All arithmetic 2-bridge link groups

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Josef Sudek, *Evening Walk* (1956)

Thou shalt always begin with an easy example

Definition

A **Lie group** is a smooth manifold G equipped with smooth maps $\mu : G \times G \to G$, $e : \{1\} \to G$, and $\iota : G \to G$ satisfying the usual group axioms.

- **Important examples** are the classical matrix groups: $\mathrm{SL}(2, \mathbb{C})$, $\mathrm{PSL}(4, \mathbb{R})$, $U(67)$, $\mathrm{Sp}(238, \mathbb{C})$, etc etc.
- But we can define these groups over other rings/fields easily where we can no longer use Lie theory.
- How do we make sense of, e.g., $\mathrm{PSL}(3048, \mathbb{Z})$?

Not a definition

A **scheme** over a ring R is a topological space with an atlas of local charts; each chart is the intersection of the zerosets of polynomials in several variables with coefficients in R.

The actual definition is more complicated.

We want to allow different rings on each chart, we don't want to prioritise a coordinate system, and we need to deal with R not having a nice topology—no classical partitions of unity like you need for differential geometry!

Classical introduction: David Mumford, *The red book of varieties and schemes*, Springer (1999).
Modern introduction: Ravi Vakil, *The rising sea: Foundations of algebraic geometry*, online notes (2024).

Example

$\mathrm{SL}(2,\mathbb{Z})$ is the set of points $(a,b,c,d) \in \mathbb{Z}^4$ satisfying the equation $ad - bc - 1 = 0$. It only consists of one piece so is called **affine**.

This example is very nice (reduced, irreducible, separable) so we can think of it as a 'variety over $\mathbb{Z}$'. This runs into psychological problems but it leads us to look at $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{C}) \cap \mathbb{Z}^4$ and ask what it means to intersect a variety with a subring of its defining field.

Thou shalt always pass through the Hom functor

- If something is defined in terms of equations over a ring R, and $R \to S$ is a map, then we can ask for solutions in S^n as well as in R^n.
- Knowing relations between all possible 'sets of points' for all ring morphisms $R \to S$ is enough to recover all algebraic data.
- **Yoneda's lemma**: schemes X are equivalently functors $X : \mathrm{Ring} \to \mathrm{Set}$ where a ring R is sent to the set $X(R)$ of solutions of a bunch of polynomials in R^n.
- This is the **functor of points**.

Technical definition: $X(R) = \mathrm{Hom}(\mathrm{Spec}(R), X)$—e.g. if k is a field, $\mathrm{Spec}\, k$ is a point, so $X(k)$ is the set of maps $(\mathrm{Spec}\, k = \bullet) \to X$. Dually, we get maps $\mathcal{O}_X \to k$, i.e. evaluation maps ending up in k, i.e. points in k^n.

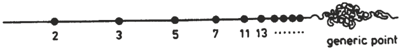
TECHNICAL INTERLUDE!!!!!!!

Let's look at Spec $\mathbb{Z}$ ('the affine variety $\mathbb{Z}^0$')...

As a topological space,

Spec $\mathbb{Z}$ is the set of prime ideals of $\mathbb{Z}$:

Example C. Spec $(\mathbb{Z})$. $\mathbb{Z}$ is a P.I.D. like $k[X]$, and Spec $(\mathbb{Z})$ is usually visualized as a line:



(Mumford's picture.)

But in terms of the functor of points,

A $\mathbb{Z}$-valued point of Spec $\mathbb{Z}$ is a map Spec $\mathbb{Z} \to$ Spec $\mathbb{Z}$. Passing to the dual, we look for ring maps $\mathbb{Z} \to \mathbb{Z}$. But there is only one such map. Hence there is only one $\mathbb{Z}$-valued point in Spec $\mathbb{Z}$.

Thou shalt get to the point

We wanted $\mathbb{Z}$-analogues of Lie groups.

Definition

An **algebraic group** over a ring R is a scheme X defined over R (so there is a map $X \to \operatorname{Spec} R$) together with morphisms $\mu : X \times_{\operatorname{Spec} R} X \to X$, $e : \operatorname{Spec} R \to X$, and $\iota : X \to X$ which satisfy the usual group axioms when restricted to $X(S)$ for all R-algebras S.

See *Stacks Project*, tag 022S (also SGA III of course).

Definition

An **arithmetic group** is the group $X(\mathbb{Z})$ where X is an algebraic group.
A **thin group** in an algebraic group is an infinite-index subgroup of the integer points of its Zariski closure.

For detailed motivation see A. Kontorovich, D. Darren Long, A. Lubotzky, A.W. Reid, "What is... a thin group?", `https://math.rice.edu/~ar99/WhatIs16.pdf`

Theorem (Maclachlan–Reid, Thm 8.22 and Thm 10.3.7)

A Kleinian group $\Gamma \leq \mathrm{PSL}(2,\mathbb{C})$ is arithmetic (resp. thin) iff: (i) it is finite (resp. infinite) covolume, (ii) the invariant trace field $k\Gamma^{(2)} = \mathbb{Q}(\{\mathrm{tr}^2\, g : g \in \Gamma\})$ has exactly one non-real field embedding into $\mathbb{C}$, and (iii) for all field embeddings $\rho : k\Gamma^{(2)} \to \mathbb{R}$ the algebra $A_0\Gamma^{(2)} = \{\sum \rho(a_i)\gamma_i : a_i \in k\Gamma^{(2)}, \gamma_i \in \Gamma\}$ is isomorphic to the Hamiltonian quaternions.

Theorem (Maclachlan–Martin, 1999)

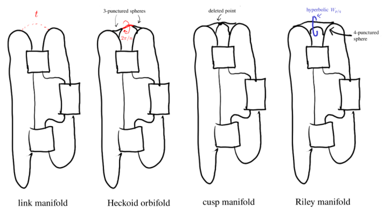
There are only finitely many conjugacy classes of arithmetic or thin Kleinian groups generated by two parabolic or elliptic elements.

What are they?

Theorem (Conj. of Agol (2001), proved Aimi–Akiyoshi–Lee–Ohshika–Parker–Sakui–Sakuma–Yoshida (2020))

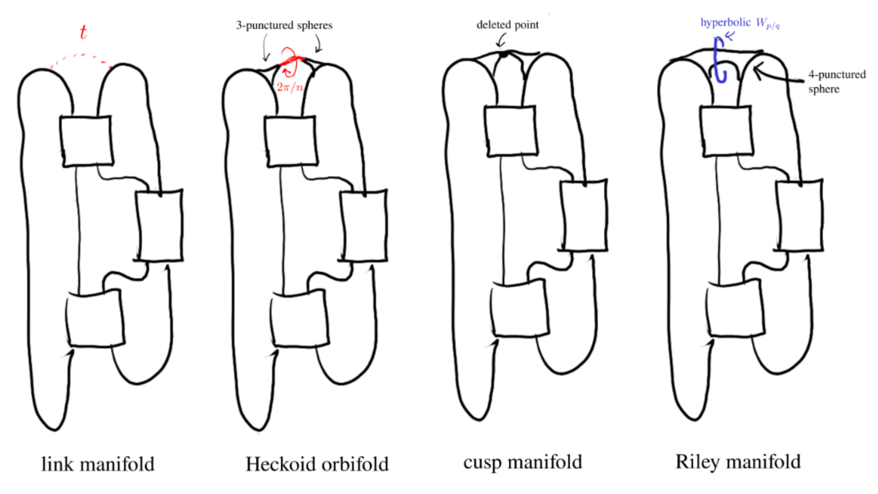
If X and Y are parabolic, and $G = \langle X, Y \rangle$ is Kleinian and non-Fuchsian, then G falls into one of the following mutually exclusive categories:

1. *Split as a free product $\langle X \rangle * \langle Y \rangle$:*
   1.1 *Groups in $\overline{\mathcal{R}}$.*
2. *Don't split:*
   2.1 *Heckoid groups: $\langle X, Y : W^n = 1 \rangle$ for $n > 1$*
   2.2 *2-bridge link complements: $\langle X, Y : W = 1 \rangle$*
   2.3 *Quotient orbifolds of (2.1) and (2.2) by involutions.*



A.J.E., G.J. Martin, J. Schillewaert, "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". arXiv:2204.11422

Thou shalt draw the usual picture



A.J.E., G.J. Martin, J. Schillewaert, “Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds”. arXiv:2204.11422

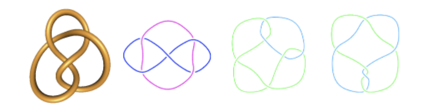
Thou shalt not try to give technical proofs

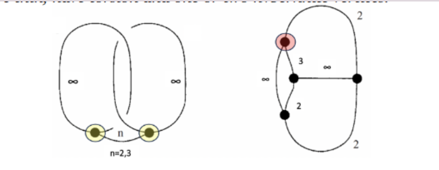
Theorem

Out of the Kleinian groups generated by 2 parabolics:

- *Gehring–Maclachlan–Martin (1998): exactly 4 are arithmetic*
- *E.–Martin–Schillewaert (2024): exactly 3 are thin*

E.–Martin–Schillewaert (to appear): Out of the Kleinian groups generated by 2 parabolics or elliptics, approx. 150 are thin .





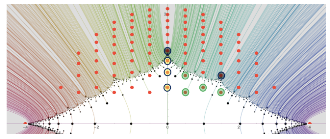
Thou shalt at least try to read the leftovers

Written notes with references to further reading:
`https://aelzenaar.github.io/kg/heckoid_talk.pdf`

- A.J.E., G.J.M., and J.S., *Approximations of the Riley slice.* Expo. Math., 2023 (arXiv 2021).
- —, G.J.M., and J.S., *Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds. 2021–22 MATRIX annals.* Springer, 2024 (arXiv 2022).
- —, J. Gong, G.J.M, and J.S., *Bounding deformation spaces of Kleinian groups with two generators* (arXiv 2024).
- —, G.J.M., and J.S., *On thin Heckoid and generalised triangle groups in* $\mathrm{PSL}(2,\mathbb{C})$ (arXiv 2024).

Sketch of proof of enumeration

1. Maclachlan–Martin together with miscellaneous computational number theorists provide bounds on the degree of the trace field , based on the properties of the embedding.
2. Our work on bounds for $\mathcal{R}^{p,q}$ (A.J.E.–Gong–Martin–Schillewaert, 2024) show that there is a bounded region in which the parameter ρ can live. There are only finitely many monic polynomials with roots in this region of bounded degree.

   (Image courtesy Y. Yamashita.)
3. For all these finitely many points, we require certificates of (i) being discrete and free (A.J.E.–G.J.M.–J.S., 2021) to eliminate some as splitting as free products, and (ii) certificates of discrete and non-free: turns out all the latter are Heckoid, and this theory is understood (work of Japanese school in 2000s and 2010s in parabolic case, Chesebro–Martin–Schillewart in elliptic case).

Tunnel number 1 knots and links

Conjecture

A classification of all Kleinian groups $\langle X, Y \rangle$ which are non-Fuchsian and which are quasiconformally rigid can be obtained by replacing, in Agol's theorem case (2), '2-bridge link complement' with 'geometrically finite hyperbolic 3-manifold with no conformal boundary and one or two rank 2 cusps that admits an unknotting tunnel'. Note that such manifolds do not necessarily embed into $\mathbb{S}^3$.

- Restrict first to groups uniformising 3-orbifolds embeddable in $\mathbb{S}^3$—this corresponds to restricting to tunnel number one links. Classification of tunnel number one links in $\mathbb{S}^3$, including their involutions, is known (Adams, Kuhn, Reid, Morimoto, Sakuma, Yokota) From Cho–McCullough (2007) there is correspondence between these links and representations of genus two Schottky groups obtained by gluing 2-handles.
- Then attempt to modify Sakuma et. al. (perhaps starting with original work of Lee–Sakuma (2012) on epimorphisms of 2-bridge knot groups, looks feasible) to obtain a proof of the special case.
- In general it is not clear how to proceed as the various results that go into the two-bridge theorem strongly depend on embedding the knot into $\mathbb{S}^3$. The ambient manifold is, instead, obtained by a 'genus two Dehn filling' of $\mathbb{S}^3$ (i.e. a manifold with Heegaard genus 2).