

# WHAT DO KNOTS HAVE TO DO WITH ALGEBRA?

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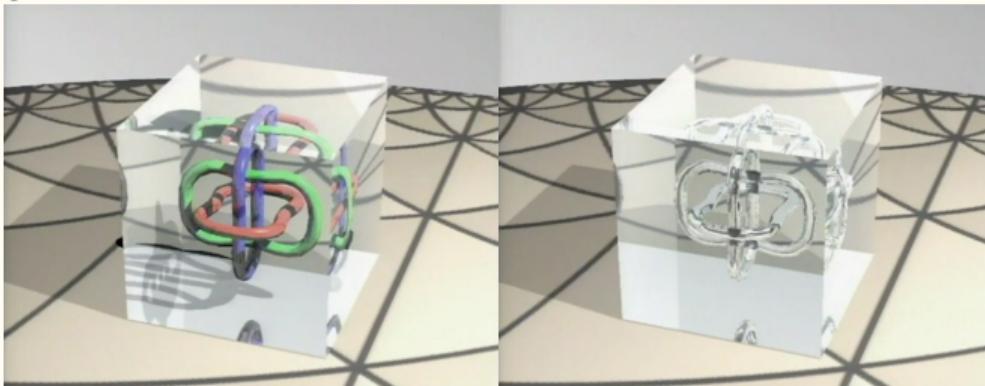
27 Nov. 2025



László Mololy-Nagy, *Kinetisches Konstruktives System* (1922)  
Bauhaus-Archiv Berlin

# KNOTS

If  $k$  is a knot, then  $\mathbb{S}^3 \setminus k$  is a smooth oriented 3-manifold.



Gunn and Maxwell, *Not Knot*: <https://www.youtube.com/watch?v=4aN6vX7qXPQ>

$\mathbb{S}^3 \setminus k$  can be described combinatorially by a **knot diagram**.

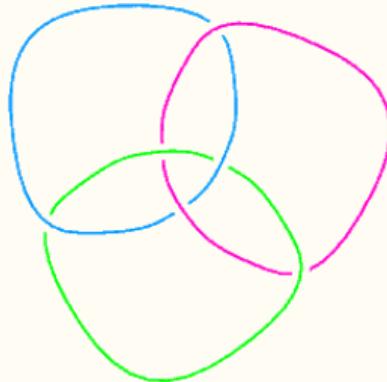
Why are these manifolds interesting to topologists?

Gordon/Luecke (1989)

Knots with one component are determined by their complements up to homeomorphism.

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<https://katlas.org/wiki/L6a4>

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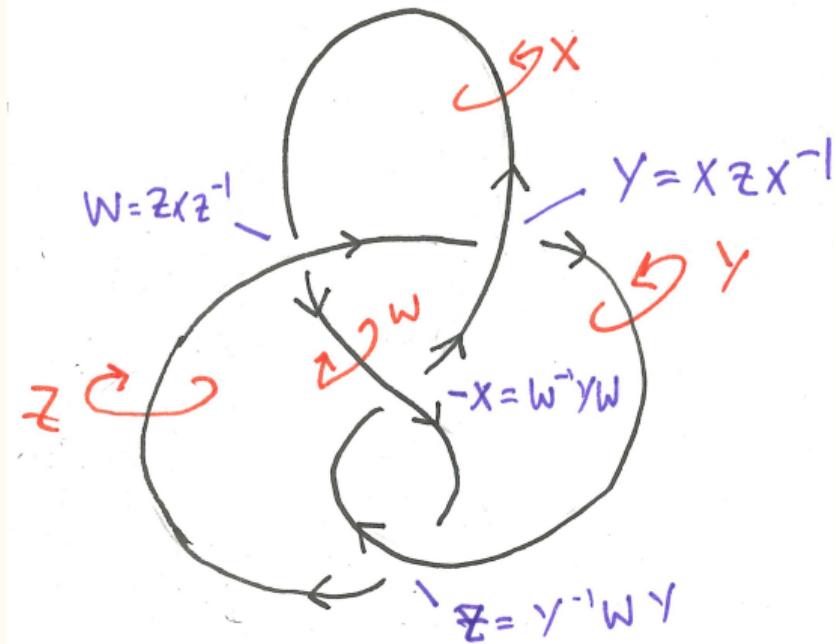
# KNOTS: GROUPS FROM DIAGRAMS

Knot complements can be described combinatorially by diagrams.

Wirtinger (c.1905)

There is an algorithm that sends a diagram of a knot  $k$  to a presentation for  $\pi_1(\mathbb{S}^3 \setminus k)$ , where all generators are conjugate:

generators	relations
diagram arcs	crossings



$$\langle X, Y, Z, W : X^{-1}W^{-1}YW, Y^{-1}XZX^{-1}, Z^{-1}Y^{-1}WY, W^{-1}ZXZ^{-1} \rangle$$

# REPRESENTATIONS OF KNOT GROUPS TO $\text{PSL}(2, \mathbb{F}_p)$

To understand a group  $G$ , we look at its representations.

Theorem (Riley, *Math. Comp.* (1971))

For any knot  $k$ , the surjective representations  $\pi_1(\mathbb{S}^3 \setminus k) \rightarrow \text{PSL}(2, \mathbb{F}_p)$  ( $p$  prime) where every Wirtinger generator is order<sup>1</sup>  $p$  can be classified into conjugacy classes by performing at most

$$\left(\frac{p-1}{2}\right)^{n-1} \frac{(p+1)^{n-1} - 1}{p}$$

experiments, where  $n$  is the minimal number of bridges for a diagram of  $k$ .

The method of proof is purely combinatorial, relying only on information about the types of relators that can appear in a Wirtinger presentation.

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<sup>1</sup>The condition on the order is purely for simplification and there are more general results in the same paper.

# REPRESENTATIONS OF KNOT GROUPS TO $\text{PSL}(2, \mathbb{C})$

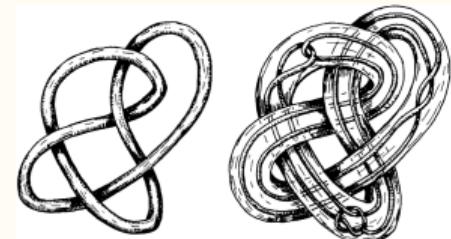
## Theorem (Thurston, c. 1978)

If  $k$  is a link then  $\mathbb{S}^3 \setminus k$  admits a Riemann metric of constant sectional curvature  $-1$  if and only if it has no:

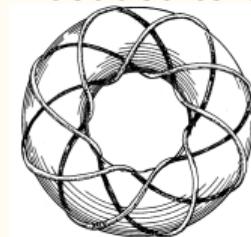
- embedded essential spheres;
- embedded essential discs;
- embedded essential tori;
- embedded essential annuli.

The hyperbolic structure on  $\mathbb{S}^3 \setminus k$  induces a faithful, discrete representation

$$\pi_1(\mathbb{S}^3 \setminus k) \rightarrow \text{Isom}^+(\mathbb{H}^3) = \text{Conf}(\mathbb{S}^2) = \text{PSL}(2, \mathbb{C}).$$



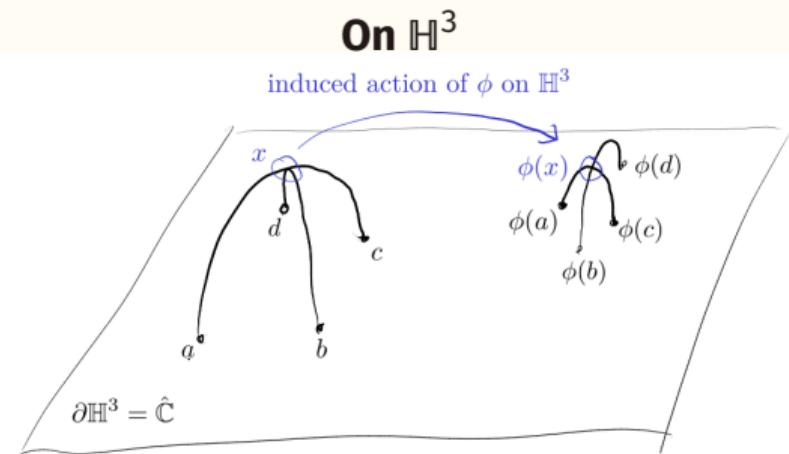
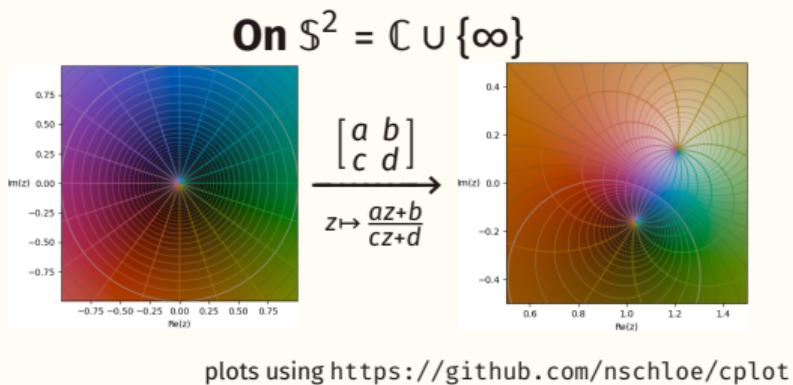
Embedded torus



Embedded annulus

Thurston, Bull. AMS (1982)  
images by George Francis

# PSL(2, $\mathbb{C}$ )-ACTIONS



E., J. Schillewaert, G.J. Martin, 2021-22 MATRIX annals, Springer (2024).

**Meaning of traces.** If  $g \in \pi_1(\mathbb{S}^3 \setminus k)$  is homotopic to a loop on a component of  $k$  (e.g. a meridian or longitude), then  $\text{tr}^2$  of the image is 4. The image of every other element has a positive minimal translation length in  $\mathbb{H}^3$  and corresponds to a geodesic loop of positive length,  $\text{tr}^2 \notin [0, 4]$ .

3-elements. Let  $E_3$  be the one which contains  $t$ . Then  $c_1$  is transformed into a segment by the deformation

$$S \rightarrow S + (E_3).$$

In this way all the circuits in  $S \cdot E_3$  can be eliminated.

When the circuits have been eliminated there will be at least one segment in  $S \cdot E_3$ , say  $u$ , which, together with a segment of  $m$ , bounds a 2-element  $C_2$ , on  $E_3$ , containing no other component of  $S$ . If  $E_3$  is defined as before, with  $N(C_2, M)$  and  $N(u, S)$  taking the place of  $N(t, M)$  and  $N(p, S)$ , the segment  $u$  is eliminated from  $S \cdot E_3$  by the deformation

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Reiterating this process, we obtain a non-singular deformation of



$S$  into a surface  $S^1$ , which does not meet  $E_3$  except in  $t$ , and this deformation leaves  $c$  unaltered. It is now obvious that the first step in the deformation  $c \rightarrow c'$  can be realized by a non-singular deformation of  $S^1$ , and the lemma follows from induction on the number of steps in  $c \rightarrow c'$ .

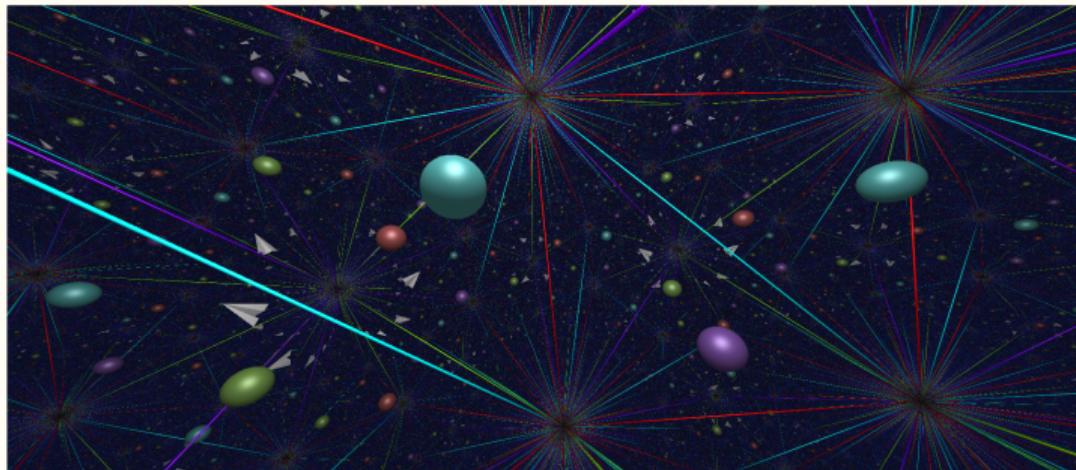
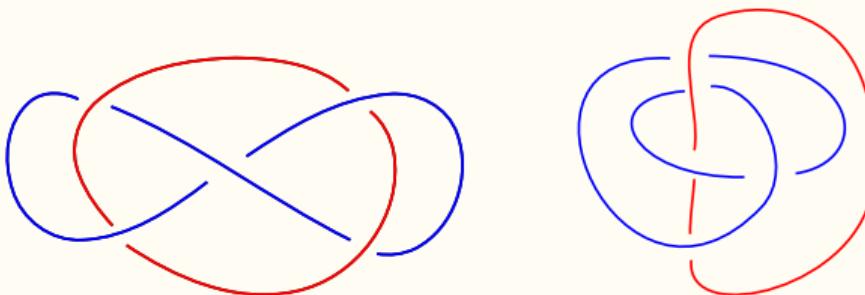
If a circuit in  $M$  is contained in a 3-element in  $M$  it will be called an *elementary circuit*. A circuit which bounds a (singular) 2-cell but which is not an elementary circuit will be called a *self-linking* circuit. The simplest type of self-linking circuit is illustrated by the diagram, the manifold being the residual space of a circuit  $m$  in Euclidean space, and  $s$  being a self-linking circuit.

We shall need two lemmas about punctured spheres.<sup>†</sup> We first recall from T.M. pp. 319–20, that any two punctured spheres are equivalent if they have the same number of boundary 2-spheres.<sup>‡</sup>

<sup>†</sup> Cf. T.M. § 2. When we refer to a punctured sphere or to any other bounded manifold, it is to be assumed that the boundary is non-singular.

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J.H.C. Whitehead, "A certain open manifold whose group is unity". *Quart. J. Math.*, 1935.



M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, *SnapPy*, a computer program for studying the geometry and topology of 3-manifolds, <http://snappy.computop.org>

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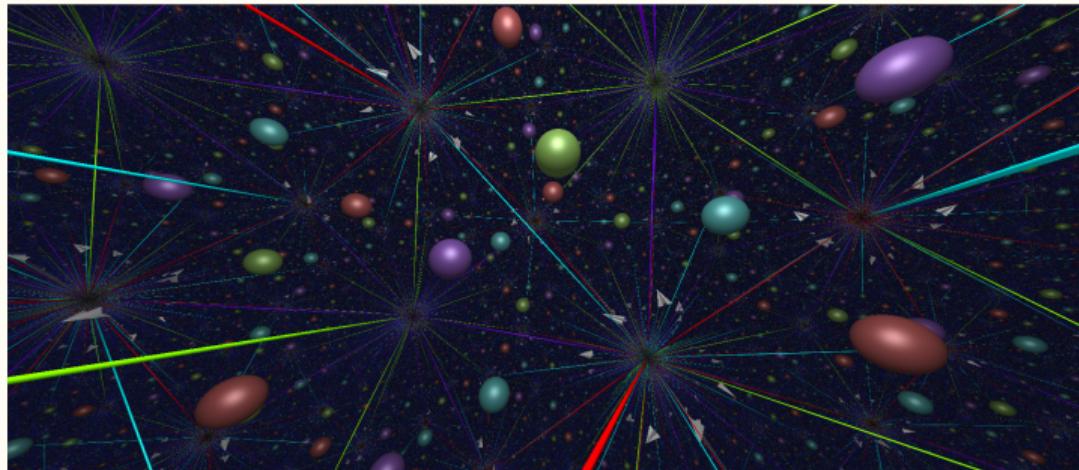
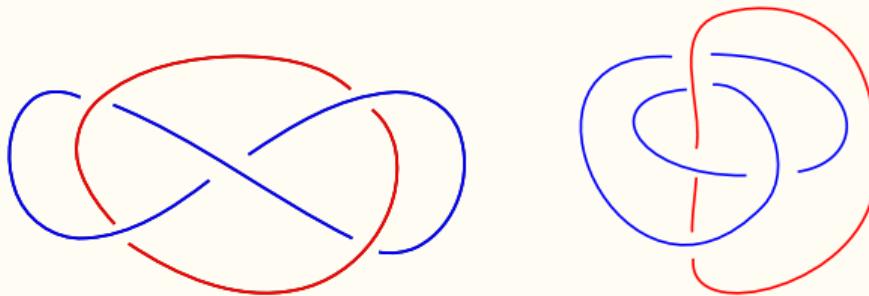
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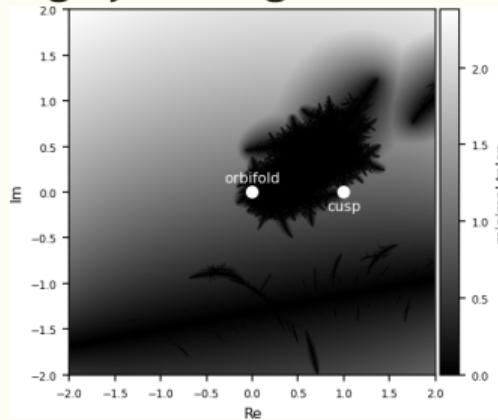


M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, <http://snappy.computop.org>

- Knot complements are one important class of geometric structures with interesting groups.
  - ▶ **Mostow rigidity theorem:** the corresponding subgroups of  $\mathrm{PSL}(2, \mathbb{C})$  are rigid—if  $\mathbb{S}^3 \setminus k = \mathbb{H}^3/G$ , and  $H < \mathrm{PSL}(2, \mathbb{C})$  is isomorphic to  $G$ , then  $\mathbb{H}^3/G = \mathbb{H}^3/H$  (up to isometry).

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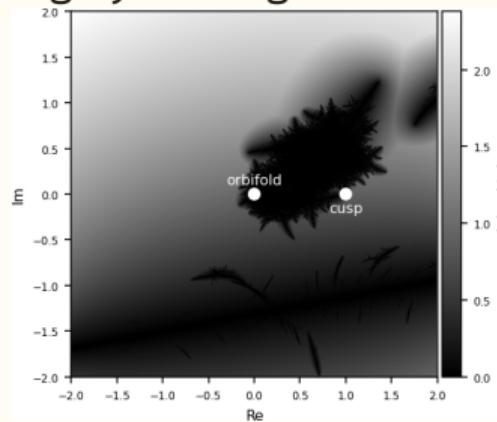
- Surface groups are important classes of geometric structures, but now are highly non-rigid:



← A linear slice through different geometric structures  $\pi_1(S_{2,0}) \rightarrow \text{PSL}(2, \mathbb{C})$  where  $S_{2,0}$  is the genus 2 surface.

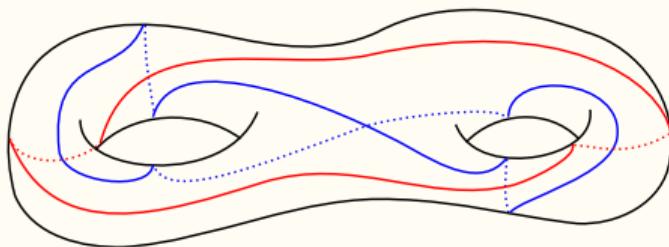
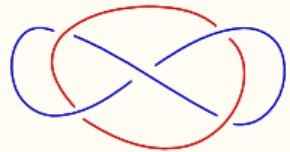
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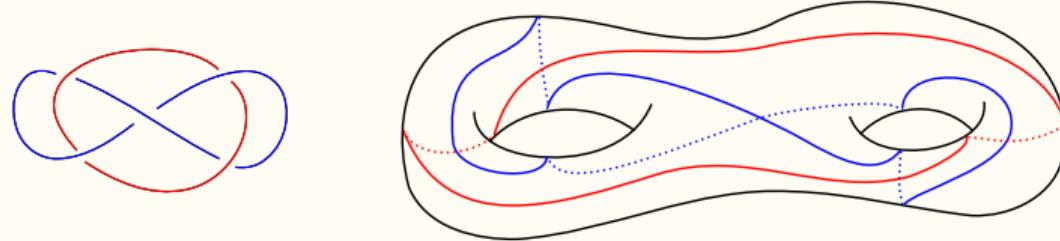
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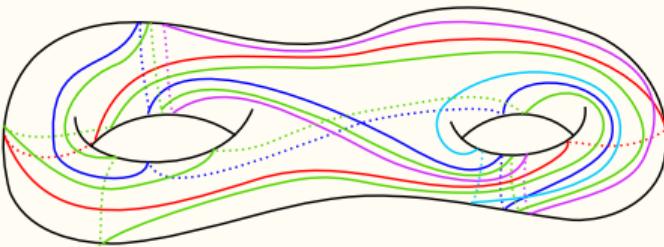
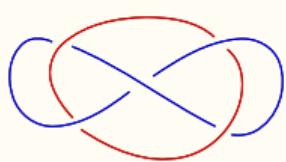
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- How do these different kinds of  $\mathrm{PSL}(2, \mathbb{C})$ -representations fit together?  
Consider embeddings  $k \rightarrow S_{2,0}$ .





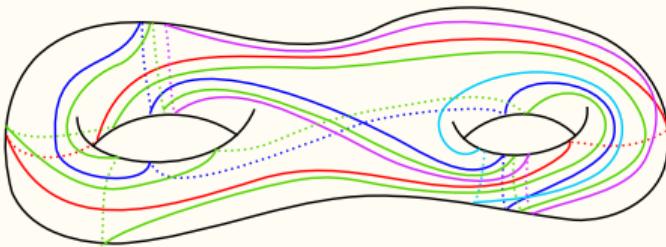
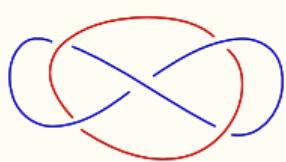
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dual graph:  
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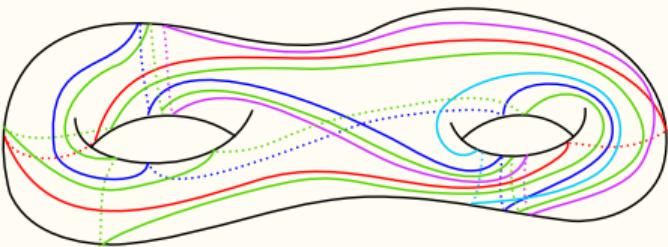
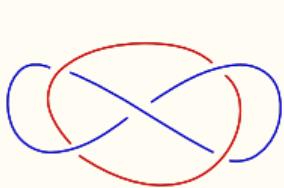


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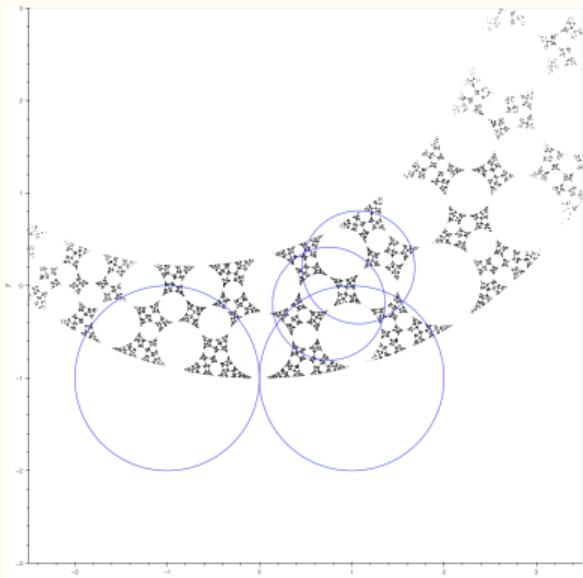
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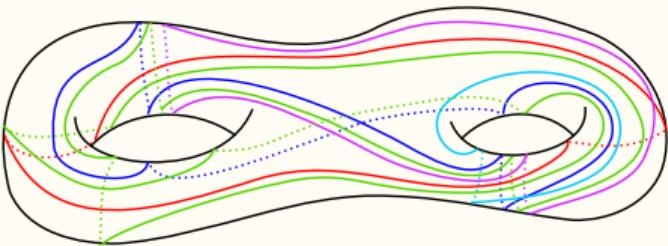
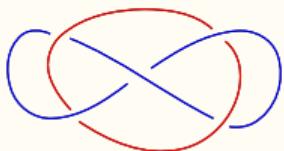
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$$\rho(Y) = \begin{bmatrix} 1 - i & 1 \\ 1 & 1 + i \end{bmatrix}$$





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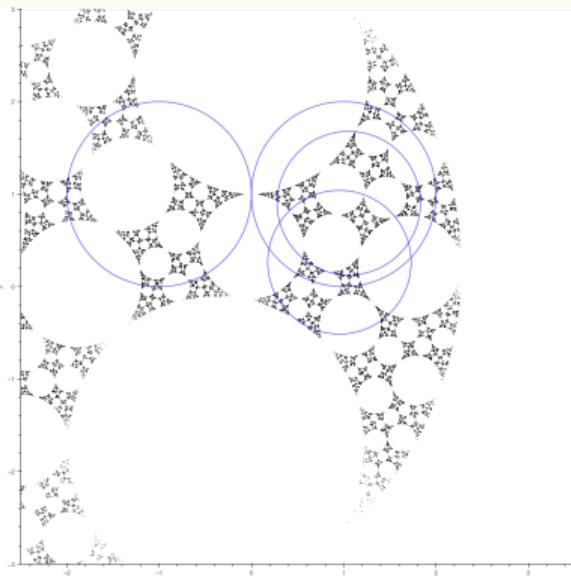
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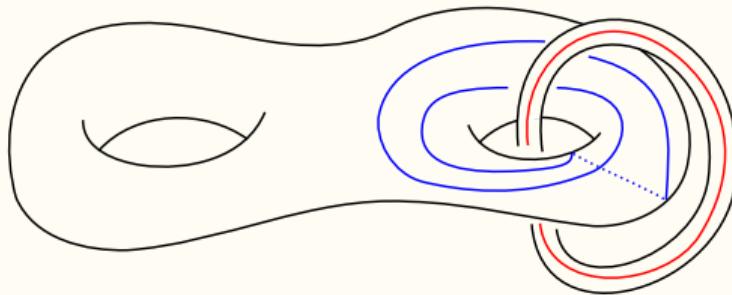
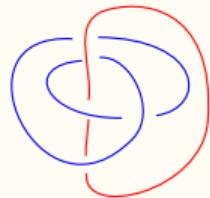
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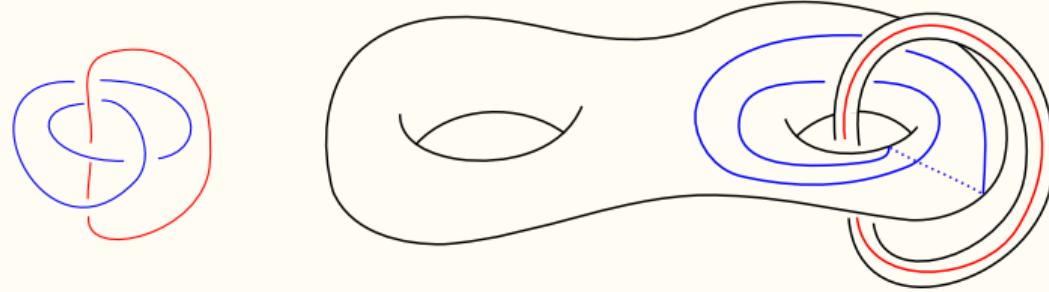
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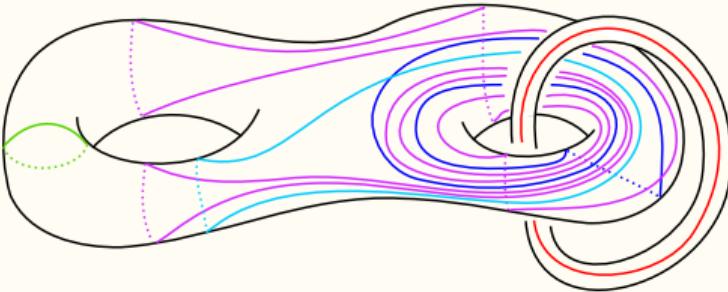
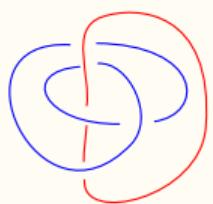
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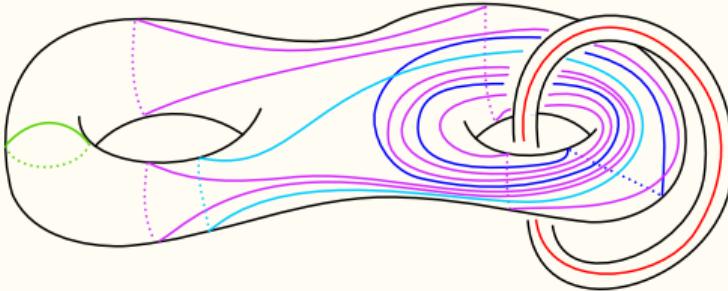
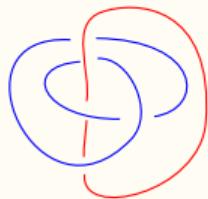
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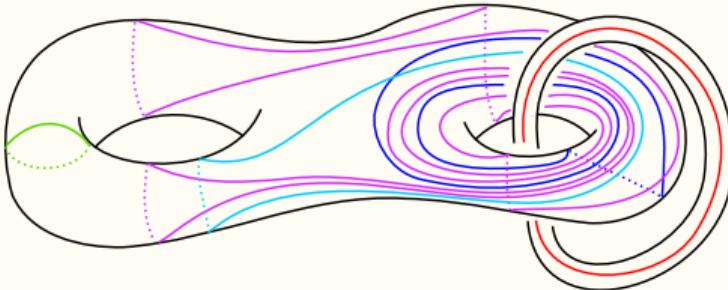
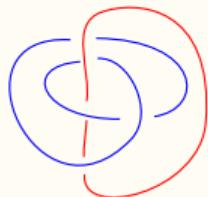
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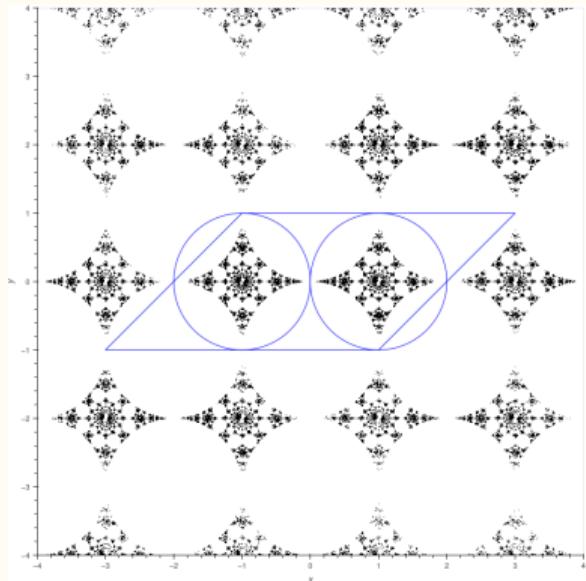
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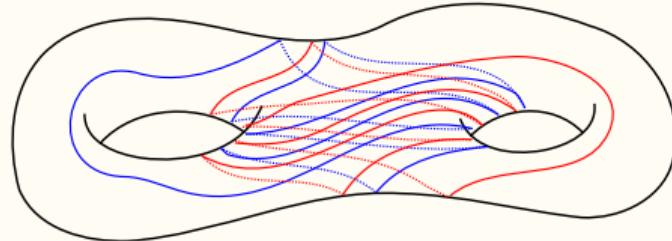
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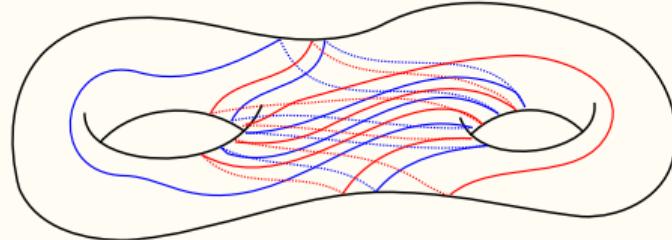
$$\rho(P) = \begin{bmatrix} 1 & 10 + 2i \\ 0 & 1 \end{bmatrix}, \quad \rho(Q) = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \quad \rho(M) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



3/8 braid  
embedding  
(as per Jeroen's talk)

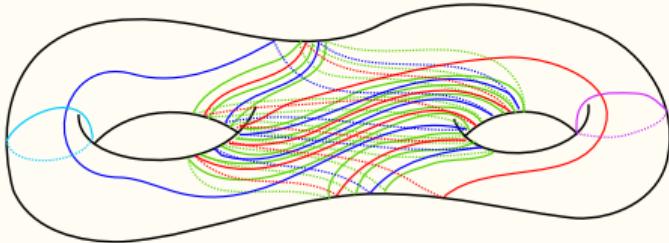


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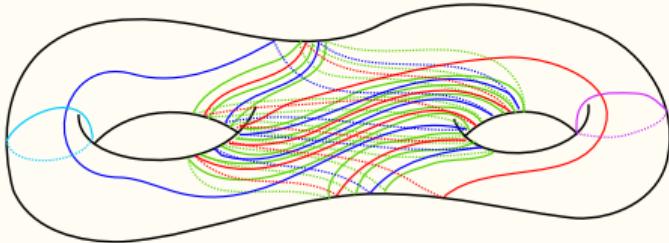


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green:  $W_{3/8}$  pink:  $Y$  blue:  $X$

dual graph:  
3 non-intersecting  
curves, each meet-  
ing the knot at  
most once

3/8 braid  
embedding  
(as per Jeroen's talk)



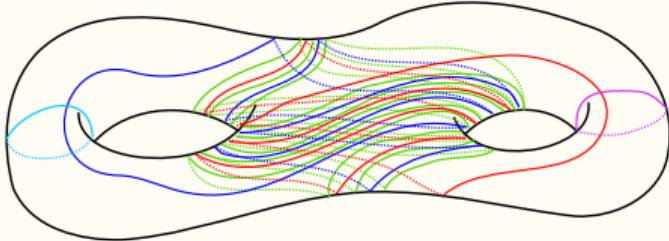
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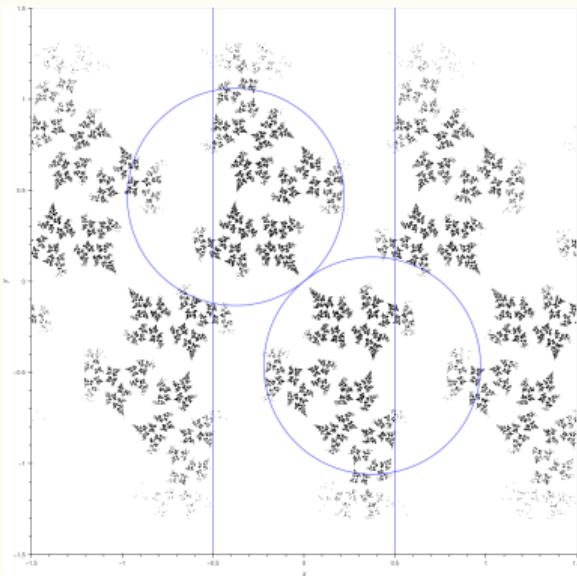
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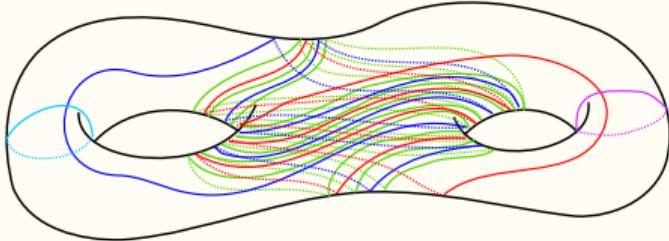
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 $\text{tr}^2 = 4$  for all these words.

$$\rho(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 & 0 \\ 1.05642 + 1.30324i & 1 \end{bmatrix}$$



3/8 braid  
embedding  
(as per Jeroen's talk)



dual graph:  
3 non-intersecting  
curves, each meet-  
ing the knot at  
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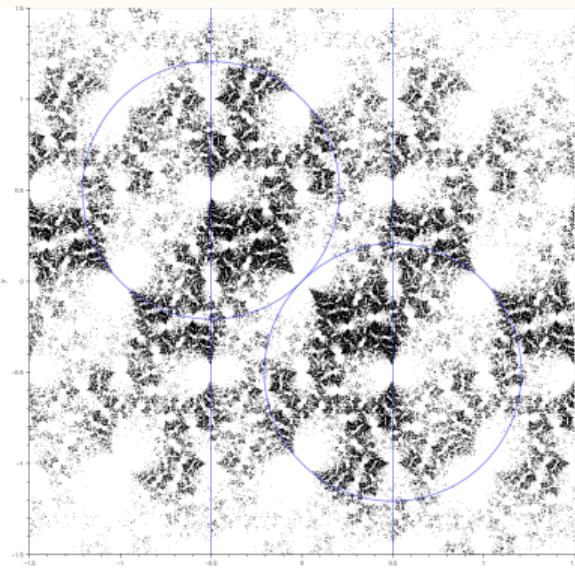
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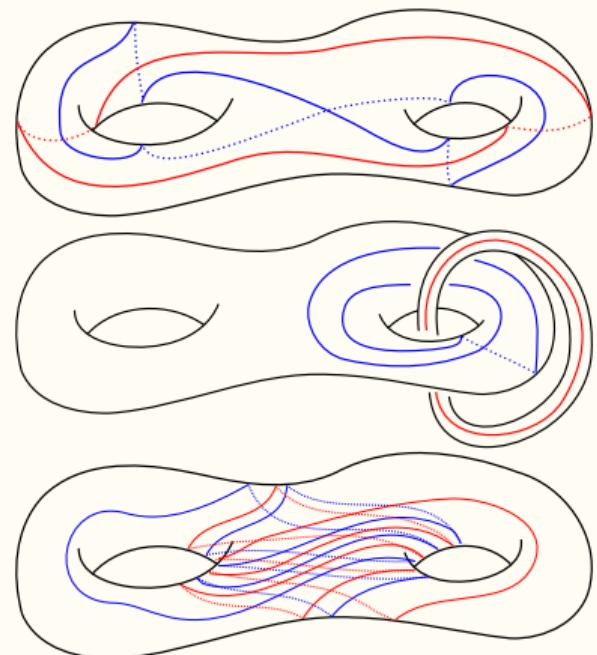
$$\rho(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 & 0 \\ 1+i & 1 \end{bmatrix}$$

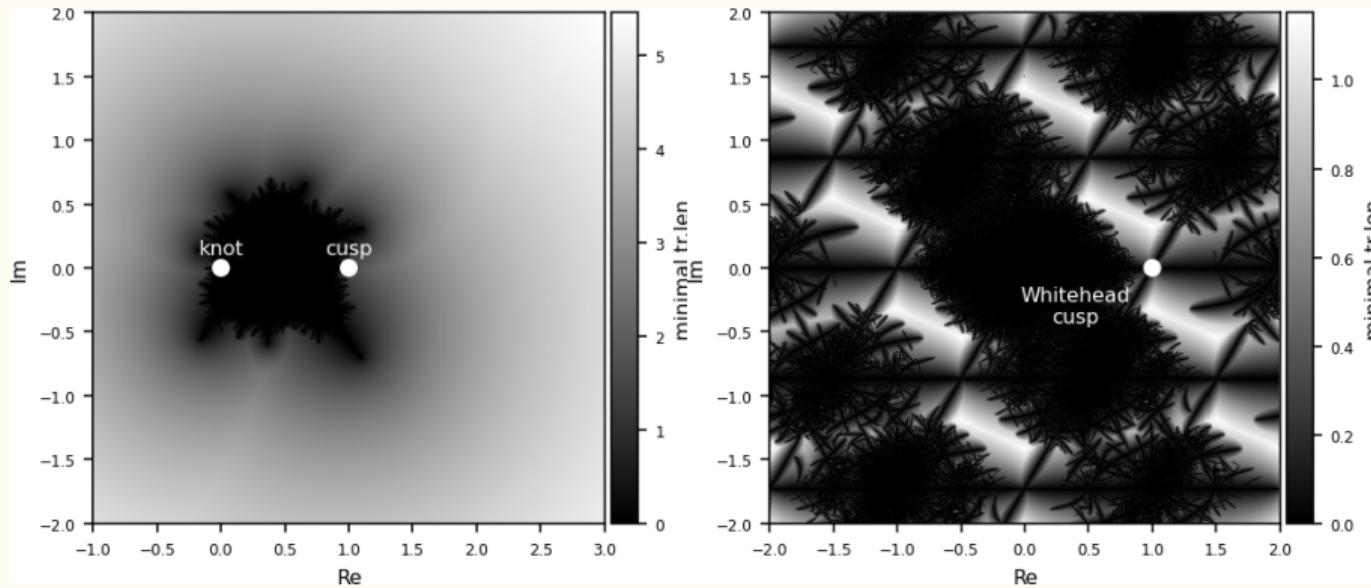


# WHAT IS THE BEST CHOICE OF EMBEDDING?

- Out of the 3, only 1 embedding of the Whitehead link into  $S_{2,0}$  gave us the link group (or any cofinite group at all) back as a rep: the braid embedding.
- Necessary (but not sufficient) condition: the dual curves to the embedding need to bound embedded discs in the complement handlebody.
- When this condition is not satisfied, if  $k$  is complicated (many crossings) then we still get lattices as well as cusp groups.
- *Open question:* characterise the lattices obtained topologically in terms of the combinatorics of  $k$ .



These methods give us reps.  $G \rightarrow \text{PSL}(2, \mathbb{C})$  that capture the combinatorics of knots (so nice geometry) & are *extremal* in representation spaces.



Two slices through  $\text{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \text{PSL}(2, \mathbb{C}))$ . The ‘cusp’ point is the same group in each picture.

E., “From disc patterns in the plane to character varieties of knot groups”, arXiv:2503.13829 [math.GT]

## Problem

If  $G$  is a group and  $\rho_0, \rho_1 : G \rightarrow \mathrm{PSL}(2, \mathbb{C})$  are reps, can you find a path of reps  $\rho_t$  with actions that ‘smoothly interpolate’ from  $\rho_0$  to  $\rho_1$ ?

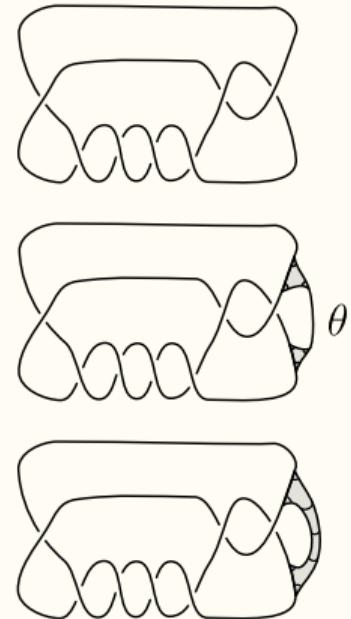
The obstruction is an understanding of paths of *indiscrete* reps. The extremal groups we discussed (called ‘cusp groups’) can act as nice endpoints for smooth curves of indiscrete representations. We have constructed such paths in a few settings, using a variety of geometric techniques:

- between compression body groups with different numbers of handles  
arXiv:2411.17940 [math.GT], video
- between ‘fully augmented link groups’ which are rigid, and cusp groups which lie on the boundary of a big deformation space (to appear shortly)
- between 2-bridge link groups and the holonomy groups of manifolds where an upper unknotting tunnel has been drilled (joint with Chesebro and Purcell, in preparation)

As an example of such a theorem, we can show that, for highly twisted 2-bridge knots  $\kappa$ , the Heckoid groups are part of a smooth curve of geometries that deform  $\kappa$  into the corresponding cusp.

### Theorem (E., 2025)

Let  $\kappa$  be 2-bridge with slope  $[2m_n, 2a_n, \dots, 2m_1, 2a_1]$ . If  $m_n \geq 4$  and  $a_n, m_{n-1}, \dots, m_1, a_1 \geq 7$ , then there is a family of cone manifolds  $M_\theta$  ( $\theta \in (0, 2\pi]$ ) supported on the link complement  $M_{2\pi} = \mathbb{S}^3 \setminus \kappa$  with only singularity an ideal cone arc along the upper unknotting tunnel of  $\kappa$ . The limit as  $\theta \rightarrow 0$  is well-defined, and is a maximal cusp point on the boundary of genus 2 Schottky space. The upper unknotting tunnel of  $\kappa$  is isotopic to a geodesic in the complete hyperbolic structure on  $M_{2\pi}$ .

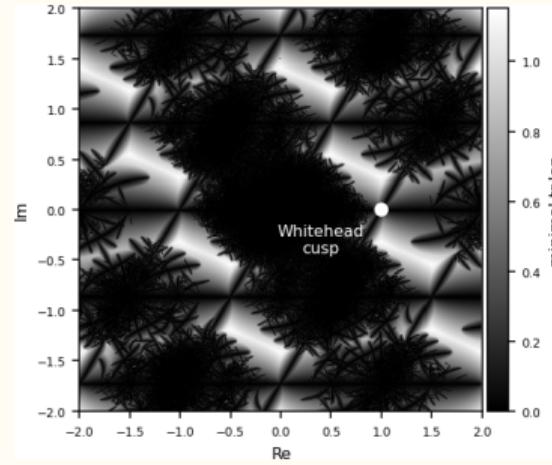
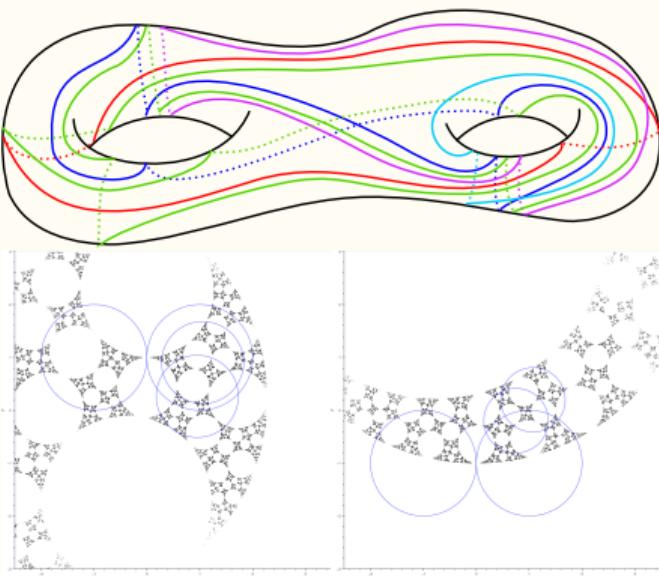


...for highly twisted 2-bridge knots  $\kappa$ , the Heckoid groups are part of a smooth curve of geometries that deform  $\kappa$  into the corresponding cusp...

## Proof.

1. exhibit such a path for the unknot & its cusp.
2. drill and fill curves to realise the element of the mapping class group that produces the Farey curve on  $S_{0,4}$ .
3. apply Gromov-Thurston  $2\pi$  theorem (which is OK for cone manifolds) to show that the resulting manifolds are hyperbolic. ◻

- We do not apply Gromov-Thurston to the cone arc (the cone deformation itself isn't done by Dehn filling). After doubling, this arc should be 'short'.
- Using entirely different but more specialised techniques, the slope restrictions can be removed (j/w Chesebro, Purcell).
- Our constructions here work to produce cone deformations for a very large family of links and other hyperbolic manifolds.
- We have a necessary condition for 'bad embeddings'  $\kappa \rightarrow S_{2,0}$ : the MCG element it corresponds to must have 'short generators' (in terms of cusp length).



- E., *From disc patterns in the plane to character varieties of knot groups.* arXiv:2503.13829 [math.GT]
- E., *Changing topological type of compression bodies through cone manifolds.* arXiv:2411.17940 [math.GT]
- J. Purcell, *Hyperbolic knot theory.* AMS, 2020.
- A. Marden, *Hyperbolic manifolds.* Cambridge, 2016.
- D. Mumford, C. Series, D. Wright, *Indra's pearls.* Cambridge, 2002.