What do knots have to do with algebra?

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NZMS Colloquium, Waikato University
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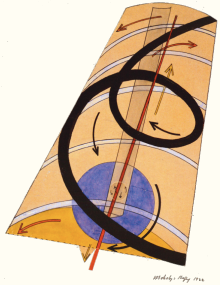
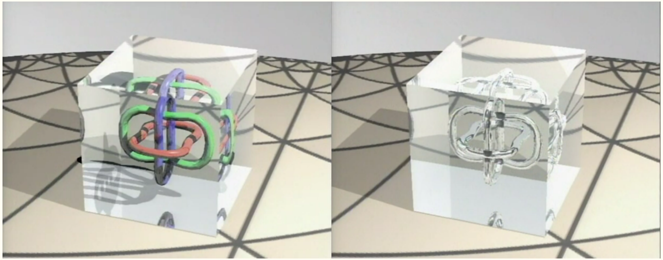


László Moloy-Nagy, *Kinetisches Konstruktives System* (1922)
Bauhaus-Archiv Berlin

If k is a knot, then $\mathbb{S}^3 \setminus k$ is a smooth oriented 3-manifold.



Gunn and Maxwell, *Not Knot*: `https://www.youtube.com/watch?v=4aN6vX7qXPQ`

$\mathbb{S}^3 \setminus k$ can be described combinatorially by a **knot diagram**.

Why are these manifolds interesting to topologists?

Gordon/Luecke (1989)

Knots with one component are determined by their complements up to homeomorphism.

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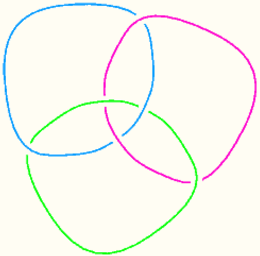

https://katlas.org/wiki/L6a4

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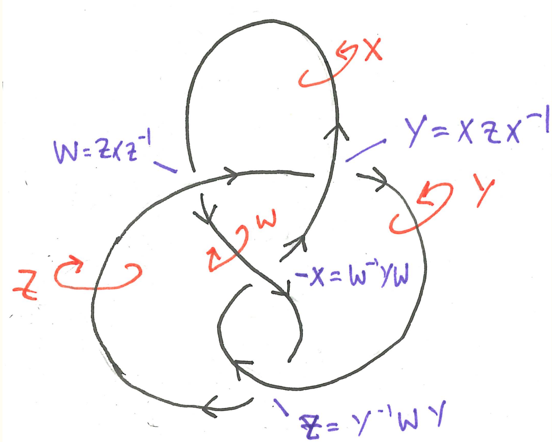
Knots: groups from diagrams

Knot complements can be described combinatorially by diagrams.

Wirtinger (c.1905)

There is an algorithm that sends a diagram of a knot k to a presentation for $\pi_1(\mathbb{S}^3 \setminus k)$, where all generators are conjugate:

generators	**relations**
diagram arcs	crossings



$$\langle X, Y, Z, W : X^{-1}W^{-1}YW,\ Y^{-1}XZX^{-1},\ Z^{-1}Y^{-1}WY,\ W^{-1}ZXZ^{-1} \rangle$$

Representations of knot groups to $\mathrm{PSL}(2, \mathbb{F}_p)$

To understand a group G, we look at its representations.

Theorem (Riley, *Math. Comp.* (1971))

For any knot k, the surjective representations $\pi_1(\mathbb{S}^3 \setminus k) \to \mathrm{PSL}(2, \mathbb{F}_p)$ (p prime) where every Wirtinger generator is order[1] p can be classified into conjugacy classes by performing at most

$$\left(\frac{p-1}{2}\right)^{n-1} \frac{(p+1)^{n-1} - 1}{p}$$

experiments, where n is the minimal number of bridges for a diagram of k.

The method of proof is purely combinatorial, relying only on information about the types of relators that can appear in a Wirtinger presentation.

[1]The condition on the order is purely for simplification and there are more general results in the same paper.

Representations of knot groups to $\mathrm{PSL}(2,\mathbb{C})$

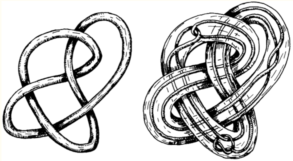
Theorem (Thurston, c. 1978)

If k is a link then $\mathbb{S}^3 \setminus k$ admits a Riemann metric of constant sectional curvature -1 if and only if it has no:

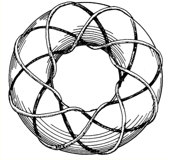
- *embedded essential spheres;*
- *embedded essential discs;*
- *embedded essential tori;*
- *embedded essential annuli.*

The hyperbolic structure on $\mathbb{S}^3 \setminus k$ induces a faithful, discrete representation

$$\pi_1(\mathbb{S}^3 \setminus k) \to \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{Conf}(\mathbb{S}^2) = \mathrm{PSL}(2,\mathbb{C}).$$



Embedded torus

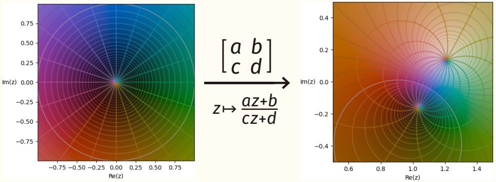


Embedded annulus

Thurston, *Bull. AMS* (1982)
images by George Francis

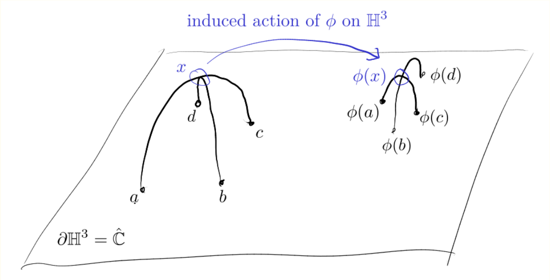
$\mathrm{PSL}(2,\mathbb{C})$-ACTIONS

On $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$



plots using `https://github.com/nschloe/cplot`

On $\mathbb{H}^3$



E., J. Schillewaert, G.J. Martin, *2021-22 MATRIX annals*, Springer (2024).

Meaning of traces. If $g \in \pi_1(\mathbb{S}^3 \setminus k)$ is homotopic to a loop on a component of k (e.g. a meridian or longitude), then tr^2 of the image is 4. The image of every other element has a positive minimal translation length in $\mathbb{H}^3$ and corresponds to a geodesic loop of positive length, $\mathrm{tr}^2 \notin [0,4]$.

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3-elements. Let E_3 be the one which contains t. Then c_1 is transformed into a segment by the deformation

$$S \to S + (E_3).$$

In this way all the circults in $S.E_2$ can be eliminated.

When the circuits have been eliminated there will be at least one segment in $S.E_2$, say u, which, together with a segment of m, bounds a 2-element C_2, on E_2, containing no other component of S. If E_3 is defined as before, with $N(C_2, M)$ and $N(u, S)$ taking the place of $N(t, M)$ and $N(p, S)$, the segment u is eliminated from $S.E_2$ by the deformation

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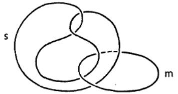
Reiterating this process, we obtain a non-singular deformation of



S into a surface S^1, which does not meet E_2 except in l, and this deformation leaves c unaltered. It is now obvious that the first step in the deformation $c \to c'$ can be realized by a non-singular deformation of S^1, and the lemma follows from induction on the number of steps in $c \to c'$.

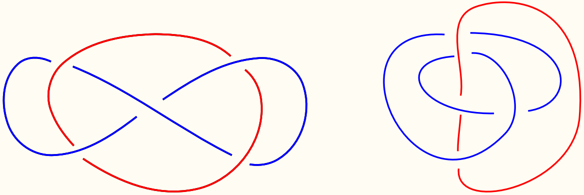
If a circuit in M is contained in a 3-element in M it will be called an *elementary circuit*. A circuit which bounds a (singular) 2-cell but which is not an elementary circuit will be called a *self-linking* circuit. The simplest type of self-linking circuit is illustrated by the diagram, the manifold being the residual space of a circuit m in Euclidean space, and s being a self-linking circuit.

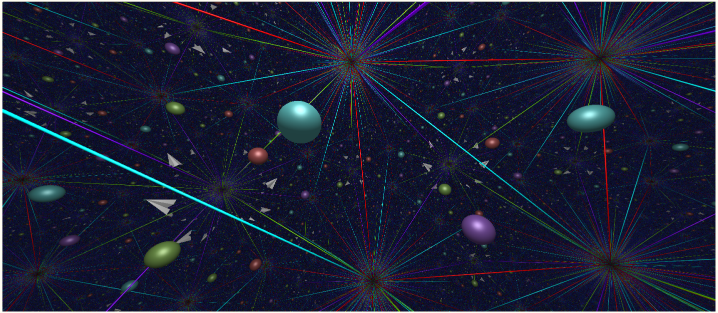
We shall need two lemmas about punctured spheres.† We first recall from T.M. pp. 319–20, that any two punctured spheres are equivalent if they have the same number of boundary 2-spheres.‡

† Cf. T. M. § 2. When we refer to a punctured sphere or to any other bounded manifold, it is to be assumed that the boundary is non-singular.

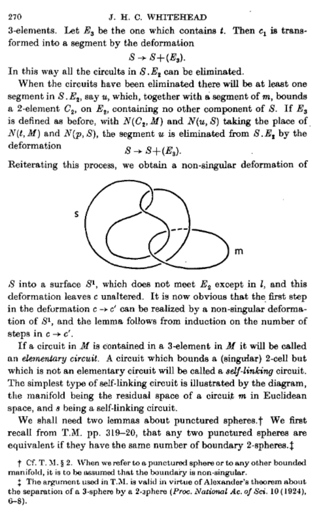
‡ The argument used in T.M. is valid in virtue of Alexander's theorem about the separation of a 3-sphere by a 2-sphere (*Proc. National Ac. of Sci.* 10 (1924), 6–8).

J.H.C. Whitehead, "A certain open manifold whose group is unity". *Quart. J. Math.*, 1935.





M. Culler, N. M. Dunfield, M. Goerner, and J. R. Weeks, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, `http://snappy.computop.org`



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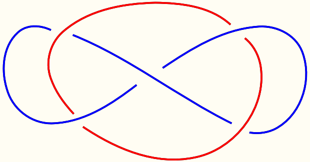

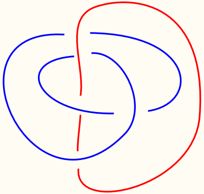

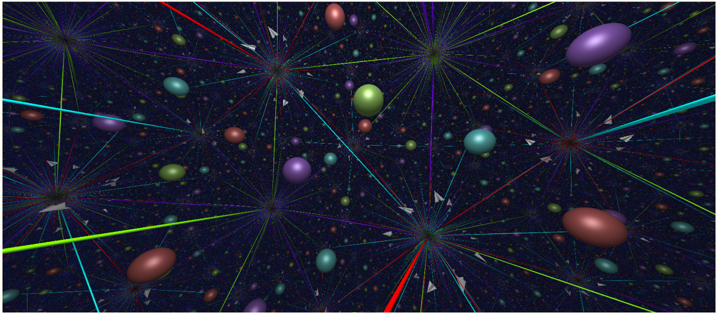

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- Knot complements are one important class of geometric structures with interesting groups.
  - **Mostow rigidity theorem**: the corresponding subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are rigid—if $\mathbb{S}^3 \setminus k = \mathbb{H}^3/G$, and $H < \mathrm{PSL}(2,\mathbb{C})$ is isomorphic to G, then $\mathbb{H}^3/G = \mathbb{H}^3/H$ (up to isometry).

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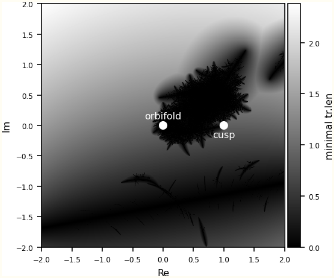

← A linear slice through different geometric structures $\pi_1(S_{2,0}) \to \mathrm{PSL}(2,\mathbb{C})$ where $S_{2,0}$ is the genus 2 surface.

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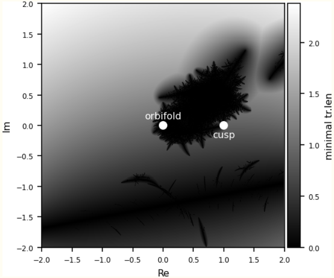

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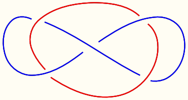
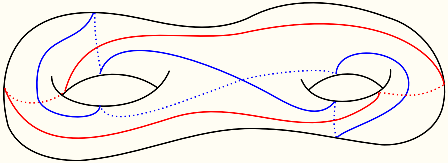
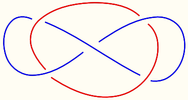
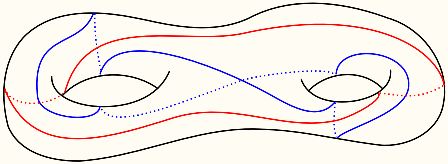
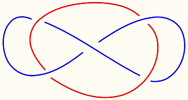
- How do these different kinds of $\mathrm{PSL}(2,\mathbb{C})$-representations fit together? Consider embeddings $k \to S_{2,0}$.

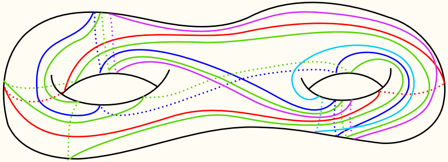




$\pi_1(\text{genus 2 handlebody}) = \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$





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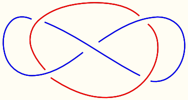

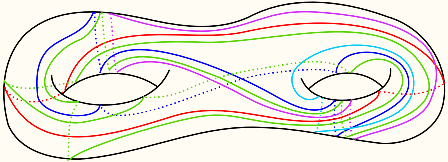

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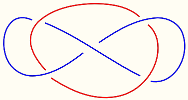

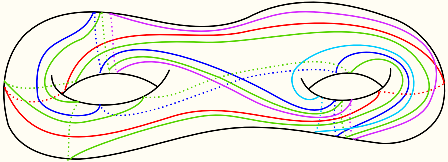

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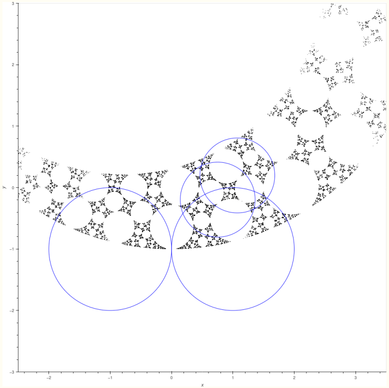
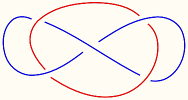
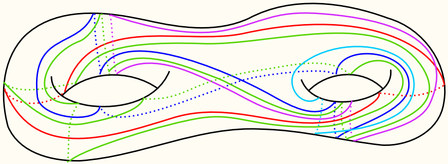
$$\rho(X) = \begin{bmatrix} 0.9222 + 0.8789i & 1.1360 - 0.6069i \\ 0.8640 + 1.3931i & -0.6501 - 1.6651i \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 - i & 1 \\ 1 & 1 + i \end{bmatrix}$$







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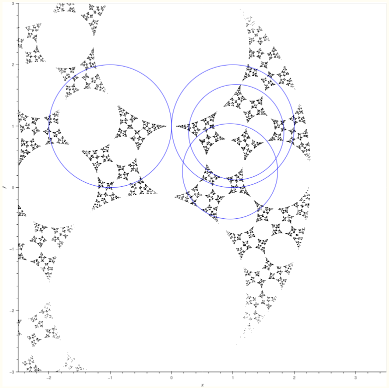

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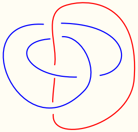
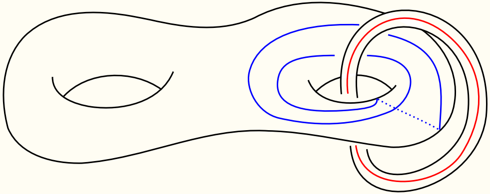
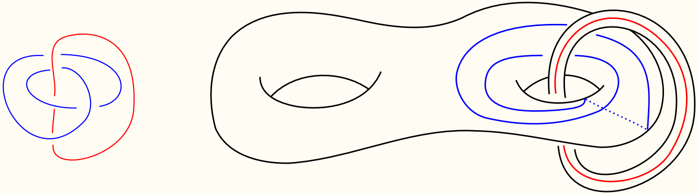
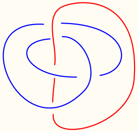
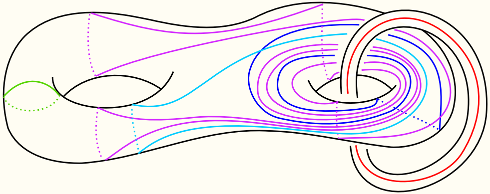






$$\pi_1(\text{compr. body}) = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} = \langle P, Q, M : [P, Q] = 1 \rangle$$





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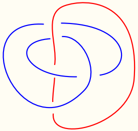

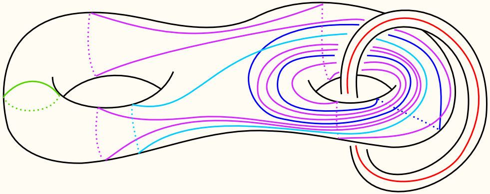

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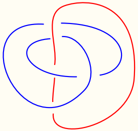

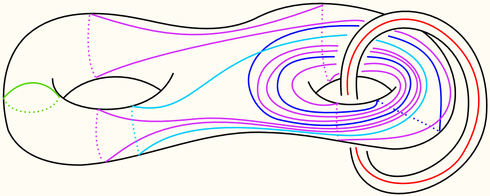

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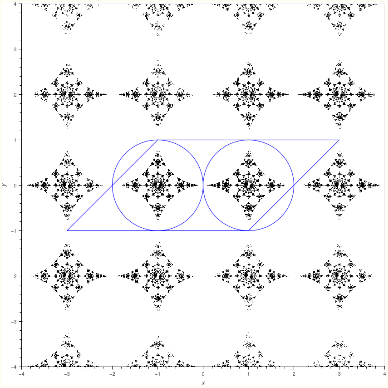
green: M pink: $QMQ^{-1}Q^{-1}P^{-1}MQPQ$ blue: QM

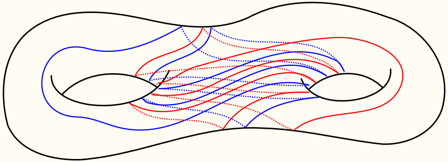
$\exists$ 1 interesting rep
$\rho : \langle P, Q, M : [P, Q] = 1 \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$
for all these words.

$$\rho(P) = \begin{bmatrix} 1 & 10 + 2i \\ 0 & 1 \end{bmatrix}, \ \rho(Q) = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \ \rho(M) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



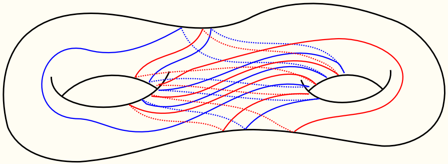
3/8 braid embedding

(as per Jeroen's talk)



3/8 braid embedding

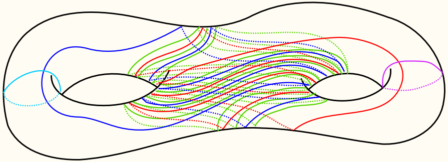
(as per Jeroen's talk)



π_1(genus 2 handlebody) = $\mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

3/8 braid embedding

(as per Jeroen's talk)



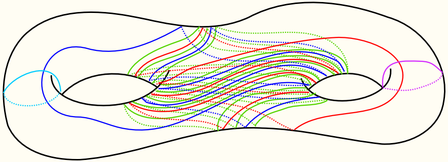
dual graph:
3 non-intersecting curves, each meeting the knot at most once

$\pi_1(\text{genus 2 handlebody}) = \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

green: $W_{3/8}$ pink: Y blue: X

3/8 braid embedding

(as per Jeroen's talk)



dual graph: 3 non-intersecting curves, each meeting the knot at most once

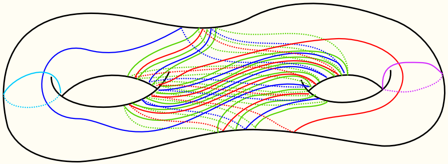
π_1(genus 2 handlebody) = $\mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

green: $W_{3/8}$ pink: Y blue: X

$\exists$ 2 discrete reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

3/8 braid embedding

(as per Jeroen's talk)



dual graph:
3 non-intersecting curves, each meeting the knot at most once

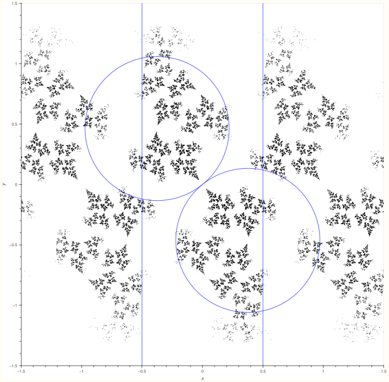
$\pi_1(\text{genus 2 handlebody}) = \mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

green: $W_{3/8}$ pink: Y blue: X

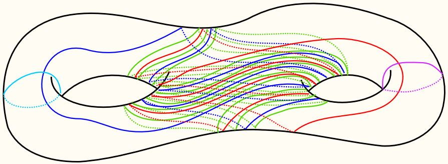
$\exists$ 2 discrete reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

$$\rho(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 & 0 \\ 1.05642 + 1.30324i & 1 \end{bmatrix}$$



3/8 braid embedding
(as per Jeroen's talk)



dual graph: 3 non-intersecting curves, each meeting the knot at most once

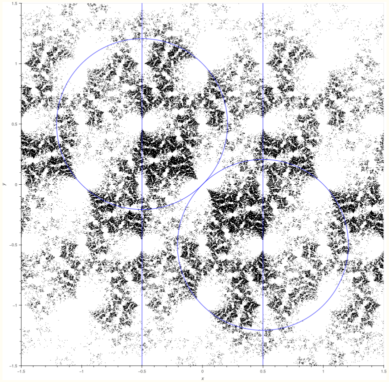
π_1(genus 2 handlebody) = $\mathbb{Z} * \mathbb{Z} = \langle X, Y \rangle$

green: $W_{3/8}$ pink: Y blue: X

$\exists$ 2 discrete reps $\rho : \langle X, Y \rangle \to \mathrm{PSL}(2, \mathbb{C})$ so that $\mathrm{tr}^2 = 4$ for all these words.

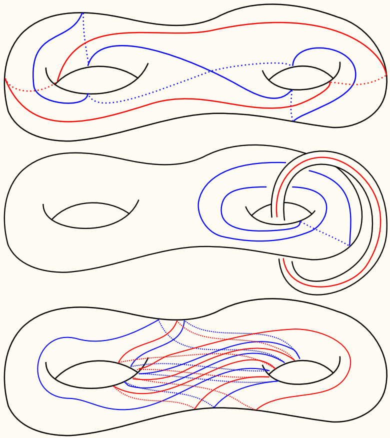
$$\rho(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\rho(Y) = \begin{bmatrix} 1 & 0 \\ 1 + i & 1 \end{bmatrix}$$

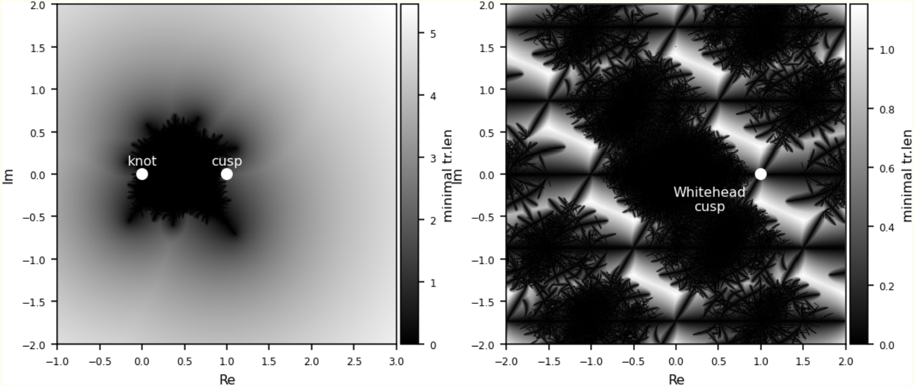


What is the best choice of embedding?

- Out of the 3, only 1 embedding of the Whitehead link into $S_{2,0}$ gave us the link group back as a representation: the braid embedding.
- Necessary (but not sufficient) condition: the dual curves to the embedding need to bound embedded discs in the complement handlebody.
- When this condition is not satisfied, if k is complicated (many crossings) then we still get lattices as well as cusp groups.
- *Open question*: characterise the lattices obtained topologically in terms of the combinatorics of k.



These methods give us reps. $G \to \mathrm{PSL}(2,\mathbb{C})$ that capture the combinatorics of knots (so nice geometry) & are *extremal* in representation spaces.



Two slices through $\mathrm{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \mathrm{PSL}(2,\mathbb{C}))$. The 'cusp' point is the same group in each picture.

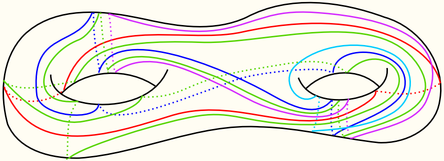
E., "From disc patterns in the plane to character varieties of knot groups", `arXiv:2503.13829 [math.GT]`

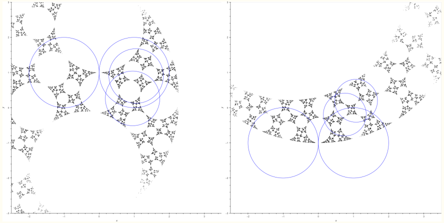
Problem

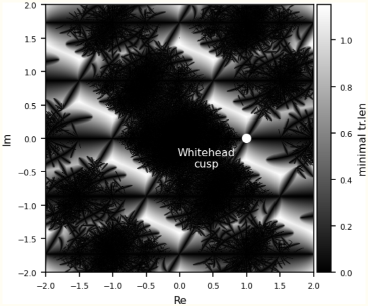
If G is a group and $\rho_0, \rho_1 : G \to \mathrm{PSL}(2, \mathbb{C})$ are reps, can you find a path of reps ρ_t with actions that 'smoothly interpolate' from ρ_0 to ρ_1?

The obstruction is an understanding of paths of *indiscrete* reps. The extremal groups we discussed (called 'cusp groups') can act as nice endpoints for smooth curves of indiscrete representations. We have constructed such paths in a few settings, using a variety of geometric techniques:

- between compression body groups with different numbers of handles
  `arXiv:2411.17940 [math.GT]`, video
- between 'fully augmented link groups' which are rigid, and cusp groups which lie on the boundary of a big deformation space (to appear shortly)
- between 2-bridge link groups and the holonomy groups of manifolds where an upper unknotting tunnel has been drilled (joint with Chesebro and Purcell, in preparation)







- E., *From disc patterns in the plane to character varieties of knot groups.* `arXiv:2503.13829 [math.GT]`
- E., *Changing topological type of compression bodies through cone manifolds.* `arXiv:2411.17940 [math.GT]`
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