MTH 1020 Week 12 tutorial

1. Three worked integrals
2. Integration

Theorem (Archimedes)

The volume of the cone with height h and radius r is one-third the volume of the cylinder it lies in.

Proof.

Slice the cone into discs.
By similar triangles, the disc at distance t from the tip of the cone has radius $t/h \cdot r$. Hence it has area $\pi(\frac{tr}{h})^2$.
Therefore the volume of the cone is

$$\int_0^h \pi \left(\frac{tr}{h}\right)^2 dt = \frac{\pi r^2}{h^2} \int_0^h t^2 \, dt = \frac{\pi r^2}{h^2} \cdot \frac{1}{3} h^3 = \frac{\pi}{3} r^2 h.$$

$\square$

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Let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$. Then

$$\begin{aligned}\int \frac{1}{1+x^2}\,dx &= \int \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta \\ &= \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta \\ &= \int d\theta = \theta = \arctan x.\end{aligned}$$

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$$\begin{aligned} u &= \theta & v &= \sin\theta \\ du &= d\theta & dv &= \cos\theta \, d\theta \end{aligned}$$

and we end up with

$$\begin{aligned} \int \theta\cos\theta \, d\theta &= \theta\sin\theta - \int \sin\theta \, d\theta \\ &= \theta\sin\theta + \cos\theta \\ &= x \arcsin x + \cos \arcsin x \\ &= x\arcsin x + \sqrt{1 - x^2}. \end{aligned}$$

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Let $w = 1 - x^2$, so $dw = -2x\, dx$ and

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}}\, dx &= \int \frac{\frac{1}{-2x}x}{\sqrt{w}}\, dw \\ &= -\frac{1}{2}\int \frac{1}{\sqrt{w}}\, dw \\ &= -\sqrt{w} = -\sqrt{1-x^2}. \end{aligned}$$

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Substituting this back into (*) gives

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2}.$$

Key concepts (integration)

1. Definite integral as a Riemann sum. Indefinite integral as anti-differentiation.
2. Fundamental theorem of calculus.
3. Area calculation.
4. Techniques of integration:
    1. Substitution (chain rule)
    2. By parts (product rule)
    3. By partial fractions
    4. By trig identities

A special challenge integral

$$\int \sqrt{\tan x}\, dx$$

MTH 1020 Week 12 tutorial

1. Get into groups of 3-4 people who all prepared a different question in advance.
2. Write your **preferred name** and **ID number** on the whiteboards so I can take attendance
3. Present your prepared question to each other as I come around, you should only take about 5min each for this.
4. Then get started on the other questions **in your groups**.
5. **At the end:** please erase the boards and return any markers etc that you used (you do not need to return the handouts)