THE COMBINATORICS OF FAREY WORDS AND THEIR TRACES

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Abstract. The set of Kleinian groups which are free on two parabolic generators is parameterised by the closed Riley slice of Schottky space. A 'Farey word' is a word in such a group which represents a non-boundary-parallel geodesic that can be pinched down to a puncture; in the interior of the Riley slice such a word is loxodromic, and the pinching process corresponds to deforming the word to be parabolic. Keen and Series showed that the geometry of the Riley slice is detected by the real loci of the trace polynomials of these words. We study these trace polynomials from a combinatorial viewpoint, and give a recursion formula for them which enables efficient calculation of the polynomials without performing matrix multiplication; we also present some intriguing examples to show that there is much still to be learned about them.

1. Introduction

A **Kleinian group** is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$; these groups have been intensively studied for a long time in association with hyperbolic geometry and conformal geometry [Bea83; Mar16; Mas87]. The **closed Riley slice** is the deformation space of Kleinian groups which are free on two parabolic generators. After normalisation, every group on two parabolic generators may be written as

$$\Gamma = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \right\rangle;$$

the closed Riley slice, which we denote by $\overline{\mathcal{R}}$, is therefore naturally identified with the set of $z \in \mathbb{C}$ such that Γ is discrete and free; in this guise, it forms the exterior of the set approximated by the points of Figure 1 (along with the boundary of that set). The interior of $\overline{\mathcal{R}}$ is the **Riley slice**, which can also be characterised as the set of $z \in \mathbb{C}$ such that Γ is free, discrete, and the Riemann surface $\Omega(\Gamma)/\Gamma$ ($\Omega(\Gamma)$ being the domain of discontinuity of Γ, as defined in [Mas87, E.2]) has homeomorphism type a 4-times punctured sphere.

A detailed historical account of the Riley slice, together with background information for the non-expert and motivating applications, can be found in our proceedings paper [EMS22a]. Though it was first defined in the mid-20th century, recent work on the Riley slice follows on from a 1994 paper of Keen and Series [KS94] (with some corrections by Komori and Series [KS98]). In this paper, they constructed a foliation of the Riley slice via a two-step process:

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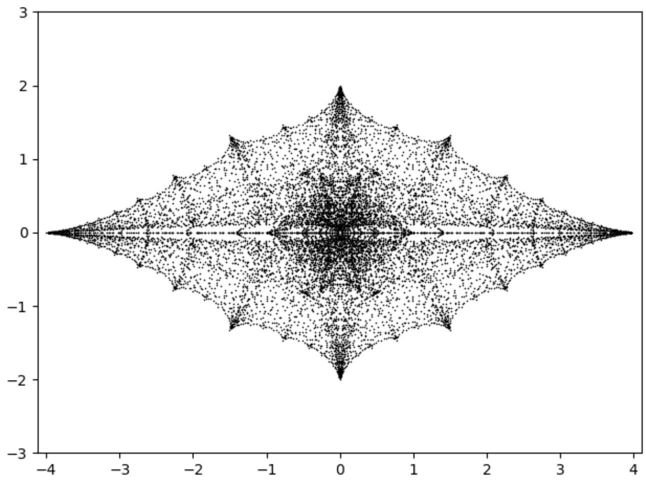
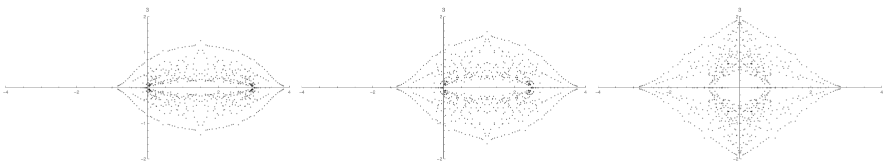


FIGURE 1. An approximation to the exterior of the Riley slice.



(A) $2\pi/3$, $2\pi/3$ (B) $2\pi/3$, $2\pi/4$ (C) $2\pi/3$, $2\pi/\infty$

FIGURE 2. Approximations to the exteriors of elliptic Riley slices with the indicated cone angles. See also Figure 8 of [EMS22b].

(1) Firstly, a lamination of $\mathscr{R}$ is defined, with leaves indexed by $\mathbb{Q}/2\mathbb{Z}$. These leaves consist of certain branches of preimages of $(-\infty, -2)$ under a family of polynomials $\Phi_{p/q}$ ($p/q \in \mathbb{Q}$) and are called the **rational pleating rays**.
(2) Then, by a completion construction similar to the completion of $\mathbb{Q}$ via Dedekind cuts, the lamination is extended to a foliation whose leaves are indexed by $\mathbb{R}/2\mathbb{Z}$. The leaves adjoined at this step are called the **irrational pleating rays**.

The polynomials $\Phi_{p/q}$ are constructed as trace polynomials of certain words in the group, which we call **Farey words**; these words enumerate all but one of the non-boundary-parallel closed curves on the sphere, and the distinguished branches on the preimages of $(-\infty, -2)$ correspond to curves in the Riley slice along which the lengths of these curves change in a particularly natural way; in particular, the geometric limit of $\Omega(\Gamma)/\Gamma$ as z travels down the (p/q)-pleating ray towards the preimage of -2 is precisely a pair of three-times punctured spheres, where the additional pair of punctures appears as the loxodromic word $W_{p/q}$ is pinched to a parabolic word with trace -2. The group corresponding to the geometric limit is known as a **cusp group**, and the set of points corresponding to cusp groups are dense in the boundary of the slice by a result of McMullen [McM91]. The cusp points were studied in detail by David Wright [Wri05].

Punctured spheres can be viewed as a limiting case of cone-pointed spheres as the cone angles tend to zero. It is therefore natural to study the case of spheres with four cone points, which have corresponding groups (after normalisation)

$$\Gamma = \left\langle X = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{bmatrix}, Y_z = \begin{bmatrix} \beta & 0 \\ z & \beta^{-1} \end{bmatrix} \right\rangle;$$

where α and β are roots of unity, and Γ is discrete, free, and isomorphic to $\mathbb{Z}_a * \mathbb{Z}_b$ (where a and b are the respective orders of X and Y_z[1]). The deformation space of these groups is called the (a,b)**-Riley slice**, and we denote it by $\mathcal{R}^{a,b}$; in our paper [EMS22b], we show that these spaces admit foliations which are defined in a completely analogous way to the Keen-Series foliation of the parabolic Riley slice. In fact, the polynomials which appear in the construction of this foliation are the trace polynomials of the same Farey words as the parabolic case (i.e. the only changes are in the coefficients of the matrices, not in the construction of the word whose trace is taken).

The goal of this paper is to study the combinatorial properties of these trace polynomials, which we call **Farey polynomials**, in both the parabolic and elliptic cases. Our main result is Theorem 4.7, which gives a recursion formula for the Farey polynomials that is independent of the geometric interpretation of the matrices (it works even if α and β are not roots of unity, in which case the groups are not discrete); in particular, it provides further evidence that the Riley slice exteriors can be studied in a completely dynamical way. This formula may be used to generate pictures of the Riley slice much more quickly than working directly from the definition of the Farey polynomials, as it removes the need for performing slow matrix multiplications; a working software implementation of this in `Python` may be found online [Elz21]. Similar recursions for the commutators $[X^n, Y]-2$ and the traces $\operatorname{tr}^2(X^n)-4$ were found by Alaqad, Gong, and Martin in [AGM21], and for the analogues of the Farey polynomials for the Maskit slice in [MSW02, pp. 283–285].

Notation. Throughout, $\Gamma = \langle X, Y \rangle$ is discrete and free with

$$X = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{bmatrix} \quad \text{and} \quad Y_z = \begin{bmatrix} \beta & 0 \\ z & \beta^{-1} \end{bmatrix}$$

where $z \in \mathbb{C} \setminus \{0\}$ and α, β are of norm 1. When discussing words we usually do not care about the precise value of z, and to simplify notation we will just write Y for Y_z. We use the convention $x := X^{-1}$, $y := Y^{-1}$ (and we use similar conventions throughout without comment).

1.1. **Structure of the paper.** There are five sections to this paper beyond this introduction, of varying lengths. Section 2 gives a combinatorial definition of the Farey words, using cutting sequences; we take care to avoid requiring any knowledge of geodesic coding; then in Section 3 we list some definitions and results about Farey sequences from classical number theory. Our main result (Theorem 4.7) is proved in Section 4; in the following section, we apply the work of Chesebro and his collaborators [Che+20; Che20] to show a formal analogy between the Farey polynomials and the Chebyshev polynomials. Finally in Section 6 we discuss some applications to the approximation of irrational pleating rays and cusps, and give some computational results which show that there are interesting connections with dynamical systems and number theory for future work to explore. All of the tables have been placed at the end of the paper to avoid breaking the flow of the text, as many are quite large.

[1]There is a technical detail we sweep under the rug here with our notation: since the ambient group is $\mathrm{PSL}(2,\mathbb{C})$ and so $I = -I$, we need to take for example $\alpha = \exp(\pi i/a)$ rather than $\alpha = \exp(2\pi i/a)$.

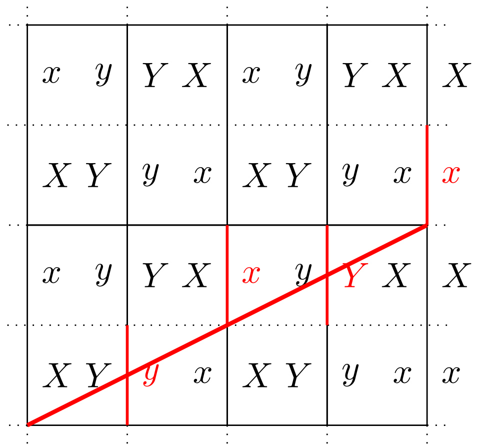


FIGURE 3. The cutting sequence of the 1/2 Farey word.

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2. Cutting sequences and Farey words

We define the Farey word[2] of slope p/q via cutting sequences, following [KS94] and [EMS22b, Section 5]. Consider the marked tiling of $\mathbb{R}^2$ shown in Figure 3, and let $L_{p/q}$ be the line through $(0,0)$ of slope p/q; now define $S_{p/q} = L_{p/q} \cap 2\mathbb{Z}^2$. Then the **Farey word of slope** p/q, $W_{p/q}$, is the word of length $2q$ such that the ith letter is the label on the right-hand side of the ith vertical line segment crossed by $L_{p/q}$ (i.e. the label to the right of the point $(p/q)i$); if $(p/q)i$ is a lattice point then this definition is ambiguous and by convention we take the label on the north-west side. In other words, the ith letter of $W_{p/q}$ is determined by the parity of $\operatorname{ceil}(p/q)i$ with the convention that $\operatorname{ceil} n = n + 1$ for integral n.

Example 2.1. As an example of the construction process, from Figure 3 we can read off that $W_{1/2} = yxYX$; we include a list of the Farey words with $q \leq 12$ as Table 1.

There are various symmetries visible in the Farey words; for instance, they are alternating products of $X^{\pm 1}$ and $Y^{\pm 1}$ which always end in X (this is obvious from the definition). The following slightly less trivial symmetry will be useful in the sequel:

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[2]We name these words and the related polynomials after John Farey Sr. as they are closely related to the so-called Farey sequences of rational numbers which we will discuss briefly later in this paper; with regard to this attribution, we quote from the historical notes to Chapter III of Hardy and Wright [HW60, pp. 36–37]: "The history of 'Farey series' is very curious... [their properties] seem to have been stated and proved first by Haros in 1802... Farey did not publish anything on the subject until 1816. [...] Mathematicians generally have followed Cauchy's example in attributing the results to Farey, and the series will no doubt continue to bear his name. Farey has a notice of twenty lines in the *Dictionary of national biography* where he is described as a geologist. As a geologist he is forgotten, and his biographer does not mention the one thing in his life which survives."

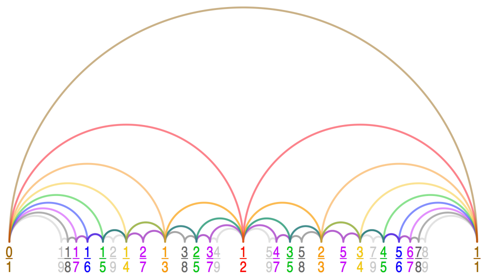


FIGURE 4. The Farey tessellation of $[0, 1]$. (Image by Cmglee — CC BY-SA 4.0. `https://commons.wikimedia.org/w/index.php?curid=59832325`.)

Lemma 2.2. *Let $W_{p/q}$ be a Farey word; then the word consisting of the first $2q-1$ letters of $W_{p/q}$ is conjugate to X or Y according to whether the qth letter of $W_{p/q}$ is $X^{\pm 1}$ or $Y^{\pm 1}$ (i.e. according to whether q is even or odd respectively)*

Proof. This identity comes from considering the rotational symmetry of the line of slope p/q about the point (q, p); it is clear from the symmetry of the picture that the first $p-1$ letters of $W_{p/q}$ are obtained from the $(p+1)$th to $(2p-1)$th letters by reversing the order and swapping the case (imagine rotating the line by 180 degrees onto itself and observe the motion of the labelling). $\square$

The **Farey polynomial of slope** p/q is defined by $\Phi_{p/q} := \operatorname{tr} W_{p/q}$; this is a polynomial in z of degree q, with coefficients rational functions of α and β. If we wish to emphasise the dependence on a and b, we write $\Phi^{a,b}_{p/q}$.

Example 2.3. We list the Farey polynomials $\Phi^{a,b}_{p/q}$ with $q \leq 4$ in Table 2. This illustrates some of the difficulty in studying these polynomials: they are clearly very symmetric, but quickly become too unwieldy to write explicitly and so actually guessing what the symmetries *are* in general is hard. We also list the first few 'Fibonacci' Farey polynomials $\Phi^{\infty,\infty}_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}$ (as usual, $\mathsf{fib}(1) = 1$, $\mathsf{fib}(2) = 1$, and $\mathsf{fib}(n) := \mathsf{fib}(n-1) + \mathsf{fib}(n-2)$) in Table 3.

3. FAREY THEORY

Our recursion formula, like that for the Maskit slice in [MSW02, pp. 283–285], is a recursion down the **Farey diagram**. We therefore take a quick break from the Keen–Series theory to recall some of the notation and ideas, which can be found for instance in [GKP94, §4.5] or [HW60, Chapter III].

Recall that $\mathrm{PSL}(2,\mathbb{Z})$ acts as a group of isometries on $\mathbb{H}^2$. The ideal triangle spanned by

$$(1/0, 1/1, 0/1) = (\infty, 1, 0)$$

bounds a fundamental domain for this action, and the simplicial complex formed by the tessellations of this triangle under $\mathrm{SL}(2,\mathbb{Z})$ is called the **Farey triangulation** of $\mathbb{H}^2$; we denote this by $\mathcal{D}$. The portion of $\mathcal{D}$ 'above' the segment $[0, 1]$ is depicted in Figure 4. The vertices of $\mathcal{D}$ are exactly the points of $\hat{\mathbb{Q}}$, and the 2-faces are triangles with vertex triples of the form $\{p/q, (p+r)/(q+s), r/s\}$ where $ps - qr = \pm 1$. This second assertion follows immediately from the definition of $\mathcal{D}$ as a tessellation: if $\{a/b, c/d, e/f\} \in \mathcal{D}(2)$, then there exists some matrix

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$$

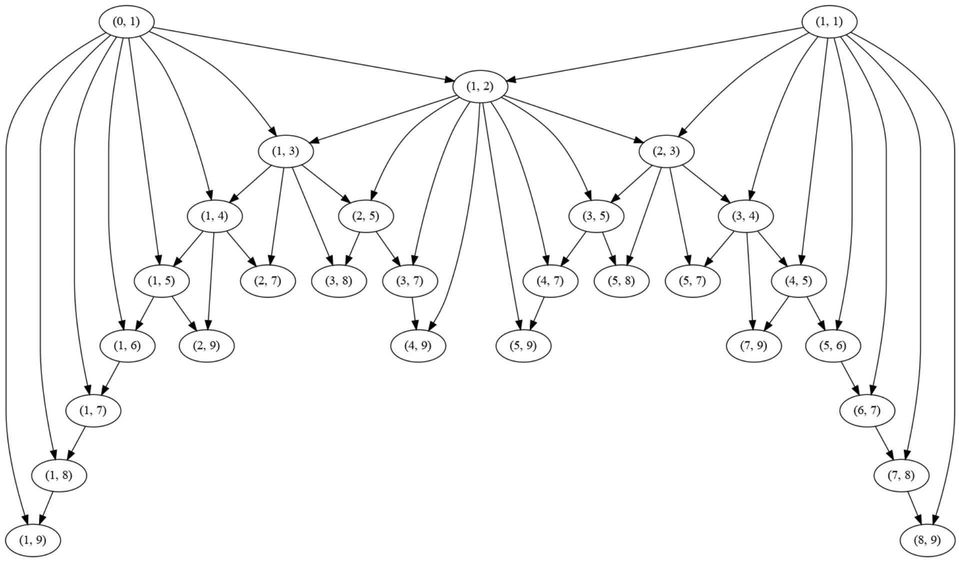


FIGURE 5. The Farey addition graph.

which acts on the original triangle as

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix},$$

giving immediately that $(a/b, c/d, e/f) = (p/q, (p+r)/(q+s), r/s)$. The operation

$$(p/q, r/s) \mapsto (p+r)/(q+s)$$

will be fundamental to our later study; it is called the **mediant** or **Farey addition** operation. We write $(p/q) \oplus (r/s)$ for the Farey addition of p/q to r/s. We will be careful to only combine p/q and r/s in this way if they satisfy the determinant condition $ps - qr = \pm 1$ as above (in which case they are both in least terms); we call two such fractions **Farey neighbours**. Farey addition has many useful properties: for instance, the Farey sum of two Farey neighbours is also a Farey neighbour of each summand (an easy calculation). We will also need the following lemma which appears in [HW60, §3.3]:

Lemma 3.1. *If p/q and r/s are Farey neighbours with $p/q < r/s$, then $p/q < (p/q) \oplus (r/s) < r/s$, and $(p/q) \oplus (r/s)$ is the unique fraction of minimal denominator between p/q and r/s. More precisely, let u/v be any fraction in $(p/q, r/s)$; then there exist two positive integers λ, μ such that*

$$u = \lambda p + \mu r \text{ and } v = \lambda r + \mu s$$

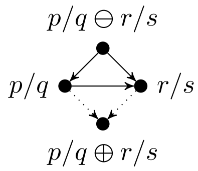
(so of course the minimal denominator is obtained when $\lambda = \mu = 1$). □

The **Farey graph** is the digraph with vertices $\hat{\mathbb{Q}}$ and directed edges from p/q and r/s to $p/q \oplus r/s$ (Figure 5). This is just the 1-skeleton of $\mathcal{D}$ with an added orientation.

It will be convenient, finally, to have the notation $p/q \ominus r/s$ for the fraction $(p-r)/(q-s)$; we shall only use this when it is known that $(p-r)/(q-s)$ and r/s are Farey neighbours (which is implied by neighbourliness of p/q and r/s).

4. A RECURSION FORMULA TO GENERATE FAREY POLYNOMIALS

In this section, we will give a recursion formula for the Farey polynomials. This recursion will be a recursion 'down the Farey graph', in the sense that its input will be the Farey polynomials at the vertices of a triangle $(p/q, p/q \ominus r/s, r/s)$ and its output will be the Farey polynomial at $p/q \oplus r/s$:



We begin by finding a similar recurrence for the Farey words; we will then produce a recurrence for the polynomials using standard trace identities and elbow grease. One might guess, for instance by analogy with the Maskit slice [MSW02, p. 277], that $W_{p/q}W_{r/s} = W_{p/q\oplus r/s}$. It is easy to check whether or not this is true:

Example 4.1. We use the convention $x := X^{-1}$, $y := Y^{-1}$:

- $W_{1/2} = yxYX$, $W_{1/1} = YX$,
$$W_{1/2}W_{1/1} = yxYXYX, \text{ and}$$
$$W_{1/2\oplus 1/1} = W_{2/3} = yxyXYX.$$
- $W_{1/3} = yXYxYX$, $W_{2/5} = yXYxyXyxYX$,
$$W_{1/3}W_{2/5} = yXYxYXyXYxyXyxYX, \text{ and}$$
$$W_{1/3\oplus 2/5} = W_{3/8} = yXYxYXyxYxyXyxYX.$$

Example 4.1 shows that our guess is almost correct; the corrected statement is:

Lemma 4.2. *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then $W_{p/q\oplus r/s}$ is the word $W_{p/q}W_{r/s}$ with the sign of the $(q+s)$th exponent swapped.*

Proof. The situation is diagrammed in Figure 6 for convenience. To simplify notation, in this proof we write $h(i)$ for the height $(\frac{p}{q} \oplus \frac{r}{s})i$. Observe that $h(i)$ is integral only at $i = 0$ and $i = 2q + 2s$ (at both positions trivially the letters in $W_{(p/q)\oplus(r/s)}$ and $W_{p/q}W_{r/s}$ are identical) and at $i = q + s$. The lemma will follow once we check that that at the positions $i \notin \{0, 2q + 2s\}$,

$$\text{if } 0 < i \leq 2q \text{ then } (p/q)i < h(i) < \mathrm{ceil}(p/q)i \tag{4.3}$$

and

$$\text{if } 0 < i < 2s \text{ then } (r/s)i + 2q < h(i + 2p) < \mathrm{ceil}[(r/s)i + 2p] : \tag{4.4}$$

indeed, these inequalities show that at every integral horizontal distance the height of the line corresponding to $W_{p/q\oplus r/s}$ is meeting the same vertical line segment as the line corresponding to $W_{p/q}$ or $W_{r/s}$, and so the letter chosen is the same except at $i = q + s$ since at this position the height of the line of slope $(p/q) \oplus (r/s)$, being integral, is rounded up to $h(i) + 1$ while the height of the line of slope r/s is non-integral so is rounded up to the integer $h(i)$.

Observe now that the inequalities Equations (4.3) and (4.4) are equivalent to the following: there is no integer between $(p/q)i$ and $(p/q \oplus r/s)i$ (exclusive) if $0 < i \leq 2q$, and there is no integer between $(r/s)i + 2q$ and $h(i + 2p)$ if $0 < i < 2s$. But these follow from Lemma 3.1. Indeed, the lemma shows that no integer lies between p/q and $(p/q) \oplus (r/s)$; suppose a/b is a rational between $(p/q)i$ and $h(i)$, then $a = i(\lambda p + \mu(p + r))$ and $b = (\lambda q + \mu(q + s))$ for some positive λ, μ; suppose $a/b \in \mathbb{Z}$, so $\lambda q + \mu(q + s)$ divides $i(\lambda p + \mu(p + r))$. By the case $i = 1$, $\lambda p + \mu(p + r)$ and $\lambda q + \mu(q + s)$ are coprime, so $\lambda q + \mu(q + s)$ divides i; but $\lambda q + \mu(q + s) \geq 2q$. The case of $(r/s)i + 2q$ and $h(i + 2p)$ is proved in a similar way. $\square$

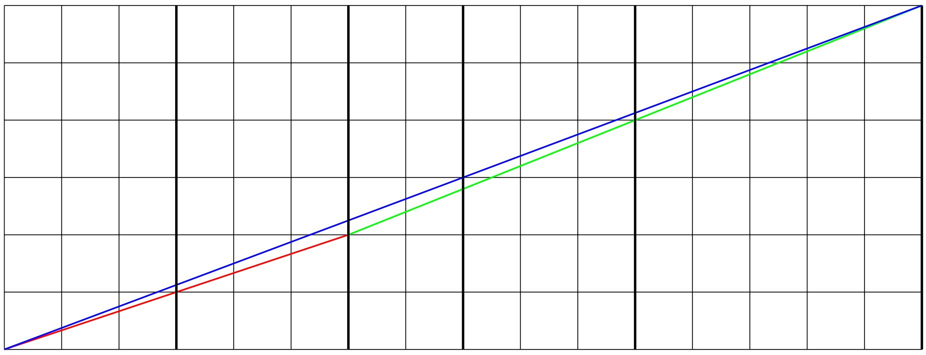


FIGURE 6. Farey addition versus word multiplication for $W_{p/q}$ (red) and $W_{r/s}$ (green).

Lemma 4.2 is not new; for instance, it appears as Propostion 4.2.4 of [Zha10]. However, the following consequences for the Farey polynomials are comparatively nontrivial to prove and have not appeared before in the literature.

Lemma 4.5 (Product lemma). *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identity holds:*

$$\operatorname{tr} W_{p/q}W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} = \begin{cases} 2+\alpha^2+\frac{1}{\alpha^2} & \text{if } q+s \text{ is even,} \\ \alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta} & \text{if } q+s \text{ is odd.} \end{cases}$$

Proof. Trace is invariant under cyclic permutations, thus (applying Lemma 4.2) we can write

$$\operatorname{tr} W_{p/q}W_{r/s} = \operatorname{tr} AB \text{ and } \operatorname{tr} W_{p/q\oplus r/s} = \operatorname{tr} AB^{-1},$$

where B is the $(q+s)$th letter of $W_{p/q}W_{r/s}$ and A is the remainder of the word but with the final letters cycled to the front. Now we know that $\operatorname{tr} AB = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1}$ (see the useful list of trace identites found in Section 3.4 of [MR03]), so it suffices to check that $\operatorname{tr} A \operatorname{tr} B = 2+\alpha^2+\frac{1}{\alpha^2}$ if $q+s$ is even and $\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}$ otherwise.

Case I. $q+s$ is even. Observe now that B is $X^{\pm 1}$ if $q+s$ is even; then $\operatorname{tr} B = 2\Re\alpha$. The identity to show is therefore $\operatorname{tr} A = (2+\alpha^2+\frac{1}{\alpha^2})/(2\Re\alpha)$; recalling that $|\alpha| = 1$ and using the double angle formulae we have

$$\frac{2}{2\Re\alpha}+\frac{\alpha^2}{2\Re\alpha}+\frac{\alpha^{-2}}{2\Re\alpha} = \frac{1}{\cos\theta}+\left(\alpha-\frac{1}{2\cos\theta}\right)+\left(\overline{\alpha}-\frac{1}{2\cos\theta}\right) = 2\cos\theta$$

where $\theta = \operatorname{Arg}\alpha$. Thus we actually just need to show $\operatorname{tr} A = 2\cos\theta$. As an aside, this shows that A is parabolic if X is, and is elliptic if X is.

Case II. $q+s$ is odd. In this case, B is $Y^{\pm 1}$ and so $\operatorname{tr} B = 2\Re\beta$; we therefore wish to show that $\operatorname{tr} A = (\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta})/(2\Re\beta)$; again using trigonometry we may simplify the right side,

$$\frac{\alpha\beta}{2\Re\alpha}+\frac{\alpha/\beta}{2\Re\alpha}+\frac{\beta/\alpha}{2\Re\alpha}+\frac{1/(\alpha\beta)}{2\Re\beta} = 2\cos\theta$$

and so again we need only show that $\operatorname{tr} A = 2\cos\theta$ where $\theta = \operatorname{Arg}\alpha$.

Both cases then reduce to the identity $\operatorname{tr} A = \operatorname{tr} X$. It will be enough to show that A is conjugate to X; by construction of A, this is equivalent to showing that in $W_{p/q\oplus r/s}$ the $(q+s+1)$th to $(2q+2s-1)$th letters are obtained from the first $q+s-1$ letters by reversing the order and swapping the case. But this is just Lemma 2.2. ∎

In the case that X and Y are parabolics and $\alpha = \beta = 1$, the two formulae unify to become:

$$\operatorname{tr} W_{p/q} W_{r/s} = 4 - \operatorname{tr} W_{p/q \oplus r/s}.$$

We may similarly prove the following 'quotient lemma':

Lemma 4.6 (Quotient lemma)**.** *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identity holds:*

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \ominus r/s} = \begin{cases} 2 + \beta^2 + \frac{1}{\beta^2} & \text{if } q - s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q - s \text{ is odd.} \end{cases}$$

Proof. We begin by setting up notation. By Lemma 2.2 we may write $W_{p/q} = UAuX$ with $A = X^{\pm 1}$ if q is even and $A = Y^{\pm 1}$ if q is odd; similarly, write $W_{r/s} = VBvX$ with B one of $X^{\pm 1}$ or $Y^{\pm 1}$. Then

$$W_{p/q} W_{r/s}^{-1} = UAuXxVbv = UAuVbv;$$

by Lemma 4.2, we have also that $W_{r/s} W_{p/q \ominus r/s}$ is $W_{p/q}$ with the sign of the exponent of the qth letter swapped; explicitly,

$$W_{p/q \ominus r/s} VBvX = UauX \implies W_{p/q \ominus r/s} = UauXxVbv = UauVbv.$$

Our goal is therefore to compute $\operatorname{tr} UAuVbv + \operatorname{tr} UauVbv$; performing a cyclic permutation again, this is equivalent to $\operatorname{tr} A(uVbvU) + \operatorname{tr} a(uVbvU)$. In this form, this becomes

$$\operatorname{tr} A(uVbvU) + \operatorname{tr} a(uVbvU) = \operatorname{tr} A \operatorname{tr} uVbvU = \operatorname{tr} A \operatorname{tr} b.$$

Consider now the cases for the product $\operatorname{tr} A \operatorname{tr} b$:

	q odd	q even
s odd	$\operatorname{tr}^2 Y$	$\operatorname{tr} X \operatorname{tr} Y$
s even	$\operatorname{tr} X \operatorname{tr} Y$	$\operatorname{tr}^2 X$.

If $p/q, r/s$ are Farey neighbours then it is not possible for both q and s to be even since $ps - rq \equiv 1 \pmod 2$. Further, $q - s$ is odd iff exactly one of p and q is odd, otherwise $q - s$ is even. Thus we see that if $q - s$ is even then

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \ominus r/s} = \operatorname{tr}^2 Y = (\beta + 1/\beta)^2$$

and if $q - s$ is odd then

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \ominus r/s} = \operatorname{tr} X \operatorname{tr} Y = (\alpha + 1/\alpha)(\beta + 1/\beta)$$

which are the claimed formulae. □

Using the product and quotient lemmata, we can prove the desired recursion formula for the trace polynomials.

Theorem 4.7 (Recursion formulae)**.** *Let p/q and r/s be Farey neighbours. If $q+s$ is even, then*

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q \oplus r/s} + \Phi_{p/q \ominus r/s} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2. \tag{4.8}$$

Otherwise if $q + s$ is odd, then

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q \oplus r/s} + \Phi_{p/q \ominus r/s} = 2\left(\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right). \tag{4.9}$$

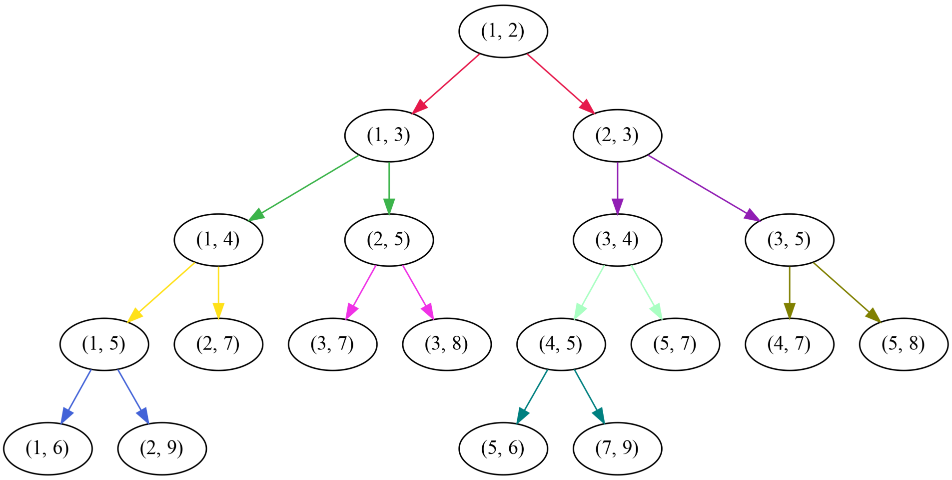


FIGURE 7. The induced colouring of the Stern-Brocot tree.

Proof. Suppose $q+s$ is even; then $q-s$ is also even, so

$$\begin{aligned}\Phi_{p/q}\Phi_{r/s} &+ \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} \\ &= \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q \oplus r/s} + \operatorname{tr} W_{p/q \ominus r/s} \\ &= \operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \oplus r/s} + \operatorname{tr} W_{p/q \ominus r/s} \\ &= 2 + \alpha^2 + \frac{1}{\alpha^2} + 2 + \beta^2 + \frac{1}{\beta^2}\end{aligned}$$

where in the final step we used Lemma 4.5 and Lemma 4.6. Similarly, when $q-s$ is odd then $q+s$ is also odd and

$$\begin{aligned}\Phi_{p/q}\Phi_{r/s} &+ \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} \\ &= \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q \oplus r/s} + \operatorname{tr} W_{p/q \ominus r/s} \\ &= \operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \oplus r/s} + \operatorname{tr} W_{p/q \ominus r/s} \\ &= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\end{aligned}$$

as desired. □

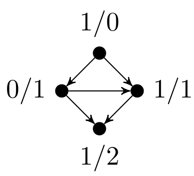
Again in the parabolic case the two formulae unify and the recursion identity becomes

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} = 8. \tag{4.10}$$

We observe as an aside that if we just draw the edges of the Farey graph corresponding to Farey neighbours which appear as products in some iterate of the recursion, then we obtain a nice colouring of the Stern-Brocot tree (Figure 7).

We also make the following useful convention/definition:

Definition 4.11. Observe that $0/1$ and $1/0$ are Farey neighbours in $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Thus, applying Equation (4.8) *formally* to the diamond



we obtain

$$\Phi_{0/1}\Phi_{1/1} + \Phi_{1/0} + \Phi_{1/2} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2;$$

substituting for $\Phi_{1/1}$, $\Phi_{1/2}$, and $\Phi_{0/1}$ from Table 2 we get the following expression for $\Phi_{1/0}$, which we henceforth take to be a definition:

$$\begin{aligned}\Phi_{1/0} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z\right)\left(\alpha\beta + \frac{1}{\alpha\beta} + z\right) - 2 \\ - \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z - z^2 \\ = 2.\end{aligned}$$

Observe that $\Phi_{1/0}^{-1}((-\infty, -2]) = \emptyset$, so this is compatible with the Keen–Series theory; it is also a polynomial of degree q (here, $q = 0$) with constant term 2, which all agrees with the properties of the higher-degree polynomials. On the other hand, it is not monic!

We also define $\Phi_{p/q}$ for all $p/q \in \mathbb{Q}$ via this method.

5. Some properties of the recurrence

Recall that the **Chebyshev polynomials** (of the first kind) are the family of polynomials T_n defined via the recurrence relation

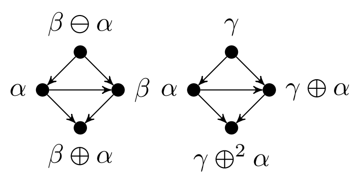
$$\begin{aligned}T_0(x) &= 1\\ T_1(x) &= x\\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x).\end{aligned}$$

It is well-known that these polynomials satisfy the product relation

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$$

for $m, n \in \mathbb{Z}_{\geq 0}$. Compare this relation with the relation Equation (4.10) developed above for the parabolic Farey polynomials (but note that the Chebyshev product rule holds for all m, n and the identities for the Farey polynomials hold only for Farey neighbours).

We may apply the theory of 'Farey recursive functions' [Che+20; Che20] in order to explain this analogy. The following diagram may be useful for translating the notation of that paper (right) into the notation we use here (left):



Definition 5.1 (Definition 3.1 of [Che+20])**.** Let R be a (commutative) ring, and suppose $d_1, d_2 : \hat{\mathbb{Q}} \to R$. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is a (d_1, d_2)**-Farey recursive function** if, whenever $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours,

$$\mathcal{F}(\beta \oplus \alpha) = -d_1(\alpha)\mathcal{F}(\beta \ominus \alpha) + d_2(\alpha)\mathcal{F}(\beta). \tag{5.2}$$

Observe that the relation Equation (4.10) looks essentially of this form; to make this clearer, we rewrite it slightly as

$$\Phi(\beta \oplus \alpha) = 8 - \Phi(\beta \ominus \alpha) - \Phi(\alpha)\Phi(\beta) \tag{5.3}$$

(where we set $\beta = p/q$, $\alpha = r/s$, and changed from subscript notation to functional notation). In our case, then, $d_1(\alpha)$ is constantly 1 and $d_2 = -\Phi$. Note also that our relation is not homogeneous. We therefore adapt the definition of [Che+20] to the following:

Definition 5.4. Let R be a (commutative) ring, and suppose $d_1, d_2, d_3 : \hat{\mathbb{Q}} \to R$. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is a (d_1, d_2)**-Farey recursive function** if, whenever $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours,

$$\mathcal{F}(\beta \oplus \alpha) = -d_1(\alpha)\mathcal{F}(\beta \ominus \alpha) + d_2(\alpha)\mathcal{F}(\beta) + d_3(\alpha). \tag{5.5}$$

If d_3 is the zero function, we say that the relation of Equation (5.5) is **homogeneous**, otherwise it is **non-homogeneous**.

The relevant generalisations of the existence-uniqueness results of [Che+20, Section 4] follow easily (the same proofs work, with the usual property that the space of non-homegenous solutions is the sum of a particular solution and the space of homogeneous solutions).

In our case, there is an obvious explicit solution to the non-homogeneous Farey polynomial recursion, Equation (5.3): namely, the map Φ which sends every $\alpha \in \mathbb{Q}$ to the constant polynomial $2 \in \mathbb{Z}[z]$. It therefore remains to solve the corresponding homogeneous equation.

5.1. A Fibonacci-like subsequence of the homogeneous Farey polynomials.

Notation. In this section, we work exclusively with the parabolic Farey polynomials, $\Phi^{\infty,\infty}_{p/q}$. We will reuse the symbols α and β for rational numbers (since the roots of unity used to define X and Y become 1 in this situation and so there is no need for special notation for them).

In Table 4, we list the first few **homogeneous Farey polynomials** for a particular set of seed values: that is, the polynomials Φ^h which solve the homogeneous recursion relation

$$\Phi^h(\beta \oplus \alpha) = -\Phi^h(\beta \ominus \alpha) - \Phi^h(\alpha)\Phi^h(\beta) \tag{5.6}$$

with the initial values $\Phi^h(0/1) = 2 - z$, $\Phi^h(1/0) = 2$, and $\Phi^h(1/1) = 2 + z$.

The polynomials with numerator 1 listed in the table have very nice properties: immediately one sees that the constant terms alternate in sign and increase in magnitude by 4 each time; also, we have that $\Phi^h_{1/q}(1)$ cycles through the values $3, -5, 2$, $\Phi^h_{1/q}(2)$ cycles through the values $4, -2, -4, 2$; and when we evaluate at 3 and 4 we get a 6-cycle and an arithmetic sequence of step 4 respectively. When we consider $\Phi^h(1/q)(5)$, though, we obtain more interesting behaviour: this is OEIS sequence A100545[3] and satisfies the Fibonacci-type relation

$$\Phi^h_{1/q}(5) = 3\Phi^h_{1/(q-1)}(5) - \Phi^h_{1/(q-2)}(5) \quad \text{ with } \Phi^h_{1/1}(5) = 7, \Phi^h_{1/2}(5) = 19.$$

Of course, from the way that we defined the Φ^h such types of relations ought to be expected. In this section, we use the standard diagonalisation technique to explain the behaviour of the sequence $a_q := \Phi^h(1/q)(z)$ for fixed $z \in \mathbb{C}$. From Equation (5.6), we have that

$$a_q = -(2 - z)a_{q-1} - a_{q-2}. \tag{5.7}$$

We may rewrite this equation in matrix form as the following:

$$\begin{bmatrix} 0 & 1 \\ -1 & z-2 \end{bmatrix} \begin{bmatrix} a_{q-2} \\ a_{q-1} \end{bmatrix} = \begin{bmatrix} a_{q-1} \\ a_q \end{bmatrix}. \tag{5.8}$$

[3] `http://oeis.org/A100545`

One easily computes that the eigenvalues of the transition matrix are

$$\lambda^{\pm} = \frac{z-2\pm\alpha}{2}$$

(where $\alpha = \sqrt{z^2-4z}$) with respective eigenvectors

$$v^{\pm} = \begin{bmatrix} z-2\mp\alpha \\ 2 \end{bmatrix}$$

(note the alternated sign). Thus the transition matrix may be diagonalised as

$$\frac{-1}{2\alpha}\begin{bmatrix} z-2-\alpha & z-2+\alpha \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{z-2+\alpha}{2} & 0 \\ 0 & \frac{z-2-\alpha}{2} \end{bmatrix}\begin{bmatrix} 2 & 2-z-\alpha \\ -2 & z-2-\alpha \end{bmatrix} \tag{5.9}$$

and so a_q is the first coordinate of

$$\frac{-1}{2\alpha}\begin{bmatrix} z-2-\alpha & z-2+\alpha \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \left(\frac{z-2+\alpha}{2}\right)^q & 0 \\ 0 & \left(\frac{z-2-\alpha}{2}\right)^q \end{bmatrix}\begin{bmatrix} 2 & 2-z-\alpha \\ -2 & z-2-\alpha \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix};$$

expanding out, we get

$$\begin{aligned} a_q = 2^{-1-q}\alpha^{-1}\Bigg(& \left(a_0\left(z-2+\alpha\right)-2a_1\right)\left(z-2-\alpha\right)^q \\ & + \left(a_0\left(2-z+\alpha\right)+2a_1\right)\left(z-2+\alpha\right)^q \Bigg). \end{aligned} \tag{5.10}$$

We may also characterise the z for which $\Phi^h_{1/q}(z)$ is cyclic: this occurs precisely when the diagonal matrix of Equation (5.9) is of finite order, i.e. whenever both $(z-2\pm\alpha)/2$ are roots of unity.

As an application of the theory above, we have seen that the Chebyshev polynomials also satisfy a second-order recurrence relation with transition matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix}.$$

If we set $z = 2x+2$, then we get back our transition matrix from Equation (5.8). Thus our sequence $\Phi^h_{1/q}(z)$ for fixed z is of the form $W_q(\frac{z-2}{2})$ where W_q is the qth Chebyshev polynomial in the sequence beginning with $W_0 = 2x$ and $W_1 = 2x+4$.

Finally, we consider the solution of the non-homogeneous equation for $\Phi_{1/q}$. Above, we observed that there is a constant solution to the global recursion relation on the entire Stern-Brocot tree; we therefore guess that there is a similar solution to this recursion. Such a solution f will satisfy

$$8 = f(z) + (2-z)f(z) + f(z)$$

and arithmetic gives $f(z) = 8/(4-z)$. Combining this with Equation (5.10) above gives us the following general solution to the non-homogeneous relation:

$$\begin{aligned} a_q = \frac{8}{4-z} + 2^{-1-q}\alpha^{-1}\Bigg(& (\lambda(z-2+\alpha)-2\mu)(z-2-\alpha)^q \\ & + (\lambda(2-z+\alpha)+2\mu)(z-2+\alpha)^q \Bigg). \end{aligned}$$

In our case, we have $a_0 = \Phi_{1/0}(z) = 2$ and $a_1 = \Phi_{1/1}(z) = 2+z$. Solving the resulting system of equations gives

$$(\lambda,\mu) = \left(\frac{2z}{z-4}, \frac{2z-z^2}{z-4}\right)$$

and hence

$$a_q = \frac{8}{4-z} + \frac{2^{-q}z}{z-4}\left((-2+z-\sqrt{z^2-4z})^q + (-2+z+\sqrt{z^2-4z})^q\right)$$

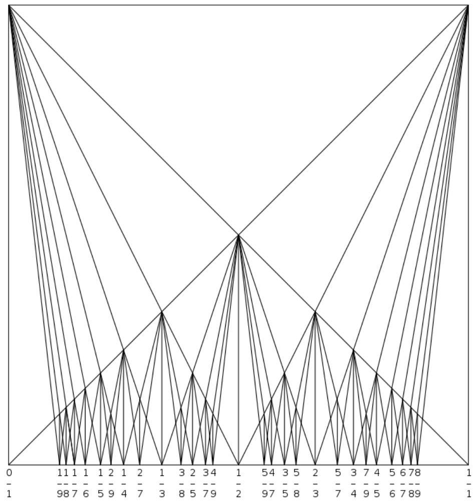


FIGURE 8. An embedding of the Farey graph into $\mathbb{H}^2$. (By Hyacinth — Own work, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=63343697`.)

5.2. **Solving the homogeneous recursion relation in general.** In the previous section, we computed a closed form formula for $\Phi^h_{1/q}(n)$ using standard techniques from the theory of second-order linear recurrences. We now tackle the general problem of finding a closed-form formula for $\Phi^h_{p/q}(n)$; in order to do this, we use the theory of Section 6 of [Che+20] but with a slight modification: in that paper, the authors define a special case of Farey recursive function, a **Farey recursive function of determinant** d (where $d : \hat{\mathbb{Q}} \to R$), to be a Farey recursive function $\mathcal{F}$ with $d_1 = d$ and $d_2 = \mathcal{F}$. That is, they replace the recurrence of Equation (5.2) with

$$\mathcal{F}(\beta \oplus \alpha) = -d(\alpha)\mathcal{F}(\beta \ominus \alpha) + \mathcal{F}(\alpha)\mathcal{F}(\beta).$$

This is very similar to our situation, except that instead of $d_2 = \mathcal{F}$ we have $d_2 = -\mathcal{F}$. To reflect this, we shall call a Farey recursive function satisfying a relation of the form

$$\mathcal{F}(\beta \oplus \alpha) = -d(\alpha)\mathcal{F}(\beta \ominus \alpha) - \mathcal{F}(\alpha)\mathcal{F}(\beta). \tag{5.11}$$

a Farey recursive function of ***anti*-determinant** d. We shall work for the time being in this setting (i.e. we shall work with the general function $\mathcal{F}$ rather than the particular example Φ) in order to restate in sufficient generality the theorem which we need (Theorem 6.1 of [Che+20]).

Let $\alpha \in \mathbb{Q}$. The **boundary sequence** $\partial(\alpha)$ is defined inductively by the process of 'continuing to expand down the Farey graph by constant steps'. More precisely, let $\beta \oplus^k \gamma$ denote $((\beta \underbrace{\oplus \gamma) \oplus \cdots) \oplus \gamma}_{k \text{ iterates}}$ for $\beta, \gamma \in \hat{\mathbb{Q}}$ and let γ_L, γ_R be the unique Farey neighbours such that $\alpha = \gamma_L \oplus \gamma_R$; then we set

$$\partial(\alpha) := \{\gamma_L \oplus^k \alpha : k \in \mathbb{Z}_{\geq 0}\} \cup \{\gamma_R \oplus^k \alpha : k \in \mathbb{Z}_{\geq 0}\}.$$

If we allow the Farey graph to embed in the Euclidean upper halfplane by sending $\mathbb{Q} \ni p/q \mapsto (p/q, 1/q) \in \mathbb{H}^2$, then except for the exceptional cases $\alpha = 1/0$ and $\alpha = n/1$ for $n \in \mathbb{Z}$ the subgraph spanned by $\partial(\alpha)$ corresponds to a Euclidean triangle containing α in its interior, see Figure 8 and Figure 3 of [Che+20]; for example, the triangle spanned by $\partial(1/n)$ is the triangle with vertices $0, (1/2, 1/2), 1$.

It will be useful to have specific names for the terms in each of the two subsequences and so we set, for $k \in \mathbb{Z}$,

$$\beta_k := \begin{cases} \gamma_L \oplus^{-k-1} \alpha & \text{if } k < -1 \\ \gamma_L & \text{if } k = -1 \\ \gamma_R & \text{if } k = 0 \\ \gamma_R \oplus^k \alpha & \text{if } k > 0. \end{cases} \tag{5.12}$$

For every $\alpha \in \mathbb{Q}$, define

$$M_\alpha = \begin{bmatrix} 0 & 1 \\ -d(\alpha) & \mathcal{F}(\alpha) \end{bmatrix}. \tag{5.13}$$

Given any Farey neighbour β of α, we have

$$M_\alpha^n \begin{bmatrix} \mathcal{F}(\gamma \oplus^0 \alpha) \\ \mathcal{F}(\gamma \oplus^1 \alpha) \end{bmatrix} = \begin{bmatrix} \mathcal{F}(\gamma \oplus^n \alpha) \\ \mathcal{F}(\gamma \oplus^{n+1} \alpha) \end{bmatrix}$$

and so the recursion Equation (5.11) is equivalent to a family of second-order linear recurrences, one down $\partial(\alpha)$ for each α.

We may now state the following theorem:

Theorem 5.14 (Adaptation of Theorem 6.1 of [Che+20])**.** *Let $d : \hat{\mathbb{Q}} \to R$ be a multiplicative function (in the sense that $d(\gamma \oplus \beta) = d(\gamma)d(\beta)$ for all pairs of Farey neighbours $\beta, \gamma \in \mathbb{Q}$) to a commutative ring R, such that $d(\hat{\mathbb{Q}})$ contains no zero divisors. Suppose that $\mathcal{F}$ is a Farey recursive function with anti-determinant d. Given $\alpha \in \mathbb{Q}$, define M_α as in Equation* (5.13) *and $(\beta_k)_{k \in \mathbb{Z}}$ as in Equation* (5.12)*.*

Then, for all $n \in \mathbb{Z}$,

$$M_\alpha^n \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathcal{F}(\beta_n) \\ \mathcal{F}(\beta_{n+1}) \end{bmatrix} & n \geq 0, \\ \begin{bmatrix} \frac{1}{d(\beta_{-1})} \mathcal{F}(\beta_{-1}) \\ \mathcal{F}(\beta_0) \end{bmatrix} & n = -1, \text{ and} \\ \begin{bmatrix} \frac{1}{d(\beta_{-1}) d_\alpha^{-n-1}} \mathcal{F}_{\beta_n} \\ \frac{1}{d(\beta_{-1}) d_\alpha^{-n-2}} \mathcal{F}_{\beta_{n+1}} \end{bmatrix} & n < -1. \end{cases}$$

We proceed to prove Theorem 5.14 by exactly the same argument as given in [Che+20]. The key point is the following lemma, which is the analogue of the discussion directly preceeding the statement of Theorem 6.1 in that paper.

Lemma 5.15. *With the setup of Theorem 5.14, we have*

$$M_\alpha^{-1} \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{bmatrix} \frac{d(\beta_0)}{d(\alpha)} \mathcal{F}(\beta_{-1}) \\ \mathcal{F}(\beta_0) \end{bmatrix}$$

$$M_\alpha^{-2} \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{bmatrix} \frac{1}{d(\alpha) d(\beta_{-1})} \mathcal{F}(\beta_{-2}) \\ \frac{1}{d(\beta_{-1})} \mathcal{F}(\beta_{-1}) \end{bmatrix}$$

Proof. The formula involving M_α^{-1} comes directly from computing the product on the left via the definition and simplifying with the formula

$$\mathcal{F}(\beta_1) = -d(\beta_0)\mathcal{F}(\beta_{-1}) - \mathcal{F}(\alpha)\mathcal{F}(\beta_0)$$

which is almost exactly the same as Equation (8) of [Che+20] — the single sign change cancels exactly with the sign change between the 'determinant' and 'anti-determinant' recurrences so we get the same overall formula for the M_α^{-1} product as they do in Equation (11) of their paper.

The formula for M_α^{-2} comes from applying the analogues of Equations (9) and (10) of their paper,

$$\mathcal{F}(\beta_{-2}) = -d(\beta_{-1})\mathcal{F}(\beta_0) - \mathcal{F}(\alpha)\mathcal{F}(\beta_{-1})$$
$$d(\alpha) = d(\beta_{-1})d(\beta_0)$$

and simplifying; again the minus signs cancel and we get the same formula. □

Proof of Theorem 5.14. The formula for $n \geq 0$ holds for all Farey recursive formulae as noted above; the formulae for $n = -1$ and $n = -2$ are just the formulae of Lemma 5.15; and we proceed to prove the formula for $n < -2$ by induction. Assume that the formula holds for some fixed $n \leq -2$; then from the definitions we have

$$\mathcal{F}(\beta_{n-1}) = -\mathcal{F}(\alpha)\mathcal{F}(\beta_n) - d(\alpha)\mathcal{F}(\beta_{n+1})$$

and so we can compute

$$\begin{aligned} M_\alpha^{n-1}\begin{bmatrix} F(\beta_0) \\ F(\beta_1) \end{bmatrix} &= M_\alpha^{-1}M_\alpha^{n}\begin{bmatrix} F(\beta_0) \\ F(\beta_1) \end{bmatrix} \\ &= \frac{1}{d(\alpha)}\begin{bmatrix} -\mathcal{F}(\alpha) & -1 \\ d(\alpha) & -0 \end{bmatrix}\begin{bmatrix} \frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}F(\beta_n) \\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-2}}F(\beta_{n+1}) \end{bmatrix} \\ &= \frac{1}{d(\alpha)}\begin{bmatrix} -\frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}\left(\mathcal{F}(\alpha)\mathcal{F}(\beta_n) + d(\alpha)\mathcal{F}(\beta_{n+1})\right) \\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-2}}\mathcal{F}(\beta_n) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{d(\beta_{-1})d(\alpha)^{-n}}\mathcal{F}(\beta_{n-1}) \\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}\mathcal{F}(\beta_n) \end{bmatrix} \end{aligned}$$

which is the desired result. □

Corollary 5.16 (Adaptation of Corollary 6.2 of [Che+20])**.** *Let Φ^h be a family of homogeneous Farey polynomials (i.e. a family solving Equation* (5.6) *for some starting values). Then, for some $\alpha \in \mathbb{Z}$, if M_α is the matrix*

$$\begin{bmatrix} 0 & 1 \\ -1 & \Phi^h(\alpha) \end{bmatrix}$$

and if $(\beta_k)_{k\in\mathbb{Z}}$ are the boundary values about α as in Equation (5.12)*, then for all $n \in \mathbb{Z}$ we have*

$$M_\alpha^n \begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix} = \begin{bmatrix} \Phi^h(\beta_n) \\ \Phi^h(\beta_{n+1}) \end{bmatrix}.$$

Proof. This follows directly from Theorem 5.14 with the observation that the anti-determinant of Φ^h is the constant function $d(\gamma) = 1$ for all $\gamma \in \mathbb{Q}$. □

Thus to determine Φ^h_α for all $\alpha \in \mathbb{Q}$ it suffices to compute and diagonalise the M_α matrices, using the techniques of Section 5.1. (Of course, we need to diagonalise in the ring of rational functions over $\mathbb{Q}$ rather than the ring of polynomials over $\mathbb{Z}$.) More precisely, we need to compute M_{α_i} for some family (α_i) of rationals with the property that the boundary sets $\partial(\alpha_i)$ cover $\mathbb{Q}$. (In Section 5.1, we did this computation for $\partial(0/1)$.)

In any case, from Corollary 5.16 we immediately have a qualitative result:

Theorem 5.17. *For any $\gamma \in \mathbb{Q}$, there exists a sequence $\ldots, \gamma_{-1}, \gamma_0 = \gamma, \gamma_1, \gamma_2, \ldots$ of rational numbers such that $\Phi^h_{\gamma_n}(z)$ is a sequence of Chebyshev polynomials $W_n(\Phi^h_\gamma(z)/2)$. (Namely, let γ_{-1} be a neighbour in the Stern-Brocot tree of γ and take the sequence (γ_k) to be precisely the sequence (β_k) of Equation* (5.12) *with $\alpha := \gamma \ominus \gamma_{-1}$.)* □

Remark. Of course, the boundary sequence (γ_k) constructed here is just a geodesic line Λ in the Stern-Brocot tree rooted at γ, defined by choosing one vertical half-ray in the tree starting from γ (where 'vertical' refers to the embedding of Figure 7) and then extending that in the Farey graph in the obvious way by repeated Farey arithmetic with the same difference. There are clearly two such natural choices for Λ given a fixed γ (γ has three neighbours, but two correspond to the same geodesic), and a single natural choice is obtained by taking the unique neighbour of γ which lies above.

We easily compute that the eigenvalues of M_α are

$$\lambda^{\pm} = \frac{1}{2}\left(\Phi^h_\alpha \pm \sqrt{(\Phi^h_\alpha)^2 - 4}\right).$$

Let $x = \Phi^h_\alpha$ and $\kappa = \sqrt{x^2-4}$ (this is the analogue of the constant α from Section 5.1); then the respective eigenvectors are

$$v^{\pm} = \begin{bmatrix} x \mp \kappa \\ 2 \end{bmatrix}.$$

We therefore may diagonalise M_α as

$$M_\alpha = -\frac{1}{4\kappa}\begin{bmatrix} x-\kappa & x+\kappa \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}(x+\kappa) & 0 \\ 0 & \frac{1}{2}(x-\kappa) \end{bmatrix}\begin{bmatrix} 2 & -x-\kappa \\ -2 & x-\kappa \end{bmatrix};$$

in particular, $\Phi^h(\beta_n)$ is the first component of

$$\begin{aligned} &M_\alpha^n \begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix} \\ &\quad = -\frac{1}{4\kappa}\begin{bmatrix} x-\kappa & x+\kappa \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2^n}(x+\kappa)^n & 0 \\ 0 & \frac{1}{2^n}(x-\kappa)^n \end{bmatrix}\begin{bmatrix} 2 & -x-\kappa \\ -2 & x-\kappa \end{bmatrix}\begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix} \end{aligned}$$

computing this, we have

$$\Phi^h(\beta_n) = \frac{\begin{array}{c}(\Phi^h(\beta_0)(x+\kappa) - 2\Phi^h(\beta_1))(x-\kappa)^n \\ +\,(\Phi^h(\beta_0)(\kappa - x) + 2\Phi^h(\beta_1))(x+\kappa)^n\end{array}}{2^{1+n}\kappa}.$$

In particular, we have proved the following quantitative improvement of Theorem 5.17:

Theorem 5.18. *Let β_0 and β_1 be Farey neighbours, and let $\alpha = \beta_1 \ominus \beta_0$. Then we have a closed form formula for $\Phi^h(\beta_n)$ ($n \in \mathbb{Z}$), namely*

$$\Phi^h_{\beta_n} = \frac{\left(\Phi^h_{\beta_0}\left(\Phi^h_\alpha + \kappa\right) - 2\Phi^h_{\beta_1}\right)\left(\Phi^h_\alpha - \kappa\right)^n + \left(\Phi^h_{\beta_0}\left(\kappa - \Phi^h_\alpha\right) + 2\Phi^h_{\beta_1}\right)\left(\Phi^h_\alpha + \kappa\right)^n}{2^{1+n}\kappa}.$$

where $\kappa = \sqrt{\left(\Phi^h_\alpha\right)^2 - 4}$. □

This gives a 'local' closed form solution for the recursion around any $\alpha \in \mathbb{Q}$; a 'global' solution corresponds to a collection of these solutions, each local to a particular geodesic in the graph and which are compatible on intersections. Unfortunately, our original recurrence relied on knowning only three initial values globally in the graph; while this local formula relies on knowing three initial values which are local on the particular geodesic.

6. Approximating irrational pleating rays

As we mentioned in the introduction to [EMS21], a version of this theory can be used to give approximations to irrational pleating rays. In order to do this, we must deal with the theory of infinite continued fractions.

It is well-known that every irrational $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ has a unique simple continued fraction approximation of the form

$$\lambda = [a_1, a_2, \dots, a_n, \dots] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}$$

which can be computed efficiently by repeated application of the Euclidean algorithm (see, for example, §10.9 of [HW60]). We now show that this exhibits λ as a limit of a sequence of rationals 'down the Farey tree'.

We first recall a standard fact from classical number theory.

Proposition 6.1 ([HW60, Theorem 162])**.** *Every rational number can be expressed as a finite simple continued fraction in exactly two ways, one with an even and one with an odd number of convergents (number of sequence elements a_n). These are of the form*

$$[a_1, \dots, a_{N-1}, a_N, 1] \text{ and } [a_1, \dots, a_{N-1}, a_N + 1]$$

respectively, for some N. □

Proposition 6.2. *Suppose that $p/q = [a_1, \dots, a_{N-1}, a_N, 1]$; define*

$$\frac{r_1}{s_1} = [a_1, \dots, a_{N-1}, a_N] \text{ and } \frac{r_2}{s_2} = [a_1, \dots, a_{N-1}].$$

Then r_1/s_1 and r_2/s_2 are Farey neighbours and $p/q = (r_1/s_1) \oplus (r_2/s_2)$.

Proof. That the Farey sum is as claimed follows from Theorem 149 of [HW60]: that is, if $p_n/q_n = [a_1, \dots, a_n]$ then

$$p_n = a_n p_{n-1} + p_{n-2} \text{ and } q_n = a_n q_{n-1} + q_{n-2}.$$

Indeed, take $p/q = p_{N+1}/q_{N+1} = [a_1, \dots, a_{N-1}, a_N, 1]$, then $a_{N+1} = 1$ so $p_{N+1} = 1p_N + p_{N-1}$ and $q_{N+1} = q_N + q_{N-1}$.

That the two are Farey neighbours is exactly Theorem 150 of [HW60], which actually gives slightly more information:

$$p_N q_{N-1} - p_{N-1} q_N = (-1)^{N-1}.$$ □

In the previous section, we indicated how to compute in closed form the sequence of Farey polynomials corresponding to the Farey fractions

$$\frac{p_1}{q_1}, \frac{p_2}{q_2} = \frac{p_1}{q_1} \oplus \left(\frac{p_2}{q_2} \ominus \frac{p_1}{q_1}\right), \dots, \frac{p_n}{q_n} = \frac{p_1}{q_1} \oplus^{n-1} \left(\frac{p_2}{q_2} \ominus \frac{p_1}{q_1}\right)$$

where p_1/q_1 and p_2/q_2 are Farey neighbours. That is, we gave a way to compute the Farey polynomials down a branch of the Farey tree with constant difference (for instance, we gave the example of $\Phi_{1/q}$, where $\frac{1}{q} = \frac{1}{0} \oplus^q \frac{0}{1}$). The study of partial fraction decompositions here gives, in general, different sequences: the constant addition sequence rooted at an element ξ in the tree is the sequence which constantly chooses the *left* branch when moving down from ξ (with respect to the embedding of Figure 7), while the sequence corresponding to continually adding the previous two items in the tree (and therefore building a continued fraction decomposition) corresponds to the sequence which is eventually constantly moving *rightwards*. For this reason, we might (rather foolishly, as it turns out) expect there to also exist a nice way to compute closed-form expressions for sequences of Farey polynomials corresponding to finite convergents of infinite continued fraction decompositions, and therefore for there to be a reasonable way to approximate irrational pleating rays and attempt to compute expressions for the analytic functions of which they are subsets of zero-sets.

6.1. **Examples, and an intriguing dynamical system.** The recursion relation of interest comes, as always, from applying the Farey polynomial operator to the recursion relation $a_n = a_{n-1} \oplus a_{n-2}$; doing this we obtain

$$\Phi_{a_n} = \Phi_{a_{n-1} \oplus a_{n-2}} = 8 - \Phi_{a_{n-1} \ominus a_{n-2}} - \Phi_{a_{n-1}} \Phi_{a_{n-2}}.$$

Observe that $a_{n-1} \ominus a_{n-2}$ is just a_{n-3} and replace the cumbersome notation Φ_{a_k} with x_k to get the relation

$$x_n = 8 - x_{n-3} - x_{n-2} x_{n-1}. \tag{6.3}$$

Unfortunately, this is a non-linear recurrence relation — this means that it is harder to study than the linear left-recursion studied in the previous section, and we are unable to find a closed-form solution. (We believe that if one exists then it will be in terms of combinatorial objects, e.g. Stirling numbers.) However, it does at least give a computationally feasible method for approximating irrational cusp points.

Example 6.4 (The Fibonacci polynomials)**.** Let us compute an approximation to the pleating ray with asymptotic angle π/ϕ, where ϕ is the golden ratio $(\sqrt{5}+1)/2$. It is well-known that

$$\phi = [1, 1, ...] = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\ddots}}}$$

and so by Proposition 6.2 we get that $1/\phi$ is approximated by

$$[0, 1] = 1, [0, 1, 1] = 1/2, [1, 1, 1] = 2/3, ... ;$$

in other words, by the Fibonacci fractions $\mathsf{fib}(q-1)/\mathsf{fib}(q)$. Being efficiency-minded, we have already computed the corresponding polynomials, which are listed in Table 3; the inverse images of -2 under the first 16 such polynomials are shown in Figure 9, where the $1/\pi$-cusp is approximated by the 'corner' point, at the top-left of each picture. Quite a good approximation seems to be given after only a few pictures, which is good since after about 15 terms the polynomial coefficients and degrees become too big for both Mathematica and MATLAB to study without hassle. We also note that eventually the cusp point occurs at quite a 'sharp' point in the set; perhaps it would be interesting to use this to distinguish the cusp computationally from the other points in the inverse image.

Example 6.5 ($\sqrt{2}$)**.** For our next trick, we consider another irrational number with a nice continued fraction decomposition:

$$\sqrt{2} = [1, 2, 2, ... , 2, ...]$$

so we may approximate $1/\sqrt{2}$ by the Farey sequence beginning

$$1, 2/3, 5/7, 7/10, 12/17, ...$$

The corresponding Farey polynomials are higher degree and so longer to write down than the Fibonacci polynomials, but it is easy enough for the computer to plot the preimages of -2 for the first 10 or so polynomials; we give the preimage for $\Phi_{19/27}$ in Figure 10.

Having seen some examples, we consider the dynamical system a little more closely. Define the transition map $f : k^3 \to k^3$ by $f(x^1, x^2, x^3) = (x^2, x^3, 8 - x^1 -$

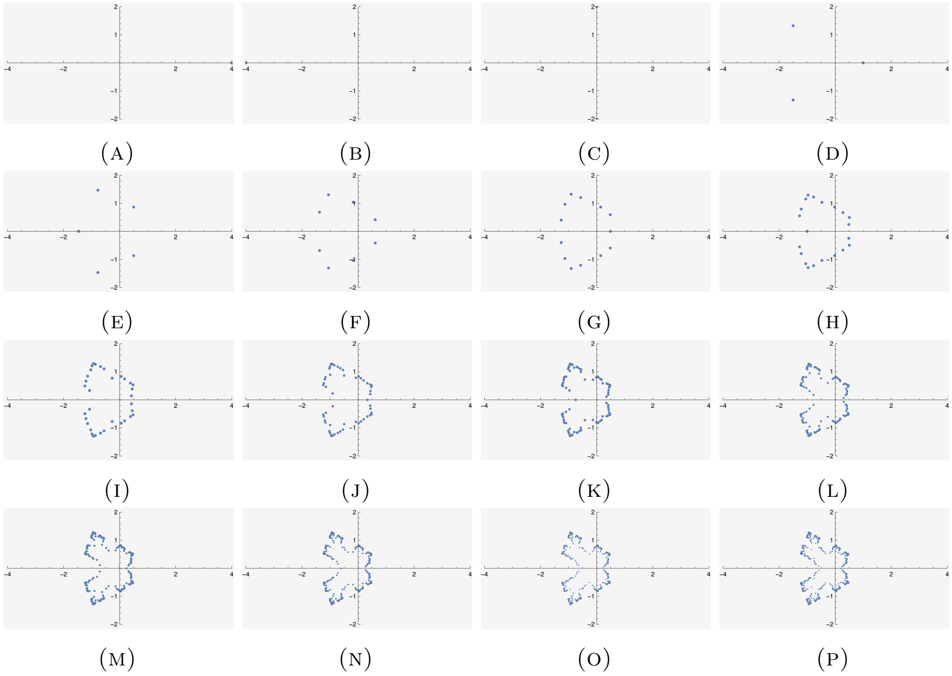


FIGURE 9. The zeros of $\Phi_{\mathsf{fib}(p-1)/\mathsf{fib}(p)} + 2$ for $p \in \{1, \dots, 16\}$.

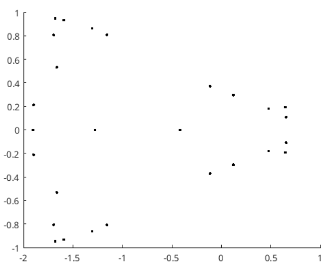


FIGURE 10. $\left(\Phi^{\infty,\infty}_{19/27}\right)^{-1}(-2)$, approximating the $1/\sqrt{2}$ cusp point.

x^2x^3). It is easy to check that f has exactly two fixed points, $(-4,-4,-4)$ and $(2,2,2)$, and that its differential is given by

$$D_{x^1,x^2,x^3} f(\xi^1,\xi^2,\xi^3) = f(x^1,x^2,x^3) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -x^2 & -x^3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{bmatrix};$$

at the two fixed points the Jacobian has eigenproperties and determinants as in Table 5. We also give three slices through the phase diagram for this system in Figure 11.

By replacing each x_i with $y_i := x_i - 2$ (since 2 is a fixed point), we may replace Equation (6.3) with the homogeneous relation

$$y_i = -y_{i-3} - y_{i-2}y_{i-1} - 2(y_{i-2} + y_{i-1}); \tag{6.6}$$

with the initial conditions $(y_1, y_2, y_3) = (-z, z, z^2)$, the result is a sequence of **reduced Farey polynomials** $\phi_{p/q} := \Phi^{\infty,\infty}_{p/q} - 2$ which we studied in [EMS21] (in that paper, these polynomials were called $Q_{p/q}$). We were surprised to observe,

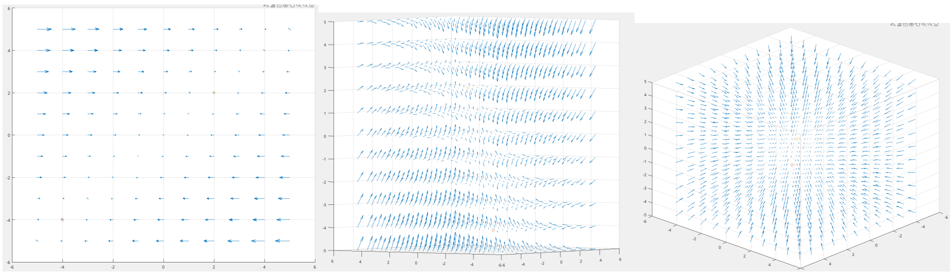


FIGURE 11. The real dynamics of the 'right-recursion' system.

through computing a few examples, that the alternating sum of the coefficients of these polynomials seem to be squares (Table 6)!

More precisely, we believe the following.

Conjecture 6.7. *There exist functions $\epsilon : \hat{\mathbb{Q}} \to \{-1, +1\}$, $k : \hat{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$, $R : \hat{\mathbb{Q}} \to \mathbb{Z}[z]$ such that*

(1) *if $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours, then $\epsilon(\alpha \oplus \beta) = \epsilon(\alpha)\epsilon(\beta)$;*
(2) *if $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours, then $k(\alpha \oplus \beta) = k(\alpha) + k(\beta)$;*
(3) *if $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours, then either $R(\alpha\oplus\beta) = R(\alpha)R(\beta)+R(\alpha\ominus\beta)$ or $R(\alpha\oplus\beta) = R(\alpha)R(\beta)-R(\alpha\ominus\beta)$ depending on the position of $\alpha\oplus\beta$ in $\hat{\mathbb{Q}}$ (e.g. in the case of the geodesic $\mathsf{fib}(p)/\mathsf{fib}(p+1)$, the rule with a minus sign occurs iff $\mathsf{fib}(p)$ is even; the 'bad points' where this minus sign rule occurs are shown in red in Figure 12).*
(4) *for all $\alpha \in \hat{\mathbb{Q}}$, the reduced Farey polynomial ϕ_α is of the form*
$$(-1)^{\epsilon(\alpha)} z^{k(\alpha)} R(\alpha)^2.$$

The functions ϵ, k, and R are defined by (1)–(3), given the initial conditions in the following table:

α	$\epsilon(\alpha)$	$k(\alpha)$	$R(\alpha)$
$0/1$	-1	$+1$	1
$1/1$	$+1$	$+1$	1
$1/2$	$+1$	0	z

This conjecture can probably be proved by an application of Farey induction applied to Equation (6.6), but really the interest to us is the following observation: our initial recurrence relation is uniform on $\hat{\mathbb{Q}}$ (that is, the recurrence rule is the same no matter the position in the Farey graph), but from it there naturally arises a recurrence rule which depends on some *arity* structure in the graph (i.e. the recurrence relation in (3) of Conjecture 6.7). The reader may recall that this is the *second* such recurrence relation which has appeared in this paper (the elliptic Farey polynomial recurrence also had parity).

We end this paper with a few questions which we would very much like answers to.

Questions 6.8.

(1) Do other arity-dependent recurrence relations on the Farey graph occur in nature?
(2) Is there an analogue of the theory of Farey recursive functions with constant difference that is applicable to the elliptic Farey polynomial recurrence?
(3) Is there a geometric proof for Theorem 4.7 or Conjecture 6.7?
(4) As a stepping stone to (3), is there a geometric reason (i.e. in terms of geodesics on the marked sphere) for arity to matter for the elliptic Farey

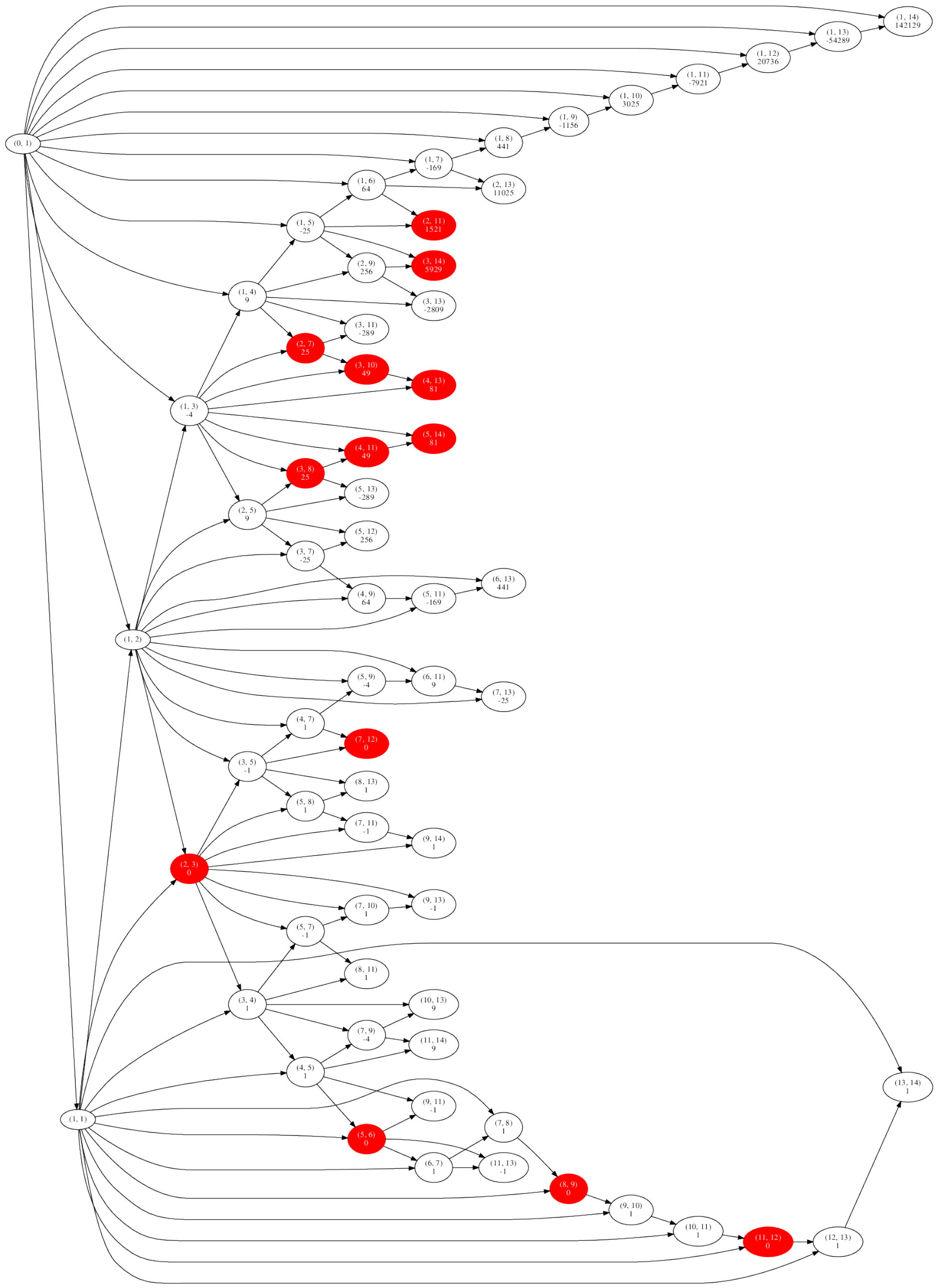


FIGURE 12. Points in the Farey graph coloured according to which rule of Conjecture 6.7(iii) is followed at that point. In red are points $\alpha \oplus \beta$ such that $R(\alpha \oplus \beta) \neq R(\alpha)R(\beta) + R(\alpha \ominus \beta)$; at these points the rule $R(\alpha \oplus \beta) = R(\alpha)R(\beta) - R(\alpha \ominus \beta)$ holds. Warning: at some points, $0 = R(\alpha \ominus \beta)$ and so both rules hold! The full picture may be viewed at the URL `https://aelzenaar.github.io/farey/badpoints.ps`.

polynomial recurrence? That is, take two adjacent triangles in the Farey graph; what is it about the arrangements of the corresponding geodesics which causes their lengths to depend on each other by either the odd relation or the even relation? (Perhaps a starting point would be to learn the language of Jørgensen's theory, as described in [Aki+07].)

(5) Is it possible to find a number-theoretic interpretation of the Farey polynomials or the reduced Farey polynomials, like the number-theoretic interpretations of the Riley polynomials [Bow98; Che20]?
(6) Is there a way of determining computationally which roots of $\Phi^{a,b}_{p/q} + 2$ correspond to the cusp points? Computation shows that the shape of the sets $\left(\Phi^{\infty,\infty}_{p/q}\right)^{-1}(-2)$ as $q \to \infty$ seems to become quite regularly shaped (possibly with polygonal convex hull), with two extremal points corresponding to the cusps (see Figure 9 and Figure 10). If this can be made precise and shown to hold in general, then it would provide such a method for finding cusps.
(7) Can one give closed forms for the factors of $\phi^{\infty,\infty}_{p/q} = \Phi^{a,b}_{p/q} - 2$?
(8) How quickly does the sequence of Farey polynomials coming from the continued fraction decomposition approximate the irrational pleating ray?

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TABLE 1. Farey words Word(p/q) for small q.

p/q	Word(p/q)
0/1	yX
1/1	YX
1/2	$yxYX$
1/3	$yXYxYX$
2/3	$yxyXYX$
1/4	$yXyxYxYX$
3/4	$yxyxYXYX$
1/5	$yXyXYxYxYX$
2/5	$yXYxyXyxYX$
3/5	$yxYXYxyXYX$
4/5	$yxyxyXYXYX$
1/6	$yXyXyxYxYxYX$
5/6	$yxyxyxYXYXYX$
1/7	$yXyXyXYxYxYxYX$
2/7	$yXyxYxyXyXYxYX$
3/7	$yXYxyXYxYXyxYX$
4/7	$yxYXyxyXYxyXYX$
5/7	$yxyXYXYxyxYXYX$
6/7	$yxyxyxyXYXYXYX$
1/8	$yXyXyXyxYxYxYxYX$
3/8	$yXYxYXyxYxyXyxYX$
5/8	$yxYXYxyxYXyxyXYX$
7/8	$yxyxyxyxYXYXYXYX$
1/9	$yXyXyXyXYxYxYxYxYX$
2/9	$yXyXYxYxyXyXyxYxYX$
4/9	$yXYxyXYxyXyxYXyxYX$
5/9	$yxYXyxYXYxyXYxyXYX$
7/9	$yxyxYXYXYxyxyXYXYX$
8/9	$yxyxyxyxyXYXYXYXYX$
1/10	$yXyXyXyXyxYxYxYxYxYX$
3/10	$yXyxYxyXyxYxYXyXYxYX$
7/10	$yxyXYXyxyxYXYxyxYXYX$
9/10	$yxyxyxyxyxYXYXYXYXYX$
1/11	$yXyXyXyXyXYxYxYxYxYxYX$
2/11	$yXyXyxYxYxyXyXyXYxYxYX$
3/11	$yXyxYxYXyXYxYxyXyXYxYX$
4/11	$yXYxYXyXYxyXyxYxyXyxYX$
5/11	$yXYxyXYxyXYxYXyxYXyxYX$
6/11	$yxYXyxYXyxyXYxyXYxyXYX$
7/11	$yxYXYxyxYXYxyXYXyxyXYX$
8/11	$yxyXYXYxyxyXYXyxyxYXYX$
9/11	$yxyxyXYXYXYxyxyxYXYXYX$
10/11	$yxyxyxyxyxyXYXYXYXYXYX$
1/12	$yXyXyXyXyXyxYxYxYxYxYxYX$
5/12	$yXYxyXyxYXyxYxyXYxYXyxYX$
7/12	$yxYXyxyXYxyxYXyxYXYxyXYX$
11/12	$yxyxyxyxyxyxYXYXYXYXYXYX$

TABLE 2. Farey polynomials $\Phi^{a,b}_{p/q}(z)$ for small q.

p/q	
$0/1$	$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}-z$
$1/1$	$\alpha\beta+\frac{1}{\alpha\beta}+z$
$1/2$	$2+\left(\alpha\beta-\frac{\alpha}{\beta}-\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}\right)z+z^2$
$1/3$	$\dfrac{1}{\alpha\beta}+\alpha\beta+\left(3-\dfrac{1}{\alpha^2}-\alpha^2-\dfrac{1}{\beta^2}-\beta^2+\dfrac{\alpha^2}{\beta^2}+\dfrac{\beta^2}{\alpha^2}\right)z$
	$+\left(\alpha\beta-2\dfrac{\alpha}{\beta}-2\dfrac{\beta}{\alpha}+\dfrac{1}{\alpha\beta}\right)z^2+z^3$
$2/3$	$\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}+\left(-3+\alpha^2+\dfrac{1}{\alpha^2}-\dfrac{1}{\alpha^2\beta^2}-\alpha^2\beta^2+\beta^2+\dfrac{1}{\beta^2}\right)z$
	$+\left(-2\alpha\beta-\dfrac{2}{\alpha\beta}+\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right)z^2-z^3$
$1/4$	$2+\left(\dfrac{\alpha}{\beta^3}-\dfrac{\alpha^3}{\beta^3}+\dfrac{2}{\alpha\beta}-3\dfrac{\alpha}{\beta}+\dfrac{\alpha^3}{\beta}+\dfrac{\beta}{\alpha^3}-3\dfrac{\beta}{\alpha}+2\alpha\beta-\dfrac{\beta^3}{\alpha^3}+\dfrac{\beta^3}{\alpha}\right)z$
	$+\left(6-\dfrac{2}{\alpha^2}-2\alpha^2-\dfrac{2}{\beta^2}+3\dfrac{\alpha^2}{\beta^2}-2\beta^2+3\dfrac{\beta^2}{\alpha^2}\right)z^2$
	$+\left(\dfrac{1}{\alpha\beta}-3\dfrac{\alpha}{\beta}-3\dfrac{\beta}{\alpha}+\alpha\beta\right)z^3+z^4$
$3/4$	$2+\left(\dfrac{1}{\alpha^3\beta^3}-\dfrac{1}{\alpha\beta^3}-\dfrac{1}{\alpha^3\beta}+\dfrac{3}{\alpha\beta}-2\dfrac{\alpha}{\beta}-2\dfrac{\beta}{\alpha}+3\alpha\beta-\alpha^3\beta-\alpha\beta^3+\alpha^3\beta^3\right)z$
	$+\left(6-\dfrac{2}{\alpha^2}-2\alpha^2-\dfrac{2}{\beta^2}+\dfrac{3}{\alpha^2\beta^2}-2\beta^2+3\alpha^2\beta^2\right)z^2$
	$+\left(\dfrac{3}{\alpha\beta}-\dfrac{\alpha}{\beta}-\dfrac{\beta}{\alpha}+3\alpha\beta\right)z^3+z^4$

TABLE 3. Farey polynomials of the form $\Phi^{\infty,\infty}_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}(z)$ for small q.

$\frac{\mathsf{fib}(q-1)}{\mathsf{fib}(q)}$	
$0/1$	$2 - z$
$1/1$	$2 + z$
$1/2$	$2 + z^2$
$2/3$	$2 - z - 2z^2 - z^3$
$3/5$	$2 + z + 2z^2 + 3z^3 + 2z^4 + z^5$
$5/8$	$2 + 4z^4 + 8z^5 + 8z^6 + 4z^7 + z^8$
$8/13$	$2 - z - 2z^2 - 5z^3 - 12z^4 - 22z^5 - 32z^6 - 44z^7 - 54z^8$
	$- 53z^9 - 38z^{10} - 19z^{11} - 6z^{12} - z^{13}$
$13/21$	$2 + z + 2z^2 + 7z^3 + 14z^4 + 31z^5 + 64z^6 + 124z^7 + 214z^8$
	$+ 339z^9 + 498z^{10} + 699z^{11} + 936z^{12} + 1148z^{13} + 1216z^{14}$
	$+ 1064z^{15} + 746z^{16} + 409z^{17} + 170z^{18} + 51z^{19} + 10z^{20} + z^{21}$
$21/34$	$2 + z^2 + 8z^4 + 24z^5 + 68z^6 + 192z^7 + 516z^8 + 1256z^9 + 2834z^{10}$
	$+ 5912z^{11} + 11460z^{12} + 20816z^{13} + 35598z^{14} + 57248z^{15}$
	$+ 86446z^{16} + 122560z^{17} + 163199z^{18} + 203952z^{19} + 238564z^{20}$
	$+ 259704z^{21} + 260686z^{22} + 238320z^{23} + 195694z^{24} + 142328z^{25}$
	$+ 90451z^{26} + 49552z^{27} + 23058z^{28} + 8952z^{29} + 2831z^{30} + 704z^{31}$
	$+ 130z^{32} + 16z^{33} + z^{34}$
$34/55$	$2 - z - 4z^2 - 10z^3 - 34z^4 - 103z^5 - 286z^6 - 791z^7$
	$- 2078z^8 - 5221z^9 - 12680z^{10} - 29824z^{11} - 67872z^{12}$
	$- 149896z^{13} - 321800z^{14} - 671896z^{15} - 1364228z^{16}$
	$- 2692102z^{17} - 5158232z^{18} - 9587668z^{19} - 17273444z^{20}$
	$- 30141702z^{21} - 50903644z^{22} - 83138942z^{23} - 131230688z^{24}$
	$- 200056876z^{25} - 294348624z^{26} - 417663240z^{27} - 571010576z^{28}$
	$- 751328456z^{29} - 950188464z^{30} - 1153232920z^{31} - 1340813030z^{32}$
	$- 1490107333z^{33} - 1578696308z^{34} - 1589182962z^{35} - 1513960786z^{36}$
	$- 1358696535z^{37} - 1142850158z^{38} - 896137319z^{39} - 651440922z^{40}$
	$- 436582355z^{41} - 268228504z^{42} - 150207744z^{43} - 76207672z^{44}$
	$- 34797892z^{45} - 14193584z^{46} - 5125756z^{47} - 1621110z^{48}$
	$- 442809z^{49} - 102556z^{50} - 19630z^{51} - 2990z^{52}$
	$- 341z^{53} - 26z^{54} - z^{55}$

TABLE 4. Selected homogeneous Farey polynomials Φ^h of slope p/q for small q, with the initial values as given.

p	q	$\Phi^h_{p/q}$
1	0	2
0	1	$-z+2$
1	1	$z+2$
1	2	z^2-6
1	3	$z^3-2z^2-7z+10$
2	3	$-z^3-2z^2+7z+10$
1	4	$z^4-4z^3-4z^2+24z-14$
3	4	$z^4+4z^3-4z^2-24z-14$
1	5	$z^5-6z^4+3z^3+34z^2-55z+18$
2	5	$-z^5+2z^4+13z^3-22z^2-41z+58$
3	5	$z^5+2z^4-13z^3-22z^2+41z+58$
4	5	$-z^5-6z^4-3z^3+34z^2+55z+18$
1	6	$z^6-8z^5+14z^4+32z^3-119z^2+104z-22$
1	7	$z^7-10z^6+29z^5+10z^4-186z^3+308z^2-175z+26$
1	8	$z^8-12z^7+48z^6-40z^5-220z^4+648z^3-672z^2+272z-30$
1	9	$z^9-14z^8+71z^7-126z^6-169z^5+1078z^4-1782z^3+1308z^2-399z+34$
1	10	$z^{10}-16z^9+98z^8-256z^7+35z^6+1456z^5-3718z^4+4224z^3-2343z^2+560z-38$

TABLE 5. Eigenproperties of the right-recurrence relation.

fixed point	eigenvalue	eigenvector
-4	$\frac{5+\sqrt{21}}{2}$	$\left(-\frac{-5+\sqrt{21}}{5+\sqrt{21}}, \frac{2}{5+\sqrt{21}}, 1\right)^t$
	-1	$(1,-1,1)^t$
	$\frac{5-\sqrt{21}}{2}$	$\left(-\frac{5+\sqrt{21}}{-5+\sqrt{21}}, -\frac{2}{-5+\sqrt{21}}, 1\right)^t$
	determinant:	-1
2	$\frac{-1+i\sqrt{3}}{2}$	$\left(\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}, 1\right)^t$
	-1	$(1,-1,1)^t$
	$\frac{-1-i\sqrt{3}}{2}$	$\left(\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}, 1\right)^t$
	determinant:	-1

TABLE 6. $\sqrt{\left|\phi_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}(-1)\right|}$ for small q.

q	$\sqrt{\left\|\phi_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}(-1)\right\|}$
1	1
2	1
3	1
4	0
5	1
6	1
7	1
8	2
9	3
10	5
11	17
12	88
13	1491
14	131225
15	195656563
16	25675032478184
17	5023488609594854052817
18	1289782332051352621319118557319008891
19	6479206853916657743711391370771009189061832501316908881763
20	835676652588773441434735633165055341787601058478245210146 00110051618636610092033658769403650

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