Lorentzian polynomials and projective varieties

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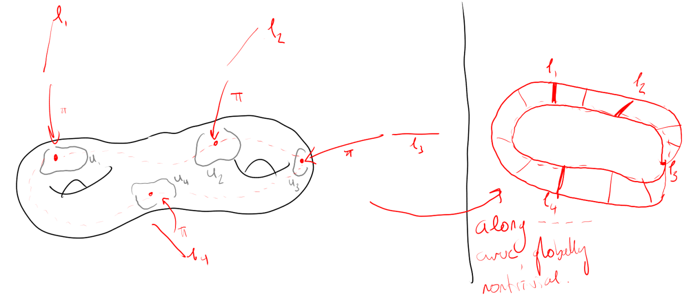
May 23, 2022

Warnings

- This talk will necessarily be sketchy.
- We always assume (usually without explicit statement) that the objects are ‘nice enough to make the theorems true’.
- Usually this means all varieties are complete and smooth over $\mathbb{C}$.

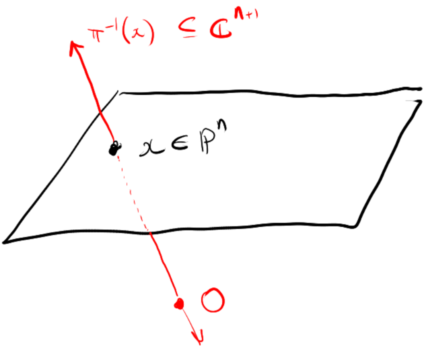
Line bundles



Line bundle over X: A space $\mathcal{E}$ with a map $\pi : \mathcal{E} \to X$ such that

- $\pi^{-1}(x)$ is a 1-dimensional vector space for all $x \in X$, and
- around each $x \in X$ there is a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic (as a vector space) to $U \times \mathbb{R}$ in a way compatible with the two projections π and $U \times \mathbb{R} \to U$.

The natural line bundle on $\mathbb{P}^n$



The tautological line bundle $\pi : \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{P}^n$ with $\pi^{-1}(x)$ the line in $\mathbb{C}^{n+1}$ defining x. The local trivialisation cover can be chosen to be the usual open cover of $\mathbb{P}^n$ by $n + 1$ affine spaces.

- Let $\mathcal{E}$ be a line bundle over X. Let (U_i) be an open cover giving a local trivialisation.
- For each U_i set $\mathcal{E}(U_i)$ to be the set of continuous sections of $\mathcal{E}$ over U_i, i.e. the set of maps $\phi : U_i \to \mathcal{E}$ such that $\pi\phi = \mathrm{id}$.
- If X is actually a variety then we replace 'continuous' with 'polynomial' and take lines to be 1D over $\mathbb{C}$, so $\mathcal{E}$ becomes a subsheaf of $\mathcal{O}_X$.
- The sheaves associated to line bundles are called **invertible sheaves**.

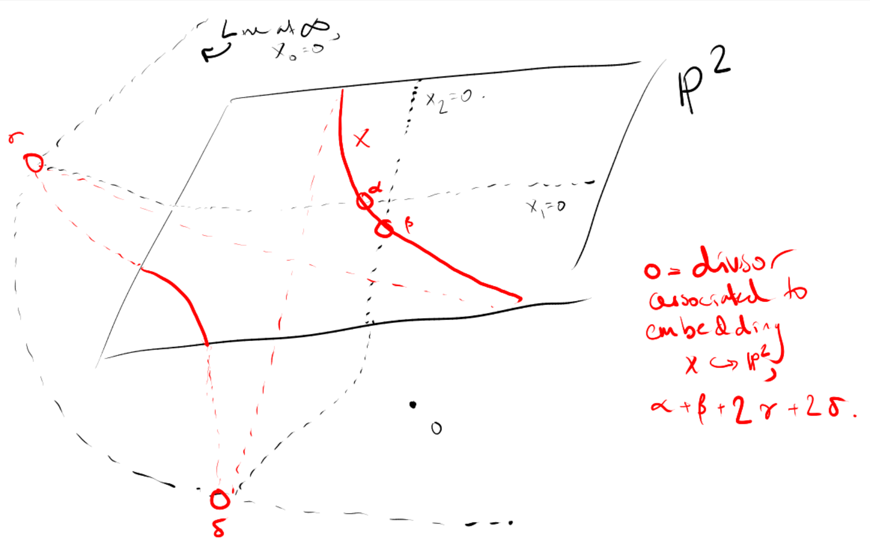
Weil divisors and line bundles

- Suppose $\mathcal{L}$ is a line bundle whose associated sheaf has global sections generated by $s_0, \ldots, s_n$.
- Then there is a natural Weil divisor associated to $\mathcal{L}$, namely the union of the zeros of the s_i.
- Conversely, if $D = \sum a_i D_i$ is a Weil divisor then we can associate an invertible sheaf $\mathcal{L}(D)$, namely the 'sheaf whose sections are functions who locally have poles at of order at worst $-a_i$'.

The dual bundle: the Serre twisting sheaf

- The dual vector bundle to the tautological bundle is the bundle $\mathcal{O}(1)$ whose fibre over x is the set of hyperplanes in $\mathbb{C}^{n+1}$ orthogonal to the line defining x (i.e. the 1-dimensional vector space of linear functionals on $\mathbb{C}^{n+1}$ which kill the line).
- Let $f : \mathbb{P}^n \to \mathbb{C}$ be a global section of $\mathcal{O}(1)$; then f is a polynomial map $\mathbb{P}^n \to \mathrm{Hom}(\mathbb{C}^{n+1}, \mathbb{C}) \simeq \mathbb{C}$; this map has to behave linearly when passing between the affine open subsets and so must be of degree 1.
- Thus the global sections of $\mathcal{O}(1)$ are the linear forms on $\mathbb{P}^n$.
- Hence there is a global set of section generators for $\mathcal{O}(1)$, the coordinate forms $x_0, \ldots, x_n$.

- Let X be a smooth variety over $\mathbb{C}$. Let $\varphi : X \to \mathbb{P}^n$ be a morphism.
- The pullback $\varphi^*\mathcal{O}(1)$ entirely determines X, since points in $\mathbb{P}^n$ are determined by the set of hyperplanes through them (= the fibres of $\varphi^*\mathcal{O}(1)$).
- The invertible sheaf $\varphi^*\mathcal{O}(1)$ has an associated Weil divisor: the intersection of $\varphi(X)$ with the standard coordinate hyperplanes of $\mathbb{P}^n$ (= the individual zero sets of the global generators of the pullback sheaf).
- **Theorem.** Conversely, if $\mathcal{L}$ is an invertible sheaf on X and if $s_0, \ldots, s_n$ generate $\mathcal{L}(X)$, then there exists a unique morphism $\varphi : x \to \mathbb{P}^n$ such that $\mathcal{L} \simeq \varphi^*\mathcal{O}(1)$ and $s_i = \varphi^*(x_i)$ for each i.


Line at ∞, $X_0=0$
$\mathbb{P}^2$
$X_2=0$
X
α
β
$X_1=0$
γ
δ
0
0 = divisor associated to embedding $X \hookrightarrow \mathbb{P}^2$,
$\alpha+\beta+2\gamma+2\delta$.

- The choice of embedding $\varphi : X \to \mathbb{P}^n$ has an associated degree $\deg \varphi$.
- $\mathbb{Z}_{>0}$-linear combinations of divisors which give embeddings also give embeddings.
- **Question.** What is $\deg(\lambda_1\varphi_1 + \cdots + \lambda_k\varphi_k)$?
- **Answer.** It is a homogeneous polynomial in the variables $\lambda_1, \ldots, \lambda_k$, called the **volume polynomial** of $\varphi_1, \ldots, \varphi_k$.

- Let $X = X_\Sigma$ for some fan Σ over M. The divisor class group of X_Σ is generated by the characters χ^m.
- Let D be a torus-invariant divisor of X_Σ. Then D is the closure of an orbit corresponding to one of the rays of Σ. The group of torus-invariant divisors is $\langle D_\tau : \tau \in \Sigma(1) \rangle$.
- **Lemma.** $\operatorname{Div} \chi^m = \langle m, u_\tau \rangle D_\tau$ where u_τ is a minimal ray generator for τ.
- **Theorem.** If $D = \sum a_\tau D_\tau$ then $\chi^m \in \mathcal{L}(D)(X_\Sigma)$ iff $\langle m, u_\tau \rangle \geq -a_\tau$ for all τ.
- **Corollary.** The set of torus-invariant Weil divisors of X_Σ is in bijection with the set of polyhedra.

Theorem

Let $D_1, \ldots, D_k$ be torus-invariant (ample) divisors on a toric X_Σ of (complex) dimension n with corresponding embeddings φ_i and polyhedra P_i. Then

$$\mathrm{Vol}(\lambda_1\varphi_1 + \cdots + \lambda_k\varphi_k) = (\lambda_1 D_1 + \cdots + \lambda_k D_k)^{\cdot n} = k!\,\mathrm{Vol}(\lambda_1 P_1 + \cdots + \lambda_k P_k).$$

Matching coefficients:

Corollary

$V(P_1, \ldots, P_n) = \frac{1}{k!}(D_1 \cdot \ldots \cdot D_n)$.

Theorem (Hodge index theorem)

Let X be a compact Kähler surface (e.g. a projective variety of dimension 2 over $\mathbb{C}$, with Kähler structure induced by pulling back the Serre twisting sheaf as above). Then the standard middle cohomology pairing

$$H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to H^4(X,\mathbb{R}) \simeq \mathbb{R}$$
$$(\alpha,\beta) \mapsto \langle \alpha,\beta\rangle := \int_X \alpha \wedge \beta$$

satisfies $\langle \alpha,\alpha\rangle < 0$ if α is taken in $H^{1,1}(\mathbb{R})$ (the orthogonal complement of the space of cohomology classes satisfying $\langle \beta,\beta\rangle > 0$.

- Then $D_1.D_2 = \langle c_1(D_1), c_1(D_2)\rangle$ is the usual intersection product, of course.

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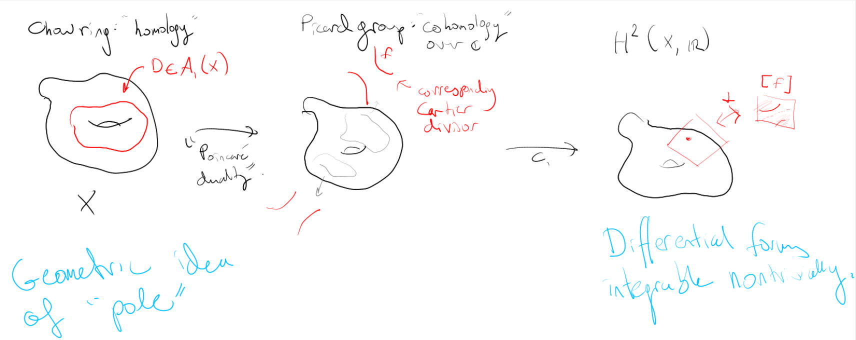
General situation



Corollary

If D_1, D_2 are divisors on X, and $D_1, D_1 > 0$, then $(D_1 \cdot D_2)^2 \geq (D_1 \cdot D_1)(D_2 \cdot D_2)$.

Now let P_1 and P_2 be full-dimensional lattice polyhedra in $\mathbb{R}^n$ and let D_1, D_2 be corresponding divisors giving toric embeddings.

$$V(P_1, P_2)^2 = \frac{1}{4}(D_1 \cdot D_2)^2 \geq \frac{1}{4}(D_1 \cdot D_1)(D_2 \cdot D_2) = V(P_1, P_1)V(P_2, P_2).$$

A similar argument gives the full Alexandrov-Fenchel inequality for k polyhedra.

In general, the volume polynomial

$$\mathrm{Vol}(\sum \lambda_i \varphi_i)$$

is Lorentzian[1]. In fact, one can prove a ‘Hodge index theorem’ over arbitrary loopless matroids and deduce that arbitrary Lorentzian polynomials satisfy the corresponding symmetry results.

[1] the divisors here are required to satisfy a positivity criterion called nefness

- Robert Lazarsfeld, *Positivity in Algebraic Geometry* (Springer)
- William Fulton, *Introduction to Toric Varieties* (Princeton)
- Claire Voisin, *Hodge Theory and Complex Algebraic Geometry* (Cambridge)
- Daniel Huybrechts, *Complex Geometry* (Springer)
- Yuriĭ D. Burago and Viktor A. Zalgaller, *Geometric Inequalities* (Springer)
- Matthew Baker, “Hodge Theory in Combinatorics” (survey article, `https://arxiv.org/abs/1705.07960`)