

COMBINATORIAL STRUCTURES IN TRACE POLYNOMIALS OF FUNCTION GROUPS

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The discreteness problem

Let G be an abstract group. Let X be a Lie group. Let $\rho : G \rightarrow X$ be a representation. When is ρ discrete? What is the moduli space of all discrete ρ ?

(Think of something like $X = \mathrm{SL}(2, \mathbb{C})$.)

Connections

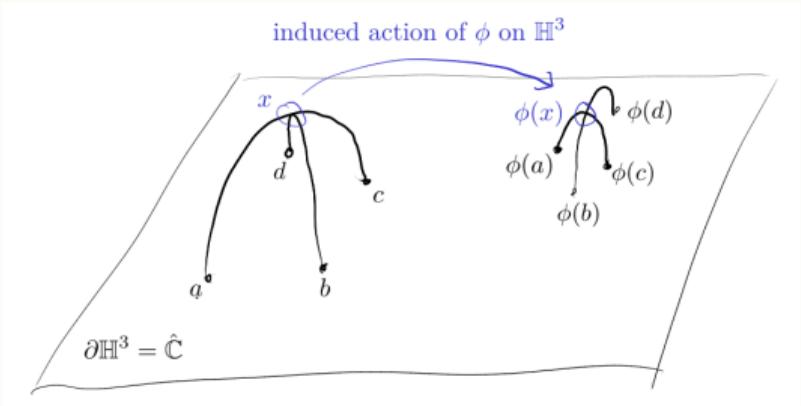
- Discrete groups often come with number-theoretic or algebraic-geometric properties (see e.g. Margulis' superrigidity theorem).
- Discrete groups are models for classically interesting objects like tilings in geometric spaces.

Elements of $\text{PSL}(2, \mathbb{C})$ act on the Riemann sphere $\hat{\mathbb{C}}$ as fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az + b}{cz + d}.$$

They also act on the upper halfspace $\mathbb{H}^3 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$ as hyperbolic isometries.

A discrete subgroup of $\text{PSL}(2, \mathbb{C})$ is called a **Kleinian group**. Its action can have complicated dynamics on the Riemann sphere, originally studied by Poincaré, Fricke, and Klein in the 1880s and 90s.

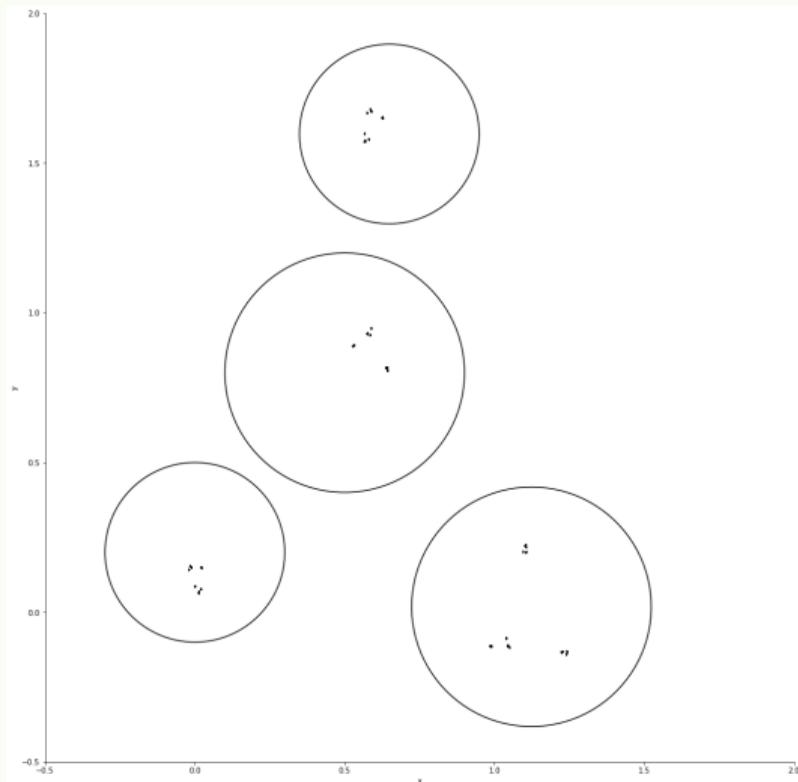


SCHOTTKY GROUPS

Pick g pairs of round discs in the plane and for each pair pick a Möbius transformation that sends the interior of one disc to the exterior of the other.

If all $2g$ discs are disjoint:

- By a ping-pong argument, the group G is free and discrete.
- The common exterior of the discs is mapped around by G to tile an open set $\Omega(G) \subset \hat{\mathbb{C}}$.
- $\Omega(G)/G$ is a compact genus g surface bounding a handlebody \mathbb{H}^3/G .

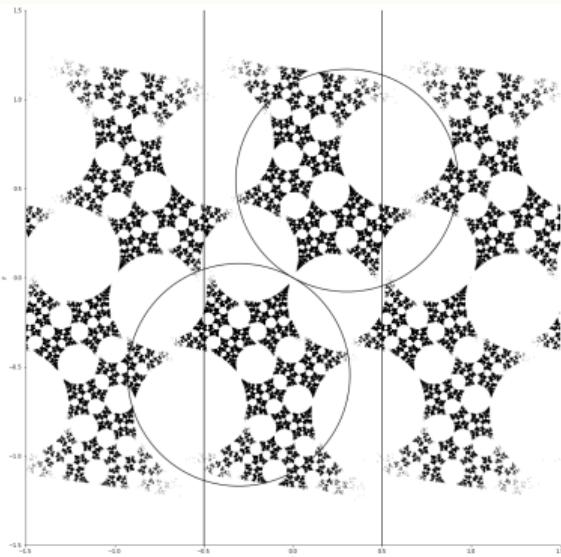


RILEY GROUPS

Let $z \in \mathbb{C}$ and construct the group

$$\Gamma = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \right\rangle.$$

- There is a neighbourhood $U \ni \infty$ such that $z \in U$ implies Γ is discrete and free.
- For $z \in U$, $\Omega(\Gamma)/\Gamma$ is a four-punctured sphere bounding a manifold $M = D \times (-1, 1)$ where D is a twice-punctured disc.
- Visual inspection suggests that the example \rightarrow is discrete: how do you give a formal certificate of this fact?



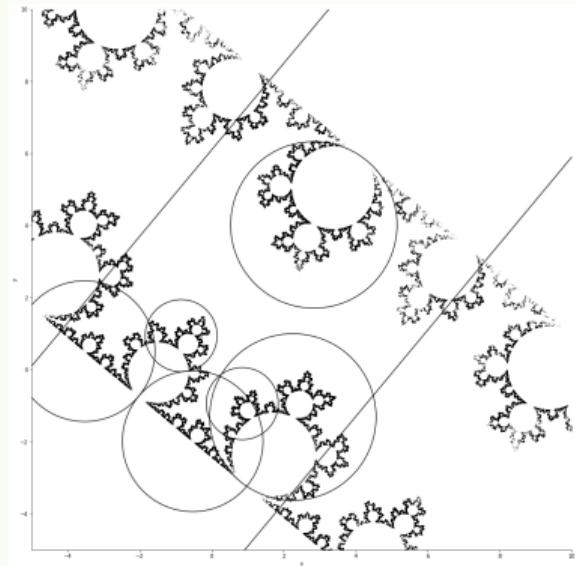
$$z = -0.78 + 1.4i$$

MAXIMALLY CUSPED COMPRESSION BODIES

Let \mathcal{F} parameterise holonomy groups of $S_2 \times (0, 1)$ so that the one end is S_2 and the other end is a pair of 3-punctured spheres. Generators are

$$P = \begin{bmatrix} -1 & \alpha \\ 0 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} \rho & \beta \\ \sigma & (1 + \beta\sigma)/\rho \end{bmatrix}$$
$$M = \begin{bmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{bmatrix} \quad N = \begin{bmatrix} r & b \\ s & (1 + bs)/r \end{bmatrix};$$

using relations in the surface group we can write all the parameters in terms of algebraic functions of α , β , and σ .



$\alpha = 6 - 5i$, $\beta = 6 - i$, and
 $\sigma = -0.24 + 0.36i$.
Is this group in \mathcal{F} ?

Theorem (E-Schillewaert–Martin, 2021 (after Keen–Series, 1994))

Let $z \in \mathbb{C}$ and define

$$\Gamma(z) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \right\rangle.$$

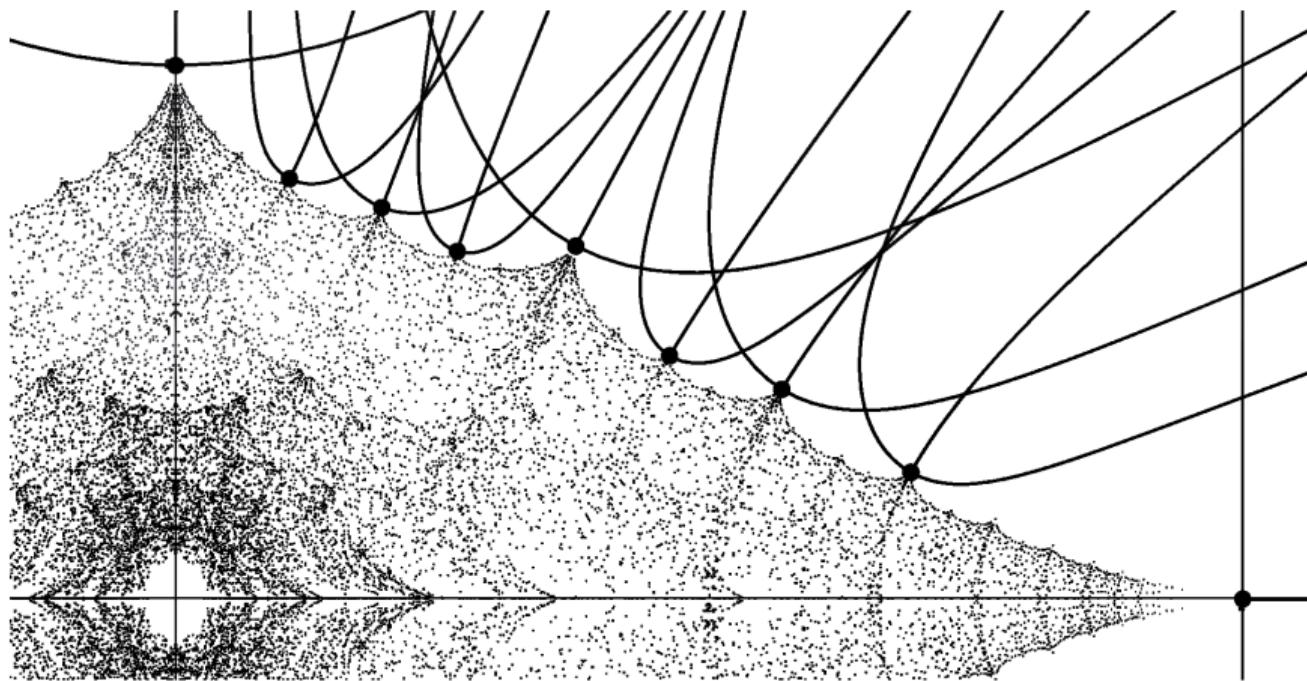
Let \mathcal{H} be the half-plane $\operatorname{Re} z < -2$.

There is a family of polynomials $\Phi_\xi \in \mathbb{Z}[z]$, indexed by $\xi \in \mathbb{Q}$, such that

- for every z such that $\Gamma(z)$ is discrete and free, there is some ξ such that $\Phi_\xi(z) \in \mathcal{H}$.
- if $z \in \mathbb{C}$ lies in a specific branch of $\Phi_{p/q}^{-1}(\mathcal{H})$ then $\Gamma(z)$ is discrete and free.

This gives a semi-algorithm to determine discrete-and-freeness for $\Gamma(z)$.

Subsequent work of E-M.-S. and Chesebro–M.-S., following Akiyoshi, Sakuma, Yamashita, and others, gives a semi-algorithm for the discreteness problem for arbitrary subgroups of $\operatorname{PSL}(2, \mathbb{C})$ with two elliptic or parabolic generators.



ENUMERATION OF ARITHMETIC GROUPS

One application is upcoming work of E.-Martin–Schillewaert:

Theorem (Announced June 2024, details to appear)

A subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is called **thin** if it is an infinite index subgroup of an arithmetic Kleinian group. It is called **Heckoid** if it is generated by two finite order elements (or parabolics) but does not split as a free product.

There are circa 150 thin Heckoid groups up to conjugacy in $\mathrm{PSL}(2, \mathbb{C})$.

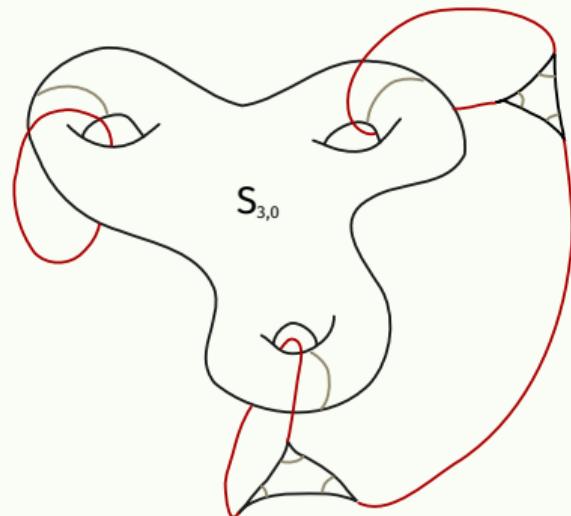
Essentially depends on (i) arithmetic work to bound the degree of invariant trace fields, and (ii) application of discreteness semi-algorithms to check whether the (finitely remaining) possible groups are discrete groups with correct isomorphism type.

Let G be a non-elementary, geometrically finite, Kleinian group which has an invariant component $\Delta \subseteq \Omega(G)$.

Let $X(G) \subseteq \mathbb{C}^n$ be a holomorphic parameterisation of the character variety of G .

Theorem (E., upcoming preprint in early 2025)

There is a family of semi-algebraic sets in $X(G)$, indexed by a simplicial complex $\mathcal{K}(\Delta/G)$ and with defining polynomials computable in terms of walks in this complex, such that the union of these sets exhausts $QH(G)$. This is a method for producing discreteness/group recognition certificates in $X(G)$.



An example of a maximally cusped function group.

Proof.

1. There is a dense lamination of $\text{QH}(G)$ where each leaf is the set of groups with a certain pleating locus on the convex core boundary. These laminations are indexed by the complex of curves $\mathcal{K}(\Delta/G)$ and the words in G corresponding to each lamination can be found by a combinatorial recursion.
2. These leaves correspond to existence of sets of peripheral subgroups in the ambient group satisfying realness conditions on their generators (i.e. $\text{tr } \in \mathbb{R} \setminus [-2, +2]$) and certain asymptotic conditions (these are the obstructions to computational effectiveness).
3. By arguments in quasiconformal analysis, these realness conditions can be weakened to inequalities, $\text{tr } \in \mathcal{CS}$, where \mathcal{CS} is the set of triples $(\text{tr } A, \text{tr } B, \text{tr } AB)$ such that A and B generate a classical Schottky group pairing their isometric circles (it's easy to give such inequalities in terms of the traces).



- Due to results of Anderson–Canary (1996) and McMullen (1998), see paper of Ohshika cited in endnotes, it is known that neighbourhoods of the boundary of a quasiconformal deformation space are not locally connected.
- Unlike in the one-dimensional case, the semi-algebraic varieties obtained via these methods may not be smooth or connected, but they give local neighbourhoods of cusps that are slightly better than the ‘ambient’ neighbourhoods in $\text{QH}(G)$ or $\text{AH}(G)$.
- Another advantage is that well-understood methods are available to deal with semi-algebraic sets computationally (e.g. quantifier elimination algorithms).

EXAMPLE: MAXIMALLY CUSPED GENUS 2 GROUP

We can explicitly construct a group in \mathcal{F} (one end genus two, the other end a pair of 3-punctured spheres) with specific peripheral structure.

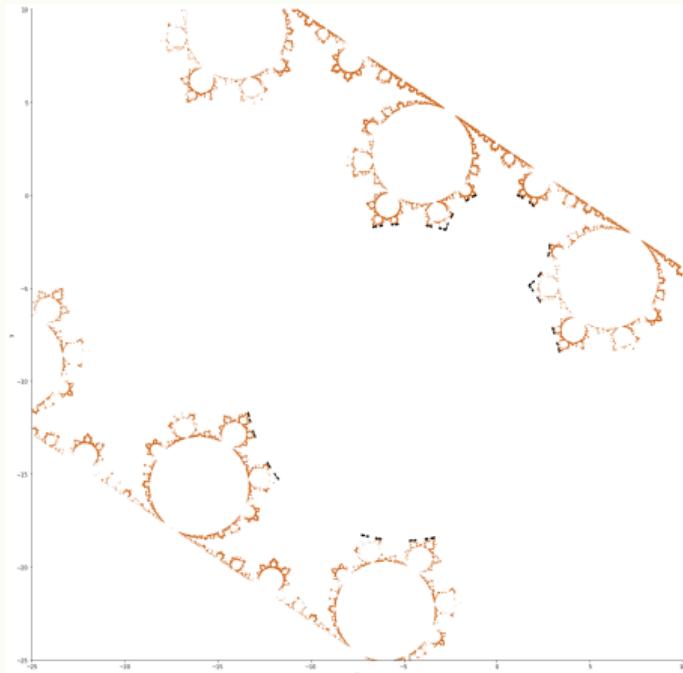
Choose as peripheral groups

$$\langle M, pNMnP \rangle \quad \text{and} \quad \langle mQ, pmQP \rangle.$$

Specify the traces $\text{tr } M = 3$, $\text{tr } mQ = -3$,
 $\text{tr } m(pNMnP) = -3$. Then:

$$\alpha = 10 - 3i\sqrt{5}, \quad \beta = \frac{5}{29}(171 + 2i\sqrt{5}),$$

$$\sigma = \frac{4}{29}(3 - 2i\sqrt{5}).$$

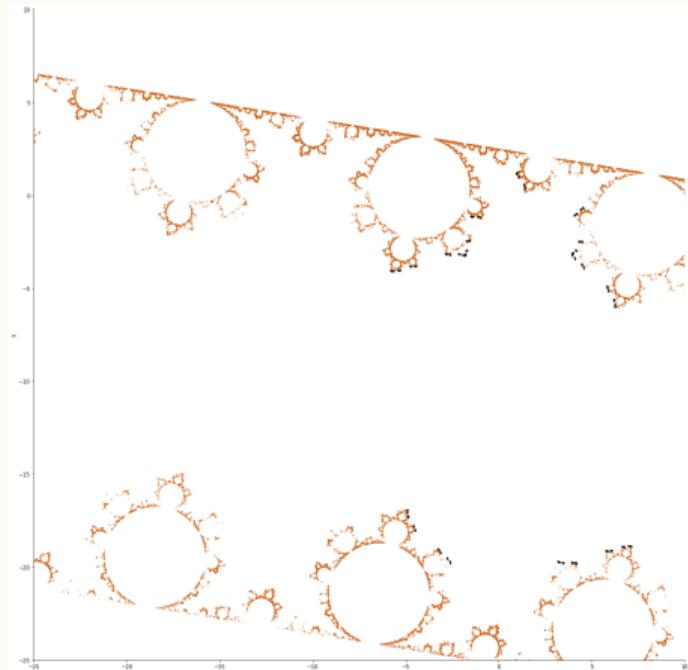


EXAMPLE: MAXIMALLY CUSPED GENUS 2 GROUP

Now suppose we make a slight deformation of the parameters in \mathbb{C}^3 from G to a new group \tilde{G} :

	$\text{tr } M$	$\text{tr } mQ$	$\text{tr } m(pNMnP)$
G	3	-3	-3
\tilde{G}	$2.5 + 2i$	$-3 - 0.5i$	$-3 - 0.5i$

Peripheral structures are quasi-Fuchsian but still discrete with correct quotient, so this new group is still in \mathcal{F} : it's discrete with known isomorphism type.



BEDTIME READING

- A.J.E., G.M., and J.S., *Approximations of the Riley slice.* *Expo. Math.*, 2023 (arXiv 2021).
- —, G.M., and J.S., *The combinatorics of the Farey words and their traces.* *Groups, Geometry, and Dynamics*, online 11/2024 (arXiv 2022).
- —, G.M., and J.S., *On thin Heckoid and generalised triangle groups in $\text{PSL}(2, \mathbb{C})$* (arXiv 2024).
- Albert Marden, *Hyperbolic manifolds*. Cambridge U. Press, 2016.
- Ken'ichi Ohshika, *Divergence, exotic convergence and self-bumping in quasi-Fuchsian spaces.* *Ann. Fac. Sci. Toulouse Math.*, 2020 (arXiv 2010).
- Title picture: Power lines at Hintere Gartenkuppe (near Saalfeld), own work, 2023.