Discontinuous subgroups of $\mathrm{Aut}(\mathbb{S}^2)$ come in real-algebraic families with stable combinatorics

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9th Australian Algebra Conference, La Trobe University

František Kupka, *Čáry, plochy, hloubka III* [Lines, surface, depth III] (1913–1923)
National Gallery Prague

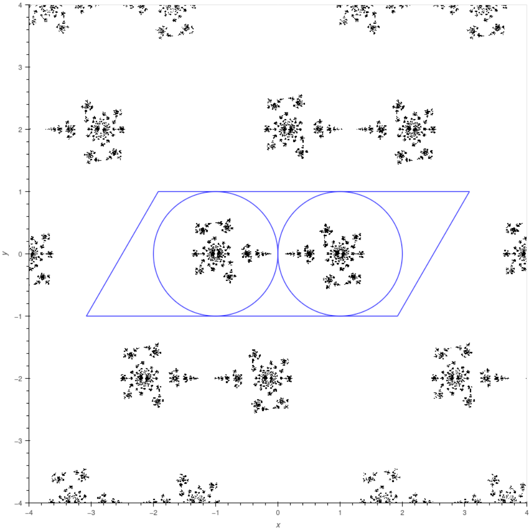
Let G be a group, usually infinite.

representations		conformal actions
$G \to \mathrm{PSL}(2,\mathbb{C})$	$\Longleftrightarrow$	$G \quad \mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$

Example

For $G = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$, a family of representations is

$$G(\alpha,\beta,\lambda) = \left\langle \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{bmatrix} \right\rangle$$



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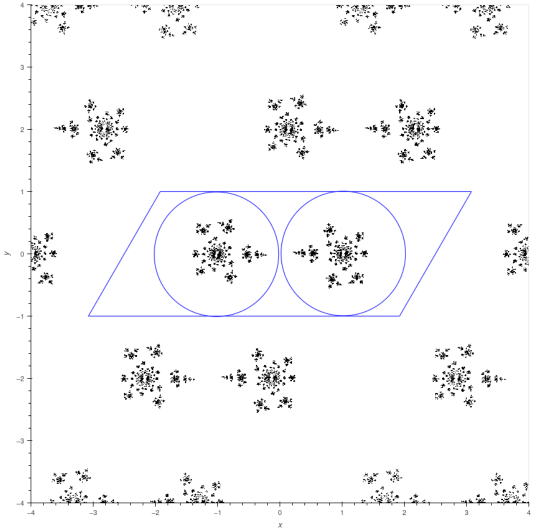

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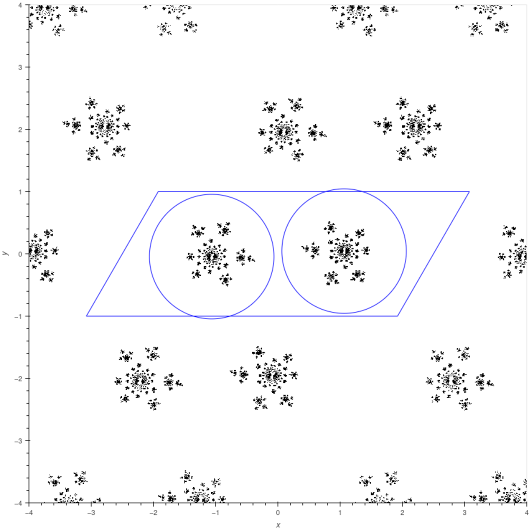

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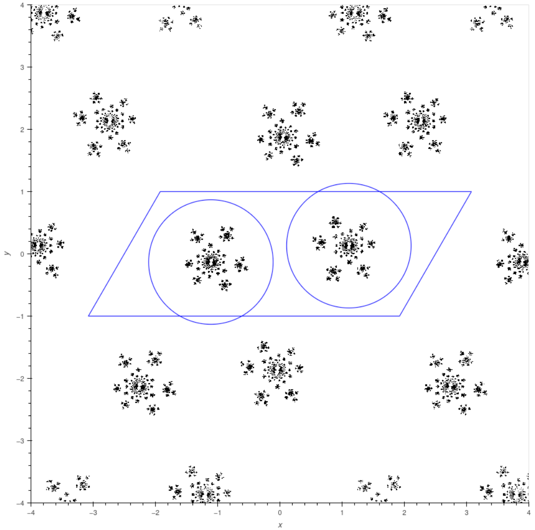

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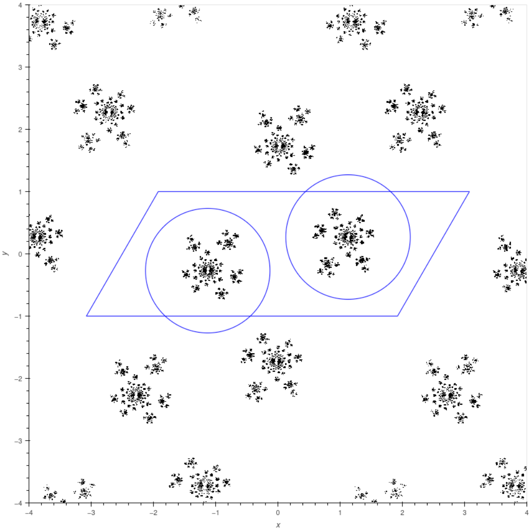

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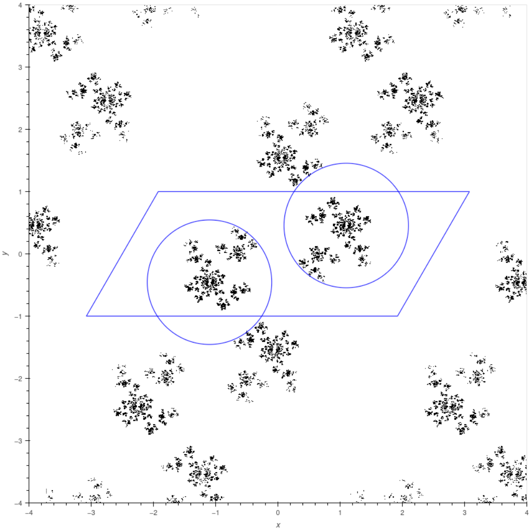

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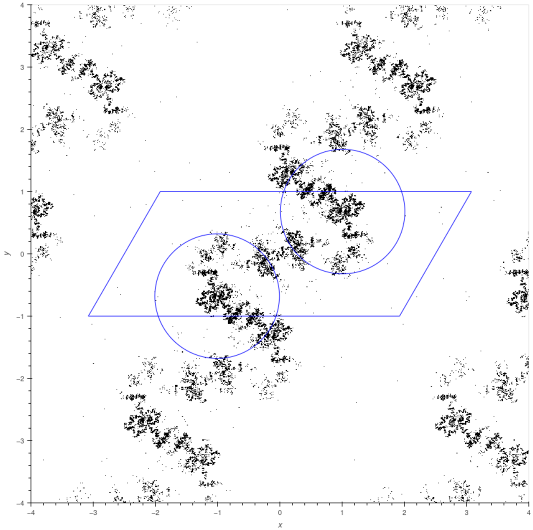

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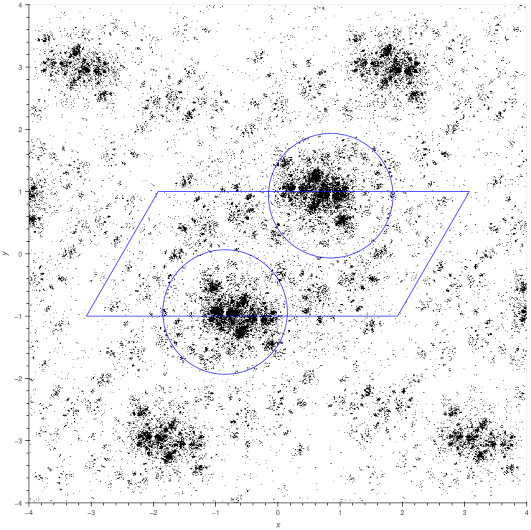

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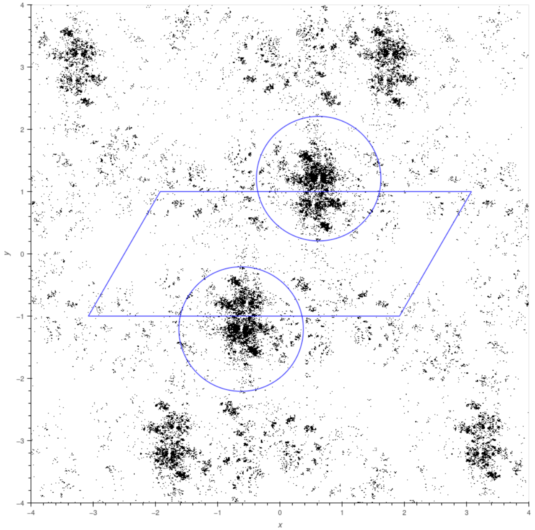

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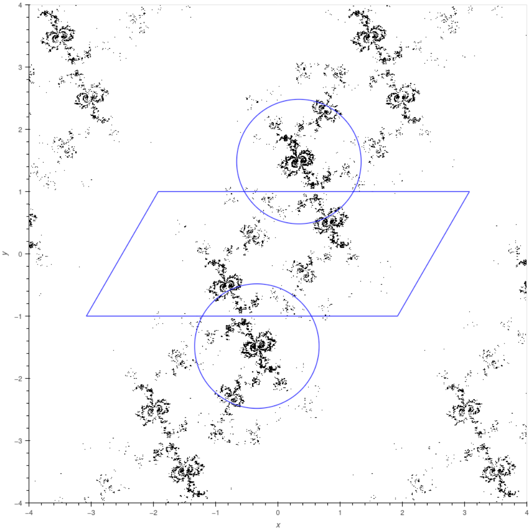

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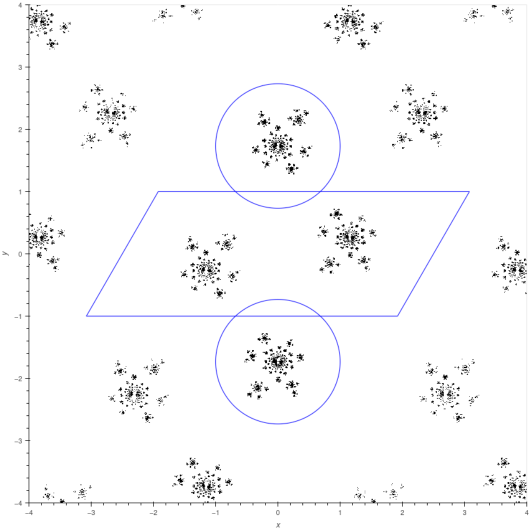

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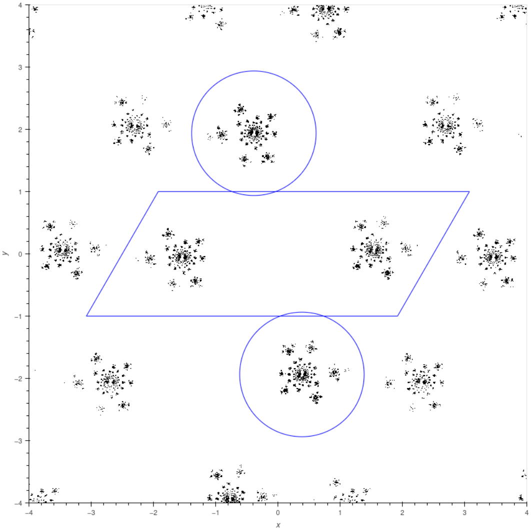

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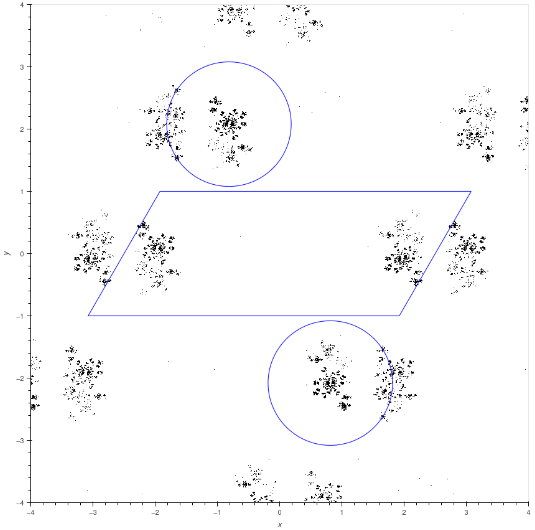

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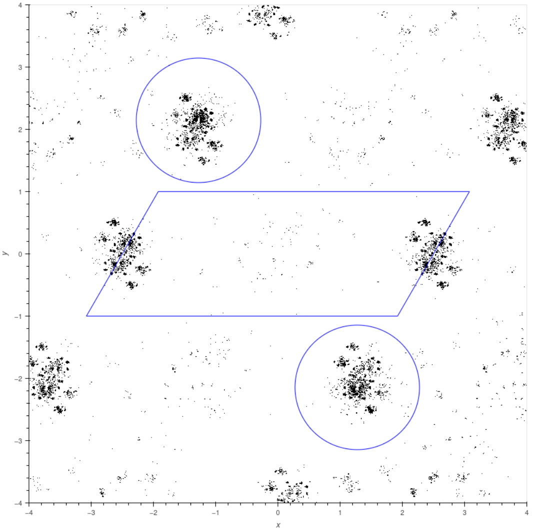

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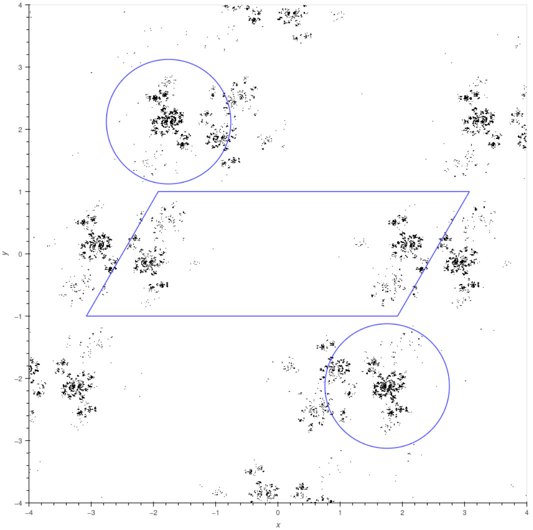

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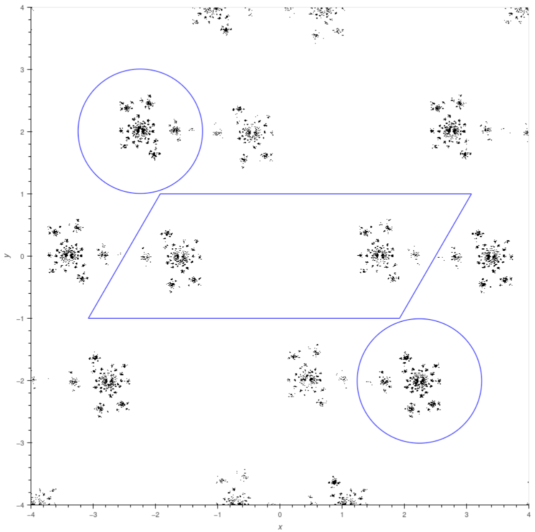

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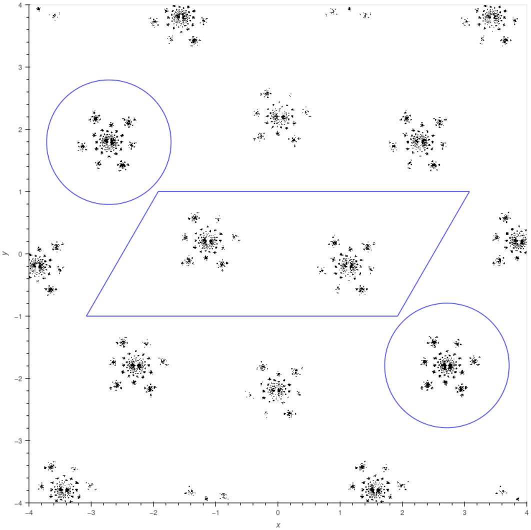

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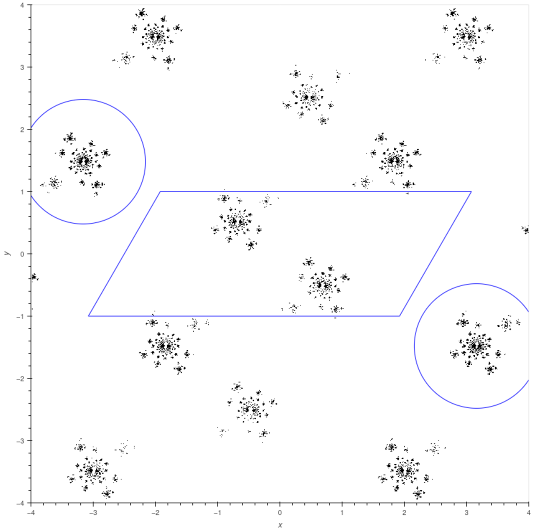

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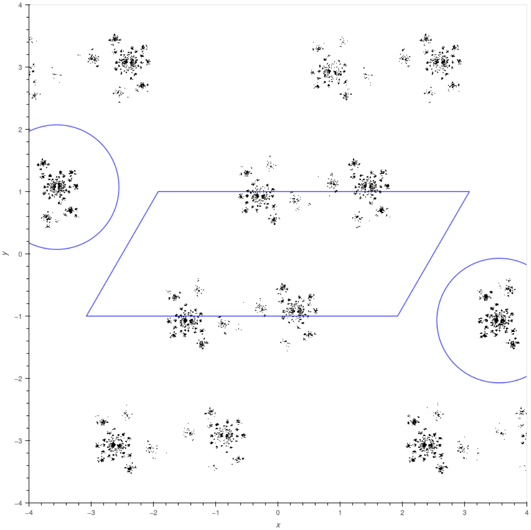

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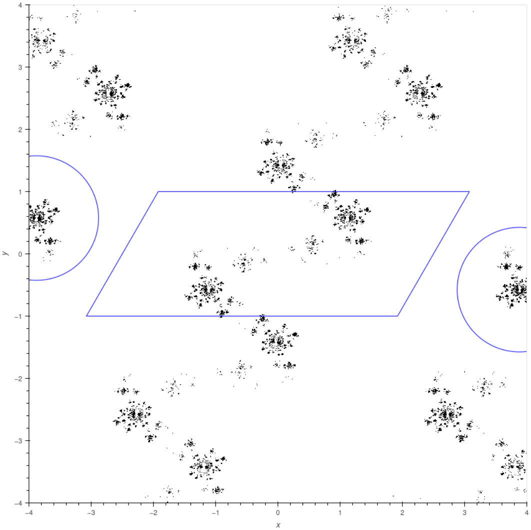


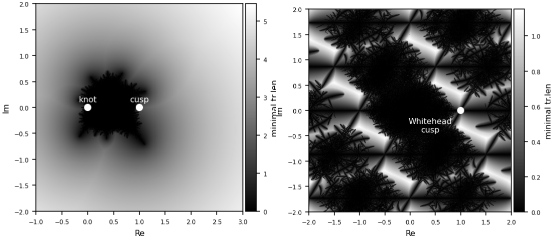
Small deformations of the rep. $G \to \mathrm{PSL}(2,\mathbb{C})$ are sometimes stable[a] (small deformations = small change in 'global behaviour').

Given a representation, how do you:

- check whether it is stable under small deformations?
- compute the global geometry (e.g. isomorphism class, quasi-isometry class)?
- compute the extent of the stable locus it lies in, if any?

[a]defn: discrete & non-empty Ω

Two slices through $\mathrm{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \mathrm{PSL}(2,\mathbb{C}))$.
White = island of stability



E., "From disc patterns in the plane to character varieties of knot groups" `arXiv:2503.13829 [math.GT]`

Theorem (Ahlfors–Bers–Maskit theorems (c.1970) + Marden isomorphism theorem (1974) + λ-lemma (early 90s) + Ending lamination theorem (conj. Thurston 1982, proved Brock, Canary, Minsky & others c.2004))

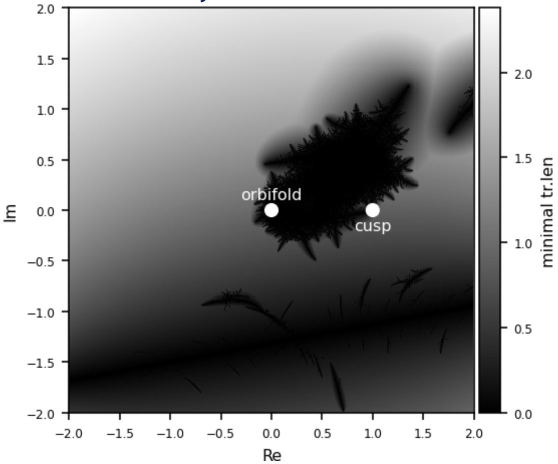
For G a group:

- *Islands of stability in* $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$ *are (equivalently):*
  - *deformation spaces of hyperbolic metrics on fixed topological* 3*-manifolds;*
  - *quasi-isometry classes of the representations;*
  - *set of all conjugates by quasiconformal maps;*
  - *maximal open sets where the limit points have not congealled into rigid circle packings, possibly with bits filled in (Ahlfors measure* 0 *conj./theorem)*
- *Every island of stability is a quotient of a product of Teichmüller spaces; so stable representations are detected by looking at Riemann surfaces* $\Omega(\rho(G))/\rho(G)$ *where* ρ *is the rep and* $\Omega(\rho(G)) \subset \hat{\mathbb{C}}$ *is maximal so that the quotient is Hausdorff.*

Problem

These high-powered theorems are far from effective. It's known that the islands of discreteness are embedded very wildly (e.g. not locally connected, see Canary, *Introductory bumponomics*, arXiv 2010); compare with the PSL(n, $\mathbb{R}$) theory, where components are fairly well understood from real algebraic point of view.

A slice through Hom($\mathbb{Z} * \mathbb{Z}$, PSL(2, $\mathbb{C}$)). White = island of stability



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Theorem (E., “Peripheral subgroups of Kleinian groups”, arXiv 2025)

There exists a computable exhaustion of any stable region in any algebraic parameterisation of $X(G)$ by semi-algebraic sets.

Strategy of proof.

1. Find a dense set of semi-algebraic subsets of the desired stable region. These are *pleating varieties* and correspond to groups with very nice coarse geometry.
2. Thicken each semi-algebraic subset to a countable set of full-dimensional open semi-algebraic subsets.
3. Observe that the union of all these semi-algebraic sets is a decomposition of the entire stable region.

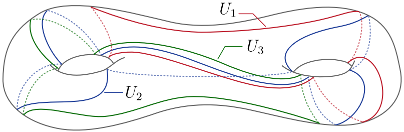
Definition

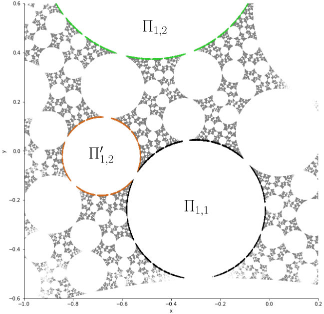
An F-peripheral subgroup of a rep. $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ is some $\Pi \subset \rho(G)$ such that

1. Π is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$
2. Π acts on a disc $\Delta \subset \hat{\mathbb{C}}$ so that G acts discontinuously on Δ and $\Delta/G = \Delta/\Pi$

The set of ρ which contain a maximal set of peripheral subgroups is called a *pleating variety*.

Pleating varieties are semi-algebraic and locally closed in the stability region.



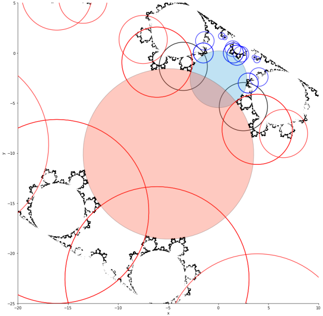


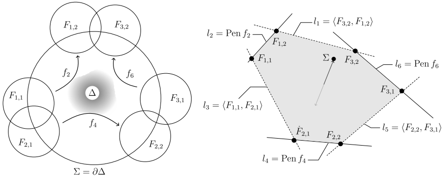
E., "Peripheral subgroups of Kleinian groups" `arXiv:2508.00297 [math.GT]`

Thickening pleating varieties

- If ρ is on a pleating variety, then the F-peripheral subgroups induce a 'canonical' fundamental domain for $\rho(G)$.
- This domain is stable under small perturbations of the generators of $\rho(G)$. we can write down semi-algebraic conditions on the pertubations that guarantee stability.
- Method: convert the 'canonical' fundamental domains into incidence structures in $\mathbb{P}^3\mathbb{R}$. Rewrite the action on $\mathbb{P}^1\mathbb{C}$ into one on $\mathbb{P}^3\mathbb{R}$, then work with geometric inequalities.

In reality, these conditions are hard to compute even though the proof is fully constructive.





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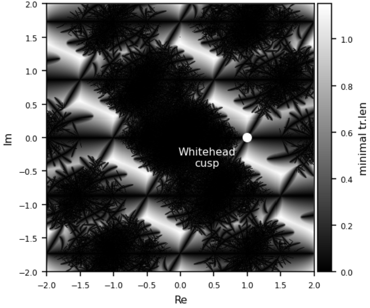
Bedtime reading



- E., G. Martin, J. Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". *2021–22 MATRIX annals*. Springer, 2024.
- E., *From disc patterns in the plane to character varieties of knot groups*. `arXiv:2503.13829 [math.GT]`
- E., *Peripheral subgroups of Kleinian groups*. `arXiv:2508.00297 [math.GT]`
- Albert Marden, *Hyperbolic manifolds*. Cambridge, 2016.
- Katsuhiko Matsuzaki and Masahiko Taniguchi, *Hyperbolic manifolds and Kleinian groups*. Oxford, 1998.