Strange circles

The Riley slice of quasi-Fuchsian space

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Joint work with
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What this talk is...

We approach a particular concrete moduli space of hyperbolic 3-manifolds (called the **(generalised) Riley slice**) via a very straight path through complex analysis and Riemann surface theory.
Much of the theory is developed in three joint preprints with G. Martin and J. Schillewaert (and my thesis), generalising work of L. Keen and C. Series (1994) and several others.

...and what it isn't

We will not talk about: the knot theory; arithmetic groups and the slice exterior; the topology of the boundary; the PDE theory; the dynamical theory.

In this talk, $\mathbb{H}^3$ will denote hyperbolic 3-space with sphere at infinity identified with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We can extend isometries of $\mathbb{H}^3$ to a conformal action on the boundary. Conversely, conformal maps on $\hat{\mathbb{C}}$ extend uniquely to isometries on the interior.

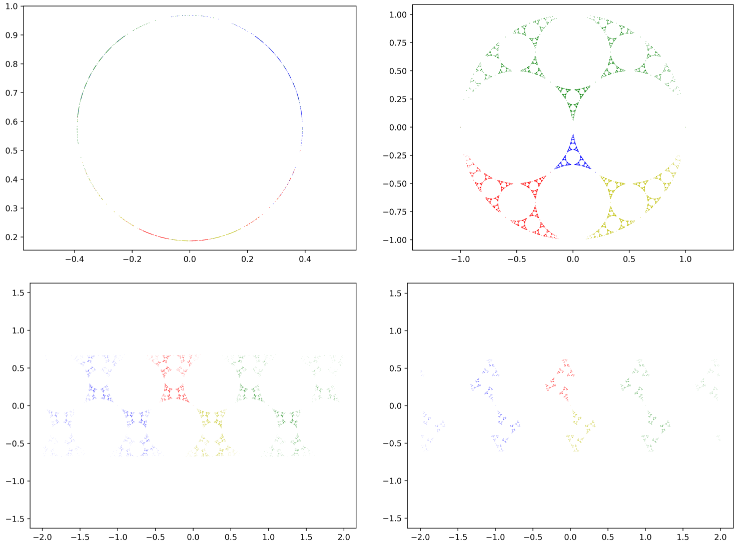
Conformal maps on $\hat{\mathbb{C}}$ are exactly the Möbius transformations; thus all of the following are naturally identified after choosing coordinates:

$$\mathbb{M} := \left\{ \begin{tikzcd} & \text{conformal maps of the sphere} \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{biholomorphic automorphisms of } \hat{\mathbb{C}}=\mathbb{PC} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \mathrm{PSL}(2,\mathbb{C}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{fractional linear transformations} \arrow[dl, leftrightarrow] \\ & \text{isometries of } \mathbb{H}^3 & \end{tikzcd} \right\}$$

We are interested in subgroups $G \leq \mathbb{M}$ which have 'geometric' actions. That is, we wish for $\mathbb{H}^3/G$ to be an orbifold. The most useful assumption on G which guarantees this is discreteness. Discrete subgroups of $\mathbb{M}$ are called **Kleinian**.

Discreteness is not enough to ensure the action on $\hat{\mathbb{C}}$ gives a nice geometric quotient, since orbits of points often accumulate on $\hat{\mathbb{C}}$ making the quotient non-Hausdorff. If we delete these points, everything works nicely:

- The **limit set** of a Kleinian group G is the set $\Lambda(G) \subseteq \hat{\mathbb{C}}$ of accumulation points of the orbits of G on $\overline{\mathbb{H}^3}$.
- The **regular set** is $\Omega(G) := \hat{\mathbb{C}} \setminus \Lambda(G)$.
- **Theorem:** $\mathcal{S}(G) := \Omega(G)/G$ is a (marked, possibly disconnected) Riemann surface.


1.0
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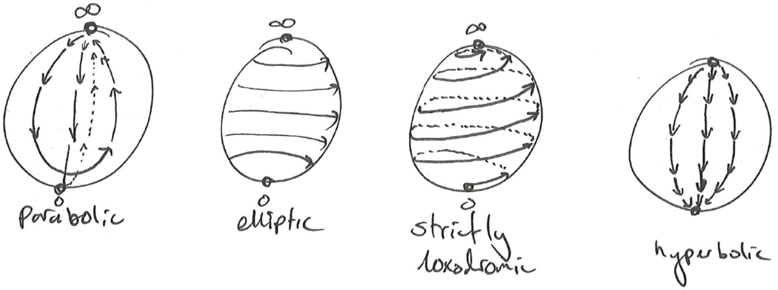
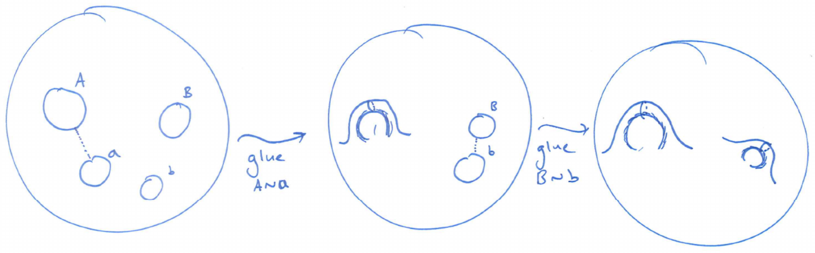
We can classify the elements of a Kleinian group according to their orbits on $\hat{\mathbb{C}}$:



The orbit types are distingushed by the tr^2 of matrix representatives for the element in $\mathrm{PSL}(2, \mathbb{C})$.

Consider four disjoint circles in the plane, all disjoint. Put the circles in pairs:

- Pick two conformal bijections f and g which act on $\hat{\mathbb{C}}$ such that $f(\operatorname{int} A) = \operatorname{ext} B$ and $g(\operatorname{int} b) = \operatorname{ext} B$. The group $\Gamma = \langle f, g \rangle$ is a Kleinian group, called the **classical Schottky group** obtained from the given data.
- (**Poincaré polyhedron theorem**) The quotient surface $\Omega(\Gamma)/\Gamma$ can be obtained by taking the common exterior of the four circles and gluing a to A and b to B with the identifications given by f and g.
- Both f and g are loxodromic.


A
B
a
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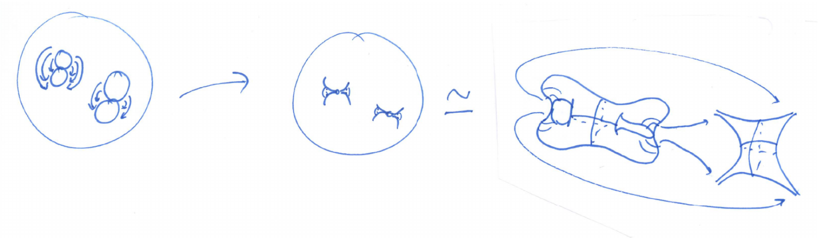
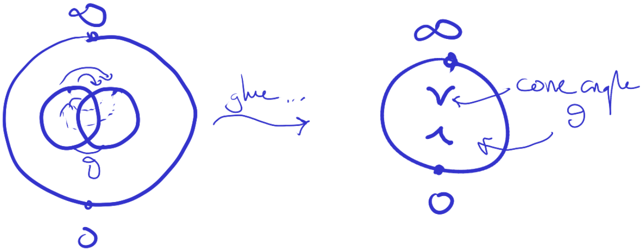
glue
A∼a
B
b
glue
B∼b

- Consider now the following deformation. We take a and A and push them together, and take b and B and push them together, such that the paired circles become tangent. As long as the two pairs remain mutually disjoint and the paired circles have the same radius, the Poincaré polyhedron theorem still holds.
- However, the Riemann surface now has *cusps* at the point of tangency, and the two generators are parabolic. Hence we obtain a 4-times punctured sphere as the quotient.



- Now push take a and A *through* each other, and do the same with b and B, so that each pair makes a well-defined non-zero angle. Again, as long as the two pairs remain mutually disjoint and the paired circles have the same radius, the Poincaré polyhedron theorem still holds.
- The Riemann surface now has cone points at the point of tangency.



What should a deformation space of Kleinian groups do?

- The underlying hyperbolic manifolds should move in a continuous way (i.e. should naturally have the geometric convergence topology).
- The (complex structures of the) boundary Riemann surfaces should move continuously (i.e. should have a Teichmüller structure).
- The matrices of the group as a subgroup of the Lie group $\mathrm{PSL}(2, \mathbb{C})$ should move holomorphically in the entries.
- The limit sets of the Kleinian group as subsets of $\hat{\mathbb{C}}$ should move holomorphically.

We will take as our starting point (3) and (4).

Definition (Mañé/Sad/Sullivan, 1983)

Let $A \subseteq \hat{\mathbb{C}}$. A **holomorphic motion** of A is a map $\Phi : \mathbb{B}^2 \times A \to \hat{\mathbb{C}}$ such that

1. For each $a \in A$, the map $\mathbb{B}^2 \ni \lambda \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is holomorphic;
2. For each $\lambda \in \mathbb{B}^2$, the map $A \ni a \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is injective;
3. The mapping $A \ni a \mapsto \Phi(0, a) \in \hat{\mathbb{C}}$ is the identity on A.

Theorem (Slodkowski's extended λ-lemma, 1991)

If $\Phi : \mathbb{B}^2 \times A \to \hat{\mathbb{C}}$ is a holomorphic motion of $A \subseteq \mathbb{C}$, then Φ has an extension to $\tilde{\Phi} : \mathbb{B}^2 \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

1. *$\tilde{\Phi}$ is a holomorphic motion of $\hat{\mathbb{C}}$;*
2. *For each $\lambda \in \mathbb{B}^2$, the map $\tilde{\Phi}_\lambda$ defined by $\hat{\mathbb{C}} \ni a \mapsto \tilde{\Phi}(\lambda, a) \in \hat{\mathbb{C}}$ is a K-quasiconformal homeomorphism with $K \leq \frac{1+|\lambda|}{1-|\lambda|}$;*
3. *$\tilde{\Phi}$ is jointly continuous in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$; and*
4. *For all $\lambda_1, \lambda_2 \in \mathbb{B}^2$, $\tilde{\Phi}_{\lambda_1}\tilde{\Phi}_{\lambda_2}^{-1}$ is K-quasiconformal with $\log K \leq \rho(\lambda_1, \lambda_2)$ (where ρ is the hyperbolic metric on $\mathbb{B}^2$).* □

We actually need an equivariant version due to Earle, Kra, and Krushkal' (1994).

- Let Φ be a one-parameter holomorphic motion of the entries of the matrices of some Kleinian group.
- The limit set moves holomorphically with the entries, and by the λ-lemma this holomorphic motion extends (equivariantly) to a motion $\tilde{\Phi}$ of $\hat{\mathbb{C}}$.
- Let the parameter move from ρ to $\tilde{\rho}$. Via $\tilde{\Phi}$, we get a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which induces an isomorphism between the group at ρ and the group at $\tilde{\rho}$: $\Gamma_{\tilde{\rho}} = f^{-1}\Gamma_{\rho}f$.

This motivates:

Definition (Maskit, 1971; Kra, 1972)

The **quasiconformal deformation space** of Γ, denoted $\mathrm{QH}(\Gamma)$, is the space of representations $\theta : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ (up to conjugacy) such that

1. θ is faithful and $\theta\Gamma$ is discrete;
2. θ is type-preserving, that is if $\gamma \in \Gamma$ is parabolic (resp. elliptic of order n) then $\theta\gamma$ is parabolic (resp. elliptic of order n); and
3. the groups $\theta\Gamma$ are all quasiconformally conjugate.

Let X be a hyperbolic Riemann surface. Then there is a diagram

$$\begin{array}{c} \mathrm{Teich}(X) \\ {\scriptstyle \mathrm{MCG}(X)}\big\downarrow \\ \mathrm{Mod}(X) \end{array}$$

where $\mathrm{Teich}(X)$ is the Teichmüller space of X, $\mathrm{Mod}(X)$ is the Riemann moduli space, and $\mathrm{MCG}(X)$ is the mapping-class group.

Theorem

Let $\Gamma = \langle \gamma_1, ..., \gamma_n \rangle$ *be a finitely generated non-elementary Kleinian group with* $\Omega(\Gamma) \neq \emptyset$*. Then there is a natural holomorphic surjection* $p : \mathrm{Teich}(\Gamma) \to \mathrm{QH}(\Gamma)$*. Further, there is a discrete subgroup*

$$\widehat{\mathrm{MCG}}(\Omega(\Gamma)/\Gamma) \leq \mathrm{MCG}(\Omega(\Gamma)/\Gamma)$$

and a natural bijection

$$\mathrm{QH}(\Gamma) \approx \mathrm{Teich}(\Omega(\Gamma)/\Gamma)/\widehat{\mathrm{MCG}}(\Omega(\Gamma)/\Gamma)$$

compatible with the two group actions.

$$\begin{tikzcd}
\mathrm{Teich}(\Omega(\Gamma)/\Gamma) \arrow[d, "\mathrm{MCG}(\Omega(\Gamma)/\Gamma)"'] \arrow[dr, "\widehat{\mathrm{MCG}}(\Omega(\Gamma)/\Gamma)"] & \\
& \widehat{\mathrm{Mod}}(\Omega(\Gamma)/\Gamma) = \mathrm{QH}(\Gamma) \arrow[dl] \\
\mathrm{Mod}(\Omega(\Gamma)/\Gamma) &
\end{tikzcd}$$

The lift $\widehat{\mathrm{Mod}}$ of Mod detects the hyperbolic structure of the interior (since it detects the underlying Kleinian group structure), not just the complex structure on the boundary. It is the subgroup of $\mathrm{MCG}(\Omega(\Gamma)/\Gamma)$ generated by the Dehn twists about loops on the surface which bound compression discs.

- The (classical/parabolic) Riley slice is the quasiconformal deformation space of Kleinian groups representing a hyperbolic 3-manifold that is the complement of a 2-tangle.
- More precisely, it parameterises Riemann surfaces homeomorphic to 4-punctured sphere, while keeping track of two deleted arcs pairing up the punctures in the interior.

Let $\rho \in \mathbb{C} \setminus \{0\}$; define groups

$$\Gamma_\rho := \left\{ X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y_\rho = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\}$$

Definition (Brenner, 1955; Chang/Jennings/Ree, 1958; Lyndon/Ullman, 1969; Lyubich/Suvorov, 1988; etc.)

The **parabolic Riley slice**, which we denote by $\mathcal{R}^{\infty,\infty}$, is the set of $\rho \in \mathbb{C}$ such that Γ_ρ is free, discrete, and $\Omega(\Gamma_\rho)/\Gamma_\rho$ is a 4-times punctured sphere.

Following Thurston's philosophy that orbifolds are easier to study than manifolds, we generalise to the case that the generators are allowed to be elliptic, so the underlying 3-manifold has cone arcs rather than deleted arcs.

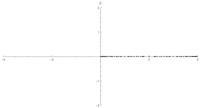
Let $\rho \in \mathbb{C} \setminus \{0\}$, and let $a, b \in \hat{\mathbb{N}}$ with $\max\{a, b\} \geq 3$; define groups

$$\Gamma_\rho^{a,b} := \left\{ X = \begin{bmatrix} e^{\pi i/a} & 1 \\ 0 & e^{-\pi i/a} \end{bmatrix}, Y_\rho = \begin{bmatrix} e^{\pi i/b} & 0 \\ \rho & e^{-\pi i/b} \end{bmatrix} \right\}$$

Write $F(n)$ for the free group on n generators.

Definition

The (a, b)-**Riley slice**, which we denote by $\mathcal{R}^{a,b}$, is the set of $\rho \in \mathbb{C}$ such that $\Gamma_\rho^{a,b}$ is isomorphic to $F(a) * F(b)$, discrete, and $\Omega(\Gamma_\rho)/\Gamma_\rho$ is a sphere with two cone points of order a and two cone points of order b.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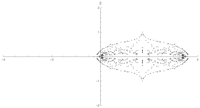
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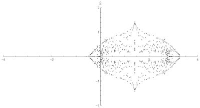
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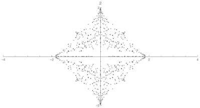
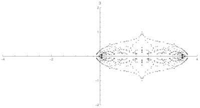

$(2, \infty)$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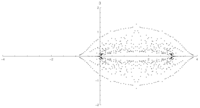
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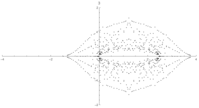
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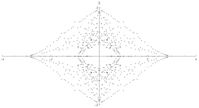
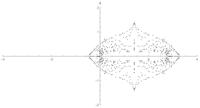

$(3, \infty)$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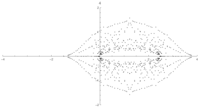
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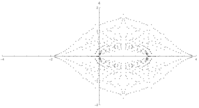
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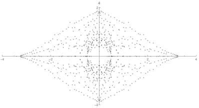
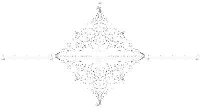
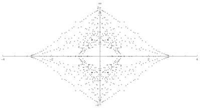

$(4, \infty)$


$(\infty, 2)$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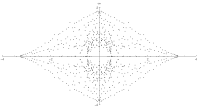
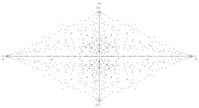

$(\infty, 4)$


(∞, ∞)

The following theorem is proved in Elzenaar, Martin, and Schillewaert (2022b). It is rather subtle, and historically the literature has glossed over the need to actually check that this is true (even in the parabolic case).

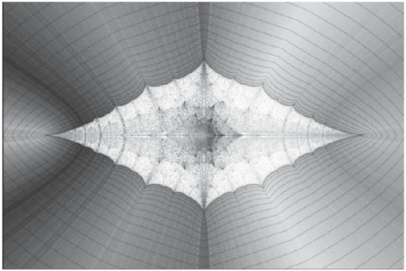
Theorem

Let $\rho \in \mathcal{R}^{a,b}$. Then

$$\mathcal{R}^{a,b} \simeq_{biholom.} \mathrm{QH}(\Gamma_\rho^{a,b}).$$

Based on the definition of the identification given e.g. in Maskit and Swarup (1989) it is not even clear that the map $\mathcal{R}^{a,b} \to \mathrm{QH}(\Gamma_\rho^{a,b})$ is even continuous!

The ‘modern’ study of the parabolic Riley slice began with the paper of Keen and Series (1994) (with corrections in Komori and Series (1998)); Keen and Series described a coordinate system for the Riley slice which exhibits its geometric structure in the sense of the Bers theory and the ending lamination theorem (image due to D. Wright taken from Gilman (2008)):



In our four preprints, we generalise the theory of Keen and Series in two different ways:

1. we give open neighbourhoods of the radial coordinate lines within which the quasi-geometry of the groups is preserved (on the lines only the conformal geometry is preserved); and
2. we generalise the whole thing to deal with elliptic generators.

We also extend some of the combinatorics needed to define the coordinate system based in part on Chesebro (2019), Chesebro, Emlen, Ke, Lafontaine, McKinnie, and Rigby (2020), and Wright (2005).

There is a natural bijection between the space $\mathcal{ML}(S)$ of measured laminations on the 4-marked sphere S which miss the marked points, and $\mathbb{R}/2\mathbb{Z} \cup \{\infty\}$. The parameterisation gives the 'slope' of the lamination.

Theorem (Keen/Series, 1994 (parabolic case); E./Martin/Schillewaert, 2022 (general case))

There is a natural bijection

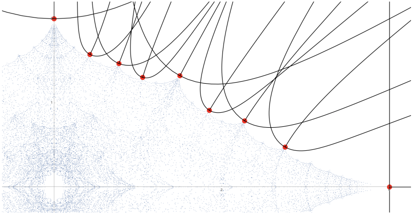
$$\mathcal{R}^{a,b} = \mathbb{R}/2\mathbb{Z} \times \mathbb{R}_{\geq 0}.$$

(Note that ∞ does not appear, for knot-theoretic reasons.)

The construction is a concrete version of the Thurston boundary of quasi-Fuchsian space. The radial coordinate gives the lamination on the boundary; the orbital coordinate is a normalised length of that lamination. The radial coordinate lines also correspond to groups whose limit sets contain circle chains with fixed combinatorics.

Theorem (E./Martin/Schillewaert, 2021)

About each radial coordinate ray there is an open neighbourhood in $\mathcal{R}^{a,b}$, consisting of groups whose limit sets contain quasi-circle chains with fixed combinatorics.



Future work includes:

- Studying the properties of the map $\Pi : \mathcal{R}^{a,b} \to \mathbb{R}/2\mathbb{Z} \times \mathbb{R}_{>0}$. For example, we are unable to prove or disprove that the coordinate system axes are Teichmüller geodesics.
- The rational radial coordinate rays are real-algebraic curves. What geometry is encoded in the algebra of the corresponding polynomials?
- Studying the dynamical properties of these polynomials (see e.g. our 2021 preprint).

Three nice introductory references:

- B. Maskit (1987). *Kleinian groups*. Grundlehren der mathematischen Wissenshaften 287. Springer-Verlag
- A. Marden (2016). *Hyperbolic manifolds. An introduction in 2 and 3 dimensions*. 2nd ed. First edition was published under the title "Outer circles". Cambridge University Press
- K. Matsuzaki and M. Taniguchi (1998). *Hyperbolic manifolds and Kleinian groups*. Oxford University Press