paper to allow for torsion generators.

5.2.6 Theorem. *The boundary $\partial\mathcal{R}^{\infty,\infty}$ is a topological circle, and the Riley slice $\mathcal{R}^{\infty,\infty}$ is the interior of its closure.* ∎

It is also believed by experts that the Riley slice has outward cusps; in the cases of the Earle slice (see [65]) and the Maskit slice (see [62]) the analogous result is known to be true [93]. Some related results were proved by Lyubich and Suvorov: if $K = \mathbb{C} \setminus \mathcal{R}^{\infty,\infty}$, then K is the closure of its interior [75, Corollary 1] and K is the closure of the set of ρ such that Γ_ρ is non-discrete; the set of ρ such that Γ_ρ has torsion is also dense in K [75, Theorem 2].

Chapter 6

Curves on the 4-marked sphere

The goal of the Keen-Series theory is to produce a foliation of the Riley slice which represents geometric properties of the quasiconformal deformation structure. We saw in the previous chapter that $\mathcal{R}$ is the quotient of Teich$(S_{0,4})$ by the Dehn twist along a curve bounding a compression disc, and in Section 1.1 and Chapter 4 we gave some geometric intuition for this: a four-marked sphere produced by a non-elementary group generated by two non-loxodromic elements is made up of two discs glued along a common curve. A dense set of fibres in the Keen-Series foliation will be curves along which the corresponding surfaces are deformed by shrinking along this gluing curve; this chapter is devoted to studying the gluing curves on the surface from a topological viewpoint.

Our discussion in this section is drawn from several sources; primarily we referenced the papers [6, 63, 66] and the monograph [7], but some parts of the theory are developed in [94] (for the 1-punctured torus case which is almost immediately applicable) and [18].

6.1 Geodesic coding on the 4-marked sphere

In discussing the coordinate system on $\mathcal{R}^{\infty,\infty}$ introduced in [63] and in generalising this to the case where we allow elliptic generators, it is necessary to be able to connect easily elements of the group Γ_ρ to geodesics on the surface $\Omega(\Gamma_\rho)/\Gamma_\rho$. The basic ideas are very classical and date back to Dehn's algorithm on surfaces (see for instance Chapter 6 of [118]) via the work of Birman and Series [18, 113] on geodesic coding on hyperbolic surfaces. Our treatment below also draws from a similar discussion for the moduli space of once-punctured tori in Chapter 1 of [7]). All of the results which we give below appear, in the parabolic case, in [63] with some corrections in [66].

Draw an ideal quadrilateral $\mathcal{P}$ on the 4-marked sphere S, as in Figure 6.1; $\mathcal{P}$ defines a polygonal complex on S with two faces. In the figure, we have labelled each face-edge pair so that every edge is labelled with a single letter in a different case on each face; this is done compatibly, so on one face the edges are labelled X, Y, W, Z and on the other x, y, w, z. Let α be a non-boundary-parallel curve on S (so $S \setminus \alpha$ is a disjoint union of two twice-marked discs); there is a unique geodesic of minimal length in the same homotopy class as α (we denote the set of all of these geodesics bounding pairs of marked points by $C(S)$). Let P be a point on this geodesic which lies in the interior of a face, and choose an orientation of the geodesic; now walk along the geodesic from P, writing down at every edge-crossing the label associated to the face you are entering. The result is a word in the symbols $\{X, x, Y, y, W, w, Z, z\}$. For a fixed $\rho \in \mathcal{R}^{a,b}$, define a map $\mathrm{Word}_{\mathcal{P}} : C(S) \to \Gamma_\rho^{a,b}$ by sending $(X, x, Y, y) \mapsto (X^{a,b}, (X^{a,b})^{-1}, Y_\rho^{a,b}, (Y_\rho^{a,b})^{-1})$, sending $(W, w, Z, z) \mapsto (1, 1, 1, 1)$, and then declaring the image of two concatenated words to be the product of the images of the adjoined words.

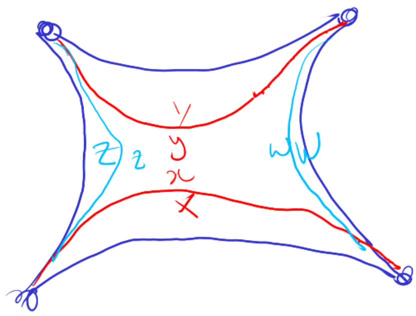


Figure 6.1: A labelled ideal quadrilateral on the 4-marked sphere.

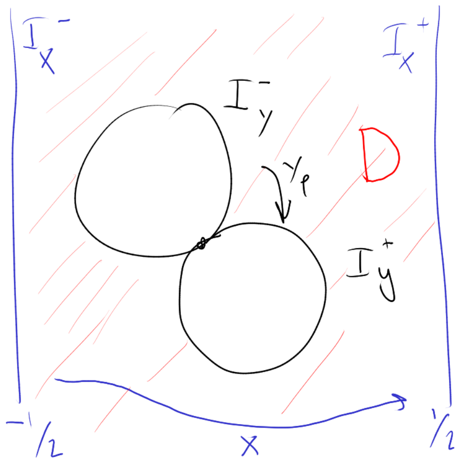


Figure 6.2: Fundamental domain for a Riley group with two parabolic generators.

The following theorem shows that, for $\rho \in \mathcal{R}^{a,b}$ with $|\rho|$ sufficiently large, there is a natural choice for $\mathcal{P}$ which reflects the group structure.

6.1.1 Theorem. *Let $I_Y^+ = S(|\rho|^{-1}, -\beta^{-1}\rho^{-1})$ and $I_Y^- = S(|\rho|^{-1}, \beta\rho^{-1})$ be the isometric circles of Y_ρ. If X is parabolic, let $I_X^\pm = \{z \in \mathbb{C} : \Im z = \pm 1\} \cup \{\infty\}$ and assume that ρ is sufficiently large that $I_Y^{\pm 1}$ lie entirely within the vertical strip $\{z \in \mathbb{C} : -1 < \Re z < 1\}$. If X is elliptic, then suppose ρ is sufficiently large that we may choose two lines $I_X^{\pm 1}$ through $(\alpha^{-1} - \alpha)^{-1}$, making an angle of $2\pi/p$ with the property that $I_Y^{\pm 1}$ are contained within one of the open cones of angle $2\pi/p$ of $\mathbb{C}$ cut out by the lines. Let $^\circ I_g^\alpha$ be the hyperbolic plane erected above I_g^α ($\alpha = \pm 1$, $g = X, Y_\rho$).*

Let $^\circ D$ be the (connected) hyperbolic polyhedron bounded by the domes $^\circ I_g^\alpha$, and let $D = \hat{\mathbb{C}} \cap \overline{^\circ D}$.

1. *The polyhedron $^\circ D$ is a fundamental polyhedron for Γ_ρ acting on $\mathbb{H}^3$.*

2. *D is a fundamental domain for Γ_ρ acting on $\Omega(\Gamma_\rho)$.*

3. *The orbifold $\mathcal{M}(\Gamma_\rho)$ has fundamental group Γ_ρ; it is the interior of a topological sphere with four marked points arranged in pairs, each pair represented by one of the two generators of the group; the two points corresponding to a generator are joined by either a deleted arc in the manifold (if*

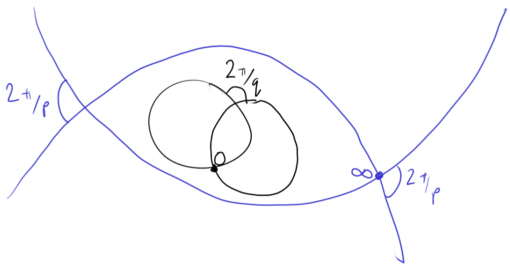


Figure 6.3: Fundamental domain for a Riley group with two elliptic generators, of orders p and q.

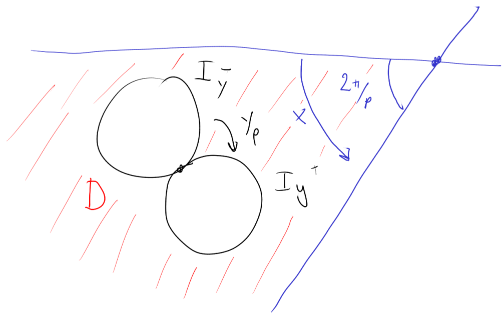


Figure 6.4: Fundamental domain for a Riley group with one elliptic and one parabolic generator.

the generator is parabolic), or an arc of singular points with a cone angle of $2\pi/n$ (if the generator is elliptic of order n).

Fundamental domains are sketched in Figures 6.2 to 6.4.

Proof. Parts (1) and (2) of the result follow from Proposition 3.5.3, since X and Y_ρ pair $I_X^\pm$ and $I_Y^\pm$ respectively, and each maps D entirely off itself. To see that the orbifold has the fundamental group Γ_ρ, we note that $\mathbb{H}^3$ is simply connected and so by the covering space theory recalled above $\Gamma_\rho = \pi_1(\mathbb{H}^3/\Gamma_\rho)$. Finally the remark about the topological type of the manifold follows from the Poincaré polyhedron theorem, as in the discussion immediately following Proposition 3.5.3. ∎

6.1.2 Corollary. *Every group in $\mathcal{R}^{a,b}$ is geometrically finite.*

Proof. By Theorem 6.1.1, there is a group in $\mathcal{R}^{a,b}$ which is geometrically finite; by [83, VI.E.7], every quasiconformal deformation of a geometrically finite group is geometrically finite. ∎

Recall that, if $f \in \mathrm{PSL}\,2, \mathbb{C}$ is loxodromic, then its **complex translation length** is the number

$$\operatorname{trlen} f := 2 \operatorname{arccosh} \frac{\operatorname{tr} f}{2};$$

the two real numbers $\Re \operatorname{trlen} f$ and $\Im \operatorname{trlen} f$ respectively give the translation length of the action of f on its axis in $\mathbb{H}^3$, and the angle which f rotates points about its axis (the so-called **holonomy** of the action). A nice exposition of these quantities is found as Section 12.1 of [76].

6.1.3 Lemma. *Suppose that $\rho \in \mathcal{R}^{a,b}$ satisfies the hypothesis of Theorem 6.1.1, and let S be the corresponding Riemann surface; let $\mathcal{P}$ be the ideal quadrilateral consisting of the projections of the isometric circles as defined in that theorem. For every $\alpha \in C(S)$, $\mathrm{Word}_{\mathcal{P}}(\alpha)$ is an element of $\Gamma_\rho^{a,b}$ which preserves the lift of α to $\Omega(\Gamma_\rho^{a,b})$; accordingly, if $\mathrm{Word}_{\mathcal{P}}(\alpha)$ is hyperbolic, then the length of α in the quotient metric is exactly the translation length of $\mathrm{Word}_{\mathcal{P}}(\alpha)$.*

Proof. This is immediate from the definition of $\mathrm{Word}(p/q)$ as a cutting sequence. ∎

6.1.4 Lemma. *Suppose that $\rho \in \mathcal{R}^{a,b}$ satisfies the hypothesis of Theorem 6.1.1, and let S be the corresponding Riemann surface, let $\mathcal{P}$ be the ideal quadrilateral consisting of the projections of the isometric circles as defined in that theorem, and let $\alpha \in C(\mathcal{S}(\Gamma_\rho^{a,b}))$. Let $\tilde\rho \in \mathcal{R}^{a,b}$ be arbitrary, and let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the quasiconformal conjugate sending Γ_ρ to $\Gamma_{\tilde\rho}$. Then $f\,\mathrm{Word}_{\mathcal{P}}(\alpha) f^{-1}$ in $\Gamma_{\tilde\rho}^{a,b}$ leaves $f\alpha$ invariant.*

Proof. This follows because f is a homeomorphism, so acts sufficiently nicely on both the loops on the surface and the elements of the fundamental group which represent them; this action is of course compatible with the conjugation action on the group, since one is defined in terms of the other. ∎

By Proposition 5.1.1, $f\,\mathrm{Word}_{\mathcal{P}}(\alpha) f^{-1}$ is the word in the two generators of $\Gamma_{\tilde\rho}^{a,b}$ given by exactly the same sequence of $X^{\pm 1}$ and $Y^{\pm 1}$ as $\mathrm{Word}_{\mathcal{P}}(\alpha)$, just with the substitution of $\tilde\rho$ for ρ. Thus we may define $\mathrm{Word}(\alpha)$ for each ρ as the word in $\Gamma_\rho^{a,b}$ corresponding to this sequence.

We now introduce an enumeration of simple closed non-boundary-parallel geodesics by $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. This is done by passing to a universal cover of S; it must be stressed that this is a purely topological process, and the cover does *not* lift the geometry of S. Cutting S along three of the four arcs in the polygonal complex gives a hexagon; identify this hexagon with the polygon in $\mathbb{R}^2$ with vertices $(0,0), (0,1), (0,2), (1,2), (1,1), (1,0)$, labelling the edges of this hexagon according to the face-edge labelling on S; then tessellate $\mathbb{R}^2$ according to the induced edge pairing (e.g. the edge labelled X is glued to that labelled x), as in Figure 6.5. (We will not worry about whether the vertices of the tiling

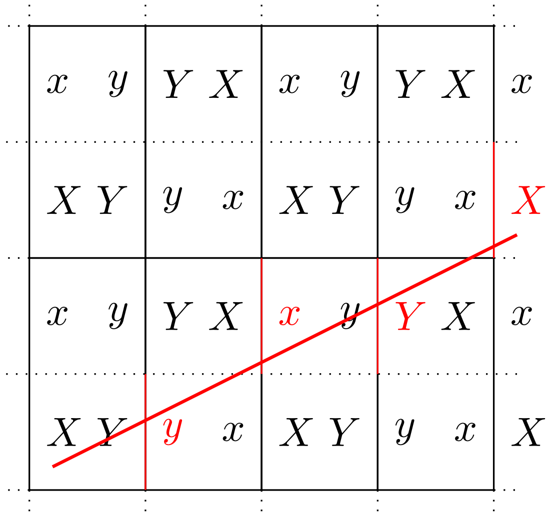


Figure 6.5: Topological cover of the 4-marked sphere induced by the polygonal complex of Figure 6.1, together with (in red) a line segment of slope 1/2 which projects exactly onto a simple closed non-boundary-parallel curve on S and which allows us to read off from the red labels that $\text{Word}(1/2) = yxYX$.

are deleted or marked in any way: when we 'lift' geodesics, they will always be lifted away from the vertex lattice.)

Define now a map $\gamma : \hat{\mathbb{Q}} \to C(S)$ as follows: if $p/q \in \hat{\mathbb{Q}}$, then pick a point $P \in (0,1)^2$ such that the line L in $\mathbb{R}^2$ of slope p/q which passes through P does not hit any points of $\mathbb{Z}^2$; then L projects down to a curve on S which bounds two marked points (namely, the curve with the same cutting sequence on S as L has on the lifted grid) and there is a unique geodesic $\gamma(p/q)$ in $C(S)$ in the same homotopy class; some examples are seen in Figure 6.6.

We wish to extend this theorem to allow irrational slopes. Observe that any nontrivial simple closed geodesic on the 4-marked sphere must bound either two discs containing resp. one and three marked points, or two discs each containing two marked points. In the latter case, there is a unique geodesic in that isotopy class of minimal length; but in the former case there is no geodesic of minimal length, and all geodesics are isotopic to the stationary curve at the marked point of the singly-marked disc. Thus no geodesics of the latter type may lie in $\mathcal{ML}_0(S)$. This shows that $C(S) = \mathcal{ML}_0(S)$ (since clearly any two geodesics of the first type must intersect).

6.1.5 Theorem. *Let S be a 4-marked sphere. Then $\mathcal{ML}(S)$ is in natural bijective correspondence with $\hat{\mathbb{R}}$ via a map $\gamma : \hat{\mathbb{R}} \to \mathcal{ML}(S)$, such that the restriction $\gamma\restriction_{\hat{\mathbb{Q}}}$ is exactly the bijective correspondence between $\hat{\mathbb{Q}}$ and $\mathcal{ML}_0(S)$ just defined.*

Proof. Define γ exactly as in the rational case: if $\lambda \in \mathbb{R}$, then $\gamma(\lambda)$ is the projection of any line passing through $(0,1)^2$ of slope λ in $\mathbb{R}^2$ which does not hit any lattice points of $\mathbb{Z}^2$ to the surface S via the side-pairing indicated in Figure 6.5. If λ is rational, then one obtains an element of $\mathcal{ML}_0(S)$ as just defined; if λ is irrational, the projection is dense in a 2-dimensional compact subset away from the singular part of S. ∎

The composition of $\gamma : \hat{\mathbb{Q}} \to C(S)$ followed by $\text{Word} : C(S) \to \Gamma_\rho^{a,b}$ gives a map $\hat{\mathbb{Q}} \to \Gamma_\rho^{a,b}$ which we also denote by Word. Observe that $\text{Word}(\infty) = 1$. (Of course, we might also define cutting sequences for irrational slopes; in this case, we obtain biinfinite words in the letters X, x, Y, y rather

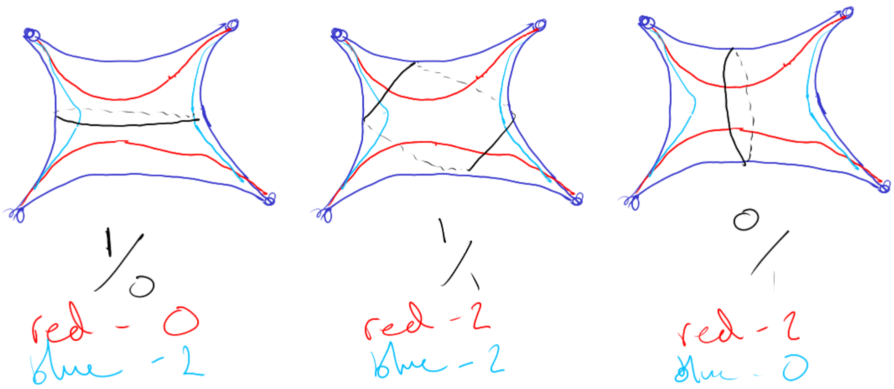


Figure 6.6: Three of the simplest rational curves on the sphere: $\gamma(0/1)$, $\gamma(1/1)$, and $\gamma(1/0)$; the red and blue arcs are the respective lifts of the vertical and horizontal lines of the lattice spanned by $\mathbb{Z}^2$, and we have indicated the intersection number of each curve with these arcs to illustrate their dependence on p/q.

than words representing group elements: see [18] for an interesting discussion of the combinatorial group theory.)

6.1.6 Lemma (Structure of Word(p/q)). *Suppose $p/q, r/s \in \hat{\mathbb{Q}}$.*[1]

1. *(Structure of image words.) Suppose $q \neq 0$. Then* Word(p/q) *is a word of length $2q$, alternating between $X^{\pm 1}$ and $(Y_\rho^{a,b})^{\pm 1}$, such that the number of times the exponent flips between ± 1 overall is the residue of $|p|$ mod $2q$.*
2. *(Defined mod 2.) If $p/q \equiv \pm r/s \pmod 2$ then* Word(p/q) = Word(r/s).
3. *(Almost injective mod 2.) If $p/q \not\equiv \pm r/s \pmod 2$ then* Word(p/q) $\neq$ Word(r/s).

Proof. (1) follows from careful study of the lattice. (2) follows from observing firstly that $p/q + 2 = (p+2q)/q$ so by (1) the two words have the same length, and then observing that the letter appearing at any given place in the word depends only on the 'height' mod 2. Similar arguments show that the lines of slope p/q and $2 - p/q$ have the same word. (3) follows immediately from (1). ∎

Remark. In [63], part (3) of Lemma 6.1.6 is incorrectly stated (Remark 2.5). This is corrected in [66] (Theorem 1.2).

The word Word(p/q) is called the **Farey word of slope** p/q; the trace tr Word(p/q) is a polynomial in the indeterminate ρ known as the **Farey polynomial of slope** p/q and we denote it here by $\Phi_{p/q}^{a,b}(\rho)$. The properties of Lemma 6.1.6 carry over to the polynomials: q is the degree of $\Phi_{p/q}^{a,b}$, and (in the parabolic case where 'sign' is meaningful) p measures the number of sign changes in the coefficient sequence. We will study the structure of the Farey polynomials more closely in Chapter 9; for the time being we simply note the following symmetry which will be useful in subsequent chapters:

6.1.7 Lemma. *Let* Word(p/q) *be a Farey word; then the word consisting of the first $2q-1$ letters of* Word(p/q) *is conjugate to X or Y according to whether the qth letter of* Word(p/q) *is $X^{\pm 1}$ or $Y^{\pm 1}$ (i.e. according to whether q is even or odd respectively)*

[1]We observe the convention here, and subsequently, that the statement "$p/q \in \hat{\mathbb{Q}}$" implicitly implies that $(p,q) = 1$ if $q \neq 0$.

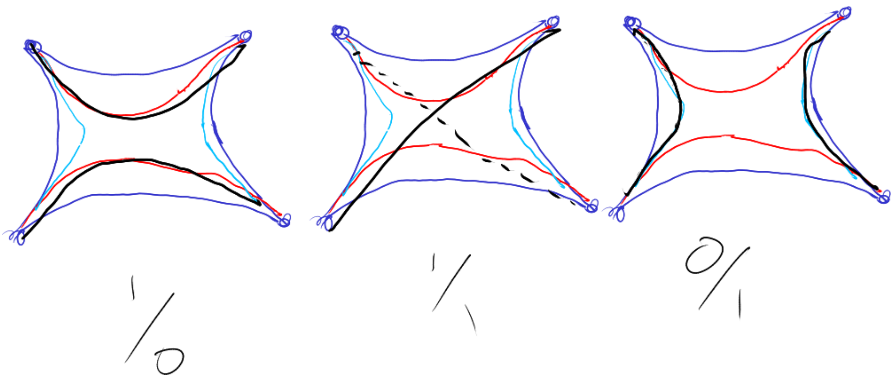


Figure 6.7: Three of the simplest rational arcs on the sphere: $\beta(0/1)$, $\beta(1/1)$, and $\beta(1/0)$. As in Figure 6.6, the interaction with the lifts of the horizontal and vertical lines in the lattice determine (or are determined by) the slope p/q.

Proof. This identity comes from considering the rotational symmetry of the line of slope p/q about the point (q, p); it is clear from the symmetry of the picture that the first $p-1$ letters of Word(p/q) are obtained from the $(p+1)$th to $(2p-1)$th letters by reversing the order and swapping the case (imagine moving the line down by ϵ to hit the point (q, p), rotating the line by 180 degrees onto itself, and then moving the line back up; and observe the motion of the labelling). ∎

6.1.8 Example. In red in Figure 6.5 we see that

$$\text{Word}(1/2) = yxYX = (Y_\rho^{a,b})^{-1}(X^{a,b})^{-1}Y_\rho^{a,b}X^{a,b};$$

this has one sign flip in the exponents (from -1 to 1) and is of length 2×2. The Farey words for $q \le 12$ are listed in Table 6.1 and the Farey polynomials for $q \le 4$ are listed in Table 6.2.

It will be useful also to consider a version of these results which allow curves through the marked points. A **rational arc** is a curve on the marked sphere which is the union of two disjoint simple arcs, each joining two marked points (and both arcs disjoint even at the endpoints, so all four marked points are covered).

6.1.9 Proposition. *Let $A(S)$ be the set of isotopy classes in $S = \mathcal{S}(\Gamma_\rho^{a,b})$ of rational arcs. Then A is in bijective correspondence with $\hat{\mathbb{Q}}$.*

Proof. The proof proceeds in precisely the same way as the case of simple closed curves, except now we consider the lines of rational slope which originate at $(0,0)$. The added complication is that each slope corresponds to two possible lines (shifted vertically by 1/2). See Figure 6.7 (and compare with Figure 6.6 to see how the curves and arcs with corresponding slopes relate). ∎

We write $\beta(p/q)$ for the rational arc of slope p/q.

Table 6.1: Farey words Word(p/q) for small q. See Example A.1.2.

p/q	Word(p/q)
0/1	yX
1/1	YX
1/2	$yxYX$
1/3	$yXYxYX$
2/3	$yxyXYX$
1/4	$yXyxYxYX$
3/4	$yxyxYXYX$
1/5	$yXyXYxYxYX$
2/5	$yXYxyXyxYX$
3/5	$yxYXYxyXYX$
4/5	$yxyxyXYXYX$
1/6	$yXyXyxYxYxYX$
5/6	$yxyxyxYXYXYX$
1/7	$yXyXyXYxYxYxYX$
2/7	$yXyxYxyXyXYxYX$
3/7	$yXYxyXYxYXyxYX$
4/7	$yxYXyxyXYxyXYX$
5/7	$yxyXYXYxyxYXYX$
6/7	$yxyxyxyXYXYXYX$
1/8	$yXyXyXyxYxYxYxYX$
3/8	$yXYxYXyxYxyXyxYX$
5/8	$yxYXYxyxYXyxyXYX$
7/8	$yxyxyxyxYXYXYXYX$
1/9	$yXyXyXyXYxYxYxYxYX$
2/9	$yXyXYxYxyXyXyxYxYX$
4/9	$yXYxyXYxyXyxYXyxYX$
5/9	$yxYXyxYXYxyXYxyXYX$
7/9	$yxyxYXYXYxyxyXYXYX$
8/9	$yxyxyxyxyXYXYXYXYX$
1/10	$yXyXyXyXyxYxYxYxYxYX$
3/10	$yXyxYxyXyxYxYXyXYxYX$
7/10	$yxyXYXyxyxYXYxyxYXYX$
9/10	$yxyxyxyxyxYXYXYXYXYX$
1/11	$yXyXyXyXyXYxYxYxYxYxYX$
2/11	$yXyXyxYxYxyXyXyXYxYxYX$
3/11	$yXyxYxYXyXYxYxyXyXYxYX$
4/11	$yXYxYXyXYxyXyxYxyXyxYX$
5/11	$yXYxyXYxyXYxYXyxYXyxYX$
6/11	$yxYXyxYXyxyXYxyXYxyXYX$
7/11	$yxYXYxyxYXYxyXYXyxyXYX$
8/11	$yxyXYXYxyxyXYXyxyxYXYX$
9/11	$yxyxyXYXYXYxyxyxYXYXYX$
10/11	$yxyxyxyxyxyXYXYXYXYXYX$
1/12	$yXyXyXyXyXyxYxYxYxYxYxYX$
5/12	$yXYxyXyxYXyxYxyXYxYXyxYX$
7/12	$yxYXyxyXYxyxYXyxYXYxyXYX$
11/12	$yxyxyxyxyxyxYXYXYXYXYXYX$

Table 6.2: Farey polynomials $\Phi^{a,b}_{p/q}(z)$ for small q. Here, $\alpha = \exp(\pi i/a)$ and $\beta = \exp(\pi i)/b$.

p/q	
$0/1$	$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z$
$1/1$	$\alpha\beta + \frac{1}{\alpha\beta} + z$
$1/2$	$2 + \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z + z^2$
$1/3$	$\frac{1}{\alpha\beta} + \alpha\beta + \left(3 - \frac{1}{\alpha^2} - \alpha^2 - \frac{1}{\beta^2} - \beta^2 + \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2}\right) z + \left(\alpha\beta - 2\frac{\alpha}{\beta} - 2\frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z^2 + z^3$
$2/3$	$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \left(-3 + \alpha^2 + \frac{1}{\alpha^2} - \frac{1}{\alpha^2\beta^2} - \alpha^2\beta^2 + \beta^2 + \frac{1}{\beta^2}\right) z + \left(-2\alpha\beta - \frac{2}{\alpha\beta} + \frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) z^2 - z^3$
$1/4$	$2 + \left(\frac{\alpha}{\beta^3} - \frac{\alpha^3}{\beta^3} + \frac{2}{\alpha\beta} - 3\frac{\alpha}{\beta} + \frac{\alpha^3}{\beta} + \frac{\beta}{\alpha^3} - 3\frac{\beta}{\alpha} + 2\alpha\beta - \frac{\beta^3}{\alpha^3} + \frac{\beta^3}{\alpha}\right) z + \left(6 - \frac{2}{\alpha^2} - 2\alpha^2 - \frac{2}{\beta^2} + 3\frac{\alpha^2}{\beta^2} - 2\beta^2 + 3\frac{\beta^2}{\alpha^2}\right) z^2 + \left(\frac{1}{\alpha\beta} - 3\frac{\alpha}{\beta} - 3\frac{\beta}{\alpha} + \alpha\beta\right) z^3 + z^4$
$3/4$	$2 + \left(\frac{1}{\alpha^3\beta^3} - \frac{1}{\alpha\beta^3} - \frac{1}{\alpha^3\beta} + \frac{3}{\alpha\beta} - 2\frac{\alpha}{\beta} - 2\frac{\beta}{\alpha} + 3\alpha\beta - \alpha^3\beta - \alpha\beta^3 + \alpha^3\beta^3\right) z + \left(6 - \frac{2}{\alpha^2} - 2\alpha^2 - \frac{2}{\beta^2} + \frac{3}{\alpha^2\beta^2} - 2\beta^2 + 3\alpha^2\beta^2\right) z^2 + \left(\frac{3}{\alpha\beta} - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + 3\alpha\beta\right) z^3 + z^4$

6.2 The geodesic representation in terms of Dehn twists

We have given two definitions of the curves $\gamma(p/q)$: one in this chapter (as the geodesic obtained by projecting a line of slope p/q from a cover to the 4-marked sphere), and one implicitly in Section 1.1 and Section 4.1, as the curve obtained when unknotting a two-bridge link while keeping track of the Dehn twist action on $\gamma(1/0)$. In this section, we check that these are the same.

Recall that we use the notation $[a_n, ..., a_1]$ to stand for the simple[2] continued fraction

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}};$$

Later, in Section 9.3, we will study these decompositions in the context of computing Farey polynomials using the Farey sums which we define in the next section.

We will be dealing with slopes of two-bridge links, which we defined to be equal to the reciprocal of the Schubert normal form; the Schubert normal form was defined in terms of a continued fraction decomposition, and so taking the reciprocal we end up with another continued fraction decomposition

$$[0, a_n, ..., a_1] = \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}}.$$

We now recall a standard fact from classical number theory.

6.2.1 Proposition ([54, Theorem 162]). *Every rational number can be expressed as a finite simple continued fraction in exactly two ways, one with an even and one with an odd number of convergents (number of sequence elements a_n). These are of the form*

$$[a_1, ..., a_{N-1}, a_N, 1] \text{ and } [a_1, ..., a_{N-1}, a_N + 1]$$

respectively, for some N. ∎

The **length** of the continued fraction decomposition of $[a_1, ..., a_{N-1}, a_N, 1] = [a_1, ..., a_{N-1}, a_N+1]$ is defined to be N. For example, the length of the decomposition

$$1 + \cfrac{1}{1 + \cfrac{1}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

is 3.

In stating the following result, we use the notation from Section 3.6B to denote Dehn half-twists: namely, if ω is a closed curve on a surface S, then we write σ_ω for the Dehn half-twist along ω. We write σ_ω^k for the k-fold application of the half-twist (or the $-k$-fold application of the inverse half-twist, if $k < 0$). Given a two-bridge knot represented by a 2-tangle inside a 4-marked 2-sphere and a pair of disjoint bridge arcs projected onto the sphere, write $\upsilon(1/0)$ for the unique nontrivial simple closed curve on the 2-sphere which does not intersect the bridge arcs, and write $\upsilon(0/1)$ for the unique nontrivial simple closed curve which intersects both arcs exactly once (so $\upsilon(0/1)$ and $\upsilon(1/0)$ agree with $\gamma(0/1)$ and $\gamma(1/0)$ respectively, as defined in Figure 4.3. Now define $\upsilon(p/q)$ for other $p/q \in \hat{\mathbb{Q}}$ via the following algorithm (c.f. Algorithm 4.1.2):

6.2.2 Algorithm. Let $[a_n, ..., a_1]$ be the unique continued fraction decomposition of p/q with $a_1 \neq 1$.

[2] The adjective 'simple' in this context means that the denominators are all taken to be 1 and that all of the a_n are positive.

1. Let $i = 1$, and let $\omega \coloneqq \upsilon(1/0)$.
2. If i is odd, replace ω with $\sigma_{\upsilon(0/1)}^{a_i}\omega$.
3. Otherwise, if i is even, replace ω with $\sigma_{\upsilon(1/0)}^{a_i}\omega$.
4. If $i = n$ then conclude that $\upsilon(p/q) = \omega$ and terminate, otherwise set i to $i + 1$ and go back to (2).

Now the main result is the following:

6.2.3 Theorem. *The two functions γ and υ agree on $\hat{\mathbb{Q}}$.*

Recall in preparation for the proof that we have a fixed ideal quadrilateral $\mathcal{P}$ on the 4-marked sphere S, and that we label the opposing sides W and Z, and X and Y (Figure 6.1).

Proof. The curve $\upsilon(p/q)$ is a non-boundary-parallel geodesic on the 4-marked sphere which cuts the quadrilateral $\mathcal{P}$ finitely many times, and so is equal to some $\gamma(r/s)$. The latter is uniquely determined by the number of times it cuts each pair of opposite edges of the quadrilateral (it cuts the W and Z edges r times each, and the X and Y edges s times each). Thus, we just need to compute the intersubsection numbers of $\upsilon(p/q)$ with these edges. Suppose that r_i/s_i is the cutting number fraction obtained at the ith step; then by induction in the case that i is even we have

$$\frac{r_{i+1}}{s_i+1} = \frac{r_i + a_{i+1}s_i}{s_i} = [0, a_i, ..., a_1] + a_{i+1} = [a_{i+1}, a_i, ..., a_1]$$

and in the case that i is odd

$$\frac{r_{i+1}}{s_i+1} = \frac{r_i}{s_i + a_{i+1}r_i} = \frac{1}{a_{i+1} + \frac{s_i}{r_i}} = [0, a_{i+1}, a_i, ..., a_1].$$

The first equality in each case follows from considering the geometric action of the Dehn twist on the curve at the ith step: the key point is that we are alternating between 'horizontal' and 'vertical' twists (see Figure 1.1 for a step-by-step example of this). In any case, this completes the proof, since it shows that the cutting fraction at the nth step is $[(0), a_n, ..., a_1] = p/q$; i.e. $\upsilon(p/q) = \gamma(p/q)$. ∎

6.3 The Farey triangulation and spines

Recall that $\mathrm{PSL}(2,\mathbb{Z})$ acts as a group of isometries on $\mathbb{H}^2$. The orbits of the ideal triangle spanned by

$$(1/0, 1/1, 0/1) = (\infty, 1, 0)$$

under $\mathrm{SL}(2,\mathbb{Z})$ form a simplicial complex tiling $\mathbb{H}^2$ which is called the **Farey triangulation**; we denote this by $\mathcal{D}$. Given a simplicial complex $\mathcal{C}$ we write $\mathcal{C}(n)$ for the set of n-faces, and we use $\preceq$ for the face relation. In particular, $\mathcal{D}(0) = \hat{\mathbb{Q}}$, and the objects of $\mathcal{D}(2)$ are triangles with vertex triples of the form $\{p/q, (p+r)/(q+s), r/s\}$ where $ps - qr = \pm 1$. This second assertion follows immediately from the definition of $\mathcal{D}$ as a tessellation: if $\{a/b, c/d, e/f\} \in \mathcal{D}(2)$, then there exists some matrix

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$$

which acts on the 'seed triangle' like

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

which gives immediately that $(a/b, c/d, e/f) = (p/q, (p+r)/(q+s), r/s)$. The operation

$$(p/q, r/s) \mapsto (p+r)/(q+s)$$

will be fundamental to our later study; it is called the **mediant operation** or **Farey addition**. We write $(p/q) \oplus (r/s)$ for the Farey addition of p/q to r/s. We will be careful to only combine p/q and r/s in this way if they satisfy the determinant condition $ps - qr = \pm 1$ as above (in which case they are both in least terms); we call two such fractions **Farey neighbours**. Farey addition has many useful properties: for instance, the mediant of two Farey neighbours is also a Farey neighbour of each summand (an easy calculation). We will also need the following lemma which appears in [54, §3.3]:

6.3.1 Lemma. *If p/q and r/s are Farey neighbours with $p/q < r/s$, then $p/q < (p/q) \oplus (r/s) < r/s$, and $(p/q) \oplus (r/s)$ is the unique fraction of minimal denominator between p/q and r/s. More precisely, let u/v be any fraction in $(p/q, r/s)$; then there exist two positive integers λ, μ such that*

$$u = \lambda p + \mu r \text{ and } v = \lambda r + \mu s$$

(so of course the minimal denominator is obtained when $\lambda = \mu = 1$). ∎

The action of $\mathrm{SL}(2, \mathbb{Z})$ on the lattice $\mathbb{Z}^2$ also descends to an action on $S_{0,4}$, since it is a subgroup of the group Γ of $\frac{1}{2}\mathbb{Z}^2$ rotations exhibiting $S_{0,4}$ as a quotient.

6.3.2 Lemma. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$, let $p, q \in \mathbb{Z}$ be coprime, and define $r, s \in \mathbb{Z}$ by*

$$\begin{bmatrix} r \\ s \end{bmatrix} = A \begin{bmatrix} p \\ q \end{bmatrix}.$$

Then $\gamma(r/s) = A\gamma(p/q)$ and $\beta(r/s) = A\beta(p/q)$.

Proof. Clearly if A sends $(p, q) \in \mathbb{Z}^2$ to (r, s) then A sends the line of slope p/q to the line of slope r/s. ∎

We now study the relation between a triangle $\Delta \in \mathcal{D}(2)$ and the corresponding triples of curves on $S_{0,4}$. It is easiest to consider the rational arcs first and then deduce the results for curves: this is based on the analogous theory for the once-punctured torus (see section 1.2 of [7]; the theory for the 4-punctured sphere will likely appear in the second volume, [8], which has not yet appeared at the time of writing).

6.3.3 Proposition.

1. *Let $\sigma \in \mathcal{D}(2)$ have vertices $p_1/q_1, p_2/q_2, p_3/q_3$. Then the rational arcs $\beta(p_i/q_i)$ are all mutually disjoint (except perhaps at marked points).[3] Further, the union of the three arcs determines a so-called **ideal triangulation** of $S_{0,4}$: the surface, cut along the three arcs, falls apart into a disjoint union of four 2-simplices (Figure 6.8). The simplicial complex spanned by the three arcs is denoted by* $\mathrm{trg}(\sigma)$.
2. *Let $\lambda \in \mathcal{D}(1)$ have vertices p/q, r/s. Then the rational arcs $\beta(p/q), \beta(r/s)$ determine an **ideal polygonal decomposition** of $S_{0,4}$ into a pair of ideal quadrilaterals; this complex is denoted by* $\mathrm{trg}(\lambda)$ *(extending* trg *to a map of face complexes reversing inclusion).*

[3]There is a technical point here: of course we can homotope one curve to cross another curve an additional two times by pulling a loop across. What we mean here is to choose representatives from each homotopy class with minimum intersection.

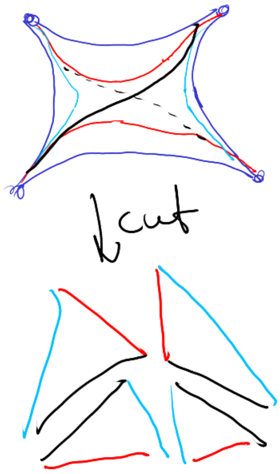


Figure 6.8: The ideal triangulation corresponding to the triangle σ with vertices $0/1, 1/1, 1/0$ (compare Figure 6.7).

Proof. The idea is to use the $\mathrm{PSL}(2, \mathbb{Z})$ action to say that it suffices to check the 'canonical triple' $(0/1, 1/1, 1/0)$. This follows immediately from Lemma 6.3.2 (the action is continuous hence preserves intersection numbers). ∎

Let $\sigma \in \mathcal{D}(2)$ be a triangle, and consider the dual complex to $\mathrm{trg}(\sigma)$ on $S_{0,4}$. This complex is also simplicial, and is a strong deformation retract of the punctured sphere (or marked sphere when the marked points are deleted); we call it the **spine** corresponding to σ, denoted $\mathrm{spine}(\sigma)$. We also define $\mathrm{spine}(\lambda)$ for $\lambda \in \mathcal{D}(1)$ analogously.

We now briefly consider the relationship between two adjacent triangles in $\mathcal{D}$ (that is, two triangles sharing a common edge). Let σ_1, σ_2 share the edge λ. Then $\mathrm{spine}(\sigma_1)$ and $\mathrm{spine}(\lambda)$ differ by the collapsing of two non-adjacent edges to form a 'cross' shape cutting the surface into four 1-punctured discs (see Figure 6.9); reversing this process gives a move from $\mathrm{spine}(\lambda)$ to $\mathrm{spine}(\sigma_2)$. This combinatorial process can be expanded into a continuous family of deformation retracts forming a 2-dimensional subcomplex of $S_{0,4} \times [-1, 1]$ analogous to the case of the 1-punctured torus in figure 1.1 of [7] which leads towards the theory of Jørgensen: this theory is out of the scope of the current work.

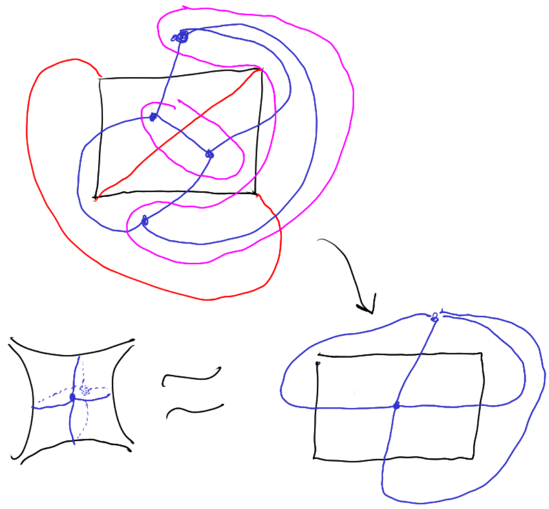


Figure 6.9: Collapsing spine(σ) (blue net) to spine(λ) for a triangle $\sigma \in \mathcal{D}(2)$ and one of its edges. The purple circles bound edges of spine(σ) which collapse to vertices of spine(λ). The dual triangulation to spine(σ) is also indicated (the red arcs).

Chapter 7

The foliation theory of Keen and Series

In the early 1990s, Keen and Series developed a coordinate system for the parabolic Riley slice $\mathcal{R}^{\infty,\infty}$ based on the same ideas as their theory for the Maskit slice described in [62], [114], and [94, from p.287]. In this section, we explain this coordinate system and indicate the extension to the case that we allow elliptic generators. To the best of our knowledge, this elliptic theory has not appeared in the literature (though it is believed by experts that the theory goes through with minimal changes from the parabolic case).

7.1 Motivation

Consider the group $\Gamma = \Gamma_{4i}^{\infty,\infty}$ with limit set shown in Figure 7.1a. In that figure, we show in red and blue the edges of the fundamental domain defined via isometric circles for Γ as in Theorem 6.1.1; clearly the isometric circles are disjoint and so by that theorem Γ lies in the Riley slice $\mathcal{R}^{\infty,\infty}$. By Lemma 3.4.2, the convex core $\mathcal{C}(\Gamma)$ is a 4-punctured sphere. Observe that the portion of the limit set shown in the figure seems to be contained within the two lines $\{z \in \mathbb{C} : \Im z = \pm 1/4\}$ and the two circles of radius $1/2$ centred at $\pm 1/2$; in fact, the entire limit set is the orbit of this portion under X. (That this is in fact the case will follow from the theory we develop later in the chapter.) Consider the two subgroups of Γ defined by

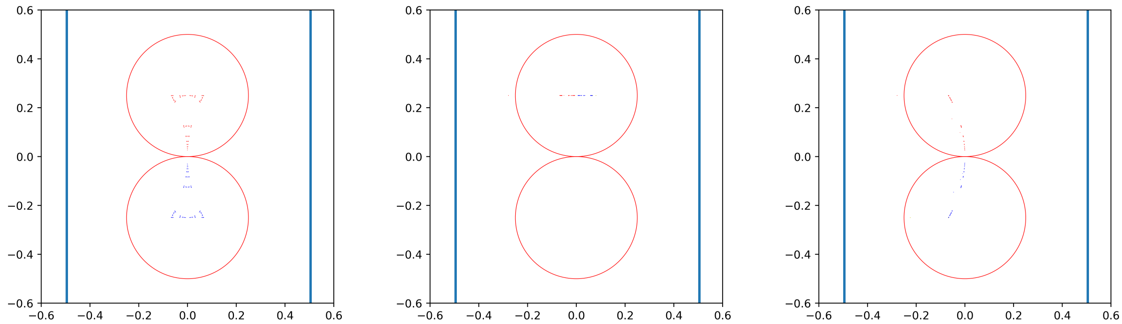
$$\Gamma_1 = \langle x, yXY \rangle, \quad \Gamma_2 = \langle xyX, Y \rangle$$

(where we use the convention $x = X^{-1}$ and $y = Y^{-1}$); these two subgroups are Fuchsian (for example, by Lemma 7.3.11 below), and have limit sets corresponding to the line $\{z \in \mathbb{C} : \Im z = 1/4\}$ and the circle $C(1/2, 1/2)$ respectively (see Figures 7.1b and 7.1c). It is easy to check that the other two bounding curves correspond to limit sets of conjugate Fuchsian subgroups to Γ_1 and Γ_2 in Γ. The boundary $\partial\,\mathrm{h.conv}\,\Lambda(\gamma)$ is the 'roof' of the hyperbolic planes above these curves, in the sense that if F is the union of the hyperbolic planes above the two horizontal lines and the hyperbolic planes above the horizontal translates of each of the circular arcs, then $\partial\,\mathrm{h.conv}\,\Lambda(\gamma)$ is the subset

$$\{(z,t) \in F : \text{ if } (z,t') \in F, \text{ then } t' \le t\}.$$

The convex core boundary $\partial\mathcal{C}(\Gamma)$ is then the quotient of this convex hull boundary by Γ. Compute the fixed points of $xyXY$:

$$\mathrm{Fix}(xyYX) = \{\pm\frac{\sqrt{3}}{4} - \frac{1}{2} + i\frac{1}{4}\}.$$



(a) Limit set of $\Gamma_{4i}^{\infty,\infty}$. (b) Limit set of $\langle x, yXY\rangle$ ($\rho = 4i, a = b = \infty$). (c) Limit set of $\langle xyX, Y\rangle$ ($\rho = 4i, a = b = \infty$).

Figure 7.1: A full limit set and two F-peripheral subgroup limit sets. The two subgroup limit sets are only small portions of the corresponding circles, so are hard to see even though we have increased the sizes of dots marking the points.

These are precisely the intersection points of the horizontal line $\Im z = 1/4$ with the circle centred at $-1/2$; thus the intersection between the corresponding hyperbolic surfaces is exactly the axis of $xyXY$ in $\mathbb{H}^3$; since $xyXY = \operatorname{Word}(1/2)$, this axis is a lift of $\gamma(1/2)$. The other pleats in $\partial\,\mathrm{h.conv}\,\Lambda(\Gamma)$ correspond to the axes of conjugates of $\operatorname{Word}(xyXY)$, and so project to the same curve on the quotient surface. This means that the bending locus of the pleated surface $\partial\mathcal{S}(\Gamma)$ is exactly the projection of this axis, and the two flat pieces are the projections of the hyperbolic domes above the limit sets of Γ_1 and Γ_2. The entire situation is visible in Figure 7.2.

Taking Γ-images of the four 'boundary' circles gives a pair of circle chains in the limit set; it turns out that a circle chain structure with the same combinatorial structure appears in the limit sets of $\Gamma_{ki}^{\infty,\infty}$ for all $k > 2$. This can be seen in Figure 7.3 (see also Figure 7.4 for an example in the $(5,\infty)$ Riley slice), which also shows that as $\rho \to 2i$ along the imaginary axis this circle chain structure degenerates to a circle packing. (Compare with the Maskit slice case, for instance the pictures in Figure 9.15 of [94].) The Keen–Series foliation of $\mathcal{R}^{\infty,\infty}$ will have, as leaves, the curves joining all groups with the same circle chain structure. In fact, in [63] (compared to the earlier [62]) the notion of circle chains is replaced by the study of the corresponding Fuchsian groups (in the example above of $\rho = 4i$, these are the groups Γ_1 and Γ_2) which project to flat pieces of the convex hull boundary; these groups will be called **F-peripheral groups**.

We have already seen that the Riley slices are homeomorphic to annuli (Corollary 5.1.6). We will strengthen this result here, and give an explicit homeomorphism between the Riley slice and an annulus such that the natural radial coordinate system of the annulus represents the geometric data of points in $\mathcal{R}^{a,b}$. More precisely, following [63, Theorem 5.4] for each pair (a,b) we will define a dense lamination Λ on $\mathcal{R}^{a,b}$ and a map

$$\Pi^{a,b} : |\Lambda| \to \mathbb{Q}/2\mathbb{Z} \times \mathbb{R}_{>0}$$

sending each ρ in a leaf of Λ to a pair (γ, ℓ) where γ is a simple closed geodesic on $\mathcal{S}(\Gamma_\rho^{a,b})$ bounding two punctures modulo Dehn twists along a curve ω bounding a compression disc (these equivalence classes of curves correspond to elements of $\mathbb{Q}/2\mathbb{Z}$ via the bijection Word of Chapter 6) and where ℓ measures the length of γ, normalised so that it is independent of the equivalence class of the curve modulo twists along ω; then we extend $\Pi^{a,b}$ to a homeomorphism

$$\Pi^{a,b} : \mathcal{R}^{a,b} \to \mathbb{R}/2\mathbb{Z} \times \mathbb{R}_{>0}.$$

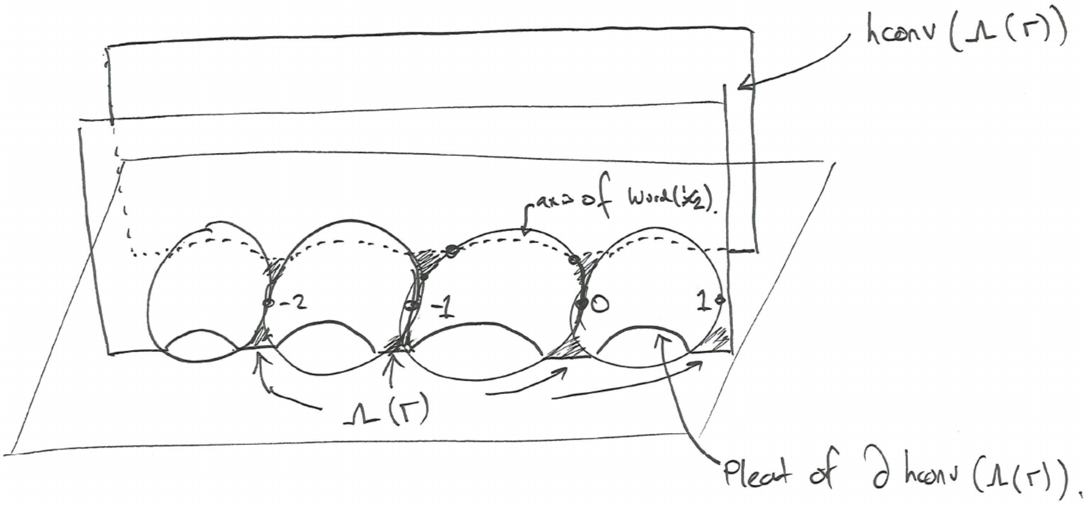


Figure 7.2: The hyperbolic convex hull, h.conv $\Lambda(\Gamma_{4i}^{\infty,\infty})$.

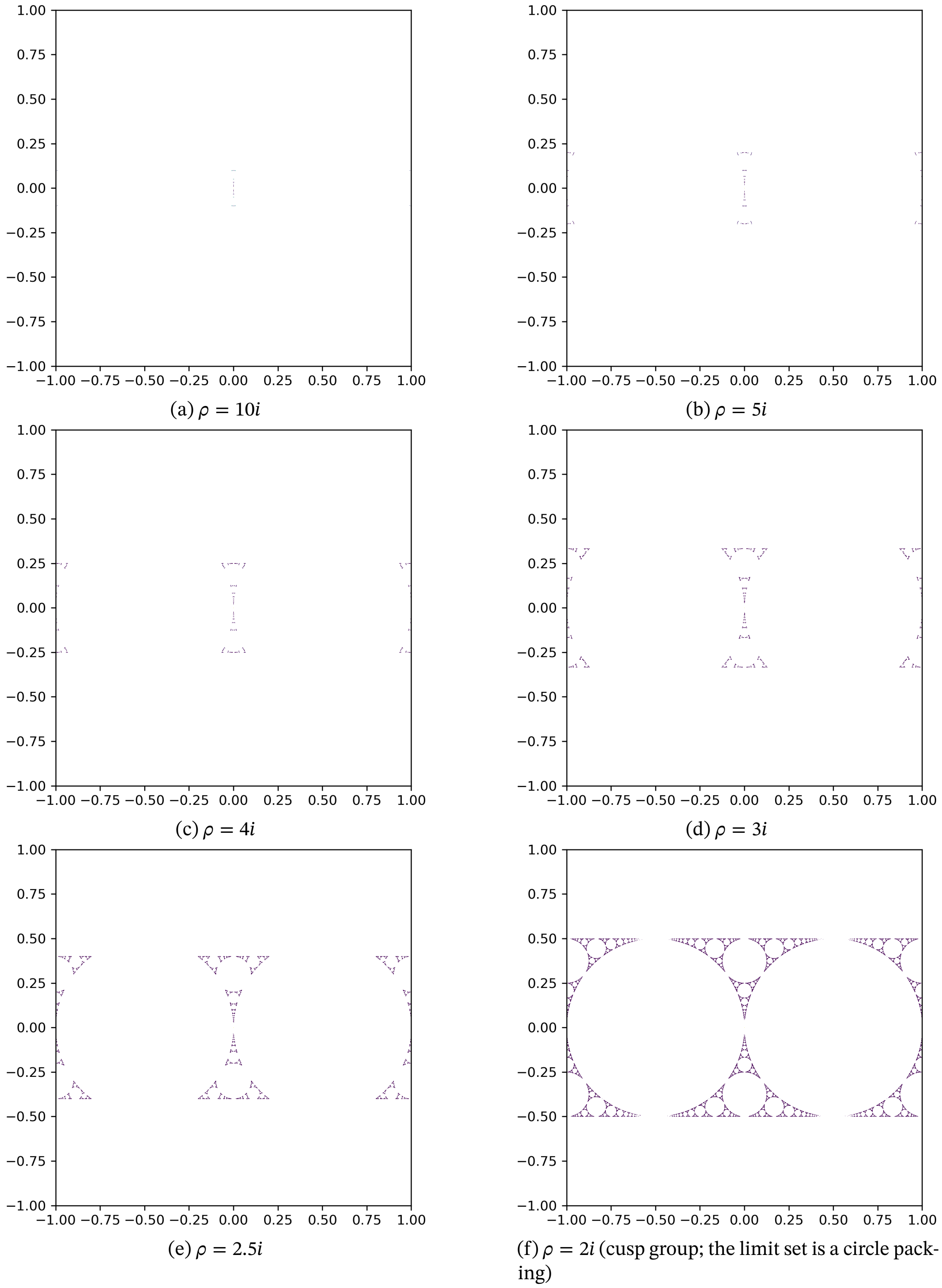
This procedure is detailed in Theorem 7.4.15; see also the motivational discussion in Section 1.1.

The **rational pleating rays** of $\mathcal{R}^{a,b}$ will correspond to the fibres of $\Pi^{a,b}$ for a single geodesic on the surface; we will see that this is exactly the curve in $\mathcal{R}^{a,b}$ such that the convex core boundary $\partial\mathcal{C}(\Gamma_\rho^{a,b})$ has two flat pieces, both 2-marked discs, glued along the projection of the axis of some Farey word (in the example above with $\rho = 4i$, we have a group lying on the 1/2 pleating ray). One disc has two marked points of order a joined by a singular arc through the 3-manifold, and the other has two joined marked points of order b. The pleating locus of $\partial\mathcal{C}(\Gamma_\rho^{a,b})$ is a single curve $\gamma(p/q)$ separating these two discs; it is precisely the curve obtained by winding up the curve defined as $\gamma(\infty)$ in Definition 4.2.1 above via Dehn twists around the compression disc that unwind the two bridge arcs into a pair of unknotted arcs (compare Figure 1.1). That this gluing is in fact the situation is proved below as Lemma 7.2.10.

Deforming $\Gamma_\rho^{a,b}$ along a rational pleating ray corresponds to pinching in or pulling out the geodesic corresponding to the bending locus (i.e. changing the length of the geodesic to obtain a new complex structure in the moduli space); along the pleating ray the corresponding Farey polynomial is real-valued, and this value (being the trace of the word representing the geodesic) gives another representation of the length of the gluing curve. In fact, the image under $\Phi_{p/q}^{a,b}$ of the rational pleating ray corresponding to the geodesic enumerated by p/q is exactly $(-\infty, -2)$. As the curve is pinched down to length 0, in the limit the word Word(p/q) becomes parabolic and we obtain a cusp group in the sense of Definition 4.2.1; there is a unique cusp group for each p/q, and they are dense in the boundary $\partial\mathcal{R}^{a,b}$.

The Keen-Series theory can be extended, by deforming past the cusp group into the Heckoid groups as in Section 4.2; recent literature has studied this from a point of view 'internal' to the Riley slice by studying the combinatorial structure of the limit sets, rather than the 'external' view we mentioned in the cited section (deforming to the boundary of the Riley slice from its exterior in $\mathbb{C}$ through knot orbifolds). Some exemplar papers include [5, 6, 81, 96].

Remark. This notion of viewing the boundary of a moduli space as being the limiting groups after pinching down geodesics represented by hyperbolic elements is quite general, for instance see [60, 85, 95]. The idea has been worked out in detail for many other moduli spaces beyond the Maskit and Riley slices we have mentioned above; for instance, the so-called **diagonal slice of Schottky space**



(a) $\rho = 10i$

(b) $\rho = 5i$

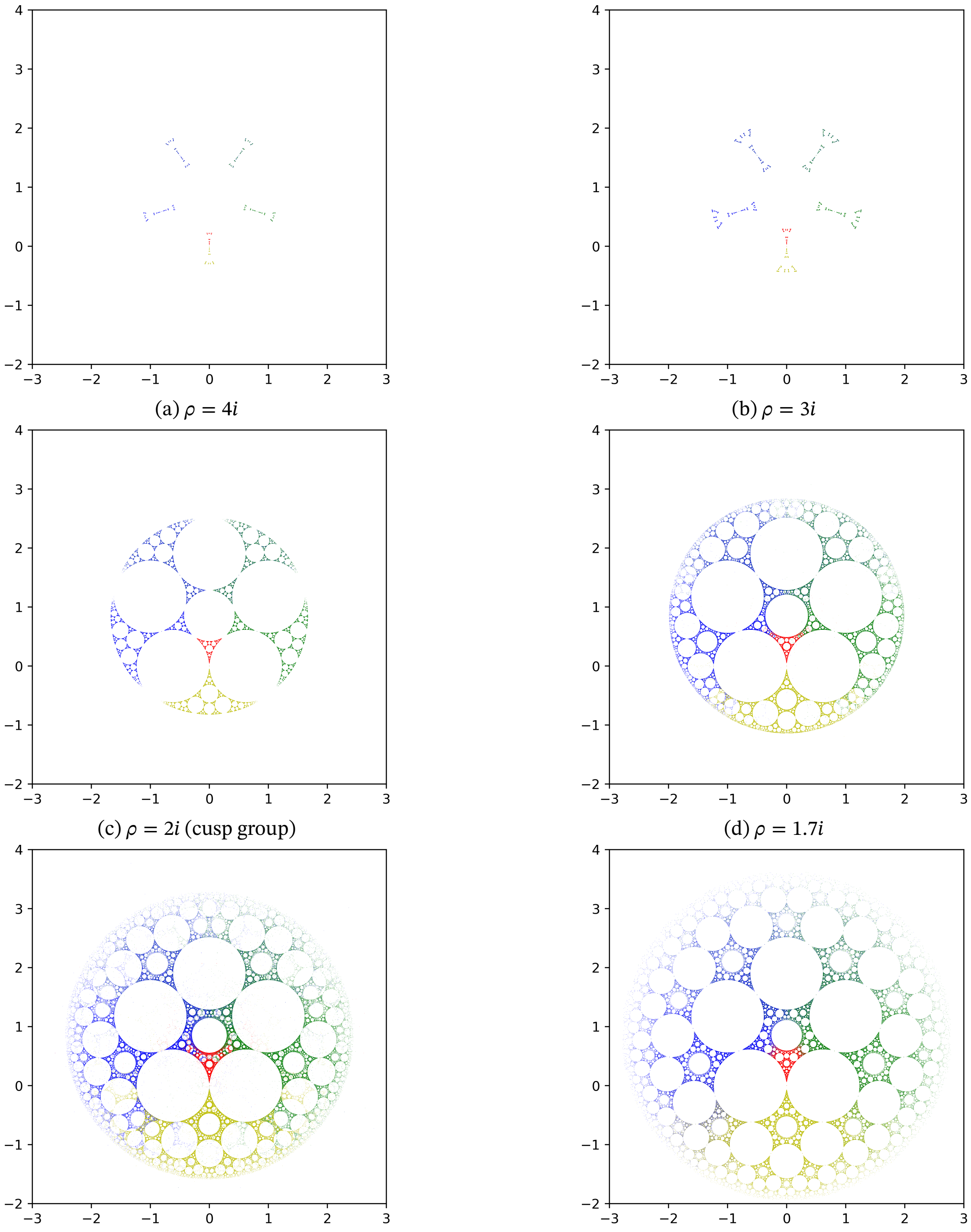
(c) $\rho = 4i$

(d) $\rho = 3i$

(e) $\rho = 2.5i$

(f) $\rho = 2i$ (cusp group; the limit set is a circle packing)

Figure 7.3: Limit sets of $\Gamma_\rho^{\infty,\infty}$ for various $\rho \in \mathbb{R}i$.



(a) $\rho = 4i$

(b) $\rho = 3i$

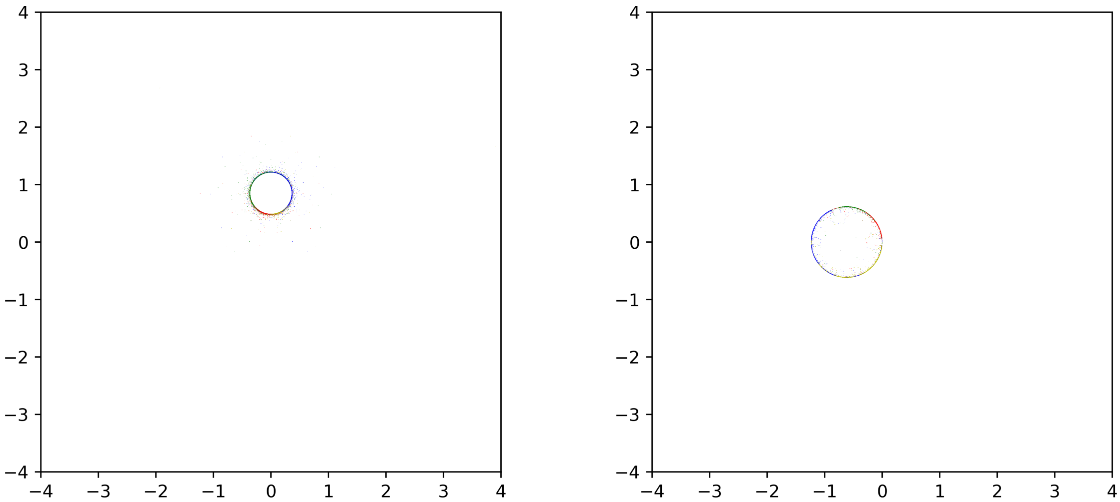
(c) $\rho = 2i$ (cusp group)

(d) $\rho = 1.7i$

(e) $\rho = 1.5i$

(f) $\rho = \sqrt{2}i$

Figure 7.4: Limit sets of $\Gamma_\rho^{5,\infty}$ for various $\rho \in \mathbb{R}i$. The limit sets (a) and (b) lie in the corresponding Riley slice; at the cusp point, shown in (c), the different portions of the limit sets collide to form a circle packing; and then limit sets (d), (e), and (f) are in the exterior of the corresponding Riley slice. To obtain the limit set circles in good definition, we plotted points corresponding to words of length up to 20, and the matrix multiplication produced some numerical error (also visible in Figure 7.5 below). To reduce the visibility of this error we reduced the opacity of the plotted points, c.f. Example A.2.2.



(a) $\langle x, yXY\rangle$ (b) $\langle xyX, Y\rangle$

Figure 7.5: Two F-peripheral subgroups of $\Gamma^{5,\infty}_{1.7i}$.

[115], the **Earle embedding** [65], and various other linear slices through complex Fenchel-Nielsen coordinates for the deformation space of quasi-Fuchsian once-punctured torus groups [67].

7.2 F-peripheral subgroups

Motivated by the previous section, we make the following definition:

7.2.1 Definition. Let Γ be a Kleinian group. A subgroup $F \leq \Gamma$ is called **F-peripheral** in Γ if

1. F is Fuchsian, and
2. F has an invariant disc Δ which contains no limit points of Γ (the **F-peripheral disc** of F).

We say that F is **strongly F-peripheral** if, as well as being F-peripheral, the boundary of the invariant disc contains no limit points of Γ which are not limit points of F; i.e.

$$\overline{\Delta} \cap \Lambda(\Gamma) = \Lambda(F).$$

Some F-peripheral subgroups of a 2-parabolic group were shown in Figure 7.1; some examples of a group with an elliptic generator may be seen in Figure 7.5 (the full limit set of this group appears above as Figure 7.4d).

In this section we will prove some technical results about F-peripheral subgroups in moduli spaces.

7.2.2 Lemma. *Let Γ_0 be an arbitrary geometrically finite Kleinian group, let $\rho : [0,1] \to \mathrm{QH}(\Gamma_0)$ be a path with $\rho(0) = \Gamma_0$, write Γ_t for $\Gamma_{\rho(t)}$, and for every t write f_t for the quasiconformal homeomorphism $f_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which conjugates Γ_0 to Γ_t.*

Suppose that $F_0 \leq \Gamma_0$ is finitely generated and strongly F-peripheral with peripheral disc Δ_0, and suppose that $F_t \coloneqq f_t F_0 f_t^{-1} \leq \Gamma_t$ is Fuchsian for every t. Then there exists some $\varepsilon > 0$ such that F_t is F-peripheral for every $t \in [0, \varepsilon]$.

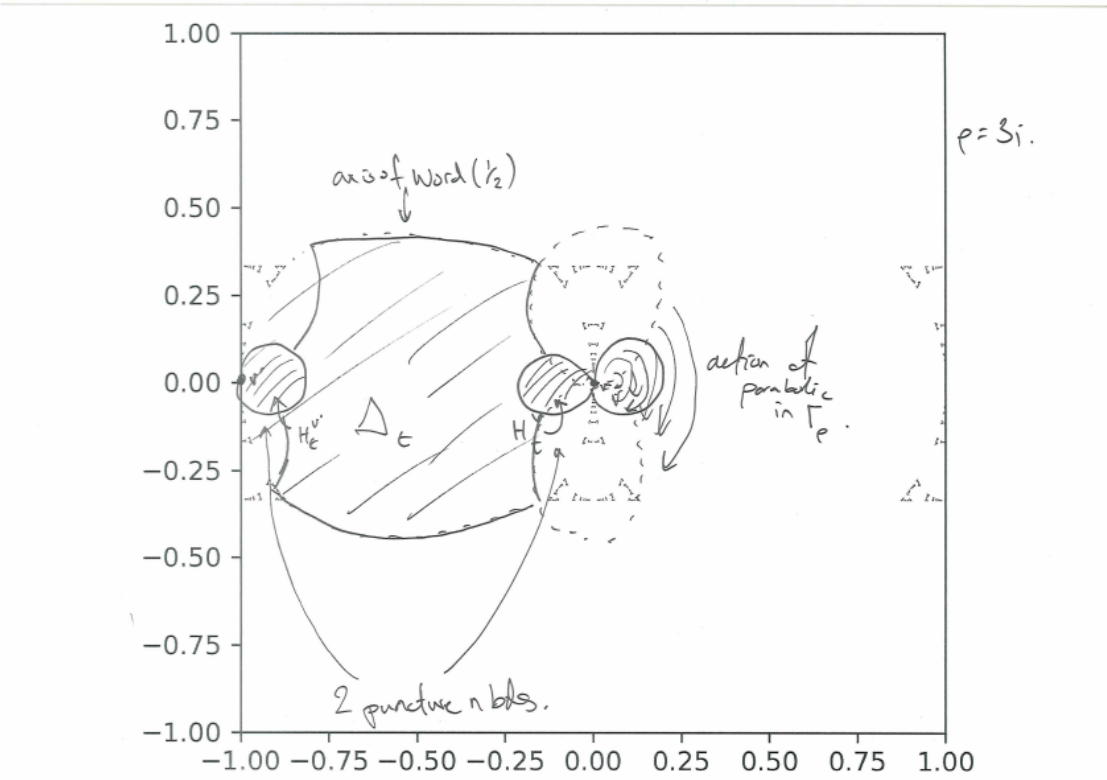


Figure 7.6: The canonical polygon of a F-peripheral group F in $\Gamma = \Gamma_{3i}^{\infty,\infty}$ showing the horoballs H_v^t for the two parabolic elements. Observe that the quotient of the roof of $\partial\,\text{h.conv}\,\Lambda(\Gamma)$ above the peripheral disc Δ by F is a disc with two punctures, with the circular edge of the disc being the projection of the axis of Word(1/2) (compare Figure 7.2).

The proof is essentially the same as that of [63, Proposition 3.1]: the case where the F-peripheral group has an elliptic generator is actually easier, since we do not need to cut out a neighbourhood of a cone point to bound the canonical polygon away from the limit set.

Proof. Without loss of generality, assume $\infty \notin \overline{\Delta_0}$. Let R_0 be the canonical Fricke polygon, which was characterised in Lemma 3.2.6; consider the holomorphic motion of $[0,1]$ induced by the quasiconformal deformation structure, namely

$$\begin{aligned} f : [0,1] \times \hat{\mathbb{C}} &\to \hat{\mathbb{C}} \\ (t,z) &\mapsto f_t(z) \end{aligned}$$

and observe that the fixed points of F_t move continuously with t and so $\Lambda(F_t)$ and thus the peripheral discs $\Delta_t = \Delta(F_t)$ move continuously with t. For t sufficiently small, there is a fundamental polygon R_t for F_t with vertices of the form $f_t v$ for v a vertex of R_0 and with sides circular arcs arbitrarily close to the sides of R_0.

We next try to bound these polygons away from the limit set $\Lambda(\Gamma)$—this is fine everywhere except for any parabolic fixed points on $\partial\Delta_t$, and so we need to cut out neighbourhoods of these, By Corollary 6.1.2, every group in $\mathcal{R}^{a,b}$ is geometrically finite; and by Lemma 3.3.4, this implies that every parabolic fixed point of each Γ_t is doubly cusped. In particular, if F_t contains any parabolic element v, we can find a horodisc H_t^v based at v in Δ_t such that $H_v^t \subseteq \Omega(\Gamma_t)$ and such that the Euclidean radius of H_t^v is continuous in t (see Figure 7.6 for an example in the Riley slice). Let $S_t = R_t \setminus \bigcup_v H_t^v$ (where the union is taken over the parabolic vertices of R_t); since $\overline{\Delta_0} \cap \Lambda(\Gamma_0) = \Lambda(\Gamma_0)$ there exists some $\delta > 0$ such that $d_{\text{Euc.}}(S_0, \Lambda(\Gamma_0)) \geq \delta$. Further, by the continuity of the polygons R_t in t there exists some ε such that

$$d_{\text{Euc.}}(S_t, \Lambda(\Gamma_t)) \geq \delta/2 > 0 \tag{7.2.3}$$

for every t.

We now show that the curve $\rho\restriction_{[0,\varepsilon]}$ parameterises only F-peripheral groups. Suppose $0 \leq t \leq \varepsilon$ is such that F_t is not F-peripheral. Then, since F_t is Fuchsian by assumption, we must have some $z \in \Lambda(\Gamma_t) \cap \Delta_t$. Since R_t is a fundamental domain for F_t and $\Lambda(\Gamma_t)$ is invariant under F_t, there is some z' in the F_t-orbit of z which lies in $\Lambda(\Gamma_t) \cap R_t$. By construction of the horodiscs, z' (being a limit point) cannot lie in any H_v^t and so $z' \in S_t$. Thus $d_{\text{Euc.}}(z', S_t) = 0$; but $z' \in \Lambda(\Gamma_t)$, contradicting Equation (7.2.3). ∎

We specialise to the Riley groups now; we formalise the notion of taking pairs of F-peripheral subgroups whose hyperbolic quotients glue along the axis of some Farey word $\mathrm{Word}(p/q)$ to give a 4-marked sphere.

7.2.4 Definition. The p/q**-circle chain** is the set $\mathcal{U}_{p/q}^{a,b}(\rho)$ of subgroups of $\Gamma_\rho^{a,b}$ generated by two elliptics or parabolics U_1, U_2, such that

1. either both U_1 and U_2 are of order a or both are of order b, or U_1 is order a and U_2 is order b (or vice versa); and
2. the product $U_1 U_2$ represents the free homotopy class of the projection of the axis of $\mathrm{Word}(p/q)$ on $\mathcal{S}(\Gamma_\rho^{a,b})$.

Remark. The two cases 'U_1 and U_2 are conjugate to the same generator' or 'U_1 and U_2 are conjugate to different generators' in Definition 7.2.4 essentially correspond to the two cases of Figure 4.3—when p/q is the slope of a link with odd length continued fraction decomposition then the curve $\gamma(p/q)$

bounds a compression disc and separates the two generators; when p/q is the slope of a link with even length continued fraction decomposition, then $\gamma(p/q)$ bounds a disc which cuts through both singular arcs and bounds two discs on the surface, each containing one marked point represented by each generator.

7.2.5 Example. This definition realises in general the ideas of the previous section: the example there was a pair of non-conjugate elements of the 1/2-circle chain,

$$\langle x, yXY\rangle, \langle xyX, Y\rangle \in \mathcal{U}_{1/2}.$$

We continue generalising the results of Section 3 of [63]. Below, the symbol ρ stands for a fixed $\rho \in \mathbb{C}$ (perhaps not in the Riley slice) such that $\Gamma_\rho^{a,b}$ is discrete (where, as always, a and b are fixed elements of $\hat{\mathbb{N}}$).

Since the marked generators of a subgroup in a circle chain set cannot be conjugate in that subgroup, we obtain the following:

7.2.6 Lemma. *Let F be a F-peripheral subgroup of $\Gamma = \Gamma_\rho^{a,b}$, lying in $\mathcal{U}_{p/q}^{a,b}$. Then the surface $\Delta(F)/F$ is a disc with two marked points.* ∎

7.2.7 Lemma. *Let F be a F-peripheral subgroup of $\Gamma_\rho^{a,b}$, lying in $\mathcal{U}_{p/q}^{a,b}$ with the two marked generators having product* $\mathrm{Word}(p/q)$*. Then every boundary hyperbolic of F is conjugate to* $\mathrm{Word}(p/q)$*.*

Proof. By Proposition 3.2.5, the number of boundary hyperbolic conjugacy classes is equal to the number of circular boundary components in the Fuchsian quotient; by Lemma 7.2.6 there is one such component. ∎

7.2.8 Lemma. *Suppose that $\Gamma = \Gamma_\rho^{a,b}$ contains two non-conjugate F-peripheral subgroups F_1 and F_2 in $\mathcal{U}_{p/q}^{a,b}$. Then both F_1 and F_2 are* strongly *F-peripheral.*

Proof. By conjugation in Γ, the products of the generators of each F_i are equal to $\mathrm{Word}(p/q)$; thus the boundary circles $\partial\Delta(F_1)$ and $\partial\Delta(F_2)$ intersect at the fixed points of $\mathrm{Word}(p/q)$. We proceed to show that F_1 is strongly F-peripheral; the same argument, swapping F_1 and F_2, shows that F_2 is strongly F-peripheral. Let σ be the arc in $\partial\Delta(F_1)$ between the two fixed points of $\mathrm{Word}(p/q)$ which is contained in $\Delta(F_2)$; it contains no limit points of Γ since F_2 is F-peripheral. By Lemma 7.2.7, every other interval of discontinuity for the action of F_1 on $\partial\Delta(F_1)$ is F_1-equivalent to σ. Since points on $\partial\Delta(F_1)$ are either in intervals of discontinuity or are limit points of $\Lambda(F_1)$, and no Γ limit points can lie on images of σ, we see that $\partial\Delta(F_1) \cap \Lambda(\Gamma) = \Lambda(F_1)$ as required. ∎

7.2.9 Lemma. *Suppose that $\Gamma = \Gamma_\rho^{a,b}$ contains two non-conjugate F-peripheral subgroups F_1 and F_2 in $\mathcal{U}_{p/q}^{a,b}$. For each i let H_i be the hyperbolic plane erected above $\Delta(F_i)$; by the definition of Poincaré extension, F_i acts as a group of hyperbolic isometries on H_i and so we may define a Nielsen region N_i for this action. This Nielsen region is precisely invariant under F_i in Γ.*

The proof of this lemma found in the parabolic case as [63, Lemma 3.4] requires the use of topology (in particular the Euler characteristic), and so the elliptic case requires the analogous differential geometry of orbifolds.

Proof. **Subclaim.** *F_i is the stabiliser in Γ of N_i.* Let F' be the stabiliser of N_i in Γ; F' therefore stabilises the boundary of $\Delta(F_i)$ (F' sends an arc through two points in N_i to another arc through two points in N_i, and since N_i is full-dimensional in H_i, for any pair of points ξ_1, ξ_2 in $\partial\Delta(F_i)$ there exist two points in N_i lying on the geodesic $[\xi_1, \xi_2]$; thus F', sending ξ_1 and ξ_2 to the endpoints of the geodesic joining the images of the two points in N_i, sends ξ_1 and ξ_2 to two other points on $\partial\Delta(F_i)$) and therefore

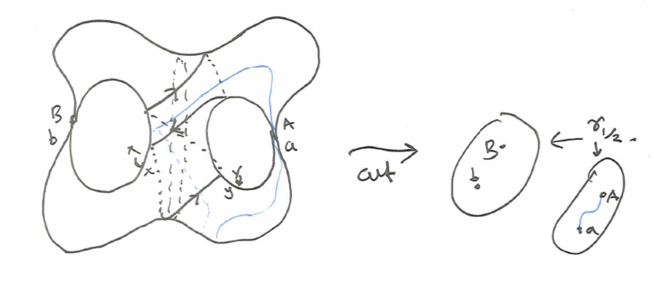


Figure 7.7: The decomposition of a surface on $\mathcal{P}_{1/2}$ by cutting along $\gamma(1/2)$ gives two discs, each containing two *paired* marked points.

stabilises the entire hemisphere H_i, so is Fuchsian. Thus, since $F' \geq F_i$, there is an induced covering map $N_i/F_i \to N_i/F'$. Since $F_i \in \mathcal{U}^{a,b}_{p/q}$, N_i/F_i is a disc with two marked points and so has Euler characteristic $-1 + 2/n$ where n is the order of the marked points (see Definition 2.2.4). Since a covering may only increase marked point orders, N_i/F' has Euler characteristic at most $-1 + 2/n$. On the other hand, the degree of a covering is a positive integer. Hence applying Proposition 2.2.5 we see that the covering is of degree 1 and so $F_i = F'$. Thus F_i is the stabiliser of N_i. *Q.E.D. (Subclaim).*

Suppose for contradiction that $U \in \Gamma \setminus F_i$ but $U(N_i) \cap N_i \neq \emptyset$. We have two cases.

1. Suppose $\Delta(F_i) = U\Delta(F_i)$. Then U stabilises $\Delta(F_i)$, and by Lemma 7.2.8 it must permute the arcs of discontinuity of F_i on the boundary. Further it is conformal on H_i and so preserves the angles of the edges of N_i on translation. These two facts imply that U stabilises the Nielsen region and thus $U \in F_i$ by the subclaim, giving the required contradiction.

2. On the other hand, suppose $\Delta(F_i) \neq U\Delta(F_i)$. The set $U(N_i) \cap N_i$ lies on the arc of intersection of the two domes H_i and UH_i. Since F_i is F-peripheral, the arc $\Delta_i \cap \partial U\Delta_i$ is an arc of discontinuity for F_i and so N_i is bounded by the arc joining the intersection points of Δ_i with $\partial U\Delta_i$; on the other hand, N_i is open, and so N_i cannot contain any points of this arc (which gives the contradiction). ∎

7.2.10 Lemma. *Suppose that $\Gamma = \Gamma^{a,b}_\rho$ contains two non-conjugate F-peripheral subgroups F_1 and F_2 in $\mathcal{U}^{a,b}_{p/q}$. Then $\partial\mathcal{C}(\Gamma)$ consists of two flat pieces, both 2-marked discs, glued along the pleating locus of the surface which consists exactly of the projection of the axis of* Word(p/q) *under* Γ. *In particular, $\rho \in \mathcal{R}^{a,b}$ and the pleating locus of $\partial\mathcal{C}(\Gamma)$ is $\gamma(p/q)$.*

The cutting procedure for a surface with pleating locus $\gamma(1/2)$ may be seen in Figure 7.7. Comparison with Figure 7.10 below shows that the curve sometimes separates paired marked points, and sometimes places paired points in the same disc. The difference is that one group comes from a two-bridge link and the other a two-bridge knot, so in one case the pleating curve is obtained from Dehn twisting a compression disc boundary and in the other case it is obtained from a curve crossing both tangle arcs.

Proof of Lemma 7.2.10. Let H_1 and H_2 be the hemispheres above the discs $\Delta(F_1)$ and $\Delta(F_2)$, and let N_1 and N_2 be the respective Nielsen regions for the hyperbolic actions on the hemispheres by the groups F_1 and F_2. By Lemma 7.2.6, each $S_i \coloneqq N_i/F_i$ is a sphere with two punctures and a hole. All of the surfaces are hyperbolic and so have curvature -1. We may also compute the Euler characteristic, namely

$$\chi(S_1) = -1 + \frac{2}{a}, \text{ and } \chi(S_2) = -1 + \frac{2}{b}$$

so by the Gauss-Bonnet theorem for orbifolds (Theorem 2.2.6) we have

$$\text{Area}(S_1) = 2\pi - \frac{4\pi}{a} \text{ and } \text{Area}(S_2) = 2\pi - \frac{4\pi}{b}.$$

By Lemma 7.2.9, $N_i/F_i = N_i/\Gamma$ for both i. Since both F_i are F-peripheral, each N_i/Γ lies in $\partial\mathcal{C}(\Gamma)$; and since the F_i are non-conjugate, they are disjoint subsets of the surface. On the other hand, we may apply Corollary 2.2.7 to see that

$$\text{Area}(\mathcal{S}(\Gamma)) = 4\pi\left(1 - \frac{1}{a} - \frac{1}{b}\right).$$

Thus the surface must be the union of the two discs. ∎

Given some $\rho \in \mathcal{R}^{a,b}$, we write $\text{pl}(\rho)$ (or $\text{pl}^{a,b}(\rho)$ if we need to emphasis a and b) for the pleating locus of $\partial\mathcal{C}(\Gamma_\rho^{a,b})$. By Lemma 7.2.10, if $\Gamma_\rho^{a,b}$ has two non-conjugate groups in $\mathcal{U}_{p/q}^{a,b}$, then $\text{pl}(\rho)$ can be identified with the rational number p/q. In the next section, we will show that one can go backwards through this equivalence.

7.3 Measured laminations and rational pleating rays

In this section, we will show that the groups in $\mathcal{R}^{a,b}$ whose convex core boundary consists of two discs glued along $\gamma(p/q)$ form an analytic curve. We continue to follow Sections 3 and 4 of [63] and the corrections in [66].

7.3.1 Definition. Let $p/q \in \mathbb{Q}$. The p/q-**rational pleating ray** is the set

$$\mathcal{P}_{p/q}^{a,b} \coloneqq \{\rho \in \mathcal{R}^{a,b} \,:\, \text{pl}(\rho) = \hat{\gamma}(p/q)\}$$

where $\hat{\gamma}(p/q)$ is the geodesic in $\mathcal{M}(\Gamma_\rho^{a,b})$ which is represented by the element $\text{Word}(p/q)$. (This word is always loxodromic since it is not in the parabolic or elliptic subgroups generated by the generators of $\Gamma_\rho^{a,b}$ or any conjugates of these: for instance this follows from Lemma 6.1.7.)

Some very nice pictures of the rational pleating rays can be found as Figure 1 of [63], Figure 0.2b of [7], and Figures 1 and 2 of the joint preprint [39] which were produced by Yasushi Yamashita.

It is immediate from the definition and Theorem 6.1.5 that $\mathcal{P}_{p/q} \cap \mathcal{P}_{r/s} = \varnothing$ iff $p/q \neq r/s$.

7.3.2 Proposition. *$\rho \in \mathcal{P}_{p/q}^{a,b}$ iff $\Gamma = \Gamma_\rho^{a,b}$ has two non-conjugate F-peripheral subgroups in $\mathcal{U}_{p/q}^{a,b}(\rho)$.*

Proof. One direction is Lemma 7.2.10, so we need to show that if $\rho \in \mathcal{P}_{p/q}^{a,b}$ then we can find two non-conjugate subgroups in the p/q circle chain. Let Σ, Σ' be the two connected components of $\partial\mathcal{C}(\Gamma) \setminus \hat{\gamma}(p/q)$. Let $\tilde{\Sigma}$ be a connected component of the lift of Σ to $\mathbb{H}^3$. By a similar argument to that in the proof of Lemma 7.2.9, the stabiliser of $\tilde{\Sigma}$ in Γ stabilises the hyperbolic plane spanned by $\tilde{\Sigma}$ and so is Fuchsian. Let F be this stabiliser. It is F-peripheral since Σ is a flat piece of the convex hull boundary and so $\text{h.conv}\,\Lambda(\Gamma)$ lies entirely on one side of the hyperbolic plane. The plane has two marked points; let U_1 and U_2 be parabolic or elliptic elements representing simple loops about each marked point with compatible orientation so that U_1U_2 represents $\gamma(p/q)$. Since F is acting on a simply connected domain to form the quotient, we have $F = \langle U_1, U_2\rangle$ and so $F \in \mathcal{U}_{p/q}^{a,b}$. Running the same argument for Σ' produces a second such group $F' \in \mathcal{U}_{p/q}^{a,b}$, and F and F' must be nonconjugate as $\Sigma \neq \Sigma'$. ∎

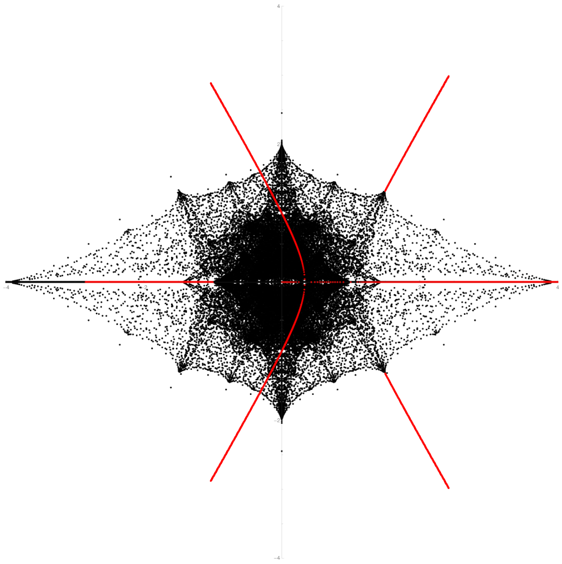


Figure 7.8: The hyperbolic locus of slope 1/3 for $\mathcal{R}^{\infty,\infty}$.

We now begin to carry out the programme described in Section 7.1, namely showing that the pleating rays are exactly the curves that correspond to pinching down the geodesic $\mathrm{Word}(p/q)$ in the complex structure.

7.3.3 Lemma. *Let $\rho \in \mathcal{P}^{a,b}_{p/q}$. Then $\mathrm{Word}(p/q)(\rho)$ is hyperbolic.*

Proof. By Proposition 7.3.2, $\mathrm{Word}(p/q)$ is conjugate to a word lying in a Fuchsian group, so has real trace, and is loxodromic. ∎

We define the **hyperbolic locus** of $\Phi^{a,b}_{p/q}$ to be the following set:

$$\widetilde{\mathcal{H}}^{a,b}_{p/q} := \{\rho \in \mathbb{C} : \Phi^{a,b}_{p/q}(\rho) \in (-\infty, -2) \cup (2, \infty)\}.$$

Note that this set does not consist only of points in the Riley slice; Figure 7.8 shows the hyperbolic locus $\widetilde{\mathcal{H}}^{\infty,\infty}_{1/3}$, and Figure 7.9 shows the hyperbolic locii of an elliptic Riley slice. By Lemma 7.3.3, $\mathcal{P}^{a,b}_{p/q} \subseteq \widetilde{\mathcal{H}}^{a,b}_{p/q}$. Based on Lemma 6.1.6, it suffices from now on to restrict ourselves to studying $p/q \in [0,1] \cap \mathbb{Q}$; from now on, we implicitly assume this inclusion.

7.3.4 Theorem. *The pleating ray $\mathcal{P}^{a,b}_{p/q}$ is the union of either one or two non-empty connected components of $\widetilde{\mathcal{H}}^{a,b}_{p/q}$. These are complex conjugate smooth 1-manifolds sets with asymptotic slope*[1] *$\pi p/q$ and $-\pi p/q$ respectively, such that the p/q-**cusp set***

$$\mathrm{Cusp}^{a,b}_{p/q} := \partial \mathcal{R}^{a,b} \cap \overline{\mathcal{P}^{a,b}_{p/q}}$$

consists of exactly two distinct complex-conjugate points.

[1]When we write that a curve k in $\mathcal{R}^{a,b}$ has 'asymptotic slope θ', we will always mean that $|\rho| \to \infty$, the argument of the tangent vector to k at ρ tends to θ.

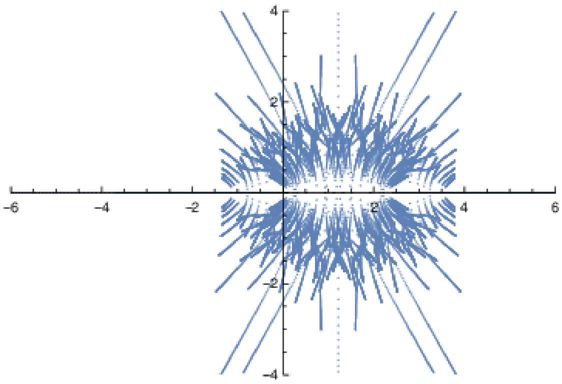


Figure 7.9: The union of the hyperbolic locii of small slope for a representative elliptic Riley slice.

Remark (A). In the parabolic case, this is stated as Theorem 2.4 of [66] (correcting Theorems 3.7 and 4.1 of [63]).

Remark (B). One can improve Theorem 7.3.4: we have already noted that $\mathcal{P}_{p/q}$ and $\mathcal{P}_{r/s}$ are disjoint if $p/q \neq r/s$ ($p/q, r/s \in \mathbb{Q} \cap [0,1]$), and this remains correct if the pleating rays are replaced with their closures [60].

We will prove a series of lemmata before giving the proof of Theorem 7.3.4 on Page 91. Most of these are adapted from the proofs given in [63, 66] with only minimal changes from the parabolic-only case.

7.3.5 Lemma. *Let $j : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a conformal or anti-conformal bijection and fix $\Gamma = \Gamma_\rho^{a,b}$ for some $\rho \in \mathcal{R}^{a,b}$. If $\mathrm{pl}(\Gamma)$ is a simple closed geodesic represented by a word $W(\rho)$ in Γ, then $\mathrm{pl}(j\Gamma j^{-1})$ is a simple closed geodesic represented by $jW(\rho)j^{-1}$.*

Proof. Since j is (anti-)conformal, it maps circles to circles. It also satisfies $\Lambda(jGj^{-1}) = j\Lambda(G)$ for any Kleinian group G, so sends F-peripheral subgroups of Γ to F-peripheral subgroups of $j\Gamma j^{-1}$. By Proposition 7.3.2, the pleating locus $\mathrm{pl}(\rho)$ is a curve represented by W in Γ iff W lies in two non-conjugate F-peripheral subgroups of Γ; by preservation of F-peripheralness this is true iff jWj^{-1} lies in two non-conjugate F-peripheral subgroups of $j\Gamma j^{-1}$, and a second application of the proposition finishes the proof. ∎

7.3.6 Lemma. *Let $p/q \in \hat{\mathbb{Q}}$ and let $j : \mathbb{C} \to \mathbb{C}$ be the complex conjugation map. Then $\mathcal{P}_{p/q}^{a,b} = j(\mathcal{P}_{p/q}^{a,b})$.*

Proof. Apply Lemma 7.3.5 to j and observe that $jXj^{-1} = X$ and $jY_\rho j^{-1} = Y_{\overline{\rho}}$. ∎

7.3A The Fuchsian case

We study the case that $\Gamma_\rho^{a,b}$ is Fuchsian separately. In this case, the entire group is F-peripheral, and $\mathrm{h.conv}\,\Lambda(\Gamma_\rho^{a,b})$ is degenerate: it is a surface, not a 3-manifold. To fit this into our strategy of considering the convex core boundary as being a pair of 2-punctured discs glued along a boundary curve represented by a Farey word, we follow the discussion at the end of Section 3 of [63] and view $\mathrm{h.conv}\,\Lambda(\Gamma_\rho^{a,b})$ as being a pleated surface with two faces (the two sides of the hyperbolic convex hull in $\mathbb{H}^3$) and a single pleating curve with bending angle π (the edge of the disc); one should think of it as being the deformation retract of the 4-times punctured sphere obtained by 'deflating' it to a disc such that the paired punctures are aligned.

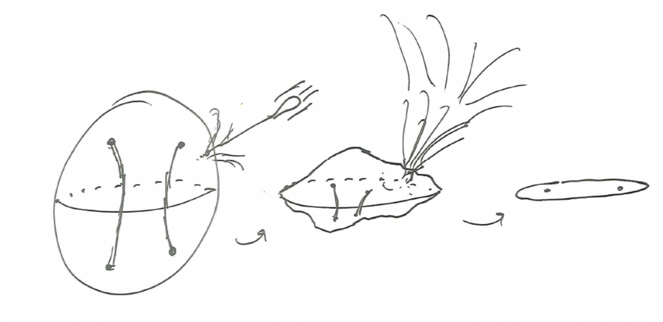


Figure 7.10: The pleated surface for a Fuchsian group in $\mathcal{R}$ is obtained by deflating the surface such that the singular arcs through the manifold retract into themselves.

For convenience, in this section we define the two (real) numbers

$$\begin{aligned} \mathbf{0} &:= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 2 + 2\cos\left(\frac{\pi}{a} - \frac{\pi}{b}\right) \\ \mathbf{1} &:= -\alpha\beta - \frac{1}{\alpha\beta} = -2 - 2\cos\left(\frac{\pi}{a} + \frac{\pi}{b}\right) \end{aligned} \tag{7.3.7}$$

where $\alpha = \exp(\pi i/a)$ and $\beta = \exp(\pi i/b)$. Observe that when $a = b = \infty$, $\mathbf{0} = 4$ and $\mathbf{1} = -4$. (These numbers will turn out to be the two cusps $\mathrm{Cusp}^{a,b}_{0/1}$ and $\mathrm{Cusp}^{a,b}_{1/1}$.)

7.3.8 Lemma. *Let $\rho \in \mathbb{R}$.*

1. *If $\rho \in (\mathbf{1}, \mathbf{0})$, then one of $Y_\rho^{-1}X$ or $Y_\rho X$ in $\Gamma^{a,b}_\rho$ is elliptic.*
2. *If $\rho \in \{\mathbf{1}, \mathbf{0}\}$, then one of $Y_\rho^{-1}X$ or $Y_\rho X$ is parabolic.*

In particular, $\rho \notin \mathcal{R}^{a,b}$ (since (1) implies that there is a relation between X and Y_ρ, and (2) implies the existence of an accidental parabolic).

Proof. The result follows immediately from directly calculating the traces of the two elements. ∎

Remark. In fact, one can improve the statement of Lemma 7.3.8. Modifications of the arguments in Section 11.4 of [13] (in particular, in the case that either X or Y_ρ is parabolic use Theorem 11.4.1 or 11.4.2 directly; if both are elliptic, modify the proof of Theorem 11.4.3 and replace Beardon's Figure 11.4.5 with our Figure 7.11), show that if $\Gamma^{a,b}_\rho$ is discrete with ρ within the bounds given in the lemma above, then $\Gamma^{a,b}_\rho$ is a triangle group. In the process one may even classify the triangle groups which appear; in recent years much work has been done to extend this classification in the case of $\mathcal{R}^{\infty,\infty}$, i.e. to specify all of the discrete groups in the exterior (a precise result, as conjectured by Agol [1] and proved in 2020 [5, 6], was stated above as Theorem 4.1.6) and to study the non-discrete groups [81].

7.3.9 Proposition. *The following are equivalent for $\rho \in \mathcal{R}^{a,b}$:*

1. *$\Gamma^{a,b}_\rho$ is Fuchsian;*
2. *$\rho \in \mathbb{R}$;*
3. *$\rho \in \mathcal{P}^{a,b}_{0/1} \cup \mathcal{P}^{a,b}_{1/1}$.*

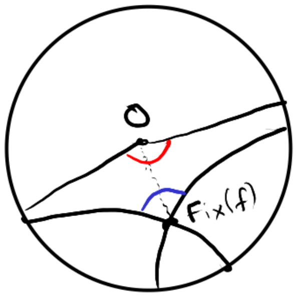


Figure 7.11: Modification of Beardon's Figure 11.4.5 in the case that both generators are known to be elliptic: we have drawn both isometric circles of f and arranged the 'cone' region for g symmetrically with respect to $\mathrm{Fix}(f)$ rather than with the midpoint of the isometric circle of f^{-1} (c.f. Figure 6.3 and Theorem 6.1.1 above); the red angle is $2\pi/b$ and the blue angle is $2\pi/a$.

Proof. It will be important to recall from Table 6.2 that

$$\Phi^{a,b}_{0/1}(\rho) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} - \rho \text{ and } \Phi^{a,b}_{1/1}(\rho) = \alpha\beta + \frac{1}{\alpha\beta} + \rho,$$

where $\alpha = \exp(\pi i/a)$ and $\beta = \exp(\pi i/b)$. We have seen that both $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ and $\alpha\beta + \frac{1}{\alpha\beta}$ are real. In particular,

$$\rho \in \mathbb{R} \iff \Phi^{a,b}_{0/1}(\rho) \in \mathbb{R} \iff \Phi^{a,b}_{1/1}(\rho) \in \mathbb{R}.$$

Now to the lemma proper. If $\Gamma^{a,b}_\rho$ is Fuchsian then $\operatorname{tr} XY_\rho = \Phi^{a,b}_{0/1}(\rho)$ is real and thus ρ is real, establishing (1) $\Longrightarrow$ (2). If ρ is real, then $\Gamma^{a,b}_\rho$ is trivially Fuchsian, establishing the converse.

Next, assume $\rho \in \mathcal{P}^{a,b}_{0/1} \cup \mathcal{P}^{a,b}_{1/1}$; then by Lemma 7.3.3, either $\Phi^{a,b}_{0/1}(\rho)$ or $\Phi^{a,b}_{1/1}(\rho)$ is real and so by the above discussion ρ is real. Conversely, suppose ρ is real. Then by Lemma 7.3.8, $\rho \notin [\mathbf{1}, \mathbf{0}]$. We now split into two cases according to whether $\rho < \mathbf{1}$ or $\rho > \mathbf{0}$; the two arguments are very similar and so we only discuss the first case. Suppose then that $\rho < \mathbf{1}$; the group $\Gamma = \Gamma^{a,b}_\rho$ is a Fuchsian group generated by two parabolics, and it acts on both the upper and lower half-planes. One can easily check that there is a fundamental domain for Γ in both half-planes consisting of the common exterior of four hyperbolic lines in $\mathbb{H}^2$ (for instance, the 2-elliptic case is seen in Figure 7.11; in general we are doing something like Theorem 6.1.1 but for Fuchsian groups). In particular, by the Poincaré polyhedron theorem, the quotient surface is a disc with two marked points, one per generator (we have just reproved Lemma 7.2.6). One also easily sees that the boundary of each disc is represented by $\mathrm{Word}(1/1) = Y_\rho X$ (in the other case, by $\mathrm{Word}(0/1) = Y_\rho^{-1} X$). It follows with an elegant inevitability that $\rho \in \mathcal{P}_{1/1}$ (resp. $\rho \in \mathcal{P}_{0/1}$). ∎

In the proof of the preceeding proposition we showed the following:

7.3.10 Corollary. *Define* $\mathbf{0}, \mathbf{1}$ *as in Equation* (7.3.7)*. Then:*

$$\mathcal{P}^{a,b}_{1/1} = (-\infty, \mathbf{1}) \qquad \mathcal{P}^{a,b}_{0/1} = (\mathbf{0}, \infty)$$
$$\mathrm{Cusp}^{a,b}_{1/1} = \mathbf{1} \qquad \mathrm{Cusp}^{a,b}_{0/1} = \mathbf{0}.$$

In particular, the two pleating rays are non-empty, 1-dimensional manifolds.

7.3B The non-Fuchsian case

We need a small technical lemma.

7.3.11 Lemma. *If $G = \langle A, B\rangle$ is a non-elementary Kleinian group with* $\operatorname{tr} A \in \mathbb{R}$, $\operatorname{tr} B \in \mathbb{R}$, *and* $\operatorname{tr} AB \in \mathbb{R}$, *then G is Fuchsian.* ∎

The proof of Lemma 7.3.11 is straightforward and left to the reader (see for example Project 6.6 of [94]).

7.3.12 Lemma. *If $p/q \notin \{0/1, 1/1\}$ then $\mathcal{P}^{a,b}_{p/q}$ is open in $\widetilde{\mathcal{H}}^{a,b}_{p/q}$.*

Proof. For convenience, in this proof we neglect to write the superscript ^{a,b} and the subscript $_{p/q}$. Suppose $\rho_0 \in \mathcal{P}$, and let K be the connected component of $\widetilde{\mathcal{H}} \cap \mathcal{R}$ containing ρ_0. Since $\mathcal{R}$ is open, K is open in $\widetilde{\mathcal{H}}$. Therefore, since $\widetilde{\mathcal{H}}$ is the inverse image of an open arc under a polynomial, there is an open arc $\alpha \subseteq K$ containing ρ_0.

By Proposition 7.3.2, Γ_{ρ_0} has two non-conjugate F-peripheral subgroups, $F_1(\rho_0)$ and $F_2(\rho_0)$, of Γ_{ρ_0} in $\mathcal{U}(\rho_0)$. For any $\rho \in \mathcal{R}$ write $F_i(\rho)$ for the quasiconformal conjugate of $F_i(\rho_0)$ lying in Γ_ρ induced by the quasiconformal conjugacy between Γ_{ρ_0} and Γ_ρ. For all $\rho \in \alpha$, $\mathrm{Word}_{p/q}(\rho) \in \Gamma_\rho$ has real trace (by definition of $\widetilde{\mathcal{H}}$). This word lies in $F_i(\rho)$ for each $\rho \in \alpha$ (by Proposition 5.1.1 the quasiconformal conjugacy preserves X and sends $Y_{\rho_0} \mapsto Y_\rho$). Hence $F_i(\rho)$ is Fuchsian: indeed, the two generators have real trace (being conjugates of elements with real trace) and $\mathrm{Word}_{p/q}(\rho)$ is the product of these two generators (by the definition of a circle chain group) and has real trace, so we may apply Lemma 7.3.11.

Now by Lemma 7.2.8, both $F_i(\rho_0)$ are strongly F-peripheral; thus by Lemma 7.2.2, there is a small open subarc of α such that all the $F_i(\rho)$ for ρ on this subarc are F-peripheral. This proves openness of $\mathcal{P}$ in $\widetilde{\mathcal{H}}$. ∎

7.3.13 Lemma. *If $p/q \notin \{0/1, 1/1\}$ then $\mathcal{P}^{a,b}_{p/q}$ is closed in $\widetilde{\mathcal{H}}^{a,b}_{p/q}$.*

Proof. For convenience, in this proof we neglect to write the superscript ^{a,b} and the subscript $_{p/q}$. Suppose $(\rho_n)_{n=1}^\infty$ is a sequence of elements in $\mathcal{P}$ and let $\rho_n \to \rho_\infty$. Let $k = \lim_{n\to\infty} \Phi(\rho_n)$; clearly this is real. Suppose that $\rho_\infty \in \widetilde{\mathcal{H}}$, so $|k| > 2$. We show that $\rho\infty \in \mathcal{P}$.

Let $F_1(\rho_1)$ and $F_2(\rho_1)$ be two non-conjugate F-peripheral groups in $\mathcal{U}(\rho_1)$. We would like to check that both of the $F_i(\rho_\infty)$ (the subgroups of Γ_{ρ_∞} corresponding to the $F_i(\rho_1)$ under the global quasiconformal conjugacy sending $\Gamma_{\rho_1} \mapsto \Gamma_{\rho_\infty}$) are F-peripheral. The problem is that we know only that $\rho_\infty \in \widetilde{\mathcal{H}}$, so it is not necessarily in $\mathcal{R}$, and so this alleged quasiconformal conjugacy does not necessarily exist! We therefore *define* the groups $F_i(\rho_\infty)$ as being algebraic limits of the $F_i(\rho_n)$ (which we may define via quasiconformal conjugacy) and check everything with these. So for each i, let $A_i(\rho_1)$ and $B_i(\rho_1)$ be elements of the correct orders (i.e. those orders which occur in the definition of circle chain groups) generating $F_i(\rho_1)$ such that $A_i(\rho_1)B_i(\rho_1) = \mathrm{Word}_{p/q}(\rho_1)$. Then (since this is just an algebraic equation), $A_i(\rho_n)B_i(\rho_n) = \mathrm{Word}_{p/q}(\rho_n)$ for every n. Further, since conjugacy is algebraic, the groups $F_i(\rho_n) = \langle A_i(\rho_n), B_i(\rho_n)\rangle$ are non-conjugate for each n. The groups also must be F-peripheral, since the choice of generating elements of the right order with product $\mathrm{Word}(p/q)(\rho_n)$ is unique in Γ_{ρ_n} up to conjugation by $\mathrm{Word}_{p/q}(\rho_n)$.

Fix i and set $F_i(\rho_\infty) \coloneqq \langle A_i(\rho_\infty), B_i(\rho_\infty)\rangle$. The two generators are of the right order and have the right product (since these are algebraic properties), and so by Lemma 7.3.11 the groups are Fuchsian. By assumption $|\Phi_{p/q}(\rho_n)| > 2$ for all n; also, observe that the peripheral discs of the $F_i(\rho_n)$ converge to one of the discs Δ bounded by the circle through the midpoint of the two fixed points of $A_i(\rho_\infty)$, the midpoint of the two fixed points of $B_i(\rho_\infty)$, and the two fixed points of $\mathrm{Word}_{p/q}(\rho_\infty)$. Since none of the converging discs contain limit points for the respective Γ_{ρ_n}, the disc Δ contains no limit points

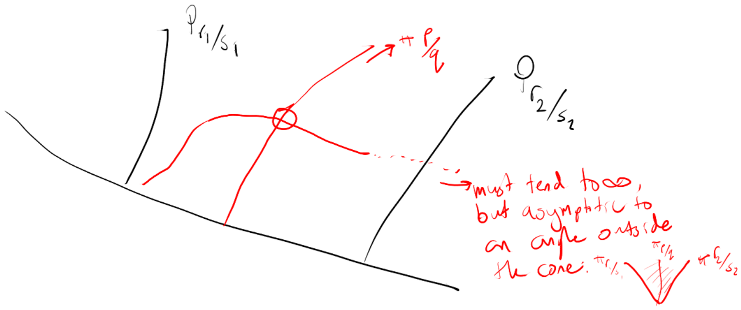


Figure 7.12: Ramifications of a singularity on a pleating ray.

for Γ_{ρ_∞}. Hence $F_i(\rho_\infty)$ is F-peripheral; the same argument for $F_{3-i}(\rho_\infty)$ and the previous observation that both groups are non-conjugate shows via Proposition 7.3.2 that $\rho_\infty \in \mathcal{P}$. ∎

7.3.14 Proposition. *$\mathcal{P}^{a,b}_{p/q}$ is a union of connected components of $\widetilde{\mathcal{H}}^{a,b}_{p/q}$.*

Proof. This is just the combination of Lemmata 7.3.12 and 7.3.13. ∎

7.3C Completing the proof of Theorem 7.3.4

7.3.15 Lemma. *The hyperbolic locus $\widetilde{H}_{p/q}$ has exactly q branches; for each $k \in \mathbb{Z}$ there is a branch which has asymptotic angle $k\pi/q$.*

Proof. As $|z| \to \infty$, $\left(\Phi^{a,b}_{p/q}(z)\right)/z^q \to 1$; and the result is clearly true for z^q. ∎

Notation. Let $\mathbb{H}^+$ and $\mathbb{H}^-$denote the upper and lower closed half-planes of $\mathbb{C}$ respectively. Given a set $A \subseteq \mathbb{C}$, write A^+ and A^- for $A \cap \mathbb{H}^+$ and $A \cap \mathbb{H}^-$ respectively.

We now proceed to prove Theorem 7.3.4 (stated on Page 86).

Proof of Theorem 7.3.4. The proof goes via induction on the Farey tree. That is, for the inductive step we assume the statement for $\alpha, \beta \in \mathbb{Q}$ and then prove it for $\alpha \oplus \beta$; the base cases are the statements for 0/1 and 1/1, which we proved as Corollary 7.3.10. (See Figure 9.2 for a graphical illustration showing the inductive paths.)

Now suppose we are interested in $p/q \in \mathbb{Q}$, and we have proved that the two neighbours of p/q in the qth Farey sequence satisfy the conclusions of the theorem; say that these neighbours are r_1/s_1 and r_2/s_2. Pick $z_1 \in \mathcal{P}_{r_1/s_1}$ and $z_2 \in \mathcal{P}_{r_2/s_2}$. By Lemma 7.3.6, we may assume that $z_1, z_2 \in \mathbb{H}^+$. Join z_1 and z_2 by an arc α lying entirely within $\mathcal{R} \cap \mathbb{H}^+$. Now by continuity of pl and the intermediate value theorem, there is a point z on α with $\mathrm{pl}(z) = (r_1/s_1) \oplus (r_2/s_2) = p/q$. Hence we see $z \in \mathcal{P}_{p/q}$ so the pleating ray is non-empty.

Now let $\mathcal{H}_{p/q}$ be the union of the two branches of $\widetilde{\mathcal{H}}_{p/q} \cap \mathbb{H}^+$ with asymptotic angles $\pi p/q$ and $-\pi p/q$ respectively (Lemma 7.3.15). There is exactly one branch of $\widetilde{\mathcal{H}}_{p/q}$ in $\mathbb{H}^+$ bounded between $\mathcal{H}^+_{r_1/s_1}$ and $\mathcal{H}^+_{r_2/s_2}$, because by Lemma 6.3.1 there are no other fractions of denominator at most q between r_1/s_1 and r_2/s_2. Similarly we see that there is one branch of $\widetilde{\mathcal{H}}_{p/q}$ in $\mathbb{H}^+$ bounded between $\mathcal{H}^-_{r_1/s_1}$ and $\mathcal{H}^-_{r_2/s_2}$. It follows from Proposition 7.3.14 that $\mathcal{P}_{p/q} = \mathcal{H}_{p/q}$.

We next check that $\mathcal{P}_{p/q}$ is non-singular. Suppose for contradiction that $\mathcal{P}_{p/q}$ contained a component of $\widetilde{\mathcal{H}}_{p/q}$ with a singular point: that is, a critical point ρ_0 of the trace polynomial $\Phi_{p/q}$, say of degree

n. Then $\Phi_{p/q}(\rho)$ expands about ρ_0 as a power series of the form

$$\Phi_{p/q}(\rho) - \Phi_{p/q}(\rho_0) = (\rho - \rho_0)^n \sum_{m=0}^{q-n} a_m(\rho - \rho_0)^m$$

and hence pre-image of a line through $\Phi_{p/q}(\rho_0)$ has $2n$ branches through ρ_0. In particular, there are at least two branches of $\widetilde{\mathcal{H}}$ of increasing trace through ρ_0 (since we assume $n \geq 1$). We have already seen that exactly one of these branches lies asymptotically between $\mathcal{P}_{r_1/s_1}$ and $\mathcal{P}_{r_2/s_2}$, so the other must cross either of these two pleating rays within the Riley slice; and hence that branch (a subset of the connected component allegedly contained in the pleating ray) cannot lie in $\mathcal{P}_{p/q}$, for branches of different slope are disjoint by the definition of the pleating rays (Figure 7.12). This is the desired contradiction.

Since the Farey polynomial has no critical points on a pleating ray, it has no local maxima or minima. In particular, it is monotone increasing or decreasing when restricted to the pleating ray. It is clear now that the closure of the pleating ray intersects $\partial\mathcal{R}$ at exactly two points: it intersects at the cusp groups where the Farey polynomial becomes -2 (so intersects at least twice) and cannot intersect anywhere else (since the interior of $\mathcal{P}_{p/q}$ lies in the Riley slice and is unbounded on two ends, being a union of two components of $\widetilde{\mathcal{H}}$, the only additional points added when taking the closure are these two). ∎

7.4 Irrational pleating rays

Recall from Theorem 6.1.5 that slight deformations of lines of rational slope which represent curves on the 4-marked sphere give foliations of the sphere which are parameterised by irrational slopes. This, along with the fact that the rational pleating rays are dense in the slice (which we will prove below as Corollary 7.4.5), indicates that we can extend our lamination of $\mathcal{R}^{a,b}$ to a foliation by considering those groups which induce a pleating locus of irrational slope. Thus we extend Definition 7.3.1 as follows:

7.4.1 Definition. Let $\lambda \in \hat{\mathbb{R}}$. The λ-**pleating ray** is the set

$$\mathcal{P}_\lambda^{a,b} := \{\rho \in \mathcal{R}^{a,b} \,:\, \mathrm{pl}(\rho) = \hat{\gamma}(\lambda)\}$$

where $\hat{\gamma}(\lambda)$ is the nearest-point retraction to $\mathcal{C}(\Gamma_\rho)$ of the curve $\gamma(\lambda)$ defined in Theorem 6.1.5. If $\lambda \in \hat{\mathbb{Q}}$ we call the pleating ray **rational**; otherwise we call it **irrational**.

7.4A Irrational properties analogous to the rational properties

In this subsection, we collect various 'niceness' results for irrational pleating rays that we have already seen for the rational ones. We essentially follow Section 6.1 of [62].

7.4.2 Lemma. *Let $\lambda, \lambda' \in \mathbb{R}$.*

1. *$\mathcal{P}_\lambda \neq \emptyset$.*
2. *$\mathcal{P}_\lambda \cap \mathcal{P}_{\lambda'} = \emptyset \iff \lambda \neq \lambda'$.*

Proof. That different pleating rays are disjoint is trivial from the definition. That $\mathcal{P}_\lambda \neq \emptyset$ follows from the same proof as that given in Theorem 7.3.4: pick p/q and r/s with $p/q < \lambda < r/s$ and use continuity of pl. ∎

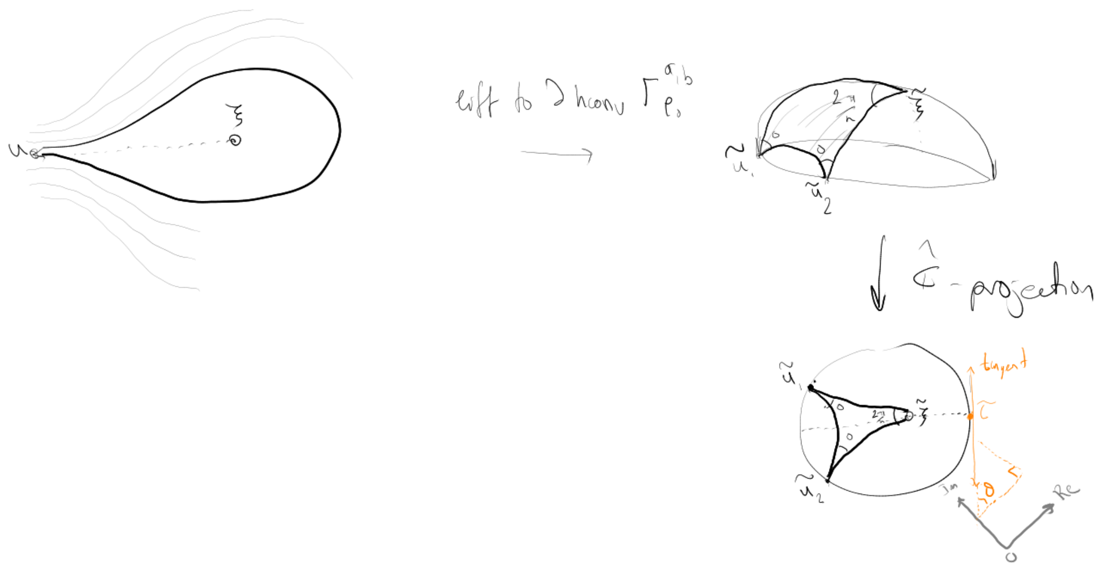


Figure 7.13: Figure for the proof of Proposition 7.4.4.

7.4.3 Lemma. $\mathcal{R}^{a,b} = \bigcup_{\lambda \in \mathbb{R}} \mathcal{P}_\lambda$.

Proof. By the discussion of the Fuchsian case above, we see that if $\rho \in \mathbb{R} \cap \mathcal{R}^{a,b}$ then ρ lies on a pleating ray. On the other hand if $\rho \in \mathcal{R}^{a,b}$ lies off the real axis, then the limit set $\Lambda(\Gamma_\mu^{a,b})$ is not the full Riemann sphere and so the pleating locus is nontrivial; by Theorem 6.1.5 the pleating locus is represented by $\gamma(\lambda)$ for some λ. ∎

Next we have an analogue/extension of Theorem 7.3.4. This is essentially a concrete application of the theory contained in Chapter 9 of [124], as we are analysing the geometrically infinite ends of the deformation space.

7.4.4 Proposition. *For $\lambda \in \mathbb{R}$, the pleating ray $\mathcal{P}_\lambda^{a,b}$ lies in the real locus of an analytic function defined on $\mathcal{R}^{a,b}$.*

Remark. We modify the function Ψ from Lemma 5.1 of [63] to obtain the desired function; a more elegant choice of function surely exists.

Proof. Assume $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ and fix $\rho_0 \in \mathcal{P}_\lambda^{a,b}$. Let ξ be one of the four marked points on the convex core $S = \mathcal{C}(\Gamma_{\rho_0}^{a,b})$, say of order n. The leaves of $\mathrm{pl}(\rho_0)$ lie away from ξ (the leaves, being geodesic, cannot pass over ξ directly, so they lie in $S \setminus \{\xi\}$; pick an open neighbourhood of the deleted point, then this is non-compact but the pleating locus of $\mathrm{pl}(\rho_0)$ is compact, and so we may shrink this open neighbourhood to avoid $\mathrm{pl}(\rho_0)$), hence ξ lies in a flat part of the lamination. By Thurston's theory of measured laminations (in particular [124, Proposition 9.5.4]) this flat part must be a monogon with a single marked point ξ in the interior. Cutting along a geodesic between ξ and the monogon vertex gives a triangle which lifts into a flat piece of $\partial\,\mathrm{h.conv}\,\Lambda(\Gamma_{\rho_0}^{a,b})$ (Figure 7.13); the lifted triangle on the induced hyperbolic metric of this flat piece has two ideal vertices $\tilde{u}_1$ and $\tilde{u}_2$, and one vertex $\tilde{\xi}$ with angle $2\pi/n$ (which may or may not be ideal, according to whether ξ is a puncture). Now define a point τ in the following way: take the diameter of the circle in $\hat{\mathbb{C}}$ bounding the support plane which the three points lie on which passes through $\tilde{\xi}$. This line cuts the circle at two points, and one of these intersubsections τ is uniquely determined by being on the same side of $[\tilde{u}_1, \tilde{u}_2]$ as $\tilde{\xi}$. Let θ be the angle of the tangent line to the circle at τ; one can determine a formula for θ in terms of the positions of u_1, u_2, τ using hyperbolic trigonometry, and in fact we may find an analytic function $\sigma : \mathbb{C}^3 \to \mathbb{C}$ which sends u_1, u_2, τ to a complex number $\sigma(u_1, u_2, \tau)$ with argument equal to θ.

The positions of $u_1, u_2, \tilde{\xi}$ and hence u_1, u_2, τ depend analytically on the value of ρ and do not collide, by the λ-lemma [37, 116] — for the ideal points one needs no further argument, but for the possible non-ideal point use the fact that $\tilde{\xi}$ lies on the axis of the corresponding generator and must form a triangle with angle $2\pi/n$ on the sphere passing through it and the other two points. In any case, we have an analytic function $\rho \mapsto \sigma(\rho)$, and this is a function defined on $\mathbb{C}$ (since it can be written entirely in terms of the trigonometry of the point configurations).

Now observe that for $\rho \in \mathcal{P}_\lambda$, θ must be the same angle ϕ as the line joining the two fixed points of the generator representing ξ (or the same angle as the translation direction, in the case of a parabolic). In particular we see that the function Ψ defined by $\Psi(\rho) = \sigma(\rho)\exp^{-i\phi(\rho)}$ is an analytic function which is real on the pleating ray $\mathrm{pl}(\lambda)$. ∎

We may now prove density of the rational pleating rays, following [62, Corollary 6.2]. Compare this with Theorem 5.2.5.

7.4.5 Corollary. *The union $\bigcup_{\lambda\in\mathbb{Q}} \mathcal{P}^{a,b}_\lambda$ is dense in $\mathcal{R}^{a,b}$.*

Proof. Suppose $\rho_0 \in \mathcal{R}^{a,b}$ lies on the irrational pleating ray $\mathcal{P}_\lambda$ for some $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. By Proposition 7.4.4 the pleating ray is smooth and so by openness of $\mathcal{R}^{a,b}$ there is a path $\rho : [0,1] \to \mathcal{R}^{a,b}$ with $\rho(0) = \rho_0$ and which is locally transverse to $\mathcal{P}_\lambda$. By Theorem 3.6.15 the map $\mathrm{pl} \circ \rho : [0,1] \to \mathbb{R}$ (where we are using Theorem 6.1.5 to view pl as a map into $\mathbb{R}$) is continuous and by transversality for small $\varepsilon > 0$ we have $(\mathrm{pl} \circ \rho)(\varepsilon) \neq \lambda$. Hence by the intermediate value theorem and density of $\mathbb{Q}$ in $\mathbb{R}$ there is some $\varepsilon' \in (0, \varepsilon]$ with $(\mathrm{pl} \circ \rho)(\varepsilon') \in \mathbb{Q}$. In particular by choosing $\rho(1)$ arbitrarily close to $\rho(0)$ we may find rational pleating rays arbitrarily close to $\rho(0)$. ∎

A simple modification to the argument of Corollary 7.4.5 gives a more quantitative result (c.f. [62, Lemma 6.3]):

7.4.6 Lemma (Rational approximation lemma)**.** *Suppose that $\rho_\infty \in \mathcal{P}_\lambda$ and that $p_n/q_n \in \mathbb{Q}$ is a sequence with $\lim_{n\to\infty} p_n/q_n = \lambda$. Then for each n there is a point $\rho_n \in \mathcal{P}_{p_n/q_n}$ such that $\lim_{n\to\infty} \rho_n = \rho_\infty$.*

Proof. Without loss of generality assume p_n/q_n is an increasing sequence. Pick an arbitrary $\rho_1 \in \mathcal{P}_{p_1/q_1}$, and pick a path $\sigma : [0,1] \to \mathcal{R}$ with $\sigma(0) = \rho_1$ and $\sigma(1) = \rho_\infty$ which is transverse to $\mathcal{P}_\lambda$ at ρ_∞ and which does not meet $\mathcal{P}_\lambda$ at any other point. As in the proof of Corollary 7.4.5, by the continuity of pl, for each n since $p_1/q_1 \leq p_n/q_n \leq \lambda$ the path σ intersects $\mathcal{P}_{p_n/q_n}$ at some point $\rho_n = \sigma(t_n)$. By deforming σ if necessary we may assume that the sequence (t_n) is monotone increasing. By the monotone convergence theorem, this sequence has a limit $t_\infty := \lim_{n\to\infty} t_n$. Now by continuity of $\mathrm{pl} \circ \sigma$,

$$\mathrm{pl}(\sigma(t_\infty)) = \lim_{n\to\infty} \mathrm{pl}(\sigma(t_n)) = \lim_{n\to\infty} p_n/q_n = \lambda$$

so $\sigma(t_\infty) \in \mathcal{P}_\lambda$ and hence $\sigma(t_\infty) = \rho_\infty$ by the assumption that σ hits $\mathcal{P}_\lambda$ exactly once. ∎

We give finally a nice consequence of the indexing of the Keen–Series foliation by $\mathbb{R}$ and the theory above, which is not important in the logical development of the theory:

7.4.7 Corollary. *The union of the rational pleating rays is of null measure in the Riley slice.*

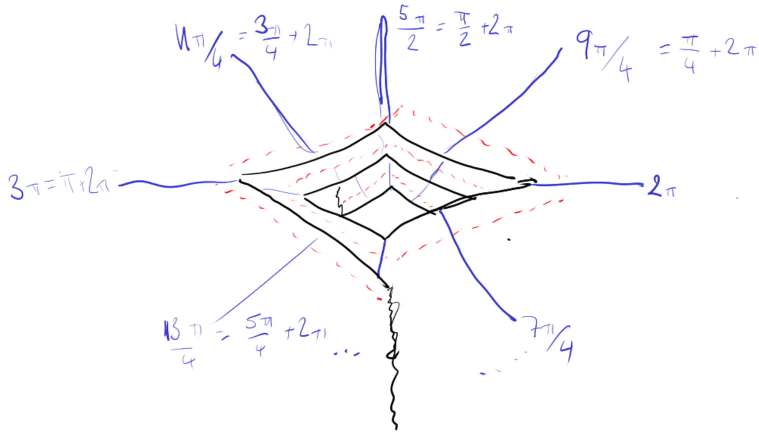


Figure 7.14: The universal cover of $\mathcal{R}^{a,b}$.

Proof. $\mathbb{Q}$ is measure zero in $\mathbb{R}$. ∎

We will use this analogy between the inclusion

$$\{\text{rational pleating rays}\} \subseteq \{\text{leaves of the Keen–Series foliation}\}$$

and $\mathbb{Q} \subseteq \mathbb{R}$ in a more fundamental way: namely, in the next section we will prove a stronger analogue of Theorem 7.3.4 than Proposition 7.4.4 by a process exactly analogous to the proof of properties of $\mathbb{R}$ via the Dedekind cut construction from $\mathbb{Q}$.

7.4B The family of normalised complex length functions

We modify the length formulae to make sense in the irrational case: there are two of these formulae, firstly the length formula for measured laminations in $\mathcal{ML}(S)$ from Definition 3.4.5 which is infinite on dense foliations, and secondly the formula which determines the length of the curve $\gamma(p/q)$ from the trace of the Farey word from Lemma 6.1.3 and which is clearly meaningless when the word is biinfinite. Since $\left|\Phi^{a,b}_{p/q}(\rho)\right| \neq 2$ for $\rho \in \mathcal{R}^{a,b}$ (there are no accidental parabolics in the interior), and since $\Phi^{a,b}_{p/q}$ is real on $\mathcal{P}_{p/q}$ by Lemma 7.3.3, there is a branch of the complex length function

$$\rho \mapsto \operatorname{trlen}(\operatorname{Word}(p/q)(\rho)) = 2 \operatorname{arccosh} \frac{\Phi^{a,b}_{p/q}(\rho)}{2}$$

on the universal cover $\widetilde{\mathcal{R}^{a,b}}$ of $\mathcal{R}^{a,b}$ (Figure 7.14) which is real on $\mathcal{P}^{a,b}_{p/q}$; we call this branch $\hat{L}_{p/q}$. This function still does not make sense in the irrational case, and as $p/q \to \lambda \in \mathbb{R} \setminus \mathbb{Q}$ one observes heuristically that $\hat{L}_{p/q}(\rho) \to \infty$. The problem is that an irrational line accumulates infinite length because it wraps 'horizontally' around the sphere infinitely many times. We therefore divide each rational length by the (finite) number of 'wraps' before taking the limit; this turns out to work.

By consideration of Figure 6.5 and the remainder of the theory in that section, we see that the intersection number $i(\gamma(p/q), \gamma(\infty))$ is $q/2$. We therefore define the **normalised complex length** of $\operatorname{Word}(p/q)$ on $\widetilde{\mathcal{R}^{a,b}}$ to be the function defined by

$$L^{a,b}_{p/q}(\rho) := \frac{\hat{L}_{p/q}(\rho)}{i(\gamma(p/q), \gamma(\infty))} = \frac{2}{q}\hat{L}_{p/q}(\rho).$$

Observe that (by definition) we have, for $\rho \in \mathcal{P}^{a,b}_{p/q}$,

$$L^{a,b}_{p/q}(\rho) = \frac{l_\rho(\gamma(p/q))}{i(\gamma(\rho), \gamma(p/q))}$$

where we identify the measured lamination with unit mass across the single bending locus $\gamma(p/q)$ with $\gamma(p/q)$ itself, and where we have chosen the branch upon projection to the the covered space, $\mathcal{R}^{a,b}$, to be the branch which has

$$\inf_{\rho \in \mathcal{P}^{a,b}_{p/q}} L^{a,b}_{p/q}(\rho) = 0.$$

(Compare all this to Lemma 6.4 of [62].) Another way of putting this is that $L^{a,b}_{p/q}$ induces by projection a multivalued function $\mathcal{R}^{a,b} \to \mathbb{C}$ where the function values at ρ differ by 2π, and we choose the branch which assignes the minimal positive length out of these options. The reason that the values differ by 2π is that a Dehn twist by the geodesic around the compression disc adds 2π to the length.

We give another characterisation of $L_{p/q}$ in terms of the bending measure on the pleated surface (c.f. [62, Proposition 6.7 and the preceeding discussion]). View $\mathrm{pl}(\rho)$ as the pleating lamination without any measure structure; use β_ρ to denote the usual bending measure on $\mathrm{pl}(\rho)$. We define a new transverse measure, the **pleating measure** π_ρ, to be the transverse measure on $\mathrm{pl}(\rho)$ given by

$$\int_I f d\pi_\rho := \int_I f d\left(\frac{\beta_\mu}{i(\gamma(\infty), \beta_\mu)}\right)$$

for measured intervals I on the surface.

7.4.8 Proposition. *Define the **pleating length** function $PL : \mathcal{R}^{a,b} \to \mathbb{R}$ by*

$$PL(\rho) = l_\rho(\pi_\rho)$$

where, as in Theorem 3.6.15, we have identified the pleating locus $\mathrm{pl}(\rho)$ *with its standard bending measure β_ρ, and where l_ρ is the lamination length as in Theorem 3.6.14. Then PL is continuous on $\mathcal{R}^{a,b}$, and*

$$PL(\rho) = L^{a,b}_{p/q}(\rho)$$

for all $\rho \in \mathcal{R}^{a,b}$.

Proof. The equality is evident by definition of the normalised complex length; the main thing to note is that the map $\rho \mapsto \pi_\rho$ is continuous by Theorems 3.6.15 and 3.6.16 and then PL is continuous by Theorem 3.6.14. ∎

In order to check that the irrational limit is meaningful, we recall from [108, Definition 14.5] the following definitions.

7.4.9 Definition. Let $U \subseteq \mathbb{C}$ be a connected and simply connected domain, and let $\mathfrak{F}$ be a family of holomorphic functions $U \to \mathbb{C}$. The family $\mathfrak{F}$ is said to be **normal** if every sequence (f_n) in $\mathfrak{F}$ contains a subsequence which converges uniformly on every compact subset of U.

The following standard theorem is then found as [108, Theorem 14.5].

7.4.10 Theorem. *A sufficient condition for the family $\mathfrak{F}$ of holomorphic functions $U \to \mathbb{C}$ to be normal in the region U is that $\mathcal{F}$ be uniformly bounded on each compact subset of K.* ∎

That this condition is met by the family of complex pleating length functions follows from the following lemma, which may be proved by following exactly the recipe of Section A.4 of [62, pp. 748–749]; we do not need the precise details of the bound so we do not give the details.

7.4.11 Lemma. *The coefficients of $\Phi^{a,b}_{p/q}$ are uniformly bounded by a quantity depending only on q.* ∎

Applying Lemma 7.4.11 and Theorem 7.4.10, we see that the family $\{L^{a,b}_{p/q}\}$ of functions $\mathcal{R} \to \mathbb{C}$ is a normal family on $\mathcal{R}$.

The final ingredient in our proof that there are well-defined limits of the rational normalised complex lengths will be the following lemma about convergence of the real part of the complex length (following the analogous [62, Lemma 6.8]).

7.4.12 Lemma. *Suppose that $p_n/q_n \to \lambda \in \mathbb{R}$ and that $\rho_0 \in \mathcal{P}^{a,b}_\lambda$. Then the real translation lengths converge to the irrational lamination length:*

$$\Re L_{p_n/q_n}(\rho_0) \to PL(\rho_0).$$

Proof. Let l be a leaf of $|\pi_\lambda|$; for every $\varepsilon > 0$, there exists some N such that for all $n \geq N$, there is a leaf l_n of $\left|\pi_{p_n/q_n}\right|$ within distance ε of l. Indeed (following the proof of [124, Proposition 8.10.3]), let α be a short arc transverse to l at some point, then π_λ acts as a measure on the space T of unit tangent vectors on α. Let x be the unit tangent vector at the intersection $\alpha \cap l$, and let U be a ε-neighbourhood of x in the space T. By the convergence $p_n/q_n \to \lambda$, there must be N large enough that for $n > N$, π_{p_n/q_n} assigns positive measure to U; in particular, U must intersect $\left|\pi_{p_n/q_n}\right|$.

Since the rational laminations are supported on geodesics, each leaf l_n has a lift $\tilde{l_n}$ into $\mathbb{H}^3$ preserved by some loxodromic g_n, a conjugate of the Farey word $\mathrm{Word}(p_n/q_n)$. Let S be the quotient surface $\mathcal{S}(\Gamma^{a,b}_{\rho_0})$; the leaf l is in the pleating locus of S, and so l lifts to a geodesic in $\mathbb{H}^3$; since the leaves l_n converge to l, we may pick the lifts compatibly so that $\tilde{l_n} \to \tilde{l}$ and hence the fixed points of the g_n (the endpoints of the $\tilde{l_n}$) converge to the endpoints of $\tilde{l}$. In particular, the axes of the g_n converge to $\tilde{l}$. and the axes of the g_n and the geodesic lifts $\tilde{l_n}$ both tend to the same curve. Replacing the axes with their projections onto $\partial\,\mathrm{h.conv}\,\Lambda(\Gamma^{a,b}_{\rho_0})$ does not change this, so (taking lengths after projecting to the quotient) $\Re\,\mathrm{trlen}(g_n)$ tends to the same value as $l_{\rho_0}(l_n)$. Normalising by dividing by $i(\gamma(\infty), \gamma(p_n/q_n))$, the latter clearly tends to $l_{\rho_0}(\pi_\lambda) = PL(\rho_0)$; the former is just $\Re L_{p_n/q_n}(\rho_0)$. ∎

We now deduce the existence of normalised complex length functions for irrational laminations, following Theorem 6.9 of [62]. If $U, V \subseteq \mathbb{C}$ are open, write $H(U, V)$ for the space of holomorphic functions $U \to V$ with the topology of uniform convergence on compact subsets.

7.4.13 Proposition. *The family $\{L^{a,b}_{p/q}\}_{p/q\in\mathbb{Q}}$ of holomorphic functions $\mathcal{R} \to \mathbb{C}$ extends to a family of holomorphic functions $\{L^{a,b}_\lambda\}_{\lambda\in\mathbb{R}}$, such that $L_\lambda(\rho) = PL(\rho)$ for each $\rho \in \mathcal{P}_\lambda$ and such that the map $\mathbb{R} \to H(\mathcal{R}^{a,b}, \mathbb{C})$ defined by $\lambda \mapsto L_\lambda$ is continuous.*

Proof. Let (p_n/q_n) be a sequence in $\mathbb{Q}$ with $\lim_{n\to\infty} p_n/q_n = \lambda$. We wish to show that the sequence of functions $f_n \coloneqq L_{p_n/q_n}$ converges uniformly on compact subsets to a function $f : \mathcal{R}^{a,b} \to \mathbb{C}$ which agrees with PL on $\mathcal{P}_\lambda$ and which is independent of the choice of sequences. Since the family indexed on $\mathbb{Q}$ is a normal family, we may replace (f_n) with a subsequence convergent on compact subsets of $\mathcal{R}^{a,b}$; define f to be the limit of this sequence. Pick $\rho_\infty \in \mathcal{P}_\lambda$. By Lemma 7.4.12, $\Re f_n(\rho_\infty) \to PL(\rho_\infty)$ and so the functions $\Re f$ and PL agree on $\mathcal{P}_\lambda$. It is easy to see that $\mathcal{P}_\lambda$ is uncountable: the rational pleating rays are clearly uncountable, so we may do an argument similar to that in the proof of Theorem 7.3.4: pick an uncountable sequence on two rational pleating rays bounding $\mathcal{P}_\lambda$ (say, two from the sequence (p_n/q_n)), ordered to be monotone increasing; for each pair join them by an arc;

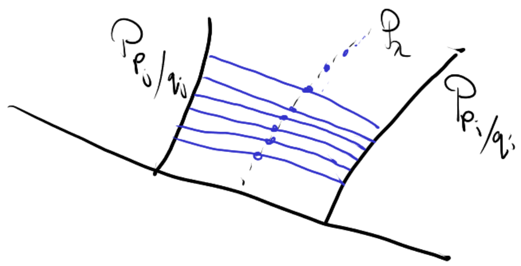


Figure 7.15: Pleating rays have uncountably many points.

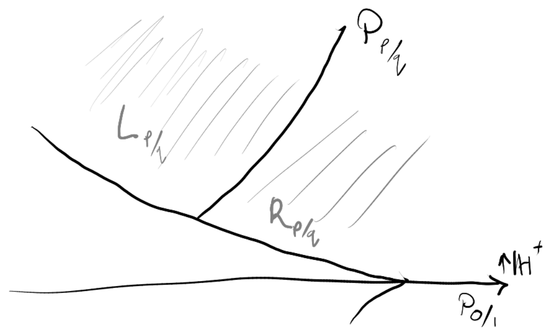


Figure 7.16: The 'cut sets' $\mathbf{L}_{p/q}$ and $\mathbf{R}_{p/q}$ in $\mathbb{H}^+$, from the proof of Theorem 7.4.14.

by continuity each of these arcs must intersect $\mathcal{P}_\lambda$ (Figure 7.15). Since all of the functions we are dealing with are holomorphic, we see that knowing $\Re f$ on this uncountable subset determines f everywhere up to a constant (by the Cauchy-Riemann equations); since *PL* is clearly independent of any kind of sequence (p_n/q_n), so too is this determination of f up to a constant. We finally check that f is real-valued at ρ_∞, which removes this dependence on a constant. By Lemma 7.4.6, there is a sequence of points $\rho_n \in \mathcal{P}_{p_n/q_n}$ tending to ρ_∞. Since $f_n(\rho_n) \in \mathbb{R}$ for each n and $f_n \to f$ uniformly around ρ_∞, $f(\rho_\infty) \in \mathbb{R}$. ∎

We now prove the main result of this subsection, an extension of Theorem 7.3.4 to include all pleating rays, whether rational or irrational. We follow the proof of [62, Theorem 7.2], adapted for the Riley slices as indicated in Theorem 5.3 of [63]. The proof goes via a Dedekind-cut-like argument (c.f. [70]).

7.4.14 Theorem. *The pleating ray $\mathcal{P}_\lambda$ is a union of two complex-conjugate connected components of the real locus of the complex pleating length L_λ in $\mathcal{R}^{a,b}$. This component contains no singularities and is asymptotic to the line of slope πk as $|\rho| \to \infty$.*

Proof. As above, we work in $\mathbb{H}^+$ and the corresponding statement for the component in $\mathbb{H}^-$ is proved in exactly the same way (c.f. the discussion of [66]).

A. Rational pleating rays cut the upper slice into two components. If $p/q \notin \{0/1, 1/1\}$ then $\mathbb{H}^+ \setminus \mathcal{P}_{p/q}$ is split into two connected components; call these $\mathbf{L}_{p/q}$ (the left component, bounded by $\mathcal{P}_{1/1}$) and $\mathbf{R}_{p/q}$ (the right component, bounded by $\mathcal{P}_{0/1}$). See Figure 7.16.

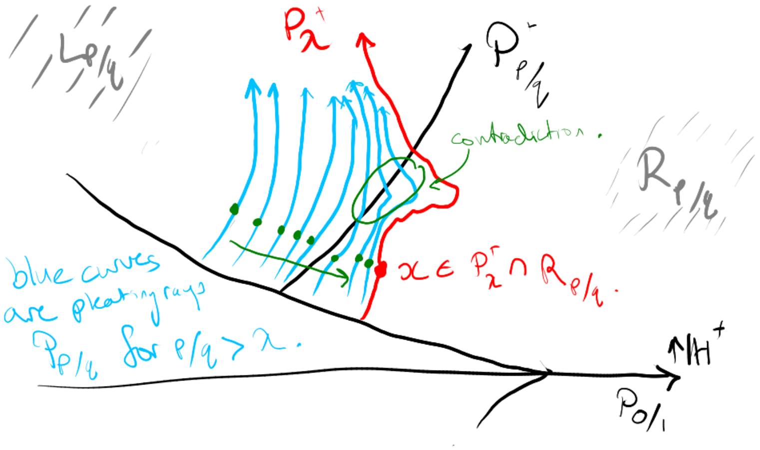


Figure 7.17: Irrational pleating rays lie on exactly one side of every rational ray.

We now claim that every $\mathcal{P}_\lambda^+$ ($\lambda \neq p/q$) lies on exactly one side of $\mathcal{P}_{p/q}$. More precisely, fix $\lambda \in \mathbb{R}$, such that $\lambda > p/q$; we claim that $\mathcal{P}_\lambda^+ \subseteq \mathbf{L}_{p/q}$. The proof is illustrated in Figure 7.17. The claim is easily shown for $\lambda \in \mathbb{Q}$, since asymptotically $\mathcal{P}_\lambda^+$ lies in $\mathbf{L}_{p/q}$ and cannot cross $\mathcal{P}_{p/q}^+$ or $\mathcal{P}_{1/1}$. If $\lambda \in \mathbb{R}$, suppose for contradiction that there is some $x \in \mathcal{P}_\lambda^+ \cap \mathbf{R}_{p/q}$. By Lemma 7.4.6, we can find a sequence of rational pleating rays containing points tending to x, with monotone decreasing slope; but $\mathbf{R}_{p/q}$ is open, so one of these rays, say of slope r/s, must intersect a small neighbourhood of x in $\mathbf{R}_{p/q}$; by the rational case we see $r/s < p/q$ but $\mathcal{P}_{r/s}^+$ intersects $\mathbf{R}_{p/q}$ giving the desired contradiction. The same argument shows that $\lambda < p/q$ implies that $\mathcal{P}_\lambda^+ \subseteq \mathbf{R}_{p/q}$.

B. Irrational pleating rays cut the upper slice into two components. We now define these cut sets for irrational λ. If $\lambda \in \mathbb{R}$ define

$$\mathbf{L}_\lambda := \bigcup_{1>r/s>\lambda} \mathbf{L}_{r/s} \quad \text{and} \quad \mathbf{R}_\lambda := \bigcup_{\lambda>r/s>0} \mathbf{R}_{r/s}.$$

Immediately from the definition and the previous discussion these sets are disjoint. In addition, $\mathcal{P}_\lambda^+ = (\mathcal{R}^{a,b} \cap \mathbb{H}^+) \setminus (\mathcal{L}_\lambda \cup \mathcal{R}_\lambda)$: one inclusion follows because $\mathcal{P}_\lambda^+$ is disjoint from both $\mathcal{L}_\lambda$ and $\mathcal{R}_\lambda$; now let $x \in (\mathcal{R}^{a,b} \cap \mathbb{H}^+)$, x lies on some pleating ray $\mathcal{P}_\mu^+$; suppose $\mu \neq \lambda$, then there exists some rational p/q between μ and λ in $\mathbb{R}$, and so x must lie in either $\mathcal{L}_\lambda$ or $\mathcal{R}_\lambda$ according to whether p/q is greater or less than λ. We see therefore that $(\mathcal{R}^{a,b} \cap \mathbb{H}^+) \setminus \mathcal{P}_\lambda^+$ has exactly two connected components, $\mathcal{L}_\lambda$ and $\mathcal{R}_\lambda$. In fact, we have shown that $\mathcal{P}_\lambda^+$ is connected, and a curve.

C. Completing the proof. Through any point of $\mathcal{P}_\lambda^+$ ($\lambda \in \mathbb{R}$) there must be an arc in the set $\mathcal{P}_\lambda^+$ along which the normalised complex length L_λ is monotone. More precisely, there is a *maximal* curve $\rho : (0,1) \to \mathcal{P}_\lambda^+$ such that $L_\lambda \circ \rho$ is monotone. Since $\mathcal{P}_\lambda^+$ is closed in the Riley slice (by B.), the limits of $L_\lambda \circ \rho(t)$ as $t \to 0$ or $t \to 1$ lie in the Riley slice exterior $\hat{\mathbb{C}} \setminus \mathcal{R}^{a,b}$.

Let π be the projection to the quotient space $\hat{\mathbb{C}}/\sim$, where $\sim$ is the equivalence relation identifying the points in $\hat{\mathbb{C}} \setminus \mathcal{R}^{a,b}$ together to form a point $\xi \in \hat{\mathbb{C}}/\sim$. Then $\hat{\mathbb{C}}/\sim$ is a sphere with two poles identified (Figure 7.18) and $\pi \circ \rho$ is a curve in the quotient extending to a Jordan curve (which we denote by the same symbols) by adding two identified endpoints at ξ.[2] Thus $\pi \circ \rho$ separates the portion

[2] Keen and Series cite Lemma 15.6 of Milnor's original preprint for *Dynamics in one complex variable* as the inspiration for this argument; for convenience, this is Lemma 17.6 in the third edition [92].

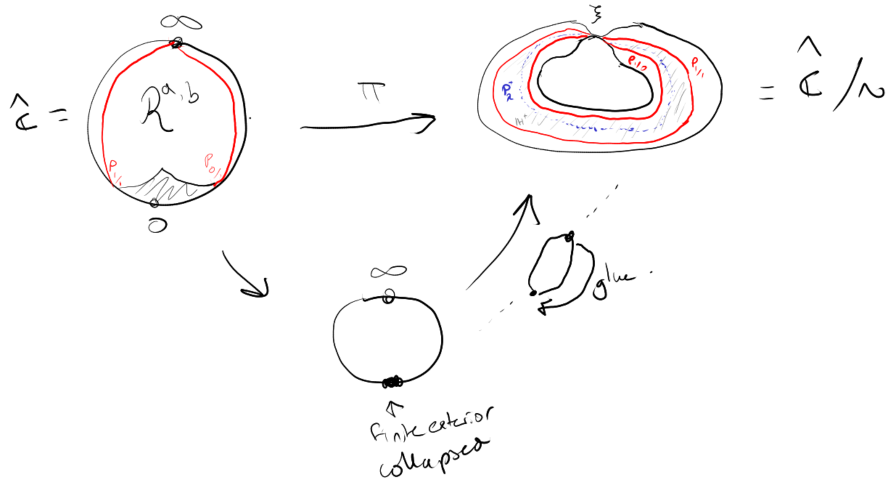


Figure 7.18: The quotient surface from part C. of the proof of Theorem 7.4.14.

of $\hat{\mathbb{C}}/\sim$ between the projections of $\mathcal{P}_{0/1}$ and $\mathcal{P}/1/1$ into two components, and ρ separates $\mathcal{R}^{a,b} \cap \mathbb{H}^+$ into two components. Suppose that $\mathcal{P}_\lambda^+$ contained a singularity; then it has a different branch which must also separate $\mathcal{R}^{a,b} \cap \mathbb{H}^+$ into two different components; but we have already checked that $\mathcal{P}_\lambda^+$ splits $\mathcal{R}^{a,b} \cap \mathbb{H}^+$ into exactly two components.

Finally, the statement about the asymptotic angle is obvious once we know the ray is a smooth curve: it can be approximated on both sides by a sequence of smooth curves with asymptotic slopes approaching $\pi\lambda$ (by Lemma 7.4.6 again) so has asymptotic slope $\pi\lambda$. ∎

7.4C The pleating ray coordinate system for $\mathcal{R}$

We now proceed to follow [63, §5] and [62, §§6–7] in order to prove the following result, giving a coordinate system on the Riley slices. The proof is exactly that given on p.743 of [62] but with some additional detail.

7.4.15 Theorem. *The map*

$$\Pi^{a,b} : \mathcal{R}^{a,b} \to \mathbb{R}/2\mathbb{Z} \times \mathbb{R}_{>0}.$$

defined by

$$\Pi^{a,b}(\rho) = (\mathrm{pl}(\rho), L_{\mathrm{pl}(\rho)}(\rho))$$

is a homeomorphism.

See Figure 1.2 for a simplified schematic depiction of this system of coordinates, and the picture due to David Wright found as Figure 1 of [63].

Proof. Surjectivity is immediate. Injectivity follows from the fact that no $\mathcal{P}_\lambda$ can contain critical points and so the normalised length functions are monotone on each ray (Theorem 7.4.14). In addition, continuity follows from continuity of pl (Theorem 3.6.15 and of the assignment of normalised length functions (Proposition 7.4.13). It remains to prove that the map is open; the reader might find it helpful to view the schematic Figure 7.19 to see the objects we use in the proof.

Since Π is bijective it has an inverse. Let $U \subseteq \mathcal{R}^{a,b}$ be an open disc, and fix $(\lambda, c) \in \Pi(U)$; since U is open and the normalised complex length functions are analytic (so open) there exists an $\varepsilon > 0$

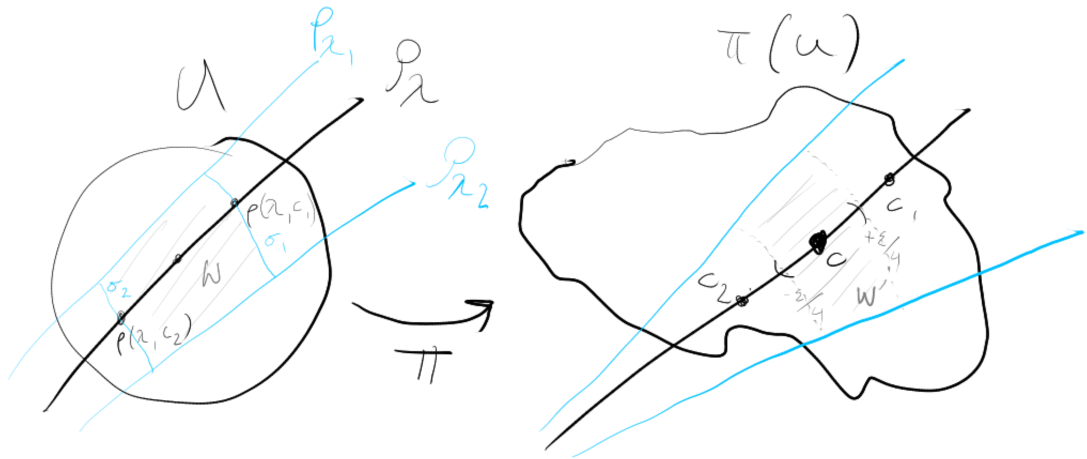


Figure 7.19: Objects in the proof of Theorem 7.4.15.

such that $(\lambda, c \pm \varepsilon/2) \in \Pi(U)$. Let the two points $c \pm \varepsilon/2$ be labelled c_1 and c_2 ($c_1 > c_2$), and draw two arcs σ_1, σ_2 in U, transverse to $\mathcal{P}_\lambda$ at c_1, c_2 respectively, with the endpoints chosen on two pleating rays $\mathcal{P}_{\lambda_1}$ and $\mathcal{P}_{\lambda_2}$ ($\lambda_1 > \lambda > \lambda_2$) with the property that $|PL(\mu) - c_1| < \varepsilon/8$ and $|PL(\mu) - c_2| < \varepsilon/8$ on σ_1 and σ_2 respectively. This is possible by continuity of PL.

Let W be the quadrilateral subset of $\mathbb{C}$ bounded between the arcs σ_1, σ_2 and the pleating rays $\mathcal{P}_{\lambda_1}, \mathcal{P}_{\lambda_2}$. This is contained entirely in U, since U is simply connected. We now show that $\Pi(W)$ contains (λ, c) within its interior; more precisely, we show that $\Pi(W)$ contains the set $K = [\lambda_1, \lambda_2] \times [c - \varepsilon/4, c + \varepsilon/4]$ which is a compact set containing (λ, c); thus $\int K$ is an open subset of $\mathbb{R}/2\mathbb{Z} \times \mathbb{R}_{>0}$ which is contained in $\Pi(U)$.

To see that $K \subseteq \Pi(W)$, suppose $t \in [\lambda_1, \lambda_2]$. By a similar 'betweeness' argument to those used in the previous proofs (i.e. continuity of pl and the intermediate value theorem), $\mathcal{P}_t$ intersects both σ_1 and σ_2. By construction of σ_1 and σ_2, if $\mu \in \sigma_1$ then $PL(\mu) > c + 3\varepsilon/8$; and if $\mu \in \sigma_2$, then $PL(\mu) < c - 3\varepsilon/8$. Because the pleating rays are disjoint and PL is monotone on each pleating ray, we see that $\Pi^{-1}(t, \mu) \in W$ for all $t \in [\lambda_1, \lambda_2]$ and all $\mu \in [c - \varepsilon/4, c + \varepsilon/4]$, which is what we wanted to show. ∎

Remark. Really, Theorem 7.4.15 is just a concrete version of the ending lamination conjecture for the quasi-Fuchsian groups in the Riley slice, and of the construction of the Thurston boundary of quasi-Fuchsian space which agrees in this special case with the Bers boundary (c.f. Section 5.11.2 of [79] and the references in the introductory paragraph to Section 3.6E). We conjecture that the Keen–Series theory will generalise quite easily to the case of general geometrically finite groups; see the disussion around Problem 10.1.2.

Chapter 8

Neighbourhoods of pleating rays

In this chapter, we show that, for every cusp ζ on the parabolic Riley slice boundary $\partial\mathcal{R}^{\infty,\infty}$, there exists an open neighbourhood of the rational pleating ray $\mathcal{P}(p/q)$ ending at ζ which consists of a connected component of $(\Phi_{p/q}^{\infty,\infty})^{-1}(\mathcal{H})$ where

$$\mathcal{H} = \{z \in \mathbb{C} : \Re z < -2\}$$

is the half-plane of $\mathbb{C}$ to the left of the vertical line through -2. Since the boundary of this neighbourhood is smooth, we get some information about the shape of the Riley slice boundary around cusp points. We will prove a similar result holds for elliptic Riley slices in our upcoming joint paper [42].

The majority of this chapter is adapted from Sections 4 and 5 of the preprint [39].

8.1 The main result and a sketch of the proof

We begin by stating our main result, and then providing some motivation for the proof.

8.1.1 Theorem (Existence of open neighbourhoods). *Let $\Phi_{p/q}$ be a Farey polynomial. Then there is a branch $\Phi_{p/q}^{-1}$ of the inverse of $\Phi_{p/q}$ such that*

$$\Phi_{p/q}^{-1}(\mathcal{H}),\ \textit{where}\ \mathcal{H} = \{\Re z < -2\},$$

is an open subset of $\mathcal{R}^{\infty,\infty}$

The bounds given in the theorem are illustrated in Figure 8.1.

The idea behind Theorem 8.1.1 is very simple: the Keen–Series theory of [63] which we discussed in Chapter 7 depends on the existence of round discs (the F-peripheral discs of Definition 7.2.1) in the ordinary sets of Riley groups which glue up along their edges to form the quotient surface $S_{0,4}$; the pleating rays are arcs in the Riley slice such that deforming the groups along these arcs preserves the 'roundness' of a given set of these peripheral discs (this was Lemma 7.2.2). In order to find neighbourhoods of these rays, we simply allow deformations in both dimensions, rather than simply the direction of the pleating ray. Of course, deformations off the pleating ray do not preserve the roundness of the F-peripheral discs; but we claim that the quasidiscs (that is, quasiconformal images of discs) obtained still 'glue up' correctly, and limits of them continue to be quasidiscs (rather than the boundary becoming space-filling)—this last property (Lemma 8.3.16) allows us to prove an analogue of Lemma 7.3.13, which is important because our proof follows a similar thread to the arguments of the previous chapter: we define a certain subset of $\mathbb{C}$, namely the set $\mathcal{N}_{p/q}$ of $\rho \in \mathbb{C}$ which admit

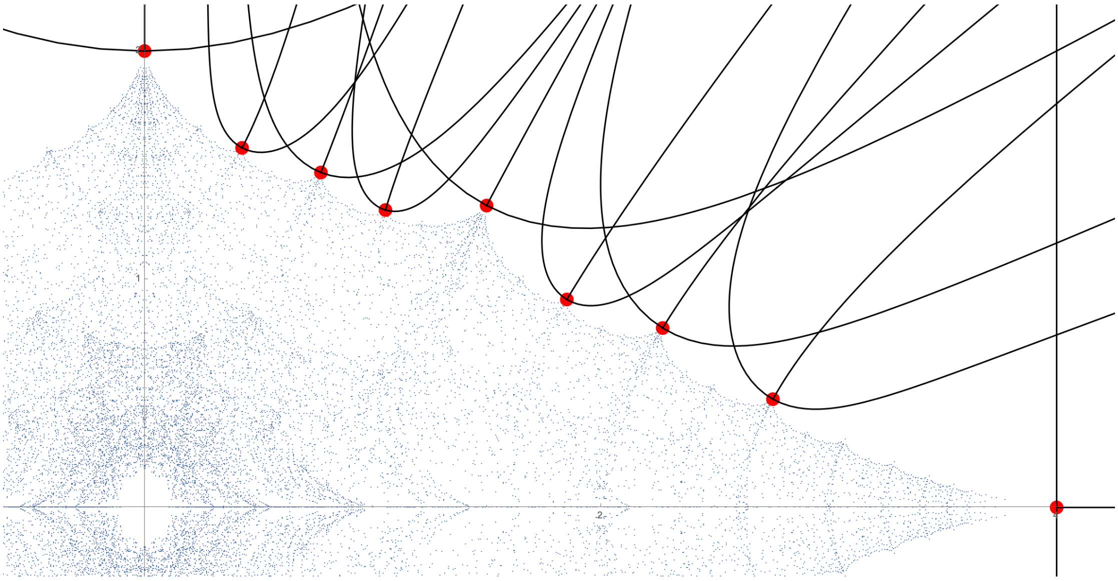


Figure 8.1: The Riley slice with neighbourhoods for our pleating ray values illustrated.

'canonical peripheral quasidiscs of slope p/q' in analogy to the non-conjugate pairs of F-peripheral discs; we then prove that any group in some $\mathcal{N}_{p/q}$ lies in the Riley slice (this is Lemma 8.3.14 below, the analogue of Lemma 7.2.10 above); and then, via an open-closed argument like Proposition 7.3.14, we see that $\mathcal{N}_{p/q}$ is precisely the set of Theorem 8.1.1.

In order to carry out this procedure, we need some information about the precise nature of the action of the quasiconformal deformations on the discs: more precisely, we will need to know that the combinatorial properties of the round peripheral discs are preserved even when we deform off the pleating ray and they turn into quasidiscs. Recall that the Riley slice $\mathcal{R}$ (which in this chapter will always be $\mathcal{R}^{\infty,\infty}$) is topologically a punctured disc in the plane (Corollary 5.1.6), and as such admits a hyperbolic metric which we denote by $\operatorname{dist}_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to [0, \infty)$.

8.1.2 Theorem. *Let α be a curve in $\mathcal{R}$ which lies a bounded hyperbolic distance from a pleating ray (that is, there exists an $M < \infty$ such that for each $\rho \in \alpha$ there is some $\nu \in \mathcal{P}_{p/q}$ with $\operatorname{dist}_{\mathcal{R}}(\rho, \nu) \leq M$). Then the quasiconformal map conjugating Γ_ρ to Γ_ν has distortion no more than e^M.*

Proof. Let $\rho \in \alpha$ and $\nu \in \mathcal{P}_{p/q}$ and let $M := \operatorname{dist}_{\mathcal{R}}(\rho, \nu)$. Let $\pi : \mathbb{B}^2 \to \mathcal{R}$ be the hyperbolic universal covering map with $\pi(0) = \rho$ and $\pi(\tanh(M/2)) = \nu$. The holomorphically parameterised family of discrete groups $\{\Gamma_{\Phi(z)} : z \in \mathbb{B}^2\}$ induces an equivariant ambient isotopy of $\hat{\mathbb{C}}$ by Theorem 3.6.9, following Proof Schema 3.6.11. If we move ρ in $\mathcal{R}$ then the motion of the fixed point set extends to a holomorphically parameterised quasiconformal ambient isotopy, equivariant with respect to the groups Γ_ρ, of the whole Riemann sphere. By part (3) of Theorem 3.6.8, the distortion of this ambient isotopy is exactly the exponential of the hyperbolic distance between the start (at 0) and finish (at $\tanh(M/2)$), that is e^M. ∎

Consider deforming a point $\rho \in \mathcal{R}$ towards the Riley slice boundary along a curve α which lies a bounded distance away from a pleating ray $\mathcal{P}_{p/q}$. Theorem 8.1.2 shows that if $\nu \in \mathcal{P}_{p/q}$ is the hyperbolic projection of ρ onto the pleating ray then the combinatorial properties of circles in the limit set of Γ_ν transfer directly to combinatorial properties of quasicircles in the limit set of Γ_ρ, since there is a uniformly bounded distortion mapping one to the other. These quasicircles bound what we will call the **F-peripheral quasidiscs** of the group Γ_ν.

Most of the information that the Keen–Series theory provides is topological and their arguments could be used almost directly if we knew these uniform bounds. However, there is no way that we can compute or even estimate the hyperbolic metric of the Riley slice near the boundary to identify a curve such as α for every rational pleating ray. What we do is guess (motivated by examining a lot of examples on the computer) that such a curve is $\alpha = (\Phi^{\infty,\infty}_{p/q})^{-1}(\{z = -2 + it : t > 0\})$, where we take the branch of the inverse of $\Phi^{\infty,\infty}_{p/q}$ with the correct asymptotic behaviour.

An important point that we need to take into account when we modify the proof of Theorem 7.3.4 is that, as mentioned above, the peripheral quasicircles could become quite entangled and eventually become space filling curves. We avoid this situation by modifying the peripheral quasidiscs as we move, so they have large scale "bounded geometry" (though the small scale geometry is uncontrolled). An important observation is that along the rational pleating ray the isometric circles of the Farey word Word(p/q) are disjoint. We deform in such a way that this property is preserved, and it is for this reason that we choose the set $\mathcal{H}$ in the theorem statement: we will prove that Word(p/q) has disjoint isometric discs when its trace lies in this region (Lemma 8.3.3), though we believe that this region can even be enlarged (see Section 8.2). Further, if we do not move too far away from the pleating ray these isometric circles do not start spinning around one another. This information allows us to construct a "nice" precisely invariant set stabilised by X and Word(p/q)—this turns out to be one of the peripheral quasidiscs which *does have* bounded geometry. Existence of this peripheral quasidisc (which we call a **canonical** peripheral quasidisc) guarantees we have the correct quotient from the action of Γ_ρ on the ordinary set; and then the open-closed argument carries through.

8.2 An aside: Lyndon and Ullman's results

Here, we recall the main result of a paper of Lyndon and Ullman [74] and examine it in the context of our pleating neighbourhoods. In the process we will make some conjectures about improvements we think are possible to make to Theorem 8.1.1. In particular, we believe that the set $\mathcal{H}$ in the theorem can be enlarged to a cone with angle $4\pi/3$.

8.2.1 Theorem (Theorem 3, [74]). *Let K denote the Euclidean convex hull of the set $B \cup \{\pm 4\}$ (B here is the disc of radius 2 about 0). Then $\mathbb{C} \setminus \mathcal{R} \subseteq K$.* ∎

See Figure 8.2 for a depiction of this bound; as a consequence, the Riley slice $\mathcal{R}$ is contained within a conic region with apex at -4, bounded by two rays tangent to the disc B. Since the two lines are orthogonal to the radii of the circle, a simple trigonometric calculation shows that the cone angle is $\pi/3$, and so the interior of the cone is the set

$$\mathcal{W} = \{z \in \mathbb{C} : \frac{-\pi}{6} < \arg(z+4) < \frac{\pi}{6}\}.$$

Let φ be the branch of $z \mapsto -(-z-4)^{3/5} - 4$ conformally mapping $\mathbb{C} \setminus \overline{\mathcal{W}}$ to the half-space $H = \{z : \Re z < -4\}$ (here, $3/5 = \pi/(2\pi - \pi/3)$). Then $H \subseteq \mathbb{C} \setminus \overline{\mathcal{W}}$, and $\varphi(H)$ is the sector

$$\{z \in \mathbb{C} : \frac{5\pi}{6} < \arg(z+4) < \frac{7\pi}{6}\}.$$

Because φ is conformal it is now straightforward to see that the distance in the hyperbolic metric of $\mathbb{C} \setminus \overline{\mathcal{W}}$ between the line $\ell_1 = -4 + i\mathbb{R}$ and the rational pleating ray $\ell_2 = (-\infty, -4]$ (which are parallel

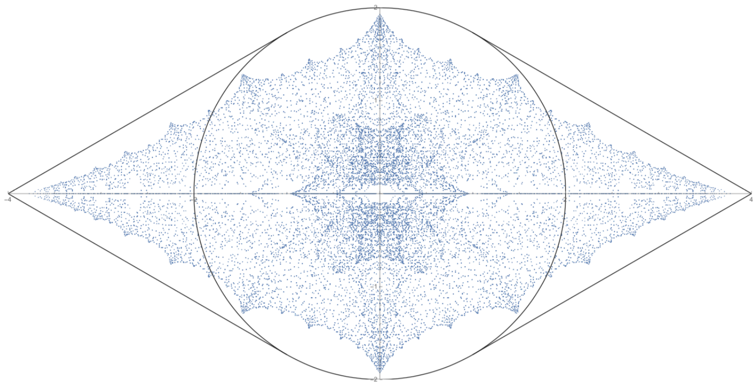


Figure 8.2: The convex hull of $B \cup \{\pm 4\}$ contains $\mathbb{C} \setminus \mathcal{R}$.

since they meet at the same point at infinity) is

$$\mathrm{dist}_{\mathbb{C}\setminus\overline{\mathcal{W}}}(\ell_1, \ell_2) = \int_0^{3\pi/10} \frac{d\theta}{\cos(\theta)} = \frac{1}{2}\ln\left[5 + 2\sqrt{5}\right] \approx 1.1241.$$

From Theorem 8.1.2 we now have the following corollary.

8.2.2 Corollary. *Let $\nu \in -4 + i\mathbb{R}$. Then there is $\rho \in (-\infty, -4]$, the rational pleating ray $\mathcal{P}_{1/1}$, so that Γ_ν and Γ_ρ are K-quasiconformally conjugate for some deformation K satisfying*

$$K \leq \sqrt{5 + 2\sqrt{5}} \approx 3.077\ldots.$$

Proof. The only thing left to observe is that the contraction principle for the hyperbolic metric shows that the hyperbolic metric of $\mathcal{R}$ is smaller than the hyperbolic metric of $\mathbb{C} \setminus \overline{\mathcal{W}}$; in particular,

$$\mathrm{dist}_{\mathcal{R}}(\rho, \nu) \leq \mathrm{dist}_{\mathcal{W}}(\rho, \nu) = \frac{1}{2}\ln\left[5 + 2\sqrt{5}\right]$$

for the point ρ closest to ν, and hence by Theorem 8.1.2

$$K \leq e^{\mathrm{dist}_{\mathcal{R}}(\rho,\nu)} \leq \sqrt{5 + 2\sqrt{5}}.$$

This proves the corollary. ∎

We believe these estimates for larger neighbourhoods of the $(-\infty, -4]$ pleating ray persist in general in the parabolic case (namely, take preimages of this cone rather than of $\mathcal{W}$), but proving this adds additional complications in the construction we give as the isometric circles of Word(p/q) may no longer be disjoint. We offer Figure 8.3, which is a slight modification of Figure 8.1, as computational support for this conjecture. Instead of looking at the branch of the inverse of $\Phi_{p/q}$ defined on $\{\Re z < -4\}$, to produce this image we compute the preimages of the conic region of opening $\frac{\pi}{3}$ given by Theorem 8.2.1.

In the elliptic case, additional difficulty arises in finding an analogue for Theorem 8.2.1 in order to even 'guess' the right cone to pull back to a neighbourhood. We will discuss this further in our upcoming joint paper [42].

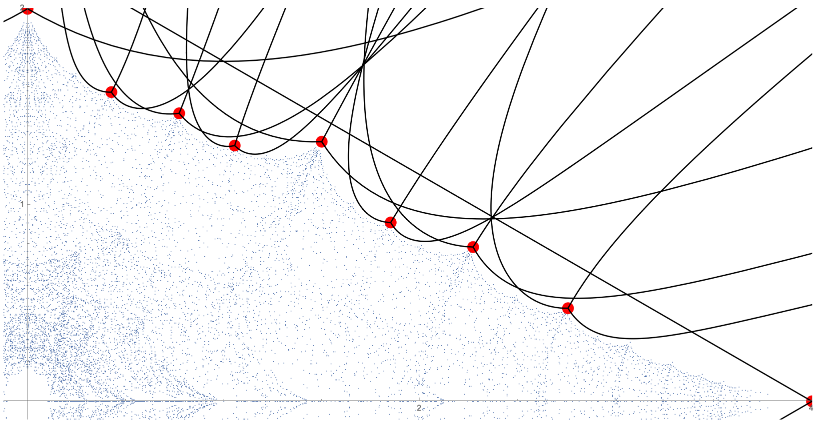


Figure 8.3: Preimages of the sector $\{\arg(z) = -\frac{\pi}{6}\} \cup \{\arg(z) = \frac{\pi}{6}\}$.

8.3 Proof of the main theorem

In this section, we carry out the proof sketch that we gave in Section 8.1. It may be useful to have a reference to a specific example. Figure 8.4 shows pictures of the geometric objects we will be interested in for two specific cusp groups.

8.3A Products of parabolics

As noted earlier (Lemma 6.1.7), an important property of a Farey word $\operatorname{Word}(p/q)$ is that it can be written as a product of parabolic elements in two essentially different ways. For $\rho \in \mathcal{R}$ there are only two conjugacy classes of parabolics, those represented by X and Y [83, VI.A]. As explained above in Chapter 7, this is just a reflection of the fact that the deletion of a non-boundary-parallel curve on the 4-punctured sphere leaves two doubly punctured discs. To find these parabolics we just look for a couple of conjugates of X and Y whose product is $\operatorname{Word}(p/q)$. Keen and Series studied the set of all such pairs (this is the data encoded in the circle chain sets $\mathcal{U}_{p/q}$ of Definition 7.2.4); in our analysis, we will only look closely at the pair $\{X, X^{-1}\operatorname{Word}(p/q)\}$. The group $\langle X, \operatorname{Word}(p/q)\rangle = \langle X, X^{-1}\operatorname{Word}(p/q)\rangle$ is generated by two parabolics, and so can therefore only be discrete and free on its generators if

$$\operatorname{tr}\big(XX^{-1}\operatorname{Word}(p/q)\big) - 2 = \operatorname{tr}(\operatorname{Word}(p/q)) - 2 \in \overline{\mathcal{R}}$$

(since a group generated by two parabolics is discrete and free iff it is conjugate to a group in the Riley slice or its boundary; the Riley parameter ρ of a group generated by two parabolics A and B is just $\operatorname{tr} AB - 2$, since $\rho + 2 = \operatorname{tr} XY_\rho$.). If $\operatorname{tr}(\operatorname{Word}(p/q)) \in \mathbb{R}$, then the traces of X, $X^{-1}\operatorname{Word}(p/q)$, and $\operatorname{Word}(p/q)$ are real (the first two are ± 2) and so $\langle X, \operatorname{Word}(p/q)\rangle$ is Fuchsian by Lemma 7.3.11. It is groups of this form (and their conjugates) which produce the *round* F-peripheral circles of [63] which we studied in Chapter 7.

That it suffices to only look at $\{X, X^{-1}\operatorname{Word}(p/q)\}$ is a consequence of the following general result:

8.3.1 Lemma. *Suppose that $u_1, u_2, v_1, v_2 \in \mathrm{PSL}(2,\mathbb{C})$ are parabolics such that $\operatorname{tr} u_1u_2 = \operatorname{tr} v_1v_2$. Then the two groups $\langle u_1, u_2\rangle$ and $\langle v_1, v_2\rangle$ are conjugate in $\mathrm{PSL}(2,\mathbb{C})$.*

The upshot of this lemma is that if we were to pick a different pair whose product was $\operatorname{Word}(p/q)$ then we get exactly the same geometry, up to a well-defined conjugation in $\hat{\mathbb{C}}$. We give a proof that

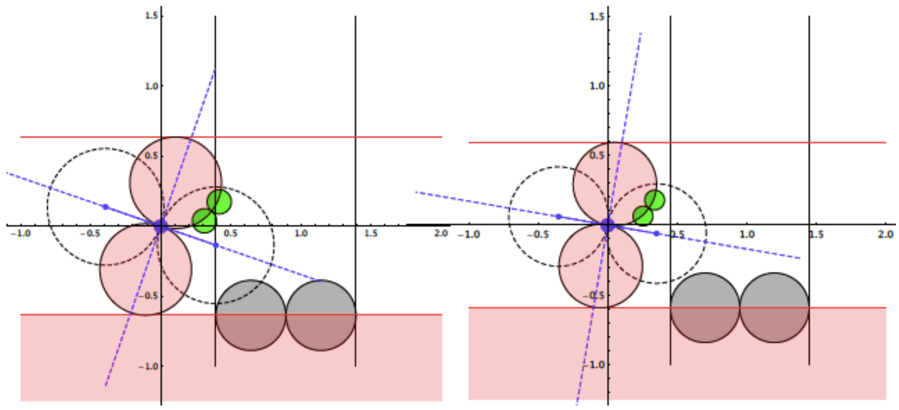


Figure 8.4: Geometric objects in the limit sets of the 3/4-cusp group (left) and the 4/5-cusp group (right). The isometric circles of Word(p/q) are shaded grey; their images under the involution $\Phi : z \mapsto 1/(\rho z)$ are shaded green. This involution defines the non-conjugate peripheral disc $\langle Y, \Phi\,\mathrm{Word}(p/q)\Phi^{-1}\rangle$. The non-conjugate peripheral discs are shaded in red (one is a lower half plane). Fixed points of the involution and its action are also illustrated.

provides slightly more information; an elementary proof that proves exactly the statement given is easy to write down (the groups can be conjugated to Γ_μ and $\Gamma_{\mu'}$, then the indicated traces are $2+\mu$ and $2+\mu'$ respectively so the groups themselves are conjugate to each other).

Proof of Lemma 8.3.1. There are involutions $\phi_u, \phi_v \in \mathrm{PSL}(2,\mathbb{C})$ so that $\phi_u u_1 \phi_u^{-1} = u_2$ and $\phi_v v_1 \phi_v^{-1} = v_2$.

$$\operatorname{tr}^2 \phi_u = \operatorname{tr}^2 \phi_v = 0 \text{ and } \operatorname{tr}^2 u_1 = \operatorname{tr}^2 v_1 = 4.$$

Also,

$$\begin{aligned}\operatorname{tr}[u_1,\phi_u] &= \operatorname{tr} u_1\phi_u u_1^{-1}\phi_u^{-1} = \operatorname{tr} u_1 u_2^{-1} = \operatorname{tr} u_1 \operatorname{tr} u_2 - \operatorname{tr} u_1 u_2 \text{ and}\\ \operatorname{tr}[v_1,\phi_v] &= \operatorname{tr} v_1\phi_v v_1^{-1}\phi_v^{-1} = \operatorname{tr} v_1 v_2^{-1} = \operatorname{tr} v_1 \operatorname{tr} v_2 - \operatorname{tr} v_1 v_2;\end{aligned}$$

the two right-hand sides are equal (since $\operatorname{tr} u_i = \operatorname{tr} v_j$ for all i, j, and the product traces are equal by assumption) so $\operatorname{tr}[u_1,\phi_u] = \operatorname{tr}[v_1,\phi_v]$. In [49] it is shown that any pair of two-generator groups with the same trace square of the generators and the same trace of the commutators are conjugate in $\mathrm{PSL}(2,\mathbb{C})$. Thus $\langle v_1,\phi_v\rangle$ and $\langle u_1,\phi_u\rangle$ are conjugate and so are their subgroups $\langle u_1,u_2\rangle$ and $\langle v_1,v_2\rangle$. ∎

8.3B Rotation angles and isometric discs

Let $f \in \mathrm{PSL}(2,\mathbb{C})$ with $\operatorname{tr} f = -2 + ti$ $(t \in \mathbb{R})$. Then f has complex translation length $\tau_f + i\theta_f$, where τ_f and θ_f are respectively the real translation length and the rotation angle f given by the formulae

$$\frac{\tau_f}{2} = \Re\left[\sinh^{-1}\left(\frac{i}{2}\sqrt{t(4i+t)}\right)\right] \text{ and } \frac{\theta_f}{2} = \Im\left[\sinh^{-1}\left(\frac{i}{2}\sqrt{t(4i+t)}\right)\right]$$

We also have the following asymptotics.

$$\text{As } t \to 0, \begin{cases}\frac{\tau_f}{\sqrt{2t}} \to 1,\\ \frac{\theta_f}{\sqrt{2t}} \to -1.\end{cases} \qquad \text{For } 0 < t < 1, \begin{cases}1 \le \frac{\tau_f}{\sqrt{2t}} \le 1.03642...,\\ -1 \le \frac{\theta_f}{\sqrt{2t}} \le -0.954\ldots.\end{cases}$$

In addition, $\theta_f \to -\pi$ as $t \to \infty$.

Suppose that f is represented by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the isometric discs of f are the two discs D_1 and D_2 given by

$$D_1 = \left\{ z \in \mathbb{C} : \left| z - \frac{a}{c} \right| \le \frac{1}{|c|} \right\}, \quad D_2 = \left\{ z \in \mathbb{C} : \left| z + \frac{d}{c} \right| \le \frac{1}{|c|} \right\}$$

The isometric circles are the boundaries of these two discs. We say that f has **disjoint isometric discs** if these discs have disjoint interior. This is clearly equivalent to the condition $|a + d| \ge 2$.

The mapping f pairs these discs in the sense that

$$f(D_1) = \hat{\mathbb{C}} \setminus \overline{D_2}.$$

Thus $\hat{\mathbb{C}} \setminus \overline{D_1 \cup D_2}$ is a fundamental domain for the action of f on $\hat{\mathbb{C}}$. Notice that when $c \neq 0$, $f(\infty) = \frac{a}{c}$ and that $f^{-1}(\infty) = -\frac{d}{c}$ are the centers of the isometric discs.

We now specialise to the case that f is the Farey word Word(p/q). Label the entries of the matrix representing Word$(p/q)(\rho)$ as follows:

$$\mathrm{Word}(p/q)(\rho) = \begin{pmatrix} a_{p/q}(\rho) & b_{p/q}(\rho) \\ c_{p/q}(\rho) & d_{p/q}(\rho) \end{pmatrix} \qquad a_{p/q}d_{p/q} - b_{p/q}c_{p/q} = 1.$$

We first observe that the entries of the matrix are not independent. Indeed:

8.3.2 Lemma. $Q_{p/q}(\rho) = a_{p/q}(\rho) + d_{p/q}(\rho) - 2 = c_{p/q}(\rho)$.

Proof. Using Lemma 6.1.7 we will show this reduces to the well known Fricke identity in $\mathrm{PSL}(2, \mathbb{C})$ (see Formula (3.15) of [76]),

$$\mathrm{tr}[A, B] = \mathrm{tr}^2 A + \mathrm{tr}^2 B + \mathrm{tr}^2 AB - \mathrm{tr}\, A\, \mathrm{tr}\, B\, \mathrm{tr}\, AB - 2.$$

We put $A = X^{-1}$ and $B = \mathrm{Word}(p/q)$. Note that, by Lemma 6.1.7 and the conjugacy invariance of trace, $\mathrm{tr}\, X^{-1}\, \mathrm{Word}(p/q)$ is either $\mathrm{tr}(X)$ or $\mathrm{tr}(Y)$ depending on whether q is even or odd (compare with the discussion in Chapter 9). In our situation both of these traces are 2. Thus, supposing q is odd (the result if q is even follows with a similar calculation),

$$\begin{aligned} c_{p/q}^2 &= \mathrm{tr}[X, \mathrm{Word}(p/q)] - 2 = \mathrm{tr}\big[X^{-1}, \mathrm{Word}(p/q)\big] - 2 \\ &= \mathrm{tr}^2(X^{-1}) + \mathrm{tr}^2(\mathrm{Word}(p/q)) + \mathrm{tr}^2(Y) - \mathrm{tr}\big(X^{-1}\big)\, \mathrm{tr}(\mathrm{Word}(p/q))\, \mathrm{tr}(Y) - 4 \\ &= 4 + (a_{p/q} + d_{p/q})^2 - 4\, \mathrm{tr}\big(a_{p/q} + d_{p/q}\big) \\ &= (a_{p/q} + d_{p/q} - 2)^2 \end{aligned}$$

Thus $c_{p/q} = \pm(a_{p/q} + d_{p/q} - 2)$. When $\rho = 1$, the positive square root occurs. Since the identity is continuous in ρ, it follows that the positive square root is the correct choice for all ρ. ∎

8.3.3 Lemma. *Let $\Re(\rho) \le -2$. Then the Farey word* Word$(p/q)(\rho)$ *has disjoint isometric discs.*

Proof. The isometric circles of Word(p/q) are the two discs

(8.3.4) $$D_1 = B\left(\frac{a_{p/q}(\rho)}{c_{p/q}(\rho)}, \frac{1}{|c_{p/q}(\rho)|} \right) \text{ and } D_2 = B\left(\frac{-d_{p/q}(\rho)}{c_{p/q}(\rho)}, \frac{1}{|c_{p/q}(\rho)|} \right).$$

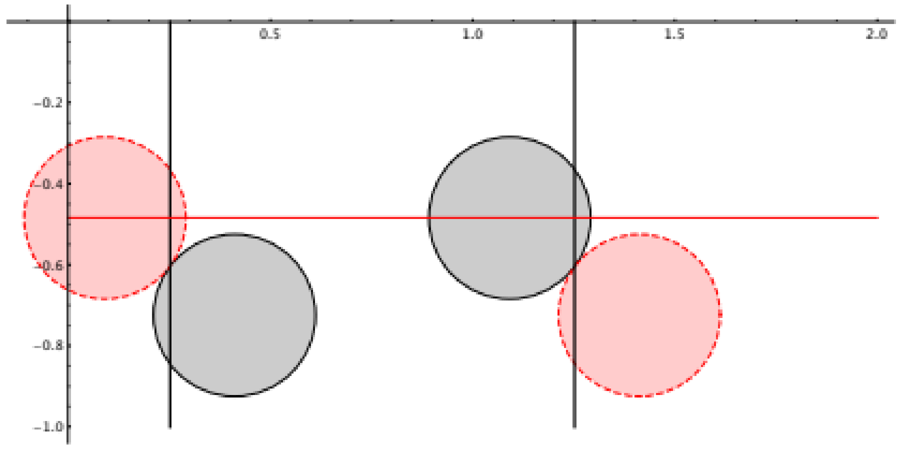


Figure 8.5: The isometric circles of Word(3/4) when tr Word(3/4) $= -2 + i$ (in grey) and their translates (in red).

We now compute with the identity of Lemma 8.3.2 that

$$\frac{a_{p/q}(\rho)}{c_{p/q}(\rho)} + \frac{d_{p/q}(\rho)}{c_{p/q}(\rho)} = \frac{2 + c_{p/q}(\rho)}{c_{p/q}(\rho)} = 1 + \frac{2}{c_{p/q}(\rho)}. \tag{8.3.5}$$

Now suppose that tr(Word$(p/q)) = -2 + it$; then $|a_{p/q} + d_{p/q}| = \sqrt{4 + t^2} > 2$, so along the path tr(Word$(p/q)) = -2 + it$ we have that the Farey word Word(p/q) has disjoint isometric discs. ∎

Lemma 8.3.2 and Equation (8.3.5) together have the following consequence.

8.3.6 Corollary. *Let* tr(Word$(p/q)) = -x + it$ *for* $x \geq 2$ *and* $t \in \mathbb{R}$. *Then the group* $\langle X, \text{Word}(p/q)\rangle$ *is discrete and free on the indicated generators.*

Proof. For clarity we drop the subscript p/q as it is fixed. Let S be the vertical strip of width one given by

$$\left\{z \in \mathbb{C} : \frac{1}{2}\left(\Re\left(\frac{a(\rho) - d(\rho)}{c(\rho)}\right) - 1\right) < z < \frac{1}{2}\left(\Re\left(\frac{a(\rho) - d(\rho)}{c(\rho)}\right) + 1\right)\right\}.$$

Using the notation of Lemma 8.3.3 for the isometric discs of Word(p/q), set $\widetilde{D_1} = D_1 - 1$ and $\widetilde{D_2} = D_2 + 1$ so each $\widetilde{D}_i$ is a translate of the respective D_i (to the left and right respectively; see Figure 8.5). Essentially following the construction on pp.1392–1393 of [74], define $\tilde{S}$ by

$$\tilde{S} = (S \cup D_1 \cup D_2) \setminus \left(\widetilde{D}_1 \cup \widetilde{D_2}\right). \tag{8.3.7}$$

Lemma 8.3.2 implies that the discs $\widetilde{D}_1$ and D_2 are tangent. Two things now follow. Firstly, the translates of $\tilde{S}$ by $n \in \mathbb{Z}$ fill the plane. Secondly, $\tilde{S}$ contains the isometric circles of Word(p/q). The Klein combination theorem [83, Theorem VII.A.13] or an argument like that in Proposition 3.5.2 now implies the result since $\hat{\mathbb{C}} \setminus (D_1 \cup D_2)$ is a fundamental domain for the action of Word(p/q). ∎

There is one further piece of information we would like out of Corollary 8.3.6: that the point of tangency of the isometric discs and their translates is a parabolic fixed point. To save space, write $h = h_{p/q}$ for the function on $\hat{\mathbb{C}}$ corresponding to the action of Word(p/q). The point of tangency can be calculated to be

$$z_\infty := \frac{a - d}{2c} - \frac{1}{2} = \frac{a - d - c}{2c} = \frac{1 - d}{c}.$$

(where we continue to use a, b, c, d for the entries of a matrix representing Word(p/q)). Then

$$h(z_\infty) = \frac{az_\infty + b}{cz_\infty + d} = \frac{a\frac{1-d}{c} + b}{c\frac{1-d}{c} + d} = \frac{1 + c - d}{c} = z_\infty + 1$$

and so with f representing X we have shown $f^{-1}h(z_\infty) = z_\infty$ so z_∞ is a fixed point of $f^{-1}h$. Since X^{-1} Word(p/q) is parabolic as previously observed we have proved the following lemma.

8.3.8 Lemma. *The point $\frac{1-d}{c} \in \partial\tilde{S}$ (with $\tilde{S}$ defined by Equation* (8.3.7)*), a point of tangency of the isometric discs of* Word(p/q) *and their unit translates, is a parabolic fixed point.* ∎

8.3C Canonical peripheral quasidiscs

In this section we show that the geometry of the peripheral quasicircles is controlled by the pairing of the isometric circles of $W_{p/q}$. We begin by studying the geometry for $\rho \in \mathcal{P}_{p/q}$, and then we allow ρ to move holomorphically off the pleating ray, inducing a quasiconformal deformation of the limit set and hence the peripheral discs, with a quasiconformality constant that we can explicitly bound.

Fix $\rho \in \mathcal{P}_{p/q}$, and as above we write $h_{p/q}(\rho)$ for the Möbius transformation represented by $W_{p/q}(\rho)$ which maps ∂D_2 onto ∂D_1 (c.f. Equation (8.3.4)). Let $z_0 \in \partial D_2$ be the unique closest point of ∂D_2 to ∂D_1. Since $h_{p/q}(\rho)$ is hyperbolic (its trace is real, since ρ is on the p/q-pleating ray) it maps z_0 to the point of ∂D_1 which is closest to ∂D_2. Let $L(\rho)$ be the Euclidean line segment joining z_0 to $h_{p/q}(\rho)(z_0)$.

We now allow ρ to move off the pleating ray; more precisely, we choose a holomorphic motion Φ : $\mathbb{D}\times A \to A$ such that A is a sufficiently small neighbourhood of the pleating ray (really the point is that we can choose A to be the neighbourhood which we claim the existence of in Theorem 8.1.1 without moving out of the Riley slice) and then take $\tilde{\rho} = \Phi(\lambda, \rho)$ for some $\lambda \in \mathbb{B}^2$. After this deformation, $L(\tilde{\rho})$ is still a line segment joining two points on the boundaries of $\partial D_1(\tilde{\rho})$ and $\partial D_2(\tilde{\rho})$ such that the endpoint $z_0(\tilde{\rho})$ on $\partial D_2(\tilde{\rho})$ is mapped by $h_{p/q}(\tilde{\rho})$ onto the other endpoint of $L(\tilde{\rho})$; but now the line segment itself does not form the projection of the axis of $h_{p/q}(\tilde{\rho})$ (though the projection of the orbit is symmetric with respect to $L(\tilde{\rho})$) and $z_0(\tilde{\rho})$ and $h_{p/q}(\tilde{\rho})(z_0(\tilde{\rho}))$ are not the closest points of the two circles.

The line segment $L_{p/q}(\tilde{\rho})$ will lie entirely in $\tilde{S}$ provided that $\tilde{\rho}$ is close enough to ρ that the isometric discs have not twisted too far around. In particular, it is enough if the absolute value of the difference between the real parts of the centers of the isometric discs exceeds twice the radius of the isometric discs. That is, if

$$\left|\Re\frac{a_{p/q}(\tilde{\rho}) + d_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})}\right| \geq \frac{2}{\left|c_{p/q}(\tilde{\rho})\right|}.$$

Using Lemma 8.3.2, we calculate that

$$\begin{aligned}\Re\frac{a_{p/q}(\tilde{\rho}) + d_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})} &= \Re\frac{c_{p/q}(\tilde{\rho}) + 2}{c_{p/q}(\tilde{\rho})} = 1 + \Re\frac{2}{c_{p/q}(\tilde{\rho})}\\ &= 1 + \frac{2}{\left|c_{p/q}(\tilde{\rho})\right|^2}\Re c_{p/q}(\tilde{\rho});\end{aligned}$$

thus we are requiring

$$\left|\left|c_{p/q}(\tilde{\rho})\right|^2 + 2\Re c_{p/q}(\tilde{\rho})\right| \geq 2\left|c_{p/q}(\tilde{\rho})\right|.$$

This is true if $\Re c_{p/q}(\tilde{\rho}) \leq -4$, so under these conditions the line segment $L_{p/q}(\tilde{\rho})$ has the property that it lies entirely in $\tilde{S}$ with its endpoints on $\partial\tilde{S}$; and as we mentioned above the endpoints of $L_{p/q}(\tilde{\rho})$ are identified by $h_{p/q}(\tilde{\rho})$.

For convenience, introduce now the notation $\Gamma_{p/q}(\tilde{\rho}) = \langle f, h_{p/q}(\tilde{\rho})\rangle$ where f and $h_{p/q}(\tilde{\rho})$ are the Möbius transformations with respective matrices X and $W_{p/q}(\tilde{\rho})$. We have identified a fundamental domain $\tilde{S}$ for the action of $\Gamma_{p/q}(\tilde{\rho})$ on $\Omega(\Gamma_{p/q}(\tilde{\rho}))$. The quotient

$$\Omega(\Gamma_{p/q}(\tilde{\rho}))/\Gamma_{p/q}(\tilde{\rho})$$

is the four-times punctured sphere $S_{0,4}$, since

$$\Gamma_{p/q}(\tilde{\rho}) = \left\langle f, h_{p/q}(\tilde{\rho})\right\rangle = \left\langle f, f^{-1}h_{p/q}(\tilde{\rho})\right\rangle$$

is a circle-pairing group generated by two parabolics. The line segment $L_{p/q}(\tilde{\rho})$ projects to a simple closed curve (though not a geodesic in general) in the homotopy class of $h_{p/q}(\tilde{\rho})$ and separates one pair of punctures from another. We remark that the projection of $L_{p/q}(\tilde{\rho})$ is smooth away from one corner (namely, the point of projection of the segment endpoints) and the angle at that corner tends to π as $\Im\tilde{\rho} \to 0$. The Schottky lift (that is, the lift into $\hat{\mathbb{C}}$ induced by viewing $S_{0,4}$ as a quotient of $\hat{\mathbb{C}}$ by a Schottky-type group) of the projection of $L_{p/q}(\tilde{\rho})$ into $S_{0,4}$ is a quasiline through ∞ (we have no control on the distortion here, even though we expect that we are a bounded hyperbolic distance from a Fuchsian group on the rational pleating ray $\mathcal{P}_{p/q}$, so there is a nice quasiline which must pass through the midpoint of $L_{p/q}(\tilde{\rho})$ for reasons of symmetry). This quasiline must be

$$\mathcal{L}_{p/q}(\tilde{\rho}) = \bigcup_{g\in\langle f, h_{p/q}(\tilde{\rho})\rangle} g(L_{p/q}(\tilde{\rho})).$$

It consists of the translates of $L_{p/q}(\tilde{\rho})$ by f^n, $n \in \mathbb{Z}$, together with images which lie in the union of the two isometric circles of $h_{p/q}$ and their integer translates. We note that

$$h_{p/q}(\tilde{\rho})(\infty) = \frac{a_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})} \text{ and } h_{p/q}(\tilde{\rho})^{-1}(\infty) = -\frac{d_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})}$$

and these are parabolic fixed points on $\mathcal{L}_{p/q}(\tilde{\rho})$ (conjugates of the fixed points of f) as well as being the centers of the isometric circles. The parabolic fixed point we earlier identified in Lemma 8.3.8, $z_\infty = \frac{1-d}{c}$, also lies in $\mathcal{L}_{p/q}(\tilde{\rho})$ and is not a conjugate of a fixed point of f (since it is not conjugate in the abstract group $\langle X, Y\rangle$ from which the rational words come, a consequence of the fact that they represent simple closed curves on the four-times punctured sphere.) The translates of the endpoints of $L_{p/q}(\tilde{\rho})$ under $\langle h_{p/q}(\tilde{\rho})\rangle$ lie on a log-spiral connecting the fixed points of $h_{p/q}(\tilde{\rho})$. This is illustrated in the examples of Figure 8.6.

If we denote by $H^{\pm}_{p/q}(\tilde{\rho})$ the components of $\mathbb{C} \setminus \mathcal{L}_{p/q}(\tilde{\rho})$, then

$$H^{\pm}_{p/q}(\tilde{\rho})/\Gamma_{p/q}(\tilde{\rho})$$

is a twice punctured disc with boundary given by a projection of $L_{p/q}(\tilde{\rho})$.

We can give some bounds on the position of the invariant quasiline; in particular, this shows that it has bounded large-scale geometry (as we discussed in Section 8.1).

8.3.9 Lemma. *The invariant quasiline $\mathcal{L}_{p/q}$ lies in the strip*

$$\left\{z \in \mathbb{C} : \Im(\frac{a_{p/q}}{c_{p/q}}) + \frac{1}{|c_{p/q}|} \leq \Im z \leq \Im(-\frac{d_{p/q}}{c_{p/q}}) - \frac{1}{|c_{p/q}|}\right\}$$

(where all objects are taken with respect to $\Gamma_{\tilde{\rho}}$).

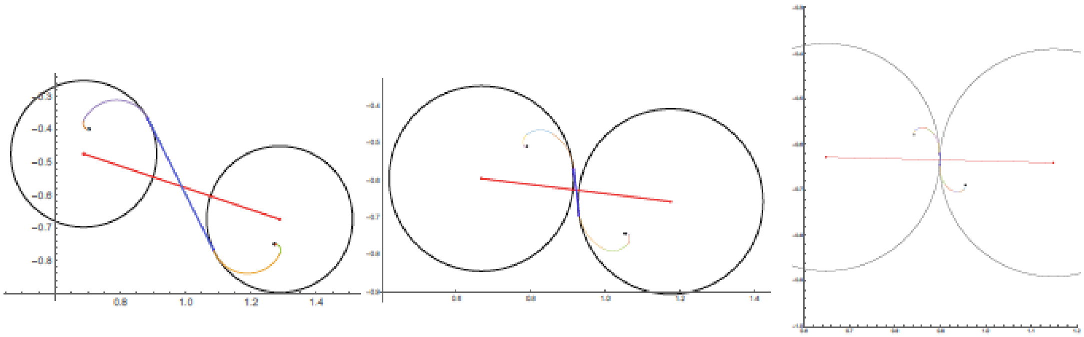


Figure 8.6: A log-spiral connecting the fixed points of $h_{3/4}(\tilde{\rho})$, the Möbius transformation representing $W_{3/4}(\tilde{\rho})$. Red lines connect the isometric circle centers, and the spirals connect the fixed points of $h_{3/4}(\tilde{\rho})$. Here, $\tilde{\rho}$ is chosen such that $\mathrm{tr}\big(W_{3/4}\big) = -2 + it$ where the values of t shown from left to right are $t = 2$, $t = 0.5$, and $t = 0.1$.

Proof. By construction $\ell_{p/q}$ lies in, and separates $\tilde{S}$. Its translates together with the translates of the isometric discs of $W_{p/q}$ separate both the ordinary set of $\langle f, h_{p/q}\rangle$ and the plane into two parts. The strip is the smallest horizontal strip containing the isometric circles of $W_{p/q}$. Note that $\Im \frac{a_{p/q}}{c_{p/q}} > \Im - \frac{d_{p/q}}{c_{p/q}})$ and that both are negative. This particular fact holds if we choose, as we may, ρ to be in the positive quadrant of $\mathbb{C}$. ∎

Our computational investigations suggest that in fact the width of this strip can be improved to where the spiral "turns over". This appears proportional to the difference of the imaginary parts of the fixed points. A consequence would be that as $\Im\tilde{\rho} \to 0$ the strip turns into a line and the quasilines $\mathcal{L}_{p/q}(\tilde{\rho})$ converge to the line through the fixed points of $h_{p/q}(\tilde{\rho})$, which is a line in the limit set of the cusp group.

8.3.10 Definition. By analogy with Keen and Series, we call the component $H_{p/q}(\tilde{\rho})$ of $\mathbb{C} \setminus \mathcal{L}_{p/q}(\tilde{\rho})$ which does not contain 0 a *canonical peripheral quasidisc* if

1. $\Lambda(\Gamma_{p/q}(\tilde{\rho})) = \overline{H_{p/q}(\tilde{\rho})} \cap \Lambda(\Gamma_{\tilde{\rho}})$, and
2. $\mathrm{tr}\big(W_{p/q}((\tilde{\rho}))\big) \in \{z = x + iy \in \mathbb{C} : x < -2\}$.

Notice that if $\tilde{\rho} \in \mathcal{R}$, then there exists *some* slope p/q such that $\Gamma_{\tilde{\rho}}$ admits the canonical peripheral quasidisc $H_{p/q}(\tilde{\rho})$, since each such group is quasiconformally conjugate to one on a pleating ray where there is such a peripheral circle. There seems to be no way of guaranteeing that the large scale geometry of the boundary quasiline is bounded, but we do know that the geometry is bounded for the special case of $\mathcal{L}_{p/q}(\tilde{\rho})$.

8.3D Completing the proof

We now give a series of lemmata imitating the proofs given for the case of a pleating ray in Chapter 7 following [63]. Set $S^*_{p/q} = \tilde{S} \cap H_{p/q}$; this is a fundamental domain of $\Gamma_{p/q}$ defined by the isometric circles of $h_{p/q}$ and the line segment $\ell_{p/q}$. Recall the parabolic cusp point given by Lemma 8.3.8 in $\partial H_{p/q}$ (and also in $S^*_{p/q}$). The following lemma is immediately clear from construction.

8.3.11 Lemma. *An F-peripheral disc in the sense of Definition 7.2.1 is a canonical peripheral quasidisc.* ∎

In fact, in this case $h_{p/q}$ is hyperbolic, with disjoint isometric discs and $\ell_{p/q}$ is a segment of the line through its fixed points (and also through isometric circles) and orthogonal to them.

Recall that in the Keen–Series theory it was important that the F-peripheral discs moved continuously with ρ; since the defining points of $L_{p/q}$ move continuously with ρ, the analogous result is true:

8.3.12 Lemma. *Fix a rational slope p/q. The quasiline $L_{p/q}$ moves continuously with ρ and the data $a_{p/q}, b_{p/q}, c_{p/q}$ and $d_{p/q}$, as does the associated fundamental domain $S^*_{p/q}$.* ∎

Remark. In fact, the defining points (vertices of $S^*_{p/q}$) move *holomorphically*, but as a set $S^*_{p/q}$ does not.

Next the analogue of Lemma 7.2.2.

8.3.13 Lemma. *Fix a rational slope p/q. The set*

$$\{\rho \,:\, \Gamma_\rho \text{ admits the canonical peripheral quasidisc } H_{p/q}\}$$

is open.

Proof. By definition $\mathrm{tr}(\mathrm{Word}(p/q)) \in \{\Re z < -2\}$. Choose a small neighbourhood of ρ so that this remains true. That is, $\mathrm{tr}(\mathrm{Word}(p/q)(\rho')) \in \{\Re z < -2\}$ for ρ' close to ρ. Each Γ_ρ is geometrically finite (see the discussion immediately preceeding Theorem 5.2.4), and therefore each parabolic fixed point is doubly cusped (Lemma 3.3.4). Let U be a horodisc neighbourhood of the parabolic fixed point in $\partial S^*_{p/q}$ (not ∞). As $\ell_{p/q} \in \partial H_{p/q}$ is in the domain of discontinuity for $\Gamma_{p/q}$ it is in the ordinary set of Γ_ρ and projects to a loop bounding a doubly punctured disc in $S_{0,4}$. It follows that $S^*_{p/q} \setminus U$ is compactly supported away from $\Lambda(\Gamma_\rho)$. This limit set moves holomorphically and so for small time t the varying $(S^*_{p/q})_t \setminus U_t$ lie in the ordinary set of Γ_{ρ_t}. The images of $(S^*_{p/q})_t \setminus U_t$ under $(\Gamma_{p/q})_t$ tessellate $(H_{p/q})_t$, apart from the deleted cusp neighbourhoods which we now put back to find a canonical peripheral quasidisc $(H_{p/q})_t$. ∎

For $p/q \in \mathbb{Q}$, let $\mathcal{N}_{p/q}$ be the set defined by

$$\mathcal{N}_{p/q} := \{\rho \in \mathbb{C} \,:\, \Gamma_\rho \text{ admits a canonical peripheral quasidisc } H_{p/q}\}.$$

We prove a version of Lemma 7.2.10, for $\mathcal{N}_{p/q}$ rather than the pleating ray $\mathcal{P}_{p/q}$.

8.3.14 Lemma. *Fix a rational slope p/q. If $\rho \in \mathcal{N}_{p/q}$, then $\rho \in \mathcal{R}$.*

Proof. We have $\Gamma_{p/q} = \langle f, h_{p/q}\rangle = \langle f, f^{-1}h_{p/q}\rangle$. As described earlier there is another group $\Gamma'_{p/q}$ generated by two parabolics in Γ_ρ whose product is also $h_{p/q}$. These groups are not conjugate in Γ_ρ but are conjugate when the $\mathbb{Z}_2$ symmetry that conjugates X to Y is added. This symmetry leaves the limit set set-wise invariant. Hence both groups are quasi-Fuchsian with canonical peripheral quasidiscs. The remainder of the argument is as in Lemma 7.2.10. Briefly, both of the sets

$$H_{p/q}/\Gamma_{p/q} = H_{p/q}/\Gamma_\rho \text{ and } H'_{p/q}/\Gamma'_{p/q} = H'_{p/q}/\Gamma_\rho$$

are two different twice punctured discs in the quotient glued along a common boundary (a translation arc of f which lies in $H_{p/q} \cap H'_{p/q}$). Then the quotient is $S_{0,4}$ and hence $\rho \in \mathcal{R}$ by definition. ∎

The following technical lemma will be used in the proof of Lemma 8.3.16.

8.3.15 Lemma. *Let Γ_ρ be discrete and $\rho \neq 0$. Then for all rational slopes p/q,*

$$|\Phi_{p/q}(\rho) - 2| \geq 1$$

unless $\Phi_{p/q}(\rho) = 2$.

Remark. This estimate is actually sharp, by considering the figure-8 knot complement group [39, Lemma 4]. It also admits a strengthening: the union of all of the inverse images of the unit disc under the polynomials $\Phi_{p/q} - 2$ fills the Riley slice complement [75, Lemma 3].

Proof of Lemma 8.3.15. Label the entries of the matrix of Word(p/q) by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose first that $c \neq 0$. Then the Shimitzu–Leutbecher inequality (Corollary 3.1.11) applied to the discrete group $\langle X, \mathrm{Word}(p/q)\rangle$ gives

$$1 \leq \operatorname{tr}[X, \mathrm{Word}(p/q)] - 2 = |c|^2 = |a + d - 2|^2$$

which is the desired result by Lemma 8.3.2. If $c = 0$, then $h_{p/q}$ is parabolic and also fixes ∞. ∎

It is in the proof of the next lemma (the analogue of Lemma 7.3.13) where we use the fact that the quasidiscs $H_{p/q}$ have bounded geometry. Without this, the invariant quasicircles for the peripheral discs could either become space-filling curves or collapse entirely. This indeed happens in general with the formation of B-groups, or the geometrically infinite groups on the boundary of $\mathcal{R}$.

8.3.16 Lemma. *Fix a rational slope p/q. Suppose that Γ_{ρ_j} admits canonical peripheral quasidiscs $H^j_{p/q}$, and that $\operatorname{tr}(\mathrm{Word}(p/q)^j) \to z_0$ with $\Re(z_0) < -2$. Then there is a subsequence ρ_{j_k} which converges to some $\rho \in \mathcal{R}$ such that Γ_ρ admits a canonical peripheral quasidisc of the same slope.*

Proof. That $\operatorname{tr}(W^j_{p/q}) \to z_0$ where $\Re(z_0) < -2$ means that $a_{p/q}, b_{p/q}, c_{p/q}$ and $d_{p,q}$ all have finite limits and that $c_{p/q} \not\to 0$ by Lemma 8.3.15 and Lemma 8.3.2; therefore we can apply Lemma 8.3.9 to conclude that the invariant lines bounding $L_{p/q}$ also have a limiting height above and below. It follows that there is a non-empty open set U such that, for j sufficiently large, $U \subset H^j_{p/q}$. Each of the groups Γ_{ρ_j} is discrete (and free) and, after passing to a subsequence if necessary, the limit group Γ_ρ is also discrete (and free). Thus the ordinary set of Γ_ρ must contain U. By Lemma 8.3.14 we have $\rho_j \in \mathcal{R}$ and hence $\rho \in \overline{\mathcal{R}}$. If $\rho \in \mathcal{R}$ we are done. Otherwise $\rho \in \partial\mathcal{R}$, and Γ_ρ has nonempty ordinary set $\Omega_\rho = \hat{\mathbb{C}} \setminus \Lambda(\Gamma_\rho)$. Since ρ lies in the boundary of $\mathcal{R}$ the quotient surface Ω/Γ_ρ can support no moduli. The group Γ_ρ is torsion-free with non-empty ordinary set (it contains U), so the quotient is a union of triply punctured spheres and the point ρ must be a cusp group (these results are all found in the paper [87]). Notice that $h_{p/q}$ will have its fixed points in the boundary of a component of the ordinary set, which are now round circles. Thus $\Gamma_{p/q}$ is Fuchsian (since it is *a priori* quasi-Fuchsian and has limit set dense in a round circle), $\operatorname{tr}(h_{p/q})$ is real and therefore $\operatorname{tr}(h_{p/q}) \in (-\infty, -2)$. But these groups lie on the pleating ray in $\mathcal{R}$ and so have F-peripheral discs. This completes the proof. ∎

We now complete the proof of Theorem 8.1.1. Consider the set $\mathcal{Z}_{p/q}$ defined by

$$\mathcal{Z}_{p/q} = \{\rho \in \mathcal{R} : \Re\Phi_{p/q}(\rho) < -2\}.$$

We show that $\mathcal{N}_{p/q}$ is a connected component of $\mathcal{Z}_{p/q}$, by showing (as in the proof of Theorem 7.3.4) that $\mathcal{N}_{p/q}$ is a non-empty clopen subset of $\mathcal{Z}_{p/q}$.

By Lemma 8.3.14, $\mathcal{N}_{p/q} \subseteq \mathcal{R}$.

We make four observations.

1. $\mathcal{N}_{p/q} \subseteq \mathcal{Z}_{p/q}$ since, by Definition 8.3.10, $\Re \operatorname{tr} \mathrm{Word}(p/q)(\rho) < -2$ for $\rho \in \mathcal{N}_{p/q}$;
2. Note that $\mathcal{N}_{p/q}$ is closed in $\mathcal{Z}_{p/q}$ by Lemma 8.3.16.
3. By definition, $\mathcal{Z}_{p/q}$ is open in $\mathbb{C}$ (it is the inverse image of an open set); since $\mathcal{N}_{p/q}$ is also open in $\mathbb{C}$ (Lemma 8.3.13) it is open in $\mathcal{Z}_{p/q}$.
4. Finally, $\mathcal{N}_{p/q} \neq \varnothing$ since (by Lemma 8.3.11) it contains the (non-empty) p/q pleating ray.

Thus $\mathcal{N}_{p/q}$ is a union of non-empty connected components of $\mathcal{Z}_{p/q}$ contained in $\mathcal{R}$. By the Keen–Series theory, there are at most two such connected components, namely the components corresponding to the pleating rays of asymptotic slopes $\pm\pi p/q$ (Theorem 7.3.4); and clearly we hit both of these components. In any case, picking a branch of the inverse of $\Phi_{p/q}$ corresponding to these arguments will give a connected component of $\mathcal{N}_{p/q}$, and such a component is the desired neighbourhood of the cusp lying inside the Riley slice.

Chapter 9

The combinatorics of the Farey polynomials

In this chapter we will give some combinatorial results on the Farey polynomials. In particular, we will give a recursion formula for the Farey polynomials and a closed form formula for certain sequences of parabolic Farey polynomials.

9.1 A recursion formula to generate Farey polynomials

Recall from Section 6.3 that we say that p/q and r/s are **Farey neighbours** if $ps - qr = \pm 1$ (so if $p/q < r/s$ then $ps - qr = 1$); if p/q and r/s are such then write $p/q \oplus r/s$ for the mediant $(p+r)/(q+s)$. It will be convenient also to have the notation $p/q \ominus r/s$ for the fraction $(p-r)/(q-s)$; we shall only use this when it is known that $(p-r)/(q-s)$ and r/s are Farey neighbours (it is easy to check that if p/q and r/s are neighbours then so are $(p-r)/(q-s)$ and r/s) and that $q - s \neq 0$. The graph of Figure 9.2 shows the fractions in $[0, 1]$ of low denominator, with directed edges from p/q and r/s to $p/q \oplus r/s$.

Notation. In this chapter, because we will give a lot of combinatorial formulae, it will be convenient to shorten Word(p/q) to $W_{p/q}$. We also slightly modify the conventions of Figure 6.5. Consider the same labelled tiling of $\mathbb{R}^2$ which we repeat in Figure 9.1, and let $L_{p/q}$ be the line through $(0, 0)$ of slope p/q; now define $S_{p/q} = L_{p/q} \cap 2\mathbb{Z}^2$. Then the Farey word $W_{p/q}$, is the word of length $2q$ such that the ith letter is the label on the right-hand side of the ith vertical line segment crossed by $L_{p/q}$ (i.e. the label to the right of the point $(p/q)i$); if $(p/q)i$ is a lattice point then this definition is ambiguous and by convention we take the label on the north-west side. In other words, the ith letter of $W_{p/q}$ is determined by the parity of ceil$(p/q)i$ with the convention that ceil $n = n + 1$ for integral n. In Figure 9.1 we give the example of $W_{1/2}$ to compare with Figure 6.5. The advantage of this definition is that we have normalised the 'problem' vertex of the triangle with vertices p/q, r/s, and $(p/q) \oplus (r/s)$ (Figure 9.3 below) to be at an integer point, which simplifies the analysis.

One might guess, e.g. by analogy with the Maskit slice [94, p. 277], that $W_{p/q} W_{r/s} = W_{p/q \oplus r/s}$. Let us check:

9.1.1 Example. We use the convention $x \coloneqq X^{-1}$, $y \coloneqq Y^{-1}$:

- $W_{1/2} = yxYX$, $W_{1/1} = YX$, $W_{1/2} W_{1/1} = yxYXYX$, and $W_{1/2 \oplus 1/1} = yxyXYX$.

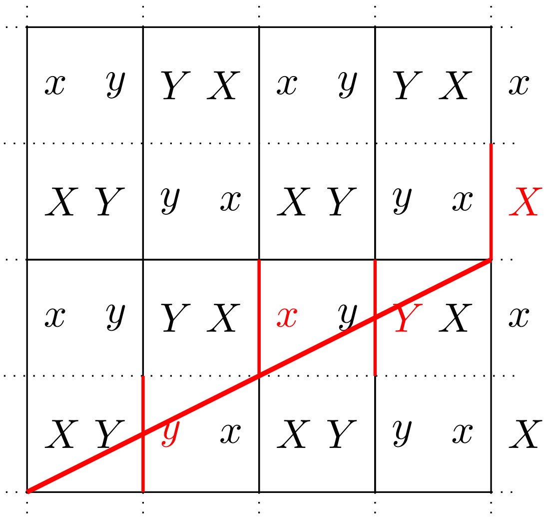


Figure 9.1: The cutting sequence of the 1/2 Farey word, with a slightly different convention to Figure 6.5.

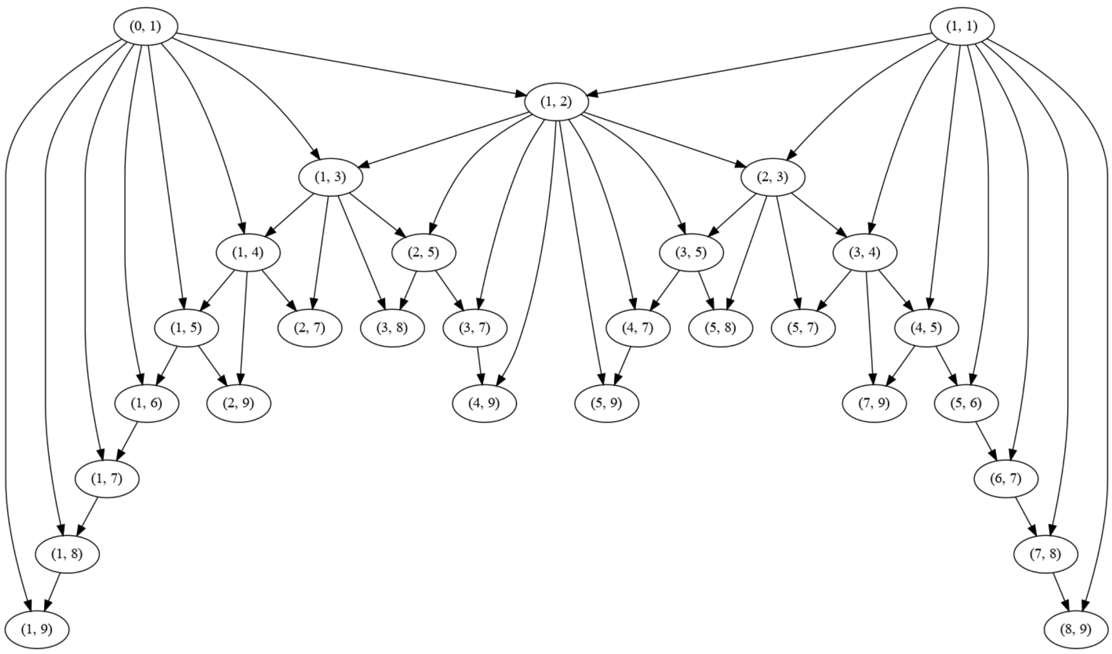


Figure 9.2: The Farey addition graph.

- $W_{1/3} = yXYxYX$, $W_{2/5} = yXYxyXyxYX$, $W_{1/3}W_{2/5} = yXYxYXyXYxyXyxYX$, and
$$W_{1/3\oplus 2/5} = W_{3/8} = yXYxYXyxYxyXyxYX.$$

This example shows that our guess is almost correct; the corrected statement is:

9.1.2 Lemma. *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then $W_{p/q\oplus r/s}$ is the word $W_{p/q}W_{r/s}$ with the sign of the $(q+s)$th exponent swapped.*

Proof. The situation is diagrammed in Figure 9.3 for convenience. To simplify notation, in this proof we write $h(i)$ for the height $(\frac{p}{q} \oplus \frac{r}{s})i$. Observe that $h(i)$ is integral only at $i = 0$ and $i = 2q + 2s$ (at both positions trivially the letters in $W_{(p/q)\oplus(r/s)}$ and $W_{p/q}W_{r/s}$ are identical) and at $i = q + s$. The lemma will follow once we check that that at the positions $i \notin \{0, 2q + 2s\}$,

$$\text{if } 0 < i \leq 2q \text{ then } (p/q)i < h(i) < \operatorname{ceil}(p/q)i \tag{9.1.3}$$

and

$$\text{if } 0 < i < 2s \text{ then } (r/s)i + 2q < h(i + 2p) < \operatorname{ceil}[(r/s)i + 2p] \ : \tag{9.1.4}$$

indeed, these inequalities show that at every integral horizontal distance the height of the line corresponding to $W_{p/q\oplus r/s}$ is meeting the same vertical line segment as the line corresponding to $W_{p/q}$ or $W_{r/s}$, and so the letter chosen is the same except at $i = q + s$ since at this position the height of the line of slope $(p/q) \oplus (r/s)$, being integral, is rounded up to $h(i) + 1$ while the height of the line of slope r/s is non-integral so is rounded up to the integer $h(i)$.

Observe now that the inequalities Equations (9.1.3) and (9.1.4) are equivalent to the following: there is no integer between $(p/q)i$ and $(p/q \oplus r/s)i$ (exclusive) if $0 < i \leq 2q$, and there is no integer between $(r/s)i + 2q$ and $h(i + 2p)$ if $0 < i < 2s$. But these follow from Lemma 6.3.1. Indeed, the lemma shows that no integer lies between p/q and $(p/q) \oplus (r/s)$; suppose a/b is a rational between $(p/q)i$ and $h(i)$, then $a = i(\lambda p + \mu(p + r))$ and $b = (\lambda q + \mu(q + s))$ for some positive λ, μ; suppose $a/b \in \mathbb{Z}$, so $\lambda q + \mu(q + s)$ divides $i(\lambda p + \mu(p + r))$. By the case $i = 1$, $\lambda p + \mu(p + r)$ and $\lambda q + \mu(q + s)$ are coprime, so $\lambda q + \mu(q + s)$ divides i; but $\lambda q + \mu(q + s) \geq 2q$. The case of $(r/s)i + 2q$ and $h(i + 2p)$ is proved in a similar way. ∎

Notation. We adopt the convention that, if we fix the Riley slice $\mathcal{R}^{a,b}$, then we write α and β for $\exp(\pi i/a)$ and $\exp(\pi i/b)$ respectively.

9.1.5 Lemma. *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identity holds:*

$$\operatorname{tr} W_{p/q}W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} = \begin{cases} 2 + \alpha^2 + \frac{1}{\alpha^2} & \text{if } q + s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q + s \text{ is odd.} \end{cases}$$

Proof. Trace is invariant under cyclic permutations, thus (applying Lemma 9.1.2) we can write

$$\operatorname{tr} W_{p/q}W_{r/s} = \operatorname{tr} AB \text{ and } \operatorname{tr} W_{p/q\oplus r/s} = \operatorname{tr} AB^{-1},$$

where B is the $(q+s)$th letter of $W_{p/q}W_{r/s}$ and A is the remainder of the word but with the final letters cycled to the front. Now we know that $\operatorname{tr} AB = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1}$ (see the useful list of trace identites found in Section 3.4 of [76]), so it suffices to check that $\operatorname{tr} A \operatorname{tr} B = 2 + \alpha^2 + \frac{1}{\alpha^2}$ if $q + s$ is even and $\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$ otherwise.

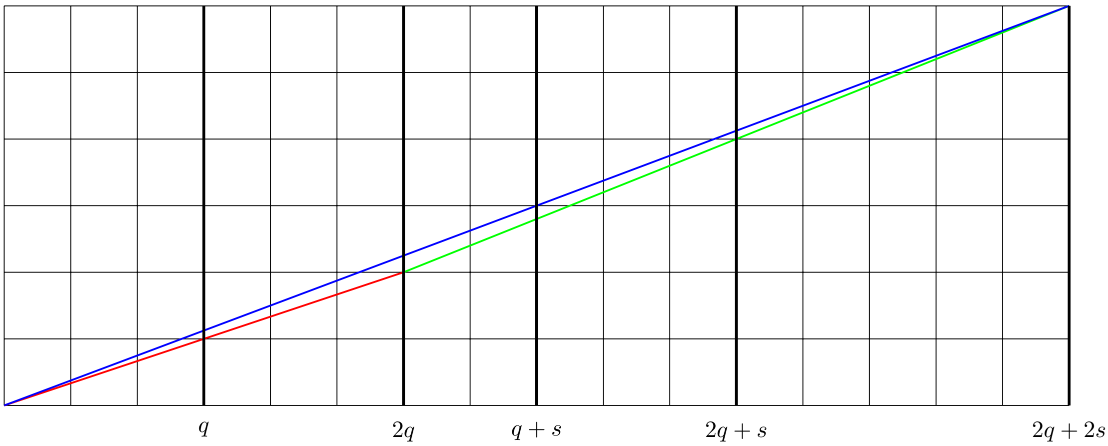


Figure 9.3: Farey addition versus word multiplication for $W_{p/q}$ (red) and $W_{r/s}$ (green).

Case I. $q+s$ **is even.** Observe now that B is $X^{\pm 1}$ if $q+s$ is even; then $\operatorname{tr} B = 2\Re\alpha$. The identity to show is therefore $\operatorname{tr} A = (2 + \alpha^2 + \frac{1}{\alpha^2})/(2\Re\alpha)$; recalling that $|\alpha| = 1$ and using the double angle formulae we have

$$\frac{2}{2\Re\alpha} + \frac{\alpha^2}{2\Re\alpha} + \frac{\alpha^{-2}}{2\Re\alpha} = \frac{1}{\cos\theta} + \left(\alpha - \frac{1}{2\cos\theta}\right) + \left(\overline{\alpha} - \frac{1}{2\cos\theta}\right) = 2\cos\theta$$

where $\theta = \operatorname{Arg}\alpha$. Thus we actually just need to show $\operatorname{tr} A = 2\cos\theta$. As an aside, this shows that A is parabolic if X is, and is elliptic if X is.

Case II. $q+s$ **is odd.** In this case, B is $Y^{\pm 1}$ and so $\operatorname{tr} B = 2\Re\beta$; we therefore wish to show that $\operatorname{tr} A = (\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta})/(2\Re\beta)$; again using trigonometry we may simplify the right side,

$$\frac{\alpha\beta}{2\Re\alpha} + \frac{\alpha/\beta}{2\Re\alpha} + \frac{\beta/\alpha}{2\Re\alpha} + \frac{1/(\alpha\beta)}{2\Re\beta} = 2\cos\theta$$

and so again we need only show that $\operatorname{tr} A = 2\cos\theta$ where $\theta = \operatorname{Arg}\alpha$.

Both cases then reduce to the identity $\operatorname{tr} A = \operatorname{tr} X$. It will be enough to show that A is conjugate to X; by construction of A, this is equivalent to showing that in $W_{p/q \oplus r/s}$ the $(q+s+1)$th to $(2q+2s-1)$th letters are obtained from the first $q+s-1$ letters by reversing the order and swapping the case. But this is just Lemma 6.1.7. ∎

In the case that X and Y are parabolics and $\alpha = \beta = 1$, the two formulae unify to become:

$$\operatorname{tr} W_{p/q} W_{r/s} = 4 - \operatorname{tr} W_{p/q \oplus r/s}.$$

We may similarly prove the following lemma:

9.1.6 Lemma. *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identity holds:*

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \ominus r/s} = \begin{cases} 2 + \beta^2 + \frac{1}{\beta^2} & \text{if } q-s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q-s \text{ is odd.} \end{cases}$$

Proof. We begin by setting up notation. By Lemma 6.1.7 we may write $W_{p/q} = UAuX$ with $A = X^{\pm 1}$ if q is even and $A = Y^{\pm 1}$ if q is odd; similarly, write $W_{r/s} = VBvX$ with B one of $X^{\pm 1}$ or $Y^{\pm 1}$. Then

$$W_{p/q}W_{r/s}^{-1} = UAuXxVbv = UAuVbv;$$

by Lemma 9.1.2, we have also that $W_{r/s}W_{p/q\ominus r/s}$ is $W_{p/q}$ with the sign of the exponent of the qth letter swapped; explicitly,

$$W_{p/q\ominus r/s}VBvX = UauX \implies W_{p/q\ominus r/s} = UauXxVbv = UauVbv.$$

Our goal is therefore to compute $\operatorname{tr} UAuVbv + \operatorname{tr} UauVbv$; performing a cyclic permutation again, this is equivalent to $\operatorname{tr} A(uVbvU) + \operatorname{tr} a(uVbvU)$. In this form, this becomes

$$\operatorname{tr} A(uVbvU) + \operatorname{tr} a(uVbvU) = \operatorname{tr} A \operatorname{tr} uVbvU = \operatorname{tr} A \operatorname{tr} b.$$

Consider now the cases for the product $\operatorname{tr} A \operatorname{tr} b$:

	q odd	q even
s odd	$\operatorname{tr}^2 Y$	$\operatorname{tr} X \operatorname{tr} Y$
s even	$\operatorname{tr} X \operatorname{tr} Y$	$\operatorname{tr}^2 X$.

If $p/q, r/s$ are Farey neighbours then it is not possible for both q and s to be even since $ps - rq \equiv 1 \pmod 2$. Further, $q - s$ is odd iff exactly one of p and q is odd, otherwise $q - s$ is even. Thus we see that if $q - s$ is even then

$$\operatorname{tr} W_{p/q}W_{r/s}^{-1} + \operatorname{tr} W_{p/q\ominus r/s} = \operatorname{tr}^2 Y = (\beta + 1/\beta)^2$$

and if $q - s$ is odd then

$$\operatorname{tr} W_{p/q}W_{r/s}^{-1} + \operatorname{tr} W_{p/q\ominus r/s} = \operatorname{tr} X \operatorname{tr} Y = (\alpha + 1/\alpha)(\beta + 1/\beta)$$

which are the claimed formulae. ∎

9.1.7 Theorem. *Let p/q and r/s be Farey neighbours. If $q + s$ is even, then*

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q\oplus r/s} + \Phi_{p/q\ominus r/s} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2. \tag{9.1.8}$$

Otherwise if $q + s$ is odd, then

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q\oplus r/s} + \Phi_{p/q\ominus r/s} = 2\left(\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right). \tag{9.1.9}$$

Proof. Suppose $q + s$ is even; then $q - s$ is also even, so

$$\begin{aligned}
\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} &= \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&= \operatorname{tr} W_{p/q}W_{r/s} + \operatorname{tr} W_{p/q}W_{r/s}^{-1} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&= 2 + \alpha^2 + \frac{1}{\alpha^2} + 2 + \beta^2 + \frac{1}{\beta^2}
\end{aligned}$$

where in the final step we used Lemma 9.1.5 and Lemma 9.1.6. Similarly, when $q - s$ is odd then $q + s$ is also odd and

$$\begin{aligned}
\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} &= \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&= \operatorname{tr} W_{p/q}W_{r/s} + \operatorname{tr} W_{p/q}W_{r/s}^{-1} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}
\end{aligned}$$

as desired. ∎

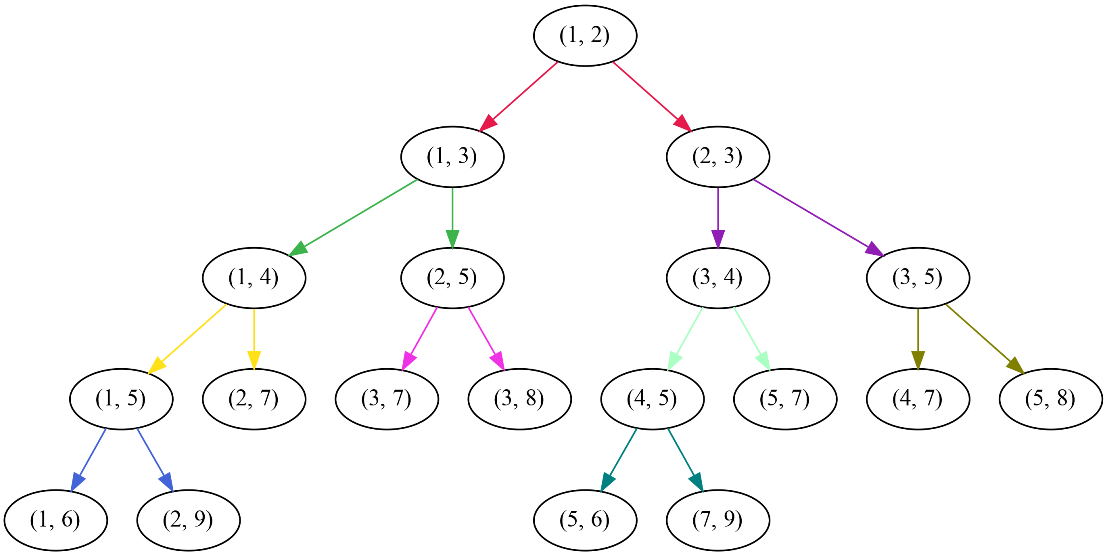


Figure 9.4: The induced colouring of the Stern-Brocot tree.

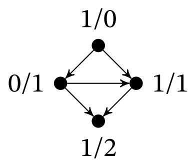
Again in the parabolic case the two formulae unify and the recursion identity becomes

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} = 8. \tag{9.1.10}$$

We observe as an aside that if we just draw the edges of the Farey graph corresponding to Farey neighbours which appear as products in some iterate of the recursion, then we obtain a nice colouring of the Stern-Brocot tree (Figure 9.4).

Based on the recurrence, we make the following useful convention/definition:

9.1.11 Definition. Observe that $0/1$ and $1/0$ are Farey neighbours in $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Thus, applying Equation (9.1.8) *formally* to the diamond



we obtain

$$\Phi_{0/1}\Phi_{1/1} + \Phi_{1/0} + \Phi_{1/2} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2;$$

substituting for $\Phi_{1/1}$, $\Phi_{1/2}$, and $\Phi_{0/1}$ from Table 6.2 we get the following expression for $\Phi_{1/0}$, which we henceforth take to be a definition:

$$\begin{aligned}\Phi_{1/0} &= 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z\right)\left(\alpha\beta + \frac{1}{\alpha\beta} + z\right) - 2 - \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right)z - z^2 \\ &= 2.\end{aligned}$$

Observe that $\Phi_{1/0}^{-1}((-\infty, -2]) = \varnothing$, so this is compatible with the Keen–Series theory; it is also a polynomial of degree q (here, $q = 0$) with constant term 2, which all agrees with the properties of the higher-degree polynomials. On the other hand, it is not monic!

We also define $\Phi_{p/q}$ for all $p/q \in \mathbb{Q}$ via this method.

9.2 Some properties of the recurrence

Recall that the **Chebyshev polynomials** (of the first kind) are the family of polynomials T_n defined via the recurrence relation

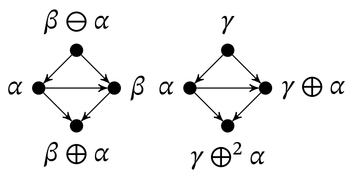
$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned}$$

It is well-known that these polynomials satisfy the product relation

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$$

for $m, n \in \mathbb{Z}_{\geq 0}$. Compare this relation with the relation Equation (9.1.10) developed above for the parabolic Farey polynomials (but note that the Chebyshev product rule holds for all m, n and the identities for the Farey polynomials hold only for Farey neighbours).

We may apply the theory of 'Farey recursive functions' [30, 31] in order to explain this analogy; in a following section, we will give a generalisation to the elliptic case. The following diagram may be useful for translating the notation of that paper (right) into the notation we use here (left):



9.2.1 Definition (Definition 3.1 of [31]). Let R be a (commutative) ring, and suppose $d_1, d_2 : \hat{\mathbb{Q}} \to R$. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is a (d_1, d_2)-**Farey recursive function** if, whenever $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours,

$$\mathcal{F}(\beta \oplus \alpha) = -d_1(\alpha)\mathcal{F}(\beta \ominus \alpha) + d_2(\alpha)\mathcal{F}(\beta). \tag{9.2.2}$$

Observe that the relation Equation (9.1.10) looks essentially of this form; to make this clearer, we rewrite it slightly as

$$\Phi(\beta \oplus \alpha) = 8 - \Phi(\beta \ominus \alpha) - \Phi(\alpha)\Phi(\beta) \tag{9.2.3}$$

(where we set $\beta = p/q$, $\alpha = r/s$, and changed from subscript notation to functional notation). In our case, then, $d_1(\alpha)$ is constantly 1 and $d_2 = -\Phi$. Note also that our relation is not homogeneous. We therefore adapt the definition of [31] to the following:

9.2.4 Definition. Let R be a (commutative) ring, and suppose $d_1, d_2, d_3 : \hat{\mathbb{Q}} \to R$. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is a (d_1, d_2)-**Farey recursive function** if, whenever $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours,

$$\mathcal{F}(\beta \oplus \alpha) = -d_1(\alpha)\mathcal{F}(\beta \ominus \alpha) + d_2(\alpha)\mathcal{F}(\beta) + d_3(\alpha). \tag{9.2.5}$$

If d_3 is the zero function, we say that the relation of Equation (9.2.5) is **homogeneous**, otherwise it is **non-homogeneous**.

The relevant generalisations of the existence-uniqueness results of [31, Section 4] follow easily (the same proofs work, with the usual property that the space of non-homegenous solutions is the sum of a particular solution and the space of homogeneous solutions).

In our case, there is an obvious explicit solution to the non-homogeneous Farey polynomial recursion, Equation (9.2.3): namely, the map Φ which sends every $\alpha \in \mathbb{Q}$ to the constant polynomial $2 \in \mathbb{Z}[z]$. It therefore remains to solve the corresponding homogeneous equation.

Table 9.1: Selected homogeneous Farey polynomials Φ^h of slope p/q for small q, with the initial values as given.

p	q	$\Phi^h_{p/q}$
1	0	2
0	1	$-z+2$
1	1	$z+2$
1	2	z^2-6
1	3	$z^3-2z^2-7z+10$
2	3	$-z^3-2z^2+7z+10$
1	4	$z^4-4z^3-4z^2+24z-14$
3	4	$z^4+4z^3-4z^2-24z-14$
1	5	$z^5-6z^4+3z^3+34z^2-55z+18$
2	5	$-z^5+2z^4+13z^3-22z^2-41z+58$
3	5	$z^5+2z^4-13z^3-22z^2+41z+58$
4	5	$-z^5-6z^4-3z^3+34z^2+55z+18$
1	6	$z^6-8z^5+14z^4+32z^3-119z^2+104z-22$
1	7	$z^7-10z^6+29z^5+10z^4-186z^3+308z^2-175z+26$
1	8	$z^8-12z^7+48z^6-40z^5-220z^4+648z^3-672z^2+272z-30$
1	9	$z^9-14z^8+71z^7-126z^6-169z^5+1078z^4-1782z^3+1308z^2-399z+34$
1	10	$z^{10}-16z^9+98z^8-256z^7+35z^6+1456z^5-3718z^4+4224z^3-2343z^2+560z-38$

9.2A A Fibonacci-like subsequence of the homogeneous Farey polynomials

In Table 9.1, we list the first few **homogeneous Farey polynomials** for a particular set of seed values: that is, the polynomials Φ^h which solve the homogeneous recursion relation

$$\Phi^h(\beta \oplus \alpha) = -\Phi^h(\beta \ominus \alpha) - \Phi^h(\alpha)\Phi^h(\beta) \tag{9.2.6}$$

with the initial values $\Phi^h(0/1) = 2 - z$, $\Phi^h(1/0) = 2$, and $\Phi^h(1/1) = 2 + z$.

The polynomials with numerator 1 listed in the table have very nice properties: immediately one sees that the constant terms alternate in sign and increase in magnitude by 4 each time; also, we have that $\Phi^h_{1/q}(1)$ cycles through the values $3, -5, 2$, $\Phi^h_{1/q}(2)$ cycles through the values $4, -2, -4, 2$; and when we evaluate at 3 and 4 we get a 6-cycle and an arithmetic sequence of step 4 respectively. When we consider $\Phi^h(1/q)(5)$, though, we obtain more interesting behaviour: this is OEIS sequence A100545[1] and satisfies the Fibonacci-type relation

$$\Phi^h_{1/q}(5) = 3\Phi^h_{1/(q-1)}(5) - \Phi^h_{1/(q-2)}(5) \quad \text{with } \Phi^h_{1/1}(5) = 7, \Phi^h_{1/2}(5) = 19.$$

Of course, from the way that we defined the Φ^h such types of relations ought to be expected. In this section, we use the standard diagonalisation technique to explain the behaviour of the sequence $a_q \coloneqq \Phi^h(1/q)(z)$ for fixed $z \in \mathbb{C}$. From Equation (9.2.6), we have that

$$a_q = -(2-z)a_{q-1} - a_{q-2}. \tag{9.2.7}$$

We may rewrite this equation in matrix form as the following:

$$\begin{bmatrix} 0 & 1 \\ -1 & z-2 \end{bmatrix} \begin{bmatrix} a_{q-2} \\ a_{q-1} \end{bmatrix} = \begin{bmatrix} a_{q-1} \\ a_q \end{bmatrix}. \tag{9.2.8}$$

[1] `http://oeis.org/A100545`

One easily computes that the eigenvalues of the transition matrix are

$$\lambda^{\pm} = \frac{z-2\pm\alpha}{2}$$

(where $\alpha = \sqrt{z^2-4z}$) with respective eigenvectors

$$v^{\pm} = \begin{bmatrix} z-2\mp\alpha \\ 2 \end{bmatrix}$$

(note the alternated sign). Thus the transition matrix may be diagonalised as

$$\frac{-1}{2\alpha}\begin{bmatrix} z-2-\alpha & z-2+\alpha \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{z-2+\alpha}{2} & 0 \\ 0 & \frac{z-2-\alpha}{2} \end{bmatrix}\begin{bmatrix} 2 & 2-z-\alpha \\ -2 & z-2-\alpha \end{bmatrix} \tag{9.2.9}$$

and so a_q is the first coordinate of

$$\frac{-1}{2\alpha}\begin{bmatrix} z-2-\alpha & z-2+\alpha \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \left(\frac{z-2+\alpha}{2}\right)^q & 0 \\ 0 & \left(\frac{z-2-\alpha}{2}\right)^q \end{bmatrix}\begin{bmatrix} 2 & 2-z-\alpha \\ -2 & z-2-\alpha \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix};$$

expanding out, we get

$$a_q = \frac{2^{-1-q}\left((a_0(z-2+\alpha)-2a_1)(z-2-\alpha)^q + (a_0(2-z+\alpha)+2a_1)(z-2+\alpha)^q\right)}{\alpha}. \tag{9.2.10}$$

We may also characterise the z for which $\Phi^h_{1/q}(z)$ is cyclic: this occurs precisely when the diagonal matrix of Equation (9.2.9) is of finite order, i.e. whenever both $(z-2\pm\alpha)/2$ are roots of unity.

As an application of the theory above, we have seen that the Chebyshev polynomials also satisfy a second-order recurrence relation with transition matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix}.$$

If we set $z = 2x+2$, then we get back our transition matrix from Equation (9.2.8). Thus our sequence $\Phi^h_{1/q}(z)$ for fixed z is of the form $W_q(\frac{z-2}{2})$ where W_q is the qth Chebyshev polynomial in the sequence beginning with $W_0 = 2x$ and $W_1 = 2x+4$.

Finally, we consider the solution of the non-homogeneous equation for $\Phi_{1/q}$. Above, we observed that there is a constant solution to the global recursion relation on the entire Stern-Brocot tree; we therefore guess that there is a similar solution to this recursion. Such a solution f will satisfy

$$8 = f(z) + (2-z)f(z) + f(z)$$

and arithmetic gives $f(z) = 8/(4-z)$. Combining this with Equation (9.2.10) above gives us the following general solution to the non-homogeneous relation:

$$a_q = \frac{8}{4-z} + \frac{2^{-1-q}\left((\lambda(z-2+\alpha)-2\mu)(z-2-\alpha)^q + (\lambda(2-z+\alpha)+2\mu)(z-2+\alpha)^q\right)}{\alpha}.$$

In our case, we have $a_0 = \Phi_{1/0}(z) = 2$ and $a_1 = \Phi_{1/1}(z) = 2+z$. Solving the resulting system of equations gives

$$(\lambda,\mu) = \left(\frac{2z}{z-4}, \frac{2z-z^2}{z-4}\right)$$

and hence

$$a_q = \frac{8}{4-z} + \frac{2^{-q}z}{z-4}\left((-2+z-\sqrt{z^2-4z})^q + (-2+z+\sqrt{z^2-4z})^q\right)$$

9.2B Solving the homogeneous recursion relation in general

In the previous section, we computed a closed form formula for $\Phi^h_{1/q}(n)$ using standard techniques from the theory of second-order linear recurrences. We now tackle the general problem of finding a closed-form formula for $\Phi^h_{p/q}(n)$; in order to do this, we use the theory of Section 6 of [31] but with a slight modification: in that paper, the authors define a special case of Farey recursive function, a **Farey recursive function of determinant** d (where $d : \hat{\mathbb{Q}} \to R$), to be a Farey recursive function $\mathcal{F}$ with $d_1 = d$ and $d_2 = \mathcal{F}$. That is, they replace the recurrence of Equation (9.2.2) with

$$\mathcal{F}(\beta \oplus \alpha) = -d(\alpha)\mathcal{F}(\beta \ominus \alpha) + \mathcal{F}(\alpha)\mathcal{F}(\beta).$$

This is very similar to our situation, except that instead of $d_2 = \mathcal{F}$ we have $d_2 = -\mathcal{F}$. To reflect this, we shall call a Farey recursive function satisfying a relation of the form

$$\mathcal{F}(\beta \oplus \alpha) = -d(\alpha)\mathcal{F}(\beta \ominus \alpha) - \mathcal{F}(\alpha)\mathcal{F}(\beta). \tag{9.2.11}$$

a Farey recursive function of ***anti*-determinant** d. We shall work for the time being in this setting (i.e. we shall work with the general function $\mathcal{F}$ rather than the particular example Φ) in order to restate in sufficient generality the theorem which we need (Theorem 6.1 of [31]).

Let $\alpha \in \mathbb{Q}$. The **boundary sequence** $\partial(\alpha)$ is defined inductively by the process of 'continuing to expand down the Farey graph by constant steps'. More precisely, let $\beta \oplus^k \gamma$ denote $((\beta \oplus \underbrace{\gamma) \oplus \cdots) \oplus \gamma}_{k \text{ iterates}}$ for $\beta, \gamma \in \hat{\mathbb{Q}}$ and let γ_L, γ_R be the unique Farey neighbours such that $\alpha = \gamma_L \oplus \gamma_R$; then we set

$$\partial(\alpha) := \{\gamma_L \oplus^k \alpha : k \in \mathbb{Z}_{\geq 0}\} \cup \{\gamma_R \oplus^k \alpha : k \in \mathbb{Z}_{\geq 0}\}.$$

If we allow the Farey graph to embed in the Euclidean upper halfplane by sending $\mathbb{Q} \ni p/q \mapsto (p/q, 1/q) \in \mathbb{H}^2$, then except for the exceptional cases $\alpha = 1/0$ and $\alpha = n/1$ for $n \in \mathbb{Z}$ the subgraph spanned by $\partial(\alpha)$ corresponds to a Euclidean triangle containing α in its interior, see Figure 9.5 and Figure 3 of [31]; for example, the triangle spanned by $\partial(1/n)$ is the triangle with vertices $0, (1/2, 1/2), 1$.

It will be useful to have specific names for the terms in each of the two subsequences and so we set, for $k \in \mathbb{Z}$,

$$\beta_k := \begin{cases} \gamma_L \oplus^{-k-1} \alpha & \text{if } k < -1 \\ \gamma_L & \text{if } k = -1 \\ \gamma_R & \text{if } k = 0 \\ \gamma_R \oplus^k \alpha & \text{if } k > 0. \end{cases} \tag{9.2.12}$$

For every $\alpha \in \mathbb{Q}$, define

$$M_\alpha = \begin{bmatrix} 0 & 1 \\ -d(\alpha) & \mathcal{F}(\alpha) \end{bmatrix}. \tag{9.2.13}$$

Given any Farey neighbour β of α, we have

$$M_\alpha^n \begin{bmatrix} \mathcal{F}(\gamma \oplus^0 \alpha) \\ \mathcal{F}(\gamma \oplus^1 \alpha) \end{bmatrix} = \begin{bmatrix} \mathcal{F}(\gamma \oplus^n \alpha) \\ \mathcal{F}(\gamma \oplus^{n+1} \alpha) \end{bmatrix}$$

and so the recursion Equation (9.2.11) is equivalent to a family of second-order linear recurrences, one down $\partial(\alpha)$ for each α.

We may now state the following theorem:

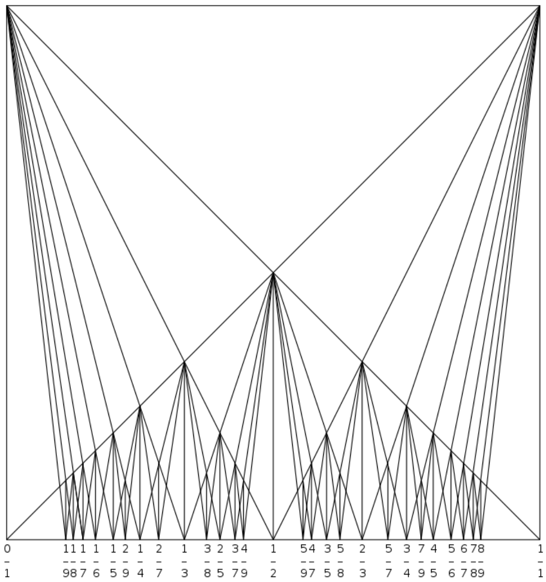


Figure 9.5: An embedding of the Farey graph into $\mathbb{H}^2$. (By Hyacinth—Own work, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=63343697`.)

9.2.14 Theorem (Adaptation of Theorem 6.1 of [31]). *Let $d : \hat{\mathbb{Q}} \to R$ be a multiplicative function (in the sense that $d(\gamma \oplus \beta) = d(\gamma)d(\beta)$ for all pairs of Farey neighbours $\beta, \gamma \in \mathbb{Q}$) to a commutative ring R, such that $d(\hat{\mathbb{Q}})$ contains no zero divisors. Suppose that $\mathcal{F}$ is a Farey recursive function with anti-determinant d. Given $\alpha \in \mathbb{Q}$, define M_α as in Equation* (9.2.13) *and $(\beta_k)_{k\in\mathbb{Z}}$ as in Equation* (9.2.12)*.*

Then, for all $n \in \mathbb{Z}$,

$$M_\alpha^n \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathcal{F}(\beta_n) \\ \mathcal{F}(\beta_{n+1}) \end{bmatrix} & n \geq 0, \\[2ex] \begin{bmatrix} \frac{1}{d(\beta_{-1})}\mathcal{F}(\beta_{-1}) \\ \mathcal{F}(\beta_0) \end{bmatrix} & n = -1, \text{ and} \\[2ex] \begin{bmatrix} \frac{1}{d(\beta_{-1})d_\alpha^{-n-1}}\mathcal{F}_{\beta_n} \\ \frac{1}{d(\beta_{-1})d_\alpha^{-n-2}}\mathcal{F}_{\beta_{n+1}} \end{bmatrix} & n < -1. \end{cases}$$

We proceed to prove Theorem 9.2.14 by exactly the same argument as given in [31]. The key point is the following lemma, which is the analogue of the discussion directly preceeding the statement of Theorem 6.1 in that paper.

9.2.15 Lemma. *With the setup of Theorem 9.2.14, we have*

$$M_\alpha^{-1} \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{bmatrix} \frac{d(\beta_0)}{d(\alpha)}\mathcal{F}(\beta_{-1}) \\ \mathcal{F}(\beta_0) \end{bmatrix}$$

$$M_\alpha^{-2} \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{bmatrix} \frac{1}{d(\alpha)d(\beta_{-1})}\mathcal{F}(\beta_{-2}) \\ \frac{1}{d(\beta_{-1})}\mathcal{F}(\beta_{-1}) \end{bmatrix}$$

Proof. The formula involving M_α^{-1} comes directly from computing the product on the left via the definition and simplifying with the formula

$$\mathcal{F}(\beta_1) = -d(\beta_0)\mathcal{F}(\beta_{-1}) - \mathcal{F}(\alpha)\mathcal{F}(\beta_0)$$

which is almost exactly the same as Equation (8) of [31]—the single sign change cancels exactly with the sign change between 'determinant' and 'anti-determinant' recurrences so we get the same overall formula for the M_α^{-1} product as they do in Equation (11) of their paper.

The formula for M_α^{-2} comes from applying the analogues of Equations (9) and (10) of their paper,

$$\begin{aligned}\mathcal{F}(\beta_{-2}) &= -d(\beta_{-1})\mathcal{F}(\beta_0) - \mathcal{F}(\alpha)\mathcal{F}(\beta_{-1})\\ d(\alpha) &= d(\beta_{-1})d(\beta_0)\end{aligned}$$

and simplifying; again the minus signs cancel and we get the same formula. ∎

Proof of Theorem 9.2.14. The formula for $n \geq 0$ holds for all Farey recursive formulae as noted above; the formulae for $n = -1$ and $n = -2$ are just the formulae of Lemma 9.2.15; and we proceed to prove the formula for $n < -2$ by induction. Assume that the formula holds for some fixed $n \leq -2$; then from the definitions we have

$$\mathcal{F}(\beta_{n-1}) = -\mathcal{F}(\alpha)\mathcal{F}(\beta_n) - d(\alpha)\mathcal{F}(\beta_{n+1})$$

and so we can compute

$$\begin{aligned}
M_\alpha^{n-1}\begin{bmatrix}F(\beta_0)\\F(\beta_1)\end{bmatrix} &= M_\alpha^{-1}M_\alpha^{n}\begin{bmatrix}F(\beta_0)\\F(\beta_1)\end{bmatrix}\\
&= \frac{1}{d(\alpha)}\begin{bmatrix}-\mathcal{F}(\alpha) & -1\\ d(\alpha) & -0\end{bmatrix}\begin{bmatrix}\frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}F(\beta_n)\\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-2}}F(\beta_{n+1})\end{bmatrix}\\
&= \frac{1}{d(\alpha)}\begin{bmatrix}-\frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}\left(\mathcal{F}(\alpha)\mathcal{F}(\beta_n) + d(\alpha)\mathcal{F}(\beta_{n+1})\right)\\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-2}}\mathcal{F}(\beta_n)\end{bmatrix}\\
&= \begin{bmatrix}-\frac{1}{d(\beta_{-1})d(\alpha)^{-n}}\mathcal{F}(\beta_{n-1})\\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}\mathcal{F}(\beta_n)\end{bmatrix}
\end{aligned}$$

which is the desired result. ∎

9.2.16 Corollary (Adaptation of Corollary 6.2 of [31]). *Let Φ^h be a family of homogeneous Farey polynomials (i.e. a family solving Equation* (9.2.6) *for some starting values). Then, for some $\alpha \in \mathbb{Z}$, if M_α is the matrix*

$$\begin{bmatrix}0 & 1\\ -1 & \Phi^h(\alpha)\end{bmatrix}$$

and if $(\beta_k)_{k\in\mathbb{Z}}$ are the boundary values about α as in Equation (9.2.12)*, then for all $n \in \mathbb{Z}$ we have*

$$M_\alpha^n\begin{bmatrix}\Phi^h(\beta_0)\\ \Phi^h(\beta_1)\end{bmatrix} = \begin{bmatrix}\Phi^h(\beta_n)\\ \Phi^h(\beta_{n+1})\end{bmatrix}.$$

Proof. This follows directly from Theorem 9.2.14 with the observation that the anti-determinant of Φ^h is the constant function $d(\gamma) = 1$ for all $\gamma \in \mathbb{Q}$. ∎

Thus to determine Φ^h_α for all $\alpha \in \mathbb{Q}$ it suffices to compute and diagonalise the M_α matrices, using the techniques of Section 9.2A. (Of course, we need to diagonalise in the ring of rational functions over $\mathbb{Q}$ rather than the ring of polynomials over $\mathbb{Z}$.) More precisely, we need to compute M_{α_i} for some family (α_i) of rationals with the property that the boundary sets $\partial(\alpha_i)$ cover $\mathbb{Q}$. (In Section 9.2A, we did this computation for $\partial(0/1)$.)

In any case, from Corollary 9.2.16 we immediately have a qualitative result:

9.2.17 Theorem. *For any $\gamma \in \mathbb{Q}$, there exists a sequence $\ldots, \gamma_{-1}, \gamma_0 = \gamma, \gamma_1, \gamma_2, \ldots$ of rational numbers such that $\Phi^h_{\gamma_n}(z)$ is a sequence of Chebyshev polynomials $W_n(\Phi^h_\gamma(z)/2)$. (Namely, let γ_{-1} be a neighbour in the Stern-Brocot tree of γ and take the sequence (γ_k) to be precisely the sequence (β_k) of Equation* (9.2.12) *with $\alpha \coloneqq \gamma \ominus \gamma_{-1}$.)* ∎

Remark. Of course, the boundary sequence (γ_k) constructed here is just a geodesic line Λ in the Stern-Brocot tree rooted at γ, defined by choosing one vertical half-ray in the tree starting from γ (where 'vertical' refers to the embedding of Figure 9.4) and then extending that in the Farey graph in the obvious way by repeated Farey arithmetic with the same difference. There are clearly two such natural choices for Λ given a fixed γ (γ has three neighbours, but two correspond to the same geodesic), and a single natural choice is obtained by taking the unique neighbour of γ which lies above.

We easily compute that the eigenvalues of M_α are

$$\lambda^\pm = \frac{1}{2}\left(\Phi^h_\alpha \pm \sqrt{(\Phi^h_\alpha)^2 - 4}\right).$$

Let $x = \Phi^h_\alpha$ and $\varkappa = \sqrt{x^2 - 4}$ (this is the analogue of the constant α from Section 9.2A); then the respective eigenvectors are

$$v^\pm = \begin{bmatrix} x \mp \varkappa \\ 2 \end{bmatrix}.$$

We therefore may diagonalise M_α as

$$M_\alpha = -\frac{1}{4\varkappa}\begin{bmatrix} x-\varkappa & x+\varkappa \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}(x+\varkappa) & 0 \\ 0 & \frac{1}{2}(x-\varkappa) \end{bmatrix}\begin{bmatrix} 2 & -x-\varkappa \\ -2 & x-\varkappa \end{bmatrix};$$

in particular, $\Phi^h(\beta_n)$ is the first component of

$$M^n_\alpha \begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix} = -\frac{1}{4\varkappa}\begin{bmatrix} x-\varkappa & x+\varkappa \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2^n}(x+\varkappa)^n & 0 \\ 0 & \frac{1}{2^n}(x-\varkappa)^n \end{bmatrix}\begin{bmatrix} 2 & -x-\varkappa \\ -2 & x-\varkappa \end{bmatrix}\begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix}$$

computing this, we have

$$\Phi^h(\beta_n) = \frac{(\Phi^h(\beta_0)(x+\varkappa) - 2\Phi^h(\beta_1))(x-\varkappa)^n + (\Phi^h(\beta_0)(\varkappa - x) + 2\Phi^h(\beta_1))(x+\varkappa)^n}{2^{1+n}\varkappa}.$$

In particular, we have proved the following quantitative improvement of Theorem 9.2.17:

9.2.18 Theorem. *Let β_0 and β_1 be Farey neighbours, and let $\alpha = \beta_1 \ominus \beta_0$. Then we have a closed form formula for $\Phi^h(\beta_n)$ ($n \in \mathbb{Z}$), namely*

$$\Phi^h_{\beta_n} = \frac{\left(\Phi^h_{\beta_0}\left(\Phi^h_\alpha + \varkappa\right) - 2\Phi^h_{\beta_1}\right)\left(\Phi^h_\alpha - \varkappa\right)^n + \left(\Phi^h_{\beta_0}\left(\varkappa - \Phi^h_\alpha\right) + 2\Phi^h_{\beta_1}\right)\left(\Phi^h_\alpha + \varkappa\right)^n}{2^{1+n}\varkappa}.$$

where $\varkappa = \sqrt{\left(\Phi^h_\alpha\right)^2 - 4}$. ∎

This gives a 'local' closed form solution for the recursion around any $\alpha \in \mathbb{Q}$; a 'global' solution corresponds to a collection of these solutions, each local to a particular geodesic in the graph and which are compatible on intersections. Unfortunately, our original recurrence relied on knowning only three initial values globally in the graph; while this local formula relies on knowing three initial values which are local on the particular geodesic.

9.3 Approximating irrational pleating rays

As mentioned in the introduction to [39], a version of this theory can be used to give approximations to irrational pleating rays. In order to do this, we must deal with the theory of infinite continued fractions which we have danced around several times (c.f. Algorithm 4.1.2, Section 6.2, and Section 6.3).

It is well-known that every irrational $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ has a unique simple continued fraction approximation of the form

$$\lambda = [a_1, a_2, ..., a_n, ...] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \ddots}}}$$

(see, for example, §10.9 of [54]). We now show that this exhibits λ as a limit of a sequence of rationals 'down the Farey tree'.

Recall from Proposition 6.2.1 that a *rational number* $p/q \in \mathbb{Q}$ has exactly two continued fraction decompositions, of the forms $[a_1, ..., a_{N-1}, a_N, 1]$ and $[a_1, ..., a_{N-1}, a_N + 1]$.

$$\frac{r_1}{s_1} = [a_1, ..., a_{N-1}, a_N] \text{ and } \frac{r_2}{s_2} = [a_1, ..., a_{N-1}].$$

9.3.1 Proposition. *With the notation displayed above, r_1/s_1 and r_2/s_2 are Farey neighbours and $p/q = (r_1/s_1) \oplus (r_2/s_2)$.*

Proof. That the Farey sum is as claimed follows from Theorem 149 of [54]: that is, if $p_n/q_n = [a_1, ..., a_n]$ then

$$p_n = a_n p_{n-1} + p_{n-2} \text{ and } q_n = a_n q_{n-1} + q_{n-2}.$$

Indeed, take $p/q = p_{N+1}/q_{N+1} = [a_1, ..., a_{N-1}, a_N, 1]$, then $a_{N+1} = 1$ so $p_{N+1} = 1p_N + p_{N-1}$ and $q_{N+1} = q_N + q_{N-1}$.

That the two are Farey neighbours is exactly Theorem 150 of [54], which actually gives slightly more information:

$$p_N q_{N-1} - p_{N-1} q_N = (-1)^{N-1}. \qquad \blacksquare$$

Earlier in this chapter, we indicated how to compute in closed form the sequence of Farey polynomials corresponding to the Farey fractions

$$\frac{p_1}{q_1}, \frac{p_2}{q_2} = \frac{p_1}{q_1} \oplus \left(\frac{p_2}{q_2} \ominus \frac{p_1}{q_1}\right), \dots, \frac{p_n}{q_n} = \frac{p_1}{q_1} \oplus^{n-1} \left(\frac{p_2}{q_2} \ominus \frac{p_1}{q_1}\right)$$

where p_1/q_1 and p_2/q_2 are Farey neighbours. That is, we gave a way to compute the Farey polynomials down a branch of the Farey tree with constant difference (for instance, we gave the example of $\Phi_{1/q}$, where $\frac{1}{q} = \frac{1}{0} \oplus^q \frac{0}{1}$). The study of partial fraction decompositions here gives, in general, different sequences: the constant addition sequence rooted at an element ξ in the tree is the sequence which constantly chooses the *left* branch when moving down from ξ (with respect to the embedding of Figure 9.4), while the sequence corresponding to continually adding the previous two items in the tree (and therefore building a continued fraction decomposition) corresponds to the sequence which is eventually constantly moving *rightwards*. For this reason, we expect there to also exist a nice way to compute closed-form expressions for sequences of Farey polynomials corresponding to finite convergents of infinite continued fraction decompositions, and therefore for there to be a reasonable way to approximate irrational pleating rays and attempt to compute expressions for the analytic functions of which they are subsets of zero-sets.

Chapter 10

Conjectures and open problems

In this chapter we list some conjectures, open problems, and other miscellaneous questions, which we believe to be of varying levels of difficulty.

10.1 General problems

Our first conjecture asks for the relation between the parabolic and the elliptic Riley slices, as subsets of $\mathbb{C}$.

10.1.1 Conjecture. *(a) $\mathcal{R}^{a,b} \supseteq \mathcal{R}^{\infty,\infty}$ for all a and b after translating so that the centres of symmetry coincide.*

(b) Even stronger, $\mathcal{R}^{\infty,\infty} = \bigcap_{(a,b)\in\mathbb{N}^2} \mathcal{R}^{a,b}$ after translating so that the centres of symmetry coincide.

(c) Stronger but orthogonal to (b), there exists a biholomorphic or quasiconformal map $\mathcal{R}^{a,b} \to \mathcal{R}^{\infty,\infty}$, which is a contraction in $\mathbb{C}$ and extends uniformly continously to the boundary.

(d) As a corollary of (c), from the fact that $\partial\mathcal{R}^{\infty,\infty}$ is a Jordan curve (Theorem 5.2.6) we obtain the same result for $\partial\mathcal{R}^{a,b}$.

We share the belief of experts that the Keen–Series theory may be extended much further (see e.g. the *Remark added in press* of [62, pp. 720–721]): the idea behind the Keen–Series theory extends to other geometrically finite groups and the difficulty is in actually carrying out the concrete work needed to check it. We state the following problem merely to sketch why the theory should extend, and how it would be done. See the final remark of Section 7.1 for some examples of other papers which use this method.

10.1.2 Problem. Fix a geometrically finite Kleinian group G, we study the geometry of $\mathrm{QH}(G)$.

By the theory of boundary parabolics for geometrically finite Kleinian groups [85, 95] (see also [15, Theorem 11]), whenever $a_1, ..., a_n$ is a system of simple disjoint non-boundary-parallel curves on the surface of G we may find a loxodromic group element g_i representing a_i for each i and we may pinch a_i to a puncture by deforming g_i to parabolicity. Observe that $a_1, ..., a_n$ is exactly giving a maximal geodesic lamination on the surface (c.f. [79, p. 166] or [124, Definition 8.7.5 and Proposition 8.7.6]). Fix some such system of geodesics T, and define the T-**rational pleating ray** $\mathrm{pl}(T)$ to be the set of G' in $\mathrm{QH}(G)$ such that the pleating locus of the convex core of G' is exactly T. Then the definition of a T-**circle chain** U_T should be something along the lines of 'the set of F-peripheral subgroups $F < G$

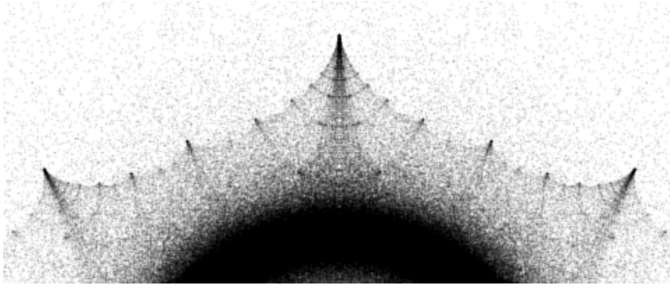


Figure 10.1: A portion of the Riley slice exterior.

such that (1) the quotient of the hyperbolic dome above the peripheral disc of F is homeomorphic to one of the subsets of $\mathcal{S}(G)$ obtained by cutting along the curves of T, (2) F is generated by elliptics or parabolics representing punctures or cone points in that surface, and (3) the curves bounding the surface in $\mathcal{S}(G)$ are represented by appropriate products of these generators.'

Now $\mathrm{pl}(T)$ should laminate $\mathrm{QH}(G)$; and a group lies on $\mathrm{pl}(T)$ iff it contains a system of non-conjugate subgroups in U_T, one for each of the surfaces obtained by cutting $\mathcal{S}(G)$ along T. Taking the completion of this lamination we should obtain a homeomorphism $\Pi : \mathrm{QH}(G) \to \mathcal{ML}(G)x\mathbb{R}^n_{>0}$ defined by the completion of the map sending any G' on a T-rational pleating ray to

$$(\mathrm{pl}(T), L(a_1), ..., L(a_n))$$

where the a_i are the curves of T realised in the complex structure from G' and where L is a suitable normalised complex length function like that discussed above in Section 7.4B. Of course the details require some non-trivial argument in each specific case.

10.2 The structure of the Riley slice exterior

Let us next consider the picture of the Riley slice exterior which was included as Figure 4.8. This plot is the set of points z such that $\Phi_{p/q}(z) = -2$, where $q \leq 80$. By McMullen's theorem [90], a subset of these points are dense in the boundary of the Riley slice. In fact:

10.2.1 Theorem. *Let $z \in \mathbb{C}$ satisfy $\Phi_{p/q}(z) = -2$ for some slope p/q. Then $z \in \mathbb{C} \setminus \mathcal{R}$.*

A proof of Theorem 10.2.1 follows the same ideas as the arguments of Section 3 of our paper [39], where we show that the semigroup under composition generated by the polynomials $Q_{p/q} \coloneqq \Phi_{p/q} - 2$ preserves the set $\mathbb{C} \setminus \overline{\mathcal{R}}$—this is a similar result but allowing boundary points.

We conjecture based on a large number of numerical calculations that Theorem 10.2.1 can be strengthened:

10.2.2 Conjecture. *The set $\{z \in \mathbb{C} : \text{there exists } p/q \in \mathbb{Q} \text{ such that } \Phi_{p/q}(z) = -2\}$ is dense in $\mathcal{R}$.*

In the following, we define the **reduced Farey polynomial of slope** p/q by $\phi_{p/q}(z) \coloneqq \Phi_{p/q}(z) + 2$.

The approximations to the Riley slice exterior have some rather intricate structure. In Figure 10.1 we show a view of such an approximation (this time produced with polynomials of slope denominator ≤ 500), zoomed in around the 1/2-cusp. Define the **extended pleating ray of slope** p/q to be the preimage of $\mathbb{R}$ defined by the same branch of the inverse of the Farey polynomial. We use $\mathcal{EP}_{p/q}$ to denote this curve. One sees immediately that there appears to be clustering around extensions of the pleating rays into the exterior; it is known that all discrete hyperbolic two-bridge knot complement groups lie on these extensions [5, 6] and at least one paper on the patterns visible has appeared [51]— in that paper, the groups we see in the exterior were called **parabolic dust** by Gilman.

Our interest here lies in the following property: the extended pleating rays appear to be paired together across a central neighbour to form a smooth curve. We list some problems which might be interesting to study, related to this observation.

10.2.3 Problem.
- What is the relationship between the two 'paired' rays? (Experimentation shows that the relationship is *not* that they are Farey neighbours.) For' the time being call such a pair a **Riley pair**.
- Suppose p/q and r/s form a Riley pair. Conjecture: There exists an analytic curve $\mathcal{EP}_{p/q \dashv\vdash r/s}$ such that $\mathcal{EP}_{p/q \dashv\vdash r/s} \cap (\mathbb{C} \setminus \mathcal{R}) = \mathcal{P}_{p/q} \cup \mathcal{P}_{r/s}$. (We cannot even guess what operation $p/q \dashv\vdash r/s$ might represent.)
- Conjecture: The curve $\mathcal{EP}_{p/q \dashv\vdash r/s}$ meets the extension of $\mathcal{P}_{(p+r)/2(q+s)}$ at a single point.

It is known that the Riley slice admits a natural hyperbolic metric that agrees with the Teichmüller metric.

10.2.4 Question. Are the pleating rays geodesics in this metric?

This will be true if the mapping Π of the Riley slice to the annulus $\mathcal{ML}(S) \times \mathbb{R}_{>0}$ of Theorem 7.4.15 is sufficiently nice (and the picture due to David Wright found as Figure 1 of [63] seems to back this up: it appears that the map is conformal, at least away from the boundary, though a search of the literature has not revealed any statements either way). In this direction, a broader question is:

10.2.5 Question. How does the coordinate system Π relate to the canonical isometric map $\mathcal{R} \to \{z \in \mathbb{C} : |z| > 2\}$.

More natural questions arise if we consider the analytic interpretation of Teichmüller geodesics, which we briefly recall from [46, §11.4]. Suppose that R is a Riemann surface and $X \in \mathrm{Teich}(R)$; pick a measured foliation $\mathcal{F}$ on X which comes from some holomorphic quadratic differential q on X. For every $k \in (0, \infty)$ there exists some $X_k \in \mathrm{Teich}(R)$, some quadratic differential q_k on X_k, and some **Teichmüller map** $f : X \to X_k$ with initial differential q, terminal differential q_k, and horizontal stretch factor k; this map corresponds to taking the metric on X and locally deforming it, stretching it by a factor of k along the leaves of F and shrinking by a factor of k along the transverses of the leaves. With this setup, the set $X_k : x \in [0, \infty)$ is a geodesic ray in $\mathrm{Teich}(R)$.

Now suppose that we naïvely try to answer Question 10.2.4 in the affirmative. Let S be a 4-times punctured sphere, and pick a non-boundary-parallel simple closed curve $\gamma(p/q)$ on S. There is a natural extension of $\gamma(p/q)$ to a foliation $\mathcal{F}(p/q)$ on S, given by lifting $\gamma(p/q)$ to a line of slope p/q (as in Chapter 6), taking the foliation on $\mathbb{R}^2$ of all parallel lines to this lift, and projecting back down. Now consider the Teichmüller geodesic corresponding to shrinking along $\mathcal{F}(p/q)$ and expanding along transverses to $\mathcal{F}(p/q)$.

10.2.6 Question. Does the resulting foliation $\mathcal{F}(p/q)$ arise as the trajectory structure of a holomorphic quadratic differential?

It is known that there is a holomorphic quadratic differential with trajectory structure *homotopic* to this, at least. Perhaps this is good enough to make the argument go through since we should be able to modify the original curve k by homotopies without changing anything except the ability to extend to a foliation via the process we outlined. In any case:

10.2.7 Question. What is the holomorphic quadratic differential with trajectory structure homotopic to (or equal to) the foliation coming from $\gamma(p/q)$?

Suppose now that $s \in \mathcal{P}_{p/q}$, and that we have managed to find a quadratic differential with trajectory structure 'close enough' to $\gamma(p/q)$ as discussed above. It is intuitive now that deforming along the Teichmüller geodesic is precisely the same process as deforming the surface so as to travel down the pleating ray. (Yes, when we deform down the geodesic we are pinching curves other than $\gamma(p/q)$; but these curves are 'closer to the punctures' so shrink in length less than $\gamma(p/q)$ and so shouldn't pinch to a point in the limit—their pinching corresponds to the fact that you get a cusp neighbourhood on the pair of 3-punctured spheres where you have a series of curves 'parallel' to the puncture which shrink in size as you go towards the puncture).

In any case, we pose the following question which relates these Teichmüller metric problems to the problem of 'paired' extended pleating rays.

10.2.8 Problem. Suppose p/q and r/s form a Riley pair. Define a suitable geometry on $\mathcal{R}$ such that there is a natural involutive isometry of $\mathcal{R}$ swapping $\mathcal{P}_{p/q}$ and $\mathcal{P}_{r/s}$, fixing the conjectured ray $\mathcal{P}_{p/q \dashv\vdash r/s}$ pointwise.

We guess that, if all of the above is correct, a suitable candidate for this geometry is the Teichmüller geometry, and that ρ may be chosen to be the reflection across the conjectured geodesic ray $\mathcal{P}_{p/q \dashv\vdash r/s}$.

10.3 Some computational problems

The Keen–Series definition of the pleating rays in [63] is difficult to compute with: since it depends on behaviour near infinity, it is difficult to apply locally at points in the Riley slice. This complicates computational study of the Keen–Series foliation. David Wright has given computational methods for computing the positions of cusp points in the Maskit and Riley slices [127]. We list some similar problems which might be amenable to study, perhaps using the Farey polynomial formulae from Chapter 9 as a start.

10.3.1 Problem. Given a point $\rho \in \mathcal{R}$:

(a) Determine whether ρ lies on a rational pleating ray.

(b) If ρ is known to lie on some rational pleating ray, determine which pleating ray it lies on.

(c) If $\rho \in \mathcal{P}_{p/q}$, determine the tangent line to $\mathcal{P}_{p/q}$ at ρ.

An answer to (c) of Problem 10.3.1 might lead to a nice solution to the problem of efficiently computing pleating rays. The Keen–Series theory tells us that $\mathcal{P}_{p/q}$ is a connected component of $\widetilde{\mathcal{H}}_{p/q}$ with certain asymptotic properties; these properties are not amenable to computation. (When q is low, we have used `MATLAB` to determine a sequence of points on $\mathcal{P}_{p/q}$ in order to animate the limit set of Γ_ρ as ρ heads to the cusp down the pleating ray;[1] however, without some further numerical analysis, we were unable to produce a useable general algorithm for producing such sequences as q becomes large.) Really, we would like to:

10.3.2 Problem. Determine other characterisations of rational pleating rays, amenable to computation.

(a) A purely algebraic or combinatorial characterisation, like that of [127].

(b) A dynamical characterisation (that is, given a point $\rho \in \mathcal{P}_{p/q}$, determine a (computable) function f such that $f^n(\rho) \in \mathcal{P}_{p/q}$ and such that as $n \to \infty$, the behaviour of f is monotone along the ray).

[1] See `https://aelzenaar.github.io/kg/animations/index.html` for some of these animations.

Knowing the dynamical behaviour of pleating rays even into the exterior of the slice would be very interesting, though of course losing discreteness makes the problem less likely to be tackleable via combinatorics of limit sets. Some limited computational experiments seem to suggest that the family of pleating ray extensions have some interesting dynamical properties

10.3.3 Conjecture. *For each p/q, define a function $f_{p/q} : \mathbb{R} \to \mathbb{C}$ by the rule '$t \mapsto \rho$, where $\Phi_{p/q}(\rho) = t$ and ρ lies in the connected component of $\Phi_{p/q}^{-1}(\mathbb{R})$ containing the p/q pleating ray'. Then $f_{p/q}(t) = f_{r/s}(t)$ if and only if $p/q = r/s$. That is, the extended pleating rays do not collide in the slice exterior.*

Finally, pictures like Figure 10.1 but for the inverse image of 0 rather than -2 seem to invite the study of questions like the following:

10.3.4 Problem. A **clumping point** (for want of a better term) of the set of Farey polynomials is a point $z \in \mathbb{C}$ such that $\Phi_{r/s}(z) = 0$ for infinitely many $r/s \in \mathbb{C} \cap [0,1]$. Let $\mathcal{C}$ be the set of all such points.

(a) $\mathcal{C} \subseteq \mathbb{R}$

(b) $z \in \mathcal{C}$ iff z is 'algebraic of surprising low degree', i.e. if there exists some r/s such that $\Phi_{r/s}(z) = 0$ but the degree of z over $\mathbb{Z}$ is strictly less than s.

(c) $\pm 2, \pm\sqrt{2} \in \mathcal{C}$

(d) The clumping points are discrete on $\mathbb{R}$; an element $\xi \in \mathbb{R}$ is an accumulation point of the space $\bigcup_{r/s \in \mathbb{Q}\cap[0,1]} \Phi_{r/s}^{-1}(-2)$ iff $\xi \in \mathcal{C}$. (Observe that this cannot be true simultaneously with Conjecture 10.2.2 above. Perhaps a better conjecture is that $\mathcal{R}$ is dense 'away from $\mathbb{R}$' in some sense.)

10.4 Factorisation properties of the Farey polynomials

We list in this section some properties of the Farey polynomials which we have determined by experiment. We hope that Chapter 9 is a first step towards the solution of the following central problem:

10.4.1 Problem. Give a closed-form formula in r, s, a, b for the coefficients of $\Phi_{r/s}^{a,b}$, or just for the polynomial itself.

Experimentation (Table 10.1) seems to show the following:

10.4.2 Conjecture. *The coefficient $\left[z^d\right] \Phi_{r/s}^{a,b}$ is piecewise polynomial in d depending on the equivalence class of s mod $2r$.*

The factorisation properties of the polynomials would also be interesting to understand. For instance, we believe that the following is true and might be fairly easy to check:

10.4.3 Conjecture. *For $\Phi_{r/s}^{\infty,\infty}$ to be reducible over $\mathbb{Z}$ it is necessary (not sufficient) that $s = \pm 1 \pmod 8$ and r is even.*

10.4.4 Problem. Determine necessary and sufficient conditions for $\Phi_{r/s}^{a,b}$ to factor over $\mathbb{Z}$.

If we plot the zero set of $\Phi_{r/s}^{\infty,\infty}(z)$ as r/s varies over $\mathbb{Q}$ from 0 to 1, it seems that the roots have much more structure than one would naïvely expect a family of polynomials of different degree to have: there appears to be a continuous flow of some sort.[2] We therefore believe that the Farey polynomials

[2]See the animation at `https://aelzenaar.github.io/kg/animations/farey_roots_ordered.mp4`

Table 10.1: Coefficients of the r/s Farey polynomial.

$$[z^0]\,\Phi_{r/s} = 2 \text{ for all } r,s$$

$$[z^1]\,\Phi_{1/s} = \begin{cases} 1 & s \equiv 1 \pmod 2 \\ 0 & s \equiv 0 \pmod 2 \end{cases}$$

$$[z^1]\,\Phi_{2/s} = -1 \text{ for all } s$$

$$[z^1]\,\Phi_{3/s} = \begin{cases} 1 & s \equiv \pm 1 \pmod 6 \\ 0 & s \equiv \pm 2 \pmod 6 \end{cases}$$

$$[z^1]\,\Phi_{4/s} = -1 \text{ for all } s$$

$$[z^1]\,\Phi_{5/s} = \begin{cases} 1 & s \equiv \pm 1 \pmod{10} \\ 1 & s \equiv \pm 3 \pmod{10} \\ 0 & s \equiv \pm 2 \pmod{10} \\ 0 & s \equiv \pm 4 \pmod{10} \end{cases}$$

$$[z^1]\,\Phi_{6/s} = -1 \text{ for all } s$$

$$[z^2]\,\Phi_{1/s} = \begin{cases} \frac{s^2}{4} & s \equiv 0 \pmod 2 \\ -\frac{(s+1)(s-1)}{4} & s \equiv 1 \pmod 2 \end{cases}$$

$$[z^2]\,\Phi_{2/s} = \begin{cases} \frac{s-1}{2} & s \equiv 1 \pmod 4 \\ -\frac{s+1}{2} & s \equiv -1 \pmod 4 \end{cases}$$

$$[z^2]\,\Phi_{3/s} = \begin{cases} \frac{(s-8)^2}{36} & s \equiv 2 \pmod 6 \\ \frac{(s+8)^2}{36} & s \equiv -2 \pmod 6 \\ \frac{(s+1)(s+17)}{36} & s \equiv 1 \pmod 6 \\ -\frac{(s+1)(s-19)}{36} & s \equiv -1 \pmod 6 \end{cases}$$

$$[z^2]\,\Phi_{4/s} = \begin{cases} -\frac{s+1}{2} & s \equiv -1 \pmod 8 \\ \frac{s-1}{2} & s \equiv 1 \pmod 8 \\ -\frac{s+7}{2} & s \equiv -3 \pmod 8 \\ \frac{s-7}{2} & s \equiv 3 \pmod 8 \end{cases}$$

$$[z^2]\,\Phi_{5/s} = \begin{cases} \frac{(s+24)^2}{100} & s \equiv -4 \pmod{10} \\ & s \equiv 4 \pmod{10} \\ & s \equiv -1 \pmod{10} \\ & s \equiv 1 \pmod{10} \\ & s \equiv -2 \pmod{10} \\ & s \equiv 2 \pmod{10} \\ & s \equiv -3 \pmod{10} \\ & s \equiv 3 \pmod{10} \end{cases}$$

$$[z^3]\,\Phi_{1/s} = \begin{cases} -\frac{s^2(s^2-4)}{48} & s \equiv 1 \pmod 2 \\ \frac{(s^2-1)(s^2-3)}{48} & s \equiv 0 \pmod 2 \end{cases}$$

$$[z^3]\,\Phi_{2/s} = \begin{cases} -\frac{(s-1)(s^2-8s+3)}{48} & s \equiv 1 \pmod 4 \\ \frac{(s+1)(s^2-4s-9)}{48} & s \equiv -1 \pmod 4 \end{cases}$$

$$[z^4]\,\Phi_{1/s} = \begin{cases} \frac{(s^2-1)(s^2-9)(2s^2-5)}{2880} & s \equiv 1 \pmod 2 \\ \frac{s^2(s^2-4)(2s^2-17)}{2880} & s \equiv 0 \pmod 2 \end{cases}$$

belong to some kind of nice algebraic family (and some experimental work with the coefficients of $\Phi^{a,b}_{r/s}$ viewed as polynomials in $\alpha = \exp(\pi i/a), \beta = \exp(\pi i/b)$ serves only to invite further speculation)

We are actually interested in the real locus $\Phi^{-1}_{p/q}(\mathbb{R})$. This locus is precisely the zero set of the polynomial $\phi_{p/q}(x, y) = \Im\Phi_{p/q}(x + yi)$ in the two real indeterminates x, y. In the parabolic case, $\phi_{0/1}(x, y) = -y$, $\phi_{1/1}(x, y) = y$, and $\phi_{1/2}(x, y) = 2xy$. Note that y divides each of these. We can apply $\Im$ to both sides of our recurrence relation to get $\phi_{p/q}\phi_{r/s} + \phi_{p/q\oplus r/s} + \phi_{p/q\ominus r/s} = 0$. Thus if some polynomial divides $\phi_{p/q}, \phi_{r/s}, \phi_{p/q\ominus r/s}$ then by induction it divides everything below that triple in the Farey graph. In particular, y divides every polynomial $\phi_{p/q}$ ($p/q \in [0, 1] \cap \mathbb{Q}$). Experimentation shows that these polynomials actually seem to factor further in many cases. In fact, it looks like the component corresponding to the pleating ray is always of even degree.

10.4.5 Problem. Determine the real algebraic geometry of the real variety $\Im\Phi_{p/q}(x + yi)$, and the subset which forms the pleating way.

Appendix A

Some remarks on computation

The author has developed, in the course of writing this thesis, some useful scripts in the computer programming languages `Mathematica` and `Python` which can compute Farey polynomials (implementing the recursive algorithm of Chapter 9) and draw nice limit sets. Many of these scripts may be found on `GitHub` by following the citation [38].

In this appendix we give some simple examples of the usage. (For a list of computer prerequisites, see the file `README.md` on the `GitHub` page.)

A.1 Farey words and polynomials

A.1.1 Example. In Section 1.1, we claimed that the figure 8 knot lies on the (extension of the) 3/5-pleating ray. Recall from Equation (4.1.4) that the figure 8 knot group is $\Gamma_{-\omega}^{\infty;\infty}$, where $\omega = \exp(2\pi i/3)$. We therefore need to check that $\Phi_{3/5}(-\omega)$ is real.

We use the following short script:

```
import farey
import mpmath as mp

# Compute the matrix of the 3/5 Farey word
# with alpha = beta = 1 and rho = -\omega
omega = mp.exp(2j*mp.pi/3)
M = farey.matrix(3,5,-omega,1,1).tolist()

# Print the imaginary part of the trace
print((M[0][0] + M[1][1]).imag)
```

Indeed, the output is `mpf('0.0')`. Of course, this is not a proof: one needs to check that it lies on the right branch of the inverse, as well. In fact, if one replaces 3/5 with 2/5 then the Farey polynomial is still real at $-\omega$—we wonder if this is just an annoying coincidence, or if it is an indication of some deeper phenomenon about the intersections of real locii (Figure A.1).

A.1.2 Example. In this example, we compute the LaTeX code for Table 6.1, entirely automatically.

```
import farey
from math import gcd

```

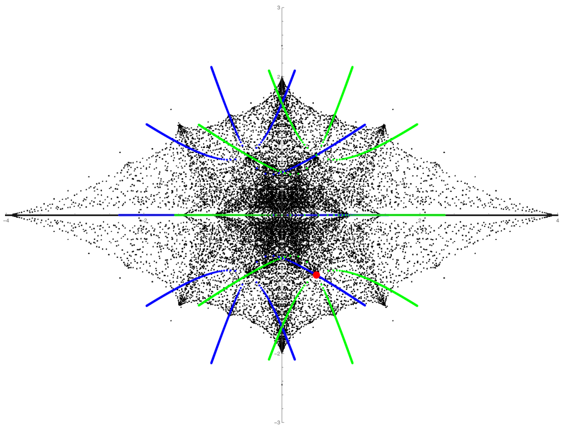


Figure A.1: The figure 8 knot (red) lies on the real locus of both the 2/5 (green) and 3/5 (blue) Farey polynomials.

```
# LaTeX preamble
print(r'''\begin{table}
  \centering
  \caption{Farey words $ \Word(p/q) $ for small $ q $.
           See \cref{ex:py_words}.\label{tab:words}}
  \begin{tabular}{r|l}
    $p/q$ &$\Word(p/q)$\\\hline''')

# Do the 0/1 and 1/1 cases separately.
print(f"    $0/1$ & ${''.join(farey.word(0,1))}$\\\\")
print(f"    $1/1$ & ${''.join(farey.word(1,1))}$\\\\")

# Now do the rest.
highest_q = 12
for q in range(2,highest_q+1):
  for p in range(1,q):
    if gcd(p,q) == 1:
      print(f"    ${p}/{q}$ & ${str().join(farey.word(p,q))}$\\\\")

# End the LaTeX code.
print(r'''  \end{tabular}
\end{table}''')
```

A.1.3 Example. The following `Mathematica` script was used to produce Figure 4.10; it also provides an example of the generation of Farey words via cutting sequences.

```
ClearAll["Global`*"];
lookupTable = {{X, x}, {y, Y}};
word[px_, qx_] := (
  p = px/GCD[px, qx]; q = qx/GCD[px, qx];
  length = 2 q;
  wd = {};
```

```
  Do[(
    height = i*p/q;
    height = If[Ceiling[height] == height, height + 1/2, height];
    AppendTo[wd,
     lookupTable[[If[Mod[i, 2] == 1, 1, 2],
      If[Mod[Ceiling[height], 2] == 1, 1, 2]]]];
    ), {i, length}];
  wd);

wordTable =
  Table[Table[{{p, q}, word[p, q]}, {p,
    Select[Range[q], GCD[#, q] == 1 &]}], {q, 20}];

wordList = Flatten[wordTable, 1];

indexWord[w_] :=
  Map[First, Select[wordList, Function[l, l[[2]] == w]]];

matrixProducts =
  Map[Function[w, Apply[Dot, w]], Map[Last, wordList]];

Map[Function[q, (
   YY = {{Exp[ \[Pi] I/q], 0}, {\[Rho], Exp[- \[Pi] I/q]}};
   yy = Inverse[YY];
   Map[Function[p, (
      XX = {{Exp[ \[Pi] I/p], 1}, {0, Exp[- \[Pi] I/p]}};
      xx = Inverse[XX];
      matrixProductsSubstituted =
       matrixProducts /. {X -> XX, Y -> YY, x -> xx, y -> yy};
      traces = Map[ Tr[N[#]] &, matrixProductsSubstituted];
      zeros = Map[Function[t, NSolve[t + 2 == 0, \[Rho]]], traces];
      plot =
       ComplexListPlot[\[Rho] /. # & /@ Flatten[zeros],
        PlotRange -> {{-4, 4}, {-2, 2}},
        PlotMarkers -> {\[Bullet], 2},
        PlotStyle -> Opacity[0.8, Black], PlotLabel -> p,
        ImageSize -> 600];
      Print[ToString@StringForm["new_riley_``_``.png", p, q]];
      Export[ToString@StringForm["new_riley_``_``.png", p, q],
       plot];
      )], {2, 3, 4, 5, 6, Infinity}])], {2, 3, 4, Infinity}]
```

A.2 Limit sets

Two algorithms are included to compute limit sets, in the `kleinian` package. One is a depth-first search, the other is a Markov search; see [94] for some pseudocode giving a simple implementation of these algorithms. (The version here is slightly more complicated: it is optimised for use on a computer with multiple CPUs, and has some other features like colouring.)

A.2.1 Example. In this example, we plot the limit set of Figure 3.1d with some nice colours.

```
import mpmath as mp
import kleinian
import matplotlib.pyplot as plt

# Filename to save the final picture to
filename = 'riley2.png'

# A list of generators for the Kleinian group
generators = [mp.matrix([[1,1],[0,1]]),mp.matrix([[1,0],[1+2j,1]])]

# Compute the limit set, with the Markov algorithm, using words
# of maximum length 8; compute 100000 words altogether.
ls = kleinian.limit_set_markov(generators,mp.matrix([0]),8,100000)

# Crop to just -2 < x < 2
ls = [t for t in ls if t[0].real > -2 and t[0].real < 2]

# Plot the limit points, with colours according to the generators
# starting each word. Each point returned by limit_set_markov is
# actually a pair (A,B) where A is the actual point in $\C$ and
# B is a small integer giving the index of the first letter of the
# word giving that point in the generator matrix (or the negative
# of the index, if the word starts with an inverse matrix),
colours = {-2: 'r', -1:'b', 1:'g', 2:'y'}
plt.scatter([t[0].real for t in ls],
            [t[0].imag for t in ls],
            c=[colours[t[1]] for t in ls],
            marker=".", s=0.1, linewidths=0)

plt.axis('equal')
plt.tight_layout()

plt.savefig(filename,dpi=500)
plt.show()
```

The naïve code of Example A.2.1 uses a lot of memory when the parameters are turned up high in order to get a good picture: the parameters there are about the limit which a small laptop can handle. We next give an example to show how this may be overcome.

A.2.2 Example. We use various high-powered `Python` data science libraries—for example, `Dask` and `Datashader`—to 'batch' the limit set computations and avoid loading everything into memory at once, even when we need to draw it. An introduction to the computer programming tools required is the book by Wes McKinney [89]. This script will produce pictures like those in Figure 7.4.

```
import mpmath as mp
import kleinian
import riley
import farey
import datashader as ds
```

```
import datashader.transfer_functions as tf
import pandas
import dask
import dask.dataframe as dd
from dask.delayed import delayed
from datashader.utils import export_image

import matplotlib.pyplot as plt
from datashader.mpl_ext import dsshow, alpha_colormap

# Orders of elliptics
p = 5
q = mp.inf

per_batch = 1000
batches = 200
depth = 30

mu = 1.5j

alpha = mp.exp(1j*mp.pi/p)
beta = mp.exp(1j*mp.pi/q)
X = mp.matrix([[1,1],[0,1]])
Y = mp.matrix([[1,0],[mu,1]])

seeds = [0]

print("Found fixed points.",flush=True)

# We are already running in parallel in each batch; if we don't do this then
# Dask launches many copies of one_batch() and we get killed by the OOM killer.
dask.config.set(scheduler='single-threaded')

def one_batch(batch):
  print(f"Running batch {batch+1}/{batches}")
  ls = kleinian.limit_set_markov([X,Y],seeds,depth,per_batch)
  df = pandas.DataFrame(data=[(float(mp.re(point[0])),
                             float(mp.im(point[0])),
                             point[1]) for point in ls],
                     columns=['x','y','colour'], copy=False)
  df['colour']=df['colour'].astype("category")
  return df

dfs = [delayed(one_batch)(batch) for batch in range(batches)]
df = dd.from_delayed(dfs)
cvs = ds.Canvas(plot_width=4000, plot_height=4000,
              x_range=(-4,4), y_range=(-3,3),
              x_axis_type='linear', y_axis_type='linear')

```

```
aggc = cvs.points(df,'x','y',ds.by('colour', ds.count()))
colours = {-2: 'red', -1:'blue', 1:'green', 2:'purple'}
img = tf.shade(aggc)

export_image(img, filename, background="white", export_path=".")
```

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Index

(X, G)-atlas, 12
(X, G)-manifold, 12
(X, G)-maps, 12
(X, G)-equivalent, 12
(X, G)-orbifold, 12
$L^{a,b}_{p/q}(\rho)$, *see* normalised complex length
$PL(\rho)$, *see* pleating length
T-circle chain, 133
T-rational pleating ray, 133
$\mathrm{Cusp}^{a,b}_{p/q}$, *see* cusp set
$\mathrm{Heck}(p/q, n)$, *see* Heckoid group
$\mathcal{ML}(S)$, *see* measured lamination
$\mathrm{Mod}(S)$, *see* mapping class group
$\Omega(G)$, *see* regular set
$\Phi_{p/q}(\rho)$, *see* Farey polynomial
$\Phi^{a,b}_{p/q}$, *see* Farey polynomial
$\mathrm{QH}(\Gamma)$, *see* quasiconformal deformation space
$\mathrm{QHom}(\Gamma)$, *see* quasiconformal representation space
$\mathrm{Teich}(S)$, *see* Teichmüller space
$\mathrm{Teich}(\Gamma)$, *see* Teichmüller space
$\mathrm{Word}(p/q)$, *see* Farey word
$\mathrm{Word}(p/q)(\rho)$, *see* Farey word
$\beta(p/q)$, *see* rational arc
λ-lemma
  equivariant, 39
  extended, 39
$\mathcal{D}$, *see* Farey triangulation
$\mathcal{EP}(p/q)$, *see* extended rational pleating ray
$\mathcal{M}(S)$, *see* Riemann moduli space
$\mathcal{P}(p/q)$, *see* rational pleating ray
$\mathcal{P}^{a,b}_{p/q}$, *see* pleating ray
$\mathcal{R}$, *see* Riley slice
$\mathcal{R}^{a,b}$, *see* Riley slice
$\mathcal{C}(G)$, *see* convex core
$\mathcal{K}(G)$, *see* Kleinian orbifold of G
$\mathcal{M}(G)$, *see* hyperbolic orbifold of G
$\mathcal{S}(G)$, *see* Riemann surface of G
π_ρ, *see* pleating measure
$\mathrm{pl}(\rho)$, 41, 85
σ_ω, *see* Dehn half-twist
$\mathrm{sing}\, O$, *see* singular locus
$\mathrm{spine}(\sigma)$, *see* spine
τ_ω, *see* Dehn twist
$\mathrm{trg}(\sigma)$, *see* ideal triangulation
$\mathrm{trlen}\, f$, *see* complex translation length
$\widetilde{\mathcal{H}}^{a,b}_{p/q}$, *see* hyperbolic locus
$^\circ\Omega(G)$, *see* free regular set
2-bridge link, *see* two-bridge link

accidental parabolic, 39
admissable loop, 49
Ahlfors measure zero theorem, 58
Ahlfors' finiteness theorem, 24
Alhfors-Bers Riemann mapping theorem, 35

Beltrami coefficient, 35
Beltrami equation, 35
bending line, 29
bending measure, 29
boundary hyperbolic element, 26
boundary sequence, 126
bounds a puncture, 24
braid region, 43

canonical Fricke polygon, 26
canonical peripheral quasidisc, 104, 105
Chebyshev polynomials, 123
circle chain, 82
clumping point, 137
complex translation length, 64
compressible surface, 17, 18
cone points, 24
conformal, 24
continued fraction decomposition, 45, 70, 130
convex core, 29

convex set, 13
cusp group, 47, 49, 58
  density of, 58
cusp manifold, 48
cusp orbifold, 51
cusp point
  of a polyhedron, 16
cusp set, 86
cycle (of a facet-pairing structure), 14
cycle relation, 15
cyclic ridge cycle (of a facet-pairing structure), 15
cylinder, 26

deck transformation, 19
Dehn half-twist, 36
Dehn twist, 3, 36
Dehn's lemma, 17
density conjecture, 58
determinant (of a Farey recursive function), 126
diagonal slice of Schottky space, 77
dihedral angle sum, 15
dihedral ridge cycle (of a facet-pairing structure), 14
discontinuous, 19
disjoint isometric discs, 109
doubly cusped region, 28

Earle embedding, 80
edge (of a polyhedron), 14
elliptic transformations, 11
ending lamination conjecture, 58, 133
Euler characteristic (orbifold), 13
extended rational pleating ray, 49
eye problem, 50

F-peripheral
  disc, 80, 103
  group, 76, 80
  quasidiscs, 105
face, 14
facet (of a polyhedron), 14
facet-pairing structure, 14
facet-pairing transformation, 14
  subproper, 15
Farey addition, 72
Farey neighbours, 72, 117
Farey polynomial, 49, 66
  homogeneous, 124
Farey recursive function, 123
  homogeneous, 123
  non-homogeneous, 123
  of anti-determinant d, 126
  of determinant d, 126
Farey triangulation, 71
Farey word, 5, 49, 66
Fatou set, 5
figure 8 knot, 1, 45
flat piece (of a pleated surface), 29
fractional linear transformations, 10
free regular set, 20
freely discontinuous, 19
Fuchsian group, 25
fundamental group of a link, 43
fundamental polyhedron, 16

Gauss-Bonnet theorem (orbifold), 13
geodesic lamination, 29
geodesic line, 11
geodesic segment, 11
geometric manifold modelled on X, 12
geometric space, 12
geometrically finite, 16

Heckoid group, 47, 48
Heckoid orbifold, 48
holomorphic motion, 39
holonomy, 64
hyperbolic halfspace, 13
hyperbolic hyperplane, 13
hyperbolic locus, 86
hyperbolic orbifold of G, 22
hyperbolic polyhedron, 13
hyperbolic space
  ball model, 9
  closure, 9
  half-space model, 9
hyperbolic structure, 36
hyperbolic transformation, 11

ideal face (of a polyhedron), 14
ideal polygonal decomposition, 72
ideal triangulation, 72
incompressible surface, 17, 18
intersection number, 30
intervals of discontinuity, 25

Jørgensen's inequality, 21

Kleinian group, 4, 20
  elementary, 21

of finite type, 24
Kleinian orbifold of G, 22
knot, 43

lamination length, 30
leaf (of a lamination), 29
length (of continued fraction decomposition), 70
limit set, 5, 21
link, 43
link complement manifold, 43
link of an ideal vertex, 16
link space of a cusp point, 16
loop theorem, 16
lower tunnel, 48
loxodromic transformations, 11

mapping class group, 36
marked points (on a Riemann surface), 24
marking (on a Riemann surface), 24
maximal (element in a group), 26
maximal dilatation, 35
measurable Riemann mapping theorem, 35
measured lamination, 29
mediant operation, 72

Nielsen region, 25
normal (family of functions), 96
normal form
see Schubert normal form, 44
normalised complex length, 95

orbifold
see (X, G)-orbifold, 12
orbifold atlas, 12
orbifold fundamental group, 20
ordinary set, 5

paired (by a facet-pairing), *see* tiled adjacently (by a facet-pairing)
parabolic dust, 135
parabolic transformations, 11
pleated surface, 29
pleating length, 96
pleating locus, 3, 29
pleating measure, 96
pleating ray, 92
irrational, 92
rational, 3, 85, 92
Poincaré extension, 10
Poincaré polyhedron theorem, 15

precisely invariant, 26

quasiconformal conjugate, 37
quasiconformal deformation space, 38
quasiconformal mapping, 35
quasiconformal representation space, 37

rational arc, 67
rational link, 43
rational pleating ray, 49
rational pleating rays, 5, 77
regular, 19
regular set, 19
related (by a facet-pairing), *see* tiled (by a facet-pairing)
relative interior, 14
ridge (of a polyhedron), 14
ridge cycle (of a facet-pairing structure), 14
Riemann moduli space, 36
Riemann surface
analytically finite, 24
of G, 22
possibly disconnected, 22
Riley manifold, 49
Riley orbifold, 51
Riley pair, 135
Riley slice, 3, 49
coordinate system theorem, 100
elliptic, 52
parabolic, 49
parabolic literature, 50
roof, 29

Schottky group
classical, 30
generalised, 32
Schubert normal form, 3, 44
Shimizu-Leutbecher lemma, 21
side (of a polyhedron), 14
side-pairing relation, 15
Siegel area formula, 13
similarity, 12
simultaneous uniformisation theorem, 25
singular locus, 12
singular point of an orbifold, 12
slope
of a two-bridge knot, 45
sphere at infinity, 9
spine, 73
Stern-Brocot tree, 122

strictly loxodromic transformation, 11
strongly F-peripheral, 80
supporting hyperplane, 14

tameness conjecture, 58
tangle, 43
Teichmüller map, 135
Teichmüller space
    of Kleinian groups, 37
    of Riemann surfaces, 36
Thurston compactification, 40
tiled (by a facet-pairing), 14
tiled adjacently (by a facet-pairing), 14
transverse measure, 29
trefoil knot, 46
tubular neighbourhood, 43
two-bridge link, 43

uniformisation of Riemann surfaces, 25
unknotting tunnel, 48
upper tunnel, 48

vertex (of a polyhedron), 14