AUTOMORPHIC FORMS, KLEINIAN GROUPS, AND PLANE CURVES

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Abstract.

1. Introduction

The theory of automorphic forms and functions goes back as far as Poincaré in the XIXth century [Poi82; Poi85], who associated these highly symmetric functions to Fuchsian groups in an effort to understand certain differential equations and their associated Riemann surfaces. Since then, automorphic forms have become major tools in subjects like analytic number theory, where the coefficients of forms associated to arithmetic groups reveal a great deal of number theoretic information [Miy89], and mathematical physics, where these functions arise in the study of differential equations related to scattering theory [LP76].

As part of a programme to understand how the structure of 'knotted' hyperbolic 3-manifolds is manifested in various related algebraic structures, we were lead to consider a simple extension of the theory of automorphic forms to Kleinian groups, which have much more complicated zoology than Fuchsian groups. More precisely, the simplest non-elementary Kleinian groups—so-called **Schottky groups**—have compact Riemann surfaces associated to them, and by the ending lamination theorem these Kleinian groups are identified up to conjugacy in $\mathrm{PSL}(2,\mathbb{C})$ by the complex structure on (that is, biholomorphism type of) the associated surface. The category of compact Riemann surfaces with holomorphic maps is equivalent to the category of smooth algebraic curves over $\mathbb{C}$ with rational maps, and so to each Schottky group we may assign in a natural way a birationality class of algebraic curves. Even better, these algebraic curves are parameterised by automorphic functions associated to the group; and so we may ask, through the lense of these parameterisations, which properties of the group can be detected in the curves. The construction of automorphic functions is not just limited to Schottky groups, and although more complicated groups may have more complicated structures bounding their corresponding 3-manifolds we may still construct these functions and obtain a parameterisation. This paper presents some of the general theory, an analysis of some groups (like Schottky groups) where a compact Riemann surface lies at infinity and hence a 'real life' algebraic curve can be found, and some experimental results about various other groups which are related to these groups by natural geometric operations. The numerical convergence of Poincaré series for Schottky groups has been studied before [Bog12; Lya22], but a detailed study of the analysis of the series and the geometry of the automorphic forms and functions so obtained does not seem to have been discussed in the literature, especially from the point

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of view of moduli theory (which requires a theory of more complicated flora than compact surfaces at infinity).

This paper is split broadly into two parts; in the first part, we present a naïve (in the sense of Halmos [Hal85, p. 246]) version of the theory of automorphic forms for geometrically finite Kleinian groups; in the second part, we analyse the plane curves parameterised by automorphic functions for various Kleinian groups. We will present some one-parameter families of groups, but the moduli theory of Kleinian groups is much more complicated than the corresponding theory of algebraic curves and so we leave a detailed analysis of this to a paper which is currently under preparation. Also under preparation is a more detailed analysis of the relationship between different Kleinian groups which lie 'above' the same reducible algebraic curve, which is a study made possible by this paper since such groups necessarily have more than two components in general and hence are not Fuchsian.

Part 1. Automorphic forms

2. A crash course on Kleinian groups

There is no comprehensive book on the modern theory of Kleinian groups and associated geometric structures; we will present some of what we will need (enough for the casual reader to follow the remainder of the paper) but the reader is encouraged to consult a combination of the following four books for more detailed information: (i) Maskit [Mas87] for the basic theory and a detailed study of geometric finiteness and other properties; (ii) Thurston [Thu79] for the detailed 3-manifold theory; and (iv) Mumford, Series, and Wright [MSW02] for a detailed case study of a variety of nice examples. One of the authors of the current paper has also co-authored an expository paper on the moduli theory of Kleinian groups, which contains more background information [EMS23].

A **Kleinian group** is a discrete subgroup of the group $\mathrm{Isom}^+(\mathbb{H}^3)$ of orientation-preserving isometries of hyperbolic 3-space (with the usual compact-open topology). The manifold $\mathbb{H}^3$ has a natural **visual boundary** $\partial\mathbb{H}^3$ which is identified with the 2-sphere; if $\mathbb{H}^3$ is modelled by the upper half-space $H = \{(z,t) \in \mathbb{R}^2 \times \mathbb{R} : t > 0\}$ then $\partial\mathbb{H}^3$ is identified with the boundary of H in $\mathbb{R}^3 \cup \{\infty\}$, the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Every isometry of $\mathbb{H}^3$ has a natural extension to a conformal map of the boundary, and conversely every conformal map of $\hat{\mathbb{C}}$ extends to a hyperbolic isometry of the interior by so-called **Poincaré extension**; in this way one constructs a natural isomorphism of topological groups between $\mathrm{Isom}^+(\mathbb{H}^3)$ and the group of conformal maps of the Riemann sphere. Standard results from undergraduate complex analysis then show that this latter group is just the group $\mathbb{M}$ of Möbius transformations—the orientation-preserving half of the group generated by all reflections in circles on the 2-sphere—which is in turn isomorphic to the group $\mathrm{PSL}(2,\mathbb{C})$ of fractional linear transformations, where a matrix acts on $z \in \hat{\mathbb{C}}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}.$$

Warning. We will assume that all Kleinian groups are torsion-free, unless otherwise stated. If one allows torsion in the group then some things which we will say become wrong, sometimes in subtle ways; most of the theory can be suitably modified to include torsion groups without the introduction of too many new ideas, but in the struggle between completeness and clarity we have tried to let clarity win (at least most of the time).

If G is a Kleinian group, then $\mathbb{H}^3/G$ is a hyperbolic 3-manifold M (that is, a 3-manifold locally modelled on $\mathbb{H}^3$) and G is identified with the holonomy group of M. One says that hyperbolic 3-manifolds are **uniformised** by Kleinian groups; there is a natural 1-1 relationship between Kleinian groups up to conjugacy in $\mathbb{M}$, and hyperbolic 3-manifolds up to isometry. One would next like $(\partial\mathbb{H}^3)/G = \hat{\mathbb{C}}/G$ to be some kind of Riemann surface 'boundary' for M, but it turns out that this quotient is in general badly behaved (it is non-Hausdorff). Let $\Lambda(G)$ be the **limit set** of G, the set of all accumulation points of orbits of G (if $|\Lambda(G)| > 2$ then this set is equal to the closure of the set of fixed points of elements of G). One can now work for a bit and show that $\Lambda(G)$ is exactly the set of points which make the quotient construction 'go wrong'; that is, if the **domain of discontinuity** of G is the complement $\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$ then $\Omega(G)/G$ is a Riemann surface, and it acts as a visual boundary for the manifold $\mathbb{H}^3/G$.

Warning. The set $\Lambda(G)$ is incredibly complicated to analyse: it is a philosophy due to Sullivan and McMullen that the study of Kleinian groups as dynamical systems is strongly analogous to the study of semigroups generated by rational maps [McM91], and under this dictionary the limit set of a Kleinian group plays the same role as the Julia set of a rational dynamical system. It is often the case that $\Omega(G)$ is highly multiply-connected (for instance, in many cases $\Lambda(G)$ is homeomorphic to a Cantor set). On the other end of the complexity scale, it could be that $\Lambda(G) = \hat{\mathbb{C}}$ and in this case $\Omega(G)$ is empty.

In a certain sense, the geometric content of the group G is entirely contained within the combinatorics of the limit set [Thu79, Chapter 8]. One precise but not very deep manifestation of this philosophical statement is the following remark:

Lemma 2.1. *If G is Kleinian and $|\Lambda(G)|$ is infinite then $\Omega(G)/G$ is a hyperbolic Riemann surface. If $|\Lambda(G)| < \infty$ then $|\Lambda(G)| \leq 2$ and $\Omega(G)/G$ is Euclidean or spherical.* □

If $|\Lambda(G)| < \infty$ then we say that G is an **elementary group**. These groups are not interesting from a geometric point of view, and so we will usually consider only non-elementary groups (but, unlike the implicit 'no torsion' assumption, we will always state 'non-elementary' when needed). Because the limit set is so important, we would like to be able to compute it; the usual method for this comes from the following dynamical lemma.

Lemma 2.2. *If G is Kleinian, then $\Lambda(G)$ and $\Omega(G)$ partition $\hat{\mathbb{C}}$. If $\xi \in \Lambda(G)$, then $G\xi$ is dense in $\Lambda(G)$. If $\mathrm{Fix}(G)$ is the set of fixed points of loxodromic elements of G and is non-empty (which is always the case if G is non-elementary), then $\Lambda(G) = \overline{\mathrm{Fix}(G)}$.* □

Example 2.3. Let us verify computationally that the group G generated by the two loxodromic elements

$$X = \begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 & 0 \\ i & 1/2 \end{bmatrix}$$

is discrete. Observe that 0 is a fixed point of Y. Hence by Lemma 2.2 the translates of 0 under G are dense in the limit set. In Figure 1 we have plotted the translates of 0 under 1 000 000 random words of length at most 20 in G—in this case the group happens to be free on the two generators so there is a bijection between infinite words in $\{X^{\pm 1}, Y^{\pm 1}\}$ and limit set elements, but this just means we don't hit limit points more than once except by chance. Since the points plotted do not look dense in $\hat{\mathbb{C}}$, we can be fairly confident that G is discrete with $\Omega(G) \neq \emptyset$. (There are theoretical results which back up this confidence.)

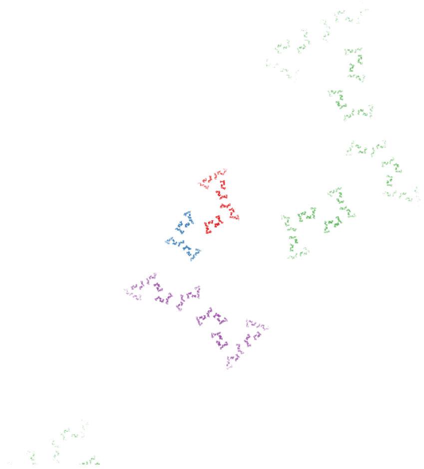


FIGURE 1. The limit set of the group defined in Example 2.3. The colours indicate the final letter in the word labelling the point, and the bounds on the plot are $|\Re z| < 4$ and $|\Im z| < 4$.

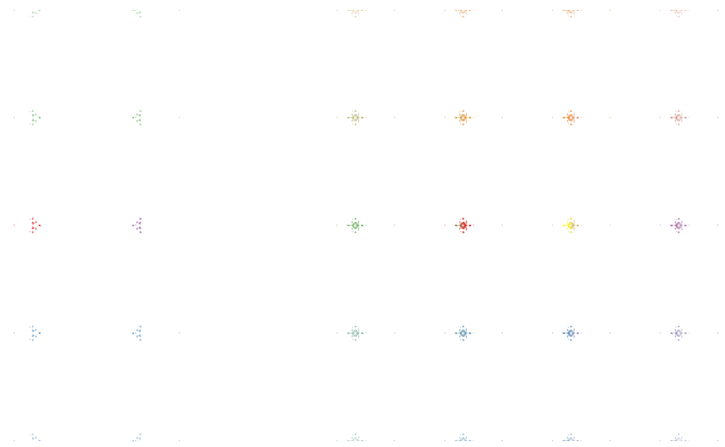
If one is given a group $G \leq \mathrm{PSL}(2, \mathbb{C})$, then checking whether G is discrete is computationally hard [Gil97]. Even if one knows that a group is discrete, trying to determine the geometric properties of the manifold and surface quotients is nontrivial. To make these two problems tractable for human study one introduces the concept of a **fundamental polyhedron** for G; this is a hyperbolic polyhedron $P \subseteq \overline{\mathbb{H}^3}$ such that the action of G on $P \cap \mathbb{H}^3$ tiles $\mathbb{H}^3$ with overlaps only on edges, and such that the action of G on $P \cap \Omega(G)$ tiles $\Omega(G)$ with overlaps only on edges (there are various technical conditions which $P \cap \Omega(G)$ ought to satisfy which do not change this intuitive picture, most related to pathological situations which will not arise in this paper; as well as the discussion in Maskit [Mas87, IV.F–IV.H] one might like to read the more lengthy geometric discussion in the large textbook of Ratcliffe [Rat94, Chapter 6]). The **Poincaré polyhedron theorem** states (roughly speaking) that, under certain combinatorial and geometric conditions:

(1) the existence of such a polygon P is a certificate of discreteness for the group G, and
(2) the manifold $(\Omega(G) \cup \mathbb{H}^3)/G$ is obtained up to isometry by taking the polyhedron P with its $\mathbb{H}^3$-geometry and identifying its faces according to the action of G.

The full version of the theorem also gives the full presentation of the group G, which can be read off from the combinatorics of the action of G on the faces of P.

If G is a Kleinian group which admits a fundamental polyhedron P with finitely many faces, then G is called **geometrically finite**. Knowledge of geometric finiteness gives us detailed information about the behaviour of G and the topology of the quotient [Mas87, Chapter VI], and so we will usually restrict ourselves to such groups. We briefly now discuss some of the topological and geometric data about the surface which can be read out of knowledge of a geometrically finite Kleinian group G. If $g \in G \setminus \{1\}$, then (assuming, as always, that g is not torsion) let $\mathrm{Fix}(g)$ denote the set of fixed points of g on $\widehat{\mathbb{C}}$. Either $|\mathrm{Fix}(g)| = 1$ or $|\mathrm{Fix}(g)| = 2$.

- If g has a single fixed point, then it is **parabolic** and it is conjugate in $\mathbb{M}$ to the translation $z \mapsto z+1$. A parabolic element is **primitive** in G if it is not a nontrivial power of some other element. The classification of geometrically



(A) The limit set of the group generated by $z \mapsto z + i$ and $z \mapsto (2z + 3/4)/(4z + 2)$, which is a non-elementary group with rank one cusp at ∞.

(B) The limit set of the group generated by $z \mapsto z + 1$, $z \mapsto z + i$, and $z \mapsto (2z + 3/4)/(4z + 2)$, which is a non-elementary group with rank two cusp at ∞.

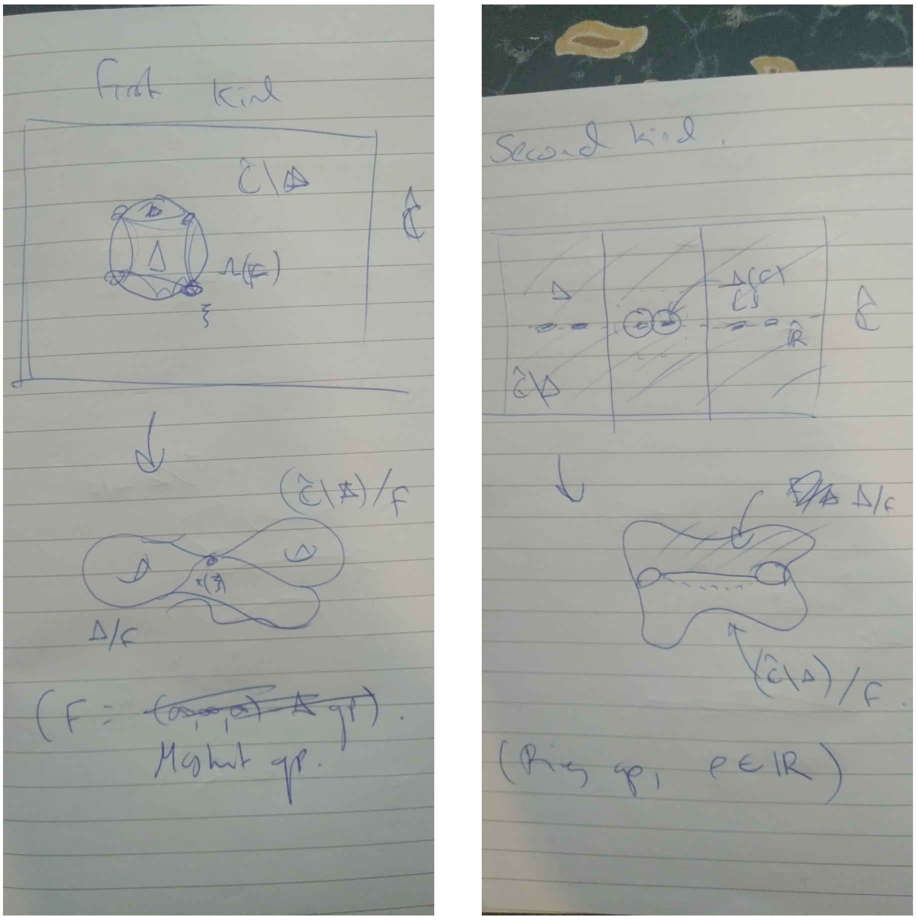
FIGURE 2. Limit sets of two non-elementary groups with cusps at ∞.

finite Kleinian groups tells us that if ξ is the fixed point of some primitive parabolic element g, then the stabiliser of ξ is a free abelian group of rank 1 or 2 generated by primitive parabolic elements (one of which is g). If ξ is stabilised by a cyclic group then we say it is a **rank 1 cusp** (Figure 2a); in this case there are two round open discs in $\Omega(G)$, tangent at ξ, such that each disc projects to a neighbourhood of a puncture in the quotient surface. Thus each primitive parabolic element of rank 1 corresponds to a pair of punctures on the surface. On the other hand, if ξ is stabilised by a rank 2 group, then there is a neighbourhood of ξ which glues up to a torus with a deleted disc; in this case we say that ξ is a **rank 2 cusp** (Figure 2b). As well as Maskit [Mas87, §VI.A], one can see Benedetti and Petronio [BP92, §D.3] for more information on the classification of cusps.

- If g has two fixed points, then it is **loxodromic** and it is conjugate in $\mathbb{M}$ to the dilation $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}^*$ with $|\lambda| \neq 1$. In this case, the fixed points of G cannot be parabolic fixed points, and in fact they are **points of approximation** (limits of isometric circles, which we discuss in a moment). Loxodromic elements, roughly speaking, correspond to either 'thick' handles on the surface, or geodesics in the 3-manifold which contract to 'long' curves on the surface.

The easiest way to construct a fundamental domain is to construct either the **Ford domain** of the group or some related set. Let g be an element of $\mathbb{M}$ which does not fix ∞; then there is a unique pair of circles C, C' in $\mathbb{C}$ with equal radius such that $gC = C'$ and g maps the exterior of C into the interior of C'. These are called the **isometric circles** of g; if g is represented by the matrix $(a\,b \mid c\,d)$ then the radius of these circles is $1/|c|$ and the two centres are $-d/c$ and a/c respectively. If G is a Kleinian group with $\infty \in \Omega(G)$, then the set of all $z \in \Omega(G)$ which do not lie on the interior of any isometric circle of an element of G is a fundamental domain for the action of G on $\Omega(G)$.

We briefly discuss now the complex structure induced on $\Omega(G)/G$ by the quotient. It is easiest to start with the case of so-called *Fuchsian* groups.



(A) The group generated by $z \mapsto (z-1)/z$ and $z \mapsto (z-1)/(-z+2)$ is a Fuchsian group of the first kind.

(B) The group generated by $z \mapsto z+1$ and $z \mapsto z/(5z+1)$ is a Fuchsian group of the second kind.

FIGURE 3. The two kinds of Fuchsian group.

Definition 2.4. A **Fuchsian group** is a pair (F, Δ) where F is a Kleinian group and Δ is a round open disc (a half-space is a round disc whose boundary contains ∞), such that F preserves the set Δ.

Given a Fuchsian group (F, Δ), the limit set $\Lambda(F)$ is contained within $\partial\Delta$. If equality occurs here, then the group is called Fuchsian **of the first kind**; otherwise, it is Fuchsian **of the second kind**. One can show that the complementary disc $\hat{\mathbb{C}} \setminus \overline{\Delta}$ is also left invariant by F. Recall that every round disc Δ admits a hyperbolic metric (that is, a Riemann metric such that there exists an isometry $f : \Delta \to \mathbb{H}^2$ where $\mathbb{H}^2$ is the usual upper half-plane $\{z \in \mathbb{C} : \Im z > 0\}$), and that the group of hyperbolic isometries of Δ is exactly the group I of conformal isometries of $\hat{\mathbb{C}}$ which preserves Δ: fIf^{-1} is just $\mathrm{PSL}(2, \mathbb{R})$, the group preserving $\mathbb{H}^2$ [KL07, Chapter 2]. In this way, every Fuchsian group (F, Δ) is (up to conjugacy) a discrete group of isometries of Δ, and is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. As a discrete group of isometries of Δ, the quotient Δ/F inherits a hyperbolic metric. If the group is of the first kind, then $\Omega(F)/F$ is a disjoint union of two such hyperbolic surfaces, possibly joined by double cusps (Figure 3a). If the group is of the second kind, then $\Omega(F)/F$ is a union of two hyperbolic surfaces along the boundaries of deleted discs, the components of $L = (\Omega(F) \cap \partial\Delta)/F$ (Figure 3b); in this situation, each half has an induced hyperbolic structure but the induced complex structure on the whole is not hyperbolic—the points of L are 'at infinity' in each of the two hyperbolic metrics, and it is possible to formalise some notion of the existence of a 'fold' along the gluing line.

We recall that every Riemann surface S of negative Euler characteristic (which includes the quotient surface $\Omega(G)/G$ for any non-elementary Kleinian group G) admits a hyperbolic structure. This is realised by the existence of a Fuchsian group $(H, \mathbb{H}^2)$ of the first kind such that S is isometric to $\mathbb{H}^2/H$ (i.e. S is realised by one of the components of the group); in fact, as in the 3-dimensional case, H is the holonomy group of S. Thus, if (F, Δ) is non-elementary and Fuchsian of the second kind, $\Omega(F)/F$ also admits a hyperbolic structure which is induced by a second Fuchsian group H. The point here is that the universal cover $\mathbb{H}^2 \to S$, realised with the deck transformation group H, factors through $\Omega(F)$. Of course, we are not using the Fuchsian-ness of F here: if G is any non-elementary Kleinian group with $\Omega(G) \neq \emptyset$, then $\Omega(G)$ admits a hyperbolic structure which is obtained by a similar construction [KL07, Chapter 7]:

$$\begin{tikzcd}
\mathbb{H}^2 \arrow[dr, "N"] \arrow[dd, "H"'] & \\
 & \Omega(G) \arrow[dl, "G \simeq H/N"] \\
\Omega(G)/G &
\end{tikzcd}$$

Thus if G is such a group, then $\Omega(G)/G$ does admit a hyperbolic structure which is induced from a hyperbolic structure on $\Omega(G)$, and if G happens to be Fuchsian of the second kind then the surface $\Omega(G)/G$ admits a decomposition into a pair of (open) hyperbolic surfaces, each coming from one of the two invariant discs of G, but with metric bearing no explicit relation to the metric arising from the whole $\Omega(G)$. (If G is Fuchsian of the first kind, then this second metric just agrees with the $\Omega(G)$-metric, and in general there is a similar decomposition which arises from the so-called **convex core** of the uniformised hyperbolic 3-manifold. We will discuss this further in a later paper on the moduli theory, in a similar way to the direction discussed in [KS94].)

3. Automorphic forms and Poincaré series

We will now define and construct automorphic forms for geometrically finite Kleinian groups and hence construct explicit meromorphic functions on the corresponding Riemann surfaces (if they are non-empty). The construction which we modify was written down by Poincaré in the first volume of *Acta Mathematica* in 1882 [Poi82]; we will mainly refer to the English translation of this paper by John Stillwell [Poi85], and the modern exposition of the theory by Lehner [Leh14; Leh64]. We will give full details, since the results in the classical literature restrict themselves to Fuchsian groups (i.e. groups such that $\Lambda(G)$ is contained in a round circle) and most modern literature is primarily interested in the arithmetic case.

Let G be a Kleinian group. We will define automorphic forms with respect to G and prove their existence; our notation will follow that used by Miyake [Miy89] in the arithmetic case. To this end, we define a map $j : G \times \mathbb{C} \to \mathbb{C}$ by the rule $j(g, z) = cz + d$, where c and d are coefficients in some choice of matrix representative[1] $(a\,b \mid c\,d)$ for g. Observe that j depends up to sign on the choice of representative in $\mathrm{SL}(2,\mathbb{C})$; thus we must always be careful to lift to $\mathrm{SL}(2,\mathbb{C})$ consistently within each group. (That such a consistent lift is possible depends only on a lack of two-torsion in the group [Cul86].) We also observe that $g'(z) = (cz+d)^{-2} = j(g,z)^{-2}$.

We now define an action of G on the space of meromorphic functions on $\Omega(G)$. If $g \in G$, $k \in \mathbb{Z}$, and $f \in M(\Omega(G))$, then we define

$$(f|_k g)(z) = j(g,z)^{-k} f(gz).$$

An **automorphic form of weight** k for G is a fixed point of this action, i.e. a meromorphic function f such that $(f|_k)g = f$ for all $g \in G$.

The usual way to produce a function which is (almost)-equivariant with respect to a group action is to take an arbitrary function and form its average over g. In this case, if H is a meromorphic function on $\Omega(G)$ then we are motivated to form the series

$$\sum_{g\in G} (H|_k g)(z).$$

We now proceed to study the convergence of this series, and in order to do this we need to make some simplifying assumptions on the Kleinian group G. For instance, we cannot sum over all of the elements of a group containing the translation $z \mapsto z+1$, since otherwise each summand appears infinitely often (if f is a translation then $j(fg,z) = j(g,z)$ for all $g \in G$) and the series will definitely not converge.

Definition 3.1. Consider a geometrically finite Kleinian group G with non-empty domain of discontinuity such that the point ∞ is not a point of approximation (equivalently, one asks that if $\infty \in \Lambda(G)$ then ∞ is a parabolic fixed point). For concision we will summarise these assumptions on G by saying that G is **pointed at infinity**.

If G is pointed at infinity, then consider the subgroup

$$G_\infty = G \cap \left\{ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} : \lambda \in \mathbb{C}^* \right\}$$

of translations in G (i.e. parabolic elements with fixed point ∞). This is not a normal subgroup of G in general, but we can still construct the system of right cosets $G_\infty \backslash G$. By the classification of geometrically finite groups, G_∞ is a free abelian group of rank 1 (giving a cylindrical cusp, Figure 2a) or 2 (giving a toroidal cusp, Figure 2b).

Definition 3.2. Fix a Kleinian group G which is pointed at infinity, let $S(G)$ be a fundamental domain for G, and let D be a complete set of coset representatives for $G_\infty \backslash G$. Let $H : \Omega(G) \to \hat{\mathbb{C}}$ be a meromorphic map with finitely many poles in $S(G)$ which is equivariant with respect to G_∞ (i.e. $H(tz) = H(z)$ for all $t \in G_\infty$, so H is well-defined on right cosets $G_\infty g \in G_\infty \backslash G$), and let $m \geq 1$. Then the series

$$\varphi_H(z) = \sum_{g\in D} H(gz)(g'(z))^m \tag{3.3}$$

is called the **Poincaré series** of degree m derived from H;[2] in the case that G is a modular group and $H \equiv 1$, it is called an **Eisenstein series**.

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[1]For typographic reasons, outside of mathematical displays we will write $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as $(a\,b \mid c\,d)$.

[2]Poincaré himself called these **theta-Fuchsian functions**, in analogy with the theta functions of Jacobi and Riemann. This terminology has not been used since, and now the term *theta function* refers specifically to functions automorphic with respect to a lattice of any dimension, i.e. meromorphic functions on torii.

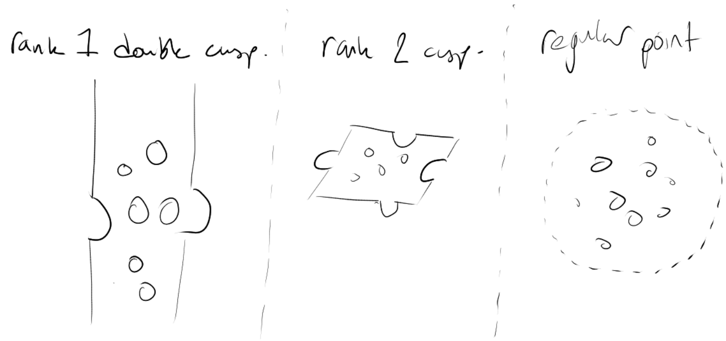


FIGURE 4. The three possibilities for a normalised fundamental domain as used in the proof of Lemma 3.5.

Lemma 3.4 (Automorphy). *If the series of Equation* (3.3) *converges in* $\Omega(G)$*, then it is an automorphic form of weight* $2m$ *for* G*.*

Proof. Suppose that $\alpha \in G$; then, by the chain rule, $g'(\alpha z) = (\alpha'(z))^{-1}(g\alpha)'(z)$. We can therefore compute,

$$\begin{aligned}\varphi_H(\alpha z) &= \sum_{g\in D} H(g\alpha z)(g'(\alpha z))^m \\ &= (\alpha'(z))^{-m} \sum_{g\in D} H(g\alpha z)((g\alpha)'(z))^m \\ &= j(\alpha, z)^{-2m} \sum_{g\in D} H(gz)(g'(z))^m\end{aligned}$$

and so φ_H is automorphic of weight $2m$. (There is a subtlety in the final equality: the two are equal because multiplying on the right by α is transitive on the space of right cosets, and then since H is equivariant with respect to G_∞ we can choose the correct coset representative.) □

Lemma 3.5 (Convergence). *If* $r \geq 2$*, and* D *is defined as in Definition 3.2, then the series*

$$\sum_{g\in D} |j(g,z)|^{-r} \tag{3.6}$$

converges uniformly on subsets of $\Omega(G)$ *bounded a positive distance away from* $\Lambda(G)$*.*

Further, the Poincaré series of Equation (3.3) *for* $m \geq 2$ *converges absolutely and uniformly in compact subsets of* $\Omega(G)$ *which do not contain any* G*-translates of poles of* H*.*

Remark. We give a version of the proof in [Leh14, II.1B] but suitably generalised to the Kleinian case, using the machinery of [Mas87, §I.C and §II.B].

Proof. Replace each coset representative in D with the representative minimising $|d/c|$ (i.e. the distance of its isometric circle centre from zero). There exists a neighbourhood U of ∞, of the form $\{z \in \mathbb{C} : |z| > \rho\} \cup \{\infty\}$, which is precisely invariant under the group elements in D: since ∞ is not a point of approximation and due to the choice of representatives both the centres and radii of the isometric circles of elements of G are universally bounded, choose ρ such that every such isometric circle lies in the disc at 0 of radius ρ, and now note that every element of D maps the exterior of one isometric circle (i.e. including U) into the interior of the other (so U is mapped strictly off itself). This should be made clearer with a picture, such as the schematic of Figure 4.

We now observe that the images of U under non-identity elements of D are all disjoint subsets of the bounded set $\hat{\mathbb{C}} \setminus U$, and hence the series

$$\sum_{g \in D \setminus 1} \operatorname{diam} g(U)$$

converges. The proofs of §I.C.7 and II.B.2–7 from [Mas87] rely only on this boundedness, and can be applied directly to see that

$$\sum_{g \in D} |j(g,z)|^{-r}$$

converges whenever $z \in \Omega(G)$ is not a preimage of ∞ (i.e. is not an isometric circle centre), and is uniform on compact sets away from preimages of ∞. The proof of convergence of the series of Equation (3.6) is completed by noting that accumulation points of preimages of ∞ are limit points [Mas87, §VI.B.6].

Finally, the convergence of Equation (3.3) follows immediately from that of Equation (3.6): the point is that holomorphic functions are uniformly bounded away from their poles. □

Remark. The main computational problem which arises is the computation of good approximations to the averages over D. For this we can just use the fact that long words in the generators have small derivative and then use any one of the number of standard algorithms in combinatorial group theory to enumerate words in order of length; for instance the various limit set algorithms in [MSW02]. Even summing across a relatively small number of words (hundreds) seems to give good convergence. Of course, the convergence of Eisenstein series (i.e. in the arithmetic setting) is an interesting field of study—the idea is to rewrite the series as a Fourier expansion and then analyse the growth rates of the coefficients, leading to deep conjectures in analytic number theory like the Ramanujan–Petersson conjecture [Miy89, Theorem 4.5.17]. We expect that a similar theory of Fourier expansions for Kleinian groups will also be interesting, especially if the Kleinian groups in question are arithmetic (in which case one might hope to find geometric data like volume information hidden in them). One also expects such a theory to exist since the Fourier expansions which arise in analytic number theory are very closely related to the representation theory of quaternion algebras, and the theory of arithmetic Kleinian groups also intersects this representation theory in a highly nontrivial way.

Example 3.7 (Necessity of ∞ not being a point of approximation)**.** Let $g \in \mathbb{M}$ be the transformation $g(z) = 2z$. The group $\langle g \rangle$ is a geometrically finite group without parabolics, so every limit point is a point of approximation; in particular, ∞ is a point of approximation. Then we have

$$\sum_{n \in \mathbb{Z}} |j(g,z)|^{-r} = \sum_{n \in \mathbb{Z}} \left|1/\sqrt{2}^{n}\right|^{-r} = \sum_{n \in \mathbb{Z}} \sqrt{2}^{nr}$$

which clearly diverges for all r. The point is that the isometric circles of g^n become large (they accumulate at ∞) and so the area bounds fail.

Lemma 3.8 (Deformations of G)**.** *Suppose $G(t)$ is a one-parameter family of Kleinian groups, satisfying the conditions*[3]

[3] We essentially list the hypotheses for the λ-lemma [AIM09, Chapter 12].

(i) *$G(t)$ is uniformly generated by matrices whose coefficients depend holomorphically on a parameter $t \in \mathbb{B}^2$, and $G(t)$ is discrete and group-isomorphic to $G(0)$ for each t;*[4]
(ii) *For every t, $G(t)$ is non-elementary, and pointed at infinity;*
(iii) *if $\gamma(t) \in G(t)$ is parabolic for some t then $\gamma(0)$ is parabolic.*

Suppose $S(G(t))$ is a family of fundamental domains for $G(t)$ which vary holomorphically with t, and suppose $H(t)$ is a family of meromorphic functions on $\Omega(G(t))$ which vary holomorphically with t and which have finitely many poles in $S(G(t))$. Let $(t_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{B}^2$. Let $G(t_\infty)$ be the algebraic limit of the $G(t_n)$ as $n \to \infty$ (so $t_\infty \in \overline{\mathbb{B}^2}$).

If $G(t_\infty)$ has the same cusp structure at infinity as $G(0)$ (which is automatic if $t_\infty \in \mathbb{B}^2$), then the sequence of functions $\varphi_{H(t_n)}$ converges uniformly to $\varphi_{H(t_\infty)}$, which is an automorphic form for $G(t_\infty)$ of the same degree as $\varphi_{H(0)}$.

Remark. It is possible for $G(t_\infty)$ to have a cusp at infinity while the $G(t)$ ($t \in \mathbb{B}^2$) do not. In this situation, the sequence $\varphi_{H(t_n)}$ will be unbounded in the ∞-norm for the same reason discussed at the start of the section (loxodromic transformations are converging to translations).

Proof. Prove the theorem □

4. Choices of coefficients and holomorphy at cusps

Here is a provocative statement: *in complex analysis, there are really only two classes of function*. By this, we mean Laurent polynomials (that is, power series with finitely many terms and with possible negative powers) and exponential functions (including all Euclidean and hyperbolic trigonometric functions). In our current theory, we can choose either (or some combination of the two) for the coefficients H of Equation (3.3). The two have some distinct behaviour in terms of the analysis.

- If the coefficient H is chosen to be a Laurent polynomial, then it can be chosen to have a pole or a zero at any point of interest. If a pole is taken in the ordinary set $\Omega(G)$, then it is automatically guranteed that the form is nonzero, without any additional work. However, it is hard to ask for a polynomial to be equivariant with respect to the group G_∞.
- Conversely, it is easy to ask for an exponential function to be automorphic with respect to G_∞, if there is a rank 1 cusp: one can choose
$$H(z) = e^{\frac{2\pi i v z}{\lambda}}$$
where $v \in \mathbb{Z}$ and $G_\infty = \langle z + \lambda \rangle$. On the other hand, if there is a rank 2 cusp, then one needs to pick H to be an elliptic function, which does make things more complex; however the theory of elliptic functions is fairly well understood, and is explained fairly explicitly in Chapter 7 of Ahfors' textbook [Ahl79]. We will not deal with rank 2 cusps in this work, but there is likely some interesting work to be done in terms of the interplay between Fourier expansions of elliptic choices for H and Fourier expansions of the Poincaré series which results.

In the moduli theory, we will often deal with double cusps with each horocycle lying in a different component of the group. We would like to pass to the natural compactification of the surface obtained by adjoining a single additional point in the obvious way (this cannot be done before the quotient by adding the parabolic fixed point, since this point would lie in every neighbourhood of any point in $\Omega(G)$

[4] Rigorously, we have a holomorphic map $G : \mathbb{B}^2 \to \mathrm{Hom}_{\mathsf{Group}}(G(0), \mathrm{PSL}(2,\mathbb{C}))$ such that $G(t)$ is a faithful discrete representation for each t; we identify each representation $G(t)$ with its image.

and so the resulting quotient is not Hausdorff); we would like our meromorphic forms to extend to this additional point in a unique way. This corresponds to asking for the meromorphic form to be meromorphic at the parabolic point, or in other words for the parabolic point to either be a pole or a removable singularity of the Poincaré series. An important point is that we *only look at limits in horocyclic neighbourhoods of the parabolic point.* Outside of these horocyclic neighbourhoods, the limiting behaviour of the function is badly behaved: since the limit set of a non-elementary Kleinian group is perfect [Mas87, p. II.D.7], every parabolic point is a limit of other elements of the limit set, in particular it is a limit of points of approximation, and hence a limit of poles of the Poincaré series (coming from the factors $|j(g,z)|^{-r}$). We recall that the series converges only if we are bounded away from the limit set, and these horocyclic neighbourhoods of parabolic fixed points are the only places we may approach the limit set without losing meromorphicity.

Lemma 4.1. *Let G be a Kleinian group with a rank 1 parabolic cusp at ∞. The Poincaré series*

$$\varphi(z) = \sum_{g \in G_\infty \backslash G} \exp\left(\frac{2\pi v g z}{\lambda}\right) |j(g,z)|^{-r}$$

has a zero at every parabolic cusp which is not G-equivalent to ∞. The series has:

(1) *if $v > 0$, a zero at ∞ (and hence every parabolic cusp) ;*
(2) *if $v < 0$, a pole at ∞;*
(3) *if $v = 0$, $\lim_{x \to_{\mathbb{R}} \infty} \varphi(ix) = 2$.*

The proof is unchanged from that given for [Leh14, Theorem II.1F].

5. Examples

We will now give some extended examples of the computation of automorphic forms, with some commentary.

5.1. **Genus two Schottky group.** We briefly recall the notion of a Schottky group. Let $C_1, \dots, C_n, C_1', \dots, C_n'$ be $2n$ mutually disjoint round circles in $\hat{\mathbb{C}}$ with a common exterior D, and for each i let $f_i \in \mathbb{M}$ be a Möbius transformation which maps C_i onto C_i' and which maps the exterior of C_i to the interior of C_i. (Since $\mathbb{M}$ is triply transitive on $\hat{\mathbb{C}}$, such maps can be found for every arrangement of disjoint circles.) In this case the set D forms a fundamental domain for $G = \langle f_1, \dots, f_n \rangle$, and the quotient $\Omega(G)/G$ is a compact Riemann surface with no marked points, of genus n. The group G is free on the n generators listed, and every element is loxodromic. A free Kleinian group which is purely loxodromic is called a **Schottky group**, and a Schottky group constructed from a physical set of generators in this way is called a **classical Schottky group**.

Warning. There exist Schottky groups which are not classical. A non-constructive proof of this was first given by Marden, but an explicit example was found by Yamamoto [Yam91].

We now take a simple genus two example, generated by two loxodromic transformations X and Y with isometric circles of radius R around ± 4 and $\pm 4i$. The isometric circle data does not determine X and Y uniquely, but if we require both to be hyperbolic then we do get uniqueness, and the matrices are easily calculated to be

$$X = \begin{bmatrix} 4/R & 16/R - R \\ 1/R & 4/R \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4/R & i(16/R - R) \\ -i/R & 4/R \end{bmatrix}.$$

The group $\langle X, Y \rangle$ is a classical Schottky group if the isometric circles of X and Y don't collide, and one can check that this is equivalent to $R \in (0, 2\sqrt{2})$. The

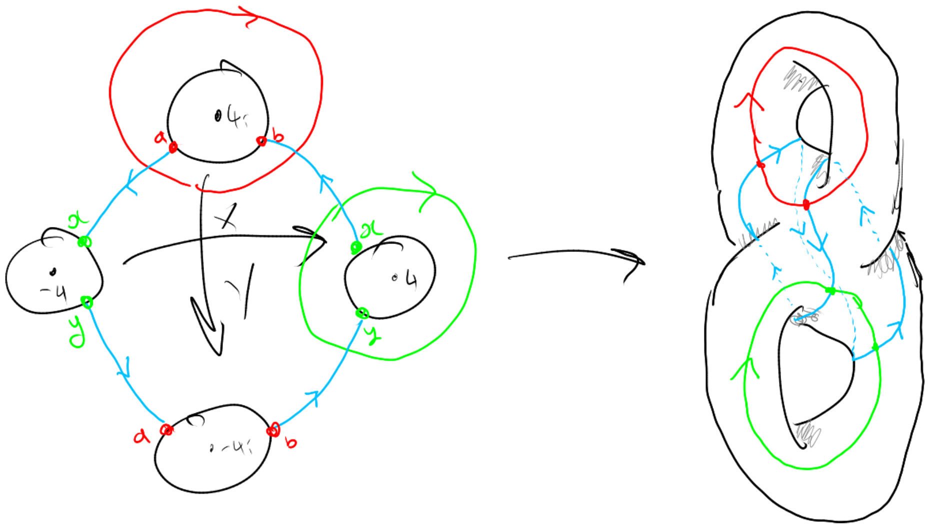


FIGURE 5. The Schottky group $\langle X, Y\rangle$ has quotient surface a compact surface of genus two. The red, green, and blue arcs in $\Omega(G)$ glue up to the corresponding coloured curves on the quotient.

gluing is shown in Figure 5. We also observe that the group is Fuchsian, since $\operatorname{tr}^2 X > 0$ and $\operatorname{tr}^2 Y > 0$; the invariant circle is the circle through the four fixed points, $\pm\sqrt{16-R^2}$ and $\pm\sqrt{R^2-16}$.

Even though the group fails to be a classical Schottky group for $R \geq 2\sqrt{2}$, it might still be discrete. The surfaces that are obtained are shown in Figure 6: when $R = 2\sqrt{2}$ the surface is a pair of punctured tori which have cusps corresponding to a doubly cusped parabolic element—namely, the loxodromic element W which preserves the blue curve in Figure 5 is pinched to become parabolic; the group now lies in the boundary of Schottky space, more precisely it lies in the Maskit embedding [KS93]). For $R \in 2\sqrt{2}$, the element W becomes elliptic, and the group is discrete iff W is finite order, i.e. the angle θ between the isometric circles of X and Y satisfies $\pi/2\theta \in \mathbb{Z}$; in this case the surface is a pair of tori with cone points of the corresponding angle (and the quotient $\mathbb{H}^3/G$ is an orbifold with a cone arc joining the corresponding points at infinity); and finally when $R \to 0$ one component vanishes completely, the other is just a torus with no markings (it is a Margulis torus, corresponding to a rank 2 cusp), and the group becomes elementary.

Remark. The elliptic groups are essentially the **Heckoid groups**, see [Aki+21; LS13; Ril92].

We now compute the Poincaré series for $G = \langle X, Y\rangle$. We take as our meromorphic coefficient function $H(z) = 1/z$, which has a single pole of order 1 at 0; hence the Poincaré series should have poles of order 1 at $g(0)$ for all $g \in G$. That this is indeed the case can be seen in Figure 7, where we give phase plots of the series for various values of R. See Appendix A.1 for the computer programme which was used to generate these figures.

We observe initially that, as expected, all of these functions have a pole of order 1 at 0 and a zero of order 1 at ∞. We expect that, at every isometric circle centre, there is a pole of order 4 coming from the factors of $|j(g,z)|^{-4}$ (we recall, $j(g,z) = 0$ exactly when z is the isometric circle centre of g); and indeed this can

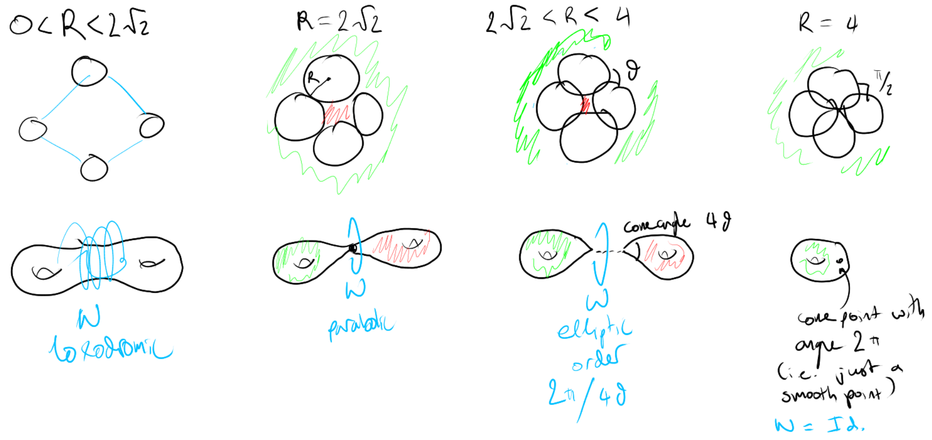


FIGURE 6. Deformations of the group $\langle X, Y\rangle$ as R goes from 0 to 4.

be seen, though not too clearly since the isometric circle centres accumulate very quickly at the points of approximation in the limit set, which are (as limits of poles of a meromorphic function) essential singularities and so the colour oscillates wildly near them.

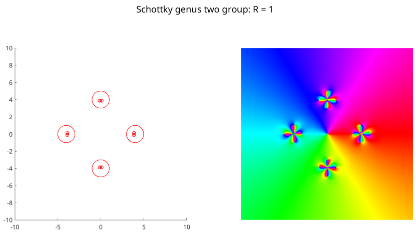
We next ask ourselves about the behaviour at the cusp group, $R = 2\sqrt{2}$. It is not clear that

5.2. **Riley groups.** In the moduli space of Schottky groups there are roughly two different kinds of deformations: 'expansion'-type deformations and 'rotation'-type deformations. By this, we mean that at each point in the space there exists a decomposition of the tangent space into one-dimensional subspaces, half of which correspond to infinitesimal deformations of the group which preserve the combinatorial structure of the convex core but which change its scale, and half of which to deformations preserving the rough scale of the convex core but modifying its structure. Consider the group generated by the two parabolic elements

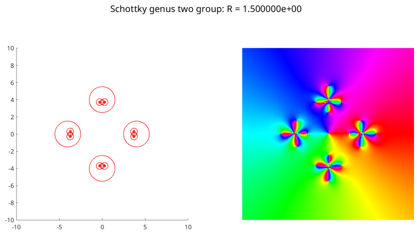
$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0\\ \rho & 1\end{bmatrix};$$

when $|\rho| \gg 0$ (actually it was originally shown by Brenner [Bre55] that $|\rho| > 4$ works, but this is still not very sharp [EMS23, §2.2]) the group is free and discrete and the quotient surface is a four-times punctured sphere; in the large scale, the Keen–Series coordinate system of this moduli space has decomposition roughly corresponding to polar coordinates in $\mathbb{C}$: if $\rho = \lambda e^{\theta}$ then λ corresponds to the scale of the convex core and θ to its combinatorics [EMS22; EMS23; KS94]. We stress that this is only a large-scale approximation to the true picture, as the circumferential part of the coordinate system becomes more 'fractal' as $|\rho|$ becomes small. In any case, in this section we give an example of automorphic forms on groups which are rotating around the circle of radius 4 of the moduli space. Since the groups have a rank 1 cusp at ∞ with parabolic subgroup $\langle z \mapsto z+1\rangle$, our coefficient should have period 1, and so we take $H(z) = \exp(2\pi i z)$ for our coefficient. The weight 2 case is shown in Figure 8 and the rank 3 case in **??**.

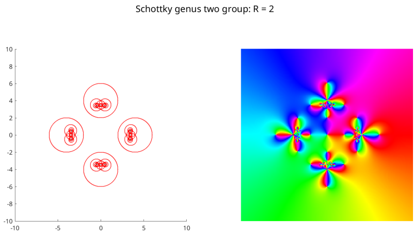
5.3. **Genus two Fuchsian group of the first kind.** We give a Fuchsian group F which glues each half-plane to a genus two surface; this group is obtained by



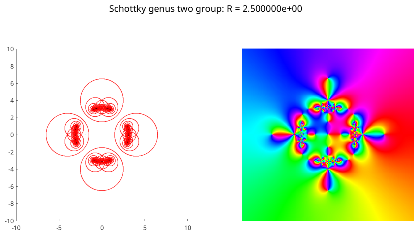
(A) $R = 1$



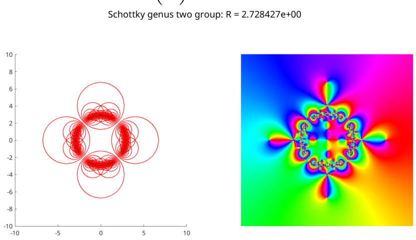
(B) $R = 1.5$



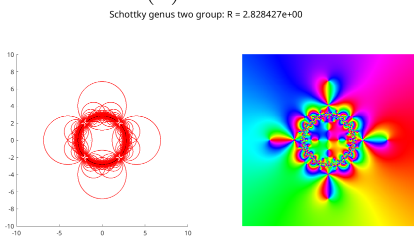
(C) $R = 2$



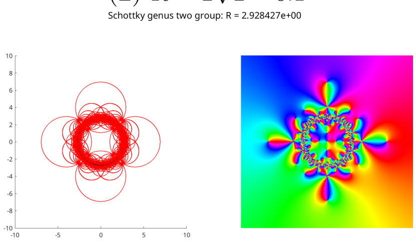
(D) $R = 2.5$



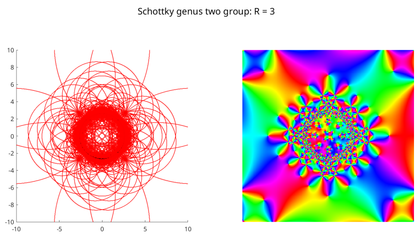
(E) $R = 2\sqrt{2} - 0.1$



(F) $R = 2\sqrt{2}$



(G) $R = 2\sqrt{2} + 0.1$



(H) $R = 3$

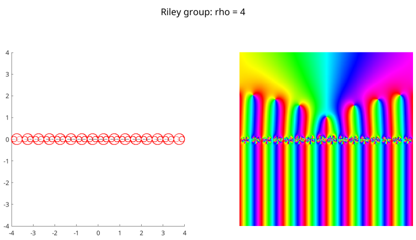
FIGURE 7. The genus two hyperbolic groups for various R. On the left of each figure we draw the isometric circles of group elements (in red) and the limit set (in black); on the right we plot the Poincaré series of weight 2 with meromorphic coefficient $H(z) = 1/z$.

writing down transformations in $\mathbb{M}$ which pair opposite sides of an octagon in $\mathbb{H}^2$ with equal side lengths and all interior angles equal to $\pi/4$.

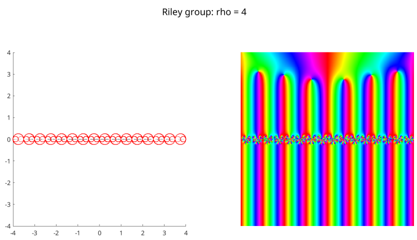
Suppose such an octagon O exists. Then it glues to a genus two compact hyperbolic surface T_2. By the Gauss-Bonnet theorem we have

$$\operatorname{Area} O = \operatorname{Area} T_2 = -\int (-1) dA = -\int K(T_2) dA = -2\pi\chi(T_2) = 4\pi.$$

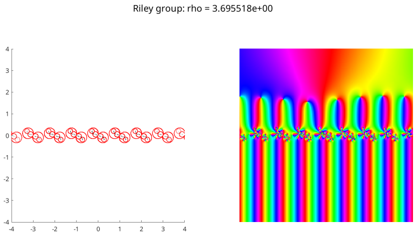
Cut O into right triangles (Figure 9a). Such a triangle Δ has area $4\pi/16 = \pi/4$. By the hyperbolic area formula, the third angle of Δ (the one at the centre) is $\pi - \pi/2 - \pi/8 - \pi/4 = \pi/8$. We will now find an explicit coordinate realisation for this triangle in $\mathbb{H}^2$, and then rotate it around the $\pi/8$ angle to form the hexagon.



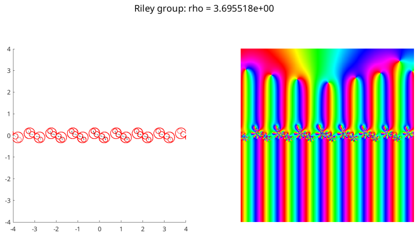
(A) $\rho = 4$



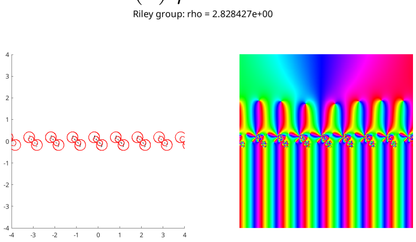
(B) $\rho = 4$



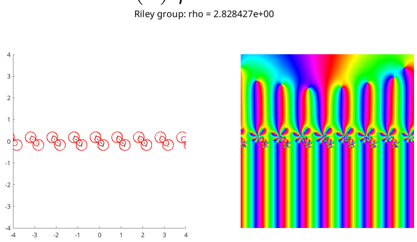
(C) $\rho = 4e^{\pi i/8}$



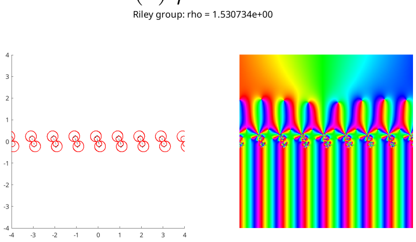
(D) $\rho = 4e^{\pi i/8}$



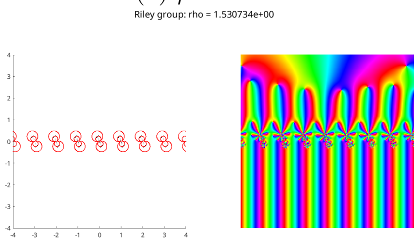
(E) $\rho = 4e^{\pi i/4}$



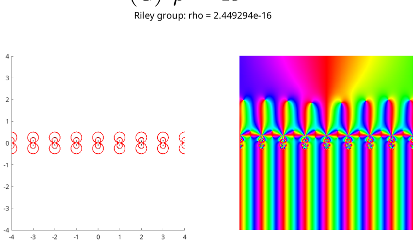
(F) $\rho = 4e^{\pi i/4}$



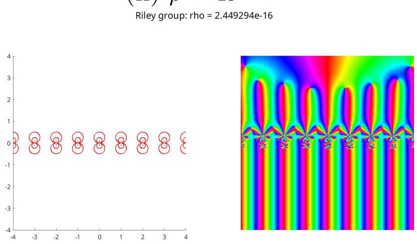
(G) $\rho = 4e^{3\pi i/8}$



(H) $\rho = 4e^{3\pi i/16}$

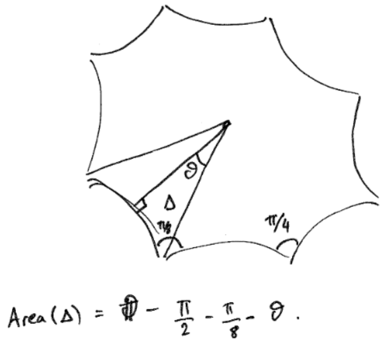


(I) $\rho = 4i$

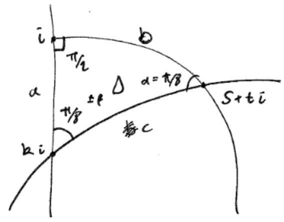


(J) $\rho = 4i$

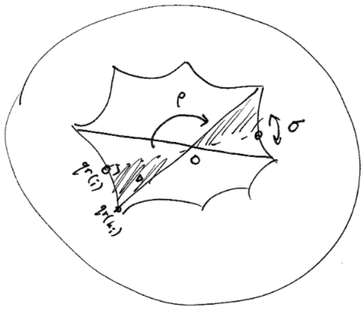
FIGURE 8. Various Riley groups with $|\rho| = 4$. On the left of each figure we draw the isometric circles of group elements (in red) and the limit set (in black); on the right we plot the Poincaré series of weight 2 (left) and weight 3 (right) with meromorphic coefficient $H(z) = e^{2\pi i z}$.



(A) A (purported) $\pi/4$-regular hexagon in $\mathbb{H}^2$.



(B) The triangle Δ.



(C) The images of Δ under ρ and σ.

FIGURE 9. The trigonometry needed to produce the octagon O explicitly.

Without loss of generality, we can assume that the three vertices of Δ lie at i, ki, and $s+ti$ where $s^2+t^2=1$, and that the right angle is at i (Figure 9b). By the trigonometry of hyperbolic right-angled triangles [Bea83, §7.11], we can compute that

$$k = \cot\frac{\pi}{8} + \sqrt{\cot^2\frac{\pi}{8} - 1}, \quad s = \sqrt{1-\tan^2\frac{\pi}{8}}, \text{ and } t = \tan\frac{\pi}{8}.$$

Now for simplicity we move from the upper half-plane model $\mathbb{H}^2$ to the disc model $\mathbb{B}^2$, conjugating $s+ti$ to 0. There is a standard map q which sends $\mathbb{H}^2 \mapsto \mathbb{B}^2$ with $i \mapsto 0$, given by

$$q(z) = \frac{2}{z+i} + i;$$

we therefore precompose q with a map sending $s+ti \mapsto i$, for instance

$$r(z) = \frac{z-s}{t}.$$

Hence in the ball model $\mathbb{B}^2$ our desired triangle is the image of Δ under $q \circ r$. Let $\rho : \Delta \to \Delta$ be the elliptic of order 8 which fixes 0, and let $\sigma : \Delta \to \Delta$ be the reflection across the geodesic joining 0 and $(qr)(i)$ in the disc (Figure 9c). Then we wish to find an isometry f of $\mathbb{B}^2$ which sends the geodesic segment joining $(qr)(ki)$ to $(\sigma qr)(ki)$ to the geodesic segment joining $(\rho^4 qr)(ki)$ to $(\rho^4\sigma qr)(ki)$ (and the other side-pairing transformations which we want will be $\rho^n f \rho^{-n}$ for $n \in \{0,1,2,3\}$). Of course, f should be a hyperbolic element of $\mathbb{M}$ which preserves Δ (so although ρ^4 does send one arc to the other it glues the endpoints in opposite order to that

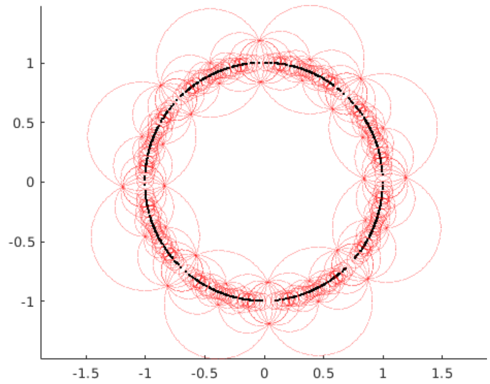


FIGURE 10. Isometric circles of the group F.

which we want). We can do this by writing down a Möbius transformation with real trace which fixes the two endpoints on the disc of the diameter through $(qr)(i)$ (i.e. $\pm(qr)(i)/|(qr)(i)|$), and then imposing the further condition that it sends $(qr)(i) \mapsto (\rho^4 qr)(i) = -(qr)(i)$. If the transformation is represented by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the desired conditions are

$$a\frac{\zeta}{|\zeta|} + b = c\frac{\zeta^2}{|\zeta|^2} + d\frac{\zeta}{|\zeta|}$$
$$-a\frac{\zeta}{|\zeta|} + b = c\frac{\zeta^2}{|\zeta|^2} - d\frac{\zeta}{|\zeta|}$$
$$a\zeta + b = -c\zeta^2 - d\zeta$$

where

$$\zeta = (qr)(i) = \frac{1-t+is}{(1+t)i-s}.$$

Let $\omega = \sqrt{2(-3+\sqrt{2}-(2i)\sqrt{2(-1+\sqrt{2})})}$. Then one solution for A is the matrix $\begin{bmatrix} 1+\sqrt{2} & \omega \\ \overline{\omega} & 1+\sqrt{2} \end{bmatrix}$. (We could take $\pm A$ or $\pm A^{-1}$ as well.) Now ρ is represented by the matrix

$$R = \begin{bmatrix} \exp(\pi i/8) & 0 \\ 0 & \exp(-\pi i/8) \end{bmatrix}$$

so the desired Fuchsian group uniformising T_2 is

$$F = \left\langle \begin{bmatrix} 1+\sqrt{2} & \omega \\ \overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & (1+i)\omega \\ (1-i)\overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1+\sqrt{2} & i\omega \\ -i\overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & (-1+i)\omega \\ (-1-i)\overline{\omega} & 1+\sqrt{2} \end{bmatrix} \right\rangle.$$

For the convenience of the reader (and as a verification that our computations are correct) we plot the isometric circles of F in Figure 10—one can see that the octogons in fact form the Ford domain of the group. We also remark that ∞ is not a limit point (essentially by construction) and so F_∞ is trivial. Moving to the automorphic functions themselves, these can be computed in the same way as the previous sections. The only difficulty is that now the groups are no longer free, and this is easily overcome either by a more sophisticated group walking algorithm

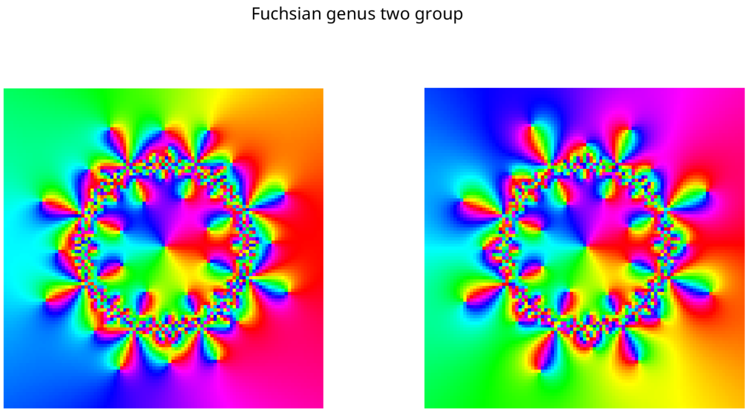


FIGURE 11. Automorphic functions on the Fuchsian group F, with meromorphic coefficients $1/z + z$ (left) and $1/z$ (right).

which follows the Cayley graph, or by simply computing all the words as if the group were free and then removing duplicates. We plot in Figure 11 two Poincaré series of degree 2 on the group F, with respective meromorphic coefficients $z \mapsto 1/z + z$ and $z \mapsto 1/z$. In Appendix A.2 we give the computer code to generate this picture.

As before, we see expected behaviour at 0 and ∞, and at isometric circle centres. Notice that we also seem to have zeros at the intersection points of the isometric circles(????)

5.4. **Various higher genus Schottky groups.** The two groups which we have seen so far have been particularly symmetric. In particular, the group of Section 5.1 admits both an anti-conformal involution (inversion in the circle containing the limit set) and a conformal involution (compose the anti-conformal involution with reflection in $\mathbb{R}$). In the second part of this paper, we will see that existence of these structures corresponds to definition over $\mathbb{R}$ and hyperellipticity, respectively. In this section, we will compute automorphic forms on some less symmetric Schottky groups:-

(1) Example 5.1: a one-parameter family of genus g Schottky groups (for arbitrary $g > 2$) which admits an anti-conformal involution but not a conformal involution;
(2) Example 5.2: a one-parameter family of genus g Schottky groups (for arbitrary $g > 2$) which admits a conformal involution but not an anti-conformal involution;
(3) A one-parameter family of genus g Schottky groups (for arbitrary $g > 2$) which admits neither anti-conformal nor conformal involutions.

Example 5.1. We will give a one-parameter family of Kleinian groups in Schottky$^\circ(n)$ (for all $n > 2$) admitting anti-conformal but not conformal involutions and whose algebraic limit lies on the boundary of Schottky(n). The group will be generated by n elements, pairing the isometric circles shown in Figure 12. Define the three

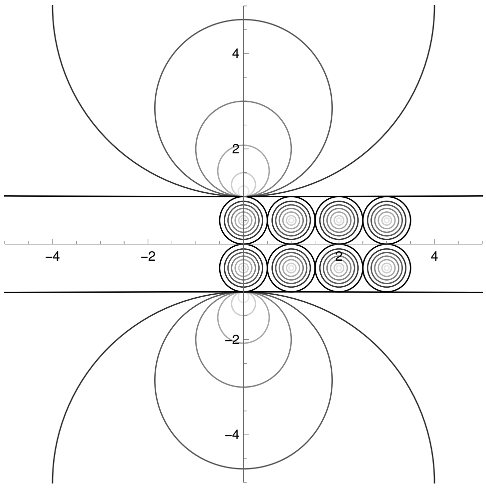


FIGURE 12. The isometric circles of the one-parameter family of groups described in Example 5.1 for $n = 5$; as $t \to 1$, the radius of the circles increases until in the limit they are tangent.

matrices

$$Y = \begin{bmatrix} -t^{-1} & -i(t+1)t^{-1} \\ i(1-t)t^{-1} & -t^{-1} \end{bmatrix}$$
$$X = \begin{bmatrix} -t^{-1} & i(t-1)(2t)^{-1} \\ 2t^{-1} & -t^{-1} \end{bmatrix}$$
$$Z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and for each $k \in \mathbb{Z}$ set $X_k := Z^k X Z^{-k}$. The group G_t is then defined by

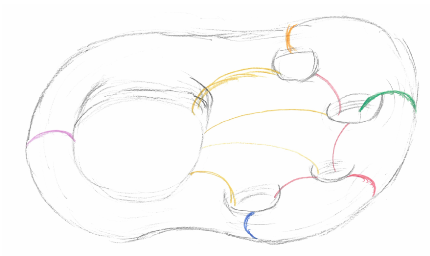
$$G_t := \langle Y, X_0, \dots, X_{n-2} \rangle.$$

The generators of G_t have been carefully chosen to not only have the correct isometric circles, but also to be hyperbolic for all t. This has the effect that the axes of the generators are all Euclidean line segments (rather than the general situation of logarithmic spirals). The anti-conformal involution is reflection in $\mathbb{R}$.

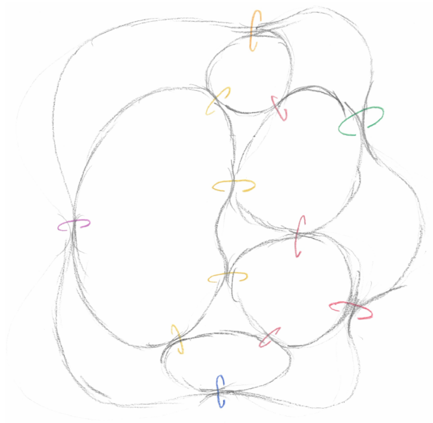
When $t = 1$, the generators all become parabolic and the quotient surface degenerates to a union of thrice-punctured spheres. This follows from a careful application of the full statement of the Poincaré polyhedron theorem, but the point is that because the degeneration is highly symmetric[5] the meridian curves of the surface are pinched to cusps. The topological deformation is depicted in Figure 13.

Example 5.2. A similar construction to Example 5.1 is used to give a one-parameter family of Kleinian groups in $\text{Schottky}^\circ(n)$ (for all $n > 2$) admitting conformal but not anti-conformal involutions and whose algebraic limit lies on the boundary of $\text{Schottky}(n)$. The group will be generated by n elements, pairing the isometric

[5]More precisely, we reach the boundary upon deforming such that the points of tangency of the circles are mapped onto each other in cycles by the circle-pairing transformations [Mas87, p. IV.I.6]; if the circles become tangent in a non-symmetric way—i.e. the axes of the transformations which pair the circles are not parallel—then it is possible to deform the fundamental domain slightly such as to remove the tangency.



(A) A representative quotient surface of the Schottky groups.



(B) A representative quotient surface of the limit group.

FIGURE 13. The quotient surfaces obtained in from the one-parameter family of groups defined in Example 5.1.

circles shown in **??**. Define the three matrices

$$Y = \begin{bmatrix} -t^{-1} & -i(t+1)t^{-1} \\ i(1-t)t^{-1} & -t^{-1} \end{bmatrix}$$
$$X = \begin{bmatrix} -t^{-1} & i(t-1)(2t)^{-1} \\ 2t^{-1} & -t^{-1} \end{bmatrix}$$
$$Z_t^k = \begin{bmatrix} 1 & k+(1-t)ki \\ 0 & 1 \end{bmatrix} W_t = \begin{bmatrix} 1 & (1/2)(1-t)(n-2)i \\ 0 & 1 \end{bmatrix}$$

(where X and Y are the same matrices as the matrices X and Y of Example 5.1). The group G_t is then defined by

$$G_t := \{ \langle W_t^{-1} Y W_t^{-1}, Z_t^{-(n-2)/2} X Z_t^{(n-2)/2}, Z_t^{-(n-2)/2+1} X Z_t^{(n-2)/2-1}, \dots, Z_t^{(n-2)/2} X Z_t^{-(n-2)/2} \rangle \quad n odd$$

Again, the generators are hyperbolic for all t. The anti-conformal involution is rotation by π about 0.

When $t = 1$, the generators all become parabolic and the quotient surface degenerates to a union of thrice-punctured spheres (in fact, it degenerates to the same group as the $t = 1$ group of Example 5.1).

5.5. **Rank two cusp groups and tori.** We shall, in this section, examine three different 'toroidal' groups:

(1) An elementary rank two cusp group.
(2) A non-elementary group with a rank two cusp.
(3) A punctured torus (i.e. a torus with a distinguished point)—in algebraic geometry, tori with distinguished points arise naturally as elliptic curves.

Example 5.3. An elementary rank two cusp group has exactly one limit point, and we must choose it away from ∞ in order for the Poincaré series to be nontrivial. We take the group generated by $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ i & 1 \end{smallmatrix}\right]$, which has fundamental domain the exterior of the four circles of diameter 1 centred at ± 1 and $\pm i$; the automorphic form with coefficient

Part 2. Plane curves

6. Terminology from algebraic geometry

We first give a brief introduction to the required concepts from algebraic geometry, as found for instance in Chapter I of Hartshorne [Har77]. A **(non-singular) algebraic curve** is a non-singular variety over $\mathbb{C}$ of dimension 1. A standard result ensures that every algebraic curve k is embeddable in $\mathbb{P}^3\mathbb{C}$, so is locally the intersection of hypersurfaces cut out by homogeneous polynomials in three variables. Under this embedding, k is identified naturally with an (abstract) compact smooth surface $S(k)$ called the **Riemann surface** of k, and this surface has a natural complex structure induced by the complex structure on $\mathbb{P}^3\mathbb{C}$. More precisely, there is a holomorphic tangent bundle $T : TS(k) \to S(k)$ of one-dimensional complex vector spaces, and the maps on each fibre corresponding to multiplication by i also vary holomorphically. It is not immediately clear that this surface is independent of the particular embedding chosen, but in a moment we will explain why this is the case.

Assigned to the curve k are several natural algebraic invariants. The one most important to us is the **function field** $\mathbb{C}(k)$, which consists of pairs (U, ϕ) where U is a dense Zariski open subset of k and ϕ is a map $U \to \mathbb{C}$ of the form

$$\phi(x) = \frac{f(x)}{g(x)}$$

where f and g are polynomial maps such that g is non-zero on U; multiplication and addition are defined pointwise on the functions ϕ, with suitable restrictions of domain. Usually we will neglect to mention the domains of definition and we will refer to ϕ as a **rational function** on k, even though it is likely not defined as a set-function on the entire curve. A function $f : k \to \ell$ (where k and ℓ are both algebraic curves) is called a **rational map** if the composition $\phi \circ f$ is a rational function on k whenever ϕ is a rational function on ℓ; we will say that k and ℓ are **birational** if f has dense image $U \subseteq \ell$ and if it has a well-defined inverse $f^{-1} : \ell \to k$ which is rational.

Theorem 6.1. *If k and ℓ are algebraic curves embedded in $\mathbb{P}^3\mathbb{C}$ with corresponding compact Riemann surfaces $S(k)$ and $S(\ell)$, then a rational map $k \to \ell$ induces a conformal map $S(k) \to S(\ell)$; this assignment is functorial and induces an equivalence of categories.* □

As a corollary of the theorem, we see that birationality of two algebraic curves is equivalent to biholomorphism of the corresponding Riemann surfaces. It is another standard result in classical algebraic geometry that two algebraic curves are birational iff they have isomorphic function fields (and that this is another equivalence of categories). Hence we have a triumvirate of useful categories,

$$\left\{\begin{matrix}\text{compact Riemann surfaces}\\ \text{with holomorphic maps}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{algebraic curves}\\ \text{with rational maps}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{function fields of curves}\\ \text{with field homomorphisms}\end{matrix}\right\};$$

it is a standard fact from complex analysis that the field of meromorphic functions on a compact Riemann surface consists only of rational functions, so one can replace 'function fields of curves' with 'fields of meromorphic functions on Riemann surfaces' here too (though this does not set up an isomorphism of sheaves, only of global sections, so one must be a little careful).

In later sections, we will be explicitly moving from the category of surfaces to the category of curves (more precisely, we will originate in the category of hyperbolic 3-manifolds which carries with it more data—but not much more data, by the ending lamination theorem—than some associated surfaces) via the construction of function fields. We will construct relations in these function fields which correspond to plane curves which are in the same birationality class, and so our next step is to say something about plane curves.

We first observe that the image of a smooth curve $k \subseteq \mathbb{P}^3\mathbb{C}$ under a birational map $f : k \to \mathbb{P}^2\mathbb{C}$ is not necessarily smooth. In the generic situation, the image $f(k)$ has finitely many nodal singularities: points $z \in f(k)$ such that there is an analytic neighbourhood U of z in $\mathbb{P}^2\mathbb{C}$ with the property that $U \cap f(k)$ is isomorphic (as a quasi-projective variety) to a neighbourhood of 0 in the variety $\mathbf{Z}(XY)$. Recall that the **degree** of $f(k)$ is the number of times that the generic line in $\mathbb{P}^2\mathbb{C}$ intersects $f(k)$ (counting multiplicity); if this number is known together with the genus of the curve (in algebraic geometry there are several different definitions of the word 'genus', but in the non-singular case they coincide with each other and with the usual topological genus of the corresponding Riemann surface), then the number of expected singularities can be counted:

Proposition 6.2 (Degree-genus formula)**.** *Let $k \subseteq \mathbb{P}^2\mathbb{C}$ be an irreducible plane curve of degree d and genus g. Then $g \leq \frac{1}{2}(d-1)(d-2)$ with equality iff k is non-singular; if k is singular and sufficiently generic then it has only nodal singularities (that is, singularities locally isomorphic to a pair of crossing lines) and the number of singularities is $n = \frac{1}{2}(d-1)(d-2) - g$.* □

We will have more to say on the subject of nodes when we discuss moduli problems; nodal curves will appear naturally on the boundaries of the algebraic moduli space. For the most part, though, the main use of this formula will be that it allows us to intelligently guess the correct degree of plane curves given only the genus: for a fixed genus g, we should look for an equation of degree

$$d = \operatorname{floor}\left(\frac{3}{2} + \sqrt{\frac{1}{4} + 2g}\right). \tag{6.3}$$

6.1. **Presentation of the function field.** Before giving the details, we sketch of the remainder of the construction. If R is a compact Riemann surface, then the field $\mathcal{M}(R)$ of meromorphic functions on R is fairly small—it is a finite extension of $\mathbb{C}$. Next, if $F/\mathbb{C}$ is finite, then F is the field of fractions of a polynomial quotient ring. In other words, if R is a compact Riemann surface then there exist finitely many $f_1, \dots, f_r \in \mathcal{M}(R)$ such that

$$\mathcal{M}(R) \simeq \operatorname{Frac} \frac{\mathbb{C}[x_1, \dots, x_r]}{\mathfrak{a}}$$

for some ideal $\mathfrak{a} \trianglelefteq \mathbb{C}[x_1, \dots, x_r]$, where the isomorphism is given by mapping $f_i \mapsto x_i$ for all i. In fact, the ideal $\mathfrak{a}$ is cyclic.

We now give a fairly complete proof of these facts—although they can be easily found in textbooks, for example [Leh64], they are constructive enough to worth explaining.

Proposition 6.4. *Let R be a compact Riemann surface, and suppose that $f, g \in \mathcal{M}(R)$ are algebraically independent meromorphic functions. Then*

(1) there is a polynomial $\phi \in \mathbb{C}[x, y]$ such that $\phi(f, g)$ is identically zero, and
(2) if $h \in \mathcal{M}(R)$ is an arbitrary meromorphic function, then there is a rational function $\phi' \in \mathbb{C}(x, y)$ such that $h(z) = \phi'(f(z), g(z))$ for all $z \in R$.

Proof. Part (1) depends on the fact that a nonzero meromorphic function on a compact Riemann surface has finitely many poles, and that it has an equal number of zeros as it does poles (counted with multiplicity). Suppose that f and g have m and n poles respectively counted with multiplicity, and consider polynomials of the form

$$\sum_{i=0}^{s}\sum_{j=0}^{t}\beta_{ij}f^i g^j; \tag{6.5}$$

such a polynomial has at most $sm+tn$ poles. Choose s and t such that $(s+1)(t+1)-1 = st+s+t > sm+tn$ (i.e. the relation has strictly more coefficients than possible poles), and pick $N = st+s+t$ distinct points $\zeta_1, \dots, \zeta_N \in R$ which are not poles of f or g and such that the points $(f(\zeta_i), g(\zeta_i)) \in \mathbb{C}^2$ are distinct for all i. We therefore have N distinct equations

$$\begin{aligned} 0 &= \sum_{i=0}^{s}\sum_{j=0}^{t}\beta_{ij}f(\zeta_1)^i g(\zeta_1)^j \\ &\vdots \\ 0 &= \sum_{i=0}^{s}\sum_{j=0}^{t}\beta_{ij}f(\zeta_N)^i g(\zeta_N)^j \end{aligned}$$

in st variables, so there exists a nonzero solution $\beta = (\beta_{ij})_{(i,j)=(0,0)}^{(s,t)}$. The summation in Equation (6.5) then defines a meromorphic function on R with at least N zeros; but by definition of f and g it has at most $sm+tn < N$; hence it cannot be a nonzero function, and the polynomial

$$\phi(x,y) = \sum_{i=0}^{s}\sum_{j=0}^{t}\beta_{ij}x^i y^j$$

has the property $\phi(f,g) \equiv 0$.

Now, for part (2), suppose that $h \in \mathcal{M}(R)$; then there exist polynomials ϕ and ψ such that $\phi(f,h) = \psi(g,h) = 0$. The intuitive idea now is that ϕ and ψ cut out algebraic hypersurfaces in $\mathbb{C}^3$, and so the intersection is a curve $\hat{k}$; let k be the projection of $\hat{k}$ onto the $h=0$ coordinate plane, so k is a plane curve birational to $\hat{k}$; thus h is a rational function of f and g given by inverting the projection. $\square$

Remark. The resulting coefficients β_{ij} are independent of the choice of points ζ_i: a plane curve has a unique irreducible equation describing it, and moving the points ζ_i does not change the locus $\{(f(t), g(t)) : t \in S(G)\}$, so the equation is not changed.

Observe that the proof of existence of the relation Equation (6.5) is constructive, as long as the meromorphic functions f and g can be computed. Even better, the computation depends only on solving a linear system.[6] This reduces the problem of writing a plane equation for a Riemann surface given by a Kleinian group to the following process.

Algorithm 6.6. Let G be a geometrically finite Kleinian group, pointed at infinity with compact non-empty quotient surface $\Omega(G)/G$, let D be a complete set of coset representatives for $G_\infty\backslash G$,[7], and let $S(G)$ be a fundamental domain for G such that no isometric circles of an element of D has centre in $S(G)$; such a domain exists, for instance the Ford fundamental domain of the group [Mas87, §II.H]. The choice of this fundamental domain implies that the poles of Equation (3.3) which arise from the factors of $g'(z)$ for $g \in G$ lie outside $S(G)$ and so the poles of the automorphic function defined by the series arise exactly from the poles of the functions H.

[6] It might be interesting to study how well-conditioned this system is, and whether the various choices made in the construction can be used to optimise this.

[7] Often, G will be free or will have a fairly easy to solve word problem, so in practice this is not an issue.

Let g be the genus of $\Omega(G)/G$, and let d be the degree bound computed in Equation (6.3). We now make the following *choices*:

- *Bidegrees of the plane curve*: Choose s and t such that the degree of the polynomial Equation (6.5), which is $(s+1)(t+1)$, is equal to d.
- *Number of asymptotes of the plane curve*: Choose n and m such that $st+s+t > sm+tn$.
- *Position of asymptotes of the plane curve*: Choose $2n+2m$ distinct (hence G-inequivalent) points of $S(G)$, $\zeta_1,\dots,\zeta_n$, $\zeta'_1,\dots,\zeta'_n$, $\xi_1,\dots,\xi_m$, $\xi'_1,\dots,\xi'_m$.

We have labelled each choice with the geometric meaning it has with respect to the plane curve which arises, but we will not discuss this until the next section.

One must also choose $N = st+s+t$ points $\upsilon_1,\dots,\upsilon_N$ which are G-inequivalent to each other and to the chosen poles, but having chosen the other data the resulting plane curve is independent of the exact choice of these points.

Now form the two quotients

$$f(z) = \frac{\sum_{g\in D}\frac{1}{\prod_{i=1}^{n}(1-g\zeta_i)}(g'(z))^2}{\sum_{g\in D}\frac{1}{\prod_{i=1}^{n}(1-g\zeta'_i)}(g'(z))^2} \text{ and } g(z) = \frac{\sum_{g\in D}\frac{1}{\prod_{i=1}^{m}(1-g\xi_i)}(g'(z))^2}{\sum_{g\in D}\frac{1}{\prod_{i=1}^{m}(1-g\xi'_i)}(g'(z))^2}.$$

and compute a non-zero vector $(\beta_{1,1},\beta 1,2,\dots,\beta_{s,t})$ in the kernel of the matrix

$$\begin{bmatrix} f(\upsilon_1)^1 g(\upsilon_1)^1 & f(\upsilon_1)^1 g(\upsilon_1)^2 & \cdots & f(\upsilon_1)^s g(\upsilon_1)^t \\ f(\upsilon_2)^1 g(\upsilon_2)^1 & f(\upsilon_2)^1 g(\upsilon_2)^2 & \cdots & f(\upsilon_2)^s g(\upsilon_2)^t \\ & & \vdots & \\ f(\upsilon_N)^1 g(\upsilon_N)^1 & f(\upsilon_N)^1 g(\upsilon_N)^2 & \cdots & f(\upsilon_N)^s g(\upsilon_N)^t \end{bmatrix};$$

the plane curve is then the vanishing set of the polynomial $\phi(x,y) = \sum_{i,j=0}^{s,t}\beta_{ij}x^i y^j$.

Appendix A. Plotting automorphic functions

A.1. **Genus two Schottky group.** The following script plots the Poincaré series of the genus two Schottky groups of Section 5.1 and then uses the MATLAB package PhasePlot [Weg] to graph them.

```
addpath('/home/alex/MATLAB/PP_Ver_2_3');

Rs = [1,1.5,2,2.5,2*sqrt(2)-0.1,2*sqrt(2),2*sqrt(2)+0.1,3];

for R = Rs
    X = [4/R, 16/R - R; 1/R, 4/R];
    Y = [4/R, 1i*(16/R-R); -1i/R, 4/R];
    x = inv(X);
    y = inv(Y);

    limseed = (-(4*R-4)+sqrt((4*R-4)^2-4*R*(R^2-16)))/(2*R);
    numgens = 4;

    % Enumeration of group elements
    gens = zeros(2,2,numgens);
    gens(:,:,1) = X;
    gens(:,:,2) = Y;
    gens(:,:,3) = x;
    gens(:,:,4) = y;

    invhash = [3,4,1,2];
```

```
uplim = 10000000;
group = zeros(2,2,uplim);
tag = zeros(1,uplim);
for i = 1:numgens
    group(:,:,i) = gens(:,:,i);
    tag(i) = i;
end

N = 8;

num = zeros(1,N);
num(1) = 1;
num(2) = numgens+1;
for n = 2:(N-1)
    fprintf('making gp:\t%d/%d --- %.3f\n',n,N-1,n/(N-1));
    inew = num(n);
    for iold = num(n-1):(num(n)-1)
        for j = 1:numgens
            if j == invhash(tag(iold))
                continue;
            end
            group(:,:,inew) = group(:,:,iold) * gens(:,:,j);
            tag(inew) = j;
            inew = inew+1;
        end
        if inew > uplim
            fprintf('**hit upper limit\n');
            break
        end
    end
    num(n+1)=inew;
end

group(:,:,all(group==0, [1 2])) = []; % trim zeros
group(:,:,end+1) = [1 0; 0 1]; % Add the identity element

grouplength = size(group,3);
fprintf('Made %d group elements  ', grouplength);


% Set up display
figure
set(gcf,'Position',[0 0 1120 626])
bounds = [-10-10i, 10+10i];
subplot(121);
axis([real(bounds(1)) real(bounds(2)) imag(bounds(1))
↪ imag(bounds(2))])
daspect([1 1 1])
subplot(122);
axis([real(bounds(1)) real(bounds(2)) imag(bounds(1))
↪ imag(bounds(2))])
daspect([1 1 1])
```

```
    xres = 200;
    yres = 200;
    zetas = zdomain(bounds(1),bounds(2),xres,yres);

    k = 2;
    mero = @(z) 1/z;
    values = zeros(xres,yres);
    count = 0;
    tot = xres*yres;
    for zi = 1:xres
        for zj = 1:yres
            count = count + 1;
            fprintf('%.3f\t\tsumming series:\t%d/%d ---
↪   %.3f\n',R,count,tot,count/tot);
            zeta = zetas(zi,zj);
            for i = 1:grouplength
                matrix = group(:,:,i);
                a = matrix(1,1); b = matrix(1,2); c = matrix(2,1); d
↪   = matrix(2,2);
                values(zi,zj) = values(zi,zj) + mero((a*zeta +
↪   b)/(c*zeta + d))*((c*zeta+d)^(-2*k));
            end
        end
    end

    sgtitle(sprintf('Schottky genus two group: R = %d',R));
    subplot(121);
    cla
    hold on
    for i = 1:grouplength
        matrix = group(:,:,i);
        a = matrix(1,1); b = matrix(1,2); c = matrix(2,1); d =
↪   matrix(2,2);
        lim = (a*limseed+b)/(c*limseed+d);
        if 1/abs(c) > 0.1
            circle(real(-d/c),imag(-d/c),1/abs(c));
        end
        plot(real(lim),imag(lim),'.k','MarkerSize',.5);
    end
    hold off
    subplot(122)
    cla
    hold on
    PhasePlot(zetas,values,'p');
    hold off
    drawnow

    saveas(gcf,sprintf('schottky2Exp.%.3f.png',R));
    close(gcf);
end
```

```
function h = circle(x,y,r)
    ct = 2*pi/(ceil(2*pi*r*100));
    th = 0:ct:2*pi;
    xunit = r * cos(th) + x;
    yunit = r * sin(th) + y;
    h =
 ↪  scatter(xunit,yunit,1,'MarkerFaceColor','r','MarkerEdgeColor','none');
    h.MarkerFaceAlpha = .3;
end
```

A.2. **Genus two Schottky group.** A modification of the script of the previous section produces automorphic functions for the genus two Fuchsian group of the second kind constructed in Section 5.3.

```
addpath('/home/alex/MATLAB/PP_Ver_2_3');

omega = sqrt(2*(-3 + sqrt(2) - 2i*sqrt(2*(-1 + sqrt(2)))));
X = [1+sqrt(2),omega;conj(omega),1+sqrt(2)];
Y = [1+sqrt(2),(1+1i)*omega/sqrt(2); (1-1i)*conj(omega)/sqrt(2),
 ↪  1+sqrt(2)];
Z = [1+sqrt(2),1i*omega; -1i*conj(omega), 1+sqrt(2)];
W = [1+sqrt(2),(-1+1i)*omega/sqrt(2); (-1-1i)*conj(omega)/sqrt(2),
 ↪  1+sqrt(2)];
x = inv(X);
y = inv(Y);
z = inv(Z);
w = inv(W);
limseed = 1;

numgens = 8;

% Enumeration of group elements

gens = zeros(2,2,numgens);
gens(:,:,1) = X;
gens(:,:,2) = Y;
gens(:,:,3) = Z;
gens(:,:,4) = W;
gens(:,:,5) = x;
gens(:,:,6) = y;
gens(:,:,7) = z;
gens(:,:,8) = w;

invhash = [5,6,7,8,1,2,3,4];

uplim = 10000000;
group = zeros(2,2,uplim);
tag = zeros(1,uplim);
for i = 1:numgens
    group(:,:,i) = gens(:,:,i);
    tag(i) = i;
end
```

```
N = 8;

num = zeros(1,N);
num(1) = 1;
num(2) = numgens+1;
for n = 2:(N-1)
    fprintf('making gp:\t%d/%d --- %.3f\n',n,N-1,n/(N-1));
    inew = num(n);
    for iold = num(n-1):(num(n)-1)
        for j = 1:numgens
            if j == invhash(tag(iold))
                continue;
            end
            group(:,:,inew) = group(:,:,iold) * gens(:,:,j);
            tag(inew) = j;
            inew = inew+1;
        end
        if inew > uplim
            fprintf('**hit upper limit\n');
            break
        end
    end
    num(n+1)=inew;
end

group(:,:,all(group==0, [1 2])) = []; % trim zeros
group(:,:,end+1) = [1 0; 0 1]; % Add the identity element

grouplength = size(group,3);
fprintf('Made %d group elements  ', grouplength);

% Now, to avoid traversing a Cayley graph in MATLAB, we do the most
 ↪  awful thing and just
% delete duplicate matrices from the list.
rsgp = reshape(group, [4, grouplength]);
% For some reason uniquetol does not work on complex values. If it
 ↪  did:
% rsgp_unique = uniquetol(rsgp.', 'ByRows',true).';
% group = reshape(rsgp_unique, [2, 2, grouplength]);
% But we need to split into real and imaginary parts and check
 ↪  uniqueness on each separately.
[~,IR,~] = uniquetol(real(rsgp.'),'ByRows',true);
[~,IC,~] = uniquetol(imag(rsgp.'),'ByRows',true);
indices = union(IR,IC);
group = group(:,:,indices);
% uniquetol workaround ends

grouplength = size(group,3);
fprintf('of which %d survive after pruning to tolerance\n',
 ↪  grouplength);
```

```
% Set up display
figure
set(gcf,'Position',[0 0 1120 626])
bounds = [-2-2i, 2+2i];
subplot(121);
axis([real(bounds(1)) real(bounds(2)) imag(bounds(1))
 ↪ imag(bounds(2))])
daspect([1 1 1])
subplot(122);
axis([real(bounds(1)) real(bounds(2)) imag(bounds(1))
 ↪ imag(bounds(2))])
daspect([1 1 1])

xres = 100;
yres = 100;
zetas = zdomain(bounds(1),bounds(2),xres,yres);

k = 2;
mero = @(z) 1/z;
mero2 = @(z) 1/z+z;
values = zeros(xres,yres);
values2 = zeros(xres,yres);
count = 0;
tot = xres*yres;
for zi = 1:xres
    for zj = 1:yres
        count = count + 1;
        fprintf('summing series:\t%d/%d ---
 ↪ %.3f\n',count,tot,count/tot);
        zeta = zetas(zi,zj);
        for i = 1:grouplength
            matrix = group(:,:,i);
            a = matrix(1,1); b = matrix(1,2); c = matrix(2,1); d =
 ↪ matrix(2,2);
            values(zi,zj) = values(zi,zj) + mero((a*zeta +
 ↪ b)/(c*zeta + d))*((c*zeta+d)^(-2*k));
            values2(zi,zj) = values2(zi,zj) + mero2((a*zeta +
 ↪ b)/(c*zeta + d))*((c*zeta+d)^(-2*k));
        end
    end
end

sgtitle(sprintf('Fuchsian genus two group'));
subplot(121);
cla
hold on
% for i = 1:grouplength
%     matrix = group(:,:,i);
%     a = matrix(1,1); b = matrix(1,2); c = matrix(2,1); d =
 ↪ matrix(2,2);
%     lim = (a*limseed+b)/(c*limseed+d);
%     if (1/abs(c) > 0.1 && abs(c) > 10^-6)
```

```
%           circle(real(-d/c),imag(-d/c),1/abs(c));
%       end
%       plot(real(lim),imag(lim),'.k','MarkerSize',.5);
% end
PhasePlot(zetas,values2,'p');
hold off
subplot(122)
cla
hold on
PhasePlot(zetas,values,'p');
hold off
drawnow

saveas(gcf,sprintf('fuchsian.png'));
close(gcf);

function h = circle(x,y,r)
    ct = 2*pi/(ceil(2*pi*r*100));
    th = 0:ct:2*pi;
    xunit = r * cos(th) + x;
    yunit = r * sin(th) + y;
    h =
 ↪  scatter(xunit,yunit,1,'MarkerFaceColor','r','MarkerEdgeColor','none');
    h.MarkerFaceAlpha = .3;
end
```

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