Two-bridge knots, genus two surfaces, and discrete groups with two generators

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Monash University,
Melbourne

Hodgsonfest,
University of Melbourne,
November 2024

A theological problem

Definition

A hyperbolic 3-orbifold (manifold) is a quotient

$$\mathbb{H}^3/\Gamma$$

where $\Gamma \leq \mathrm{PSL}(2,\mathbb{C})$ is discrete (and torsion-free).

(Even worse:)

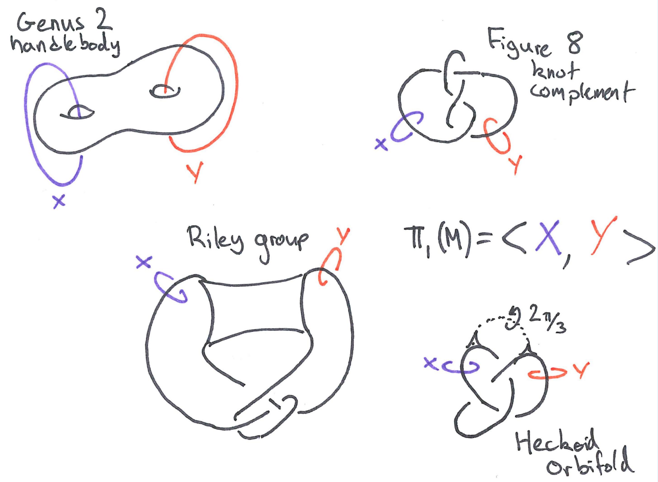
Definition

A hyperbolic 3-orbifold (manifold) is the set of cosets of

$$\mathrm{SU}(2,\mathbb{C})\backslash\,\mathrm{PSL}(2,\mathbb{C})/\Gamma$$

where $\Gamma \leq \mathrm{PSL}(2,\mathbb{C})$ is discrete (and torsion-free), with the Riemann metric coming from the Haar measure...

What do all these 3-manifolds have in common?



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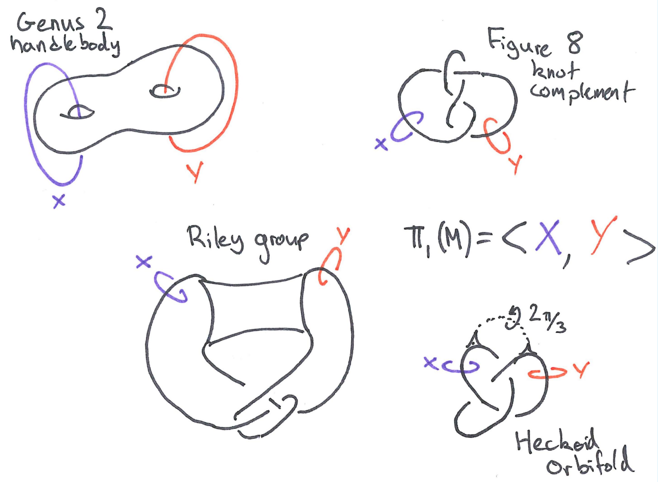
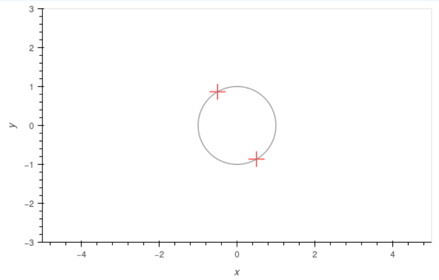
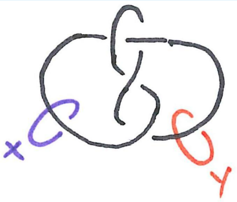


What do the holonomy groups of orbifolds with two parabolic or elliptic generators look like, and where do they live?

Warnings and disclaimers

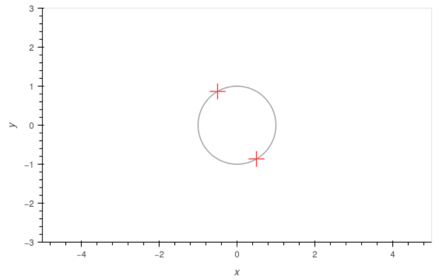
- There are no theorems in this talk. We will ignore essentially all technical detail, instead focusing on what the groups actually look like, and how they sit in their universe. If you want theorems, come to my talk in Auckland in December.
- Much new work here is joint with Gaven Martin and Jeroen Schillewaert. It was motivated by Martin's 20+ year programme to enumerate all the arithmetic groups on two generators in $\mathrm{PSL}(2,\mathbb{C})$.
- Historically many of the ideas arose in the work of a large number of authors. The most influential for us were Robert Riley, Linda Keen, Caroline Series, R.C. Lyndon, J.L. Ullman, Hirotaka Akiyoshi, Hideki Miyachi, Ken'ichi Ohshika, Makoto Sakuma, Yasushi Yamashita, and Eric Chesebro, but this is very far from an exhaustive list.

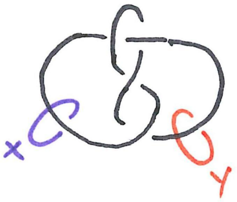




The holonomy group of the figure eight knot is

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle$$





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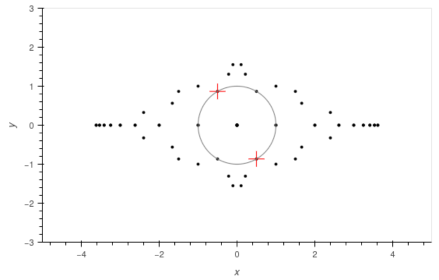
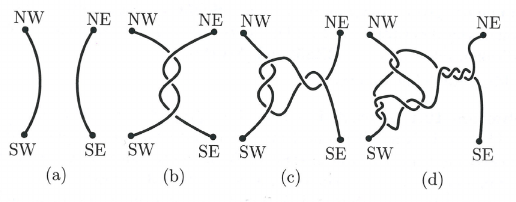
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle$$

$e^{2\pi i/3} \approx -0.5 + 0.866i$ — remember this number!



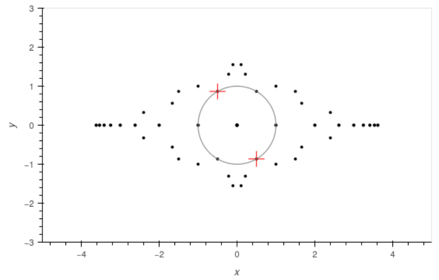


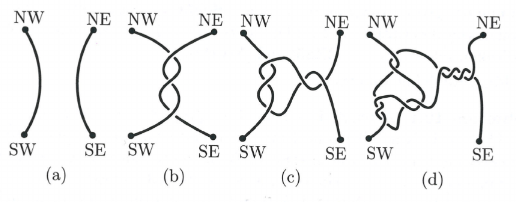
J. Purcell, *Hyperbolic knot theory*, Fig. 10.1.

Every hyperbolic 2-bridge link has a discrete holonomy group of the form

$$\Gamma_\rho = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$

for some $\rho \in \mathbb{C}$.



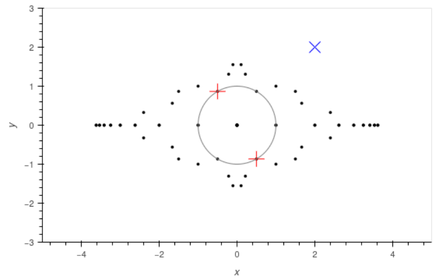


J. Purcell, *Hyperbolic knot theory*, Fig. 10.1.

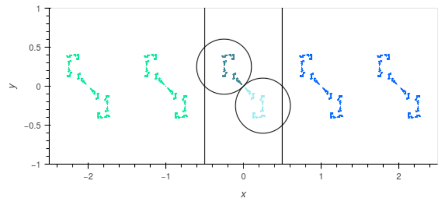
Every **non**-hyperbolic 2-bridge link's π_1 has a **non-faithful** representation of the form

$$\Gamma_\rho = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$

for some $\rho \in \mathbb{R}$.

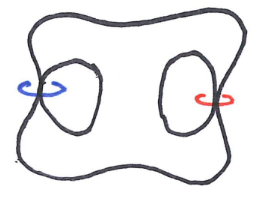


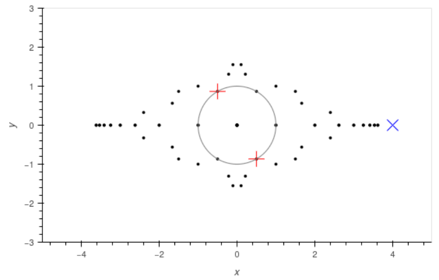
What if we make ρ bigger?



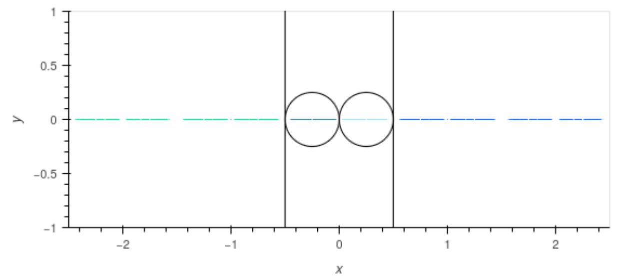
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2+2i & 1 \end{bmatrix} \right\rangle$$

The Riemann surface quotient:



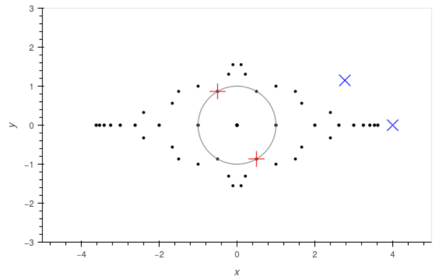


Let’s try a bunch of values of ρ.



$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (4.0 + 0.0i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete (and Fuchsian).



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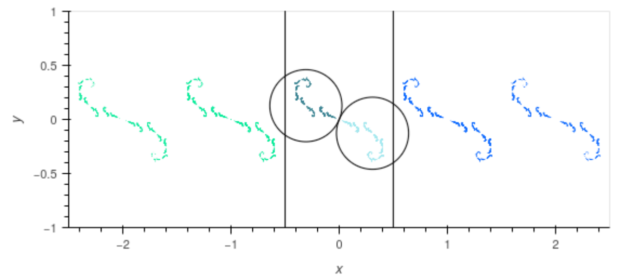
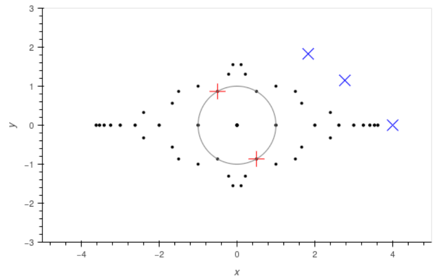


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (2.77164 + 1.14805i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete!



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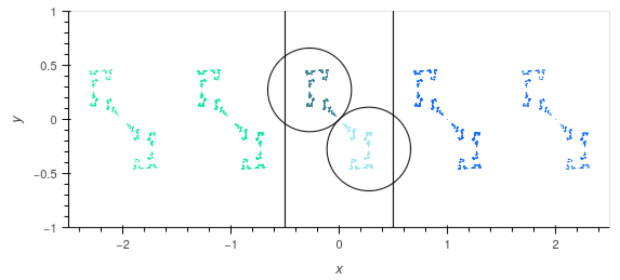
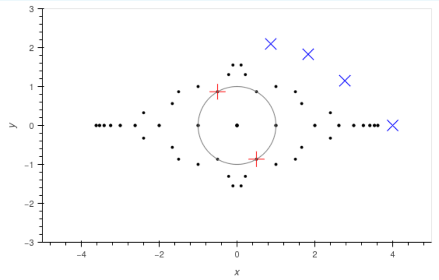


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (1.82843 + 1.82843i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete!



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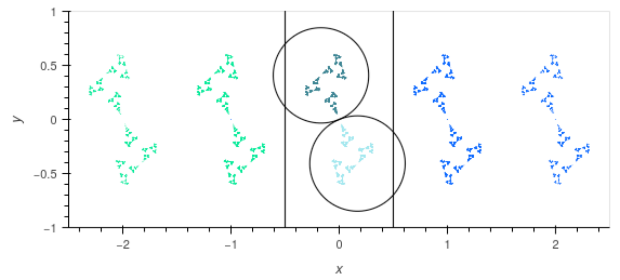
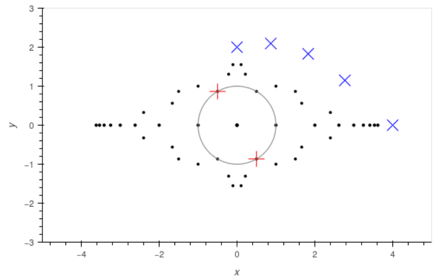


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.867907 + 2.09531i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete!



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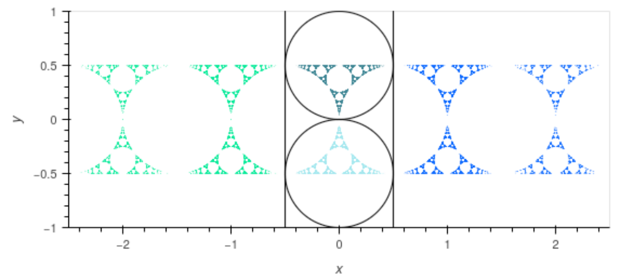

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 + 2.0i) & 1 \end{bmatrix} \right\rangle$$

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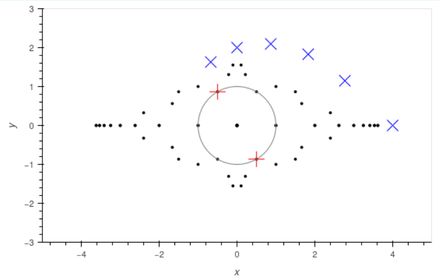

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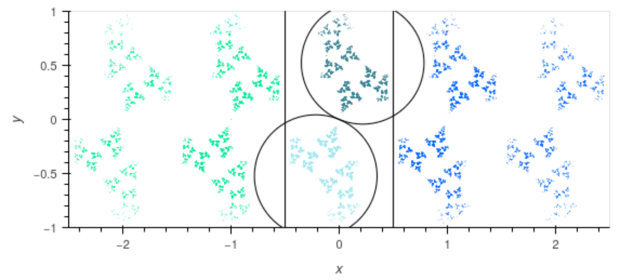
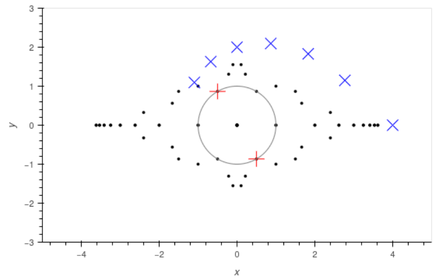


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.675028 + 1.62966i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete!



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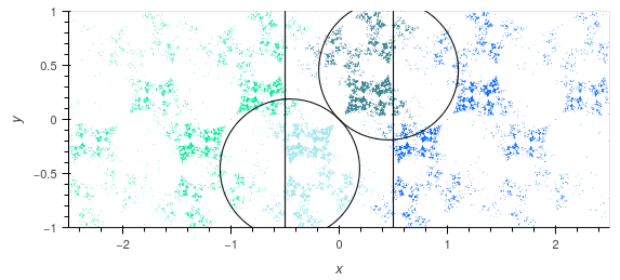
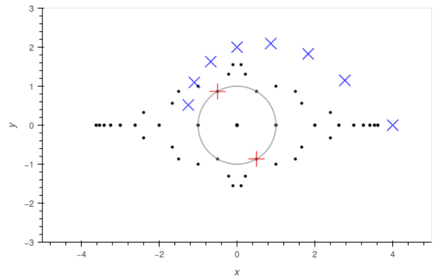


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.09638 + 1.09638i) & 1 \end{bmatrix} \right\rangle$$

Looks indiscrete.



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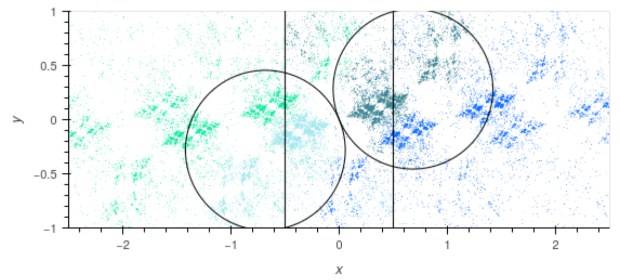
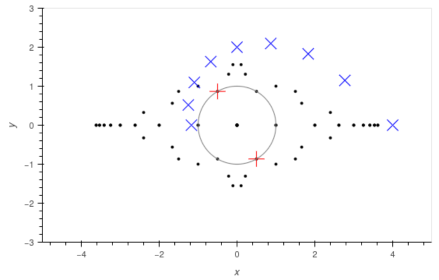


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.25116 + 0.518249i) & 1 \end{bmatrix} \right\rangle$$

Looks indiscrete.



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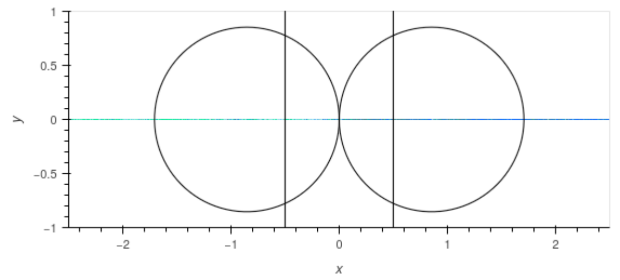
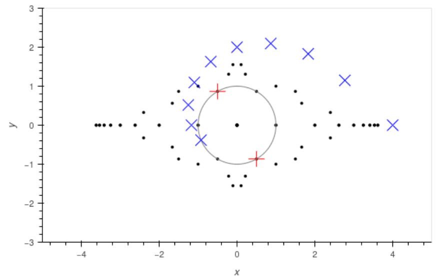


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.17157 + 0.0i) & 1 \end{bmatrix} \right\rangle$$

Almost definitely indiscrete.



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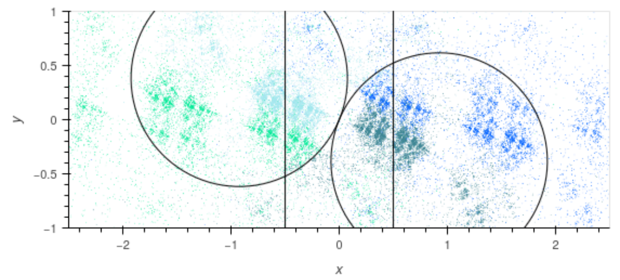
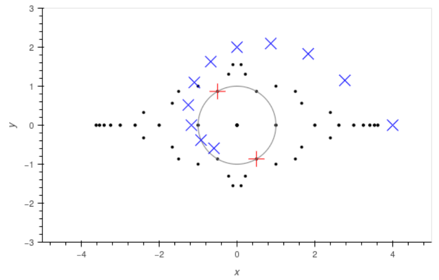


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.92388 - 0.382683i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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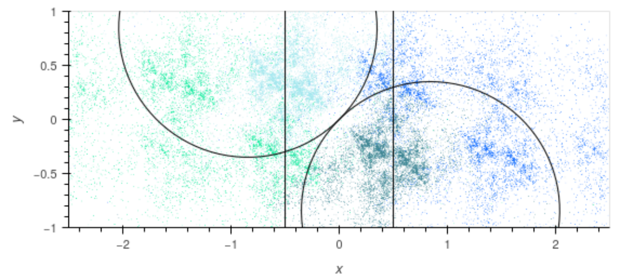
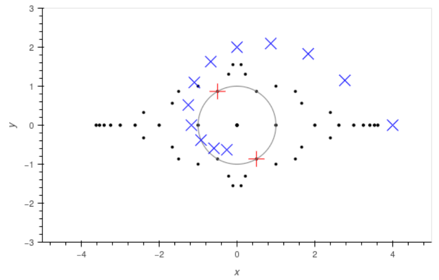


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.592359 - 0.592359i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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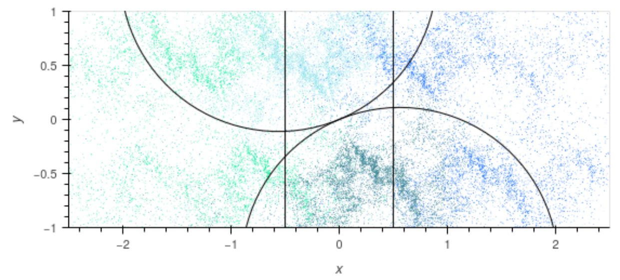
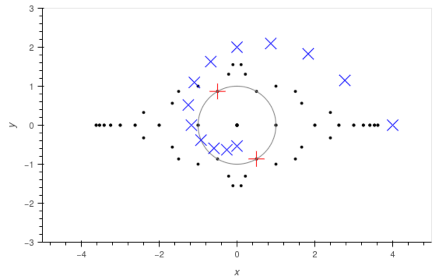


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.261516 - 0.631356i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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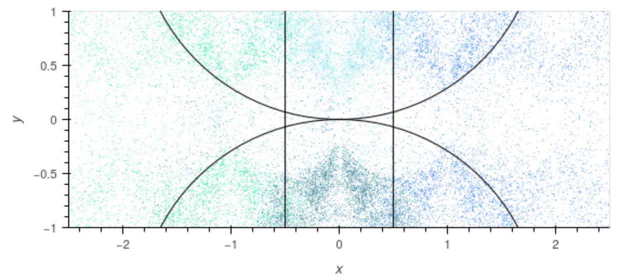
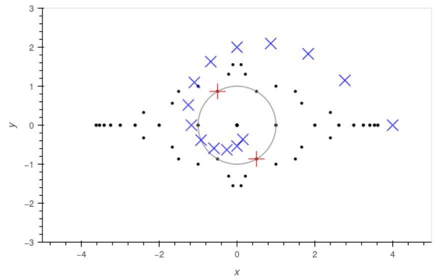


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 - 0.535898i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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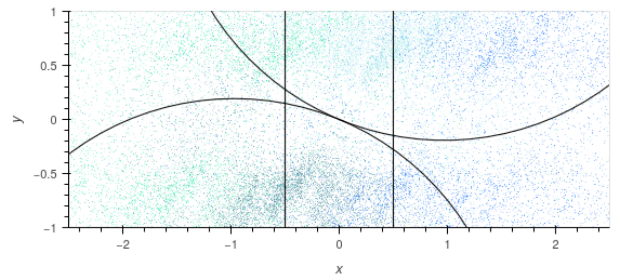
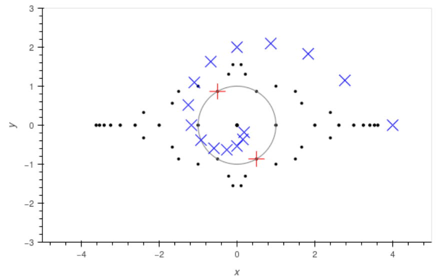


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.150949 - 0.364423i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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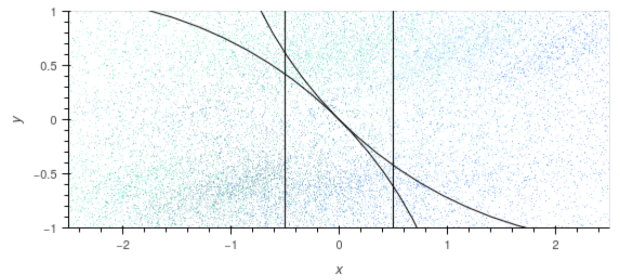
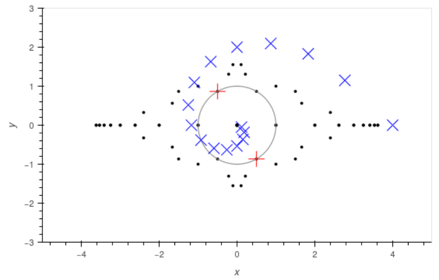


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.182676 - 0.182676i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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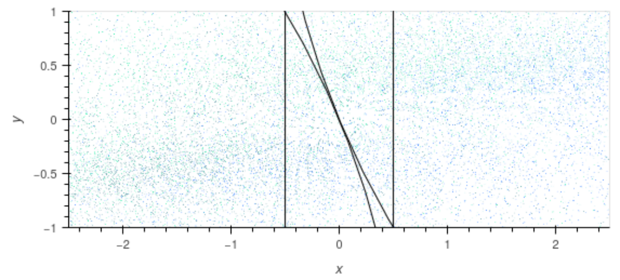
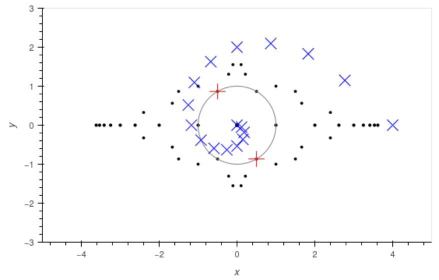


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.117348 - 0.0486072i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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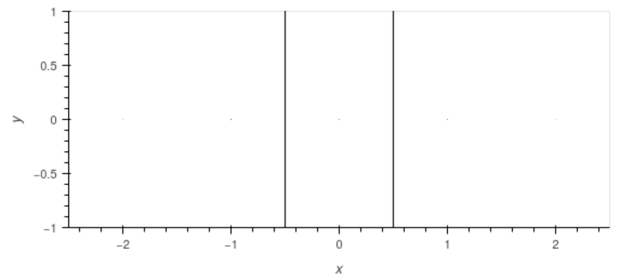
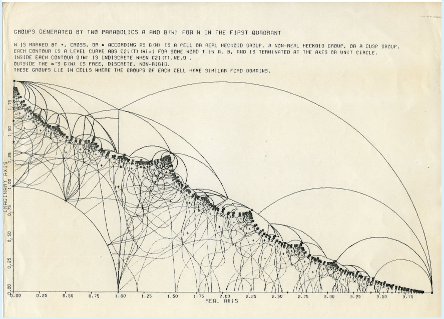


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 + 0.0i) & 1 \end{bmatrix} \right\rangle$$

Elementary group!

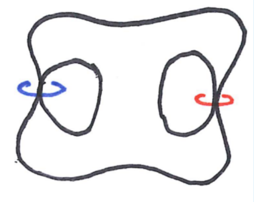
$$\Gamma_\rho = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$

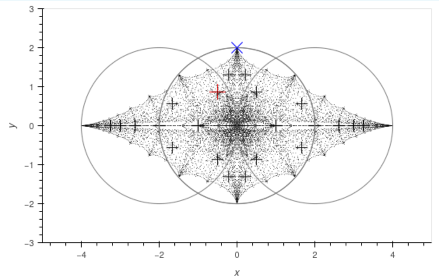


Robert Riley did these experiments in the early 1970s, and produced this plot ((+, +)-quadrant of $\mathbb{C}$).

Definition

The **Riley slice** $\mathcal{R}$ is the set of ρ such that the group Γ_ρ is discrete, and $\mathbb{H}^3/\Gamma_\rho$ is the interior of a ball with two rank one cusps drilled out.



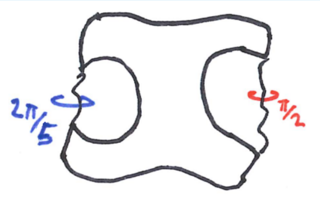


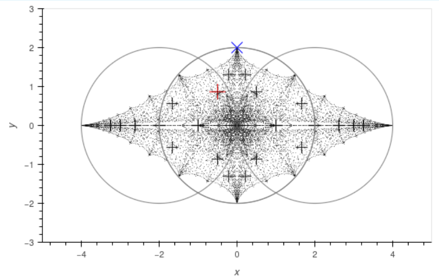
Still $\mathcal{R}$: $p = \infty, q = \infty$

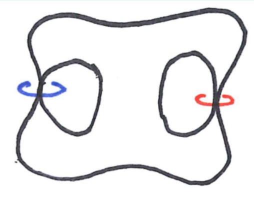
$$\Gamma_\rho = \left\langle \begin{bmatrix} e^{\pi i/p} & 1 \\ 0 & e^{-\pi i/p} \end{bmatrix}, \begin{bmatrix} e^{\pi i/q} & 0 \\ \rho & e^{\pi i/-p} \end{bmatrix} \right\rangle$$

Definition

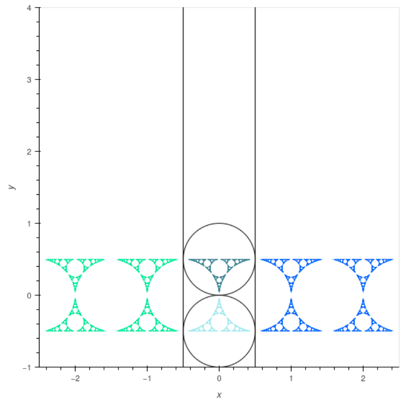
$\mathcal{R}^{p,q}$ is the set of ρ such that the group Γ_ρ is discrete, and $\mathbb{H}^3/\Gamma_\rho$ is the interior of a ball with one ideal singular arc of order p and one singular arc of order q.



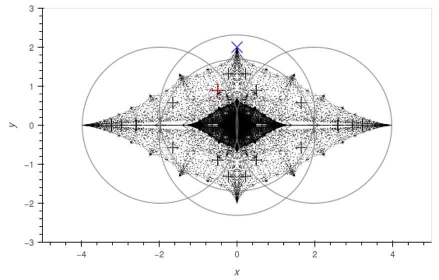




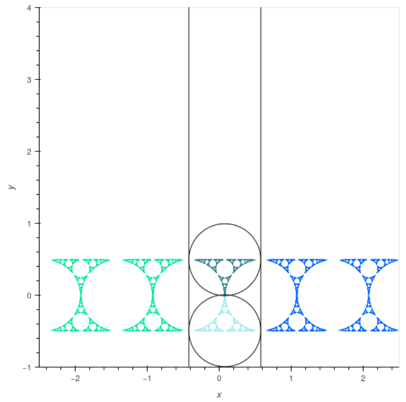
$p = \infty \qquad q = \infty$

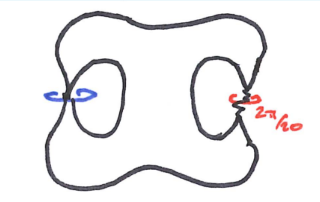


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 + 2.0i) & 1 \end{bmatrix} \right\rangle$$

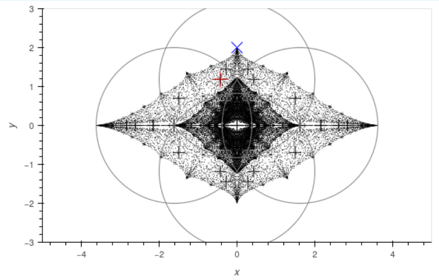


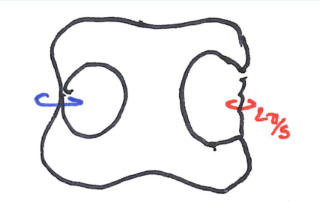
$p = \infty \qquad q = 20$



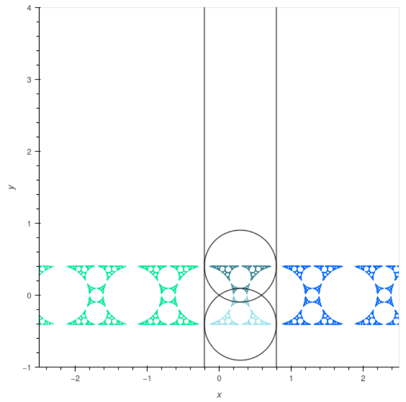


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{\pi i/20} & 0 \\ (0.0 + 2.0i) & e^{-\pi i/20} \end{bmatrix} \right\rangle$$

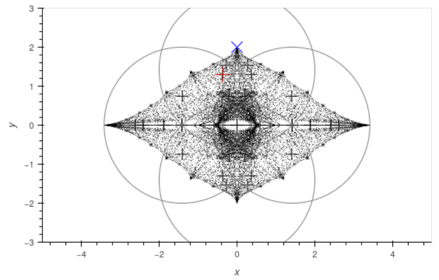


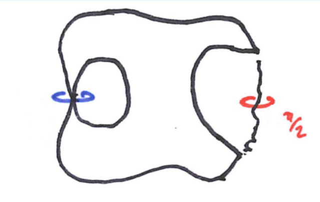


$p = \infty \qquad q = 5$

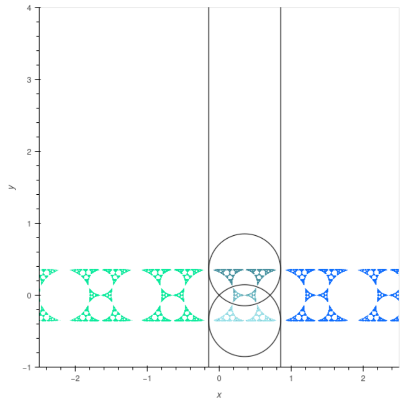


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{\pi i/5} & 0 \\ (0.0 + 2.0i) & e^{-\pi i/5} \end{bmatrix} \right\rangle$$

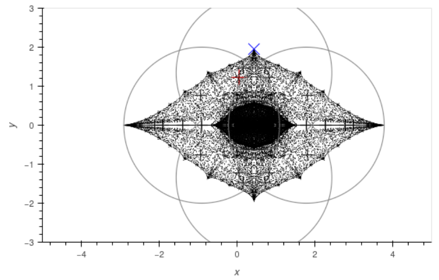




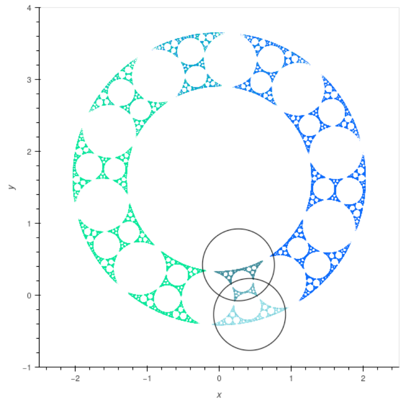
$p = \infty \qquad q = 4$

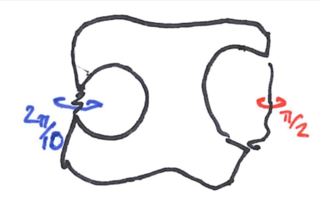


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (0.0 + 2.0i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$

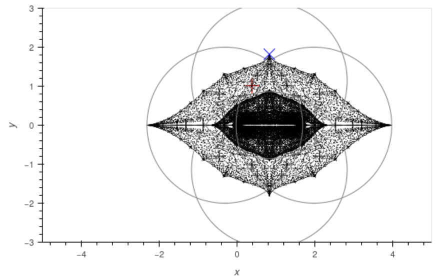


$p = 10 \qquad q = 4$

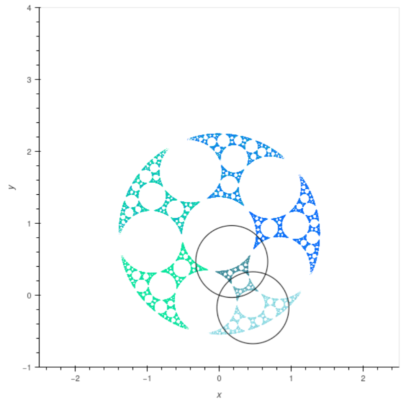


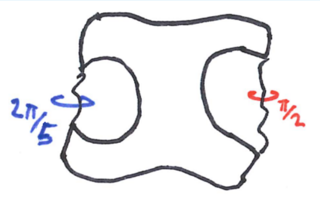


$$\left\langle \begin{bmatrix} e^{\pi i/10} & 1 \\ 0 & e^{-\pi i/10} \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (0.437016 + 1.95167i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$

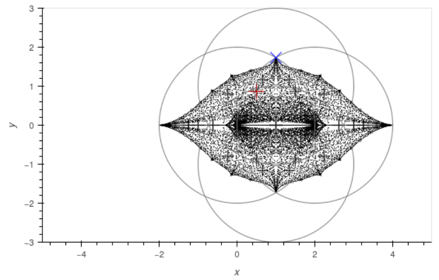


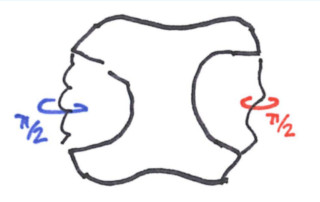
$p = 5 \qquad q = 4$



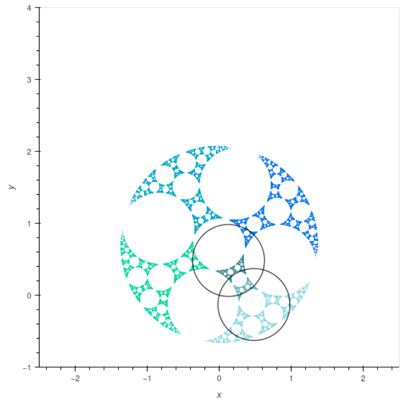


$$\left\langle \begin{bmatrix} e^{\pi i/5} & 1 \\ 0 & e^{-\pi i/5} \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (0.831254 + 1.81907i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$

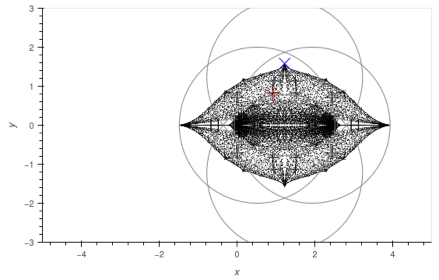




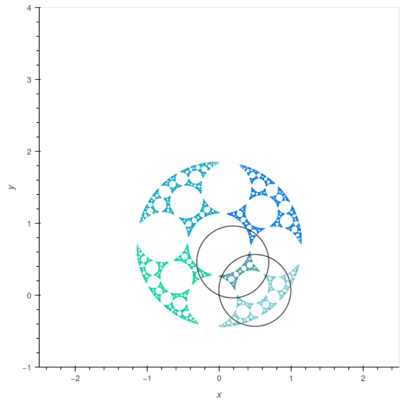
$p = 4 \qquad q = 4$

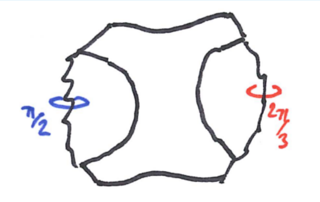


$$\left\langle \begin{bmatrix} e^{\pi i/4} & 1 \\ 0 & e^{-\pi i/4} \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (1.0 + 1.73205i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$

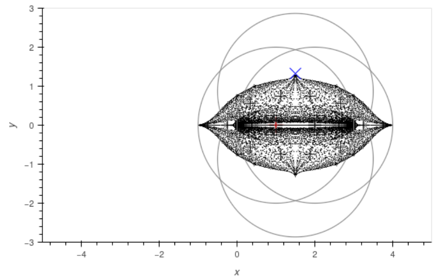


$p = 4 \qquad q = 3$

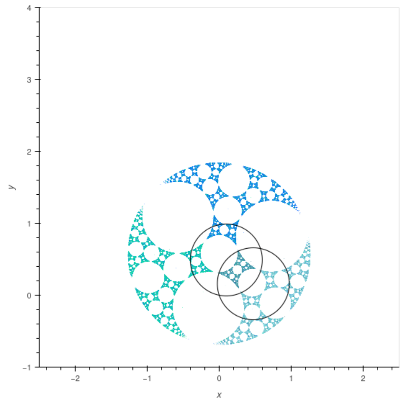


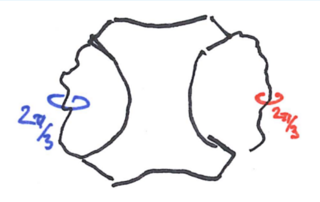


$$\left\langle \begin{bmatrix} e^{\pi i/4} & 1 \\ 0 & e^{-\pi i/4} \end{bmatrix}, \begin{bmatrix} e^{\pi i/3} & 0 \\ (1.22474 + 1.58114i) & e^{-\pi i/3} \end{bmatrix} \right\rangle$$

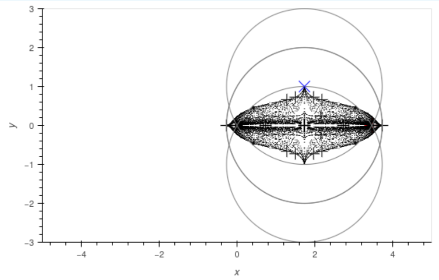


$p = 3 \qquad q = 3$

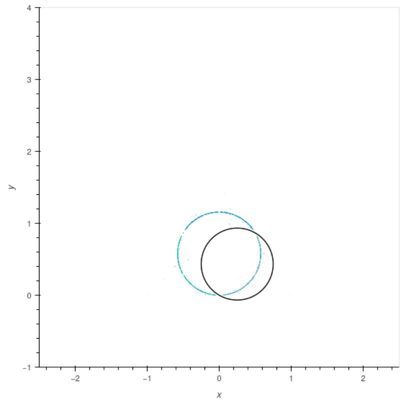


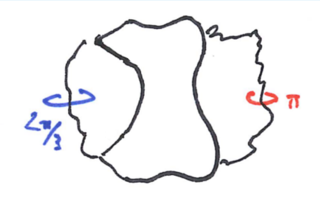


$$\left\langle \begin{bmatrix} e^{\pi i/3} & 1 \\ 0 & e^{-\pi i/3} \end{bmatrix}, \begin{bmatrix} e^{\pi i/3} & 0 \\ (1.5 + 1.32288i) & e^{-\pi i/3} \end{bmatrix} \right\rangle$$

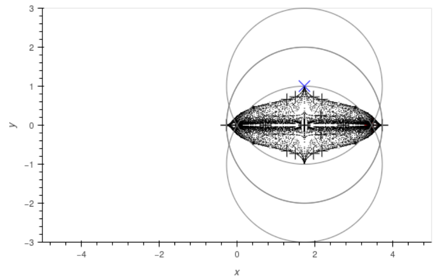


$p = 3 \qquad q = 2$



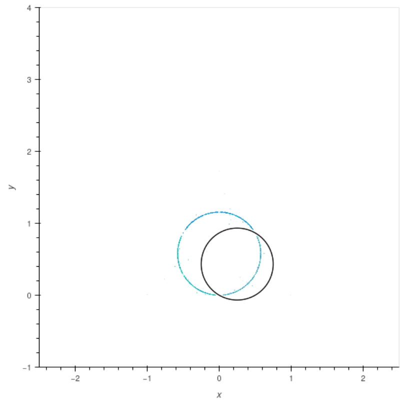


$$\left\langle \begin{bmatrix} e^{\pi i/3} & 1 \\ 0 & e^{-\pi i/3} \end{bmatrix}, \begin{bmatrix} e^{\pi i/2} & 0 \\ (1.5 + 1.32288i) & e^{-\pi i/2} \end{bmatrix} \right\rangle$$

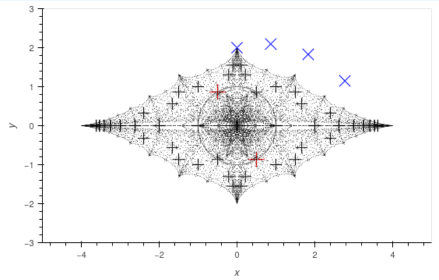


This is (up to finite index) the space of quantisations of $SL(2, \mathbb{Z}) = \mathcal{B}_3$. We (E., Gong, Martin, Schillewaert, 2024) used this theory to prove a conjecture of Morier-Genoud, Ovsienko, and Veselov about quantum rationals and Burau representations.

$p = 3 \qquad q = 2$

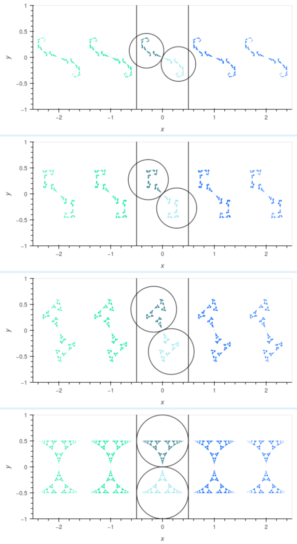


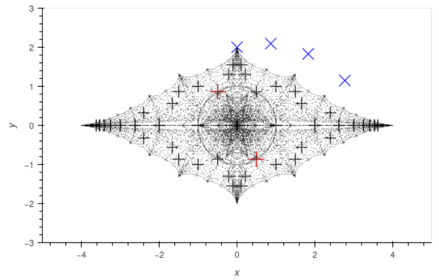
$$\left\langle \begin{bmatrix} e^{\pi i/3} & 1 \\ 0 & e^{-\pi i/3} \end{bmatrix}, \begin{bmatrix} e^{\pi i/2} & 0 \\ (1.5 + 1.32288i) & e^{-\pi i/2} \end{bmatrix} \right\rangle$$



Back to the case $p = q = \infty$ (two parabolic generators) to make things easier to visualise.

We saw earlier that going 'around' the deformation space twists the isometric circles:





Back to the case $p = q = \infty$ (two parabolic generators) to make things easier to visualise.

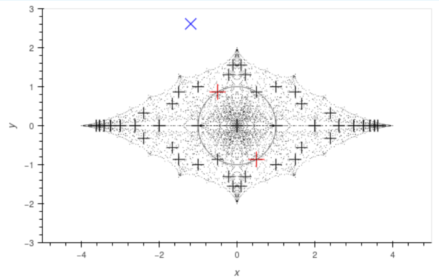
Rough idea

Going radially into the deformation space boundary preserves coarse peripheral structures.

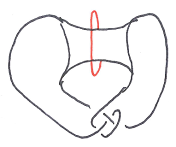
What is a coarse peripheral structure?

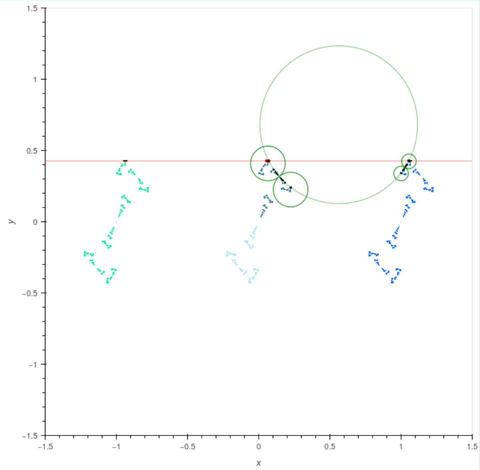
Slightly more precise idea

Teichmüller theory tells us that natural curves to the boundary of moduli space should come from preserving relative lengths of curves on the boundary surfaces of the manifolds.

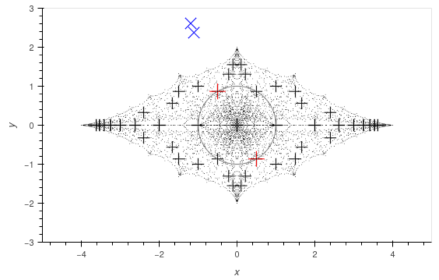


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing length: trlen $W_{3/5}$ = 9.65185

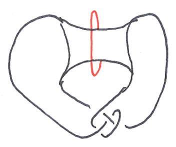


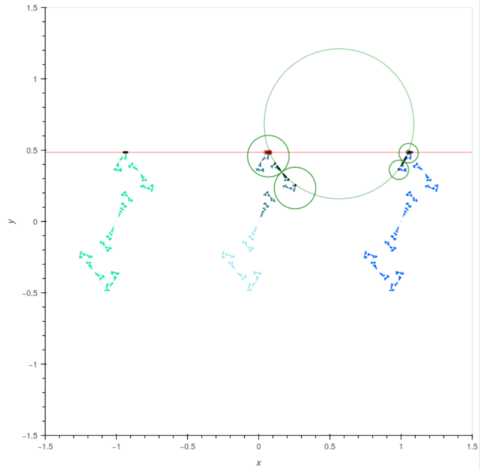


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.18884 + 2.61036i) & 1 \end{bmatrix} \right\rangle$$

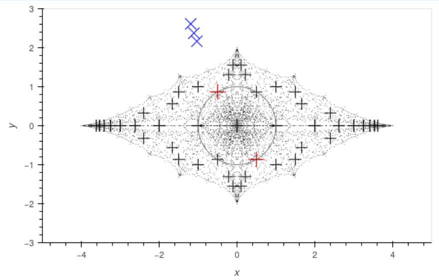


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing length: $\operatorname{trlen} W_{3/5} = 8.58525$

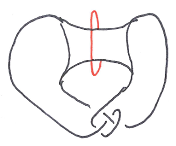


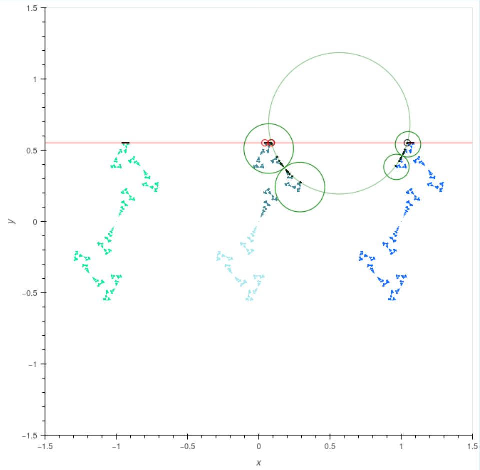


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.10637 + 2.37397i) & 1 \end{bmatrix} \right\rangle$$

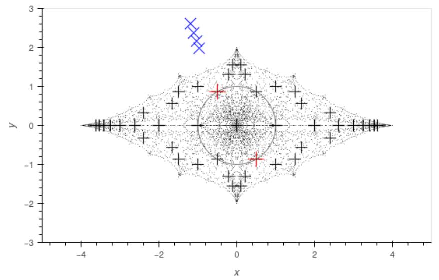


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 7.5011$

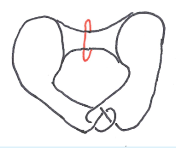


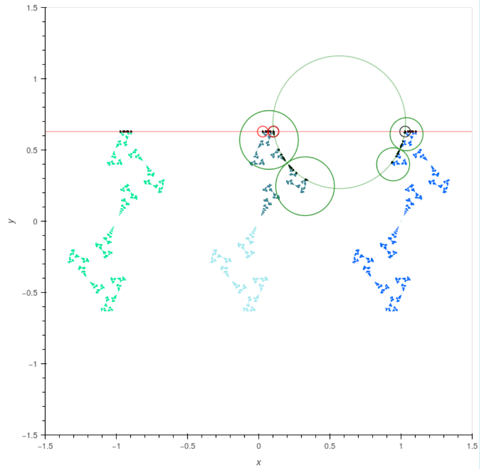


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.03184 + 2.16328i) & 1 \end{bmatrix} \right\rangle$$

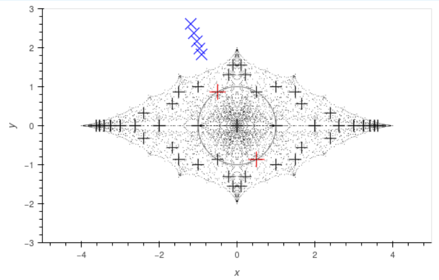


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: $\operatorname{trlen} W_{3/5} = 6.41179$

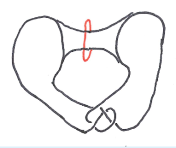


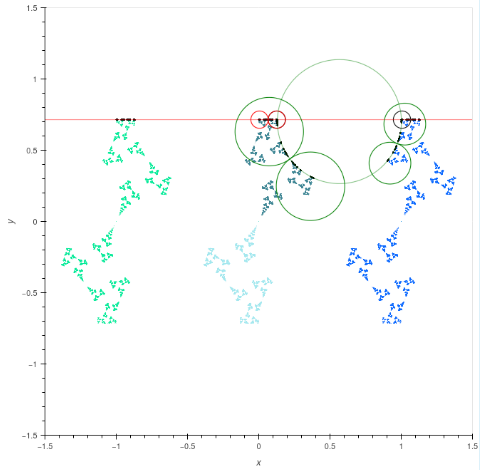


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.965971 + 1.98003i) & 1 \end{bmatrix} \right\rangle$$

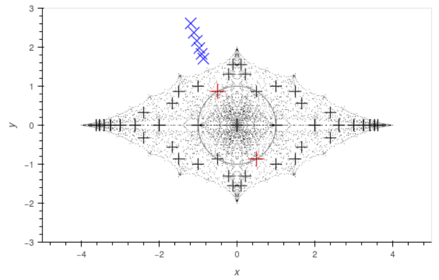


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 5.33801$

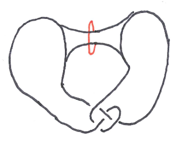


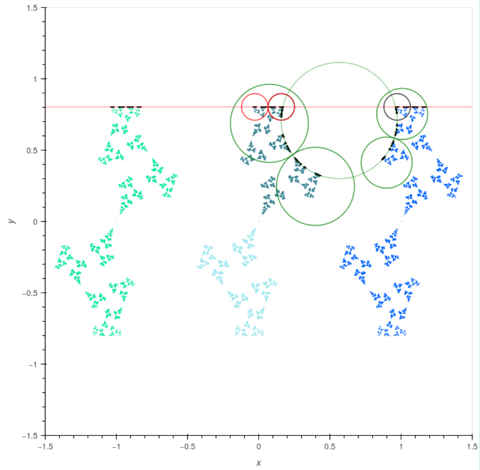


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.909739 + 1.82631i) & 1 \end{bmatrix} \right\rangle$$

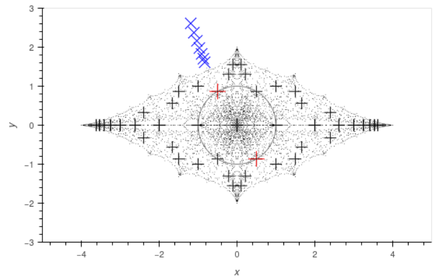


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 4.31081$

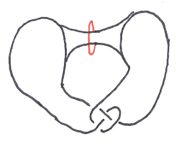


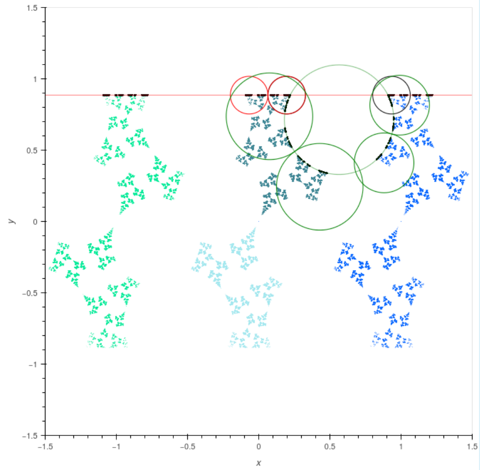


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.864178 + 1.70401i) & 1 \end{bmatrix} \right\rangle$$

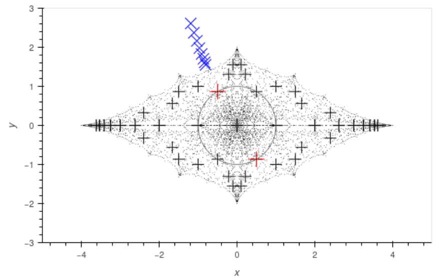


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: trlen $W_{3/5}$ = 3.36929

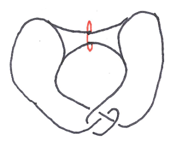


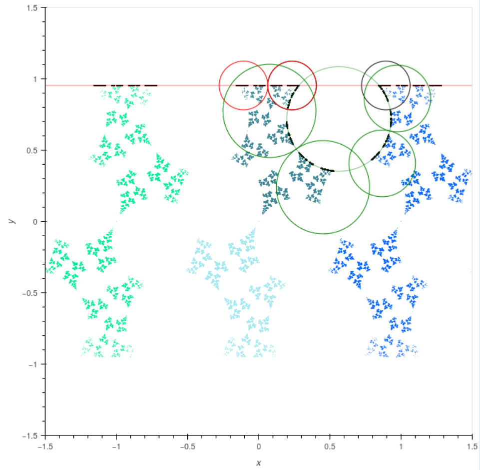


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.829848 + 1.61343i) & 1 \end{bmatrix} \right\rangle$$

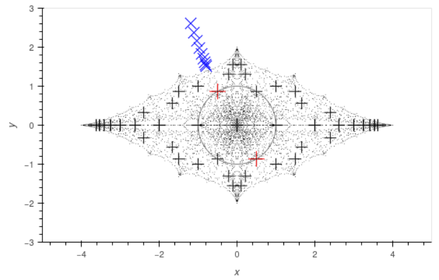


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing length: trlen $W_{3/5} = 2.55086$

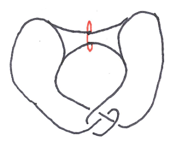


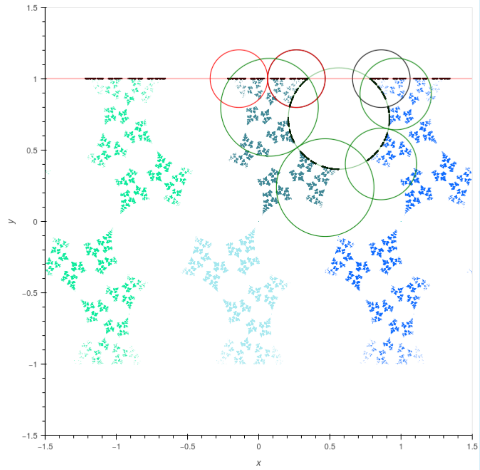


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.806173 + 1.55185i) & 1 \end{bmatrix} \right\rangle$$

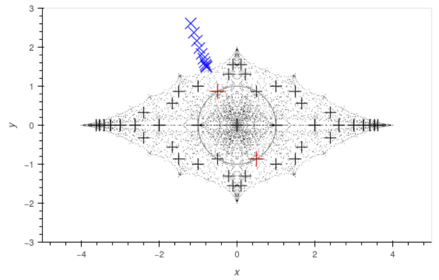


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 1.87793$

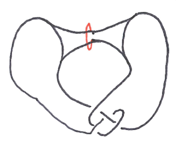


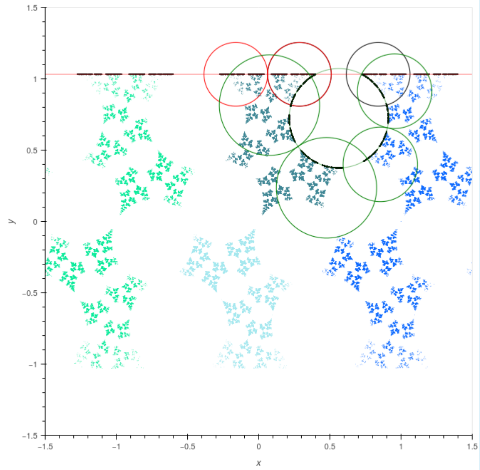


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.791287 + 1.51354i) & 1 \end{bmatrix} \right\rangle$$

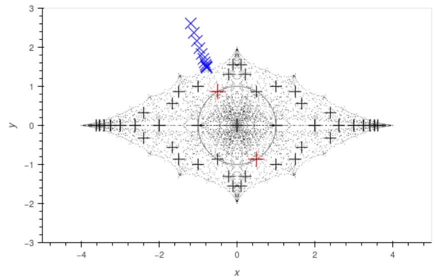


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: trlen $W_{3/5}$ = 1.35137

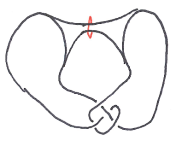


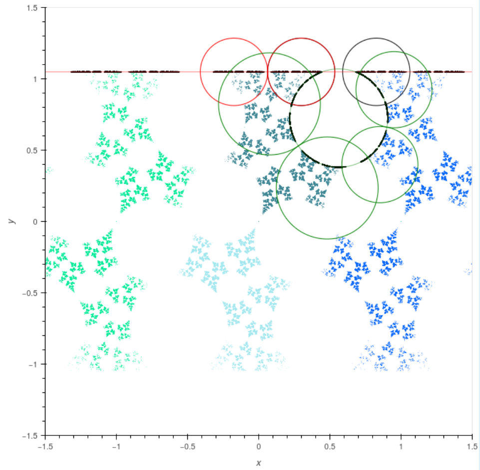


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.782668 + 1.49152i) & 1 \end{bmatrix} \right\rangle$$

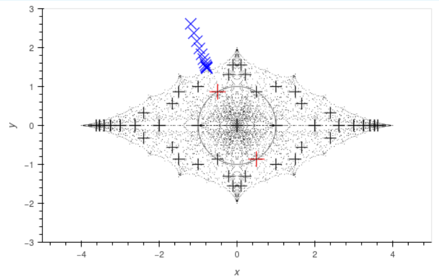


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 0.954957$

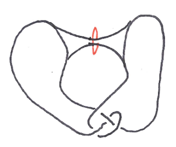


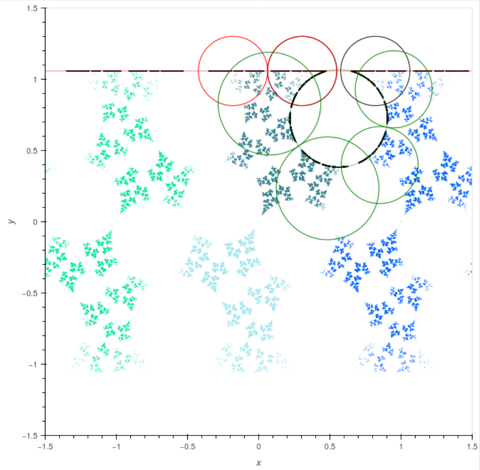


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.777994 + 1.47962i) & 1 \end{bmatrix} \right\rangle$$

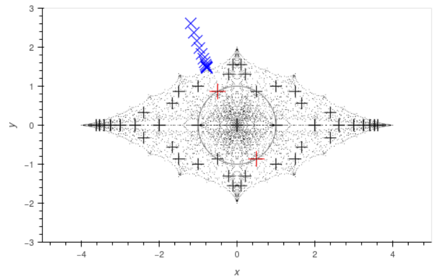


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: trlen $W_{3/5} = 0.664851$

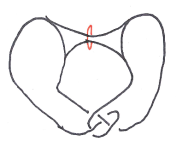


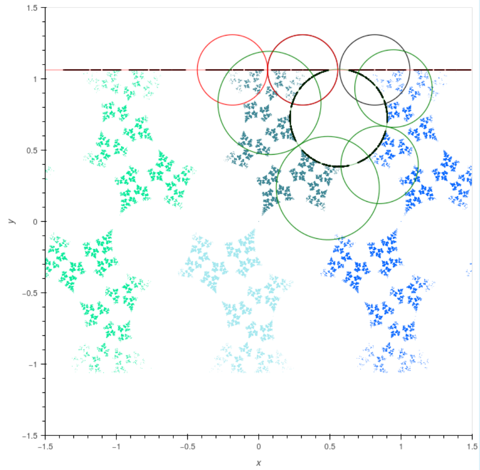


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.775579 + 1.47349i) & 1 \end{bmatrix} \right\rangle$$

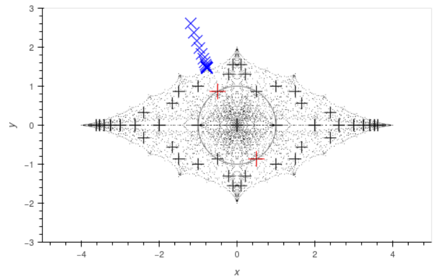


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 0.456906$

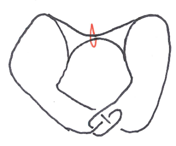


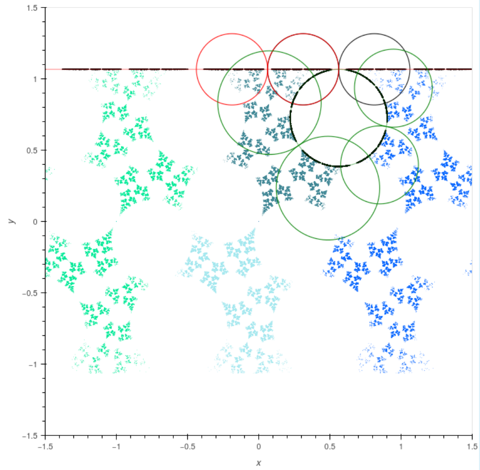


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.774378 + 1.47044i) & 1 \end{bmatrix} \right\rangle$$

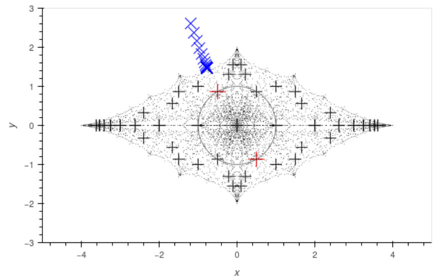


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: $\operatorname{trlen} W_{3/5} = 0.0587254$

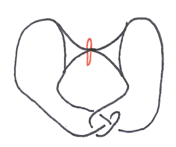


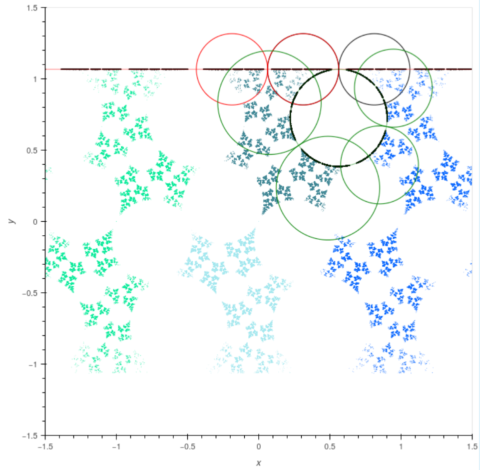


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.773319 + 1.46776i) & 1 \end{bmatrix} \right\rangle$$

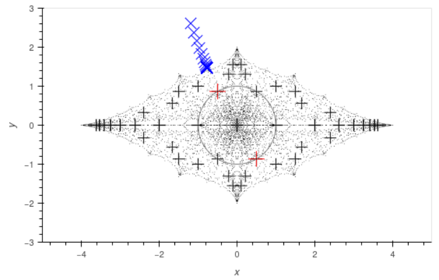


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: trlen $W_{3/5} = 0.0$



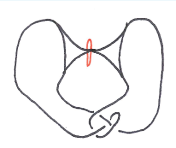


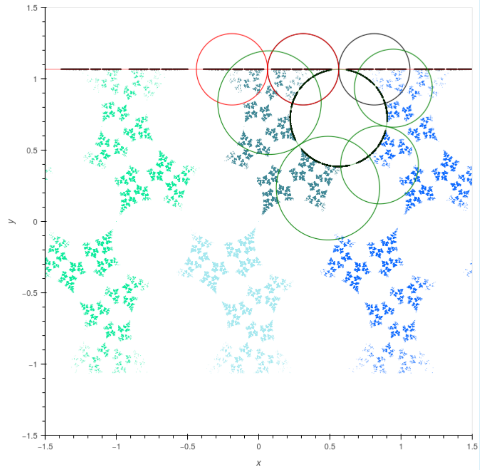
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.773301 + 1.46771i) & 1 \end{bmatrix} \right\rangle$$



The geodesic has turned into a loop around a cusp.

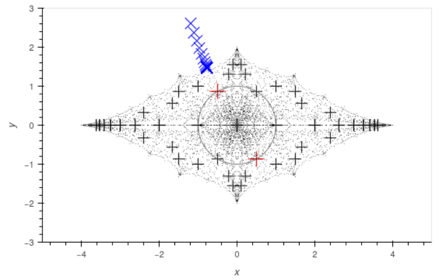
trlen $W_{3/5}$ = 0.0



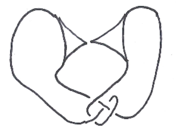


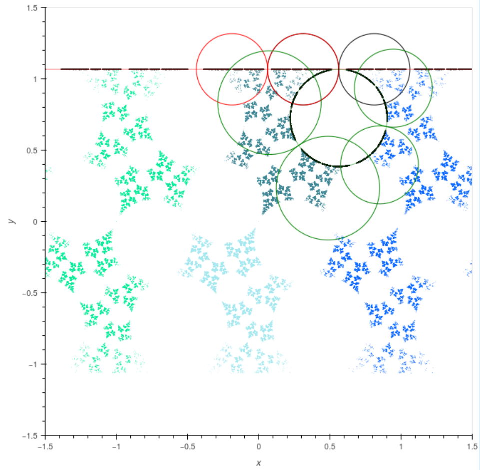
We smoothly decreased tr $W_{3/5}$ from −120 to −2.

But why stop here?

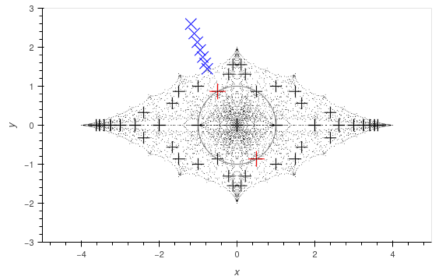


$\mathrm{trlen}\, W_{3/5} = 0\pi i$

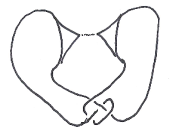


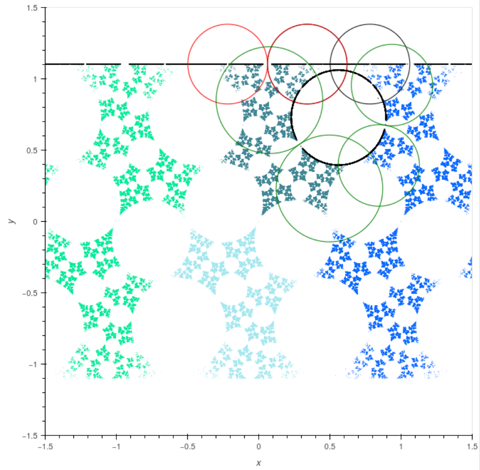


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.773301 + 1.46771i) & 1 \end{bmatrix} \right\rangle$$

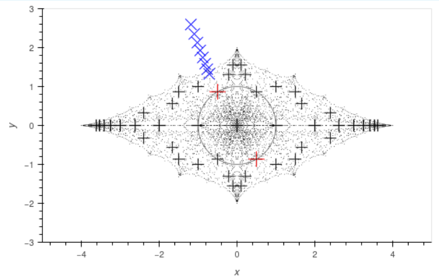


$\operatorname{trlen} W_{3/5} = 0.569\pi i$

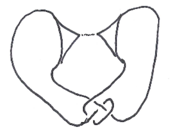


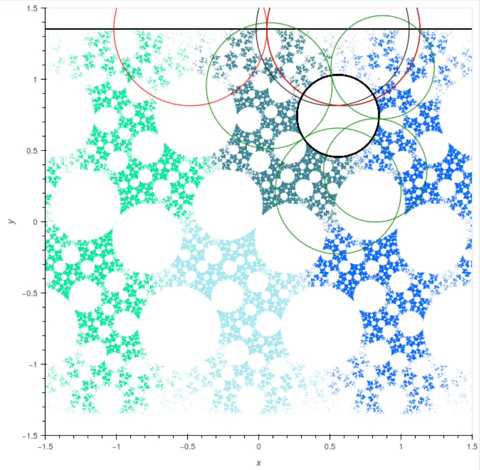


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.76416 + 1.44462i) & 1 \end{bmatrix} \right\rangle$$

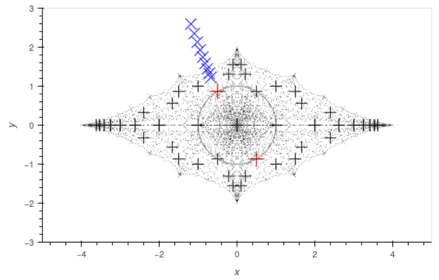


$\mathrm{trlen}\, W_{3/5} = 1.045\pi i$

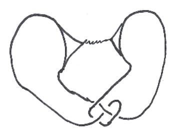


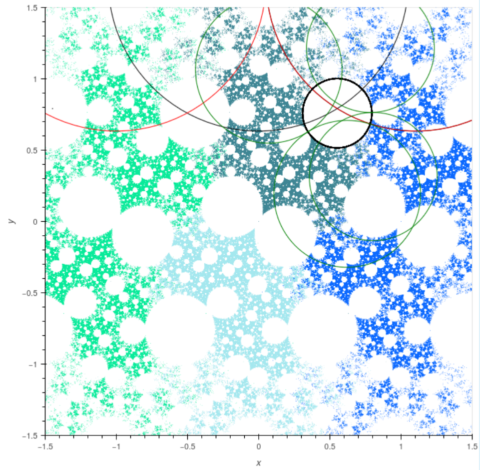


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.715145 + 1.32335i) & 1 \end{bmatrix} \right\rangle$$

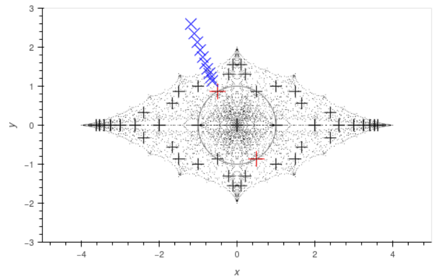


$\mathrm{trlen}\, W_{3/5} = 1.352\pi i$

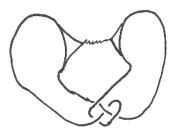


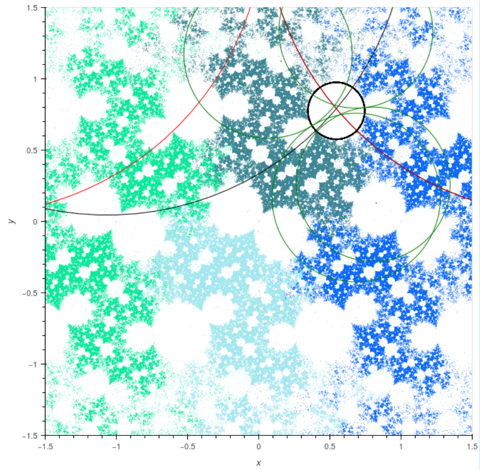


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.671951 + 1.22058i) & 1 \end{bmatrix} \right\rangle$$

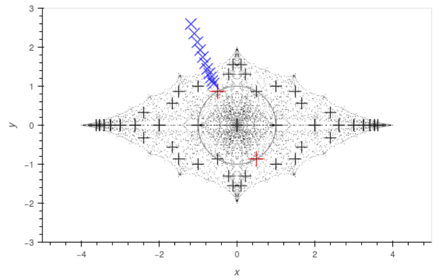


$\mathrm{trlen}\, W_{3/5} = 1.552\pi i$

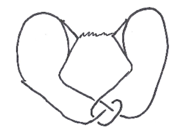


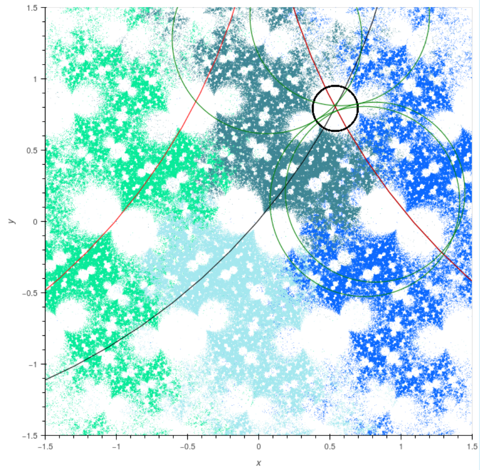


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.634562 + 1.1353i) & 1 \end{bmatrix} \right\rangle$$

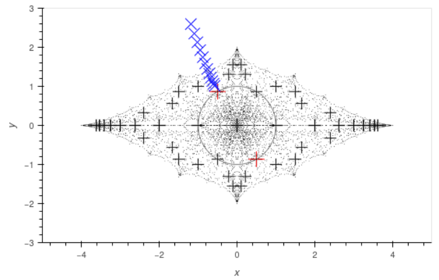


$\operatorname{trlen} W_{3/5} = 1.690\pi i$

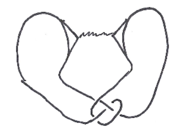


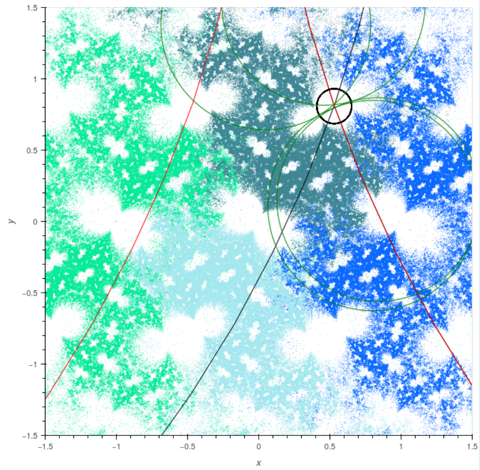


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.602914 + 1.06621i) & 1 \end{bmatrix} \right\rangle$$

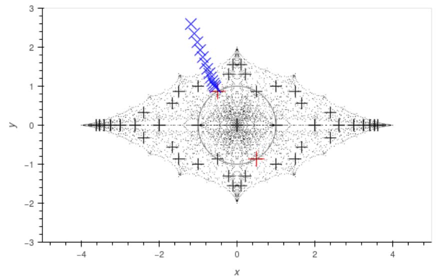


$\operatorname{trlen} W_{3/5} = 1.786\pi i$

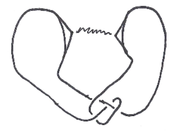


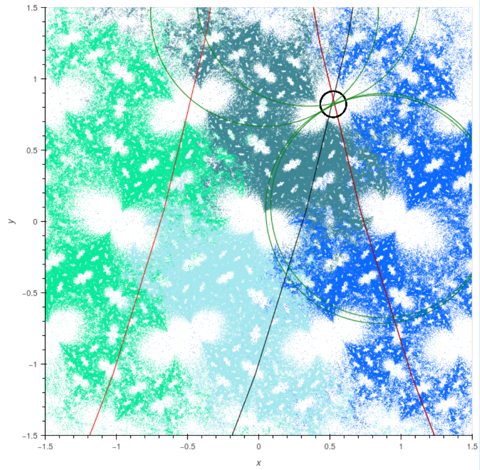


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.57683 + 1.01168i) & 1 \end{bmatrix} \right\rangle$$

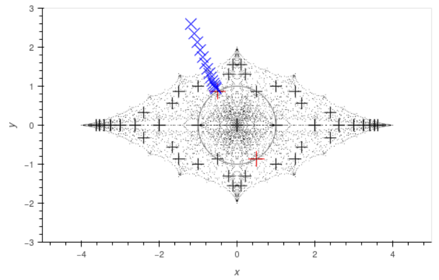


$\operatorname{trlen} W_{3/5} = 1.854\pi i$

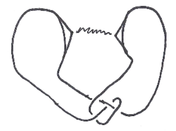


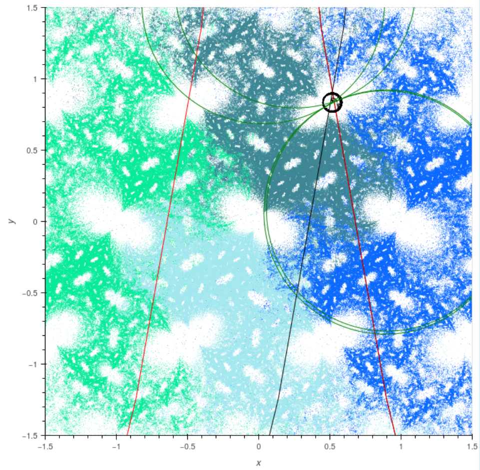


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.55597 + 0.969794i) & 1 \end{bmatrix} \right\rangle$$

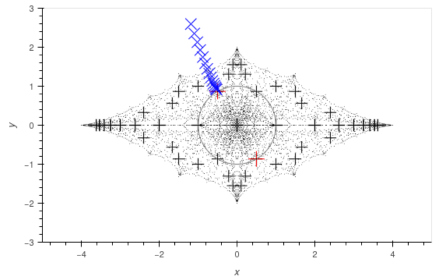


$\operatorname{trlen} W_{3/5} = 1.901\pi i$

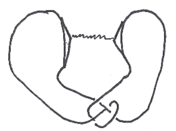


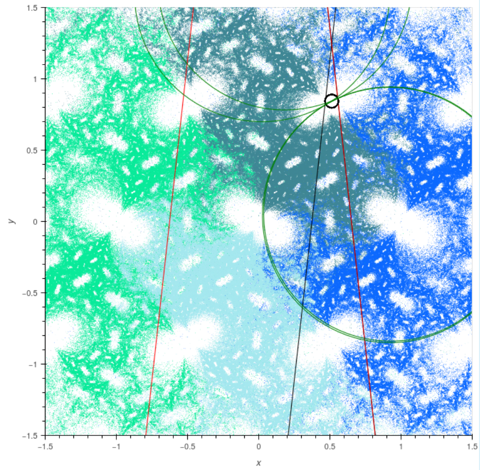


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.53981 + 0.938492i) & 1 \end{bmatrix} \right\rangle$$

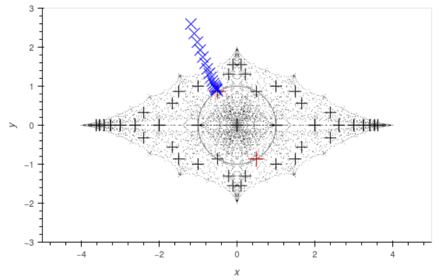


$$\operatorname{trlen} W_{3/5} = 1.934\pi i$$

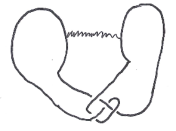


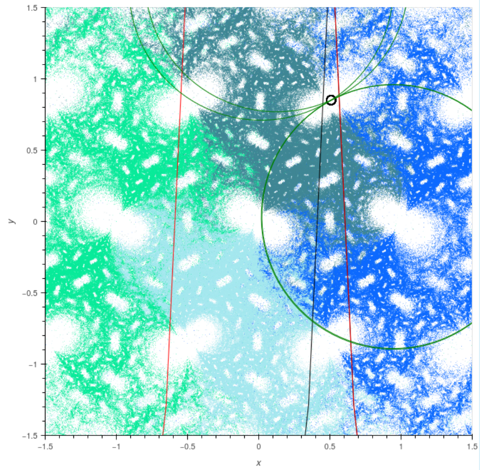


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.527683 + 0.915694i) & 1 \end{bmatrix} \right\rangle$$

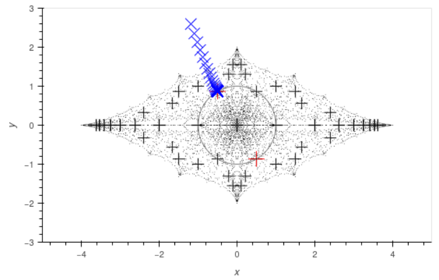


$$\operatorname{trlen} W_{3/5} = 1.956\pi i$$

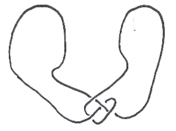




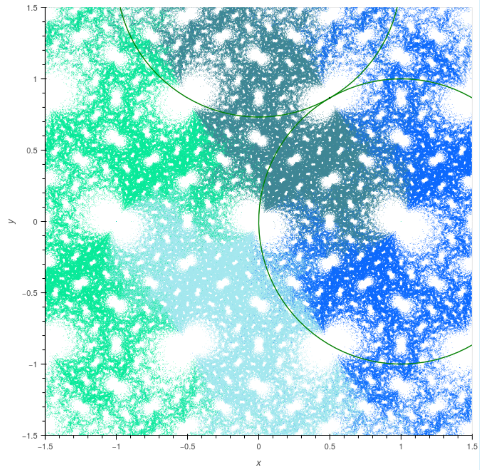
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.518852 + 0.899484i) & 1 \end{bmatrix} \right\rangle$$



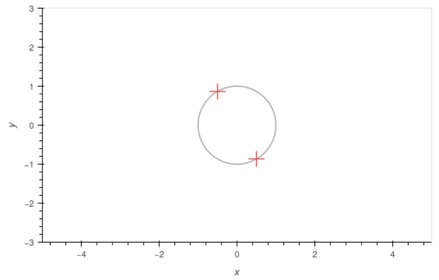
trlen $W_{3/5} = 2\pi i$

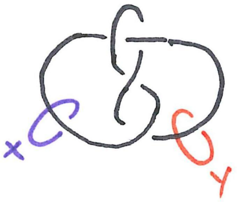


$W_{3/5}$ has become the identity.



$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.5 + 0.86603i) & 1 \end{bmatrix} \right\rangle$$



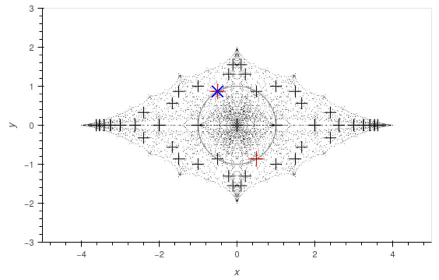


The holonomy group of the figure eight knot is

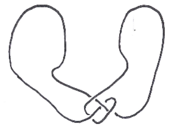
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle$$

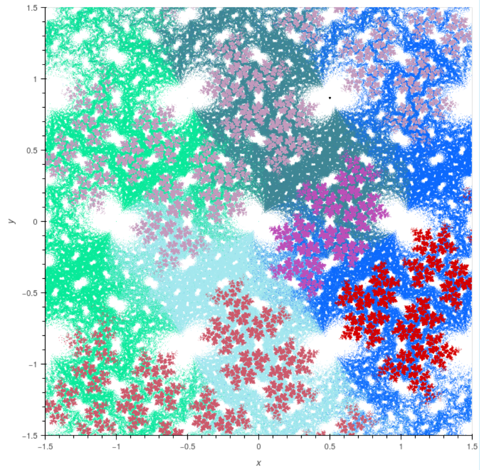
$$e^{2\pi i/3} \approx -0.5 + 0.866i$$

$$W_{3/5} = Id$$

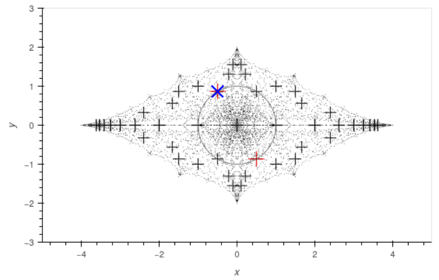


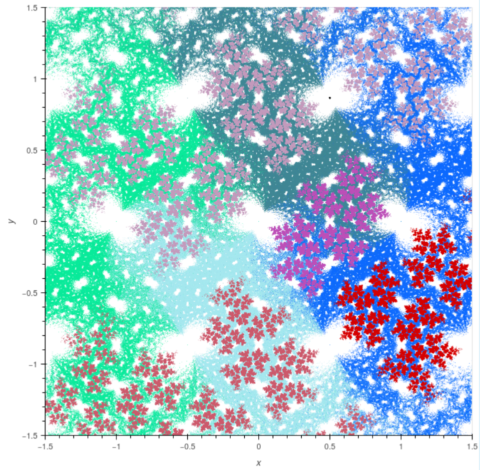
$\operatorname{trlen} W_{3/5} = 2\pi i$



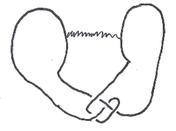


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.5 + 8.6603i) & 1 \end{bmatrix} \right\rangle$$

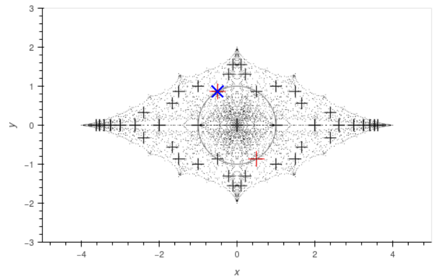


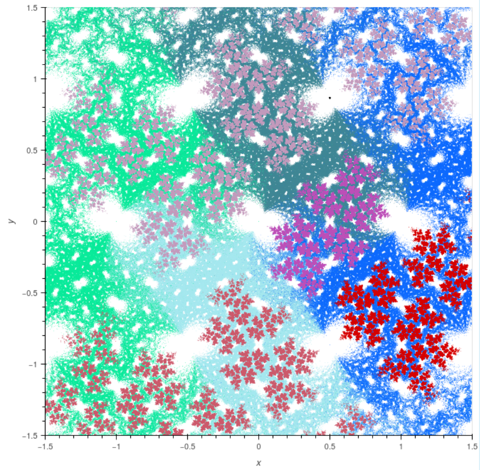


$$\operatorname{trlen} W_{3/5} = 1.9998\pi i$$

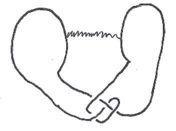


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.500096 + 0.866191i) & 1 \end{bmatrix} \right\rangle$$

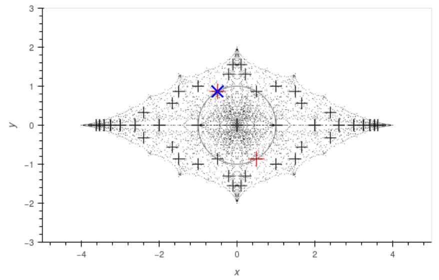




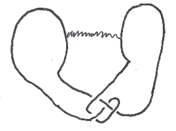
$$\mathrm{trlen}\, W_{3/5} = 1.9996\pi i$$

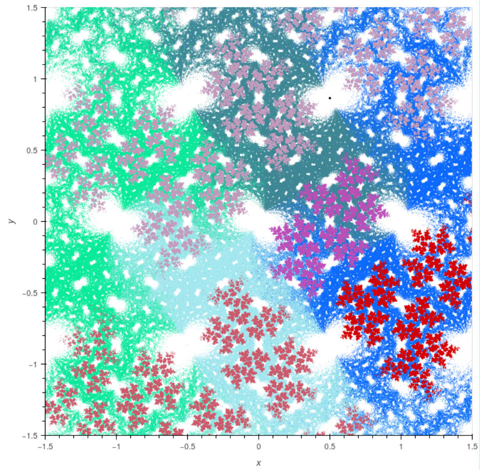


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.500166 + 0.866313i) & 1 \end{bmatrix} \right\rangle$$

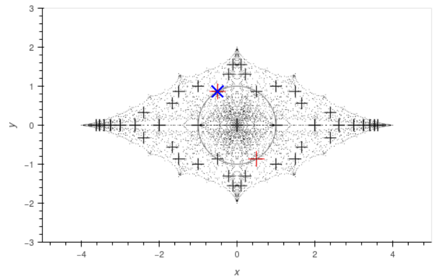


$\operatorname{trlen} W_{3/5} = 1.9982\pi i$

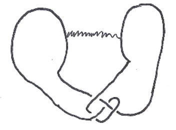


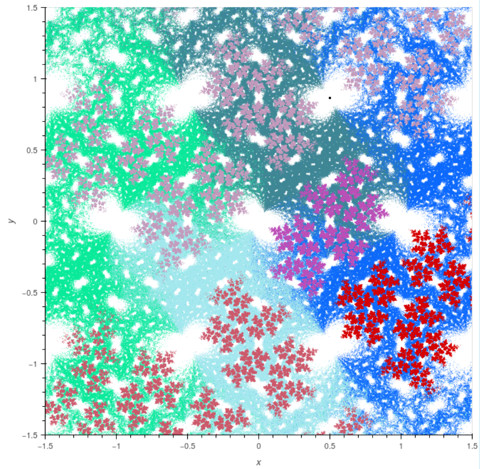


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.500802 + 0.867416i) & 1 \end{bmatrix} \right\rangle$$

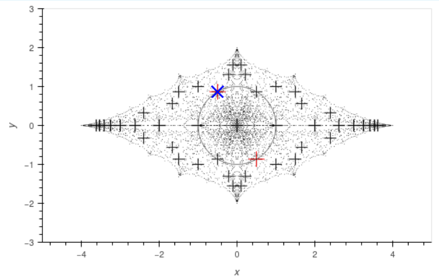


$\operatorname{trlen} W_{3/5} = 1.9971\pi i$

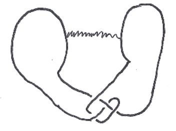


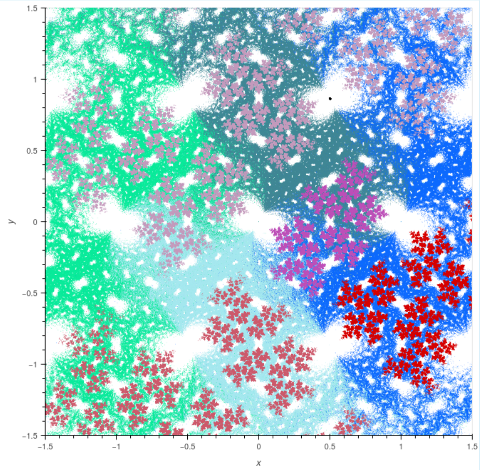


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.501316 + 0.868309i) & 1 \end{bmatrix} \right\rangle$$

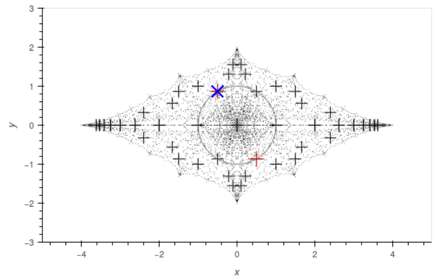


$\operatorname{trlen} W_{3/5} = 1.9935\pi i$

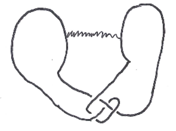


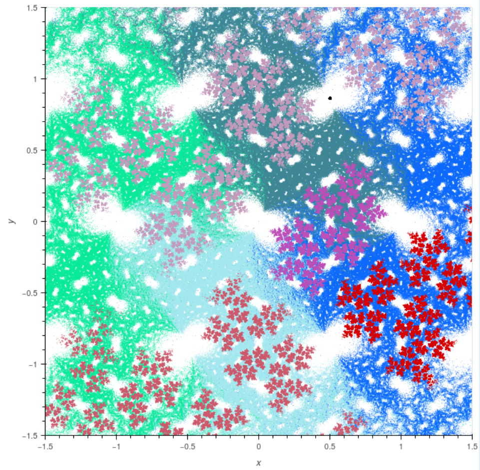


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.50213 + 0.869725i) & 1 \end{bmatrix} \right\rangle$$

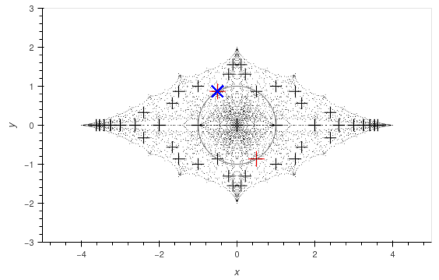


$\operatorname{trlen} W_{3/5} = 1.9924\pi i$

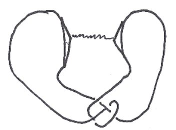


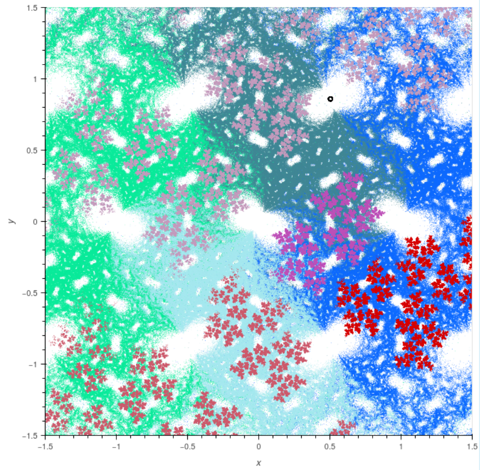


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.503398 + 0.871937i) & 1 \end{bmatrix} \right\rangle$$

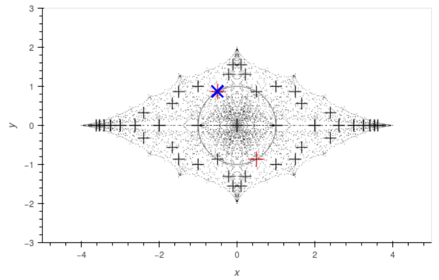


$\mathrm{trlen}\, W_{3/5} = 1.9813\pi i$

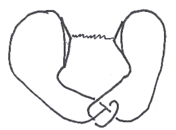


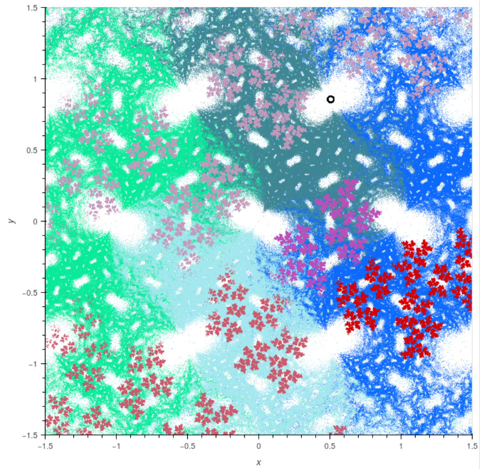


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.508269 + 0.880505i) & 1 \end{bmatrix} \right\rangle$$

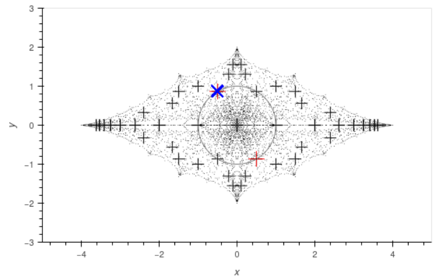


$$\operatorname{trlen} W_{3/5} = 1.9712\pi i$$

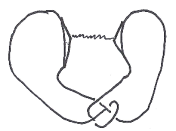


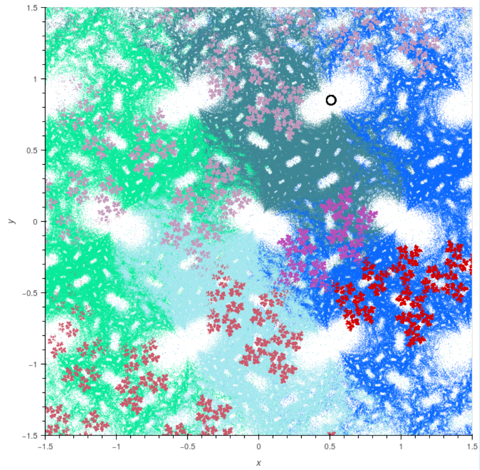


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.512595 + 0.888203i) & 1 \end{bmatrix} \right\rangle$$

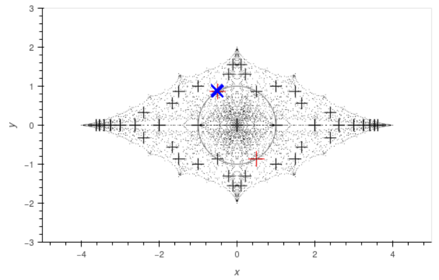


$\operatorname{trlen} W_{3/5} = 1.9560\pi i$

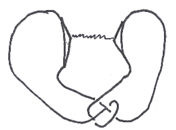


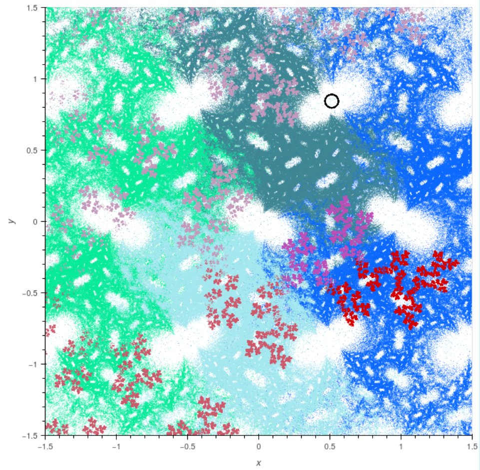


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.518852 + 0.899484i) & 1 \end{bmatrix} \right\rangle$$

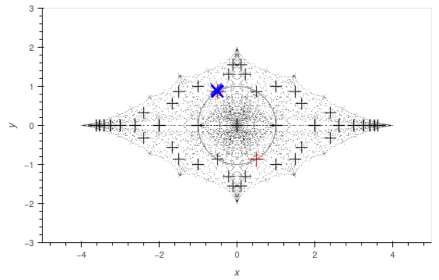


$$\operatorname{trlen} W_{3/5} = 1.9337\pi i$$

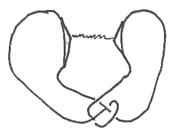


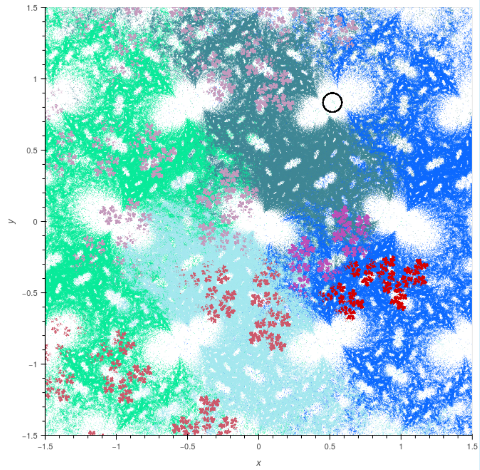


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.527683 + 0.915694i) & 1 \end{bmatrix} \right\rangle$$

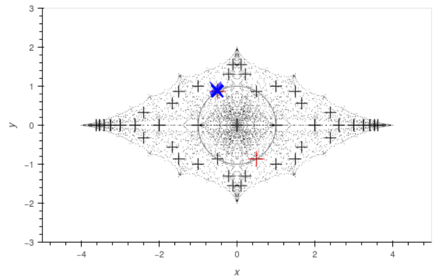


$$\operatorname{trlen} W_{3/5} = 1.9010\pi i$$

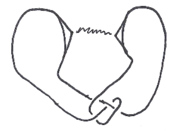


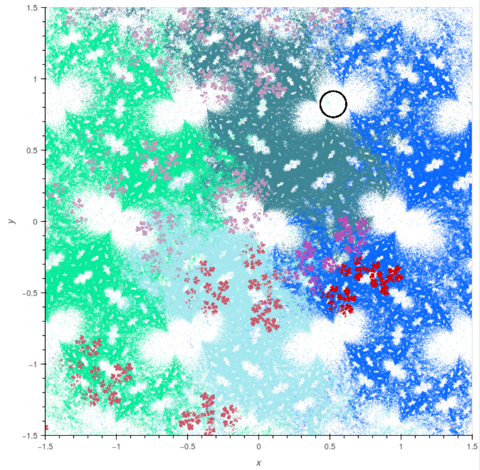


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.53981 + 0.938492i) & 1 \end{bmatrix} \right\rangle$$

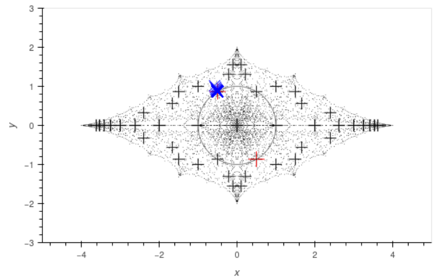


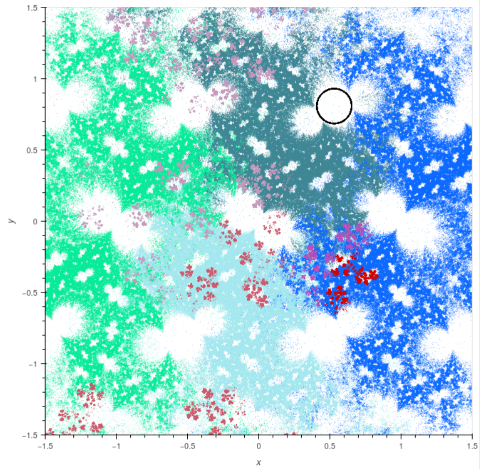
$\mathrm{trlen}\, W_{3/5} = 1.8539\pi i$



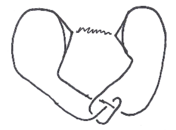


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.55597 + 0.969794i) & 1 \end{bmatrix} \right\rangle$$

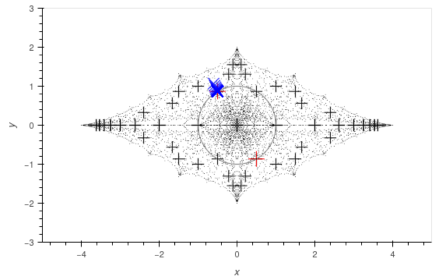




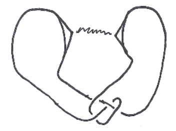
$$\operatorname{trlen} W_{3/5} = 1.7864\pi i$$

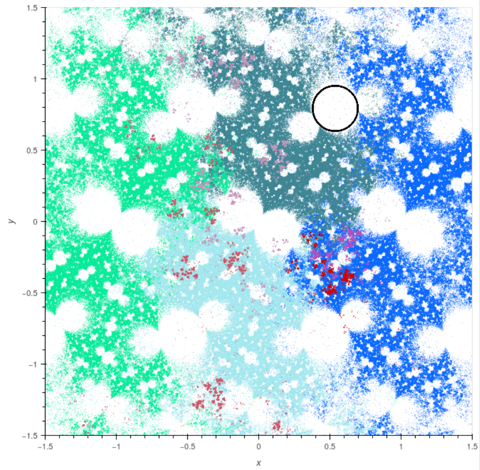


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.57683 + 1.01168i) & 1 \end{bmatrix} \right\rangle$$

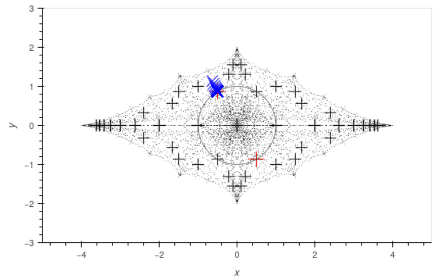


$\mathrm{trlen}\, W_{3/5} = 1.6901\pi i$

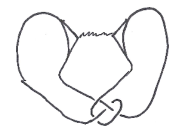


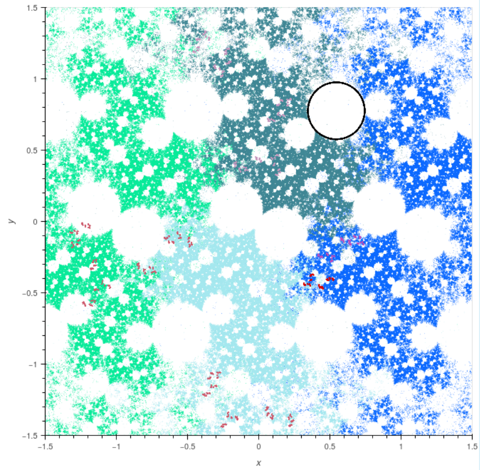


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.602914 + 1.06621i) & 1 \end{bmatrix} \right\rangle$$

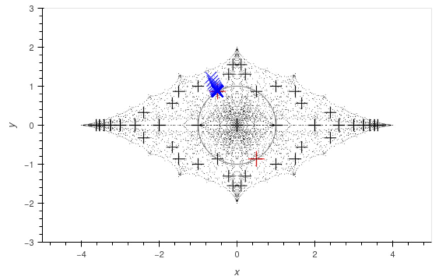


$\operatorname{trlen} W_{3/5} = 1.5525\pi i$

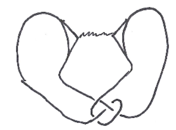


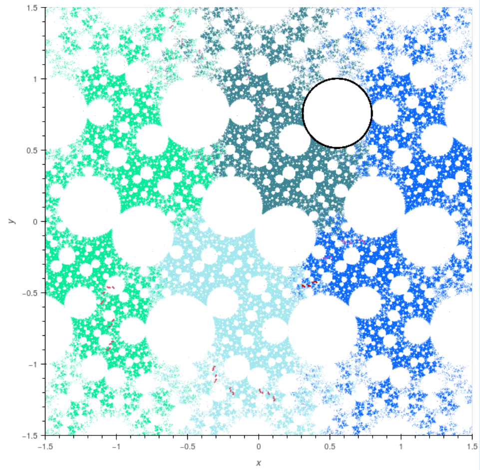


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.634562 + 1.1353i) & 1 \end{bmatrix} \right\rangle$$

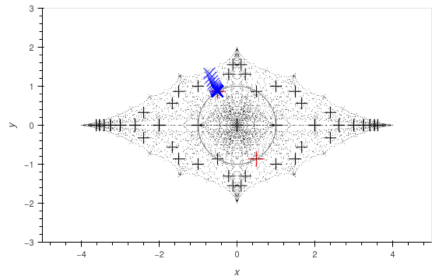


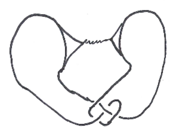
$\operatorname{trlen} W_{3/5} = 1.3525\pi i$



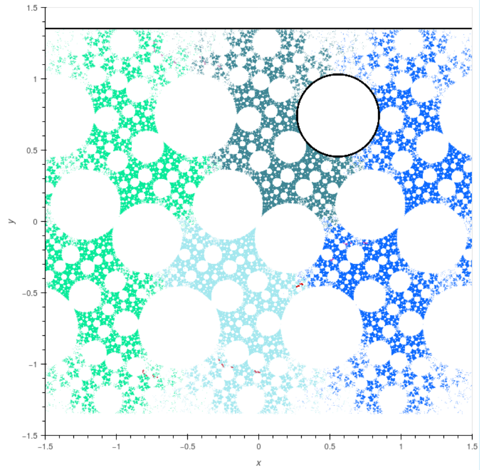


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.671951 + 1.22058i) & 1 \end{bmatrix} \right\rangle$$

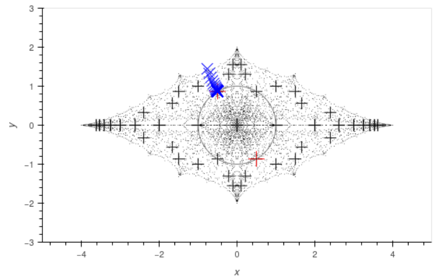




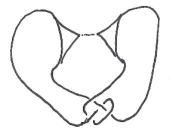
$$\operatorname{trlen} W_{3/5} = 1.0447\pi i$$

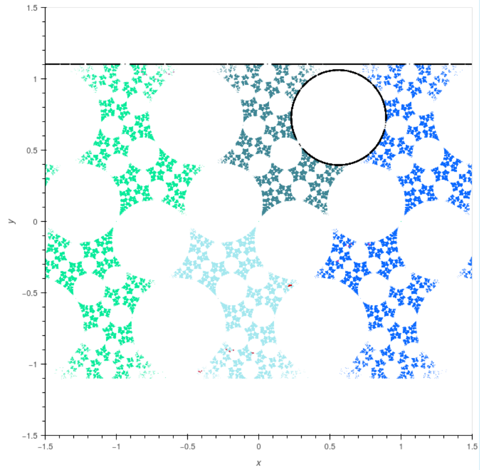


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.715145 + 1.32335i) & 1 \end{bmatrix} \right\rangle$$

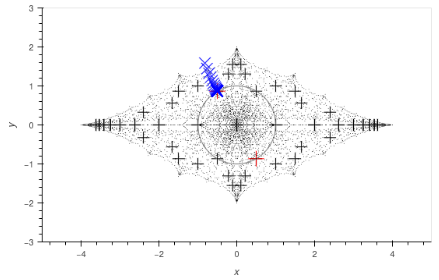


$$\operatorname{trlen} W_{3/5} = 0.4222\pi i$$

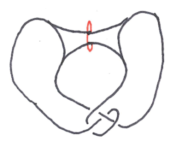


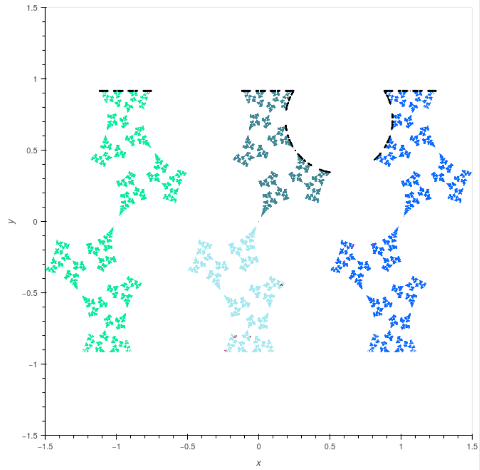


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.76416 + 1.44462i) & 1 \end{bmatrix} \right\rangle$$

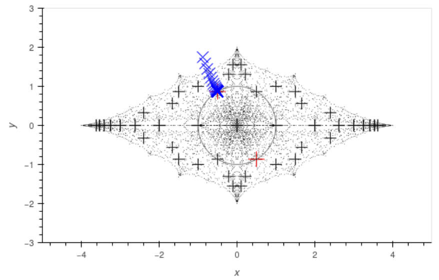


$\operatorname{trlen} W_{3/5} = 3.01983$

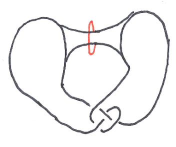


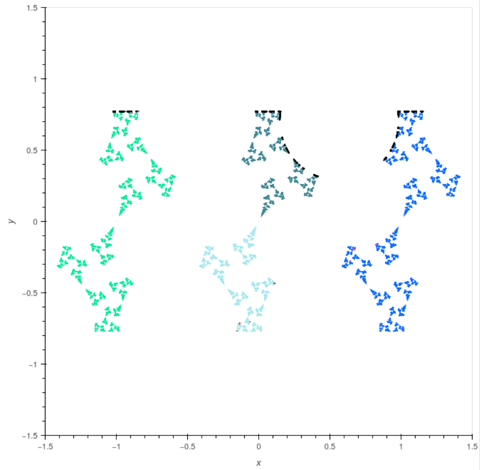


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.819013 + 1.58515i) & 1 \end{bmatrix} \right\rangle$$

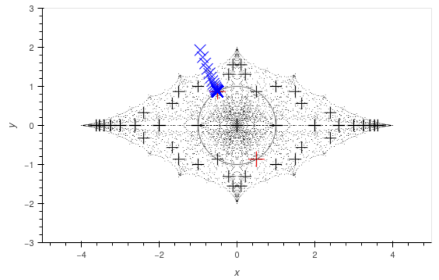


$\operatorname{trlen} W_{3/5} = 4.68338$

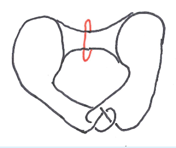


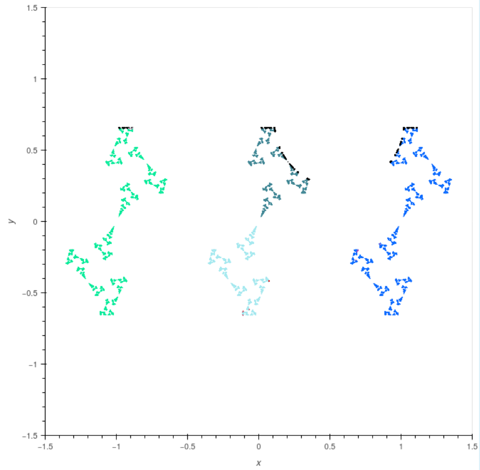


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.879751 + 1.74557i) & 1 \end{bmatrix} \right\rangle$$

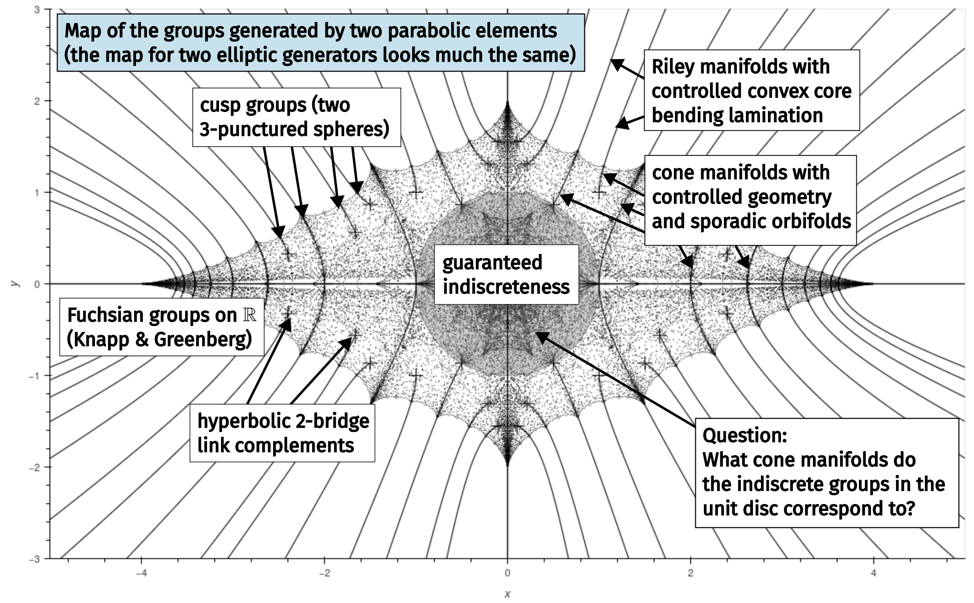


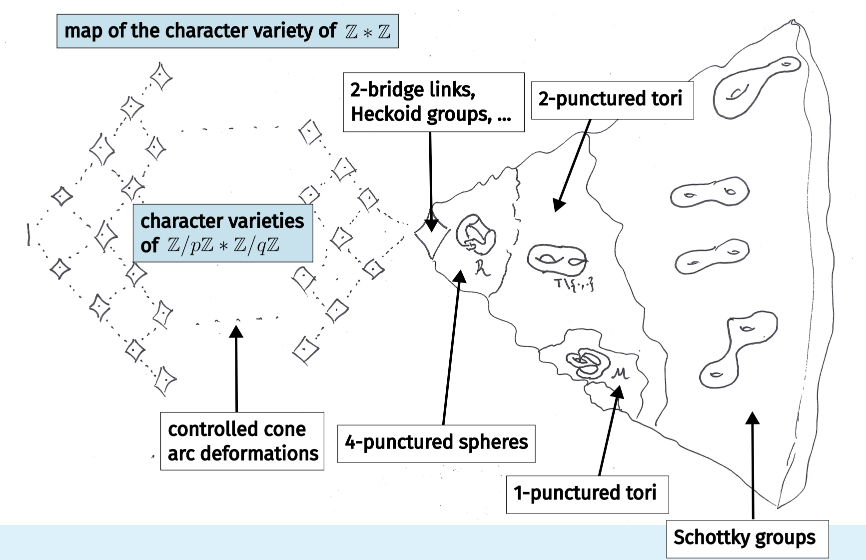
$\mathrm{trlen}\, W_{3/5} = 6.05734$





$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.946453 + 1.92636i) & 1 \end{bmatrix} \right\rangle$$


Map of the groups generated by two parabolic elements
(the map for two elliptic generators looks much the same)
cusp groups (two
3-punctured spheres)
Riley manifolds with
controlled convex core
bending lamination
cone manifolds with
controlled geometry
and sporadic orbifolds
guaranteed
indiscreteness
Fuchsian groups on ℝ
(Knapp & Greenberg)
hyperbolic 2-bridge
link complements
Question:
What cone manifolds do
the indiscrete groups in the
unit disc correspond to?
x
y
−4
−2
0
2
4
3
2
1
0
−1
−2
−3



Warning: there are lots of weird geometrically infinite things on boundaries! I only show nice groups...

Bedtime reading

- A.J.E., Gaven J. Martin, and Jeroen Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In *2021-2022 MATRIX Annals*.
- A.J.E., Jianhua Gong, G.J.M, and J.S. *Bounding deformation spaces of Kleinian groups with two generators*. arXiv preprint (2024).
- A.J.E., G.J.M., and J.S. *Deformation spaces of Kleinian groups generated by two elements of finite order*. Preprint to appear shortly.
- Eric Chesebro, G.J.M., and J.S. *2-elliptic generated Kleinian groups are Heckoid*. Preprint to appear shortly.
- Linda Keen and Caroline Series. "The Riley slice of Schottky space". Proc. LMS (1994).
- C.S. *The suggestive power of pictures*. `https://youtu.be/YOMr4eviQWs`.
- Sangbum Cho and Darryl McCullough. "The tree of knot tunnels". Geom. Topol. (2009).
- Title picture: Caspar David Friedrich, *Erinnerungen an das Riesengebirge* [Memories of the Riesengebirge], c.1835.

Riley slice theorems

Theorem (A.J.E., G.J. Martin, J. Schillewaert (2021+), after Keen, Komori, Series)

The quasiconformal deformation space $\mathcal{R}^{p,q} \subset \mathbb{C}$ of Kleinian groups which split as $\mathbb{Z}/p * \mathbb{Z}/q$ and which have quotient orbifold a ball with two ideal singular arcs admits a natural lamination by smooth curves such that:

1. *Leaves are groups with the same projective convex core bending lamination and are enumerated by the Farey triangulation.*
2. *On each leaf the groups split into peripheral subgroups in a controlled way. Extensions of leaves past $\partial\mathcal{R}^{p,q}$ pass through the Heckoid groups and end at an orbifold group with singular locus a 2-bridge link (c.f. E. Chesebro, G.J.M, J.S., to appear).*
3. *Every leaf admits a semi-algebraic neighbourhood in $\mathcal{R}^{p,q}$ that can be used to give membership certificates.*

All this can be effectively computed.

Keen and Series (1994), Komori and Series (1998): case $p = q = \infty$.

Arithmetic group applications

Theorem (E., G.J. Martin, J. Schillewaert (announced 2024, to appear))

1. *There are finitely many arithmetic groups in* $\mathrm{PSL}(2, \mathbb{C})$ *generated by two parabolic or elliptic elements.*
2. *There are finitely many thin groups in* $\mathrm{PSL}(2, \mathbb{C})$ *which do not split as free products.*
3. *The groups in (1) and (2) can be effectively tabulated.*

Proof.

Use algebraic number theory to bound maximal degree of algebraic integers that can arise in generators, and use earlier work (necessary criterion for arithmeticity) of Maclachlan and Martin to come up with a finite list of possible groups. Use joint work with J. Gong (+EMS) and E. Chesebro (+MS) to give semi-algorithm that can check all these finitely many groups to find their surface quotient topology and isomorphism type.

$\mathcal{B}_3$ REPRESENTATION APPLICATIONS

Motivated by q-rationals, S. Morier-Genoud, V. Ovsienko, and A. P. Veselov (2023) studied the family of representations given by substituting $\mu \in \mathbb{C}$ into the reduced Burau representation $\rho : \mathcal{B}_3 \to \mathrm{GL}(2, \mathbb{Z}[t^{\pm 1}])$ where

$$\rho(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho(\sigma_2) = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}.$$

(σ_1 and σ_2 are the standard Artin generators of $\mathcal{B}_3$). They conjectured that the representations are faithful in the exterior of the annulus

$$\frac{3-\sqrt{5}}{2} \leq |\mu| \leq \frac{3+\sqrt{5}}{2}.$$

Theorem (E., J. Gong, G.J. Martin, and J. Schillewaert (2024))

For $\mu \in \mathbb{C}$, let $z = \sqrt{\mu} - 1/\sqrt{\mu}$. If $3 \leq |z \pm \sqrt{z^2+3}|$ then $\mathcal{B}_3 / Z(\mathcal{B}_3) \to \mathrm{PSL}(2, \mathbb{C})$ is injective, and so is its lift $\mathcal{B}_3 \to \mathrm{SL}(2, \mathbb{C})$. These bounds are strictly better than those conjectured by Morier-Genoud, Ovsienko, and Veselov, except at one point where both the conjectured bound and our bound are tight.

Cone manifold deformations

We follow ideas of Akioshi, "Thin representations for the one-cone torus group", Topol. Appl. (2019) but applied to $\mathcal{R}$ not $\mathcal{M}$.

- On each pleating ray the groups splits as an amalgamated product of two subgroups generated by primitive p and q order elliptics. The product of the generators is the amalgamation curve γ (of course you also amalgamate the generators).
- From this we obtain a fundamental polyhedron for the larger group. This is standard application of something like Maskit's first combination theorem, which you can rephrase as a statement about gluing polyhedrons and obtaining isomorphism and side-pairing action by the resulting group generated by side-pairings regardless of discreteness.
- Travelling down the pleating ray and then the extended pleating ray is just a smooth deformation of some angles on this fundamental polyhedron. As you continue outside the deformation space, the polyhedron does not collide with itself, and so the group generated by its side pairings is the holonomy group of the cone manifold where γ has become a cone arc of steadily increasing angle; but this group is just the indiscrete group on the extended pleating ray.

Computation notes

- Figures produced in Python using Bella `https://github.com/aelzenaar/bella` (for mathematical computations) and HoloViews `https://holoviews.org/` (for visualisation)
- Algorithms for limit set calculation can be found in David Mumford, Caroline Series, and David Wright. *Indra's pearls*. Cambridge Uni. Press, 2002.
- Moduli space and pleating ray computations are based on theory in E., M., and S., "The combinatorics of Farey words and their traces". Groups, Geom. and Dyn. (accepted, preprint on the arXiv); basically we walk down a given asymptotic branch of a polynomial inverse using Newton's algorithm.
- See also: David Wright, "Searching for the cusp", in *Spaces of Kleinian groups*, Lond. Math. Soc. Lec. Notes 329, 2011.
- See also: visualisations by Emily Dumas (`https://www.dumas.io/limset/`).