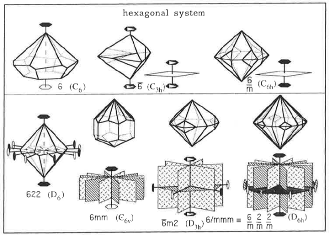
Genus two Heegaard splittings and Teichmüller spaces of Kleinian groups

Alex Elzenaar

Monash University, Melbourne, Australia

23 May 2025



D. Schwarzenbach, *Crystallography*, p. 51. Wiley, 1996.

- **Elliptic integral of the first kind:**

$$f(\omega) = \int_0^{\omega} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

- **Euler (1768):** if $a, b \in \mathbb{R}$ then there exists (an explicit) $c \in \mathbb{R}$ such that $f(a) + f(b) = f(c)$.
- **Gauss, Abel, Jacobi (c.1800–1820):** the inverse f^{-1} is single-valued and doubly periodic on $\mathbb{C}$.
- Hence f^{-1} is a holomorphic function defined on a torus.

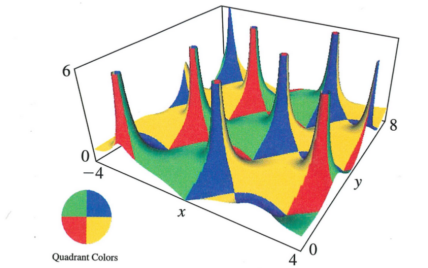
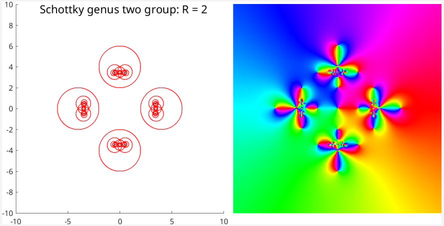


Figure 22.3.24: $\mathrm{sn}\,(x + iy, k)$ for $-4 \le x \le 4$, $0 \le y \le 8$, $k = 1 + \frac{1}{2}i$. $K = 1.5149\ldots + i0.5235\ldots$, $K' = 1.4620\ldots - i0.3552\ldots$.

NIST handbook of mathematical functions, Cambridge University Press (2010), p. 552.

- **Poincaré (c.1880)** asked: Can we define holomorphic functions on higher genus Riemann surfaces?
- **Uniformisation theorem:** Yes, there are lots of them. They lift to functions on open subsets $U \subset \mathbb{C}$ which are invariant under a group of conformal automorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.



Definition

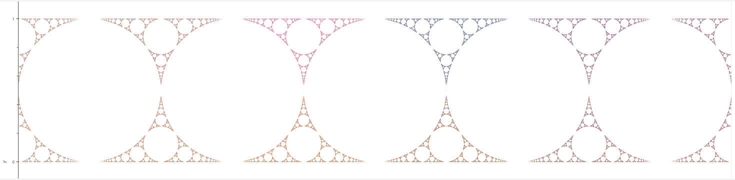
A *Kleinian group* is a discrete group of conformal automorphisms of $\hat{\mathbb{C}}$ that is not virtually Abelian.

[Virtually Abelian groups act as Euclidean or spherical isometry groups and are easy to classify.]

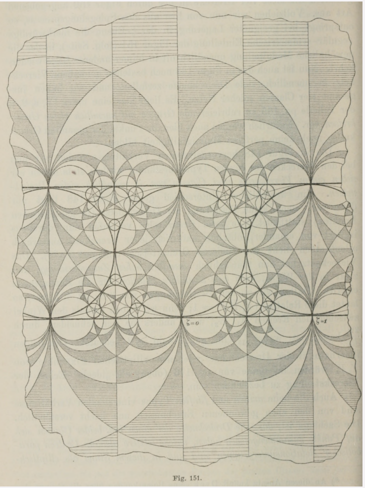
- **Bers (1960)** and **Maskit §VIII.B.3** proved: For every countable family $\{S_i\}$ of hyperbolic Riemann surfaces, there exists an open set $U \subset \hat{\mathbb{C}}$ and a Kleinian group G so that $U/G = \bigcup_i S_i$.

The tessellation shown is induced by the $(2, 3, \infty)$ triangle group (which is $\mathrm{PSL}(2, \mathbb{Z})$), normalised to act on the radius $1/2$ disc around $i/2$, and extended by adding an additional parabolic $z \mapsto z + 1$:

$$\left\langle \begin{bmatrix} -i & -1 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 1+i & 1 \\ 1 & 1-i \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$$

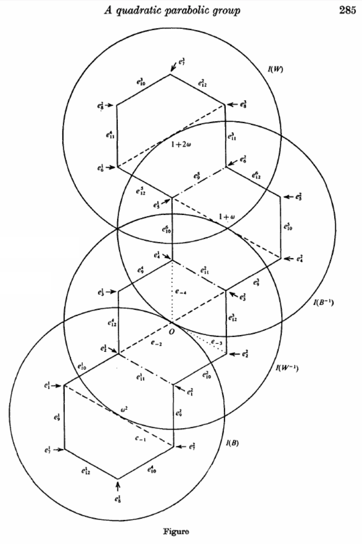


This is not a Wielenberg extension! It is an index 2 quotient of the $n = 3$ Heckoid group on the Hopf link pleating ray of $\mathcal{R}$.



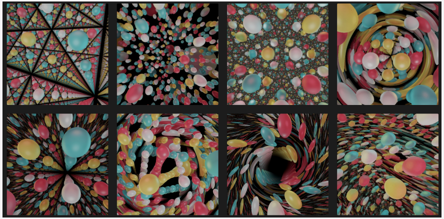
R. Fricke and F. Klein. *Vorlesungen über die Theorie der automorphen Functionen* **1**. B.G. Teubner, Leipzig (1897), p. 432.

- **State of the art of 3D topology in the 1960s:** prime decompositions; Heegaard splittings; normal surface theory; homotopy theory.
- **Marden (1974)**: Kleinian groups on $\mathbb{S}^2$ extend to discontinuous groups of hyperbolic isometries of $\mathbb{B}^3$ (observed by Poincaré). So the quotients $\mathbb{B}^3/G$ are 3-manifolds amenable to study via both complex analysis and hyperbolic geometry. How large is this family of 3-manifolds? How well behaved are its members topologically?
- **Riley (c.1975):** the complement of the figure 8 knot in the 3-sphere is a Kleinian manifold.
- In this *milieu*, **Thurston (late 1970s)**:
  - proved (c.1978) that 'most' link complements are Kleinian, and
  - conjectured that *almost every atoroidal 3-manifold is Kleinian.*



R. Riley. "A quadratic parabolic group". *Math. Proc. Camb. Phil. Soc.* **77** (1975).

- **Thurston's conjecture was resolved positively by Perelman (c.2002).** So Kleinian groups give a vast family of 3-manifolds, in fact 'most' 3-manifolds that can't be nontrivially cut along genus 1 surfaces are hyperbolic.
- **Also in the early 2000s,** Agol, Calegari, Gabai, Brock, Canary, and Minsky resolved most of the major structural conjectures about Kleinian groups (measuring how 'wild' they can be and giving families of objects that control moduli).

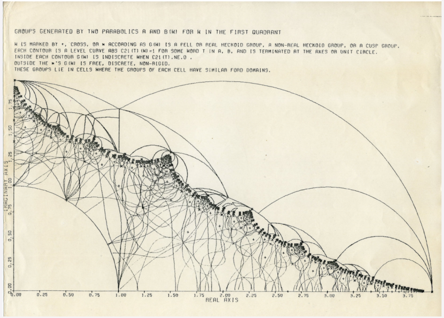


From `https://www.3-dimensional.space/`.
Resp.: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, Nil, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and Sol.

Fundamental questions

Let $X(G)$ be the $\mathrm{PSL}(2,\mathbb{C})$ character variety of G.

- How much control is there over the position of large open sets of discrete representations in $X(G)$? What is a computable family of moduli to certify membership of $\tilde{G}$ in such a locus?
- What geometric meaning do non-discrete groups in $X(G)$ have, and how much control is there over their position and distribution? Do they arise in nice families?

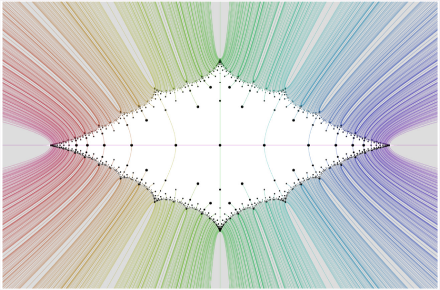


Discreteness loci in $X(F\{X,Y\})$ s.t. $\mathrm{tr}^2 X = \mathrm{tr}^2 Y = 4$.
R. Riley, unpublished (1979).

Fundamental questions

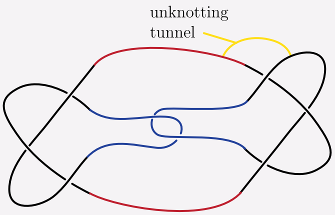
Let $X(G)$ be the $\mathrm{PSL}(2,\mathbb{C})$ character variety of G.

- How much control is there over the position of large open sets of discrete representations in $X(G)$? What is a computable family of moduli to certify membership of $\tilde{G}$ in such a locus?
- What geometric meaning do non-discrete groups in $X(G)$ have, and how much control is there over their position and distribution? Do they arise in nice families?



Discreteness loci in $X(F\{X,Y\})$ s.t. $\mathrm{tr}^2 X = \mathrm{tr}^2 Y = 4$.
Courtesy Y. Yahashita; in A.J.E., G.J. Martin, J. Schillewaert. "Approximations of the Riley slice". arXiv:2111.03230; *Expo. Math.* **41** (2023)

Let $X = X(F_2)$. Inside X there is a large open set $\mathcal{S}_2$ containing holonomy groups of genus 2 handlebodies. Also inside X lie representations of $\mathbb{S}^3$-link complements.



A.J.E. “From disc patterns in the plane to character varieties of knot groups”. arXiv:2503.13829.

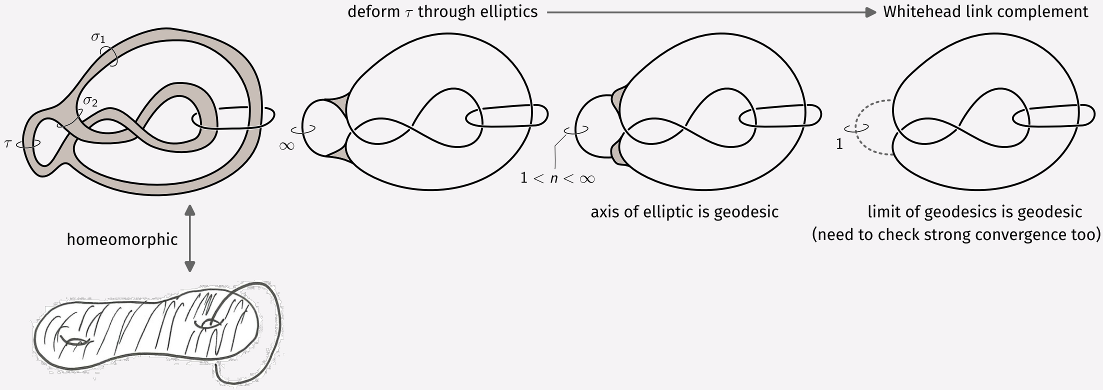
Conjecture

For every tunnel number 1 knot $\mathfrak{k}$ in $\mathbb{S}^3$, there exists a smooth path in X, through holonomy groups of cone manifolds with controlled geometry, from $\pi_1(\mathfrak{k})$ to $\partial\mathcal{S}_2$. At each point on the interior of the path the corresponding cone manifold is topologically $\mathbb{S}^3 \setminus \mathfrak{k}$, with a single (embedded) cone arc along an unknotting tunnel.

Corollary (Conjecture of Adams, c.1995)

The unknotting tunnel of a tunnel number 1 knot is isotopic, in the knot complement, to a geodesic arc.

Proof of Adams' conjecture from the cone deformation conjecture



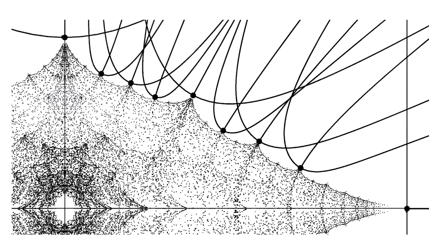
Genesis results

Theorem (E.–Martin–Schillewaert, 2021+)

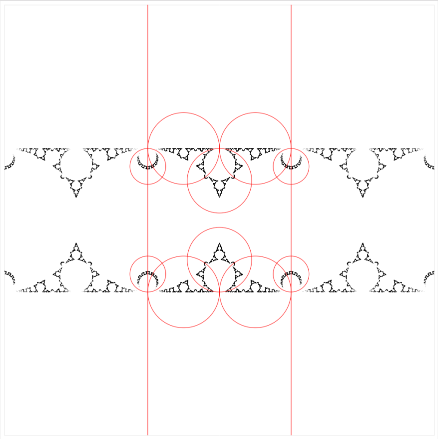
Let $X, Y \in \mathrm{PSL}(2, \mathbb{C})$ be parabolic or elliptic. If $G = \langle X, Y \rangle$:

- is discrete,
- splits as $G = \langle X \rangle * \langle Y \rangle$, and
- $\partial(\mathbb{H}^3/G)$ is a four-punctured sphere,

then there are explicit semi-algebraic regions in $X(G)$ parameterising groups with combinatorially stable fundamental domains; these regions cover the locus of free and discrete groups.



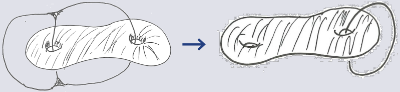
A.J.E., G.J. Martin, J. Schillewaert. "Approximations of the Riley slice". arXiv:2111.03230; *Expo. Math.* **41** (2023)



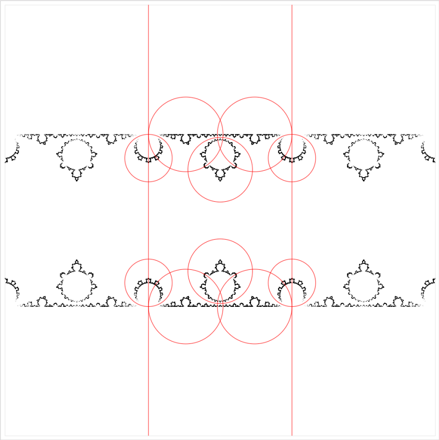
A.J.E. "Changing topological type of compression bodies through cone manifolds". arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distungshed end and an accidental parabolic rank 1 cusp g.



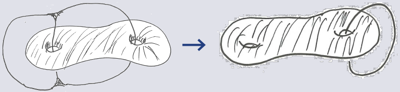
There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of 'cleaving' the rank 1 cusp by gluing a 2-handle.



A.J.E. “Changing topological type of compression bodies through cone manifolds”. arXiv:2411.17940.

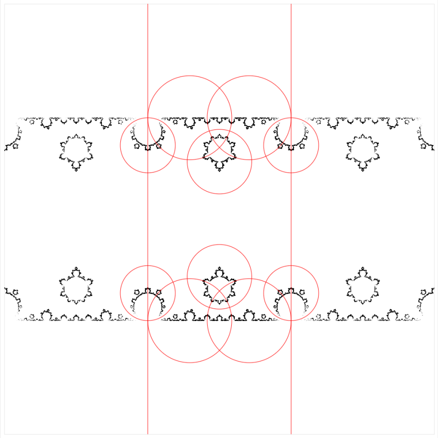
Theorem (E., 2024)

Let G be a genus 2 function group with compact distingushed end and an accidental parabolic rank 1 cusp g.



There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of ‘cleaving’ the rank 1 cusp by gluing a 2-handle.

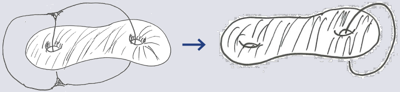
Genesis results



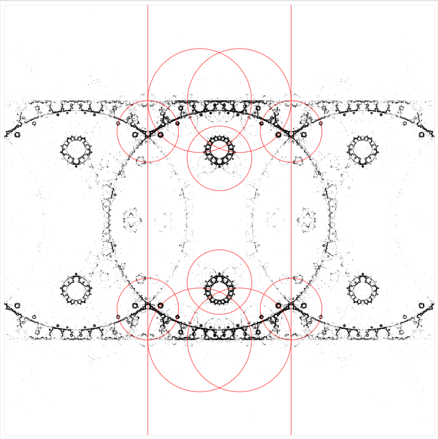
A.J.E. "Changing topological type of compression bodies through cone manifolds". arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distingushed end and an accidental parabolic rank 1 cusp g.



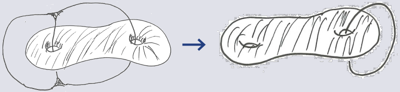
There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of 'cleaving' the rank 1 cusp by gluing a 2-handle.



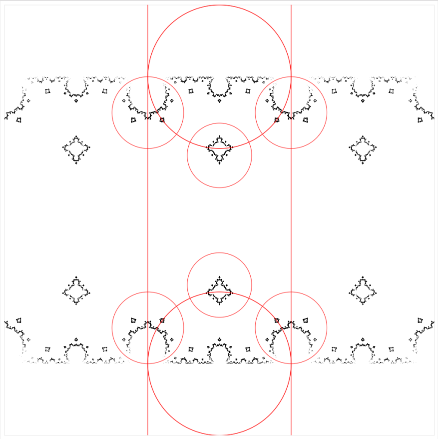
A.J.E. "Changing topological type of compression bodies through cone manifolds". arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distingushed end and an accidental parabolic rank 1 cusp g.



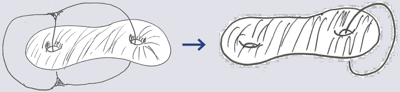
There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of 'cleaving' the rank 1 cusp by gluing a 2-handle.



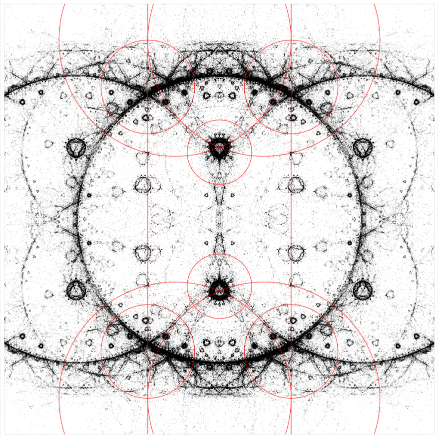
A.J.E. “Changing topological type of compression bodies through cone manifolds”. arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distingushed end and an accidental parabolic rank 1 cusp g.



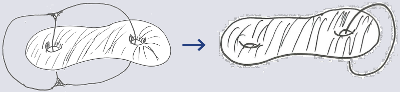
There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of ‘cleaving’ the rank 1 cusp by gluing a 2-handle.



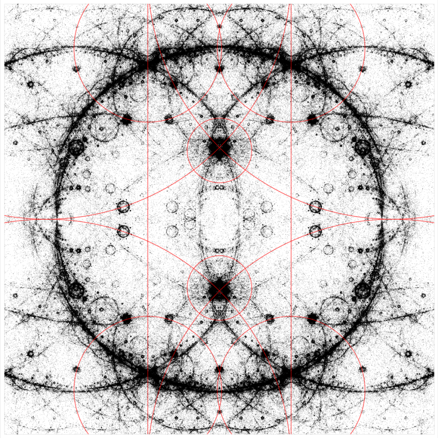
A.J.E. “Changing topological type of compression bodies through cone manifolds”. arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distinguished end and an accidental parabolic rank 1 cusp g.



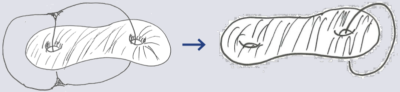
There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of ‘cleaving’ the rank 1 cusp by gluing a 2-handle.



A.J.E. "Changing topological type of compression bodies through cone manifolds". arXiv:2411.17940.

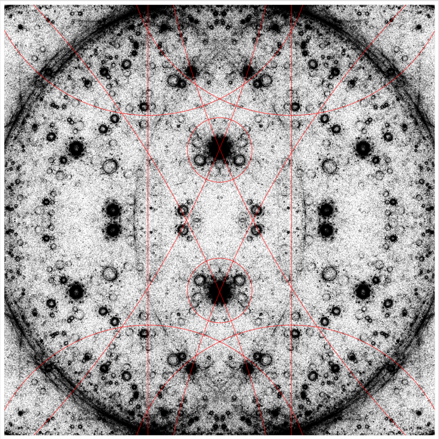
Theorem (E., 2024)

Let G be a genus 2 function group with compact distinguished end and an accidental parabolic rank 1 cusp g.



There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of 'cleaving' the rank 1 cusp by gluing a 2-handle.

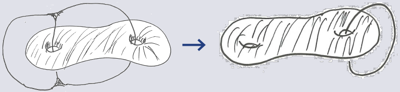
Genesis results



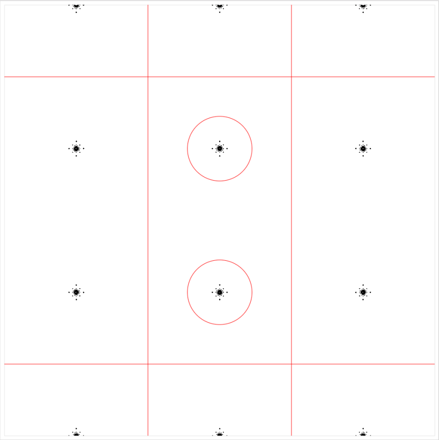
A.J.E. "Changing topological type of compression bodies through cone manifolds". arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distinguished end and an accidental parabolic rank 1 cusp g.



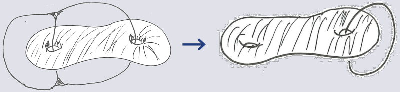
There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of 'cleaving' the rank 1 cusp by gluing a 2-handle.



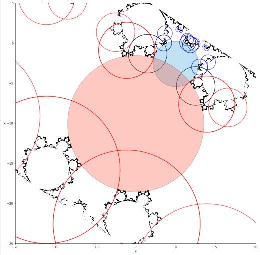
A.J.E. "Changing topological type of compression bodies through cone manifolds". arXiv:2411.17940.

Theorem (E., 2024)

Let G be a genus 2 function group with compact distinguished end and an accidental parabolic rank 1 cusp g.



There are explicit families of fundamental polyhedra in $\mathbb{H}^3$ which induce smooth families of cone manifolds deforming g through elliptic elements of decreasing holonomy angle. This is a continuous realisation of 'cleaving' the rank 1 cusp by gluing a 2-handle.



In the special case that $G \simeq F_2$ is Schottky:

Theorem (E., 2025 (to appear))

If G is discrete and geometrically finite, then there are explicit semi-algebraic regions in $X(G)$ parameterising groups with combinatorially stable fundamental domains. These regions cover the quasi-isometry locus $\mathrm{QH}(G) \subset X(G)$.

Conjecture/work in progress

Let $\mathfrak{k}$ be a hyperbolic tunnel number 1 knot with tunnel τ.

- The fundamental domains from $\overset{\text{over there}}{\Longleftarrow}$ extend to fundamental polyhedra in $\mathbb{H}^3$.
- Deforming these domains gives a smooth path of cone manifolds in $X(F_2)$ from $\mathbb{S}^3 \setminus (\mathfrak{k} \cup \tau)$ to $\mathbb{S}^3 \setminus \mathfrak{k}$. Along the path are cone manifolds homeomorphic to $\mathbb{S}^3 \setminus \mathfrak{k}$ with a singular arc along τ of continuously decreasing angle.

Five landmark works

- Henri Poincaré, *Papers on Fuchsian functions* (trans. J. Stillwell).
  Springer, 1985.
- Albert Marden, “Kleinian groups and 3-dimensional topology”.
  A crash course in Kleinian groups, Springer LNM 400, 1974.
- Robert Riley, “Seven excellent knots”.
  Low-dimensional topology, LMS LNS 48, 1982.
- William P. Thurston, “Three dimensional manifolds, Kleinian groups and hyperbolic geometry”.
  Bull. Amer. Math. Soc., 1982.
- Bernhard Maskit, *Kleinian groups*.
  Springer, 1987.