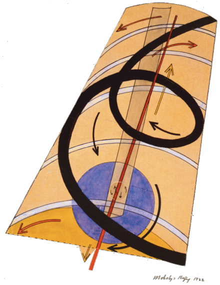
What do knots have to do with algebra?

Alex Elzenaar

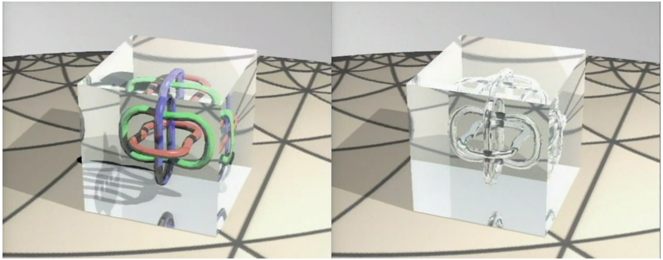
Monash University, Melbourne, Australia

NZMS Colloquium, Waikato University
26–28 Nov. 2025



László Moloy-Nagy, *Kinetisches Konstruktives System* (1922)
Bauhaus-Archiv Berlin

If k is a knot, then $\mathbb{S}^3 \setminus k$ is a smooth oriented 3-manifold.



Gunn and Maxwell, *Not Knot*: `https://www.youtube.com/watch?v=4aN6vX7qXPQ`

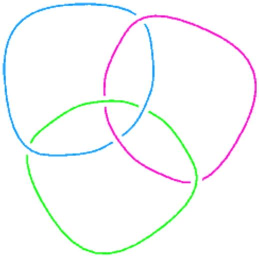
$\mathbb{S}^3 \setminus k$ can be described combinatorially by a **knot diagram**.

Why are these manifolds interesting to topologists?

Gordon/Luecke (1989)

Knots with one component are determined by their complements up to homeomorphism.

If k is a knot, then $\mathbb{S}^3 \setminus k$ is a smooth oriented 3-manifold.



https://katlas.org/wiki/L6a4

$\mathbb{S}^3 \setminus k$ can be described combinatorially by a **knot diagram**.

Why are these manifolds interesting to topologists?

Gordon/Luecke (1989)

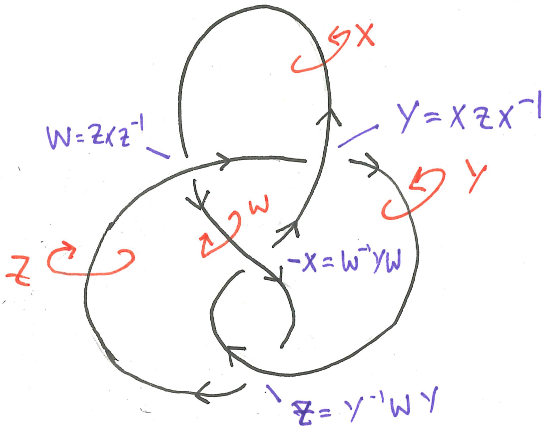
Knots with one component are determined by their complements up to homeomorphism.

Knot complements can be described combinatorially by diagrams.

Wirtinger (c.1905)

There is an algorithm that sends a diagram of a knot k to a presentation for $\pi_1(\mathbb{S}^3 \setminus k)$, where all generators are conjugate:

generators	**relations**
diagram arcs	crossings



$$\langle X, Y, Z, W : X^{-1}W^{-1}YW, Y^{-1}XZX^{-1}, Z^{-1}Y^{-1}WY, W^{-1}ZXZ^{-1} \rangle$$

To understand a group G, we look at its representations.

Theorem (Riley, *Math. Comp.* (1971))

For any knot k, the surjective representations $\pi_1(\mathbb{S}^3 \setminus k) \to \mathrm{PSL}(2, \mathbb{F}_p)$ (p prime) where every Wirtinger generator is order[1] *p can be classified into conjugacy classes by performing at most*

$$\left(\frac{p-1}{2}\right)^{n-1} \frac{(p+1)^{n-1}-1}{p}$$

experiments, where n is the minimal number of bridges for a diagram of k.

The method of proof is purely combinatorial, relying only on information about the types of relators that can appear in a Wirtinger presentation.

[1] The condition on the order is purely for simplification and there are more general results in the same paper.

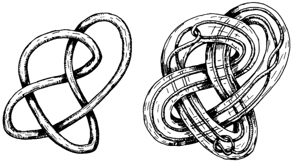
Theorem (Riley–Thurston, c. 1978)

If k is a link then $\mathbb{S}^3 \setminus k$ admits a Riemann metric of constant sectional curvature -1 if and only if it has no:

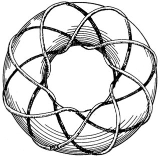
- *embedded essential spheres;*
- *embedded essential discs;*
- *embedded essential tori;*
- *embedded essential annuli.*

The hyperbolic structure on $\mathbb{S}^3 \setminus k$ induces a faithful, discrete representation

$$\pi_1(\mathbb{S}^3 \setminus k) \to \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{Conf}(\mathbb{S}^2) = \mathrm{PSL}(2, \mathbb{C}).$$

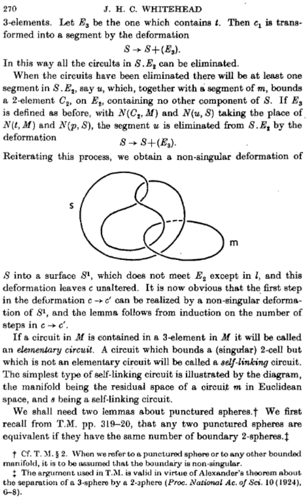


Embedded torus



Embedded annulus

Thurston, *Bull. AMS* (1982)
images by George Francis



270 J. H. C. WHITEHEAD

3-elements. Let E_3 be the one which contains t. Then c_1 is transformed into a segment by the deformation

$$S \to S + (E_3).$$

In this way all the circuits in $S.E_2$ can be eliminated.

When the circuits have been eliminated there will be at least one segment in $S.E_2$, say u, which, together with a segment of m, bounds a 2-element C_2, on E_2, containing no other component of S. If E_3 is defined as before, with $N(C_2, M)$ and $N(u, S)$ taking the place of $N(t, M)$ and $N(p, S)$, the segment u is eliminated from $S.E_2$ by the deformation

$$S \to S + (E_3).$$

Reiterating this process, we obtain a non-singular deformation of

s

m

S into a surface S^1, which does not meet E_2 except in l, and this deformation leaves c unaltered. It is now obvious that the first step in the deformation $c \to c'$ can be realized by a non-singular deformation of S^1, and the lemma follows from induction on the number of steps in $c \to c'$.

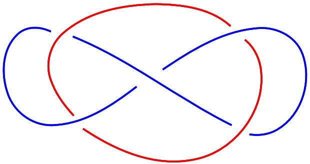
If a circuit in M is contained in a 3-element in M it will be called an *elementary circuit*. A circuit which bounds a (singular) 2-cell but which is not an elementary circuit will be called a *self-linking* circuit. The simplest type of self-linking circuit is illustrated by the diagram, the manifold being the residual space of a circuit m in Euclidean space, and s being a self-linking circuit.

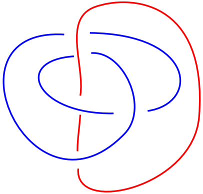
We shall need two lemmas about punctured spheres.† We first recall from T.M. pp. 319–20, that any two punctured spheres are equivalent if they have the same number of boundary 2-spheres.‡

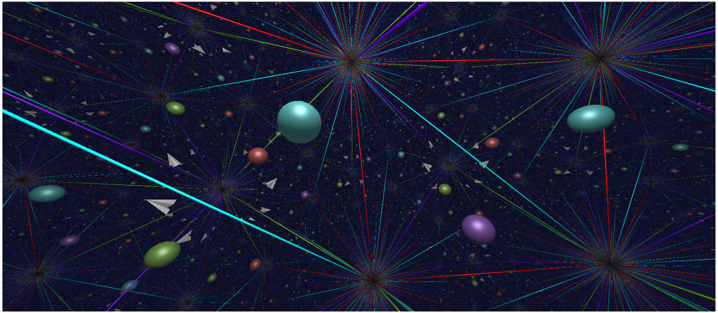
† Cf. T. M. § 2. When we refer to a punctured sphere or to any other bounded manifold, it is to be assumed that the boundary is non-singular.

‡ The argument used in T.M. is valid in virtue of Alexander's theorem about the separation of a 3-sphere by a 2-sphere (*Proc. National Ac. of Sci.* 10 (1924), 6–8).

J.H.C. Whitehead, "A certain open manifold whose group is unity". *Quart. J. Math.*, 1935.







M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, `http://snappy.computop.org`

3-elements. Let E_3 be the one which contains t. Then c_1 is transformed into a segment by the deformation

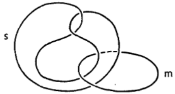
$$S \to S + (E_3).$$

In this way all the circuits in $S.E_2$ can be eliminated.

When the circuits have been eliminated there will be at least one segment in $S.E_2$, say u, which, together with a segment of m, bounds a 2-element C_2, on E_2, containing no other component of S. If E_3 is defined as before, with $N(C_2, M)$ and $N(u, S)$ taking the place of $N(t, M)$ and $N(p, S)$, the segment u is eliminated from $S.E_2$ by the deformation

$$S \to S + (E_3).$$

Reiterating this process, we obtain a non-singular deformation of



S into a surface S^1, which does not meet E_2 except in l, and this deformation leaves c unaltered. It is now obvious that the first step in the deformation $c \to c'$ can be realized by a non-singular deformation of S^1, and the lemma follows from induction on the number of steps in $c \to c'$.

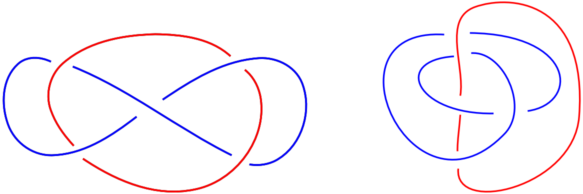
If a circuit in M is contained in a 3-element in M it will be called an *elementary circuit*. A circuit which bounds a (singular) 2-cell but which is not an elementary circuit will be called a *self-linking* circuit. The simplest type of self-linking circuit is illustrated by the diagram, the manifold being the residual space of a circuit m in Euclidean space, and s being a self-linking circuit.

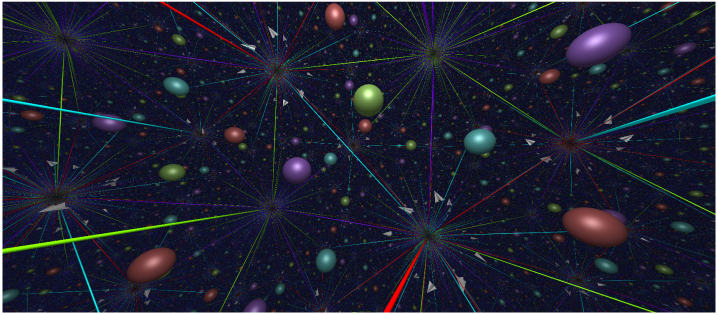
We shall need two lemmas about punctured spheres.† We first recall from T.M. pp. 319–20, that any two punctured spheres are equivalent if they have the same number of boundary 2-spheres.‡

† Cf. T. M. § 2. When we refer to a punctured sphere or to any other bounded manifold, it is to be assumed that the boundary is non-singular.

‡ The argument used in T.M. is valid in virtue of Alexander's theorem about the separation of a 3-sphere by a 2-sphere (*Proc. National Ac. of Sci.* 10 (1924), 6–8).

J.H.C. Whitehead, "A certain open manifold whose group is unity". *Quart. J. Math.*, 1935.

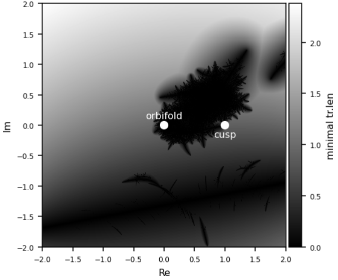




M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, `http://snappy.computop.org`

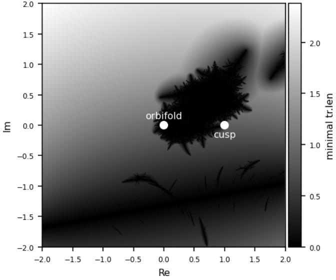
- Knot complements are one important class of geometric structures with interesting groups.
  - **Mostow rigidity theorem**: the corresponding subgroups of $\mathrm{PSL}(2, \mathbb{C})$ are rigid—if $\mathbb{S}^3 \setminus k = \mathbb{H}^3/G$, and $H < \mathrm{PSL}(2, \mathbb{C})$ is isomorphic to G, then $\mathbb{H}^3/G = \mathbb{H}^3/H$ (up to isometry).

- Knot complements are one important class of geometric structures with interesting groups.
  - **Mostow rigidity theorem**: the corresponding subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are rigid—if $\mathbb{S}^3 \setminus k = \mathbb{H}^3/G$, and $H < \mathrm{PSL}(2,\mathbb{C})$ is isomorphic to G, then $\mathbb{H}^3/G = \mathbb{H}^3/H$ (up to isometry).
- Surface groups are important classes of geometric structures, but now are highly non-rigid:



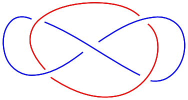
← A linear slice through different geometric structures $\pi_1(S_{2,0}) \to \mathrm{PSL}(2,\mathbb{C})$ where $S_{2,0}$ is the genus 2 surface.

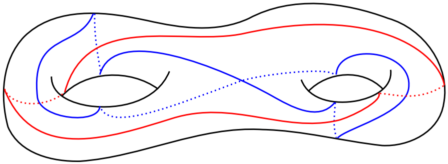
- Knot complements are one important class of geometric structures with interesting groups.
  - **Mostow rigidity theorem**: the corresponding subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are rigid—if $\mathbb{S}^3 \setminus k = \mathbb{H}^3/G$, and $H < \mathrm{PSL}(2,\mathbb{C})$ is isomorphic to G, then $\mathbb{H}^3/G = \mathbb{H}^3/H$ (up to isometry).
- Surface groups are important classes of geometric structures, but now are highly non-rigid:

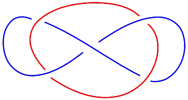


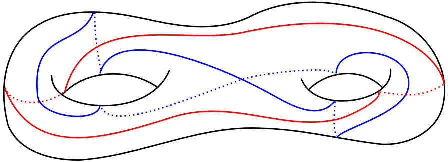
← A linear slice through different geometric structures $\pi_1(S_{2,0}) \to \mathrm{PSL}(2,\mathbb{C})$ where $S_{2,0}$ is the genus 2 surface.

- How do these different kinds of $\mathrm{PSL}(2,\mathbb{C})$-representations fit together? Consider embeddings $k \to S_{2,0}$.

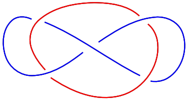


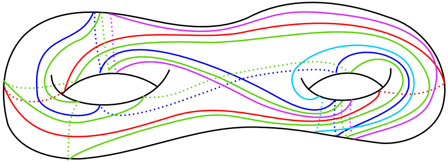






$\pi_1(\text{genus 2 handlebody}) = \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

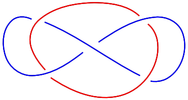


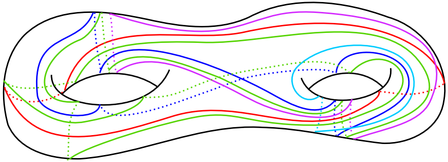


dual graph:
3 non-intersecting curves, each meeting the knot at most once

$\pi_1(\text{genus 2 handlebody}) = \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

green: $X^{-1}X^{-1}Y^{-1}XXY$ pink: $X^{-1}Y$ blue: Y



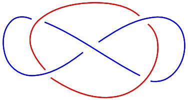


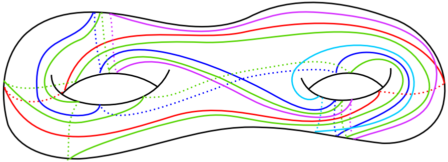
dual graph: 3 non-intersecting curves, each meeting the knot at most once

π_1(genus 2 handlebody) = $\mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

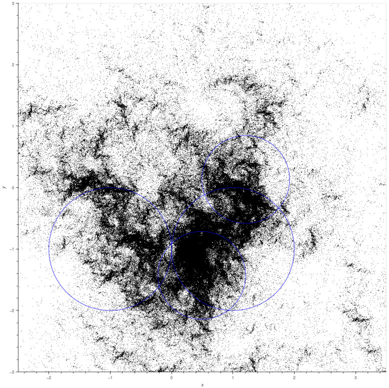
green: $X^{-1}X^{-1}Y^{-1}XXY$ pink: $X^{-1}Y$ blue: Y

$\exists$ 3 interesting reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.





dual graph:
3 non-intersecting curves, each meeting the knot at most once



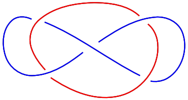
π_1(genus 2 handlebody) = $\mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

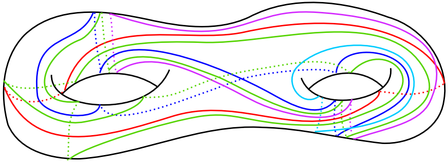
green: $X^{-1}X^{-1}Y^{-1}XXY$ pink: $X^{-1}Y$ blue: Y

$\exists$ 3 interesting reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

$$\rho(X) = \begin{bmatrix} -1.9222 - 0.8789i & 2.1360 + 0.6069i \\ 0.1360 - 1.3931i & -0.3499 + 1.6651i \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 - i & 1 \\ 1 & 1 + i \end{bmatrix}$$





dual graph:
3 non-intersecting curves, each meeting the knot at most once

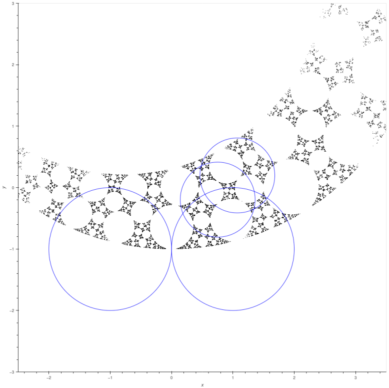
$\pi_1(\text{genus 2 handlebody}) = \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

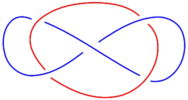
green: $X^{-1}X^{-1}Y^{-1}XXY$ pink: $X^{-1}Y$ blue: Y

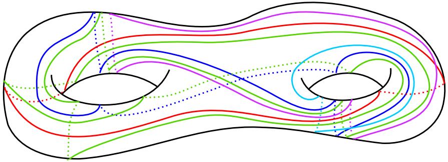
$\exists$ 3 interesting reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

$$\rho(X) = \begin{bmatrix} 0.9222 + 0.8789i & 1.1360 - 0.6069i \\ 0.8640 + 1.3931i & -0.6501 - 1.6651i \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 - i & 1 \\ 1 & 1 + i \end{bmatrix}$$







dual graph: 3 non-intersecting curves, each meeting the knot at most once

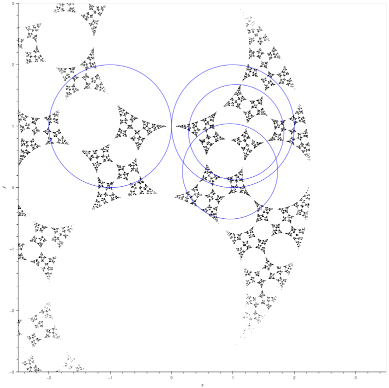
π_1(genus 2 handlebody) $= \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

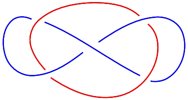
green: $X^{-1}X^{-1}Y^{-1}XXY$ pink: $X^{-1}Y$ blue: Y

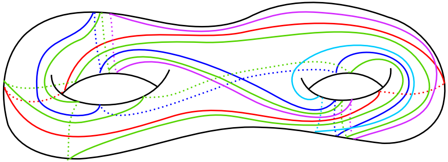
$\exists$ 3 interesting reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

$$\rho(X) = \begin{bmatrix} -0.6501 - 1.6651i & 0.8640 + 1.3931i \\ -1.1360 - 0.6069i & 0.9222 + 0.8789i \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} -1 - i & -1 \\ -1 & -1 + i \end{bmatrix}$$





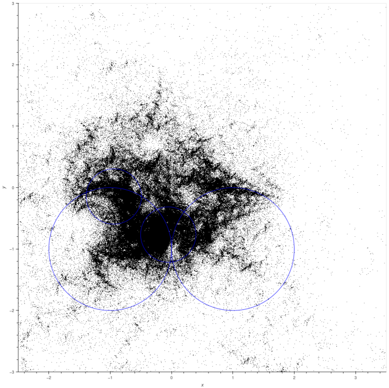


dual graph:
3 non-intersecting curves, each meeting the knot at most once

π_1(genus 2 handlebody) = $\mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

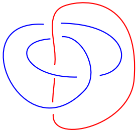
green: $X^{-1}X^{-1}Y^{-1}XXY$ pink: $X^{-1}Y$ blue: Y

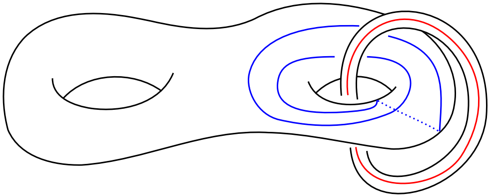
$\exists$ 3 interesting reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

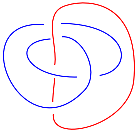
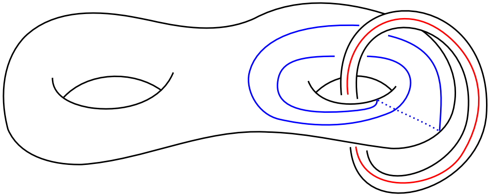


$$\rho(X) = \begin{bmatrix} -1.9222 - 0.8789i & 0.1360 - 1.3931i \\ 2.1360 + 0.6069i & -0.3499 + 1.6651i \end{bmatrix}$$

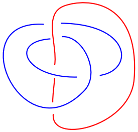
$$\rho(Y) = \begin{bmatrix} 1 - i & 1 \\ 1 & 1 + i \end{bmatrix}$$

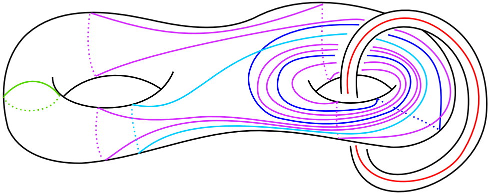




$$\pi_1(\text{compr. body}) = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} = \langle P, Q, M \,:\, [P, Q] = 1 \rangle$$

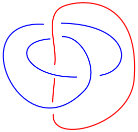


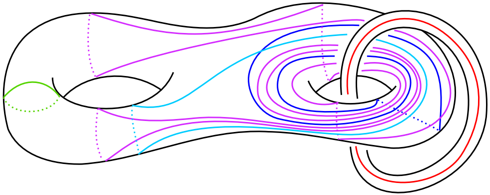


dual graph: 3 non-intersecting curves, each meeting the knot at most once

$\pi_1(\text{compr. body}) = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} = \langle P, Q, M : [P, Q] = 1 \rangle$

green: M pink: $QMQ^{-1}Q^{-1}P^{-1}MQPQ$ blue: QM



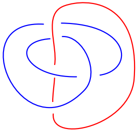


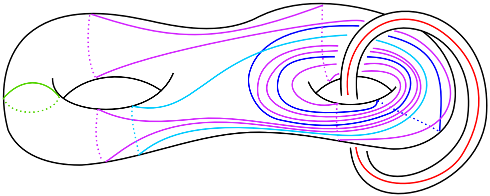
dual graph:
3 non-intersecting curves, each meeting the knot at most once

$\pi_1(\text{compr. body}) = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} = \langle P, Q, M : [P, Q] = 1 \rangle$

green: M pink: $QMQ^{-1}Q^{-1}P^{-1}MQPQ$ blue: QM

$\exists$ 1 interesting rep
$\rho : \langle P, Q, M : [P, Q] = 1 \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$
for all these words.





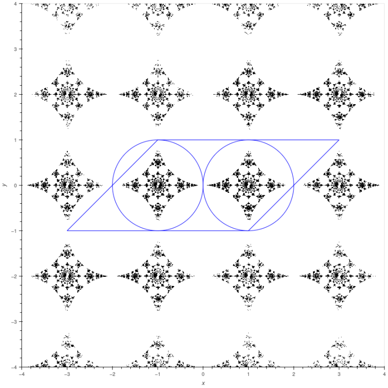
dual graph: 3 non-intersecting curves, each meeting the knot at most once

$\pi_1(\text{compr. body}) = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} = \langle P, Q, M : [P, Q] = 1 \rangle$

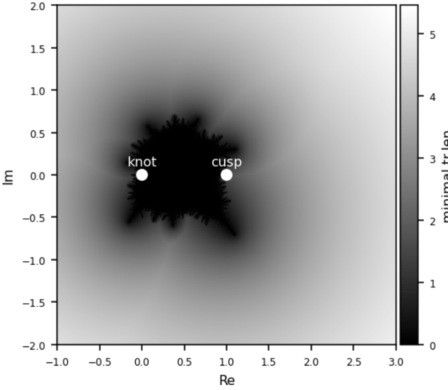
green: M pink: $QMQ^{-1}Q^{-1}P^{-1}MQPQ$ blue: QM

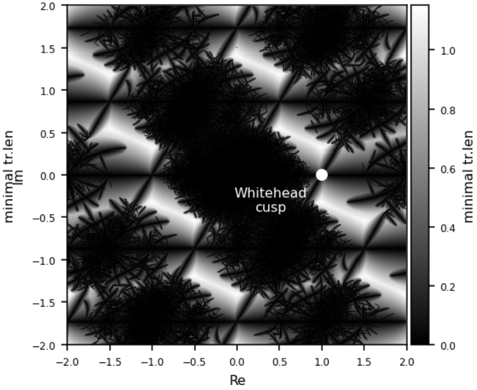
$\exists$ 1 interesting rep
$\rho : \langle P, Q, M : [P, Q] = 1 \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$
for all these words.

$$\rho(P) = \begin{bmatrix} 1 & 10 + 2i \\ 0 & 1 \end{bmatrix}, \ \rho(Q) = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \ \rho(M) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



These methods let us construct reps. $G \to \mathrm{PSL}(2, \mathbb{C})$ that capture the combinatorics of knots (so nice geometry) & are *extremal* in representation spaces.





Two slices through $\mathrm{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \mathrm{PSL}(2, \mathbb{C}))$. The 'cusp' point is the same group in each picture.

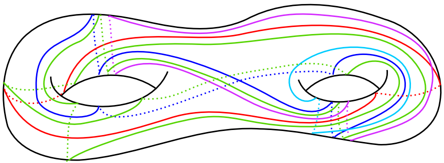
E., "From disc patterns in the plane to character varieties of knot groups", `arXiv:2503.13829 [math.GT]`

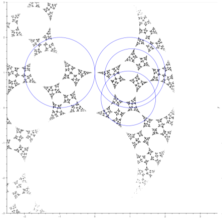
Problem

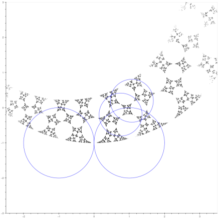
If G is a group and $\rho_0, \rho_1 : G \to \mathrm{PSL}(2, \mathbb{C})$ are reps, can you find a path of reps ρ_t with actions that 'smoothly interpolate' from ρ_0 to ρ_1?

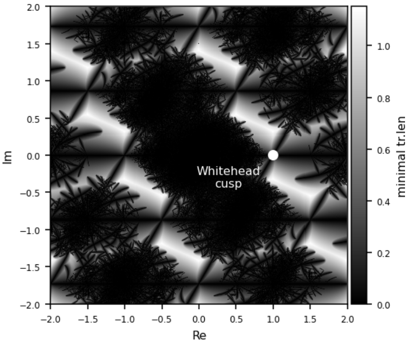
The obstruction is an understanding of paths of *indiscrete* reps. The extremal groups we discussed (called 'cusp groups') can act as nice endpoints for smooth curves of indiscrete representations. We have constructed such paths in a few settings, using a variety of geometric techniques:

- between compression body groups with different numbers of handles: `arXiv:2411.17940 [math.GT]`
- between 'fully augmented link groups' which are rigid, and cusp groups which lie on the boundary of a big deformation space (to appear shortly)
- between 2-bridge link groups and the holonomy groups of manifolds where an upper unknotting tunnel has been drilled (joint with Chesebro and Purcell, in preparation)









- E., *From disc patterns in the plane to character varieties of knot groups*. `arXiv:2503.13829 [math.GT]`
- Jessica Purcell, *Hyperbolic knot theory*. AMS, 2020.
- Albert Marden, *Hyperbolic manifolds*. Cambridge, 2016.
- David Mumford, Caroline Series, David Wright, *Indra's pearls*. Cambridge, 2002.