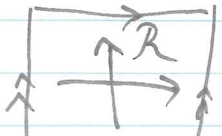
Lecture: Surface theory. 1/10

§1. TILINGS.

Rmk: Jensen said this course should show you how mathematicians "think"; the contents of the lecture is basically how I would "develop from scratch" basic surface theory, and the result is a relatively GOT. form,

T.

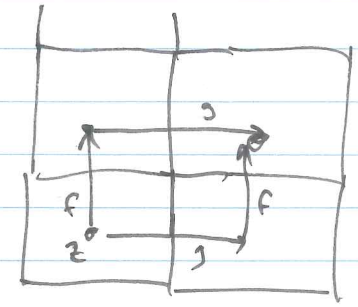
Example (from primary school).



glue opposite sides of a rectangle:

Embed R into $\mathbb{R}^2$ and define $f, g: \mathbb{R}^2 \to \mathbb{R}^2$ the translations shifting sides to each other.

Observation



for all z, $f(g(z)) = g(f(z))$

i.e. $[f,g] = 1$.

No "other" relations hold.

$\longrightarrow \langle f, g\rangle$ "$=$" $\pi_1(\mathbb{T})$.

~~Prop: Tiling ⇒ discrete: suppose $g_n \to Id$. The diam(tile) → 0.~~

Exercise: There is a homeomorphism between

$$\left\{ \mathbb{T} = R/\sim \right.$$

$\sim$ the "gluing edges" reln, i.e. $x \sim y$ iff either $x = y$ or x, y are opposite each other on the bdry.

and

$$\left\{ \mathbb{R}^2/\langle f, g\rangle \right.$$

, i.e. the quotient of $\mathbb{R}^2$ by the group action.

Tiling ⇒ discreteness.

Suppose G tiles X with tiles T of open interior.

~~Then~~ Pick $x \in \operatorname{int} T$. If G is not discrete, then exists a sequence $g_1, \ldots, g_n \to 1$ of distinct elements. So $g_1 T, g_2 T, \ldots$ are disjoint.

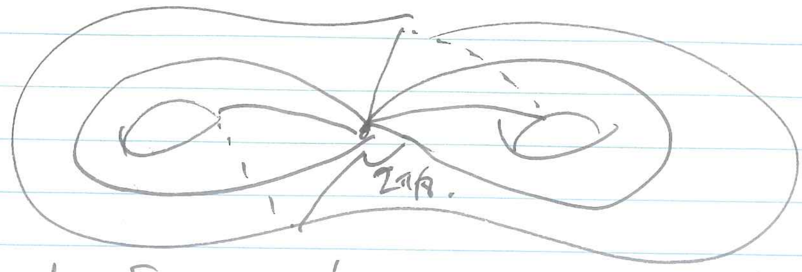
But $g_n x \in g_n T$ $\forall n$, and $g_n x \to x$

so all but finitely many $g_n x$ lie in some open nbd $x \in U \subseteq \operatorname{int} T$

$\rightsquigarrow \operatorname{int} g_n T \cap \operatorname{int} T \neq \emptyset$.

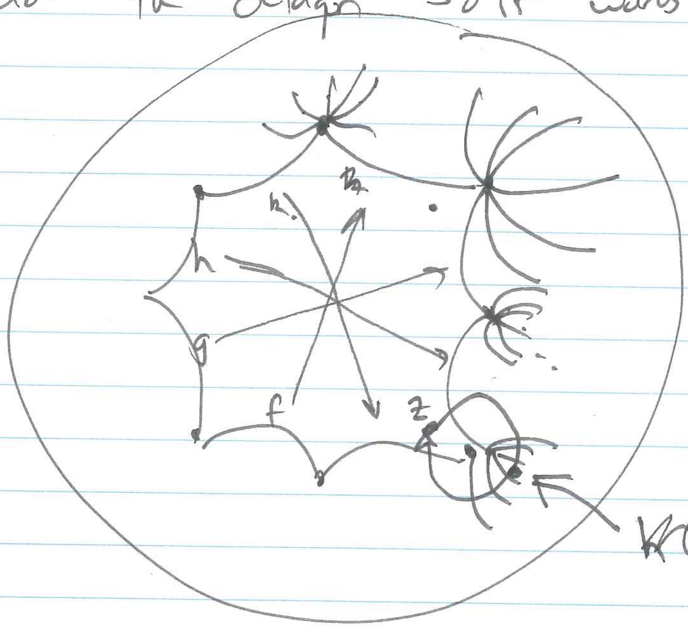
$\Rightarrow \Leftarrow$.

Example



cut $S_{2,0}$ along π_1-gens: observe the tile should have angles adding to 2π, and this is not possible in $\mathbb{R}^2$ (no octagons tile the plane).

"Bend" the octagon so it works:



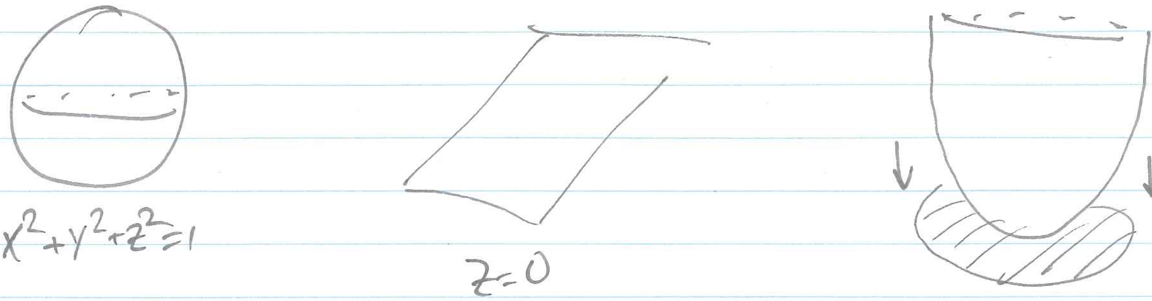
$$\cancel{kf}\, kfg^{-1}h(z)$$
$$= hg^{-1}fk(z)$$

i.e. $kfg^{-1}h\, k^{-1} f^{-1} g\, h^{-1} = 1$.

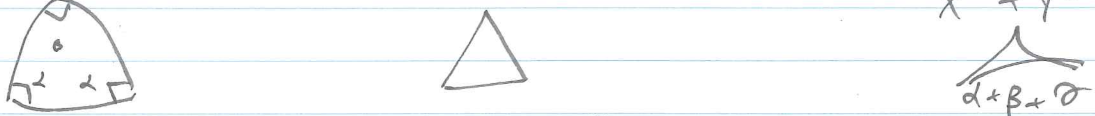
("Cayley graph is octagonal tiling".)

§2 HYPERBOLIC 2-SPACE.

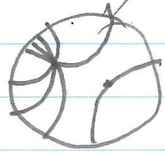
Trichotomy:



$x^2+y^2+z^2=1$	$z=0$	$x^2+y^2-z^2=1$.

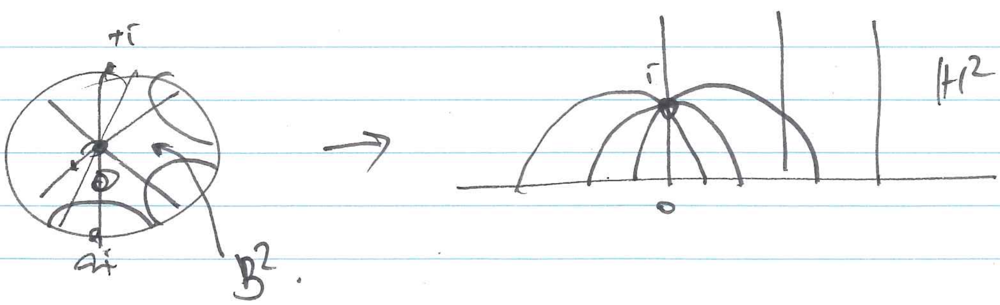


$\alpha+\beta+\gamma>\pi$	$\alpha+\beta+\gamma=\pi$	$\alpha+\beta+\gamma<\pi$.
no parallel lines	unique parallel line	∞ many parallel lines:



Geodesics:	great circles	straight lines	semicircles orthog. to the circle S^1.

There exists a unique conformal map $\mathbb{B}^2 \to \mathbb{H}^2$ which sends $0 \to i$ & $+i \to \infty$:



On $\mathbb{H}^2$, the metric is given by integrating $ds^2 = \frac{dx^2+dy^2}{y^2}$:

$$\cosh d(w,z) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}z\,\mathrm{Im}w}.$$

Isometries of $\mathbb{H}^2$:

conformal maps preserving $\mathbb{H}^2$

$\Updownarrow$

fractional linear maps with $\mathbb{R}$-coeffs:

$$z \longmapsto \frac{az+b}{cz+d} \quad \left(\begin{array}{l} a,b,c,d \in \mathbb{R} \\ ad-bc \neq 0 \end{array} \right)$$

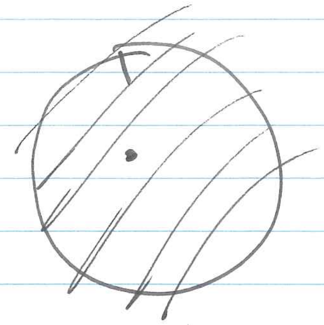
$\Updownarrow$

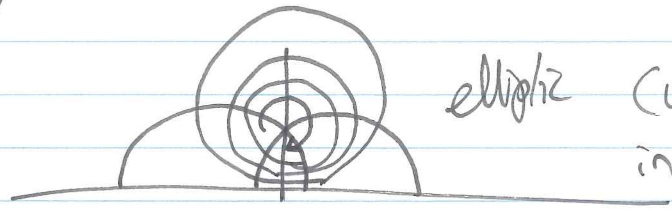
elements of $PSL(2,\mathbb{R})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

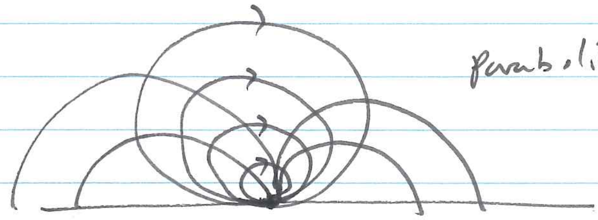
<u>Exercise:</u> compute $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \infty$.

3 kinds of isometry:

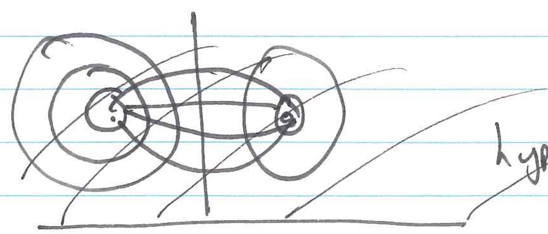




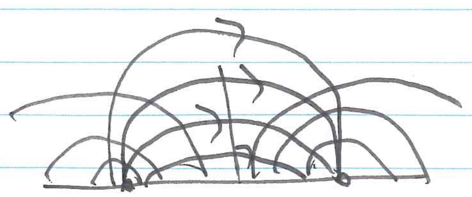
elliptic (unique f.p. in $\mathbb{H}^2$)



parabolic (unique f.p. on $\partial\mathbb{H}^2$)



hyperbolic (2 f.p. in $\mathbb{H}^2$).



hyperbolic (2 f.p. in $\partial\mathbb{H}^2$).

Exercises.

- Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a fixed pt at $z \neq \infty$, then $\begin{pmatrix} z \\ 1 \end{pmatrix}$ is an eigenvector.

- Suppose $M \in PSL(2,\mathbb{R})$ has a unique eigenvector $\begin{pmatrix} x \\ 1 \end{pmatrix}$. Show that
$$(\operatorname{tr} M)^2 = 4.$$
(and conversely, $(\operatorname{tr} M)^2 = 4 \Rightarrow$ unique fixed pt for M as a fractional linear transfm.)

- Deduce also that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvectors $\begin{pmatrix} w \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \end{pmatrix}$ then ~~and~~ $w, z \in \mathbb{R}$ iff $(\operatorname{tr} M)^2 < 4$
$w, z \in \mathbb{C} \setminus \mathbb{R}$ iff $(\operatorname{tr} M)^2 > 4$.

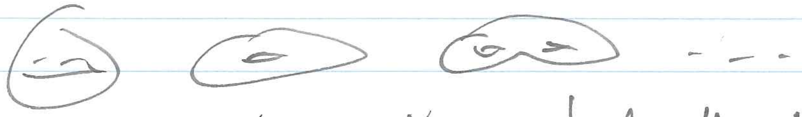
(Hint. $\underset{\text{eigenvec}}{\begin{pmatrix} x \\ 1 \end{pmatrix}} \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix} \Leftrightarrow \dfrac{ax+b}{cx+d} = x.$)

§3. UNIFORMISATION.

<u>Theorem.</u> If ~~Not~~ X is a compact, oriented* surface

* Surface means either
- Riemann surface, or
- ~~a~~ ~~Riemann surface~~
- Riemannian manifold of dim 2

then X is one of



i.e. X is topologically classified by the # of handles (genus).

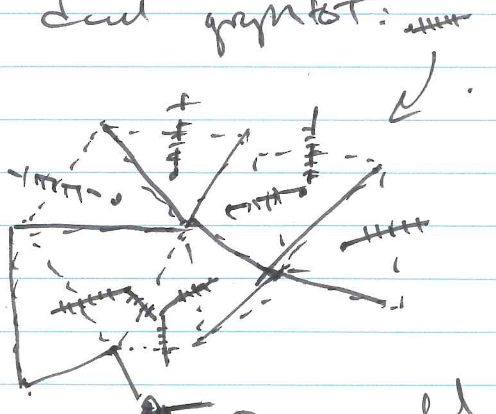
<u>Proof†.</u> Assume X is triangulated‡.

~~Lemma~~ Defn. $\chi(X) = $ # of vert. $-$ # of ed. $+$ # of faces (Euler char).

<u>Lemma.</u> If $\chi(X) = 2$, then $X = S^2$.

<u>Proof.</u> Choose a maximal tree T in the edges of the triangn. Let Γ be the dual graph to T:

i.e. verts of Γ = faces, edges = edges not in T.



Exercise: Γ is connected. (Suppose not: T ~~[illegible]~~, separates two triangs, hence contains a loop.)

† we follow Zeeman's proof as explained by Andy Putman.

‡ This is actually hard to prove but "obvious".

Suppose X has v, e, f verts, edges, faces.

Then T has v verts & e_0 edges, $e_0 \le e$.

Γ has f verts & e_1 edges, $e_1 = e - e_0$.

So
$$\begin{aligned}\chi(X) &= v - e + f \\ &= v - e_0 - e_1 + f = (v - e_0) + (f - e_1) \\ &= \chi(T) + \chi(\Gamma).\end{aligned}$$

<u>Exercise:</u> G connected $\Rightarrow \chi(G) \le 1$, sharp iff G is a tree. ⌟

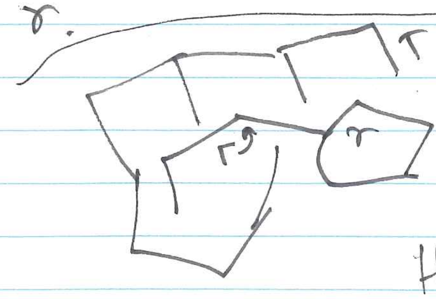
$\rightarrow \chi(T) = 1$ & $\chi(X) = \chi(\Gamma) + 1 \le 2$.

where $\chi(X) = 2 \iff \Gamma$ is a tree.

Suppose as in the lemma statement that $\chi(X) = 2$, so Γ & T are trees; then we can pick regular nbhds of Γ, T which are topological discs: 2 discs glue to a 2-sph.

□ (lemma)

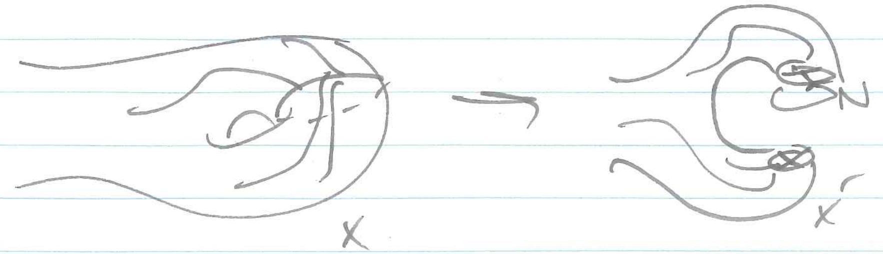
Suppose $\chi(\Gamma) < 2$. Γ is not a tree so has a loop, γ.



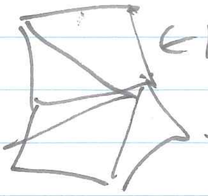
γ cannot sep the surface, since T visits every ~~vertex~~ face & is connec.

Here $\exists$ a path in T which goes from one side of γ to the other.

Cut along γ & glue in faces.



Note:



So: - # of new vertices $= V+N$.

# of new edges $= e + \underset{\substack{\uparrow \\ \text{new edges} \\ \text{from copying } \gamma}}{N} + \underset{\substack{\uparrow \\ \text{for caps.}}}{\underline{2(N-3)}}$

# of new faces $= f + 2(N-2)$.

$$\rightsquigarrow \quad \chi(X') = V+N - (e+N+2N-6) + f+2N-4$$

$$= V + \cancel{N} - e - \cancel{2N} + 6 + f + \cancel{2N} - 4$$

$$= \chi(X) \cancel{+N} + 2.$$

$$\rightarrow \quad \chi(X') > \chi(X).$$

By induction, $X' \cong S_{g-1}$.

We glued in a torus so $X = S_g$. $\square$.

Uniformisation with geometry:

Thm: Every poss. punctured surface (not nec. cpct) is (isometric / conformal) to one of

S^2 $(\chi > 0)$; $\mathbb{R}^2$ or $\mathbb{T}$ $(\chi = 0)$; or $\mathbb{H}^2/\Gamma$ $(\chi < 0)$

for $\Gamma \leq PSL(2,\mathbb{R})$ discrete. $\square$.

(Proof: hard!)

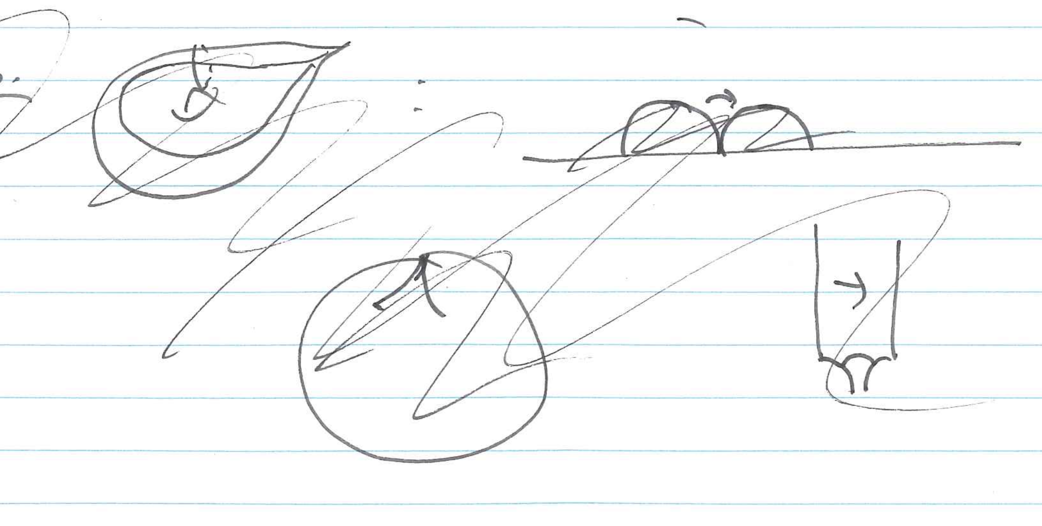
Remarks – conjectured by Klein (1883) & Poincaré (1882).

Proof by Poincaré (1907) & Koebe (").

Defn. Discrete subgps of $PSL(2,\mathbb{R})$ are Fuchsian. They are equivalently:

- holonomy gps of possibly punctured surfaces;
- actn gps of tilings of $\mathbb{H}^2$;
- disc. gps of conformal maps preserving $\mathbb{H}^2$.

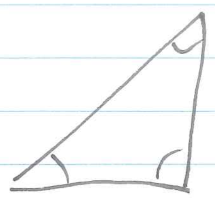
Egs.



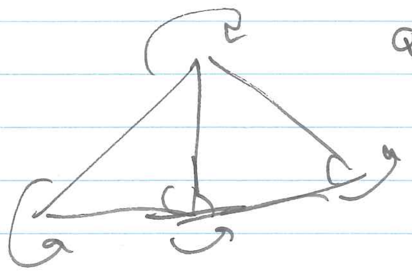
Examples.

- Δ groups. Suppose $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$.

  Then there is a hyp. Δ with angles π/p, π/q, π/r.



define $G = \langle f, g, h \rangle$ where f, g, h are the reflns in the sides. Then is an index 2 cover $\langle fg, gh, hf \rangle$ where the gens are rotations of order p, q, r:



$PSL(2,\mathbb{Z})$ is the $(2,3,\infty)$-Δ gp.

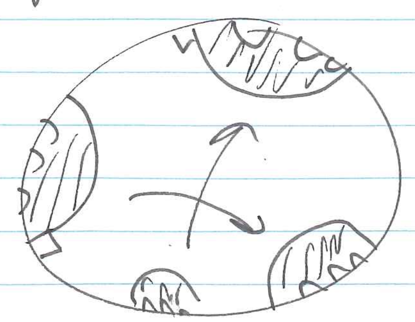
Exercise (HARD.): $PSL(2,\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$

and is gen by an order 2 elt & an order 3 elt.

(Hint: write [illegible] as a pair of cont fracs.)

Exercise: Cayley graph of a Δ gp is...?

- Schottky gps.



dual graph / Cayley graph is a tree.

APPENDIX : Definitions.

- if X is a topological space and $\sim$ is an equivalence relation,

  $X/\sim$ is the top. space with points the equivalence classes and with open sets all $U \subseteq X/\sim$ such that $\pi^{-1}(U)$ is open. ($\pi : X \to X/\sim$ projection)

  ie. the topology is the ~~smallest~~ possible to make π continuous.

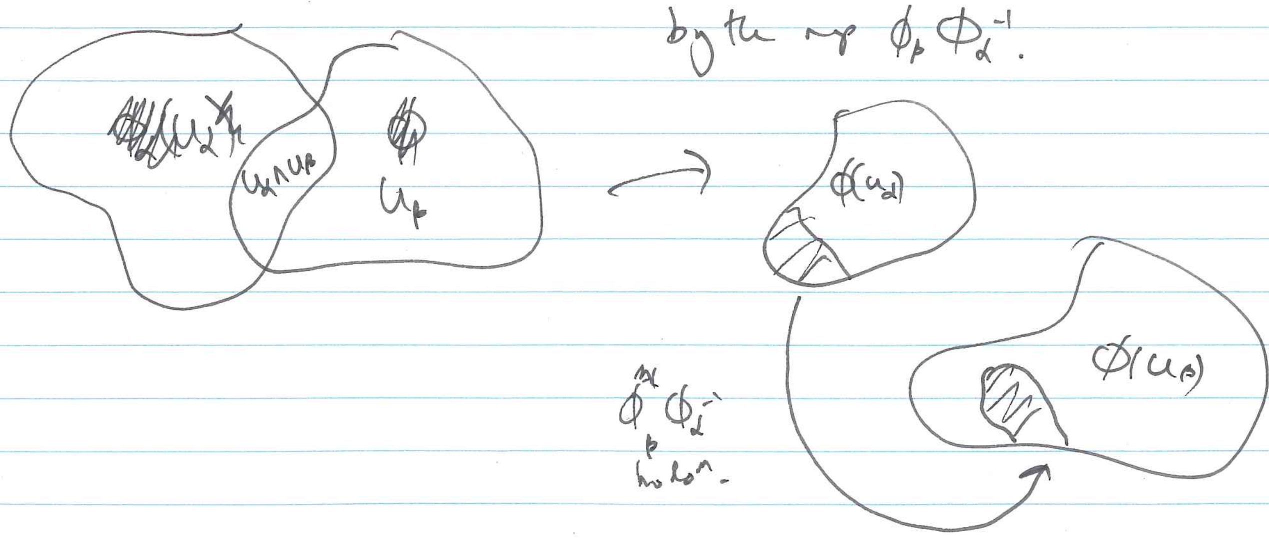
EQUIVALENT:

- If (X, d) is a metric space & $\sim$ is an eq. rel.,

  $X/\sim$ is the metric space with distance

$$d'([x],[y]) = \inf_{\substack{x' \in [x] \\ y' \in [y]}} d(x', y').$$

A RIEMANN SURFACE is a Hausdorff 2nd ctble topological space which admits an open cover $\{U_\alpha\}$ and a family of maps $\phi_\alpha : U_\alpha \to \mathbb{C}$ s.t. $\phi_\alpha(U_\alpha \cap U_\beta)$ is holomorphic to $\phi_\beta(U_\alpha \cap U_\beta)$ by the map $\phi_\beta \phi_\alpha^{-1}$.



GROUP ACTIONS

If X is a space (e.g. a set, a top. space, etc.)
then an action of G on X is a homomorphism

$$G \to \mathrm{Aut}(X).$$

e.g. X a metric space, $\mathrm{Aut}(X)$ = set of isometries,
so for all $g \in G$ we "have" an isometry.

If G acts on X ($G \curvearrowright X$) then X/G is obtained from the equivalence relation $x \equiv y$ if $\exists\, g \in G$ s.t. $gx = y$,
i.e. X/G = the set of orbits of X under G.
If G acts nicely then X/G usually has a similar kind of structure to X. (This is one of the main ideas of the course.)

If $x \in X$, $\mathrm{Stab}_G\, x = G_x = \{g \in G : gx = x\}$.