Limit sets of con

Geometric group theory of i

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*In the strip of images on the right side of the
we show a deformation from one discrete gr
conformal maps of the plane to another, thro
family of indiscrete groups that uniformise co
faces. The black dots are the images of a single
under the group action.*

Associated arXiv preprint: Alex Elzenaar. *A case study in deforming
through cone manifolds*. To appear. 2025

The hyperbolic plane

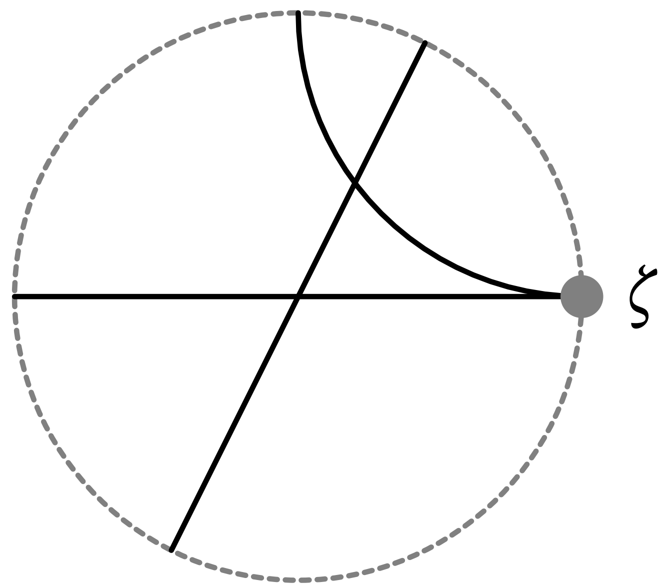
Euclid, c. 300BC, gave five postulates for geometry. His first four n
basis of modern metric geometry: they govern existence of geodesi
of circles, and well-definedness of angles. His fifth axiom is harder to

> If a straight line falling on two straight lines makes the interior
> the same side less than two right angles, the two straight lines, if
> indefinitely, meet on that side on which are the angles less than
> angles [5].

If you're confused, you're not alone: it seems like it should be a the
than a postulate. In fact, mathematicians spent over a thousand ye
deduce it from his other four axioms. They failed for one simple r
exist metric geometries in which lines can converge without meeti

Hyperbolic geometry, $\mathbb{H}^2$, is a metric geometry on the disc $\{z \in$
Writing down the explicit metric is possible but unenlightening [1,
need is that that geodesics in this geometry are circular arcs ortho
boundary circle, that this circle is a 'horizon' which is infinitely fa
every point in $\mathbb{H}^2$, and that angle measure is the usual angle measure
with a protractor. Here is a picture of three geodesics in $\mathbb{H}^2$ forming
ample to Euclid's postulate (the point ζ is on the horizon, so the tw

converging there get arbitrarily close but do not meet in $\mathbb{H}^2$):



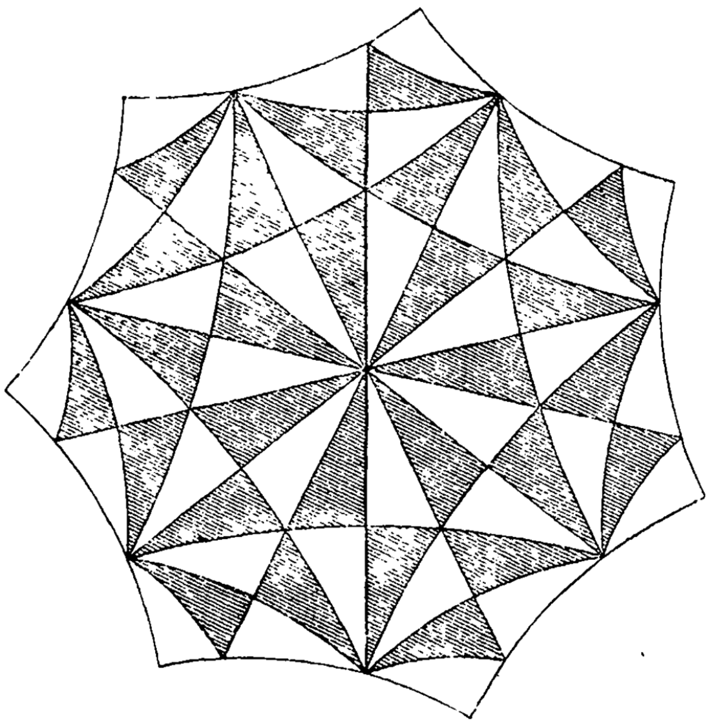
Hyperbolic triangle groups

Euclid studied lines and circles drawn in the sand and the ways you c
complicated patterns from simple ones. One of the drivers of mathe
desire to understand such construction problems: how can massiv
complexity arise from simple pieces? One model for this kind of prob
theory, and to start with we consider groups with presentation

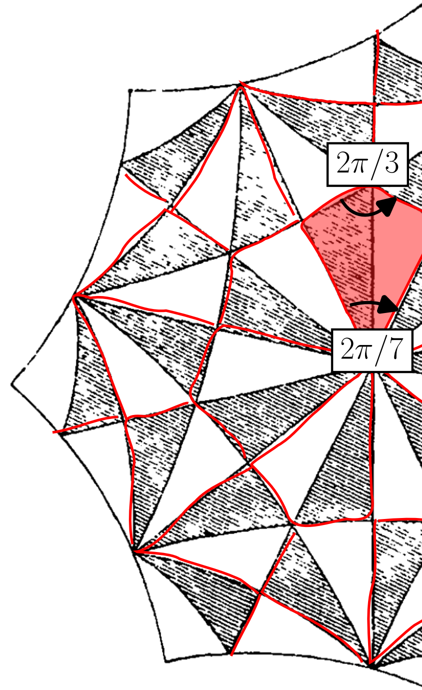
$$\langle x, y : x^p = y^q = (xy)^r = 1 \rangle.$$

These so-called *triangle groups* are actively studied: M. Conder a
present papers in this very meeting on their abstract group theory [

When p, q, and r are integers satisfying $p^{-1} + q^{-1} + r^{-1} < 1$ the
a hyperbolic triangle with angles π/p, π/q, π/r that tiles the hype
We take the example on the left, $(p, q, r) = (2, 3, 7)$, from Fricke's 1
triangle groups [6, Fig. 2]:



lei Art in jedesmal 63 *äquivalente Kreisbogendreiecke mit den Winkeln* $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$ *zerlegen lässt.* Die hierneben abgedruckte Figur 2 giebt die eine dieser beiden Eintheilungen an. Dieselbe ist augenscheinlich unsymmetrisch, und indem wir das Siebeneck an einer seiner Symmetrielinien umlegen, werden wir zur anderen Eintheilung geführt. Dass aber die 63 Dreiecke in der That äquivalent sind, sieht



The symmetry group of the tiling is the group generated by refle
sides of one triangle. Since it contains reflections, it isn't orientatio
Instead of individual triangles, look at the tiling on the right by qua
has symmetry group generated by the two indicated rotations, and
tation (▲) with $p = 3$ and $q = 7$. All tessellations of the hyperb
(p, q, r)-triangles have symmetry groups with index preserving hal
(▲).

Indiscrete groups and cone manifolds

If we draw a (geodesic-sided) triangle $\triangle$ in $\mathbb{H}^2$ with arbitrary angle
form the group of reflections in its sides and take the orientation-pr
G. However, this group G is no longer discrete, and the triangle

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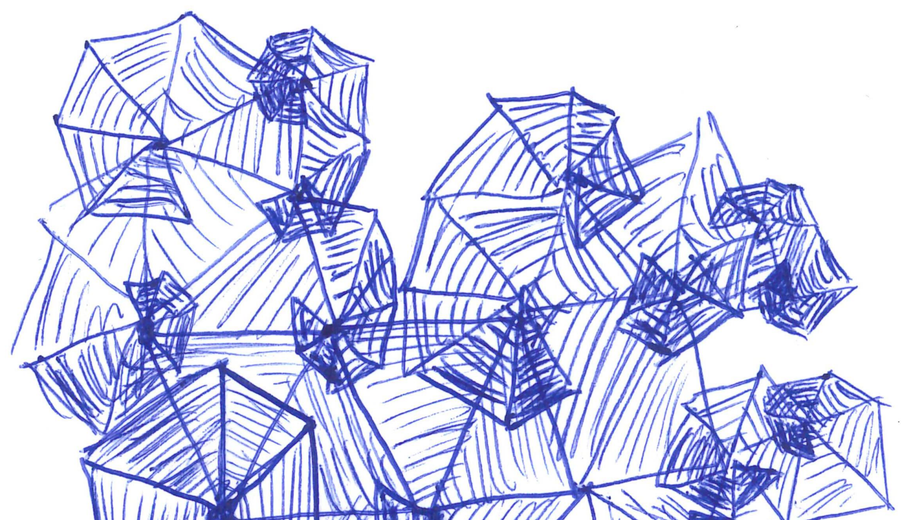
tiles $\mathbb{H}^2$: if you keep adding copies of the triangle around a
no longer 'close up' exactly. Every point of $\mathbb{H}^2$ will have in
of the triangle on top of it, all in slightly different position
produce a covering space of $\mathbb{H}^2$ where the group acts nicely
in geometric group theory [2, Chapter II.11], the metric sp
gluing copies of $\triangle$ together edge-by-edge (with singularit
still negatively curved: it is an infinitely branched non-sim
of $\mathbb{H}^2$ on which the group G acts to permute the copies of $\triangle$



now form the
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$\mathbb{C} : |z| < 1\}$.
§7.2]. All we
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ar away from
e that you get
a counterex-
wo geodesics

The quotient $\mathcal{X}_{\triangle}/G$ is a cone surface: it is a topological m
omorphic to a 2-sphere, and is smooth everywhere except
the projections of the vertices of $\triangle$. This point of view can
of triangles we can take arbitrary polygons with side-pairi
constructions like amalgamated products, HNN-extension
all up in complexes of indiscrete groups that give the cone

The situation one dimension higher, in 3D, is very intere
group G of $\mathrm{Isom}^+(\mathbb{H}^3)$ is called a *Kleinian group* and $\mathbb{H}^3/G$ is
with holonomy group G. The group G also acts on $\partial\mathbb{H}^3 = \mathbb{S}^2$
mal maps, and the projection of the maximal set where it
on $\mathbb{S}^1$ is the conformal boundary of $\mathbb{H}^3/G$. The general r
tion theory of Kleinian groups (and hence 3-orbifolds) is
the analysis of this action. If M is a cone manifold then we
holonomy group, but now it's indiscrete: it naturally acts
on a branched complex rather than $\mathbb{H}^3$ and it acts confor
complex over the sphere rather than on $\mathbb{S}^2$ itself.

can construct
ematics is the
ve systems of
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Genus two deformation

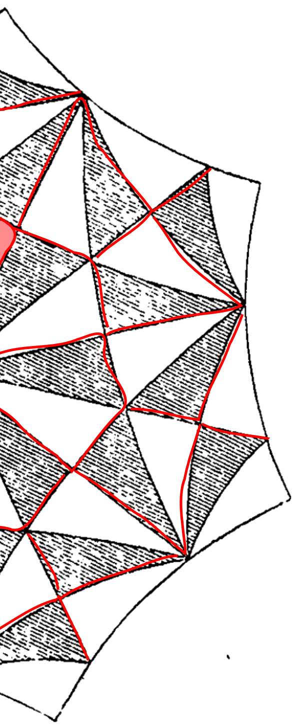
At the right running down the page, we show a family of ind
were constructed in [4] (the exact numeric parameters di
preprint). At the top, the group is discrete and uniformises t
is the exterior of the genus 2 surface minus two thrice-pu
the punctures joined in three cusps as shown; at the bottom
and uniformises the exterior of the genus 2 surface minus
termediate groups correspond to cone angles where the de
the top-most manifold becomes a singular arc of increasin
the bottom the angle becomes 2π and the singularity is go

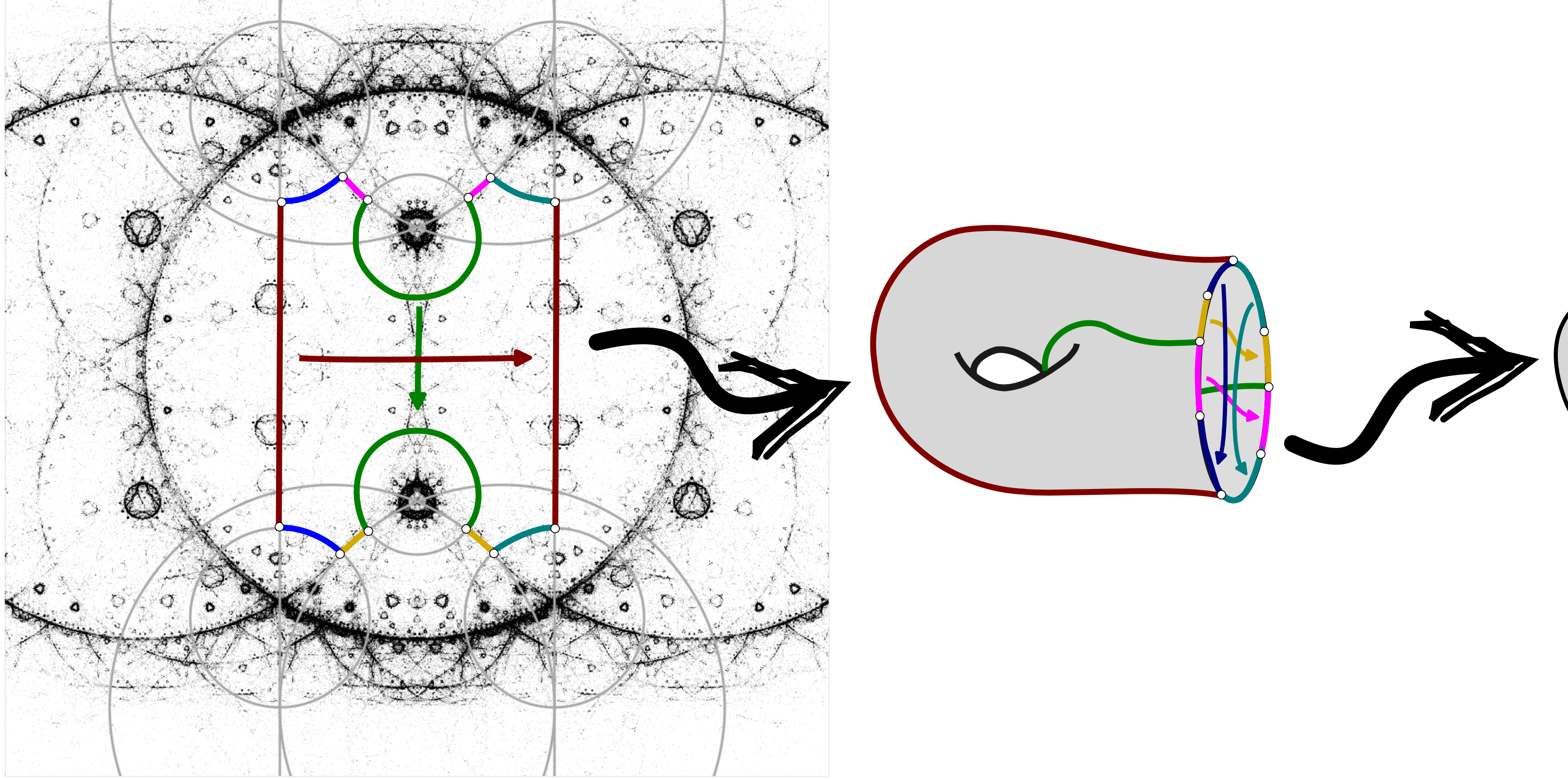
These groups are all obtained from triangle groups via
extensions and amalgamated products. The resulting gro
variant component with a fundamental domain that has tw
glues up to the genus two component as shown:

(▲)

nd D. Young
[3, 7].

n there exists
rbolic plane.
893 paper on





The two triangle group components are off the top and bo
but can you see them in the top few pictures on the right?

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matics 91. Springer, 1983.

ections in the
n-preserving.
drilaterals: it
d has presen-
olic plane by
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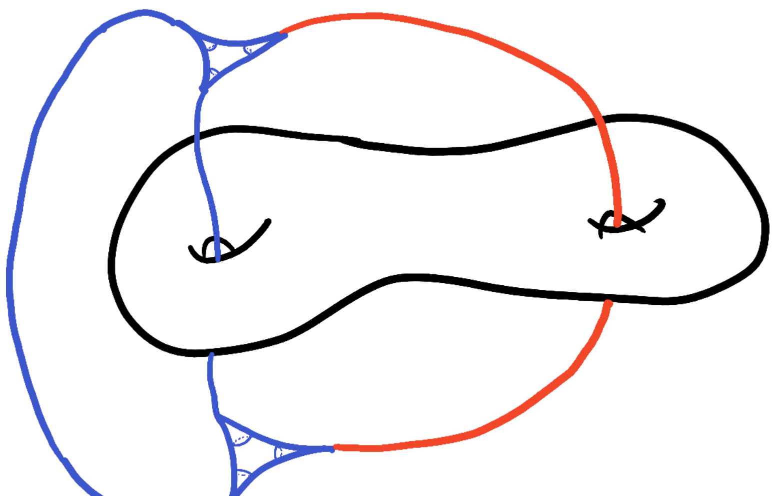

a single vertex, it will
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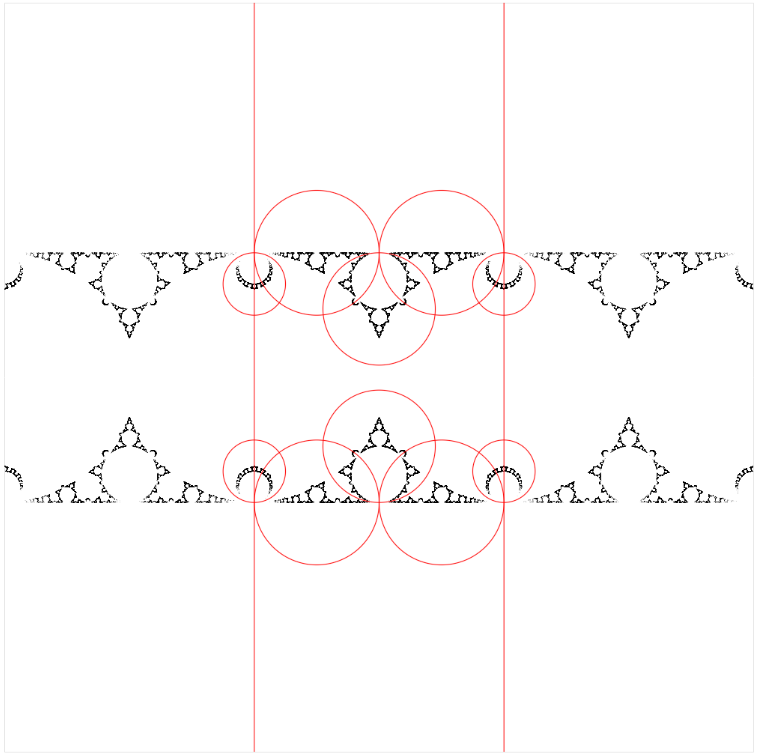
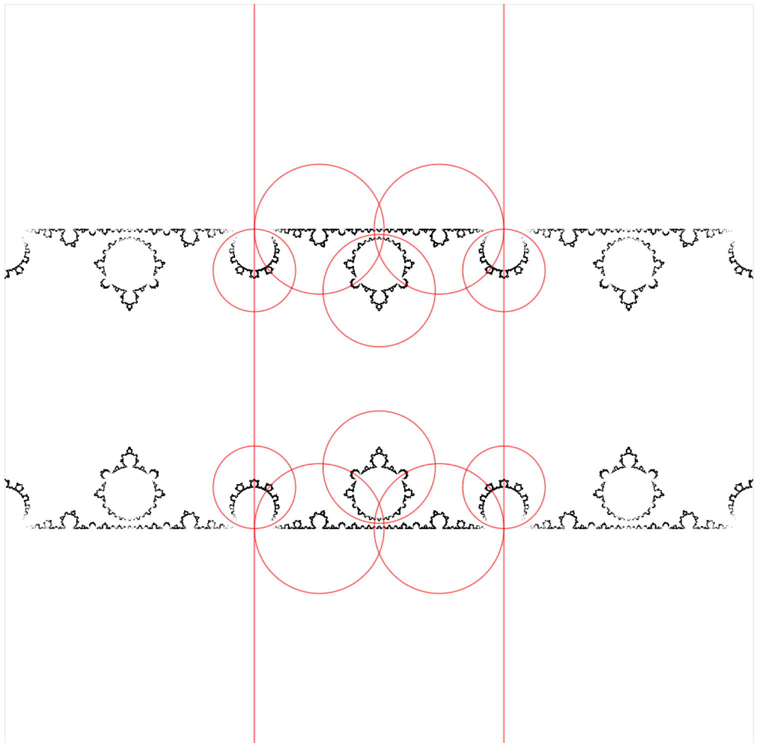
y. By standard results
pace $\mathcal{X}_{\triangle}$ obtained by
ties at the vertices) is
nply-connected cover
$\triangle$.





nanifold that is home-
three singularities at
be extended: instead
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s, and package them
manifolds structure.

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a hyperbolic orbifold
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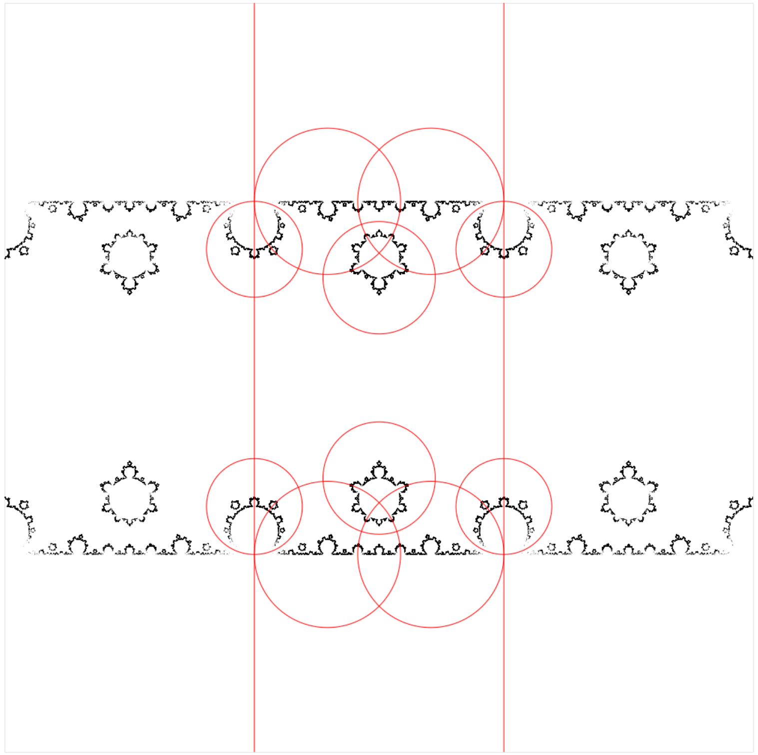

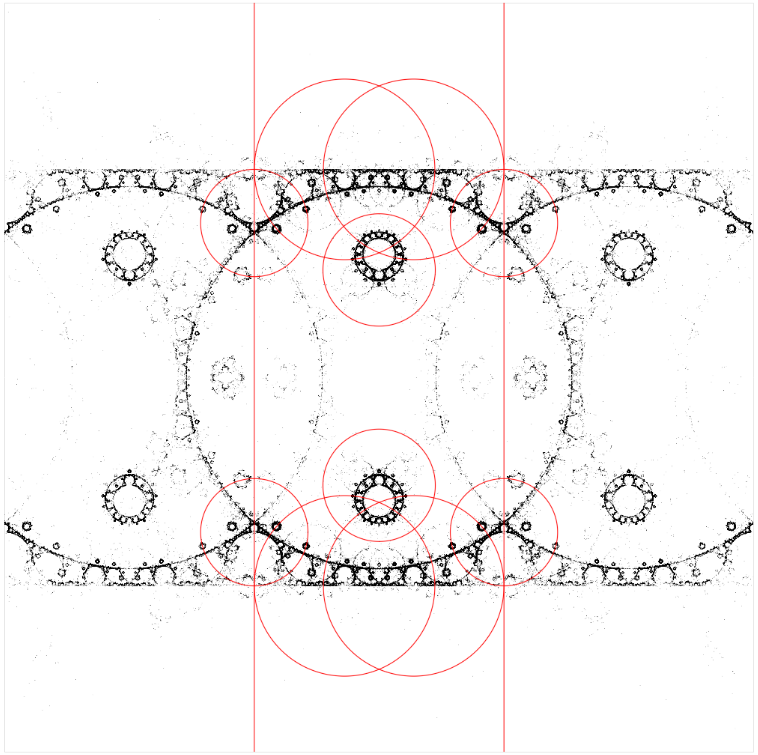

discrete groups which
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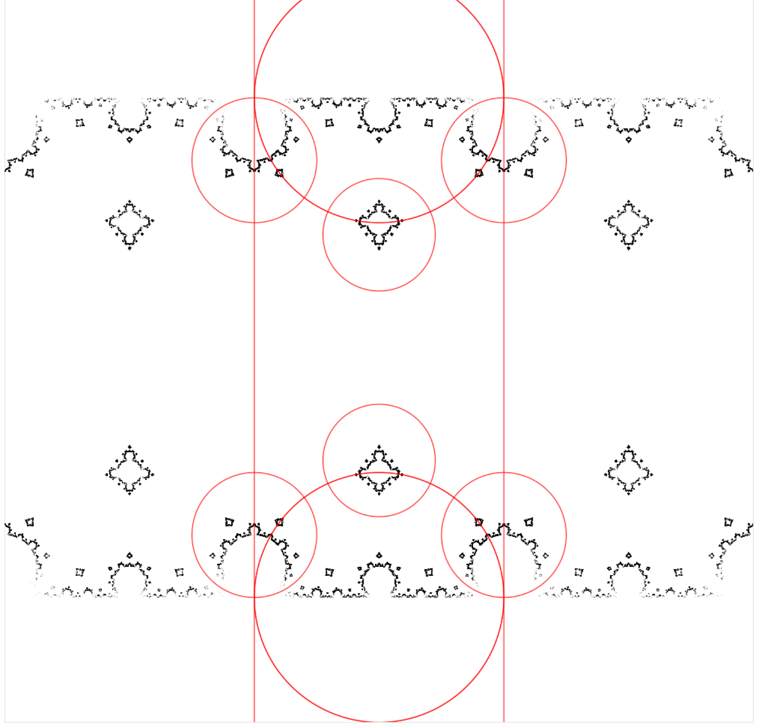

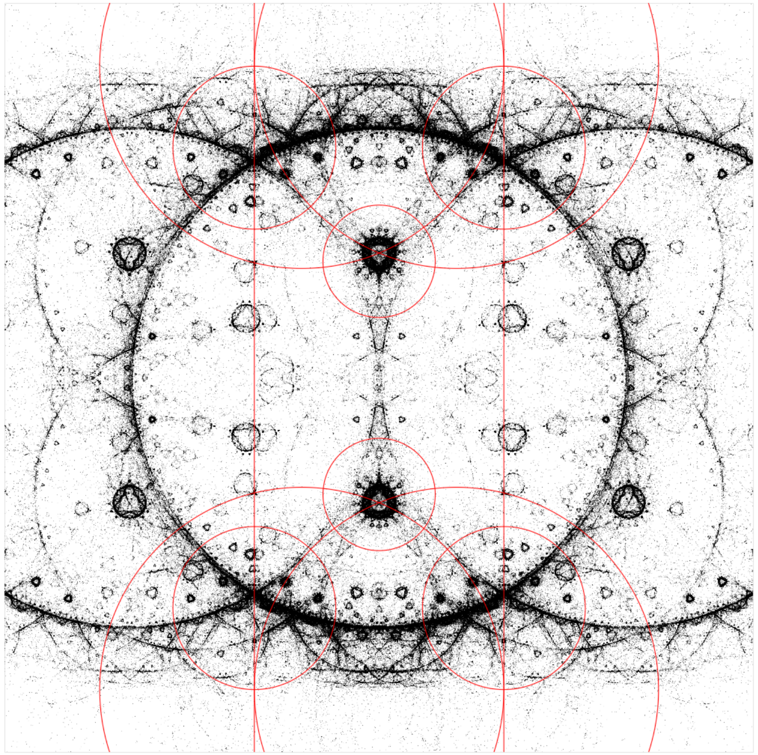

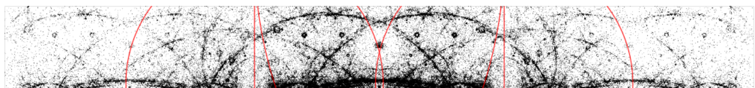

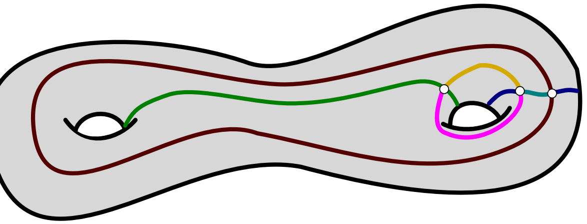

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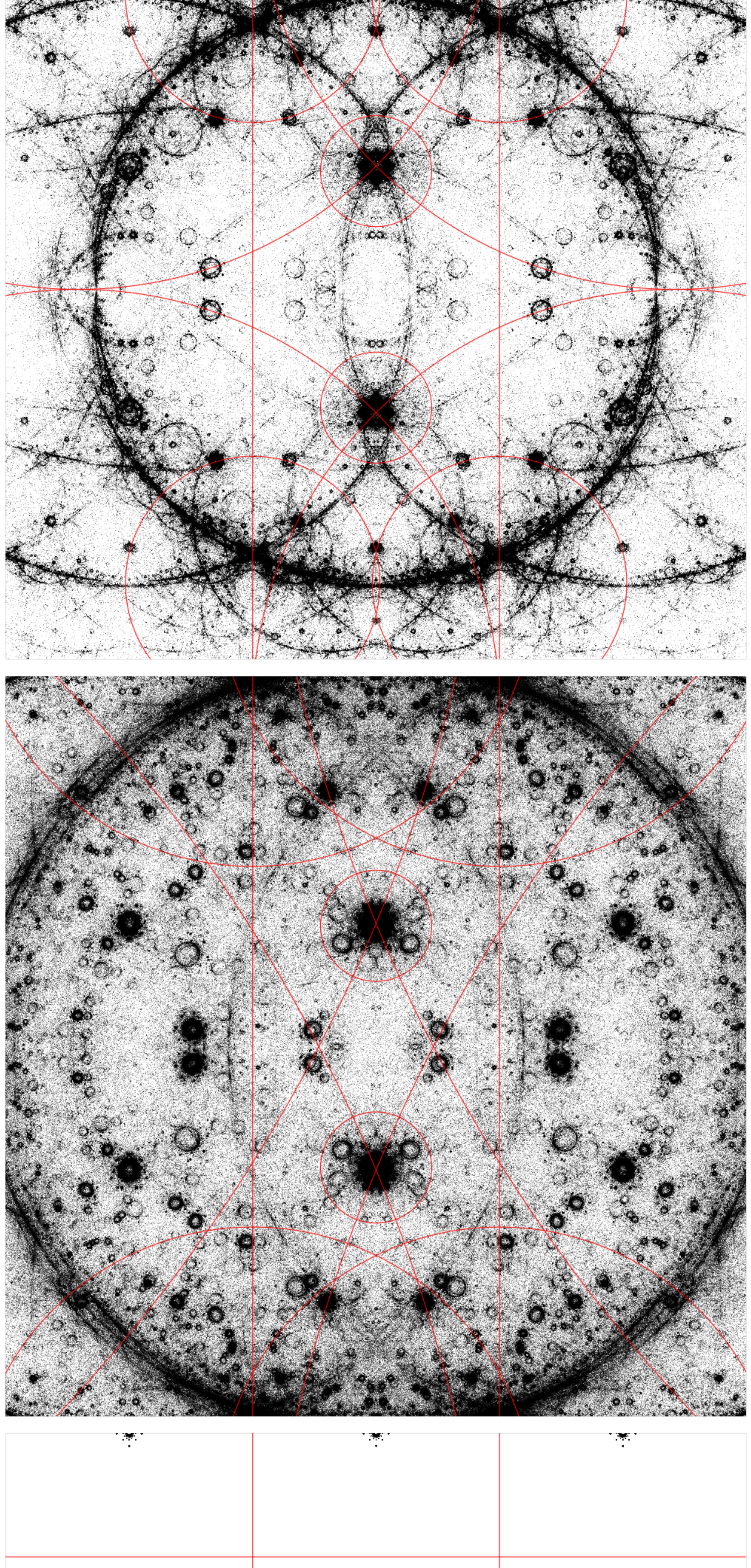

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