Variations on a theme of Wielenberg

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Abstract

We will slice up some knot groups and glue them back together.

Contents

§1. Introduction

We will place some of the work of Wielenberg on link groups in the Picard group [Wie78] in more modern language, following in part Examples 59–62 of Krushkal', Apanasov, and Gusevskii [KAG86], emphasising the point of view that these groups may be cut naturally along essential surfaces (not necessarily Fuchsian subgroups) to obtain groups of the second kind. In addition, we give some additional examples and non-examples that are suggestive (the titular *Variations*.)

Rougly speaking, there are always exactly two proofs for the results we will state below. The first method, which is found in Wielenberg, is to physically construct fundamental domains for groups and their extensions and then compute with edge cycles (i.e. apply the Poincaré polyhedron theorem in a very concrete way). The second method, which is used by Krushkal', Apanasov, and Gusevskii in their exposition of Wielenberg's paper, is essentially algebraic and relies heavily on Riley's theorem

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[Ril75] (see also [KAG86, §IV.4, problem 86]) which states that if a discrete subgroup $G \leq \mathsf{PSL}(2, \mathbb{C})$ is of the first kind and has the same presentation as the Wirtinger presentation of a link (with parabolic generators) then it is the link complement holonomy group.

The problem with both methods is the level of ingenuity and technical skill needed to actually construct examples: either one must carefully arrange circles and lines on the plane and carefully work out what the quotient manifold is (which in many cases I cannot do visually, so I must fall back to Riley's theorem); or one must set up a number of algebraic equations using topological information, hope that they have a solution, and hope that the correct (discrete) solution is easy to spot among the many indiscrete solutions.

It would be nice to have idiot-proof methods to produce infinite index extensions of surface groups (and infinite index subgroups of link groups). It is not hard to do ad-hoc constructions where a link is cut along a Seifert surface of a sublink, but there are many questions which such constructions do not answer: for example, 'what different knots can be produced by gluing peripheral surfaces in one fixed group in different ways?'. Examples below show that you can take a single group G with thrice-punctured sphere boundary components and glue them together to get different links; these must always have the same volume (equal to the convex covolume of G) and so there are only finitely many possible results since hyperbolic volume is finite-to-one, but can all of them be obtained from the same starting group G? Another way of phrasing the question: instead of giving proofs of the homeomorphism/isometry type of the supergroup $\Gamma = \langle G, f \rangle > G$ by using Poincaré's polyhedron theorem or Riley's isomorphism theorem, can you give one using the isometry type of G together with knowledge of the action of the extension element f on $\Omega(G)$ in some general way? (Of course phrased like this the answer is yes, but I want something concrete in each case, and often the action of f on $\Omega(G)/G$ is not clear and explicit to me.)

On the horizon I see the following very broad questions:

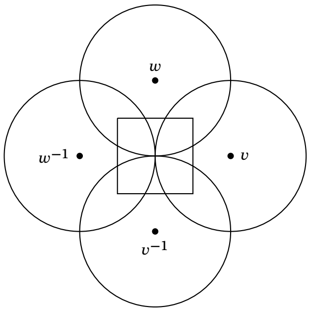
I. Groups of the second kind are very easy to construct generically, since they are flexible. But their parameter spaces are very large. Can you, instead of studying their character variety of representations $\rho : G \to \mathsf{PSL}(2, \mathbb{C})$, study their character variety of representations $\rho : G \to \Gamma$ where Γ is some discrete group of the first kind with highly controlled geometry? Given a fixed G (e.g. a Riley group), what is the correct Γ to try to embed $\mathrm{QH}(G)$ into?

II. Groups of the first kind are very annoying to construct, since they are very unforgiving: one must often fall back on tedious combinatorics. Can we construct groups of the first kind with specified quotient geometry by taking flexible groups of the second kind, and taking controlled infinite index extensions in a mechanical (or at least semi-algorithmic) way?

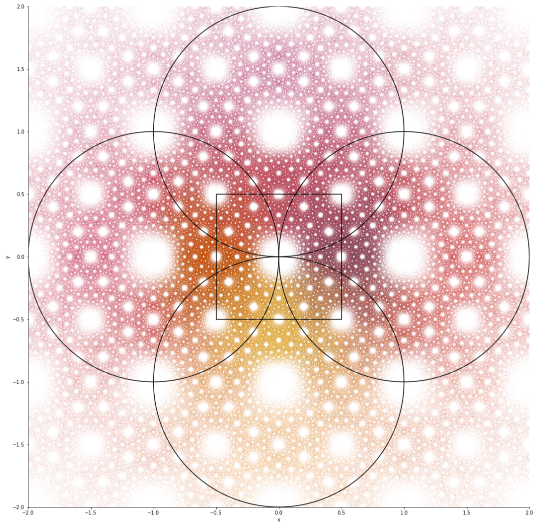
Throughout this note I have placed many questions. These range from 'I know how to do this but I can't be bothered to work out the details' (most of the time because it involves tedious algebraic manipulations), to 'give a visual or other proof of a result of Wielenberg', to 'I have no idea how to solve this and it is likely hard'.

I will close this introduction with two remarks on what will not be discussed in this note.

Remark. A close reading of Wielenberg's original paper does not reveal any arithmetic: that is, the embedding of all these groups in $\Gamma = \mathsf{PSL}(2, \mathbb{Z}(i))$ is a framing device more than it is a serious study of subgroups in Γ. Many people have subsequently studied the arithmetic, but from the point of view of the deformation theory trying to study the number theory is not much of a help (after all, the whole point is to replace rigid objects with flexible objects, and arithmetic objects are some of the most inflexible objects you can write down). If one wants to replace $\mathbb{Z}(i) = O_1$ with rings of integers of more general quadratic extensions, the possible links that arise are well-understood; see e.g. [MR03, Theorem 9.2.2] and surrounding discussion. From my point of view the most interesting questions



(a) Fundamental domain projection.



(b) An approximation to the limit set.

Figure 1: The Picard group, $\Gamma = \mathsf{PSL}(2, \mathbb{Z}(i))$.

in this direction are of the form 'what is the natural deformation space[1] which the link groups in $\mathsf{PSL}(2, O_d)$ live inside'? This is very similar in spirit to the two broad questions I and II above.

Remark. Many of the groups which we will see arise from taking geometric limits of twist links; this is studied e.g. in §4.9 of [Mar16]. The unfortunate fact is that this procedure is highly degenerate: hyperbolic 3-orbifolds exist only to allow us to draw convenient pictures of infinite groups, and this situation is one where the limit of these pictures is not reflective of the actual structure of the limit of groups. In reality rather than studying these links as geometric limits, one should study them in their natural representation space which is higher-dimensional than one would expect from just naïvely looking at the topology of the sequence as a knot theorist would. Detailed work on this is in progress but is out of scope of this exploratory document.

§2. The Picard group

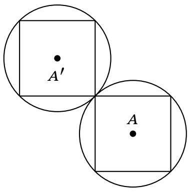
Let $\Gamma = \mathsf{PSL}(2, \mathbb{Z}(i))$ be the Picard group; this is also called the first Bianchi group $\mathsf{PSL}(2, O_1)$ [MR03, §1.4.1], see also [Fin89]. As a subgroup of $\mathbb{M}$ it is generated by the four parabolic matrices

$$t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad u = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix};$$

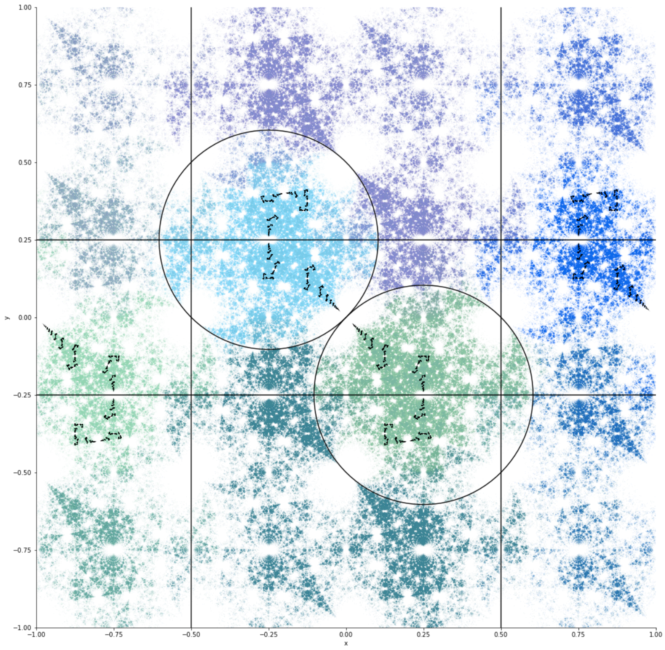
we observe that $\langle t, v \rangle$ and $\langle u, w \rangle$ are both copies of the modular group in $\mathbb{M}$. It will also be convenient to consider the two involutions

$$a = vtv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad l = uau^{-1}aua = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

[1] I mean something more general than a quasiconformal deformation space—I mean a parameterisation of a subset of a character variety with additional structures e.g. cell decompositions carrying coarse geometry, can discuss further if there is interest and I have placed some questions that implicitly head in this direction into the main text.



(a) Fundamental domain projection.



(b) An approximation to the limit set.

Figure 2: The Whitehead link group, $\pi_1(k_{\text{wh.}})$.

(2.1) Proposition ([Fin89, Theorem 4.4.1]). *A fundamental domain for the action of the Picard group on $\mathbb{H}^3$ is the set*

$$\{(z,t) \in \mathbb{C} \times \mathbb{R} : |\Re z| + |\Im z| < 1, |z \pm 1|^2 + t^2 > 1, |z \pm i|^2 + t^2 > 1\}$$

shown in figure 1a. It splits as an amalgamated product

$$\Gamma \simeq G_1 *_H G_2$$

where $H = \langle t^{-1}a^{-1}, a\rangle$ is a copy of the modular group $\mathsf{PSL}(2,\mathbb{Z})$ and where

$$G_1 = \langle lau^{-1}, H\rangle \simeq S_3 *_{\mathbb{Z}/3\mathbb{Z}} A_4$$
$$G_2 = \langle al, H\rangle \simeq S_3 *_{\mathbb{Z}/2\mathbb{Z}} D_2.$$

(2.2) Question (Warmup!). What is the topology of $O = \mathbb{H}^3/\Gamma$? How does the amalgamated product correspond to a decomposition of O?

Remark. The volume of O is computed in [MR03, §11.1].

§3. The Whitehead link

This section is Example 1 of [Wie78], also Example 59 of [KAG86].

(3.1) Proposition. *The Whitehead link group embeds into Γ as follows:*

$$\pi_1(k_{\text{wh.}}) = \langle t^2, u, vw^{-1}\rangle;$$

it is in fact generated by only u and vw^{-1}, and has fundamental domain as shown in figure 2a: the two circles A and A' are the isometric circles of vw^{-1}.

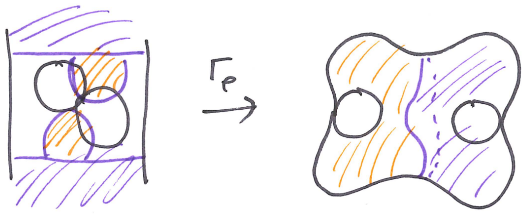


Figure 3: Left: structures on $\Omega(\Gamma_\rho)$—isometric circles of X and Y (black), and peripheral discs (purple and orange). Right: projection of peripheral structures to the quotient $\Omega(\Gamma_\rho)/\Gamma_\rho$, where to simplify the picture we do not twist the rank one cusps up (in reality the pleating locus, solid purple loop, will *never* bound a compression disc in this way.)

It is convenient to conjugate by $z \mapsto z/2$, obtaining the group

$$\left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & i/2 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ -2-2i & 1 \end{bmatrix} \right\rangle$$

It is clear (but not mentioned by Wielenberg—or elsewhere that I have seen explicitly) that there is a subgroup $G' = \langle X, Y \rangle$ uniformising a four-punctured sphere (this is the black limit set in figure 2b, but the point is that $-2-2i \in \mathcal{R}$ [EMS24]). Taking the opposite viewpoint, the Whitehead link group is obtained as a parabolic extension of a Riley group:

(3.2) Question. Describe the action of the parabolic element Z on $\Omega(G')/G'$; explicitly compute the two surfaces which are glued by Z. These surfaces must be twice-punctured discs by symmetry considerations, but it is not immediately clear to me whether $-2-2i$ lies on a pleating ray and Z identifies the two peripheral groups corresponding to that pleating ray, which would be the 'most symmetric' phenomenon possible, and in fact I doubt that this is what is going on. So what is occuring geometrically? [It cannot be a coincidence that the horizontal strip shown in figure 2b passes through the midpoints of two translates of the Farey word axis.]

(3.3) Question. Let $\rho \in \mathcal{R}$ and Γ_ρ be an arbitrary Riley group. For what $\zeta \in \mathbb{C}$ does the extension of Γ_ρ by $z \mapsto z + \zeta$ give a copy of the Whitehead link?

(3.4) Question. How many groups in $\mathcal{R}$ embed into $\pi_1(k_{\mathrm{wh.}})$? Can you deform Γ_{2+2j} around inside the link group? Can you embed the whole of $\mathcal{R}$ in there?

In an effort to understand these questions at a basic level we produce our first Variation:-

§4. Variation I: a Riley group with controlled geometry

We consider the Riley groups Γ_ρ defined by

$$\Gamma_\rho = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle.$$

A representative image (for the case that the group is on an even pleating ray/corresponds to a 2-component link) of Γ_ρ is in figure 3

define a new element and corresponding group extension

$$\Gamma_\rho^\dagger = \left\langle \Gamma_\rho, Z = \begin{bmatrix} 1 & \Im(1/\rho) \cdot 2i \\ 0 & 1 \end{bmatrix} \right\rangle;$$

when $\rho = 2 + 2i$ this agrees with $\pi_1(k_{\mathrm{wh.}})$ as modified in the previous section, and the element Z is manufactured to send the isometric circles of Y to circles with centres reflected across $\mathbb{R}$. This does not guarantee there will be parabolics induced with fixed point at 0 (this is clear from the following example); in a few paragraphs we will construct groups $\Gamma_\rho^\ddagger$ by directly adjoining such parabolics, another way of continuing the same *motif* of Wielenberg.

(4.1) Example. When $\rho = is$ for $s \in \mathbb{R}_{\geq 2}$, $\Gamma_\rho^\dagger$ is discrete and of the second kind (figure 4a): one of the peripheral discs ($\langle y, xyX\rangle$) is left untouched. In general though the supergroup is not discrete (e.g. figures 4b and 4c).

Related to this example but not to the remainder of this section as such since it comes from cusp groups and not Riley groups:

(4.2) Question (Fun and easy)**.** Take a cusp group Γ_ρ, so $\Phi_{p/q}(\rho) = -2$ for some $p/q \in \mathbb{Q}$ but the group is free and discrete. Then there exist two non-conjugate peripheral subgroups which are amalgamated products across $W_{p/q}$, and this word is parabolic [EMS24] with fixed point ξ. Let ϕ be a parabolic with fixed point ξ which conjugates the two peripheral groups onto each other. What is $\langle \Gamma_\rho, \phi\rangle$?

Remark. In some sense the problem that I have with Wielenberg's example, and the question which I am trying to hint at in the rest of this section, is that it is not clear how to generalise this question. We wish to exchange two halves of the surface by adjoining a parabolic, such that the parabolic that we adjoin forms a rank two cusp: but the only cusps available in a Riley group are X and Y, so we need to pick one of these to be the fixed point; and now the special situation in Wielenberg is that this *induces* a second additional parabolic that magically makes the other fixed point into a rank two cusp fixed point *as well*. We should not expect such a thing to happen at all.

(4.3) Question. For which ρ is $\Gamma_\rho^\dagger$ discrete? When is it cofinite?

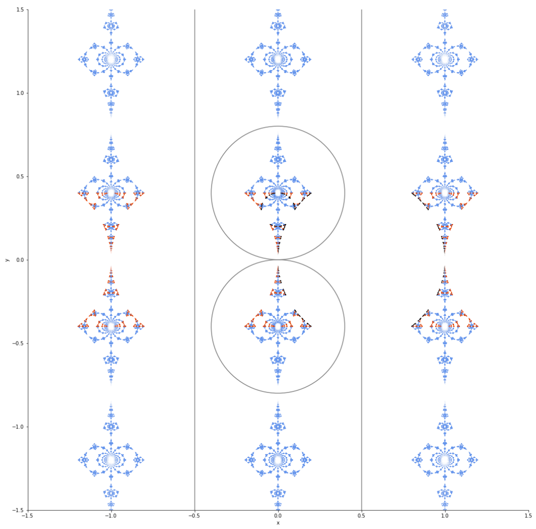
Here is how to answer question (4.3). The groups $\Gamma_\rho^\dagger$ are a 1D slice through a parameterisation of the character variety containing the (1; 2)-compression body groups [LP14] (in fact figure 4a lies on the boundary of the quasiconformal deformation space of these groups; it is not a maximal cusp). The only links which can arise in this large parameter space are tunnel number one links that have an additional rank 2 cusp drilled out from the Heegard splitting which comes from their tunnel presentation. Enumerate all of these links and you have your enumeration of cofinite groups (applying Riley's theorem to see that correct presentation means correct group). To solve the more general discreteness problem requires the development of some theory of orbifolds. Alternatively, this slice is simple enough that a Poincaré style argument may work directly (compute edge cycles in terms of ρ).

The problem with this example is that the angle between the isometric circles of Y and their Z-translates will not be correct for general ρ. As an alternative, we adjoin an element W with isometric circles a $\pi/2$-rotation about 0 of those of Y:-

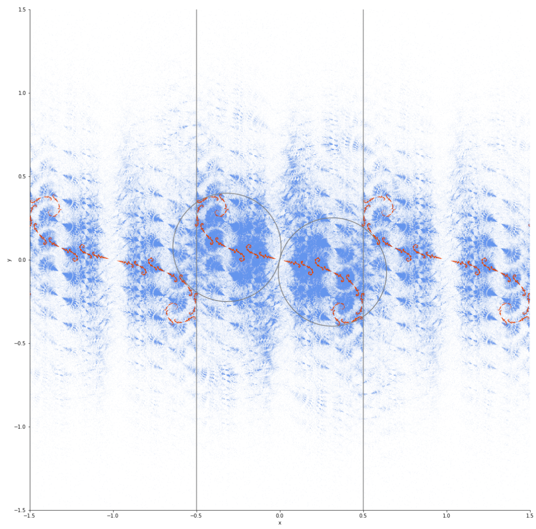
$$\Gamma_\rho^\ddagger = \left\langle \Gamma_\rho, W = \begin{bmatrix} 1 & 0 \\ -i\rho & 1 \end{bmatrix} \right\rangle;$$

we show the limit sets of these groups for the same values of ρ as $\Gamma_\rho^\dagger$ in figures 5a to 5c. They also lie in the parameter space of (1; 2)-compression body groups (but with a different normalisation)—figure 6. We can ask the following question analogous to question (4.3):

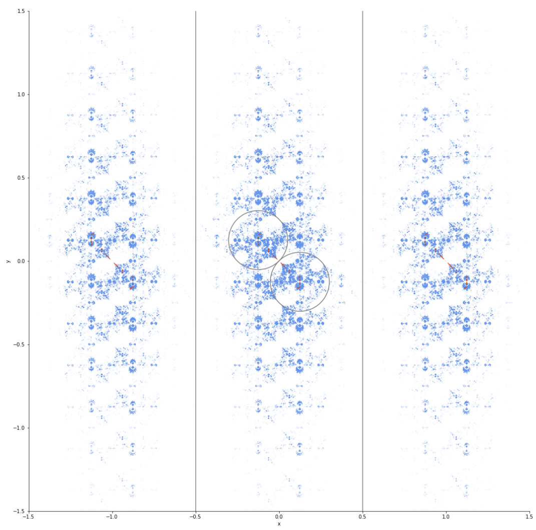
(4.4) Question. For which ρ is $\Gamma_\rho^\ddagger$ discrete? When is it cofinite?



(a) $\rho = 2.5i$ (disc.).

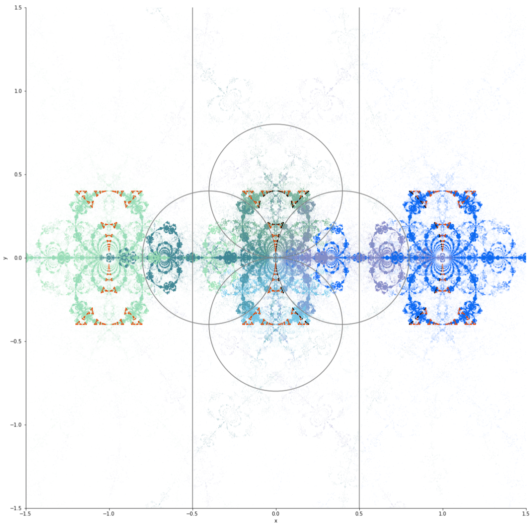


(b) $\rho = 3 + 0.7i$ (indisc.).



(c) $\rho = 4 + 4i$ (indisc.).

Figure 4: Limit sets for $\Gamma_\rho^\dagger$ (blue) and Γ_ρ (red) for various ρ.



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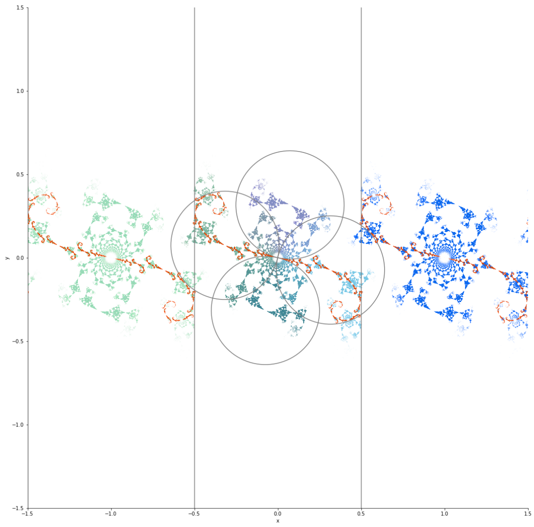

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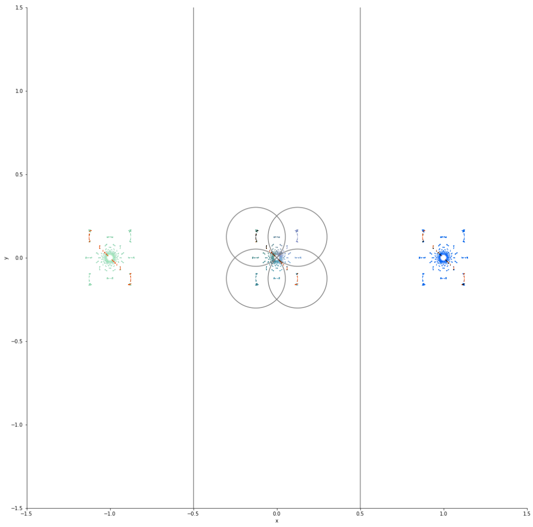


(c) $\rho = 4 + 4i$ (disc.).

Figure 5: Limit sets for $\Gamma_\rho^\ddagger$ (blue) and Γ_ρ (red) for various ρ.

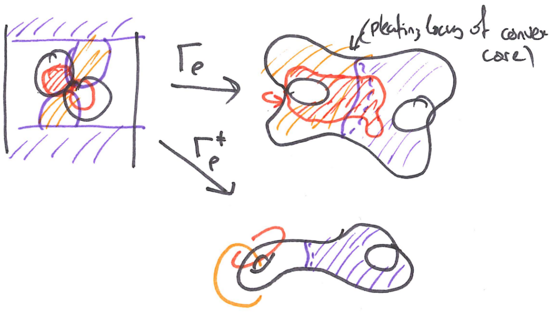


Figure 6: Structures on $\Omega(\Gamma_\rho)$ and projection to $\mathbb{H}^3/\Gamma_\rho^\ddagger$; observe that the action of the adjoined parabolic does not respect the peripheral structures.

However, we are primarily interested in the interactions between the † and ‡ 'operations':

(4.5) Question. For which ρ is $\Gamma_\rho^\ddagger = \Gamma_\rho^\dagger$? Here "equals really means equals", as groups of matrices in $\mathsf{PSL}(2,\mathbb{C})$. One example is $\rho = -2-2i$, and fourfold symmetry of $\mathcal{R}$ tells us that all of $\pm 2 \pm 2i$ (signs independent of each other) must also work (two of these will give the Whitehead link, and the other two its conjugate 2-bridge link).

Suppose you take an arbitrary 2-bridge link l where one component l_0 is unknotted and surrounds exactly two strands of the other link. Can you arrange it such that cutting along the 3-punctured disc spanned by l_0 gives a Riley group Γ_ρ with $\pi_1(l) = \Gamma_\rho^\dagger = \Gamma_\rho^\ddagger$? This procedure is studied in §12.1 of [Pur20], but I ask not just for some hyperbolic geometry but for knowledge of the structure on the resulting 4-punctured sphere. In addition, I ask not to slice along the disc to get a pair of 3-punctured discs, but to slice it to a four-punctured sphere; detailed analysis of the geometry behind question (3.2) is needed to understand exactly what the formal question is, here. As figure 6 shows, the additional parabolics do not seem to respect any peripheral structures—so how is the slicing done in terms of the surface geometry?

Of course in the above we probably need to leave $\mathsf{PSL}(2,\mathbb{Z}(i))$ behind. I suspect we need to leave the world of thin groups behind, too...

§5. The Big Four (gluing $S_{0,3}$s)

This section is Examples 2, 3, and 4 of [Wie78], also Example 61 of [KAG86].

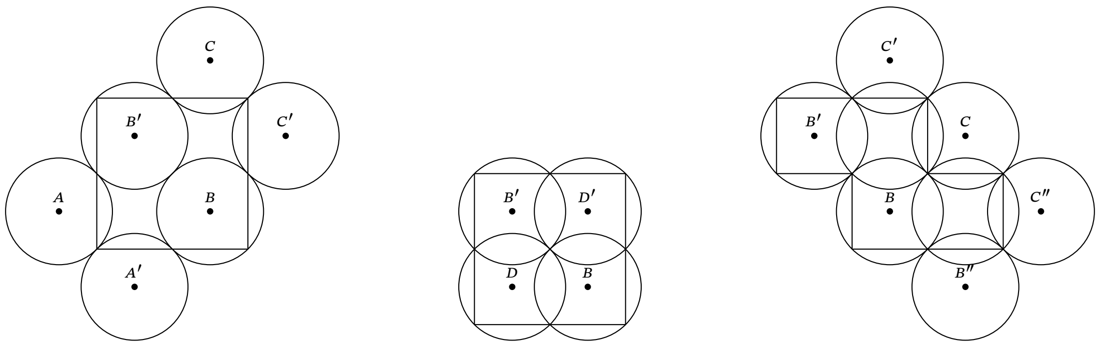
(5.1) Proposition. *A fundamental domain for the group*

$$G_1 = \left\langle t^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ u^2 = \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix},\ vw^{-1} = \begin{bmatrix} 1 & 0 \\ -1-i & 1 \end{bmatrix} \right\rangle;$$

is the set bounded by configuration of lines and circles on $\hat{\mathbb{C}}$ shown in figure 7a: t^2 and u^2 pair the sides of the square in the figure, vw^{-1} pairs its isometric circles B and B', and A (resp. C) and A' (resp. C') are paired by $u^{-2}vw^{-1}t^2$ (resp. $t^2vw^{-1}u^{-2}$). The quotient manifold $\mathbb{H}^3/G_1$ is bounded by a pair of thrice punctured spheres with a rank two cusp drilled out. The stabilisers of the thrice-punctured sphere components are

$$\langle vw^{-1}, u^{-2}vw^{-1}t^2\rangle \text{ and } \langle vw^{-1}, t^2vw^{-1}u^{-2}\rangle. \tag{5.2}$$

These two Fuchsian subgroups of G_1 are in fact peripheral subgroups, and are shown in blue in figure 8a.



(a) Fundamental domain for G_1. (b) Fundamental domain for G. (c) Fundamental domain for G'.

Figure 7: Fundamental domains for the Big Four groups.

Remark. The group G_1 lies on the boundary of the quasiconformal deformation space of $(1;2)$-compression body groups [LP14], in fact it is a maximal cusp group.

We now define a group extension $G = \langle G_1, vw\rangle$, where

$$vw = \begin{bmatrix} 1 & 0 \\ -1+i & 1 \end{bmatrix}.$$

(5.3) Proposition. *The parabolic element vw conjugates the two thrice-punctured sphere subgroups of G_1 into each other and so $\mathbb{H}^3/G$ is obtained from $\mathbb{H}^3/G_1$ by gluing together the two ends of its convex core. A fundamental domain for the action of G is shown in figure 7b, and $\mathbb{H}^3/G$ is the complement of a link made up of four unknots cyclically chained together.*

The $\mathbb{H}^3/G$ has a totally geodesic embedded thrice-punctured sphere, uniformised by the (conjugate by vw) Fuchsian subgroups whose limit sets are shown in figure 8b (of course, the limit set of G is dense in the plane, so the intricate patterns shown in this approximation are really reflecting the symmetric manner in which the Cayley graph of G in $\mathbb{H}^3$ is approaching every point on $\mathbb{S}^2$). This thrice-punctured sphere is a Seifert surface for the sublink of the three rank two cusps at the punctures, figure 9.

(5.4) Question. Draw a plausible cartoon of the quotient $\mathbb{H}^3/G_1$ and the gluing procedure that gives $\mathbb{H}^3/G$.

Following Example 4 of [Wie78] we may produce another extension of G_1: set $G' = \langle G_1, tvw^{-1}t^{-1}\rangle$, where

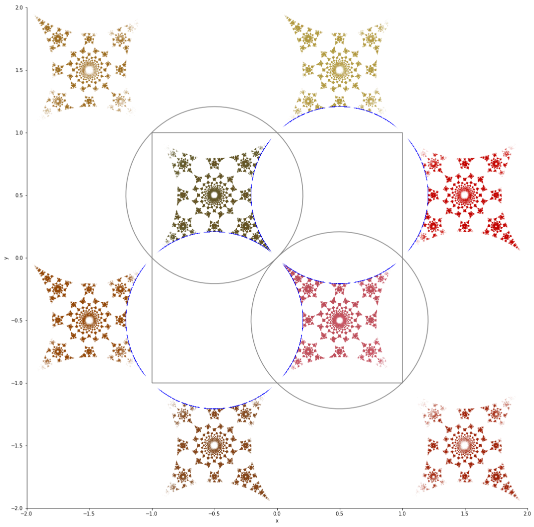
$$tvw^{-1}t^{-1} = \begin{bmatrix} -i & 1+i \\ -1-i & 2+i \end{bmatrix}.$$

We pick two other peripheral subgroups of G_1:

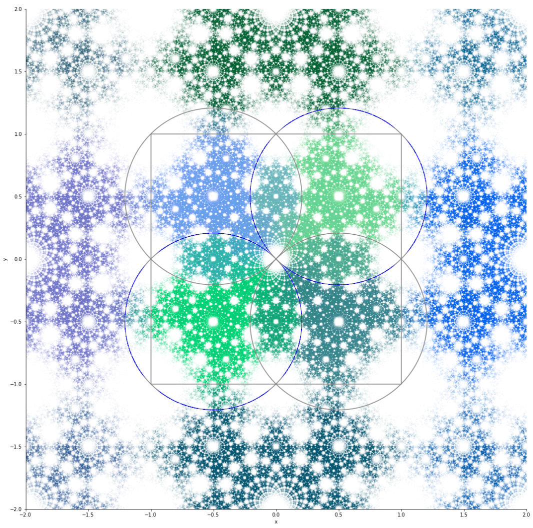
$$\langle vw^{-1}, t^2vw^{-1}u^{-2}\rangle \text{ and } \langle (vw^{-1})^{-1}t^{-2}u^2, t^2vw^{-1}t^{-2}\rangle. \tag{5.5}$$

The first peripheral subgroup in equation (5.5) is the same as one of those above in equation (5.2), and the second is a t^2-translate of the other one listed above (evident from comparing figure 8b to figure 8c); so we have just chosen a different pair of representatives for the same convex core ends.

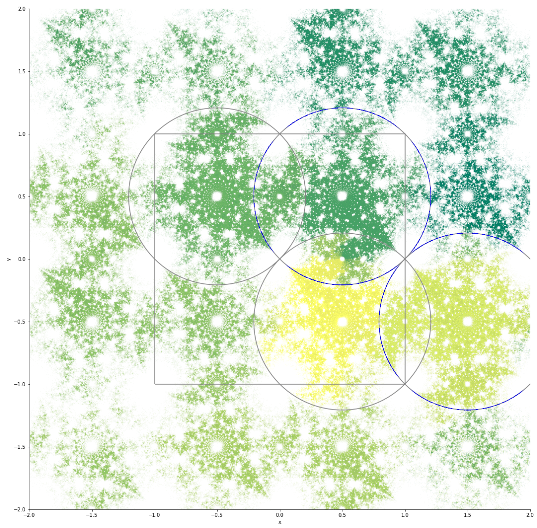
(5.6) Proposition. *The parabolic element $tvw^{-1}t^{-1}$ conjugates the two thrice-punctured sphere subgroups of G_1 listed in equation* (5.5) *into each other and so $\mathbb{H}^3/G'$ is obtained from $\mathbb{H}^3/G_1$ by gluing*



(a) Limit set of G_1.



(b) Limit set of G.



(c) Limit set of G'.

Figure 8: Limit sets for the Big Four groups.

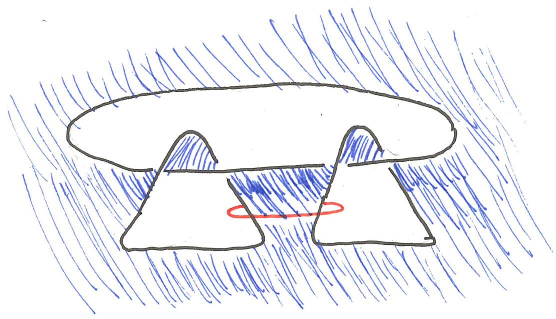


Figure 9: A Seifert surface for the alternating connect sum of two Hopf links and the additional rank 2 parabolic in the supergroup.

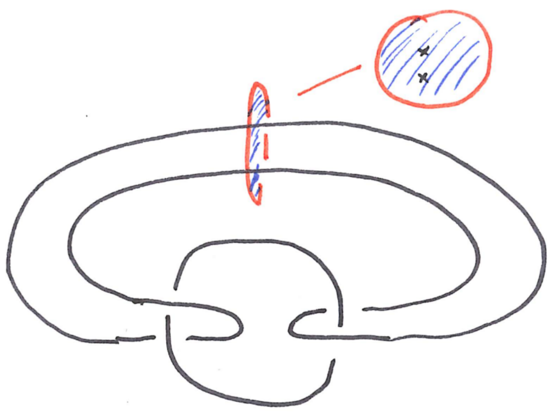


Figure 10: The quotient $\mathbb{H}^3/G'$.

together the two ends of its convex core. A fundamental domain for the action of G' is shown in figure 7c, where (i) vw^{-1} pairs B and B'; (ii) $(vw^{-1})^{-1}t^{-2}u^2$ pairs B and B''; (iii) $t^2vw^{-1}u^{-2}$ pairs C and C'; and (iv) $t^2vw^{-1}t^{-2}$ pairs C and C''. The internal circles (i.e. the circles bounding the peripheral discs of G_1) are paired by $tvw^{-1}t^{-1}$. The quotient $\mathbb{H}^3/G'$ is the complement of a link made up of three unknots cyclically chained together (figure 10).

(5.7) Question. Draw a plausible cartoon of the gluing procedure that gives $\mathbb{H}^3/G'$, compared to the procedure giving $\mathbb{H}^3/G$.

(5.8) Question. Since both $\mathbb{H}^3/G$ and $\mathbb{H}^3/G'$ are obtained by gluing the ends of the convex core of the same manifold, $\mathbb{H}^3/G_1$, they are links of equal volume. (i) What is this volume? (ii) Are there more than two manifolds with this volume? If so, can they all be obtained from $\mathbb{H}^3/G_1$ by gluing the ends of the convex core in appropriate ways? (iii) For each possible choice of pair of non-conjugate peripheral subgroups in G_1, describe which link is obtained from the extension.

(Note that there are only $3 \times 2 \times 1 = 6$ ways to glue two 3-punctured spheres together. If you fix the identification of one puncture with another, like here, there are only 2 ways left. So I think that G and G' are the only extensions of G_1 in this way, but I have not sat down to formally prove it.)

§6. Variation II: gluing thrice-punctured spheres in a Schottky group

Actually, for the same reasons discussed in Variation I, we will work with a group on the boundary of Schottky space. For fun we will work with genus 3. The parabolic element $j(z) = 1/(2iz+1)$ has isometric circles of radius 1/2 tangent at 0 with centres at $\pm i/2$. Let $k(z) = z + 1/2$, and construct $A = k^{-1}jk$, $B = kjk^{-1}$, and $C = k^4$:

$$\Gamma = \left\langle A = \begin{bmatrix} 1-i & -i/2 \\ 2i & 1+i \end{bmatrix}, B = \begin{bmatrix} 1+i & -i/2 \\ 2i & 1-i \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

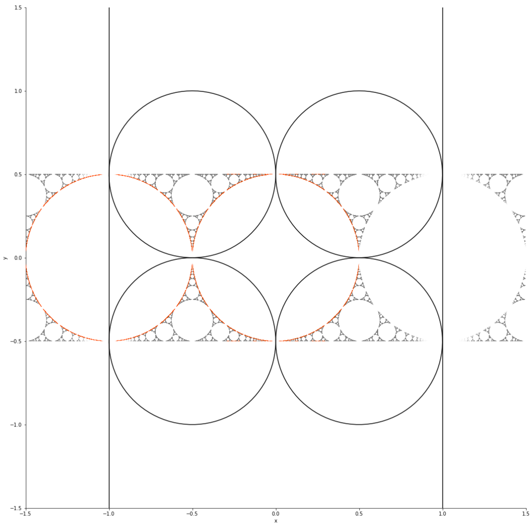
(You can manufacture such groups more generally by gluing together Fuchsian groups according to your favourite pants decomposition, modulo some stuff: it just boils down to a concrete application of Koebe–Andreev–Thurston. I just give this one as a trivial illustration that Wielenberg's construction generalises without any special insights.) This has four conjugacy classes of peripheral subgroups, represented by $P_1 = \langle A, B\rangle$, $P_2 = \langle A, C^{-1}BC\rangle$, $Q_1 = \langle C, AB^{-1}\rangle$, and $Q_2 = \langle C, A^{-1}B\rangle$. We show the limit set of Γ in figure 11a.

We now introduce two additional parabolics: $U(z) = z + i$ (which sends the interior of the peripheral disc of Q_1 to the exterior of the peripheral disc of Q_2), and $V(z) = -1/(4z+4)$ (which sends the interior of the peripheral disc of P_1 to the exterior of the peripheral disc of P_2). The three groups obtained by adjoining each of these parabolics in turn, and both at once, are shown in figures 11b to 11d.

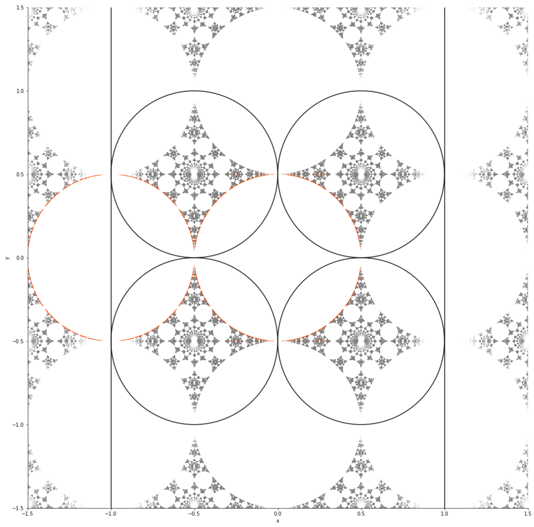
(6.1) Question. What is the topology of $\mathbb{H}^3/\langle\Gamma, U, V\rangle$? If U and V are replaced with different parabolics pairing different representative peripheral subgroups, are different quotients obtained?

In any case, one is led to the following result which is not hard to prove, it is essentially a combination theorem like those in [Mas87, Chapter VII]:

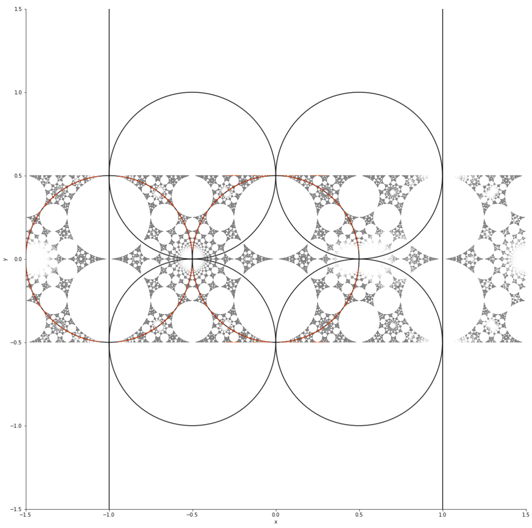
(6.2) Theorem (Mishima combination)**.** *Let G be a group such that $P, Q < G$ are F-peripheral subgroups uniformising 3-punctured spheres, with the property that $P \cap Q = \langle j\rangle$ is a primitive parabolic. Then $\langle G, \phi\rangle$, where ϕ is the parabolic with fixed point the same as the fixed point of j that sends the peripheral circle of P to the peripheral circle of Q, is discrete and has quotient manifold obtained from $\mathbb{H}^3/G$ by gluing together the convex core boundary surfaces corresponding to P and Q. No other peripheral structures are modified by this procedure.*



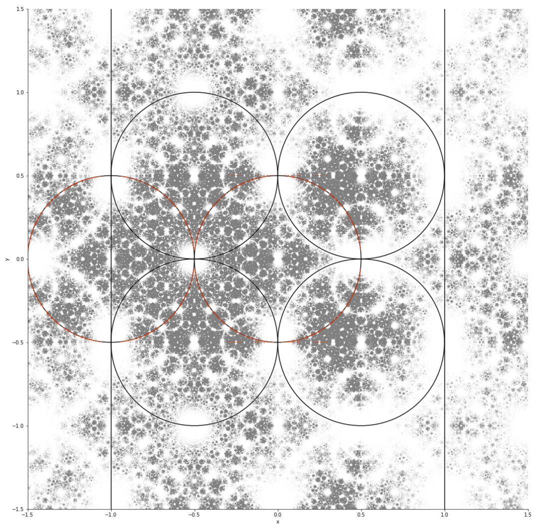
(a) Γ.



(b) $\langle \Gamma, U \rangle$.

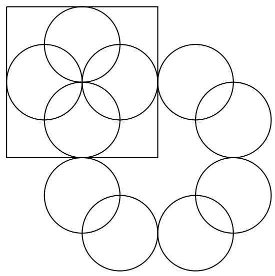


(c) $\langle \Gamma, V \rangle$.

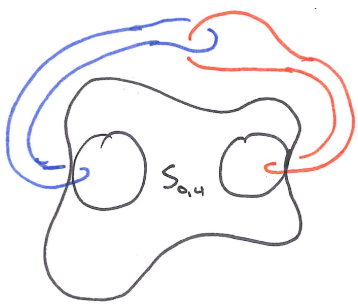


(d) $\langle \Gamma, U, V \rangle$.

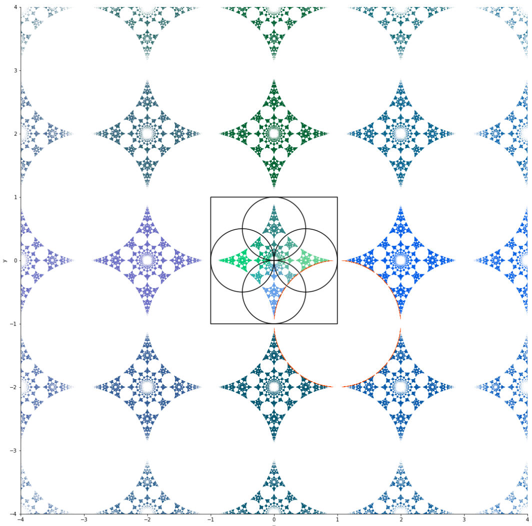
Figure 11: A maximally cusped Schottky-type group of genus 3.



(a) Fundamental domain projection.



(b) Quotient 3-fold.



(c) An approximation to the limit set.

Figure 12: The genus two group with two accidental parabolics, R_1.

(6.3) Question. Let G be a maximal cusp on the boundary of Schottky space (any genus). What links are obtained by gluing peripheral structures as in the above procedure? Conversely, let l be a link. Is $\pi_1(l)$ always obtained as an extension of some Schottky type group? [I am asking for a description of a family of maps from graph curves to links....]

§7. Six rings (gluing $S_{0,4}$s)

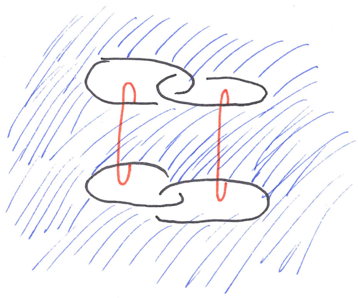
This section is Examples 5 and 6 of [Wie78], also Example 62 of [KAG86]. Set

$$R_1 = \langle t^2, u^2, v^2, w^2 \rangle.$$

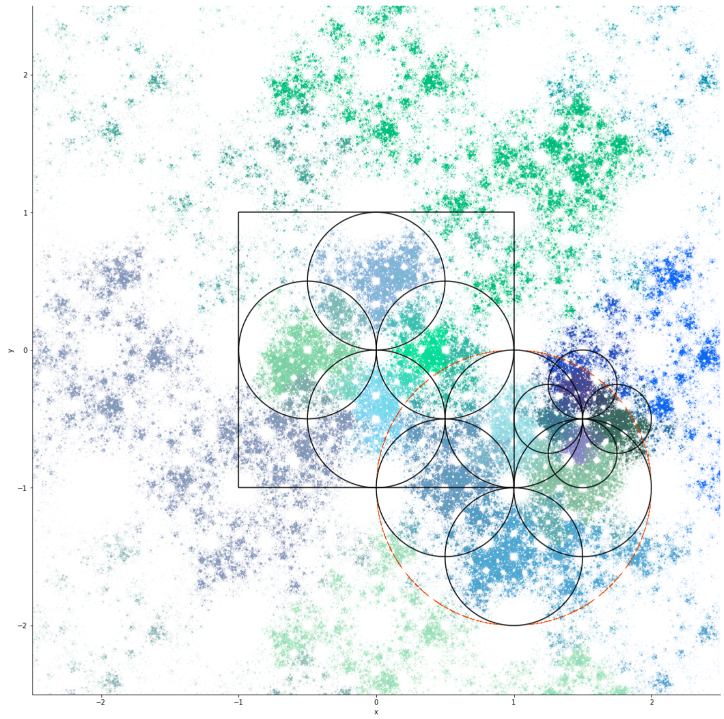
(7.1) Proposition. *A fundamental domain for the action of R_1 on $\mathbb{H}^3$ is the set shown in figure 12a; it has one end that is a four-punctured sphere (the projection of the octagon) and two rank two cusps—figure 12b. It splits as a free product $\langle t^2, u^2 \rangle * \langle v^2, w^2 \rangle$, and the $S_{0,4}$ end corresponds to the Fuchsian peripheral subgroup*

$$\langle t^2v^2, t^2u^{-2}w^{-2}t^{-2}, u^{-2}v^{-2}t^{-2}u^2, w^2, u^2 \rangle$$

which has peripheral disc D of radius 1 centred at $1 - i$.



(a) Quotient manifold.



(b) An approximation to the limit set.

Figure 13: The six-link group, R. Note that if you conjugate an element of $\mathbb{M}$ with an element $f \in \tilde{\mathbb{M}}$), the isometric circles of the result are not in general the translates of the isometric circles of the original element by f; this is why the circles in figure 13b are not the inversions of the circles in figure 12c across the orange circle.

Instead of conjugating interior of one peripheral disc onto exterior of another, we try to conjugate the interior of D onto the exterior of D.

(7.2) Lemma. *If Ψ is reflection in ∂D, then $\Psi R_1 \Psi^{-1} = (tu^{-1}aut^{-1}) R_1 (tu^{-1}aut^{-1})$, and $\Psi f \Psi^{-1} = f$ for all $f \in F$.*

(7.3) Proposition. *The group $R = \langle R_1, \Psi R_1 \Psi^{-1} \rangle$ splits as an amalgamated product $R = R_1 *_F \Psi R_1 \Psi^{-1}$. The quotient $\mathbb{H}^3/R$ is the complement of the link in figure 13a.*

(7.4) Question. Explicitly write $\mathbb{H}^3/R$ in terms of the action on $\mathbb{H}^3/R_1$ by a mapping class on $S_{0,4}$.

(7.5) Question. Replace Ψ with its composition with an $\mathbb{H}^2$-automorphism of D that projects to $\operatorname{Aut} F$.

(7.6) Question. Replace R_1 with a group where the $\operatorname{Stab}(D)$ sphere uniformiser is non-Fuchsian.

(7.7) Question. Replace R_1 with a group where $\operatorname{Stab}(D)$ is your favourite Fuchsian group of the first kind, e.g. a genus two compact surface group.

(7.8) Question. Compute the convex core volume of $\mathbb{H}^3/R_1$. What are the other groups with the same convex core volume, and how are they related?[2]

[2] This is essentially a question posed by Connie. Many other questions in this document arose from trying to understand the

§8. Variation III: in which 4 is generalised to n

In order to do this construction, we must perform some self-flagellation. We work with n punctures to begin with since there is no need to specialise early. Consider the $2n$-gon in $\mathbb{H}^2$ where the vertices alternate between angles 0 and $2\pi/n$, with equal length edges. This can be cut up into $2n$ triangles with angles $0, \pi/n, \pi/n$. In $\mathbb{H}^2$ a representative such triangle Δ has vertices ∞, $e^{i\pi/n}$, $e^{(n-1)i\pi/n}$; and the adjacent triangle in the $2n$-gon which shares its ideal vertex is the translate $T(\Delta)$ where

$$T = \begin{bmatrix} 1 & 2\cos\frac{\pi}{n} \\ 0 & 1 \end{bmatrix}.$$

Our generators will be conjugates of T^2.

We wish to embed the $2n$-gon into the standard unit disc $\mathbb{B}^2 = \{z : |z| < 1\}$. To do this we recall that the element B defined by

$$B = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$$

sends $\mathbb{H}^2$ to $\mathbb{B}^2$ with $i \mapsto 0$ and $\pm 1 \mapsto \mp i$. We actually want the corner $e^{i\pi/n}$ to go to 0 and so we will apply first the transformation

$$A = \begin{bmatrix} \sqrt{\csc\frac{\pi}{n}} & -\sqrt{\csc\frac{\pi}{n}}\cos\frac{\pi}{8} \\ 0 & \sqrt{\sin\frac{\pi}{n}} \end{bmatrix} \in \mathsf{PSL}(2,\mathbb{R})$$

which shifts that corner to i. Finally, let R be the order n rotation

$$R = \begin{bmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{bmatrix}.$$

We now construct the group

$$K_n = \langle R^k BAT^2A^{-1}B^{-1}R^{-k} : k \in \{0, ..., n-1\}\rangle.$$

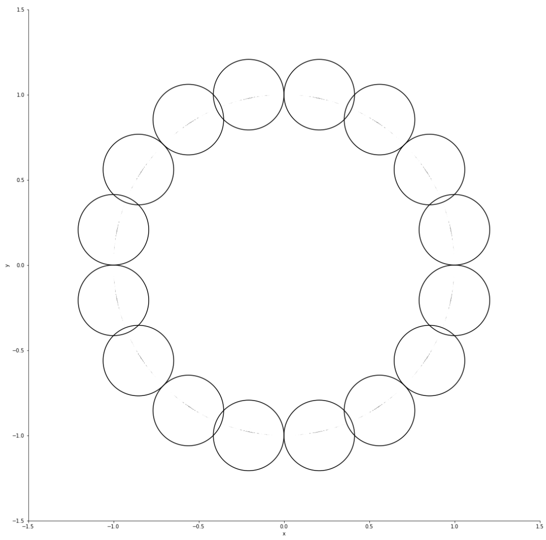
This uniformises a n-punctured sphere, with representative fundamental domain shown in figure 14a.

(8.1) Question. Let $n = 6$. The hexagon tiles the plane by the lattice $f(z) = z + 2$ and $g(z) = z + 2e^{\pi i/3}$, so we adjoin these two elements (figure 14c). Let $\rho = (1/2)\tan(\pi/6)$ be the radius of all the isometric circles of the generators of K_6, let $\xi = 1 + i\rho$, and let $\zeta = e^{i\pi/3} + e^{-i\pi/3}\rho$; these are isometric circle centres of the $k = 0$ and $k = 1$ generators. Let C (resp. D) be the circle at ξ (resp. ζ) of radius ρ. Let η be the point of $C \cap D$ which does not lie in $\mathbb{B}^2$. Then we have two additional circles of radius ρ, namely C' centred at $2\eta - \xi$ and D' centred at $2\eta - \zeta$. Let p (resp. q) be the parabolic element with isometric circles C, C' (resp. D, D').

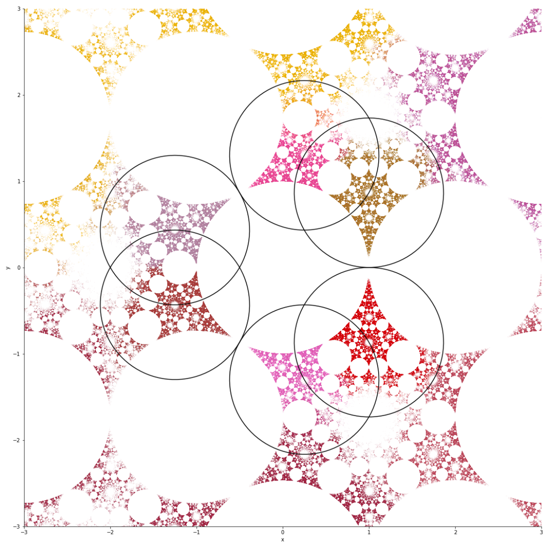
Is $K_6^\dagger = \langle K_6, f, g, p, q\rangle$ discrete? If so, construct the inversion Ψ in the unit circle and define $\langle K_6^\dagger, \Psi K_6^\dagger \Psi^{-1}$; what is the link uniformised by this group?

(8.2) Question. In the same vein as the previous, let $n = 6$; instead of taking the lattice which tiles hexagons, we let $\rho = (1/2)\tan(\pi/6)$ again be the radius of an isometric circle of the generators of K_6 and we adjoin the lattice $f(z) = z + 4i\rho$ and $g(z) = z + 4i\rho e^{\pi i/3}$. The result is the beautifically intricate limit set shown in figure 14d, which also appears to belong to a discrete group. We can ask similar questions to the previous example.

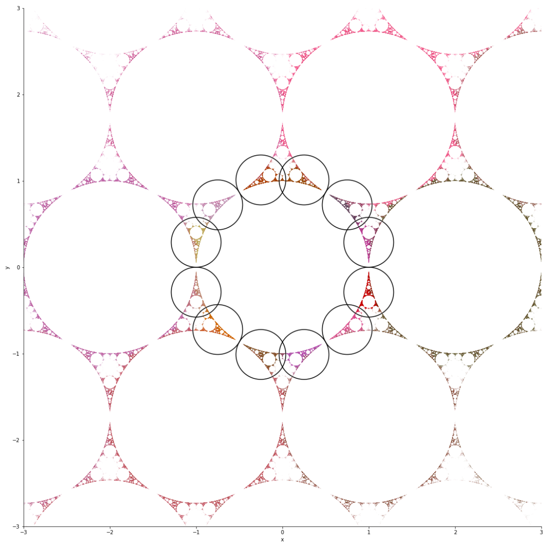
computation of convex core volumes by cutting groups up, e.g. along pleats of peripheral structures, and by gluing peripheral structures together (so the glued manifold has the same volume as the convex core of the original manifold, but is of known topology with known finite volume).



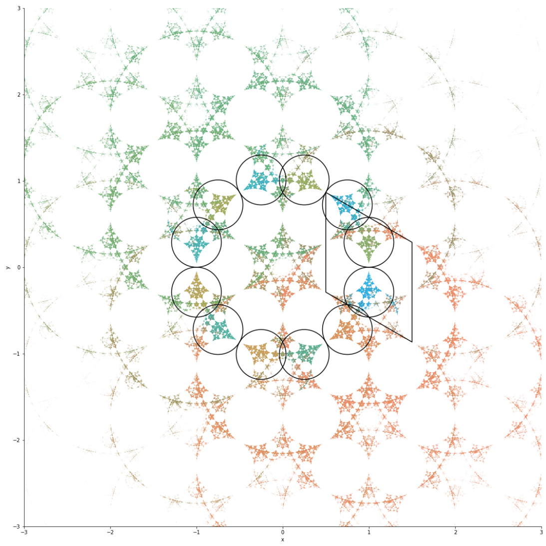
(a) The Fuchsian group K_8.



(b) Extension of K_3 by a pair of trilliptics tiling trigons.



(c) Extension of K_6 by a pair of parabolics tiling hexagons.



(d) Extension of K_6 by a pair of parabolics with translation length equal to 2 circle diameters.

Figure 14: Groups associated to K_n.

(8.3) Question. For $n = 5$ and $n > 6$, the fundamental domains do not 'tile the plane'. We can only extend by $z \mapsto z + 2$ and $z \mapsto z + 2i$ (or other lattices which will bubble off new peripheral groups not conjugate to the original K_n). We can still adjoin rank two parabolics to the edges of the $n$gon paired up by the new lattice subgroup, and perform the same procedure: just now instead of a cofinite group we get a group which still has non-torus conformal structures. We can still ask, though, exactly what manifolds are obtained and how are they related to the links for $n = 4$ and $n = 6$?

(8.4) Question. For the sake of completeness, when $n = 3$ the tiling is not by a parabolic lattice but by order three elliptics. Let $X(z) = z + 1 + i\tan(\pi/3)$ and $Y(z) = z + 1 - i\tan(\pi/3)$, then produce $\langle K_3, XRX^{-1}, YRY^{-1}\rangle$ (figure 14b); again one can ask about adjoining rank two parabolics at intersection points of the isometric circles, but the resulting structure will be orbi- not mani-fold.

§9. Borromean rings (gluing $S_{0,3}$s, *encore*)

This section is Examples 7 and 8 of [Wie78], also Example 60 of [KAG86]. We will be quick.

(9.1) Proposition. *A fundamental domain for the action of $G = \langle t^4, u^2, v\rangle$ on $\mathbb{H}^3$ is shown in figure 15a. It lies on the boundary of $(1;2)$-compression body space. Two F-peripheral subgroups are $\langle v, u^{-2}vu^2\rangle$ and $\langle vt^4, u^{-2}vt^4u^2\rangle$; they uniformise thrice-punctured spheres and are conjugated together by $s = t^{-1}u^{-1}vut$; the extension group $G' = \langle G, s\rangle$ is the Borromean rings group.*

(9.2) Question. Again, we see $(1;2)$-compression body space—this time, we have added a rank 2 cusp around a 2-bridge link with two components (by Riley's theorem whenever you pinch down curves on a handlebody through elliptics to the identity you get a link group; by [CM09] everything you get in genus 2 is a 2-bridge link). Draw a map of $(1;2)$-compression body space showing all the links you can make.

§10. Summary and anticonclusion

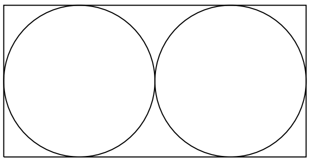
We summarised a number of constructions, which broadly speaking were of the following types:

- Take a maximally cusped group where every end is a pair of thrice-punctured spheres, and adjoin an additional parabolic which glues these ends up to form a number of rank two cusps. [HNN extension!]
- Take a function group F where one end is a n-punctured sphere and glue in a second copy of F on the 'other side' of the spherical wall. [amalgamated product!]

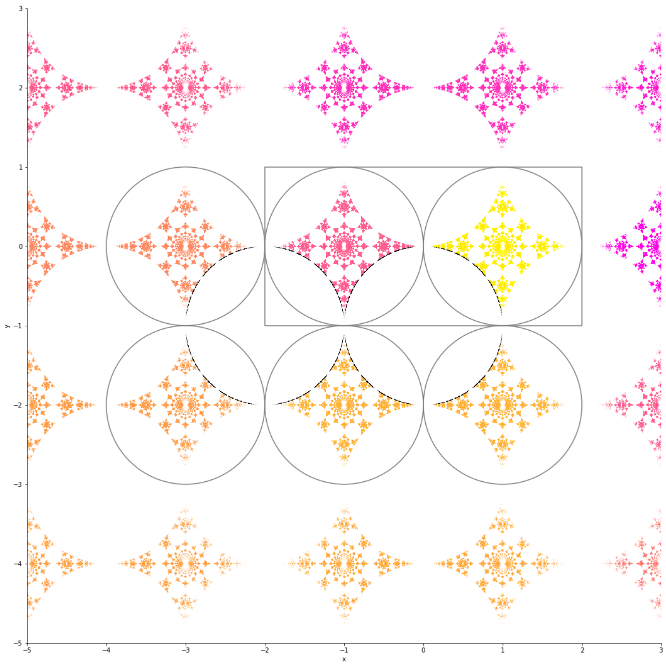
We asked highly speculative and broad questions like

- 'from the point of view of a manifold inhabitant—more precisely, the inhabitant of $\Omega(G)$—how should one view these gluing procedures?' [Draw the right picture for me!]
- 'can we replace thrice-punctured spheres with more general peripheral structures like twice-punctured discs?'
- 'how do these objects deform inside larger 3-manifolds?'
- 'how are these objects parameterised?'

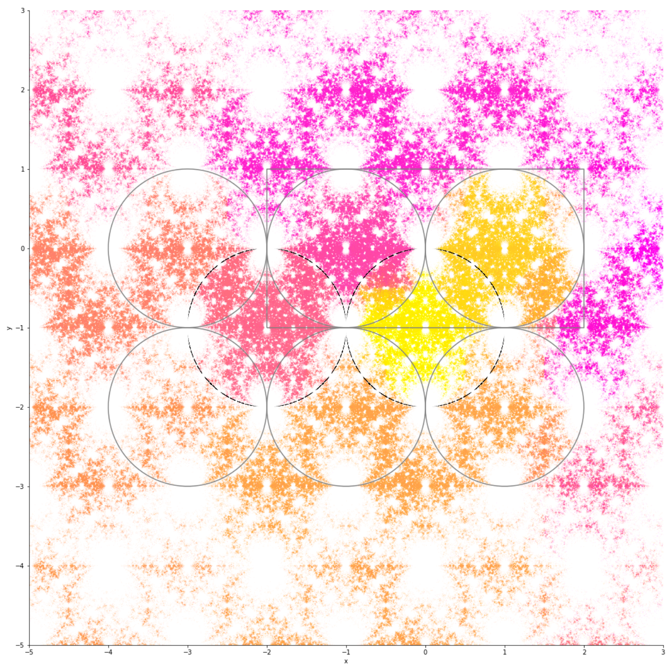
Heading in this kind of direction, one asks about constructions of parameterised families of groups obtained by cutting along surfaces in a finite-volume manifold. More precisely, let Γ be some big group of the first kind and find a holomorphic family G_t of subgroups which all extend to Γ by adjunction of a single parabolic—or similar kinds of extensions. If you take the figure eight knot it is easy to write down a once-punctured torus subgroup, namely the fundamental group of a Seifert surface, but cutting along all these surfaces gives the same punctured torus group (since the surfaces



(a) Fundamental domain projection for G.



(b) An approximation to the limit set of G.



(c) An approximation to the limit set of the extension G'.

Figure 15: A thin group G inside the Borromean ring group $G' = \pi_1(k_{\text{borr.}})$.

are all isotopic—they form the pages of a book hinged around the knot). You need some kind of homotopy obstruction to have any hope of finding different sliced up manifolds. Alternatively, find a discretely parameterised family of link groups, more generally cofinite groups, which are obtained by gluing up a discretely parameterised family of groups with surface ends. Here is a good question: can you find cocompact groups which cut along surfaces to form Schottky groups? Of course by 'find' I mean 'construct'.

Speculative questions and comments are welcomed. Any number of the questions listed above are interesting enough to work on for me, some of them are technically meaningless as stated but I think the ideas can be pushed in some interesting direction regardless (add restrictive adjectives, for example).

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