Deformations of orbifold holonomy groups

(and applications to arithmetic and algebra)

Alex Elzenaar

Monash University,
Melbourne

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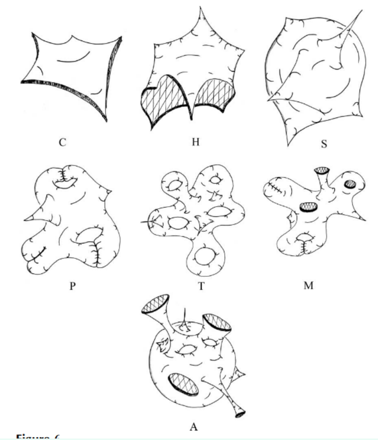
Len Lye, still frame from *Rainbow Dance*, 1936

A **hyperbolic orbifold** is the quotient of $\mathbb{H}^3$ by a discrete group Γ of isometries.

Possible coarse geometry on these objects includes:

- pieces of Riemann surface on the boundary (from the action of Γ on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$)
- deleted arcs and loops (from parabolic elements of Γ)
- cone arcs (from torsion elements of Γ)
- reflection planes (from reflections in Γ)

We will always restrict to *non-elementary, orientation-preserving, discrete* groups in $\mathrm{Isom}(\mathbb{H}^3)$.



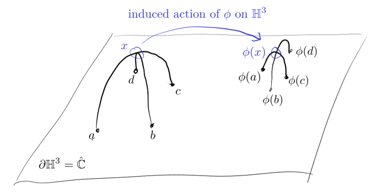
S.T. Hyde, S. J. Ramsden, and V. Robins. “Unification and classification of two-dimensional crystalline patterns using orbifolds”. *Acta Cryst.* A**70** (2014), pp.319–337.

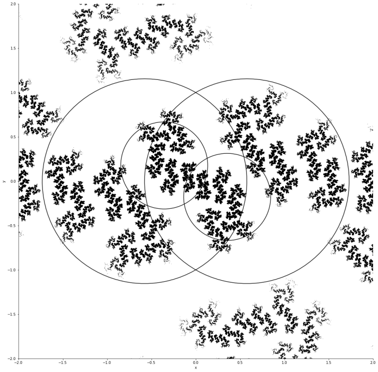
A discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ is called a **Kleinian group**. Since $\mathbb{H}^3$ is negatively curved, isometries act as conformal maps on its visual boundary, which is the Riemann sphere.

Dynamics of the action of a Kleinian group Γ are complicated. The Riemann sphere is partitioned into a limit set $\Lambda(\Gamma)$ and its complement $\Omega(\Gamma)$. The conformal boundary of $O_\Gamma = \mathbb{H}^3/\Gamma$ is $\partial O_\Gamma = \Omega(\Gamma)/\Gamma$.

The group Γ is the holonomy group of O_Γ. Each component $S \subset \partial O_\Gamma$ gives a map $\pi_1(S) \to \Gamma$ (unlikely to be injective).

How much of $\Gamma \leq \mathrm{PSL}(2, \mathbb{C})$ can be recovered from the map $\prod_{S \subset \partial O_\Gamma} \pi_1(S) \to \Gamma$?





Teichmüller theory in 30 seconds

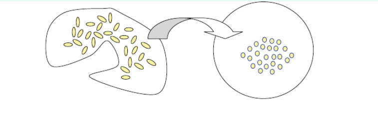
Isomorphism of Riemann surfaces: If Σ and $\tilde{\Sigma}$ are marked Riemann surfaces, and $\phi : \Sigma \to \tilde{\Sigma}$ is a conformal homeomorphism that preserves the marking, then we say that Σ and $\tilde{\Sigma}$ are isomorphic.

The **Teichmüller space** of Σ is the set (mod conformal maps) of *quasi*conformal homeomorphisms $\phi : \Sigma \to \tilde{\Sigma}$.

Sporadic examples: $T(\mathbb{T}) = T(S_{1,1}) = T(S_{0,4}) = \mathbb{H}^2$.

For everything else, the Teichmüller space is only hyperbolic after collapsing subsets where lots of disjoint curves are short (Masur and Minsky, 1999).

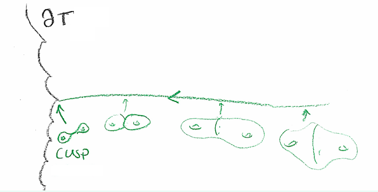
A quasiconformal map:



A measurable conformal structure transported to the usual round structure by a quasiconformal mapping

K. Astala, T. Iwaniec, G.J. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, p. 162. Princeton Uni. Press (2009)

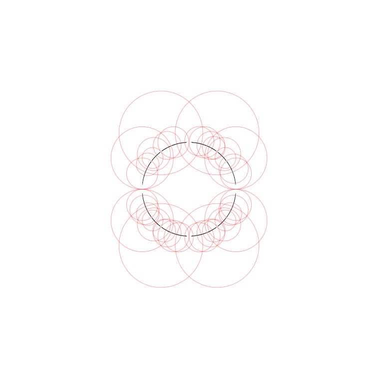
A path in $T(S_{2,0})$:



Marden–Tukia isomorphism theorem: If Γ and $\tilde{\Gamma}$ are Kleinian, non-elementary, and geometrically finite, and $\phi : \Omega(\Gamma) \to \Omega(\tilde{\Gamma})$ is a conformal map that conjugates Γ to $\tilde{\Gamma}$ where it's defined, then ϕ extends to an isometry $\tilde{\phi}$ of $\mathbb{H}^3$ which also conjugates Γ to $\tilde{\Gamma}$.

The **quasiconformal deformation space** of Γ is the set (mod conformal maps) of *quasi*conformal homeomorphisms $\phi : \Omega(\Gamma) \to \hat{\mathbb{C}}$ such that $\gamma \mapsto \phi\gamma\phi^{-1}$ is a faithful, type-preserving, discrete representation of Γ.

This space is the product of the Teichmüller spaces $\prod_{S\subset\partial O_\Gamma} T(S)$, mod the twist group of O_Γ.



Limit sets under holomorphic flow

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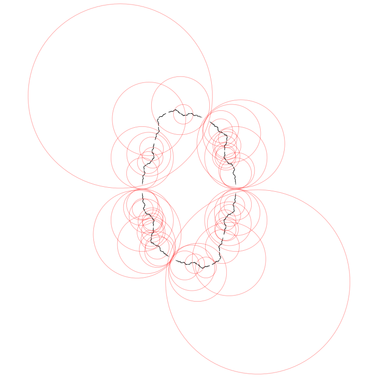

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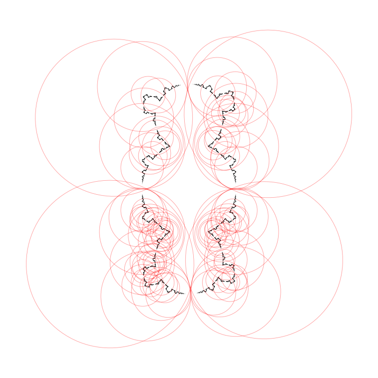

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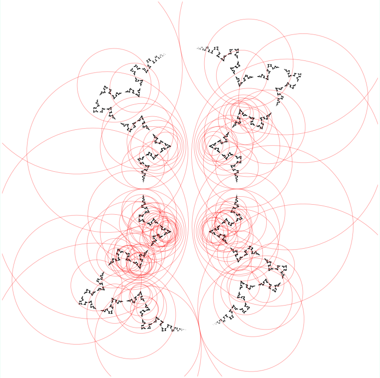

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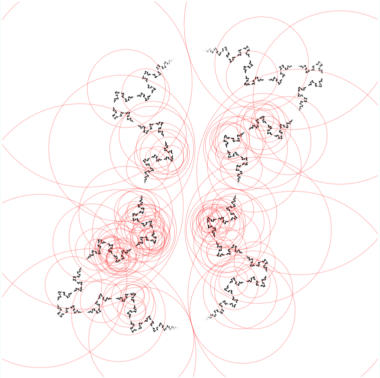

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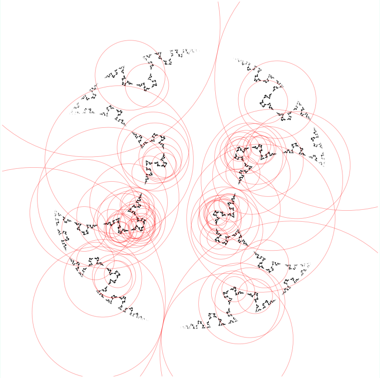


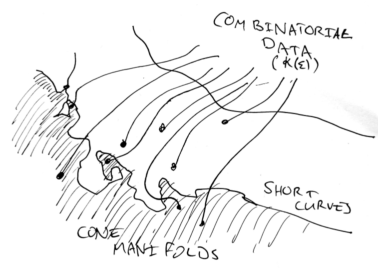
Limit sets under holomorphic flow

The quasiconformal deformation space of Γ lies inside the $\mathrm{PSL}(2,\mathbb{C})$ character variety $X(\Gamma)$ as an open but very wild set.

Giving a concrete realisation of this set (e.g. for discreteness testing or group/orbifold recognition) is practically impossible.

We have two kinds of effective theorems:

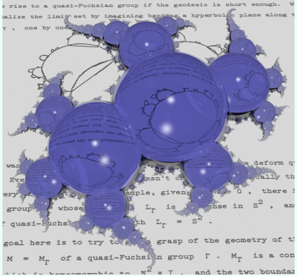
1. Give coarse bounds on the deformation space which are far away from the boundary but easy to compute. (Skipped in today's talk, slides in appendix.)
2. Give locii tending to the boundary within the quasiconformal deformation space which are 'local', based on controlling short curves.



The conformal boundary ∂O_Γ does not have hyperbolic geometry, but it is homotopic to a totally geodesic pleated surface embedded in O_Γ.

The **convex core** of O_Γ is the quotient

$$\mathrm{CC}(O_\Gamma) = \left(\mathrm{conv}_{\mathbb{H}^3} \Lambda(\Gamma)\right) / \Gamma.$$

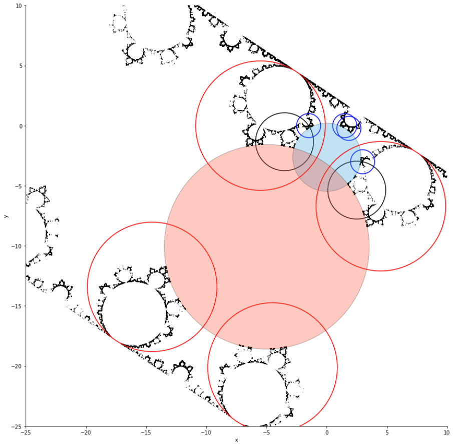


$\mathrm{CC}(O_\Gamma)$ is a coarse invariant of the orbifold, conjectured to recover the whole thing.

J. Brock and E. Dumas. *Bug on notes of Thurston*, 2006.
`https://www.dumas.io/poster/`

A. Fomenko. *A retraction of a space onto a subspace of it*, 1974.

A **F-peripheral subgroup** of Γ is a subgroup Π which leaves invariant a round disc $\Delta \subset \Omega(G)$.

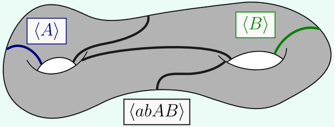


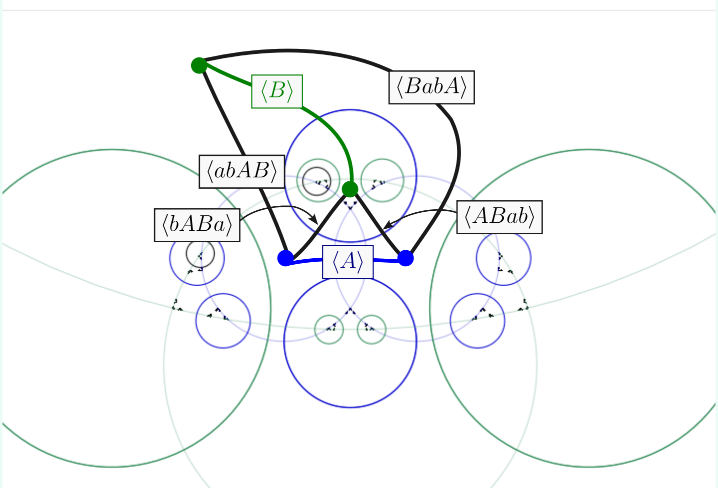
periph'er|y̌, n. Bounding line **esp. of round surface**; external boundary or surface. Hence ~AL a., ~alLY adv. [f. LL f. Gk PERI(*pheria* f. *pherō* bear) circumference]

The concise Oxford dictionary, 5th ed., p. 904.

Here, peripheral in the sense of the limit set of Γ, and in the sense of giving the hyperbolic structure to a peripheral hyperbolic plane (flat piece of $\partial CC(O_\Gamma)$) in O_Γ.

What do the peripheral groups of a Schottky group look like?





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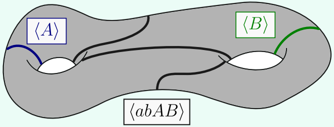

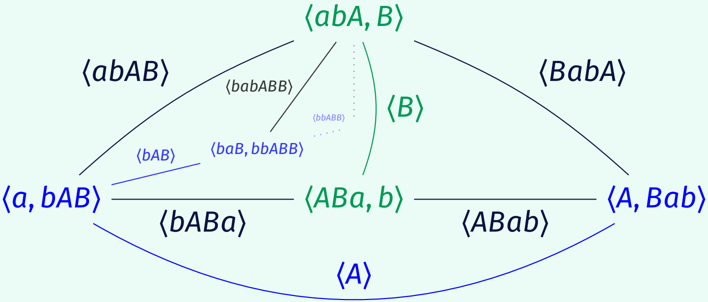

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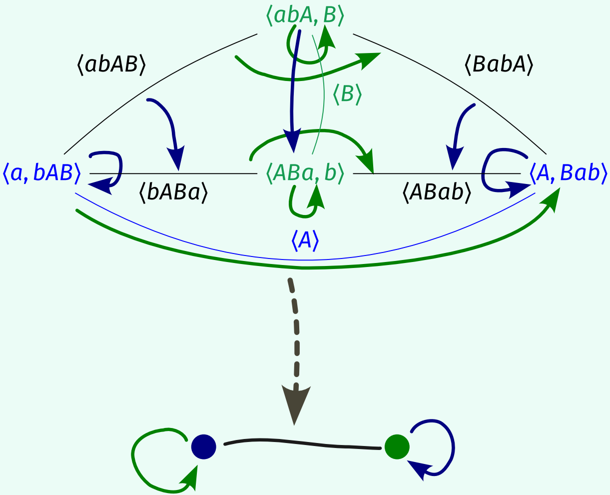
Action of Γ on this graph encodes algebraic information: it is not quite an amalgamated product but it's close.

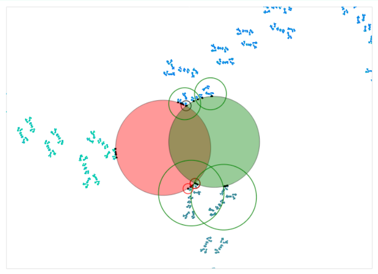
In an intriguing 1978 paper Wielenberg studied the actions of external elements $n \in \mathrm{PSL}(2, \mathbb{C})$ on the peripheral subgroup graphs of some groups Γ. You get extensions $\hat{\Gamma} = \langle \Gamma, n \rangle$ where convex core boundary pieces are glued together and rank two parabolic subgroups (knot components) pop out.

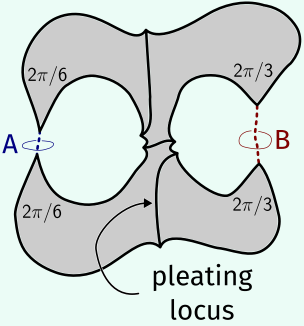


Let $\Gamma = \langle A, B \rangle$ be a Kleinian group where A and B are elliptic or parabolic and $\partial \mathcal{O}_\Gamma \neq \emptyset$. If $\partial CC(\mathcal{O}_\Gamma)$ is pleated along a simple closed curve, then we have a system L of 3 curves on the abstract genus 2 surface (pull back along the rep $\pi_1(S_{0,2}) \to \Gamma$ we talked about at the start of the talk).

Very special situation: Γ admits two (maximal!) F-peripheral subgroups which are non-conjugate and which uniformise the flat pieces of $\partial CC(\mathcal{O}_\Gamma)$. We call these an ***L*-circle chain** if they exist.

The locus of all groups in the deformation space admitting an L-circle chain is called the ***L*-pleating variety**.





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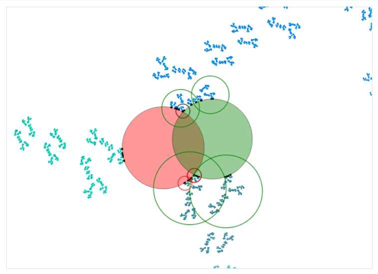

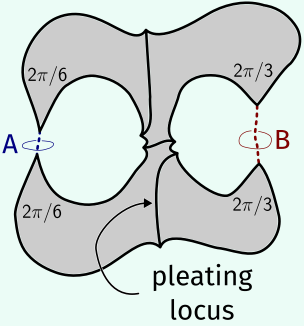

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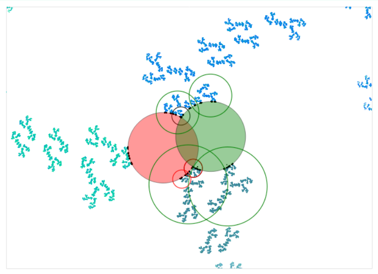

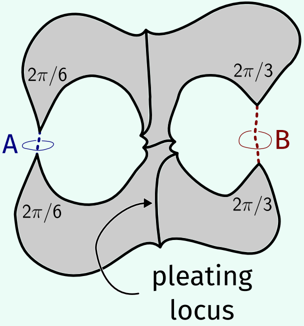

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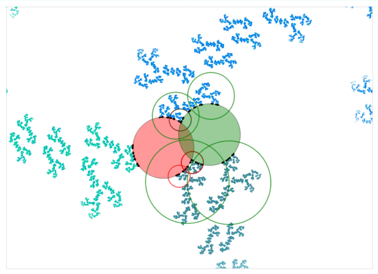

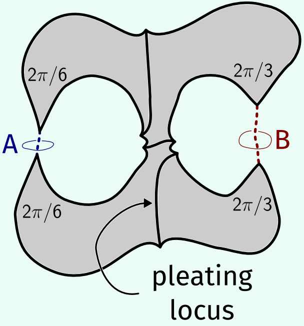

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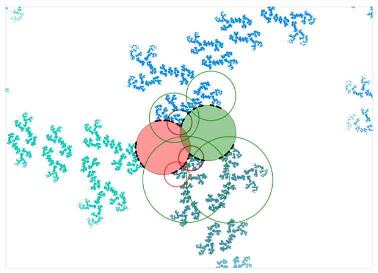

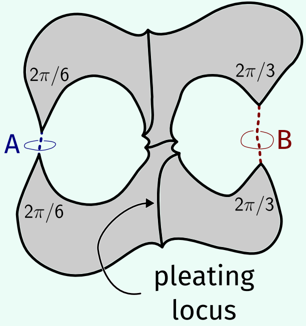

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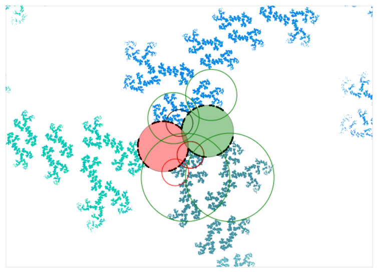

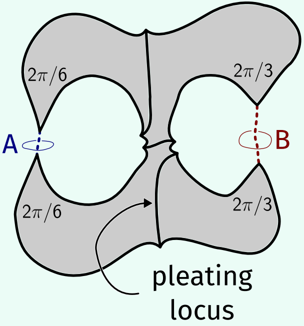

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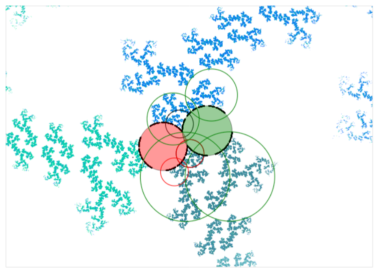

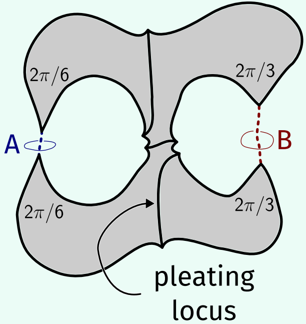

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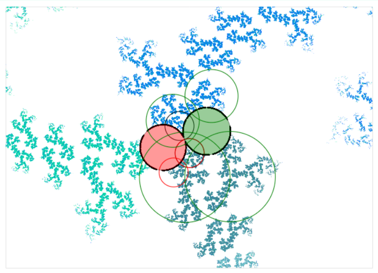
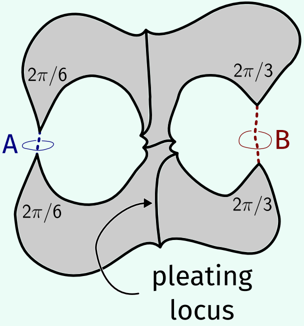
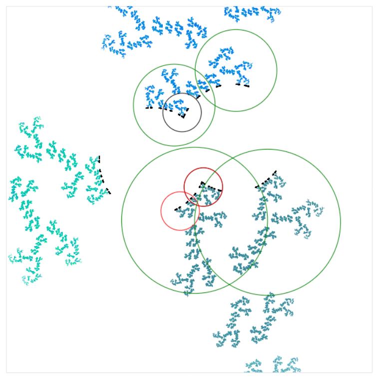




Let Γ be a group with an L-circle chain and holomorphically deform Γ to $\tilde{\Gamma}$. By the λ-lemma, small deformations stay inside $\mathrm{QH}(\Gamma)$. But we can also apply the λ-lemma to the F-peripheral subgroups; after deforming they are quasi-Fuchsian but still peripheral (and still give peripheral surfaces with the same topology).

Their existence is still a certificate of discreteness (it shows $\Omega(\tilde{\Gamma}) \neq \varnothing$). Bounding the deformation space of these peripheral groups and pulling back to $\mathrm{QH}(\Gamma)$ gives computable open sets *provably inside deformation space*.



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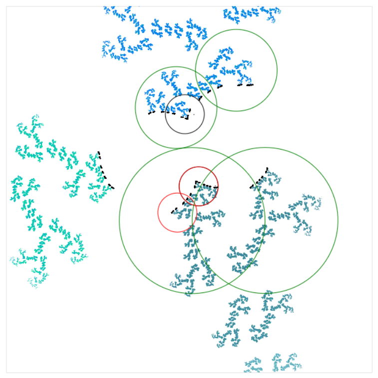

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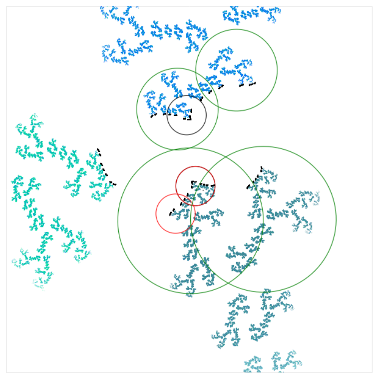

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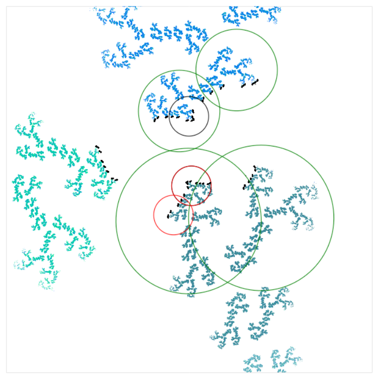

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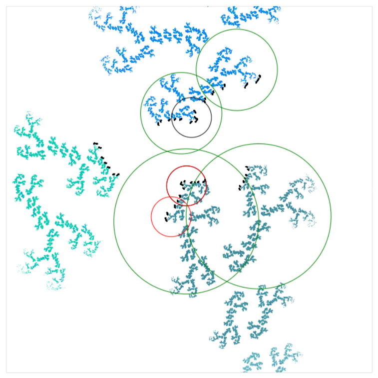

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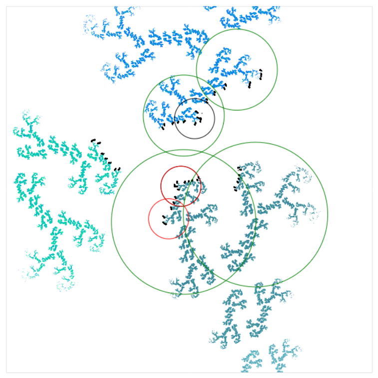

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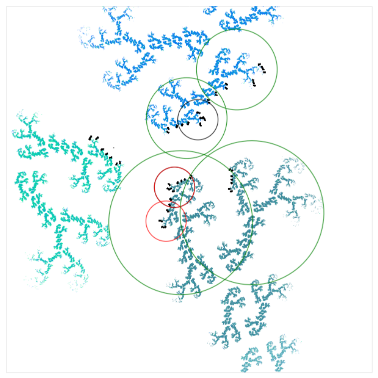

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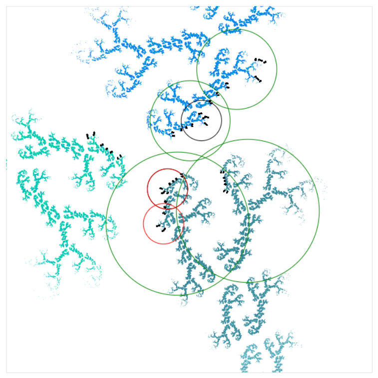
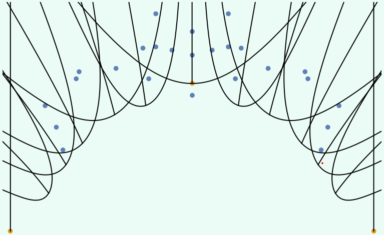
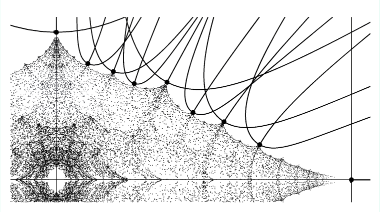


Summary: if we can bound the original deformation space by algebraic inequalities, then we know how far we can squish peripheral subgroups so that the convex core does not collapse; i.e. how far we can move off pleating varieties before we hit $\partial\,\mathrm{QH}(\Gamma)$.





Theorem (E.–Martin–Schillewaert, 2021+)

In the case of holonomy groups isomorphic to $\mathbb{Z}/a\mathbb{Z} * \mathbb{Z}/b\mathbb{Z}$, there are computable semi-algebraic sets filling out the (1 complex dimensional) deformation space.

This gives a countable list of inequalities to check. If lucky, this will give a certificate of membership within human lifetime. In practice it works(!).

An arithmetic group is an algebraic group with only integers as coefficients in its defining polynomials (so coordinates will be algebraic integers). Arithmetic Kleinian groups are finite covolume. A thin group is an infinite index subgroup of an arithmetic group. Their 3-folds are infinite volume.

Theorem (E.–Martin–Schillewaert, Maclachlan–M., Chesebro–M.–S.)

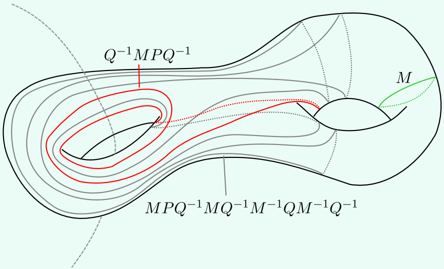
There are ~150 thin groups in $\mathrm{PSL}(2, \mathbb{C})$ *that are generated by two elliptic elements and don't split as free products. These groups can be listed explicitly.*

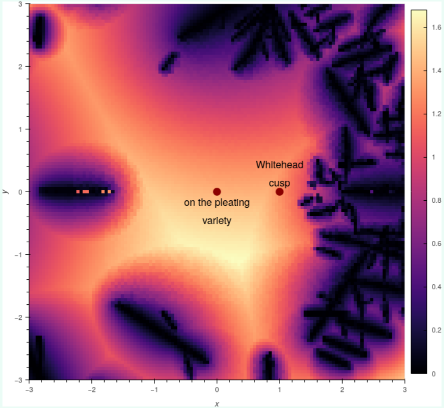
Sketch of proof.

Maclachlan–M. after Flammang–Rhin give a finite list of possible parameters which could be arithmetic or thin groups. Enumerate all of these possibilities and check whether they are in the closure of the deformation space of 4-marked sphere groups (i.e. split as free products) or not, using certificates of freeness (E.–M.–S.) and non-freeness (Chesebro–M.–S.).

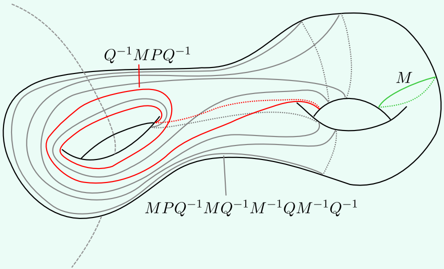
A generalisation... drilling the Whitehead link

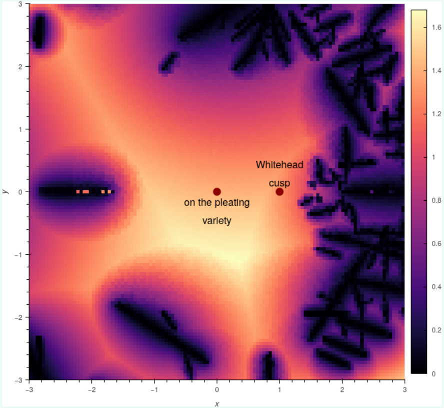
$$G = \left\langle P = \begin{bmatrix} 1 & t(2i\sqrt{3}-1)+i(1-t)\sqrt{4\sqrt{5}-1}+3 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \frac{1}{2}t(2i\sqrt{3}-1)+\frac{i}{2}(1-t)\sqrt{4\sqrt{5}-1}-\frac{3}{2} \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} \frac{1}{2}(t-3) & \frac{1}{4}t^2-\frac{3}{2}t+\frac{5}{4} \\ 1 & \frac{1}{2}(t-3) \end{bmatrix} \right\rangle$$





$$G = \left\langle P = \begin{bmatrix} 1 & f(t) \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & g(t) \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} h(t) & h(t)^2 - 1 \\ 1 & h(t) \end{bmatrix} \right\rangle$$





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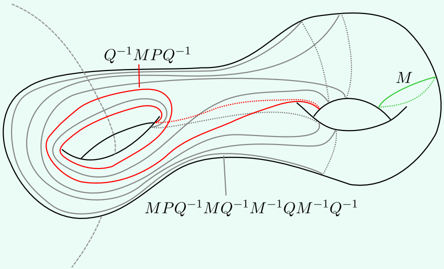

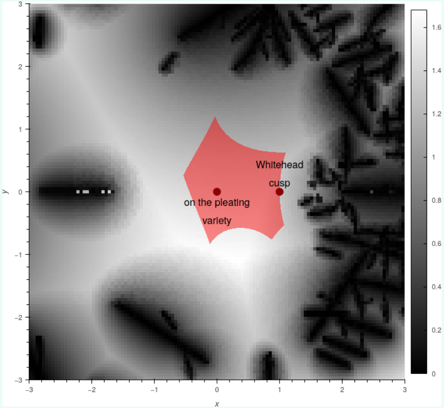

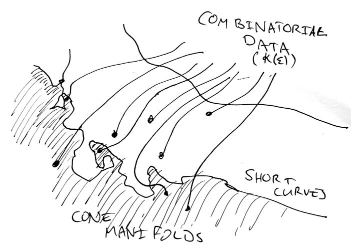

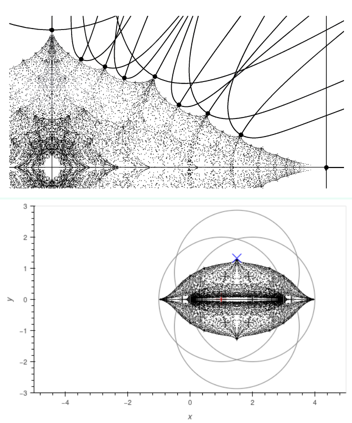

Len Lye, still frame from *Rainbow Dance*, 1936.

Bedtime reading

- A.J.E., Gaven Martin, and Jeroen Schillewaert, "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021–22 MATRIX annals*. Springer, 2024, pp. 31–74.
- —, G.J.M., and J.S. *Deformation spaces of Kleinian groups generated by two elements of finite order*. Preprint to appear shortly.
- —. *Peripheral subgroups of function groups*. Preprint to appear early 2025.
- Eric Chesebro, G.J.M., and J.S. *2-elliptic generated Kleinian groups are Heckoid*. Preprint to appear shortly.
- Linda Keen and Caroline Series. "The Riley slice of Schottky space". *Proc. LMS* (1994).

Coarse bounds

Following Lyndon–Ullman, we have coarse bounds on the entire deformation space of $\mathbb{Z}/a\mathbb{Z} * \mathbb{Z}/b\mathbb{Z}$ at once. We give just one example, $(a, b) = (2, 3)$.

A possible quantisation of $\mathrm{SL}(2, \mathbb{Z})$ which is related to the q-rational integers of Morier-Genoud–Ovsienko–Veselov is

$$\mathrm{SL}(2, \mathbb{Z})_q = \left\langle \begin{bmatrix} -q & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ q & -q \end{bmatrix} \right\rangle < \mathrm{GL}(2, \mathbb{Z}[q^{\pm 1}]).$$

When $q = -1$, you get $\mathrm{SL}(2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \mathcal{B}_3$.

Question. For which values of $q \in \mathbb{C}$ is the substitution map $\mathrm{SL}(2, \mathbb{Z})_q \to \mathrm{SL}(2, \mathbb{C})$ faithful?

O. Jones. *The grammar of ornament*, Bernard Quaritch, 1910. Taffel XXX, No. 42.

Coarse bounds

Theorem (E.–Gong–Martin–Schillewaert)

The realisations for $q \in \mathbb{C}$ of $\mathrm{SL}(2,Z)_q$ are faithful within a closed semialgebraic region strictly containing the region conjectured by Morier-Genoud, Ovsienko, and Veselov, except at one point where both bounds are tight; our region is

$$3 \leq \left|(q^{1/2} - q^{-1/2}) \pm \sqrt{(q^{1/2} - q^{-1/2})^2 + 3}\right|.$$

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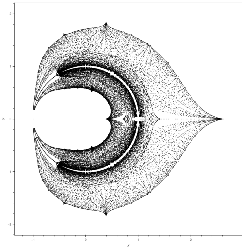

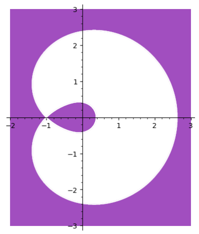

(Purple shading is the interior of the bounded region.)