Combinatorial structures in trace polynomials of function groups

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The discreteness problem

Let G be an abstract group. Let X be a Lie group. Let $\rho : G \to X$ be a representation. When is ρ discrete? What is the moduli space of all discrete ρ?

(Think of something like $X = \mathrm{SL}(2, \mathbb{C})$.)

Connections

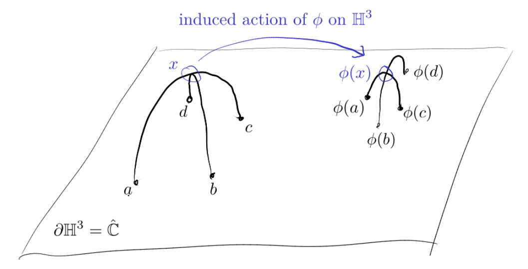
- Discrete groups often come with number-theoretic or algebraic-geometric properties (see e.g. Margulis' superrigidity theorem).
- Discrete groups are models for classically interesting objects like tilings in geometric spaces.

Elements of $\mathrm{PSL}(2, \mathbb{C})$ act on the Riemann sphere $\hat{\mathbb{C}}$ as fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az + b}{cz + d}.$$

They also act on the upper halfspace $\mathbb{H}^3 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$ as hyperbolic isometries.

A discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is called a **Kleinian group**. Its action can have complicated dynamics on the Riemann sphere, originally studied by Poincaré, Fricke, and Klein in the 1880s and 90s.

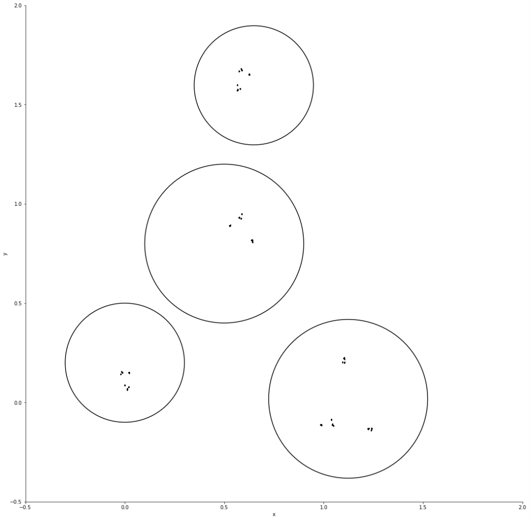


Schottky groups

Pick g pairs of round discs in the plane and for each pair pick a Möbius transformation that sends the interior of one disc to the exterior of the other.

If all $2g$ discs are disjoint:

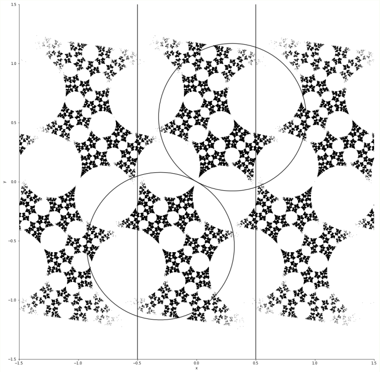
- By a ping-pong argument, the group G is free and discrete.
- The common exterior of the discs is mapped around by G to tile an open set $\Omega(G) \subset \hat{\mathbb{C}}$.
- $\Omega(G)/G$ is a compact genus g surface bounding a handlebody $\mathbb{H}^3/G$.



Let $z \in \mathbb{C}$ and construct the group

$$\Gamma = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \right\rangle.$$

- There is a neighbourhood $U \ni \infty$ such that $z \in U$ implies Γ is discrete and free.
- For $z \in U$, $\Omega(\Gamma)/\Gamma$ is a four-punctured sphere bounding a manifold $M = D \times (-1, 1)$ where D is a twice-punctured disc.
- Visual inspection suggests that the example→ is discrete: how do you give a formal certificate of this fact?



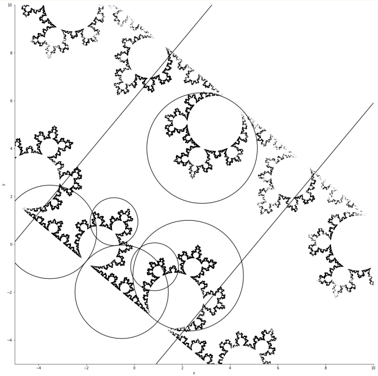
$z = -0.78 + 1.4i$

Maximally cusped compression bodies

Let $\mathcal{F}$ parameterise holonomy groups of $S_2 \times (0, 1)$ so that the one end is S_2 and the other end is a pair of 3-punctured spheres. Generators are

$$P = \begin{bmatrix} -1 & \alpha \\ 0 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} \rho & \beta \\ \sigma & (1 + \beta\sigma)/\rho \end{bmatrix}$$
$$M = \begin{bmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{bmatrix} \quad N = \begin{bmatrix} r & b \\ s & (1 + bs)/r \end{bmatrix};$$

using relations in the surface group we can write all the parameters in terms of algebraic functions of α, β, and σ.



$\alpha = 6 - 5i$, $\beta = 6 - i$, and $\sigma = -0.24 + 0.36i$.
Is this group in $\mathcal{F}$?

Theorem (E.–Schillewaert–Martin, 2021 (after Keen–Series, 1994))

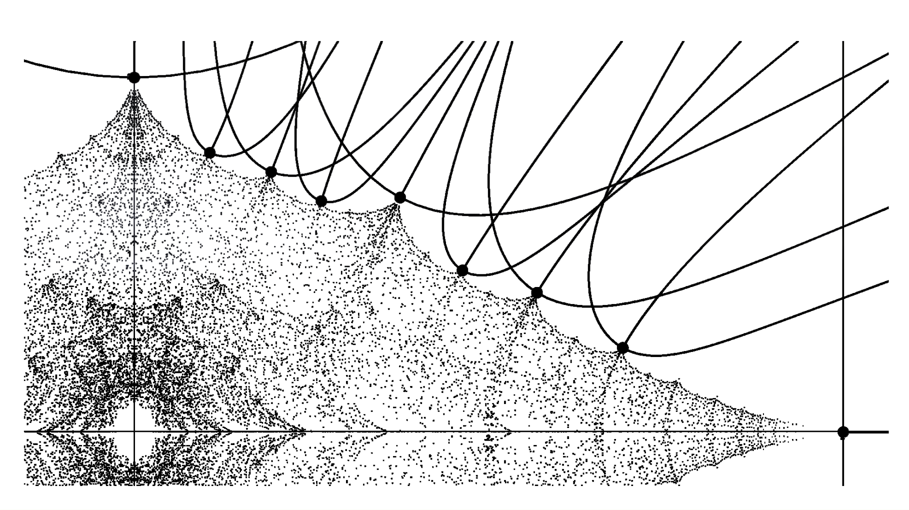
Let $z \in \mathbb{C}$ and define

$$\Gamma(z) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \right\rangle.$$

Let $\mathcal{H}$ be the half-plane $\operatorname{Re} z < -2$.
There is an family of polynomials $\Phi_\xi \in \mathbb{Z}[z]$, indexed by $\xi \in \mathbb{Q}$, such that

- *for every z such that $\Gamma(z)$ is discrete and free, there is some ξ such that $\Phi_\xi(z) \in \mathcal{H}$.*
- *if $z \in \mathbb{C}$ lies in a specific branch of $\Phi_{p/q}^{-1}(\mathcal{H})$ then $\Gamma(z)$ is discrete and free.*

This gives a semi-algorithm to determine discrete-and-freeness for $\Gamma(z)$.
Subsequent work of E.–M.–S. and Chesebro–M.–S., following Akiyoshi, Sakuma, Yamashita, and others, gives a semi-algorithm for the discreteness problem for arbitrary subgroups of $PSL(2, \mathbb{C})$ with two elliptic or parabolic generators.



One application is upcoming work of E.–Martin–Schillewaert:

Theorem (Announced June 2024, details to appear)

A subgroup of $\mathrm{PSL}(2,\mathbb{C})$ *is called* ***thin*** *if it is an infinite index subgroup of an arithmetic Kleinian group. It is called* ***Heckoid*** *if it is generated by two finite order elements (or parabolics) but does not split as a free product.*

There are 147 thin Heckoid groups up to inner automorphism in $\mathrm{PSL}(2,\mathbb{C})$.

Essentially depends on (i) arithmetic work to bound the degree of invariant trace fields, and (ii) application of discreteness semi-algorithms to check whether the (finitely remaining) possible groups are discrete groups with correct isomorphism type.

Definition

A **function group** is a pair (G, Δ) where G is a Kleinian group and Δ is a G-invariant topological component of $\Omega(G)$.

Theorem (E., upcoming preprint in early 2025)

Let G be a geometrically finite function group, let $X(G)$ be the parameter space in $\mathbb{C}^n$ containing the quasiconformal deformation space $\mathrm{QH}(G)$.

There is a family of semi-algebraic sets in $X(G)$, indexed by a simplicial complex $\mathcal{K}(S)$ and with defining polynomials computable in terms of walks in this complex, such that the union of these sets exhausts $\mathrm{QH}(G)$.

Proof.

1. There is a dense lamination of $\mathrm{QH}(G)$ where each leaf is the set of groups with a certain pleating locus on the convex core boundary. These laminations are indexed by the complex of curves $\mathcal{K}(S)$ and the words in G corresponding to each lamination can be found by a combinatorial recursion.
2. These leaves correspond to existence of sets of peripheral subgroups in the ambient group satisfying realness conditions on their generators (i.e. $\operatorname{tr} \in \mathbb{R} \setminus [-2, +2]$) and certain asymptotic conditions (these are the obstructions to computational effectiveness).
3. By arguments in quasiconformal analysis, these realness conditions can be weakened to inequalities, $\operatorname{tr} \in \mathcal{CS}$, where $\mathcal{CS}$ is the set of triples $(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$ such that A and B generate a classical Schottky group pairing their isometric circles (it’s easy to give such inequalities in terms of the traces).

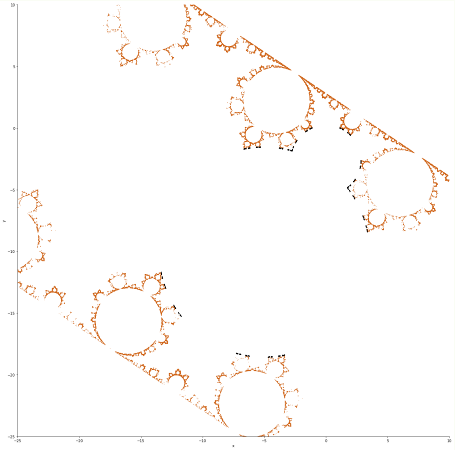
- Due to results of Anderson–Canary (1996) and McMullen (1998), see paper of Ohshika cited in endnotes, it is known that neighbourhoods of the boundary of a quasiconformal deformation space are not locally connected.
- Unlike in the one-dimensional case, the semi-algebraic varieties obtained via these methods may not be smooth or connected, but they give local neighbourhoods of cusps that are slightly better than the 'ambient' neighbourhoods in QH(G) or AH(G).
- Another advantage is that well-understood methods are available to deal with semi-algebraic sets computationally (e.g. quantifier elimination algorithms).

We can explicitly construct a group in $\mathcal{F}$ (one end genus two, the other end a pair of 3-punctured spheres) with specific peripheral structure.
Choose as peripheral groups

$$\langle M, pNMnP\rangle \quad \text{and} \quad \langle mQ, pmQP\rangle.$$

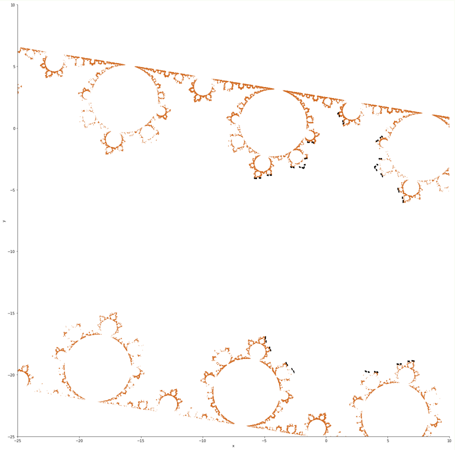
Specify the traces $\operatorname{tr} M = 3$, $\operatorname{tr} mQ = -3$, $\operatorname{tr} m(pNMnP) = -3$. Then:

$$\alpha = 10 - 3i\sqrt{5}, \quad \beta = \frac{5}{29}(171 + 2i\sqrt{5}),$$
$$\sigma = \frac{4}{29}(3 - 2i\sqrt{5}).$$



Now suppose we make a slight deformation of the parameters in $\mathbb{C}^3$ from G to a new group $\tilde{G}$:

	$\operatorname{tr} M$	$\operatorname{tr} mQ$	$\operatorname{tr} m(pNMnP)$
G	3	-3	-3
$\tilde{G}$	$2.5+2i$	$-3-0.5i$	$-3-0.5i$



Peripheral structures are quasi-Fuchsian but still discrete with correct quotient, so this new group is still in $\mathcal{F}$: it's discrete with known isomorphism type.

Bedtime reading

- A.J.E., G.M., and J.S., *Approximations of the Riley slice*. Expo. Math., 2023 (arXiv 2021).
- —, G.M., and J.S., *The combinatorics of the Farey words and their traces*. Groups, Geometry, and Dynamics, online 11/2024 (arXiv 2022).
- —, G.M., and J.S., *On thin Heckoid and generalised triangle groups in* $\mathrm{PSL}(2,\mathbb{C})$ (arXiv 2024).
- Albert Marden, *Hyperbolic manifolds*. Cambridge U. Press, 2016.
- Ken'ichi Ohshika, *Divergence, exotic convergence and self-bumping in quasi-Fuchsian spaces*. Ann. Fac. Sci. Toulouse Math., 2020 (arXiv 2010).
- Title picture: Power lines at Hintere Gartenkuppe (near Saalfeld), own work, 2023.