MTH 1020 Week 9 tutorial

1. Feedback on A2
2. A work through of an A2 question
3. Reminder of continuity etc
4. Problem sheets!

Feedback on Asst. 2

- **Please scan or use a 'PDF scanner' on your phone rather than the camera.** If what you write is not readable because of blur or shadows then you will lose marks.
- Many people had trouble with the question 'if two planes are not parallel then they intersect in a line". People reverted to vague arguments about 'curves' (what is a curve?) or 'flatness' or tried to use some notion of 'dimension' (which we have not done in this course, so just like in A1 where people tried to use calculus to prove an inequality it should be a hint that you are doing the wrong thing).
- In Q5, many people stated an answer without any proof.

Theorem

If P and Q are two non-parallel planes in $\mathbb{R}^3$, then they intersect in a line.

1. Let P and Q be planes that are not parallel.
2. Without loss of generality, we can assume P is the xy-plane (why?). This means that P and Q have equations

$$P = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$$
$$Q = \{(x, y, z) \in \mathbb{R}^3 : n_1 x + n_2 y + n_3 z = b\}.$$

   where $\vec{n} = (n_1, n_2, n_3)$ is a nonzero vector orthogonal to Q, and $b \in \mathbb{R}$.
3. Since P and Q are not parallel, it cannot be that $\vec{n} = (0, 0, n_3)$: one of n_1 or n_2 is non-zero. We proceed in two cases; the first case is that $n_1 \neq 0$ (the case for $n_2 \neq 0$ is very similar).

Standing assumptions

Standing assumptions: planes are $P = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$, $Q = \{(x,y,z) \in \mathbb{R}^3 : n_1 x + n_2 y + n_3 z = b\}$; $n_1 \neq 0$.

Suppose that $\vec{v} = (x,y,z) \in P \cap Q$.

1. Since $\vec{v} \in P$, $z = 0$. Since $\vec{v} \in Q$, $b = n_1 x + n_2 y$. Since $n_1 \neq 0$, $x = (n_2 y - b)/n_1$.
2. Therefore,
$$\vec{v} = \left(\frac{b - n_2 y}{n_1}, y, 0\right) = \left(\frac{b}{n_1}, 0, 0\right) + y\left(-\frac{n_2}{n_1}, 1, 0\right).$$
3. Let L be the line
$$L = \left\{\left(\frac{b}{n_1}, 0, 0\right) + t\left(-\frac{n_2}{n_1}, 1, 0\right) : t \in \mathbb{R}\right\}.$$

   We just showed that if $\vec{v} \in P \cap Q$, then $\vec{v} \in L$; i.e. $P \cap Q \subset L$.

Standing assumptions

Planes are $P = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$,
$Q = \{(x, y, z) \in \mathbb{R}^3 : n_1x + n_2y + n_3z = b\}$; $n_1 \neq 0$.

Previous slide

We defined $L = \left\{ \left(\frac{b}{n_1}, 0, 0 \right) + t \left(-\frac{n_2}{n_1}, 1, 0 \right) : t \in \mathbb{R} \right\}$ and showed that $P \cap Q \subset L$.

We will now show that $L \subset P \cap Q$.

1. Every element of L lies in P, since the z-ordinate of every point in L is 0.
2. Also, if $t \in \mathbb{R}$, then
$$\frac{b}{n_1}n_1 - t\frac{n_2}{n_1}n_1 + tn_2 + 0n_3 = b - tn_2 + tn_2 = b :$$
i.e. every point of L satisfies the equation for Q in (2), and so $L \subset Q$.
3. Therefore $L \subset P \cap Q$.

Theorem

If P and Q are two non-parallel planes in $\mathbb{R}^3$, then they intersect in a line.

Proof.

We just showed that if $n_1 \neq 0$ and L is the line

$$L = \left\{ \left(\frac{b}{n_1}, 0, 0 \right) + t \left(-\frac{n_2}{n_1}, 1, 0 \right) : t \in \mathbb{R} \right\}.$$

then $L \subset P \cap Q$, and $P \cap Q \subset L$. So $P \cap Q = L$.
The case $n_2 \neq 0$ is essentially the same but you get

$$L = \left\{ \left(\frac{b}{n_2}, 0, 0 \right) + t \left(1, -\frac{n_1}{n_2}, 0 \right) : t \in \mathbb{R} \right\}.$$

□

Theorem

The function $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous everywhere on $\mathbb{R}$ *but not differentiable at* $x = 0$*.*

Proof of continuity.

Since x and $\sin(1/x)$ are continuous for all $x \neq 0$, their product is continuous for all $x \neq 0$. Now observe that, for all x,

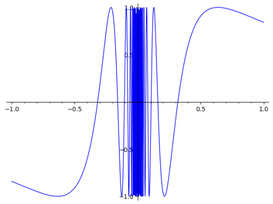
$$-x \leq x \sin(1/x) \leq x.$$

As $x \to 0$, $-x \to 0$. So $x \sin(1/x) \to 0$ (by the **squeeze theorem**), i.e. $\lim_{x \to 0} f(x) = 0 = f(0)$. □

Proof of (non)-differentiability.

When $x \neq 0$, we can apply the usual differentiation rules to find $f'(x) = \sin(1/x) - (1/x)\cos(1/x)$. We now show that f is not differentiable at 0. In other words we show that $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. We substitute for f:

$$\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h\sin(1/h) - 0}{h} = \lim_{h \to 0} \sin(1/h).$$



This limit does not exist, because as $h \to 0$ the function $\sin(1/h)$ oscillates infinitely often between ± 1, and does not approach a single value. $\square$

Key concepts so far

1. What is a limit?
2. **Definitions:** continuity, differentiability
3. Usual differentiation laws (chain and product laws, implicit differentiation) and derivatives of basic functions (x^n, $\sin x$, $\cos x$, $\exp x$, $\log x$, $|x|$)
4. **Intermediate value theorem:** if $f : [a, b] \to \mathbb{R}$ is a function, $f(a) < 0$, and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$
5. Extrema: definition of critical point, inflection point, concavity.

MTH 1020 Week 9 tutorial

1. Get into groups of 3-4 people who all prepared a different question in advance.
2. Write your **preferred name** and **ID number** on the whiteboards so I can take attendance
3. Present your prepared question to each other as I come around, you should only take about 5min each for this.
4. Then get started on the other questions **in your groups**.
5. **At the end:** please erase the boards and return any markers etc that you used (you do not need to return the handouts)