All arithmetic 2-bridge link groups

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Josef Sudek, *Evening Walk* (1956)

Thou shalt always begin with an easy example

Definition

A **Lie group** is a smooth manifold G equipped with smooth maps $\mu : G \times G \to G$, $e : \{1\} \to G$, and $\iota : G \to G$ satisfying the usual group axioms.

- **Important examples** are the classical matrix groups: $\mathrm{SL}(2, \mathbb{C})$, $\mathrm{PSL}(4, \mathbb{R})$, $U(67)$, $\mathrm{Sp}(238, \mathbb{C})$, etc etc.
- But we can define these groups over other rings/fields easily where we can no longer use Lie theory.
- How do we make sense of, e.g., $\mathrm{PSL}(3048, \mathbb{Z})$?

Not a definition

A **scheme** over a ring R is a topological space with an atlas of local charts; each chart is the intersection of the zerosets of polynomials in several variables with coefficients in R.

The actual definition is more complicated.

We want to allow different rings on each chart, we don't want to prioritise a coordinate system, and we need to deal with R not having a nice topology—no classical partitions of unity like you need for differential geometry!

Classical introduction: David Mumford, *The red book of varieties and schemes*, Springer (1999).
Modern introduction: Ravi Vakil, *The rising sea: Foundations of algebraic geometry*, online notes (2024).

Example

$\mathrm{SL}(2,\mathbb{Z})$ is the set of points $(a,b,c,d) \in \mathbb{Z}^4$ satisfying the equation $ad - bc - 1 = 0$. It only consists of one piece so is called **affine**.

This example is very nice (reduced, irreducible, separable) so we can think of it as a 'variety over $\mathbb{Z}$'. This runs into psychological problems but it leads us to look at $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{C}) \cap \mathbb{Z}^4$ and ask what it means to intersect a variety with a subring of its defining field.

- If something is defined in terms of equations over a ring R, and $R \to S$ is a map, then we can ask for solutions in S^n as well as in R^n.
- Knowing relations between all possible 'sets of points' for all ring morphisms $R \to S$ is enough to recover all algebraic data.
- **Yoneda's lemma**: schemes X are equivalently functors $X : \mathrm{Ring} \to \mathrm{Set}$ where a ring R is sent to the set $X(R)$ of solutions of a bunch of polynomials in R^n.
- This is the **functor of points**.

Thou shalt be careful with homonyms

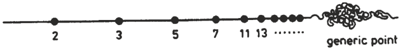
TECHNICAL INTERLUDE!!!!!!!

Let's look at Spec $\mathbb{Z}$ ('the affine variety $\mathbb{Z}^0$')...

As a topological space,

Spec $\mathbb{Z}$ is the set of prime ideals of $\mathbb{Z}$:

Example C. Spec $(\mathbb{Z})$. $\mathbb{Z}$ is a P.I.D. like $k[X]$, and Spec $(\mathbb{Z})$ is usually visualized as a line:



(Mumford's picture.)

But in terms of the functor of points,

A $\mathbb{Z}$-valued point of Spec $\mathbb{Z}$ is a map Spec $\mathbb{Z} \to$ Spec $\mathbb{Z}$. Passing to the dual, we look for ring maps $\mathbb{Z} \to \mathbb{Z}$. But there is only one such map. Hence there is only one $\mathbb{Z}$-valued point in Spec $\mathbb{Z}$.

We wanted $\mathbb{Z}$-analogues of Lie groups.

Definition

An **algebraic group** over a ring R is a scheme X defined over R (so there is a map $X \to \operatorname{Spec} R$) together with morphisms $\mu : X \times_{\operatorname{Spec} R} X \to X$, $e : \operatorname{Spec} R \to X$, and $\iota : X \to X$ which satisfy the usual group axioms when restricted to $X(S)$ for all R-algebras S.

See *Stacks Project*, tag 022S (also SGA III of course).

Definition

An **arithmetic group** is the group $X(\mathbb{Z})$ where X is an algebraic group.
A **thin group** in an algebraic group is an infinite-index subgroup of the integer points of its Zariski closure.

For detailed motivation see A. Kontorovich, D. Darren Long, A. Lubotzky, A.W. Reid, "What is... a thin group?", `https://math.rice.edu/~ar99/WhatIs16.pdf`

Theorem (Maclachlan–Reid, Thm 8.22 and Thm 10.3.7)

A Kleinian group $\Gamma \leq \mathrm{PSL}(2, \mathbb{C})$ *is arithmetic (resp. thin) iff: (i) it is finite (resp. infinite) covolume, (ii) the invariant trace field* $k\Gamma^{(2)} = \mathbb{Q}(\{\mathrm{tr}^2 g : g \in \Gamma\})$ *has exactly one non-real field embedding into* $\mathbb{C}$*, and (iii) for all field embeddings* $\rho : k\Gamma^{(2)} \to \mathbb{R}$ *the algebra* $A_0\Gamma^{(2)} = \{\sum \rho(a_i)\gamma_i : a_i \in k\Gamma^{(2)}, \gamma_i \in \Gamma\}$ *is isomorphic to the Hamiltonian quaternions.*

Theorem (Maclachlan–Martin, 1999)

There are only finitely many conjugacy classes of arithmetic or thin Kleinian groups generated by two parabolic or elliptic elements.

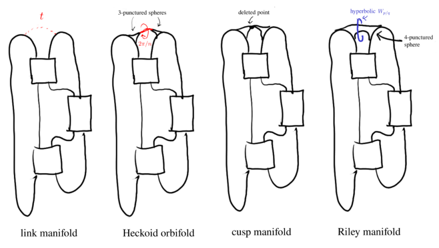
Question: what are they?

Thou shalt use Riley's picture

Theorem (Conj. of Agol (2001), proved Aimi–Akiyoshi–Lee–Ohshika–Parker–Sakui–Sakuma–Yoshida (2020))

If X and Y are parabolic, and $G = \langle X, Y \rangle$ is Kleinian and non-Fuchsian, then G falls into one of the following mutually exclusive categories:

1. *Split as a free product $\langle X \rangle * \langle Y \rangle$:*
   1.1 *Groups in $\overline{\mathcal{R}}$.*
2. *Don't split:*
   2.1 *Heckoid groups: $\langle X, Y : W^n = 1 \rangle$ for $n > 1$*
   2.2 *2-bridge link complements: $\langle X, Y : W = 1 \rangle$*
   2.3 *2-fold quotients of (2.1) and (2.2).*



A.J.E., G.J. Martin, J. Schillewaert, "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". arXiv:2204.11422

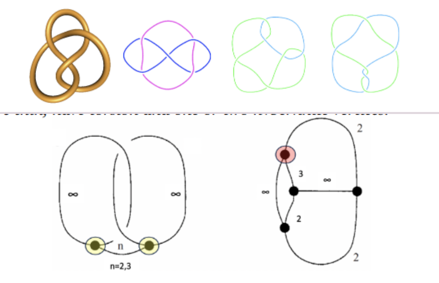
Thou shalt not try to give technical proofs

Theorem

Out of the Kleinian groups generated by 2 parabolics:

- *Gehring–Maclachlan–Martin (1998): exactly 4 are arithmetic*
- *E.–Martin–Schillewaert (2024): exactly 3 are thin*

E.–Martin–Schillewaert (to appear): Out of the Kleinian groups generated by 2 parabolics or elliptics, approx. 150 are thin .



Bedtime reading

Written notes with references to further reading:
`https://aelzenaar.github.io/kg/heckoid_talk.pdf`

- A.J.E., G.M., and J.S., *Approximations of the Riley slice.* Expo. Math., 2023 (arXiv 2021).
- —, G.M., and J.S., *Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds. 2021–22 MATRIX annals.* Springer, 2024 (arXiv 2022).
- —, G.M., and J.S., *On thin Heckoid and generalised triangle groups in* $\mathrm{PSL}(2,\mathbb{C})$ (arXiv 2024).
- Vladimír Burgus and Jan Mlčoch, *Czech photography of the 20th century.* Kant, 2010.