Lecture 2a: Conformal maps.

We recap:

Thm (Poincaré Klein Koebe):

If S is a surface* with genus g and n punctures/~~[illegible]~~ discs then set $\chi = 2-2g-n$.

If $\chi(S) = 2$, then $S = S^2$.

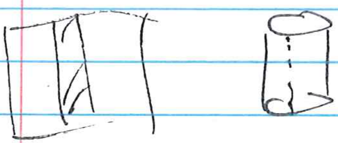
If $\chi(S) = 0$ or 1 then $S = \mathbb{R}^2/\Gamma$ where Γ is a discrete gp of Euclidean isoms.

If $\chi(S) < 0$, then $S = \mathbb{H}^2/\Gamma$ for Γ a discrete gp of hyperbolic isoms.

e.g.  plane $= S^2 \setminus \{pt\}$

$g=0$, $n=1$:

$\chi = 2-1 = 1$.



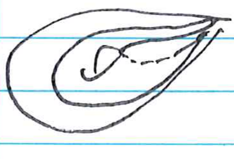
cylinder $= S^2 \setminus 2\,pts$ $\quad \chi = 0$.

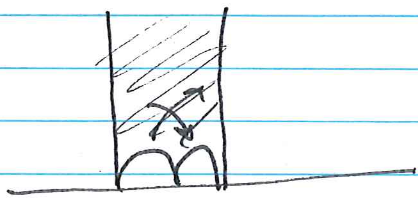
$\mathbb{R}^2/\langle z \mapsto z+1 \rangle$.

 torus: $g=1$: $\chi = 0$.

$\mathbb{R}^2/\langle z \to z+1,\ z \to z+i \rangle$.

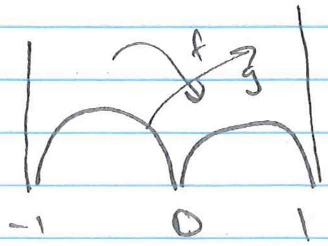
 torus $\setminus$ 1 pt: $g=1$, $n=1$. $\chi = -1$.





$\hat{\mathbb{R}}$.

Fundamental domain cont.



Map sending $-1 \to 0$, $\infty \to 1$: $f(z) = \dfrac{az+b}{cz+d}$ $\quad ad-bc=1$.

$0 = \dfrac{-a+b}{-c+d} \rightsquigarrow a=b.$

$\dfrac{a\cdot\infty+b}{c\infty+d} = 1 \iff \dfrac{a}{c} = \dfrac{a+b/\infty}{c+d/\infty} = 1$

so $a=c$.

$ad-bc=1$: $a=b=c$, so $ad-a^2=1$.

$\cancel{a^2 - ad + 1 = 0}$.

~~Also: a+d=1, so~~ $\cancel{a = \frac{d \pm \sqrt{d^2-4}}{2}}$.

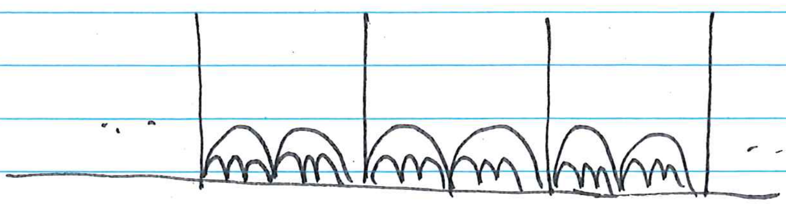
Pick arbitrarily $a=1$, so $d-1=1$

$\rightsquigarrow d=2$.

e.g. take $f(z) = \dfrac{z+1}{z+2}$.

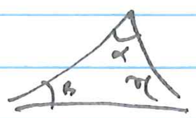
Similarly, take $g(z) = \dfrac{z-1}{-z+2}$. (note $\det\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}=3$, so the elt of PSL is $\frac{1}{\sqrt{3}}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$.)

Tiling induced by $G = \langle f, g\rangle$:



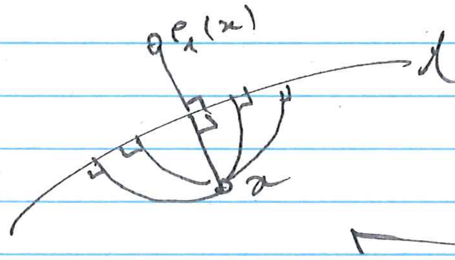
§ Triangle gps

Thm. If $p,q,r \in \mathbb{N}$ satisfy $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$, there is a hyp. Δ with angles $\pi/p, \pi/q, \pi/r$:



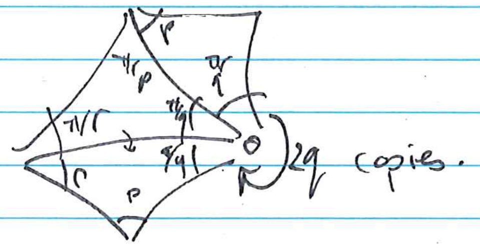
$$\alpha + \beta + \gamma < \pi$$
$$\frac{\alpha}{\pi} + \frac{\beta}{\pi} + \frac{\gamma}{\pi} < 1.$$
$$\alpha = \pi/p, \quad \beta = \pi/q, \quad \gamma = \pi/r.$$

Up till now, all elts have been orientn-presvg. Introduce reflections in the obvious way:



$\rho_\ell(x)$ = the unique pt which is on the orthogonal line to ℓ of minimal dist & opposite to x.

Defn. If $p,q,r \in \mathbb{Z}$ satisfy $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, the triangle gp $\Delta(p,q,r)$ is the gp gen by reflns in the sides of the Δ.



$\Delta(p,q,r) = \langle \rho_1, \rho_2, \rho_3 \rangle$

relns: $(\rho_1\rho_2)^p = (\rho_2\rho_3)^q = (\rho_3\rho_1)^r = 1$.

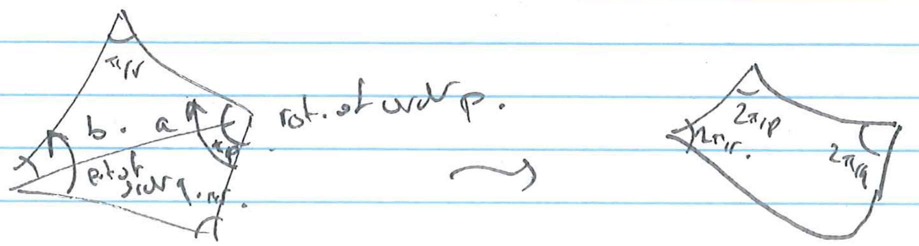
Orientn-presvg case: if G is the gp gen by all products of pairs of elts of $\Delta(p,q,r)$ then G is gen by ρ_{ij} for $i,j \in \{1,2,3\}$.

Actually have $\rho_3\rho_1 = [(\rho_1\rho_2)(\rho_2\rho_3)]^{-1}$. Then the presn is

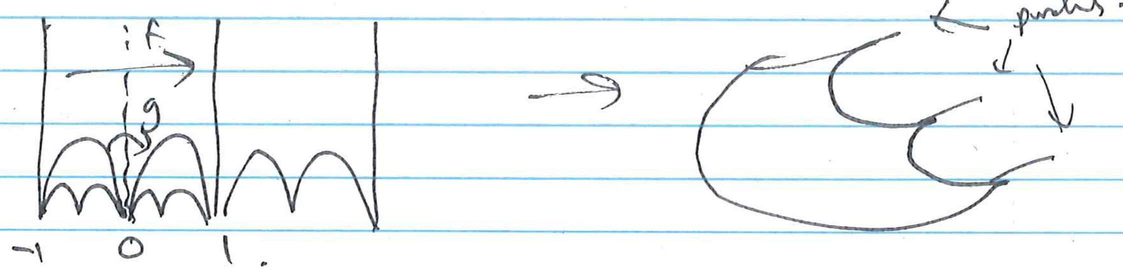
$$\langle a, b : a^p = b^q = (ab)^r = 1 \rangle.$$

Tiles.

~~Fund. domains:~~



Special example: $p=q=r=\infty$:

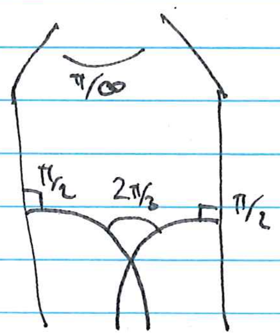


$f = z \to z+2$.

$g = z \to \frac{z}{2z+1}$, i.e. $\Delta(\infty,\infty,\infty) = \left\langle \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix} \right\rangle$.

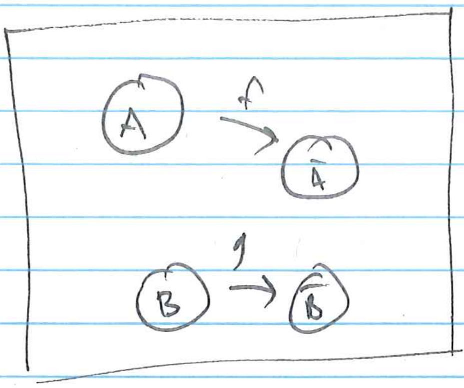
Another eg: $p=2,\ q=3,\ r=\infty$: $\mathbb{Z}$

$$PSL(2,\mathbb{Z}) = \left\langle S=\begin{pmatrix}1&1\\0&1\end{pmatrix}, T=\begin{pmatrix}0&-1\\1&0\end{pmatrix} : S^\infty = T^2 = (ST)^3 = 1 \right\rangle.$$



§ ~~Sheets~~ Bead groups.

More generally take the full group of conformal automorphisms of $\hat{\mathbb{C}}$, ie. $PSL(2,\mathbb{C})$.

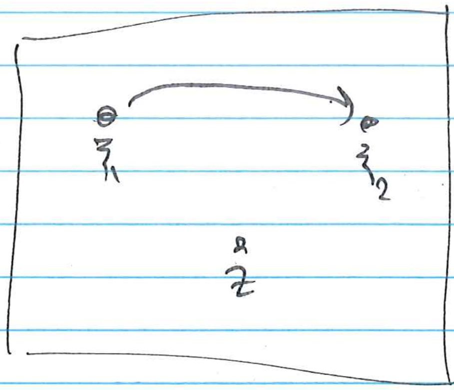


f sends ext(A) to int$(\hat{A})$ and $A \to \hat{A}$

g sends ext(B) to int$(\hat{B})$ and $B \to \hat{B}$.

By ping-pong lemma, $\langle f,g \rangle$ is free. It's called a Schottky group.

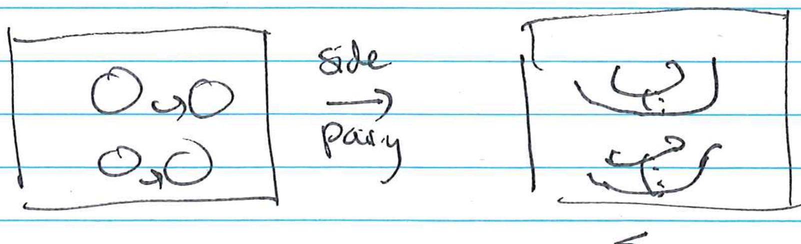
WARNING Try to compute $\hat{\mathbb{C}}/\langle f,g\rangle$:



$\text{Fix}(f) = \{\xi_1, \xi_2\}$. What is the nbhd of $[\xi_2]$ in the projection? Given any other pt z, there exists a seque $f^n z \to \xi_2$. Hence every nbhd of ξ_2 contains a translate of z; ie $[z] \in$ every nbhd of $[\xi_2]$. So $\hat{\mathbb{C}}/\langle f,g\rangle$ is not Hausdorff!

Delete all bad pts, call the set Ω.

Now

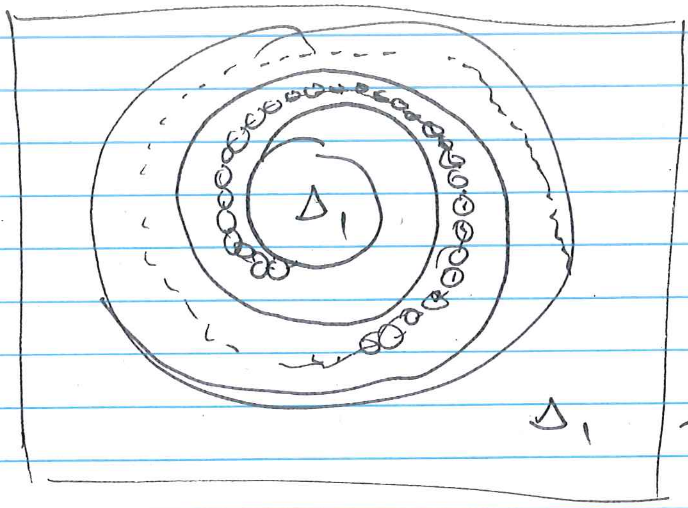


Σ_2 $= \Omega/\Gamma$. (Poincaré polyhedron thm.)

WARNING 2: Even though the quotient is Σ_2, Γ is not $\pi_1(\Sigma_2)$.

In general, $\Omega = \hat{\mathbb{C}} / \{$closure of set of attracting fixed pts.$\}$, and Ω/G is a surface (possibly disconnected).

Final ex:



Let $\Gamma_{Accola} = \langle \rho_S : S$ a circle in $\rangle$.

Lemma. A fund. dom. for Γ is the exterior of all circles. The quotient surface has two components which are discs (proj. of Δ_1 & Δ_2)

~~Theorem (Ahlfors) If G is f.d., finitely gen, then~~

~~Ω/G has finitely many punctures &~~ finite genus.

~~Ω/G is obtained by deleting~~

~~finitely~~ many pts from a cpct surface.

Theorem (Ahlfors). If G is discrete, $\Omega \neq \emptyset$, and G is finitely generated, then Ω/G is a cpct surface minus finitely many pts.

In particular, never a disc (deleted disc is not conformal to a puncture.)

Cor. Γ_{Accola} is not f.g., but is discrete!