Algebraic plane curves from Schottky groups

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§1. Basic definitions

The basic definitions from the theory of Kleinian groups which we will use can be found in the textbook by Maskit [12]. We include some of the main ones here, for completeness.

A **Kleinian group** is a discrete subgroup of $\mathbb{M}$, the group of Möbius transformations on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (the one-point compactification of $\mathbb{C}$). By identifying elements of $\mathbb{M}$ with fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$ with $|ad - bc| = 1$ we obtain a natural isomorphism in the category of topological groups between $\mathbb{M}$ and $\mathrm{PSL}(2, \mathbb{C})$. The **domain of discontinuity** of a Kleinian group G is the set $\Omega(G) \subseteq \hat{\mathbb{C}}$ at which G acts discontinously, in the sense that for every $x \in \Omega(G)$ there is a neighbourhood U such that $gU \cap U \neq \varnothing$ for all but finitely many $g \in G$; the **limit set** of G is the set $\Lambda(G)$ of accumulation points of orbits of G, that is the set of $x \in \hat{\mathbb{C}}$ such that there exists a point x_0 and a sequence of distinct $g_0, g_1, \ldots \in G$ with $\lim_{n\to\infty} g_n x_0 = x$. The sets ΛG and $\Omega(G)$ partition $\hat{\mathbb{C}}$ and $\Lambda(G)$ is closed (in fact, it is the closure of the set of fixed points of the infinite-order

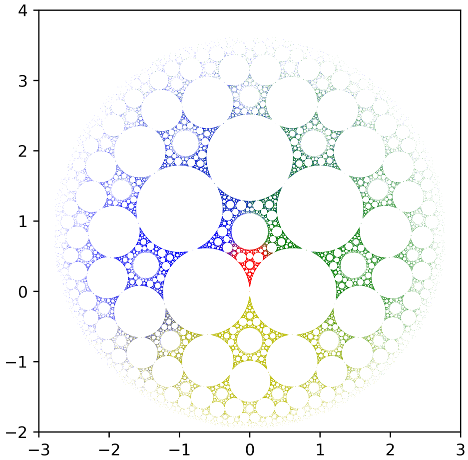


Figure 1: An intricate Kleinian group limit set.

elements of G), but beyond this the structure of the two sets can be incredibly wild; an example limit set from the authors' collection is depicted in Fig. 1.

We say G is **non-elementary** if $|\Lambda(G)| \geq 3$ (in which case $\Lambda(G)$ is actually uncountable) and in this case the quotient $\Omega(G)/G$ is a countable union of marked Riemann surfaces which each carry a hyperbolic metric. A **fundamental domain** for G is an open subset $D \subseteq \Omega(G)$ with the following properties: (i) D is disjoint from all its G-translates; (ii) for every $z \in \Omega(G)$ there exists some $g \in G$ with $gz \in \overline{D}$; (iii) ∂D consists of limit points of G together with a countable collection of curves (**sides**) which all lie (possibly sans endpoints) in $\Omega(G)$; (iv) for every side s of D there exists a side s' of D and a nontrivial **side-pairing transformation** $g \in G$ with $gs = s'$, such that $s'' = s$ and the side-pairing transformation of s' is g^{-1}; (v) the spherical diameter of a sequence of sides of D tends to zero, and the sides accumutate only at limit points; and (vi) for every compact $K \subseteq \Omega(G)$ there exist only finitely many $g \in G$ with $gD \cap K \neq \varnothing$. With these conditions, $\Omega(G)/G = D/G$.

A **Fuchsian group** is a pair (G, Δ) where G is a Kleinian group and Δ is a round open disc in $\hat{\mathbb{C}}$ such that (i) $G\Delta \subseteq \Delta$ and (ii) $\Lambda(G) \subseteq \partial\Delta$. In this case, G acts as a discrete group of isometries of the natural hyperbolic metric on Δ, and Δ/G is a connected hyperbolic Riemann surface. If $\Lambda(G) = \partial\Delta$ then the group is Fuchsian **of the first kind** (in which case the quotient $\Omega(G)/G$ consists of exactly two components, one from Δ and one from $\Omega(G) \setminus \Delta$) and otherwise the group is Fuchsian **of the second kind**.

1.1 Example. The **modular group** $\mathrm{PSL}(2, \mathbb{Z})$ is a Fuchsian group; it is discrete, and preserves the upper half-plane $\mathbb{H}^2$.

The dynamics of non-identity elements of $\mathrm{PSL}(2, \mathbb{C})$ on $\hat{\mathbb{C}}$ are very straightforward. They fall into three classes:

- **parabolic elements**: conjugate in $\mathbb{M}$ to $z \mapsto z + 1$, and with $\mathrm{tr}^2 = 4$ in $\mathrm{PSL}(2, \mathbb{C})$; these have a single fixed point and act as translations around it.
- **elliptic elements**: conjugate in $\mathbb{M}$ to $z \mapsto \lambda z$ ($|\lambda| = 1$), and with $\mathrm{tr}^2 \in [0, 4)$ in $\mathrm{PSL}(2, \mathbb{C})$; these have two fixed points and act as rotations around each circle in the concentric family of circles with centres at the fixed points.

Figure 2: The dynamics of elements in a Kleinian group.

- **loxodromic elements**: conjugate in $\mathbb{M}$ to $z \mapsto \lambda z$ ($|\lambda| \neq 1$), and with $\mathrm{tr}^2 \in \mathbb{C} \setminus [0, 4]$ in $\mathrm{PSL}(2, \mathbb{C})$; these have two fixed points and act to push points along paths which join one (repelling) fixed point to the other (attracting) fixed point. If $\mathrm{tr}^2 \in \mathbb{R}$ then these paths are circular arcs and the element is called **hyperbolic**; otherwise, the paths are logarithmic spirals and the element is **strictly loxodromic**.

§1.1. Brief remark on moduli problems

Algebraic geometers are used to working with the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n marked points. This object can also be studied from the point of view of Kleinian groups, but the most natural method of construction passes through Techmüller space; in this section we briefly explain why, in order to give some context to a few discussions we will have later on in the paper. The survey paper [3] was written as an elementary introduction to the deformation theory and the reader should refer to it for further references.

The point, essentially, is that we would like a moduli space of Kleinian groups to have the following property: small pertubations of the coefficients of the matrices in a group G should give a group G' in the same moduli space. By 'pertubation', we mean a **holomorphic motion**, which one should think of as a complex-analytic version of a homotopy:

1.2 Definition. Let $A \subseteq \hat{\mathbb{C}}$. A **holomorphic motion** of A is a map $\Phi : \Delta \times A \to \hat{\mathbb{C}}$ (where Δ is the unit disc in $\mathbb{C}$) such that

1. For each $a \in A$, the map $\Delta \ni \lambda \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is holomorphic;
2. For each $\lambda \in \Delta$, the map $A \ni a \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is injective;
3. The mapping $A \ni a \mapsto \Phi(0, a) \in \hat{\mathbb{C}}$ is the identity on A.

Now a holomorphic motion of the coefficients of a generator of some group G will give rise to a holomorphic motion on the limit set $\Lambda(G)$ since the fixed points of G depend algebraically on the generator coefficients; the λ-lemma tells us that we can extend this motion in a nice enough manner to the open set $\Omega(G)$ and therefore obtain a deformed complex structure, as long as the limit set (roughly speaking) does not collide with itself:

1.3 Theorem (Equivariant extended λ-lemma)**.** *Let $A \subseteq \hat{\mathbb{C}}$ have at least three points, and let Γ be a group of conformal motions preserving A. Let $\Phi : \Delta \times A \to \hat{\mathbb{C}}$ be a holomorphic motion on A, and suppose that for each $\gamma \in \Gamma$ and each $\lambda \in \Delta$ there is a conformal map $\theta_\lambda(\gamma)$ such that*

$$\Phi(\lambda, \gamma(z)) = \theta_\lambda(\gamma)(\Phi(\lambda, z)) \tag{1.4}$$

for all $z \in A$. Then Φ can be extended to a holomorphic motion $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that (1.4) holds for all $z \in \hat{\mathbb{C}}$, and such that

1. *For each $\lambda \in \Delta$, the map $\tilde{\Phi}_\lambda$ defined by $\hat{\mathbb{C}} \ni a \mapsto \tilde{\Phi}(\lambda, a) \in \hat{\mathbb{C}}$ is a quasiconformal homeomorphism;*
2. *$\tilde{\Phi}$ is jointly continuous in $\Delta \times \hat{\mathbb{C}}$; and*
3. *For all $\lambda_1, \lambda_2 \in \Delta$, $\tilde{\Phi}_{\lambda_1} \tilde{\Phi}_{\lambda_2}^{-1}$ is quasiconformal.*

So the natural deformations of complex structure on the Riemann surfaces in a moduli space of Kleinian groups will be quasiconformal. (The non quasi-initiated should just think of quasiconformal homeomorphisms as being generalisations of conformal maps: a conformal map sends infinitesimal circles to infinitesimal circles, and a quasiconformal map sends infinitesimal circles to infinitesimal ellipses of bounded eccentricity.) One might recognise this notion on surfaces as one of several equivalent definitions of the **Teichmüller space** of the surface; and in fact if G is Fuchsian and the quotient surface is compact then the following definition does give the usual Teichmüller space of Riemann surfaces.

1.5 Definition. The **quasiconformal deformation space** of a Kleinian group Γ, denoted by $\mathrm{QH}(\Gamma)$, is the space of representations $\theta : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ (up to conjugacy) such that

1. θ is faithful and $\theta\Gamma$ is discrete;
2. θ is type-preserving, that is if $\gamma \in \Gamma$ is parabolic (resp. elliptic of order n) then $\theta\gamma$ is parabolic (resp. elliptic of order n); and
3. the groups $\theta\Gamma$ are all quasiconformally conjugate (i.e. there exists some quasiconformal $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ depending on θ such that, as functions, $\theta\Gamma = f\Gamma f^{-1}$).

The space is equipped with a natural metric defined in roughly the same way as the classical Teichmüller metric (the distance between two deformations is defined to be the log of the maximal dilatation of the composition of the two quasiconformal homeomorphisms). In the special case that Γ is Fuchsian, we call $\mathrm{QH}(\Gamma)$ a **quasi-Fuchsian space**.

The relation between this space and the moduli space $\mathcal{M}_{g,n}$ comes from a subgroup of the modular group of the quotient surface.

1.6 Theorem. *Let Γ be a finitely generated non-elementary Kleinian group with $\Omega(\Gamma) \neq \varnothing$ and let $S = \Omega(\Gamma)/\Gamma$. Then there is a normal subgroup $N \trianglelefteq \mathrm{Mod}(S)$ such that $\mathcal{M}(S) = \mathrm{QH}(\Gamma)/(\mathrm{Mod}(S)/N)$ (where here $\mathrm{Mod}(S)$ is the usual mapping class group of the surface).*

In the case that Γ has a finite-sided fundamental domain, the group N is the subgroup generated by Dehn twists along simple closed curves which bound compression discs. The point is that we are now allowing conjugacy by hyperbolic isometries; so the moduli space can no longer detect deformations that are trivial on the interior of the hyperbolic manifold.

§1.2. Example: Schottky groups

Let $C_1, \dots, C_n, C'_1, \dots, C'_n$ be $2n$ mutually disjoint round circles in $\hat{\mathbb{C}}$ with a common exterior D, and for each i let $f_i \in \mathbb{M}$ be a Möbius transformation which maps C_i onto C'_i and which maps the exterior of C_i to the interior of C_i. (Since $\mathbb{M}$ is triply transitive on $\hat{\mathbb{C}}$, such maps can be found for every arrangement of disjoint circles.) In this case the set D forms a fundamental domain for $G = \langle f_1, \dots, f_n \rangle$, and the quotient $\Omega(G)/\mathbb{C}$ is a compact Riemann surface with no marked points, of genus n. The group G is free on the n generators listed, and every element is loxodromic. A free Kleinian group which is purely loxodromic is called a **Schottky group**, and a Schottky group constructed from a physical set of generators in this way is called a **classical Schottky group**. Non-classical groups can be constructed as quasi-conformal deformations of classical Schottky groups, the totality of these groups forms the genus n **Schottky space**, a space of complex dimension $3n - 3$ which lies in between the classical Teichmüller space and the classical moduli space of genus n surfaces. For any fixed n we can construct a Fuchsian group in genus n Schottky space by choosing the $2n$ circles to all have diameters on the real line, and so Schottky space is a quasi-Fuchsian deformation space.

We will now construct a vast family (all the boring ones, in some sense) of Schottky groups in terms of matrices. The construction follows from the next result: the existence part is a simple consequence of the characterisation of 'loxodromic element' which we gave above (more precisely, an application of the intermediate value theorem).

1.7 Lemma. *If $f \in \mathbb{M}$ is loxodromic and ∞ is not a fixed point of f, then there exist a pair of round circles $C, C' \subseteq \mathbb{C}$ of the same radius such that $f(C) = C'$ and f maps the exterior of C into the interior of C'. Let f be represented by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{PSL}(2, \mathbb{C})$ (note, the condition $f(\infty) \neq \infty$ is equivalent to $c \neq 0$). The equations for the two circles are given by $f'(z) = 0$ and $f^{-1}(z) = 1$, i.e. $|cz + d| = 1$ and $|-cz + a| = 1$, so the two circles have centres $-d/c$ and a/c and radius $1/|c|$.*

If we pick n loxodromic elements $f_1, \dots, f_n$ such that the $2n$ isometric circles of the f_i are all sufficiently far apart, then the data of the circles and the circle-pairing transformations gives us a genus n classical Schottky group with very little computational effort.

Suppose that the transformation f_i is represented by the matrix $\begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$. Then we impose the following conditions, for all $1 \leq i < j \leq n$, to ensure the isometric circles of f_i and f_j are disjoint:

$$\left| \frac{a_i}{c_i} - \frac{a_j}{c_j} \right| > \frac{1}{|c_i|} + \frac{1}{|c_j|}$$

$$\left| \frac{a_i}{c_i} + \frac{d_j}{c_j} \right| > \frac{1}{|c_i|} + \frac{1}{|c_j|}$$

$$\left| \frac{d_i}{c_i} - \frac{d_j}{c_j} \right| > \frac{1}{|c_i|} + \frac{1}{|c_j|}$$

and the following condition to ensure that the two isometric circles of f_i are disjoint:

$$\left|\frac{a_i}{c_i} + \frac{d_i}{c_i}\right| > \frac{2}{|c_i|} \iff |a_i + d_i| > 2.$$

In particular, we do not need to specify loxodromicity explicitly and only the $3\binom{n}{2} + n = 3n(n-1)/2 + n = \frac{1}{2}(3n^2 - n)$ conditions listed need to be satisfied for the group to be Schottky. We call the open subset of $\mathbb{C}^{3n}$ which is cut out by the relations $\text{Schottky}^\circ(n)$. By abuse of notation we will often write $G \in \text{Schottky}^\circ(n)$ to mean that G is the group generated by the relevant matrices, and we will write $S(G)$ for the fundamental domain of G which is the common exterior of the isometric circles.

Remark. These inequalities give bounds on the quasiconformal deformation space, but these bounds are incredibly bad (for instance, they miss all of the groups generated by loxodromics with trace in the disc of radius two). Much work has been done to study groups which lie closer to the boundary, and groups on the boundary itself. These groups are of significant interest in geometric function theory and geometric topology. An elementary treatment of the special case of genus two can be found in [15].

§2. Equivalence of Riemann surfaces and algebraic curves

The following result is well-known (see for instance [4, §IV.11]).

2.1 Proposition. *The category of nonsingular algebraic curves over $\mathbb{C}$ with rational morphisms is equivalent to the category of smooth compact Riemann surfaces with conformal maps.*

The point is that a curve k is determined up to birationality by its function field $k(\mathbb{C})$, and a Riemann surface S is determined by its field of meromorphic functions $\mathfrak{M}(S)$; and one can show that every field that arises in one way also arises in the other. The function field of a curve is always generated over $\mathbb{C}$ by two elements, and a representative plane curve in $\mathbb{C}^2$ is given by an algebraic relation between those elements. Hence, in order to write down an equation for a Riemann surface S it is enough to find two (sufficiently general) meromorphic functions on S and a polynomial which relates them; such a procedure is essentially elementary, and was written down by Poincaré in the first volume of *Acta Mathematica* in 1882 [16]; we will mainly refer to the English translation of this paper by John Stillwell [17], and the modern exposition of the theory by Lehner [11].

§2.1. Existence of meromorphic functions

An algebraic geometer, when asked to prove that there exist meromorphic (that is, rational) functions on a curve, will probably cite the Riemann-Roch theorem; this theorem allows the computation of the dimensions of vector spaces of such functions, and it is essentially the method used by Riemann to prove this existence result [18]. Poincaré's major contribution in 1882 was to construct explicit meromorphic functions on the curve, as quotients of particular automorphic forms.

We recall that automorphic forms, which nowadays are primarily studied by number theorists [13, Chapter 2], are functions $\phi : \Omega(G) \to \mathbb{C}$ such that $\phi(gz) = (cz+d)^m \phi(z)$ whenever $g \in \mathbb{M}$ is represented by the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{2.2}$$

Fix a Kleinian group G with non-empty domain of discontinuity, and let $S(G)$ be a fundamental domain for G. Let $H : \Omega(G) \to \hat{\mathbb{C}}$ be an arbitrary meromorphic map with at least one pole in $S(G)$ (the purpose of this requirement is to ensure that the function we construct is non-zero). We can construct from H a **Poincaré series**[1] of degree m with respect to G:

$$\varphi_H(z) = \sum_{g\in G} H(gz)(g'(z))^m \tag{2.3}$$

It is well-known that the series converges uniformly in $\Omega(G)$ with possible essential singularities at $\Lambda(G)$ (these results in [17, 11] are given for Fuchsian groups, but it is easy to check that they work for general Kleinian groups). We can easily check that φ_H is an automorphic form, using the result that if g is given by the matrix of Eq. (2.2) then $g'(z) = (cz+d)^{-2}$: we see that

$$\begin{aligned}\varphi_H(gz) = \sum_{h\in G} H(hgz)[h'(gz)]^m &= \sum_{h\in G} H(hgz)\left[\frac{(hg)'(z)}{g'(z)}\right]^m \\ &= (cz+d)^{-2m}\sum_{h\in G} H(hgz)[(hg)'(z)]^m = (cz+d)^{-2m}\varphi_H(z)\end{aligned}$$

(where in the final equality we use that the sum over $h \in G$ might as well be the sum over all $hg \in G$). The point now is that if we have two automorphic forms α, β with the same m and different poles, then the quotient α/β of the two forms is meromorphic on $\Omega(G)$ and is equivariant with respect to G: $(\alpha/\beta)(gz) = (\alpha/\beta)(z)$ since the automorphy factors cancel. This equivariance is exactly the condition needed for α/β to decend to a meromorphic function on the surface $\Omega(G)/G$.

The construction of a relation between two meromorphic functions (and hence a plane equation for the curve corresponding to the Riemann surface) is now an elementary argument from complex analysis, and the full details may be found in [11, §V.6]. The whole process goes as follows. Pick four independent Poincaré series f_1, f_2, g_1, g_2 with respect to G, form $f = f_1/f_2$ and $g = g_1/g_2$, and suppose f and g have $n > 1$ poles and $m > 1$ poles respectively in $S(G)$. Choose $s, t \in \mathbb{Z}$ such that

$$(s+1)(t+1) - 1 > ns + mt$$

and let $A = (a_{ij})_{i,j=0}^{s,t}$ be a $(s+1)\times(t+1)$-matrix of indeteminates. Choose $\nu = (s+1)(t+1)-1$ distinct points $z_1, ..., z_\nu \in S(G)$ away from the poles of f and g, and form the system of equations

$$0 = \sum_{i=0}^{s}\sum_{j=0}^{t} a_{ij}(f(z_k))^i(g(z_k))^j$$

for $1 \le k \le \nu$. By dimension count, this linear system in the entries A has a solution, and the meromorphic function (polynomial in f and g)

$$\Psi(z) = \sum_{i=0}^{s}\sum_{j=0}^{t} a_{ij}(f(z))^i(g(z))^j$$

is identically zero (since it otherwise has more poles than zeros); in particular, Ψ is an equation for the curve of the Riemann surface of $\Omega(G)/G$. and the function field of the curve is $\mathrm{Frac}\,\mathbb{C}[f,g]/\Psi$.

[1] Poincaré himself called these **theta-Fuchsian functions**, in analogy with the theta functions of Jacobi and Riemann which we briefly mention below in Section 6.

§2.2. Some elementary remarks

We first comment on the necessity of counting the poles of the Poincaré series (in order to know the numbers of poles of the two functions in order to fit the coefficients of Ψ to enough points). The only possibility for a pole of $\varphi_H(z)$ which does not come from H is the existence of a point $z \in \Omega(G)$ and a group element g such that $c(g)z + d(g) = 0$. Observe that such a z is exactly the centre of the isometric circle of g (see Lemma 1.7). It is now a standard result in the theory of Kleinian groups that the common exterior of the isometric circles of G is a fundamental domain for the group [12, §II.H], and so one can always choose the fundamental domain such that these poles of φ_H lie in its exterior. In practice, if the group is defined sufficiently nicely, one can simply choose the fundamental domain to be the common exterior of the isometric circles of the generators since all of the other isometric circles will be contained within them (for instance, this is the case for the classical Schottky groups constructed in Section 1.2).

We now turn to some elementary properties of the plane curves which are obtained by the process outlined in the preceeding section.

Asymptotes. By assumption, the two meromorphic functions f and g have no common poles (and each pole is of multiplicity 1). Hence a point at infinity on the curve $\mathbf{Z}(\Psi)$ always arises as the image of a point $w \in \Omega(G)$ such that exactly one of $f(w)$ and $g(w)$ is infinite. Without loss of generality, $f(w)$ is finite and $\infty = g(w)$. Then $\mathbf{Z}(\Psi)$ has an asymptote along the line $X = f(w)$. In particular, the corresponding projective curve hits exactly two points on the line at infinity, namely $P = [0 : 1 : 0]$ and $Q = [1 : 0 : 0]$ (we choose homogeneous coordinates so that the line at infinity is the line where the third coordinate is zero); P has multiplicity equal to t times the number of poles of g and Q multiplicity equal to s times the number of poles of f.

Singularities. We recall some results on algebraic curves, which can be found for instance in [7, §IV.3]. First, every curve can be embedded into $\mathbb{P}^3$; and every curve in $\mathbb{P}^3$ is birational to a curve in $\mathbb{P}^2$ with at most nodes for singularities. Recall, a **node** is a double point of the curve with distinct tangent directions. We ask two natural questions here.

1. Is the plane curve $\mathbb{Z}(\Psi)$ constructed above a curve with at most nodes for singularities?
2. What are the nodes of the curve $\mathbb{Z}(\Psi)$, anyway?
3. If so, can we naturally construct an embedding into $\mathbb{P}^3$? (Not every nodal plane curve comes from a smooth curve in $\mathbb{P}^3$—see [7, Ex. IV.3.7]— but we started with a nonsingular Riemann surface so such an embedding does exist.)

We have given a parameterisation of $\mathbb{Z}(\Psi)$ by a vastish open subset of $\mathbb{C}$. We first check that the only singularities are possible multiple points. Any other singularity is a point $z \in \mathbb{C}$ where $f'(z) = g'(z) = 0$. Such a point might indeed occur, but small deformations to either f or g (i.e. to the respective meromorphic functions H used in the construction) will fix this problem. Next suppose that there is a double point with equal tangent directions; the tangent line at $(f(z), g(z))$ is the line of slope $g'(z)/f'(z)$, so such a double point must come from $w, z \in \mathbb{C}$ such that

$g'(w)f'(z) = g'(z)f'(w)$. Again small deformations will fix the problem. (Essentially what we are saying is that you can always pick a better projection from $\mathbb{P}^3$ and in fact nice projections are the generic case.)

Now the nodes come from $z, w \in \mathbb{C}$ such that $f(z) = f(w)$ and $g(z) = g(w)$...

Lifting to a curve in 3-space. If $h = h_1/h_2$ is a meromorphic quotient of two more automorphic forms, then h is a rational function of f and g. The triplet f, g, h now parameterises a subset $\{(f(z), g(z), h(z)) : z \in \Omega(G)\}$ of $\mathbb{C}^3$ which is birational to the plane curve we have constructed. This space curve is nonsingular iff (f, g, h) is injective on the fundamental domain of G, and if there is no z where f', g', and h' all vanish. Knowing where f' and g' vanish *a priori*, we can construct h specifically with non-vanishing derivative at those points... right??

Real points. The question of real points is normally phrased in the following way:

2.4 Question. Let X be some construction which takes complex parameters $t = (t_i)_{i\in I}$ and returns a geometric object $X(t) \subseteq \mathbb{C}^J$. If t is chosen in $\mathbb{R}^I$, how much of $X(t)$ lies in $\mathbb{R}^J$?

There is a problem with the question (apart from its obvious vagueness) in this setting. The obvious specialisation is

2.5 Question. Let G be a Kleinian group with real coefficients, and compute some equation Ψ using the process outlined above with meromorphic functions that have only real zeros and poles. Compute $\mathbf{Z}(\Psi) \cap \mathbb{R}^2$.

Observe that for such a question to be meaningful we do indeed need the group G to satisfy something like having real coefficients, since the coefficients of G appear in the expansions for all the meromorphic functions. The problem is that a Kleinian group with real coefficients is Fuchsian with invariant domain $\mathbb{H}^2$ (or $-\mathbb{H}^2$), and in particular $\Lambda(G) \subseteq \mathbb{R}$. Thus the obvious way to compute some real points by taking real x and computing $(f(x), g(x))$ is not particularly nice unless we have some way of computing $\mathbb{R} \setminus \Lambda(G)$; in any case we are living in a tiny subset of $\Omega(G)$ and it is probably hopeless to get a picture of $\mathbf{Z}(\Psi) \cap \mathbb{R}^2$ by just looking at the images of $\mathbb{R}$. (In general, constructions in Kleinian groups which are symmetric with respect to a fixed circle will lead to Fuchsian structures... so even doing constructions via complex conjugation will fail.)

§3. Genus 2

Since the genus 1 case is degenerate (i.e. the groups are elementary; we deal with it briefly below in Section 4) we begin with genus 2 Schottky groups. Despite being the simplest non-elementary compact surface groups, the structure of the space of these groups is incredibly intricate: for example, the boundary of their deformation space already contains representatives of all two-bridge knot groups [2, 1]. Nonetheless, there is a lot of additional structure which we can study.

Every genus 2 surface is **hyperelliptic**, in the sense that they admit meromorphic functions with exactly two poles. The existence of such functions gives a lot of additional structure in both the algebraic and analytic categories:-

1. **Algebraic geometry.** A general algebraic curve is hyperelliptic if there is a degree 2 morphism $f : C \to \mathbb{P}^1$. If C is genus 2, then the complete linear system associated to the canonical divisor provides such a morphism and this morphism is in fact unique up to composition with an automorphism of $\mathbb{P}^1$; necessarily (by Hurwitz' theorem) it has six ramification points $p_1, \dots, p_6 \in C$ called the **Weierstrass points** of C. This data determines a plane embedding for C, given by the equation

$$y^2 = (x - f(p_1)) \cdots (x - f(p_n)) =: \phi(x).$$

   Observe that in this form there is a natural involution $(x, y) \mapsto (x, -y)$, called the **hyperelliptic involution**. See for example [7, Ex. IV.1.7 and Ex. IV.2.2].

2. **Complex analysis.** If S is a Riemann surface of genus $g \geq 2$ and $\mathcal{H}^1(S)$ is the vector space of holomorphic 1-differentials on S, then a point $P \in S$ is a Weierstrass point if there exists some nonzero $\omega \in \mathcal{H}^1(S)$ with a zero at P of order at least $\dim \mathcal{H}^1(S)$. The surface is hyperelliptic if and only if it has exactly $2g - 2$ Weierstrass points, in which case there is a unique (up to $\mathrm{Aut}(\mathbb{P}^1)$-action) meromorphic map $f : S \to \mathbb{C}$ with exactly two poles with ramification points exactly the hyperelliptic curves; the Galois group of the covering map f is $\mathbb{Z}/2$ and the involution swapping the sheets is the hyperelliptic involution. A version of Hurwitz' theorem states that any automorphism of a compact Riemann surface has at most $2g + 2$ fixed points, so any involution with at least 6 fixed points must be the hyperelliptic involution. The complex analytic point of view is vigorously analysed in [4, §III.7].

Knowledge of the hyperelliptic structure in the surface in any of these equivalent forms is therefore enough to give the algebraic structure (in terms of an equation).

§3.1. General Fuchsian Schottky groups on four circles

Suppose we are given four circles C_1, C_1', C_2, C_2', all mutually disjoint and orthogonal to a fixed fifth circle D, and suppose we set up a Schottky group on the circles $G = \langle X, Y\rangle$ with $X(C_1) = C_1'$ and $Y(C_2) = C_2'$ such that X and Y are both hyperbolic (i.e. $\mathrm{tr}^2 X$ and $\mathrm{tr}^2 Y$ both lie in $(4, \infty)$). With this setup, G is a Fuchsian group which acts on both the interior Δ and exterior Δ' of D; each of Δ/G and Δ'/G is an open hyperbolic surface, and since G is Fuchsian of the second kind (the limit set is strictly contained inside the four defining circles) the global quotient $S = \Omega(G)/G$ is obtained by gluing the two hyperbolic surfaces together along their boundary to form a genus two surface (i.e. the quotient surface of G is the Schottky double of two three-holed spheres uniformised by G).

In this situation, we can compute the hyperelliptic involution on S using the hyperbolic geometries on the two halves. The idea is that such an involution lifts to an involution on $\Omega(S)$ which is equivariant with respect to G; the full details of the hyperelliptic structure were worked out by Gilman and Keen [6], from which we draw heavily from in the following discussion.

We must split into two cases, according to whether the axes of X and Y in Δ intersect or not. (The axis of a hyperbolic element h with fixed points ζ, ξ in a Fuschsian group acting on a disc Δ is the hyperbolic line $\mathrm{Ax}(h) = [\zeta, \xi]$.) These correspond to the two different ways of cutting a genus 2 torus into mirror-symmetric pieces: either across the 'throat' (crossing axes, Fig. 3) or through both holes (disjoint axes, Fig. 4).

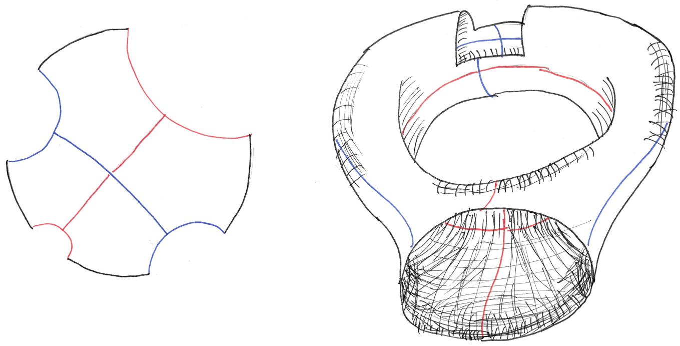


Figure 3: Gluing behaviour of a genus 2 Schottky group with crossed axes.

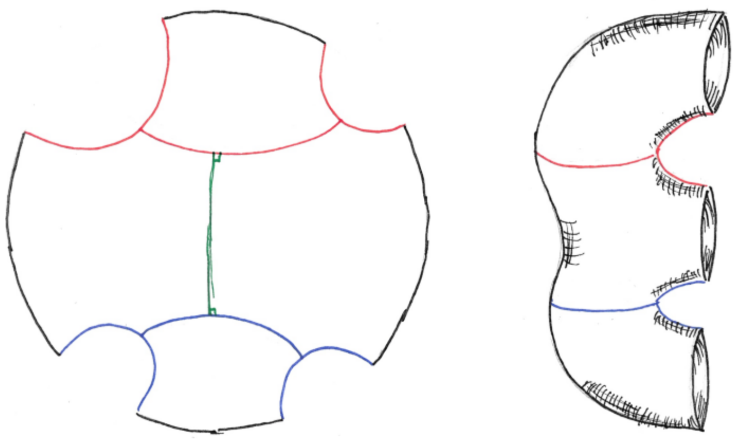


Figure 4: Gluing behaviour of a genus 2 Schottky group with disjoint axes.

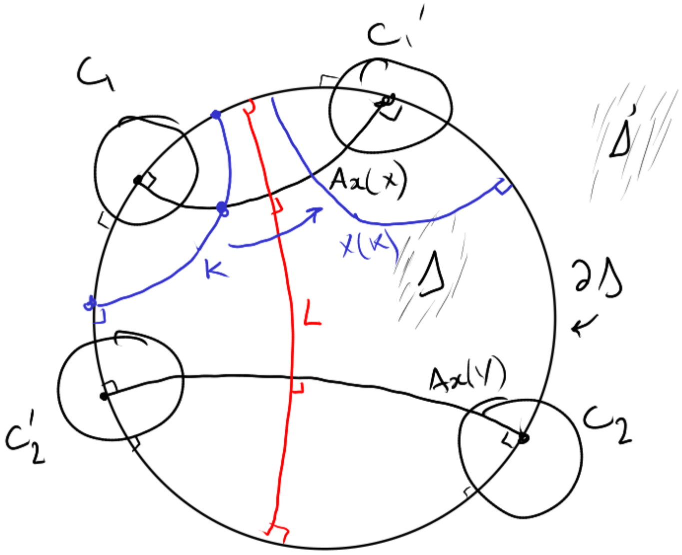


Figure 5: Weierstrass points on a Schottky domain with disjoint hyperbolic axes.

3.1 Proposition (Disjoint axes)**.** *If* $\mathrm{Ax}(X) \cap \mathrm{Ax}(Y) = \varnothing$*, then let* L *be the hyperbolic line in* Δ *which is the common perpendicular to* $\mathrm{Ax}(X)$ *and* $\mathrm{Ax}(Y)$*. Let* σ_L *and* σ_Δ *be the sphere inversions across* L *and* Δ*. Then* $E = \sigma_\Delta \sigma_L \in \mathbb{M}$ *is an elliptic transformation of order* 2 *which preserves* $\Omega(G)$ *and descends to a conformal automorphism* j *on* $\Omega(G)/G$ *which fixes exactly* 6 *points; in particular* j *is the hyperelliptic involution on the quotient surface.*

Proof. We will give a version of the construction found in §4 of [6], but we can simplify matters as we only want to get a subset of the information about the geometry that Gilman and Keen require. In particular, we do not need to compute Gilman-Maskit generators for the group, the ones we have work fine. The reader should now refer to Fig. 5. It is easy to check that E is an involution (the point is that $\sigma_{\partial\Delta}$ and σ_L commute because the circles are orthogonal) and that $EGE = G$, so E descends to the quotient as a conformal map $j : \Omega(G)/G \to \Omega(G)/G$. A fixed point of j is the projection of a point $P \in \Omega(G)$ such that $E(P) = g(P)$ for some $g \in G$; we will find six fixed points, which shows that j is indeed the hyperelliptic involution by our initial discussion. Two points are the two fixed points of E; these points are clearly the points $L \cap \partial\Delta$. All other fixed points of j must also lie on $\partial\Delta$ since E permutes Δ and Δ' while G preserves both. We first show that there are two points $P \in \partial\Delta$ such that $E(P) = X(P)$; the same argument works to find two points such that $E(P) = Y(P)$ (and it will be clear from the construction that a all six constructed fixed points are distinct and hence are a complete set of Weierstrass points). Note that $E\restriction_{\partial\Delta} = \sigma_L$, and let K be the geodesic line orthogonal to $\mathrm{Ax}(X)$ such that $|\,(K \cap \mathrm{Ax}(X), X(K) \cap \mathrm{Ax}(X)) = \mathrm{Ax}(X) \cap L$; in other words, find the point z which gets pushed by X along its axis to the symmetric point with respect to L and then erect an orthogonal geodesic to L above z. This geodesic K satisfies $\sigma_L(K) = X(K)$ by construction, so its two endpoints are fixed points of E.

Before moving to the parallel case, we compute E in terms of X and Y. The easiest way to see that two disjoint geodesic lines L, M in $\mathbb{H}^2$ have a unique common orthogonal line is to assume (as we can, up to isometry) that L is the imaginary axis $x = 0$in the upper half-plane model, and M is the upper half of the circle $(x - x_0)^2 + y^2 = r^2$ (where $r < |x_0|$ of course). Then any orthogonal to L will come from a circle centred at 0, $x^2 + y^2 = s^2$, and clearly such a circle is orthogonal to the circle defining M iff it passes through $(x_0, x_0 + r)$. This defines the circle uniquely and gives us a very simple process for computing E. Let Q be such that $QXQ^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some (unique up a factor of ± 1) $\lambda \in \mathbb{C}$, and replace X and Y with QXQ^{-1} and QYQ^{-1}. This modifies the axis of X to be on the imaginary axis. Consider the axis of (the new) Y. It is the upper half of the circle M joining the fixed points of Y with centre their midpoint. (By assumption none of these points lie at ∞, so one can assume this is a proper Euclidean circle.) In fact, we are only interested in the point of M lying directly above this midpoint. We can calculate:

$$\text{if } Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then the centre of } M \text{ is } \frac{a-d}{2c}$$

and the radius of M is

$$\left| \frac{\sqrt{(d-a)^2 + 4bc}}{2c} \right|;$$

so the point we need is

$$x_0 = \frac{a-d}{2c} + i \left| \frac{\sqrt{(d-a)^2 + 4bc}}{2c} \right|$$

(and by construction this is of the form $x + yi$ for $x, y \in \mathbb{R}$). Now the radius of the circle defining the line L is the positive number s such that

$$(\operatorname{Re} x_0)^2 + (\operatorname{Im} x_0)^2 = s^2.$$

In other words, $s = |x_0|$. This allows us to write down the formula for $Q\sigma_L Q^{-1}$: it is the circle inversion with centre 0 through the circle of radius $|x_0|$:-

$$Q\sigma_L Q^{-1}(z) = \frac{|x_0|^2}{\overline{z}}.$$

By construction, the reflection $Q\sigma_\Delta Q^{-1}$ is just complex conjugation. Hence

$$QEQ^{-1}(z) = Q\sigma_\Delta \sigma_L Q^{-1} = Q\left(\frac{|x_0|^2}{\overline{Q^{-1}(z)}} \right)$$

which determines E.

3.2 Proposition (Incident axes). *If* $\operatorname{Ax}(X) \cap \operatorname{Ax}(Y) = \{p\}$ *then let* $E \in \mathbb{M}$ *be the elliptic transformation of order* 2 *which fixes* p *and preserves* Δ. *Then* E *preserves* $\Omega(G)$ *and descends to a conformal automorphism* j *on* $\Omega(G)/G$ *which fixes exactly* 6 *points; in particular* j *is the hyperelliptic involution on the quotient surface.*

Proof. This time we follow the construction of §8 of [6] and again we do not need their elaborate machinery. The six Weierstrass points are now the six fixed points of the three elliptic transformations E, EX, and EY.

In this case, the computation of E is very straightforward. Write down the matrix Q conjugating $\partial\Delta$ to $\mathbb{R}$ and sending Δ to the upper half-plane; then one simply finds $a, b, c, d \in \mathbb{R}$ satisfying the three conditions

1. lies in $\mathrm{PSL}(2, \mathbb{R})$: $ad - bc = 1$;
2. fixes $Q(P)$:- $cQ(p)^2 + (d-a)Q(p) - b = 0$;
3. has $\mathrm{tr}^2 = 0$:- $(a+d)^2 = 0$.

We have that QEQ^{-1} is represented by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Remark. We remark very quickly that for small holomorphic deformations of G the constructions of this section give us the Weierstrass points in the quasi-Fuchsian case: the elliptic involution will be the quasi-conformal conjugate of E, whose matrix we can compute because E can be written only in terms of the coefficients of the generators of G (i.e. E is defined in terms of the limit set of G, so no application of the λ-lemma is needed). Of course, this construction only works if the exact formula for the deformations of X and Y in terms of the original matrices are known.

Remark. The constructions of this section give the lifts of the Weierstrass points to the covering space $\Omega(G)$, and so to find the points below on the algebraic curve one still must descend via the parameterisation; so the detailed computation of the meromorphic functions cannot be bypassed. Knowledge of the Weierstrass points as computed here is still useful though, as it gives us a canonical form for the curve which is made.

Remark. C.f. [8] for general Schottky domains???

§3.2. A Fuchsian example

Let us take a simple genus 2 example, generated by two loxodromic transformations X and Y with isometric circles of radius R around ± 4 and $\pm 4i$. The isometric circle data does not determine X and Y uniquely, but if we require both to be hyperbolic then we do get uniqueness, and the matrices are easily calculated to be

$$X = \begin{bmatrix} 4/R & 16/R - R \\ 1/R & 4/R \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4/R & i(16/R - R) \\ -i/R & 4/R \end{bmatrix}.$$

The group $\langle X, Y\rangle$ is a classical Schottky group if the isometric circles of X and Y don't collide, and one can check that this is equivalent to $R \in (0, 2\sqrt{2})$. The gluing is shown in Fig. 6. We also observe that the group is Fuchsian, since $\mathrm{tr}^2 X > 0$ and $\mathrm{tr}^2 Y > 0$; the invariant circle is the circle through the four fixed points, $\pm\sqrt{16 - R^2}$ and $\pm\sqrt{R^2 - 16}$.

Even though the group fails to be a classical Schottky group for $R \geq 2\sqrt{2}$, we can still carry out the procedure to compute a curve as long the poles and points to which we fit the coefficients continue to lie in the common exterior of the defining circles. The surfaces that are obtained are shown in Fig. 7: when $R = 2\sqrt{2}$ the surface is a pair of punctured tori which have cusps corresponding to a doubly cusped parabolic element—namely, the loxodromic element W which preserves the blue curve in Fig. 6 is pinched to become parabolic; the group now lies in the boundary of Schottky space, more precisely it lies in the Maskit embedding [9]). For $R \in 2\sqrt{2}$, the element W becomes elliptic, and the group is discrete iff W is finite order, i.e.

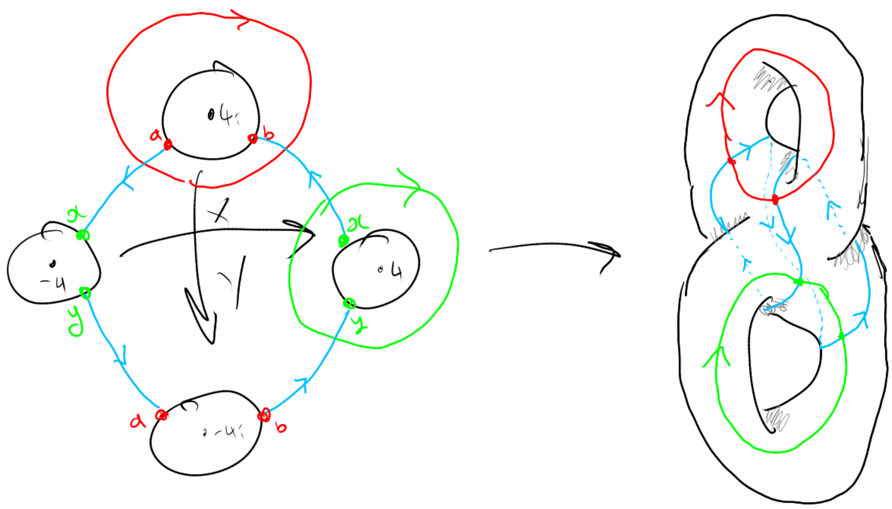


Figure 6: The Schottky group $\langle X, Y\rangle$ has quotient surface a compact surface of genus 2. The red, green, and blue arcs in $\Omega(G)$ glue up to the corresponding coloured curves on the quotient.

the angle θ between the isometric circles of X and Y satisfies $\pi/2\theta \in \mathbb{Z}$; in this case the surface is a pair of tori with cone points of the corresponding angle (and the quotient $\mathbb{H}^3/G$ is an orbifold with a cone arc joining the corresponding points at infinity); and finally when $R \to 0$ one component vanishes completely, the other is just a torus with no markings, and the group becomes elementary.

Remark. The elliptic groups are essentially the **Heckoid groups**, see [2, 19, 10].

Choose rational functions

$$H_\varepsilon(z) = \frac{1}{z-\varepsilon} \text{ for } \varepsilon \in \{\pm 1, \pm i\} \tag{3.3}$$

and define $f = H_1/H_{-1}$, $g = H_i/H_{-i}$; we have $n = m = 2$ and so we want

$$(s+1)(t+1) - 1 > 2(s+t) \iff st > s+t$$

i.e. $s = 2$, $t = 3$ works; pick eleven random points in $\mathbb{C}$, say the eleven scaled roots of unity $(1/2)\exp(2n\pi i/11)$ for $n = 1, \ldots, 11$. We get 11 equations,

$$0 = \sum_{i=k}^{s}\sum_{j=\ell}^{t} a_{ij} \left(\frac{\sum_{g\in G} \frac{((1/2)c(g)\exp(2\pi i/n)+d(g))^{-4}}{g((1/2)\exp(2n\pi i/11))-1}}{\sum_{g\in G} \frac{((1/2)c(g)\exp(2\pi i/n)+d(g))^{-4}}{g((1/2)\exp(2n\pi i/11))+1}} \right)^{k} \left(\frac{\sum_{g\in G} \frac{((1/2)c(g)\exp(2\pi i/n)+d(g))^{-4}}{g((1/2)\exp(2n\pi i/11))-i}}{\sum_{g\in G} \frac{((1/2)c(g)\exp(2\pi i/n)+d(g))^{-4}}{g((1/2)\exp(2n\pi i/11))+i}} \right)^{\ell},$$

one for each n.

Now the point is to compute good approximations to the averages over G. For this we can just use the fact that long words in the generators have small derivative and then use any one of the number of standard algorithms in combinatorial group theory to enumerate words; for instance the various limit set algorithms in [15].

With this setup, we can use the `MATLAB` code listed in Appendix A to compute the equations for various values of R. With our setup, we can let R range from 0 to

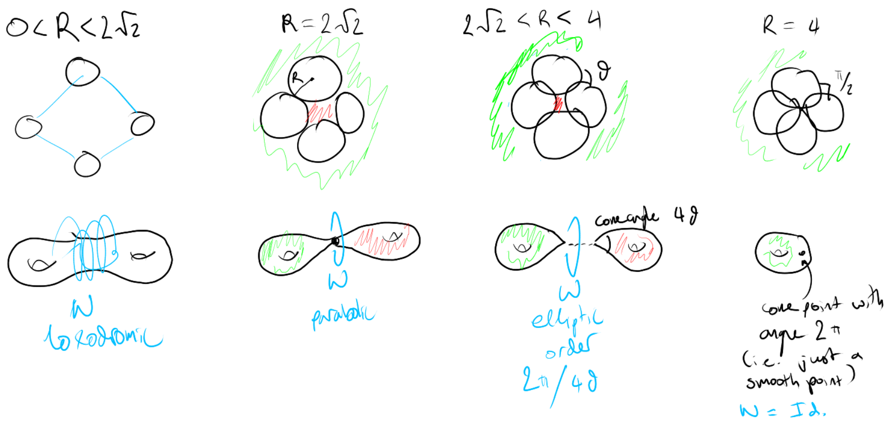


Figure 7: Deformations of the group $\langle X, Y \rangle$ as R goes from 0 to 4.

3 (at $R = 3$ the generating circles hit the poles of the parameterising functions and the theory breaks down); within this range, we would like to sample behaviour at $0 < R < 2\sqrt{2}$ (in Schottky$^{\circ}(2)$), at $R = 2\sqrt{2}$ (where we should be able to observe a singularity in the plane curve arising from the cusp), and for various $2\sqrt{2} < R < 3$ (where we should see finite-order singularities). The polynomials which arise are listed in Table 1, and the plane curves themselves are shown in Fig. 8.

Insert computation of Weirstrass points here for hyperbolic region........

In theory, everything should break down at the parabolic point $R = 2\sqrt{2}$. This is not immediately obvious from the pictures, though, because everything we are doing is only up to some approximation and so we might be finding solutions where none exist. The point is that on non-compact Riemann surfaces there are many many holomorphic functions. In the case that we have non-puncture boundary components, one can solve the Dirichlet problem [5, Theorem 22.7] to see that there are meromorphic functions for every continuous real-valued map on the boundary. In the case that there are punctures, there exist many holomorphic functions like exponential maps and which don't all lie in a finite extension of $\mathbb{C}$. In other words, we have computed two elements of the function field, but it is no longer the case that they are generators: we have a proper subfield of $\mathcal{M}(S)$. The curve we are computing should therefore be a compactification of S. It is natural to ask if this compactification is independent of the choice of f and g. We can get different f and g by chosing different poles.

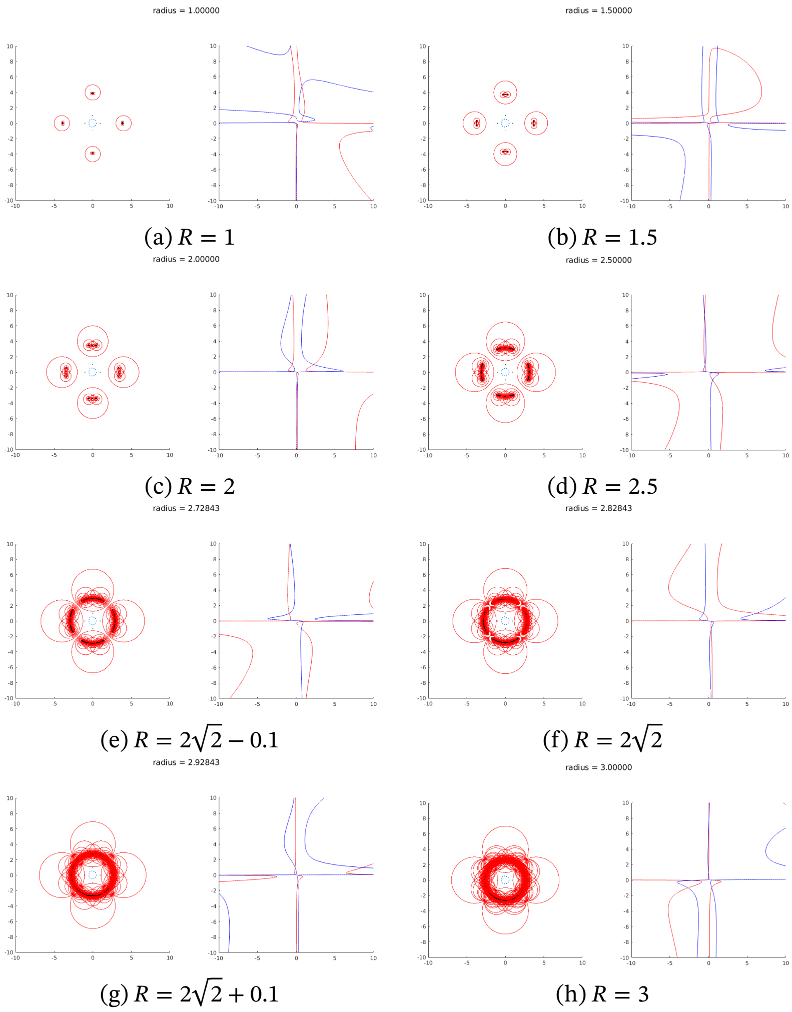
Chose different poles in the para case and check if Weierstrass points are the same.

What does meromorphicity at the cusp mean here (if we have that), does it imply uniqueness?.

§3.3. A quasi-Fuchsian example

We consider another one-parameter family of groups in Schottky$^{\circ}(2)$, those generated by the matrices

$$X = \begin{bmatrix} 4/R & 16/R - R \\ 1/R & 4/R \end{bmatrix} \text{ and } Y = \begin{bmatrix} 4i/R & -16/R - R \\ 1/R & 4i/R \end{bmatrix}.$$



(a) $R = 1$ (b) $R = 1.5$

(c) $R = 2$ (d) $R = 2.5$

(e) $R = 2\sqrt{2} - 0.1$ (f) $R = 2\sqrt{2}$

(g) $R = 2\sqrt{2} + 0.1$ (h) $R = 3$

Figure 8: The genus 2 hyperbolic groups for various R. On the left of each figure we draw the isometric circles of group pointwiseelements (in red), the limit set (in black), the poles (green), and the points $z_1, \ldots, z_{11}$ (in blue).

Table 1: Plane curves from the hyperbolic genus 2 surface.

R	$\Psi(X,Y)$
1	$(0.001623 - 0.004022i)X^0Y^0 + (0.064275 - 0.057562i)X^0Y^1 + (0.016108 + 0.084766i)X^0Y^2 + (-0.001153 + 0.004181i)X^0Y^3 + (0.017619 - 0.081196i)X^1Y^0 + (-0.167178 + 0.666387i)X^1Y^1 + (0.271875 - 0.630955i)X^1Y^2 + (-0.035312 + 0.075208i)X^1Y^3 + (-0.003345 - 0.001613i)X^2Y^0 + (-0.059328 - 0.100399i)X^2Y^1 + (-0.108107 + 0.043735i)X^2Y^2 + (-0.003637 - 0.000750i)X^2Y^3$
1.5	$(0.002191 - 0.002391i)X^0Y^0 + (-0.077999 - 0.073112i)X^0Y^1 + (0.049104 - 0.094964i)X^0Y^2 + (-0.002864 - 0.001522i)X^0Y^3 + (-0.060784 + 0.008269i)X^1Y^0 + (0.668170 - 0.154621i)X^1Y^1 + (-0.681445 + 0.077405i)X^1Y^2 + (0.060056 - 0.012505i)X^1Y^3 + (0.005331 + 0.000747i)X^2Y^0 + (-0.028530 + 0.116597i)X^2Y^1 + (0.065832 + 0.100375i)X^2Y^2 + (-0.004776 + 0.002484i)X^2Y^3$
2	$(0.002585 - 0.008629i)X^0Y^0 + (-0.142654 - 0.002200i)X^0Y^1 + (-0.048493 - 0.134177i)X^0Y^2 + (-0.007318 + 0.005252i)X^0Y^3 + (-0.034654 + 0.034935i)X^1Y^0 + (0.245896 - 0.637272i)X^1Y^1 + (-0.522598 + 0.439854i)X^1Y^2 + (0.021760 - 0.044135i)X^1Y^3 + (0.003055 - 0.002859i)X^2Y^0 + (0.076720 + 0.068254i)X^2Y^1 + (0.089502 + 0.050339i)X^2Y^2 + (-0.001709 + 0.003819i)X^2Y^3$
$5/2 = 2.5$	$(-0.023698 - 0.011458i)X^0Y^0 + (-0.050416 + 0.300104i)X^0Y^1 + (-0.304255 + 0.005760i)X^0Y^2 + (0.007851 + 0.025125i)X^0Y^3 + (0.015529 + 0.032545i)X^1Y^0 + (-0.627751 + 0.092125i)X^1Y^1 + (-0.183390 + 0.607394i)X^1Y^2 + (-0.029909 - 0.020143i)X^1Y^3 + (-0.000049 - 0.000900i)X^2Y^0 + (0.008283 - 0.052674i)X^2Y^1 + (0.053319 - 0.000451i)X^2Y^2 + (0.000883 + 0.000180i)X^2Y^3$
$2\sqrt{2} - 0.1$	$(-0.028042 - 0.024654i)X^0Y^0 + (-0.184998 + 0.393729i)X^0Y^1 + (-0.398807 - 0.173781i)X^0Y^2 + (-0.004803 + 0.037028i)X^0Y^3 + (-0.010366 + 0.035403i)X^1Y^0 + (-0.457333 - 0.312265i)X^1Y^1 + (-0.137598 + 0.536405i)X^1Y^2 + (-0.031149 - 0.019762i)X^1Y^3 + (-0.000370 - 0.000995i)X^2Y^0 + (0.025621 - 0.026033i)X^2Y^1 + (0.036429 + 0.002656i)X^2Y^2 + (0.000378 + 0.000992i)X^2Y^3$
$2\sqrt{2}$	$(-0.021115 - 0.001855i)X^0Y^0 + (0.093079 + 0.251620i)X^0Y^1 + (-0.262062 + 0.057441i)X^0Y^2 + (0.012957 + 0.016775i)X^0Y^3 + (0.013627 + 0.032677i)X^1Y^0 + (-0.644147 - 0.096714i)X^1Y^1 + (0.429068 + 0.490082i)X^1Y^2 + (-0.034864 + 0.006164i)X^1Y^3 + (-0.000169 - 0.000217i)X^2Y^0 + (0.015597 - 0.042407i)X^2Y^1 + (0.027283 - 0.036018i)X^2Y^2 + (0.000275 + 0.000025i)X^2Y^3$
$2\sqrt{2} + 0.1$	$(-0.005810 + 0.009076i)X^0Y^0 + (0.107615 - 0.087161i)X^0Y^1 + (0.003603 + 0.138438i)X^0Y^2 + (0.003129 - 0.010312i)X^0Y^3 + (0.025354 - 0.024162i)X^1Y^0 + (0.149760 + 0.672904i)X^1Y^1 + (0.608966 - 0.323092i)X^1Y^2 + (-0.001910 + 0.034971i)X^1Y^3 + (0.001688 - 0.002415i)X^2Y^0 + (-0.063385 + 0.015013i)X^2Y^1 + (-0.029745 - 0.057950i)X^2Y^2 + (-0.000740 + 0.002852i)X^2Y^3$
3	$(-0.014670 + 0.006005i)X^0Y^0 + (0.092715 - 0.093179i)X^0Y^1 + (-0.067555 + 0.112760i)X^0Y^2 + (0.002219 - 0.015696i)X^0Y^3 + (0.019544 - 0.008155i)X^1Y^0 + (0.333089 + 0.597697i)X^1Y^1 + (0.661195 + 0.176099i)X^1Y^2 + (-0.003105 + 0.020949i)X^1Y^3 + (0.005080 - 0.012609i)X^2Y^0 + (-0.089851 + 0.074659i)X^2Y^1 + (0.050306 - 0.105435i)X^2Y^2 + (-0.010975 + 0.008021i)X^2Y^3$

Like the groups of Section 3.2, these groups are generated by transformations with isometric circles of radius R at $\pm 4i$ and ± 4; however, these groups are no longer Fuchsian, since Y has holonomy. The complex translation length of a loxodromic element g is computed to be

$$\ell(g) = 2 \operatorname{arccosh} \frac{\operatorname{tr} g}{2}$$

and the holonomy of g is $\operatorname{Im} \ell(g)$; here, we have $\operatorname{tr} Y = 8i/R$ and so Y has holonomy $2 \operatorname{Im} \operatorname{arccosh}(4i/R) = \pi$: the isometric circles of Y are mapped onto each other with a half-twist. Let us see how this affects the curve.

We now easily write down the conformal map on the coefficients which sends prev sec to this one, this indu

§4. The genus one Schottky case

Consider a group of the form $\langle f \rangle$ for some loxodromic element $f \in \mathbb{M}$ (so the group has exactly two limit points and is elementary). Up to conjugation we may assume that f fixes ∞ and hence is of the form $f(z) = az + b$ with $a \neq 1$. In this form, f is gluing opposide sides of an annulus and the group glues the infinite cylinder $\mathbb{R}^2 \setminus \{0\}$ up to an (incomplete) affine torus (see [20, pgphs 3.1.10, 3.3.4]). We can still carry out the construction of meromorphic functions; $c = 0$ in Eq. (2.3) for every $g \in \langle f \rangle$, and $d = \sqrt{a^n}$ when $g(z) = f^n(z)$, so we have the formula

$$\begin{aligned}
\varphi_H(z) &= \sum_{n \in \mathbb{Z}} H\left(a^n z + a^{n-1} b + a^{n-2} b + \cdots + b\right) \sqrt{a^n}^{-2m} \\
&= \sum_{n \in \mathbb{Z}} H\left(a^n z + b \frac{a^n - 1}{a - 1}\right) a^{-mn}
\end{aligned}$$

§5. Behaviour of coefficients under deformation

As is well-known, holomorphic movement of the parameters of a Kleinian group gives rise to quasiconformal movement of the complex structure on the quotient surface and therefore corresponds to movement in the Teichmuller space of the surface. The extended λ-lemma gives an estimate on the dilatation of the quasiconformal maps which arise: if $\Phi : \mathbb{D} \times A \to \hat{\mathbb{C}}$ is the holomorphic motion of the limit set in some domain $A \subseteq \mathbb{C}$ (here, $\mathbb{D}$ is the unit disc) then the extension $\tilde{\Phi} : \mathbb{D} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfies the property that for $\lambda_1, \lambda_2 \in \mathbb{D}$, $\tilde{\Phi}(\lambda_1, \cdot) \circ \tilde{\Phi}(\lambda_2, \cdot)^{-1}$ is K-quasiconformal with

$$\log K \leq \rho_{\mathbb{D}}(\lambda_1, \lambda_2).$$

In the construction of a polynomial equation above, the coefficients (a_{ij}) depend linearly on $(s+1)(t+1) - 1$ values of the meromorphic functions f, g, and the limit set arises as the set of essential singularities of f and g; on the other hand, the complex structure of the underlying surface has a complicated dependence on the (a_{ij}).

§6. A remark on theta functions

An alternative method for obtaining an algebraic curve from a Riemann surface is to go via the Riemann theta functions. These functions are defined on the Jacobian of a Riemann surface, and we therefore pause to recall how we can fit the Jacobian into the automorphic picture. More precisely, we will recall the various different definitions of the Jacobian.

1. **Algebraic geometry.** The Jacobian variety $J(C)$ of an algebraic curve C is the space of degree zero line bundles on C.

2. **Cohomological.** The Jacobian of a Riemann surface S is the quotient space $H^0(\Omega^1_S)^*/H_1(S)$, where Ω^1_S is the sheaf of holomorphic differentials on S and $H_1(S)$ embeds into $H^0(\Omega^1_S)^*$ via the map $\gamma \mapsto \int_\gamma$.

3. **Complex analysis.** The Jacobian space $J(S)$ of a Riemann surface S is the set of holomorphic 1-forms on S

Now observe that the holomorphic 1-forms on $\Omega(G)/G$ are exactly of the form $\mu(z)dz$ for some $\mu(z) : \Omega(G) \to \mathbb{C}$ satisfying

$$\mu(\gamma(z))\gamma'(z) = \mu(z)$$

for all $\gamma \in G$, i.e. they are automorphic 1-forms. (This comes directly from writing down the transition laws for differential forms.)

Now we find sufficiently many automorphic 1-forms via Poncaré series........

See the appendix to [14]

§A. Genus 2 MATLAB script

```
R = 2*sqrt(2);
fprintf('radius = %0.5f\n',R);
X = [4/R,16/R-R;1/R,4/R];
Y = [4/R,-1i*(R-16/R);-1i/R,4/R];
x = inv(X);
y = inv(Y);
limseed = sqrt(16-R^2);


% Enumeration of group elements

gens = zeros(2,2,4);
gens(:,:,1) = X;
gens(:,:,2) = Y;
gens(:,:,3) = x;
gens(:,:,4) = y;

invhash = [3,4,1,2];

uplim = 10000;
group = zeros(2,2,uplim);
tag = zeros(1,uplim);
for i = 1:4
    group(:,:,i) = gens(:,:,i);
    tag(i) = i;
end

N = 400;

num = zeros(1,N);
```

```
num(1) = 1;
num(2) = 5;
for n = 2:(N-1)
   inew = num(n);
   for iold = num(n-1):(num(n)-1)
       for j = 1:4
           if j == invhash(tag(iold))
               continue;
           end
           group(:,:,inew) = group(:,:,iold) * gens(:,:,j);
           tag(inew) = j;
           inew = inew+1;
       end
       if inew > uplim
           break
       end
   end
   num(n+1)=inew;
   if inew > uplim
       N=n;
       break
   end
end

fprintf('Made %d group elements\n',min(uplim,(num(N)-1)));


% Set up display

figure
set(gcf,'Position',[0 0 1120 626])
subplot(121);
axis([-10 10 -10 10])
daspect([1 1 1])
subplot(122);
axis([-10 10 -10 10])
daspect([1 1 1])

epsilons = [1,-1,1i,-1i];
s = 2; t = 3;
zetas = 0.5*exp(2i*pi*(1:11)/11);

M = zeros((s+1)*(t+1)-1,(s+1)*(t+1));

% Compute the values of f and g at the marked points in zeta and
↪  set up the matrix
% whose kernel will be the coeffients in A

for zetaindex = 1:length(zetas)
    zeta = zetas(zetaindex);
```

```
    summands = zeros(s+1,t+1);
    for k = 0:s
        for ell = 0:t
            poincvalues = [0,0,0,0];
            for epsindex = 1:4
                epsilon = epsilons(epsindex);
                for i = 1:min(uplim,(num(N)-1))
                    matrix = group(:,:,i);
                    a = matrix(1,1); b = matrix(1,2); c =
↪   matrix(2,1); d = matrix(2,2);
                    poincvalues(epsindex) = poincvalues(epsindex)
↪   + ((c*zeta+d)^(-4))/((a*zeta+b)/(c*zeta+d) - epsilon);
                end
            end
            summands(k+1,ell+1) =
↪   poincvalues(1)^k/poincvalues(2)^k*poincvalues(3)^ell/poincvalues(4)^ell;
            M(zetaindex,:) = reshape(summands.',1,[]);
        end
    end
end

assert(rank(M)==(s+1)*(t+1)-1);
K = null(M);
A = (reshape(K,[t+1,s+1])).';

% Plot limit set on left and curve on right.

sgtitle(sprintf('radius = %0.5f',R));
subplot(121);
cla
hold on
for i = 1:min(uplim,(num(N)-1))
    matrix = group(:,:,i);
    a = matrix(1,1); b = matrix(1,2); c = matrix(2,1); d =
↪   matrix(2,2);
    lim = (a*limseed+b)/(c*limseed+d);
    if 1/abs(c) > 0.1
        circle(real(-d/c),imag(-d/c),1/abs(c));
    end
    plot(real(lim),imag(lim),'.k','MarkerSize',.5);
    plot(real(epsilons),imag(epsilons),'.','Color',[0 0.5
↪   0],'MarkerSize',2);
    plot(real(zetas),imag(zetas),'.','Color',[30/255 144/255
↪   255/255],'MarkerSize',2);
end
hold off
subplot(122)
cla
hold on
fimplicit(@(x,y) real(curveEquation(x,y,A,s,t)),[-10 10],'r')
```

```
fimplicit(@(x,y) imag(curveEquation(x,y,A,s,t)),[-10 10],'b')
hold off
set(gcf,'Position',[0 0 1120 626])
drawnow

for k = 0:s
    for ell = 0:t
        fprintf('(%f + %f I) X^%d Y^%d +
 ↪  ',real(A(k+1,ell+1)),imag(A(k+1,ell+1)),k,ell);
    end
end
fprintf('\n');


function z = curveEquation(x,y,A,s,t)
    z=0;
    for k = 0:s
        for ell = 0:t
            z = z + A(k+1,ell+1) .* x.^k .* y.^ell;
        end
    end
end

function h = circle(x,y,r)
    ct = 2*pi/(ceil(2*pi*r*100));
    th = 0:ct:2*pi;
    xunit = r * cos(th) + x;
    yunit = r * sin(th) + y;
    h =
 ↪  scatter(xunit,yunit,1,'MarkerFaceColor','r','MarkerEdgeColor','none');
    h.MarkerFaceAlpha = .3;
end
```

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