Approximations of the Riley slice

Alex Elzenaar, Gaven Martin & Jeroen Schillewaert *

Abstract

Adapting the ideas of L. Keen and C. Series used in their study of the Riley slice of Schottky groups generated by two parabolics, we explicitly identify 'half-space' neighbourhoods of pleating rays which lie completely in the Riley slice. This gives a provable method to determine if a point is in the Riley slice or not. We also discuss the family of Farey polynomials which determine the rational pleating rays and their root set which determines the Riley slice; this leads to a dynamical systems interpretation of the slice.

Adapting these methods to the case of Schottky groups generated by two elliptic elements in subsequent work facilitates the programme to identify all the finitely many arithmetic generalised triangle groups and their kin.

1 Introduction

The usual route to the Riley slice is through the theory of Kleinian groups, or more precisely Schottky groups, generated by two parabolic elements in $\mathrm{PSL}(2,\mathbb{C})$. This problem has a long history begining with Sanov [51] in 1947, Brenner [10] and Chang, Jennings, and Ree [11], and Lyndon and Ullman [32]; a detailed literature survey and history of the Riley slice together with some explanation of the background material may be found in our proceedings article [16]. The geometric group theory which is relied upon relies

*AE+JS, Department of Mathematics, The University of Auckland,
email: aelz176@aucklanduni.ac.nz,j.schillewaert@auckland.ac.nz
GM, Institute for Advanced Study, Massey University, Auckland, New Zealand.
email: g.j.martin@massey.ac.nz
Parts of this work appear in the MSc thesis of the first author. Work of authors partially supported by the New Zealand Marsden Fund and by a University of Auckland FRDF grant. We thank the anonymous referees for their useful remarks and suggested improvements.
Keywords. Kleinian groups, Teichmüller space, hyperbolic geometry.
MSC classification (2020). 20H10, 30F40, 30F60, 57K32

upon various versions of what are now known as combination theorems and these are explained in some detail in Maskit's book [41]; this book also includes the basic theory of Kleinian groups, which we briefly discuss now in order to fix notation.

Recall that a Kleinian group may be equivalently defined as (a) a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, or (b) a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. The relationship between these two definitions comes from the fact that isometries of hyperbolic 3-space are uniquely characterised by their actions on the sphere at infinity: namely, there is a natural bijection between $\mathrm{Isom}^+(\mathbb{H}^3)$ and the group of conformal automorphisms of S^2. After identifying S^2 with the Riemann sphere $\hat{\mathbb{C}}$, we may characterise the conformal automorphisms as none other than the Möbius transformations, the maps of the form $z \mapsto \frac{az+b}{cz+d}$ ($a, b, c, d \in \mathbb{C}$ with $ad-bc \neq 0$). Performing one final identification, of $\hat{\mathbb{C}}$ with $\mathbb{PC}^1$, we see that the Möbius transformations are in natural correspondence with $\mathrm{SL}(2,\mathbb{C})$ via the identification

$$\left(z \mapsto \frac{az+b}{cz+d}\right) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Observe finally that $\mathrm{SL}(2,\mathbb{C})/\{\pm I\} = \mathrm{PSL}(2,\mathbb{C})$, but in the world of Möbius transformations, as long as the transformation is not of order 2, we can always multiply both the numerator and denominator through by $\sqrt{-1}$ if necessary to normalise the determinant of any matrix representative to 1 without changing the geometry of the map—in other words, we can always lift non-involutive elements from $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$ without issue as long as we are careful to always pick representatives of determinant 1. In fact we can always lift an entire discrete group, as long as it has no two-torsion [14].

The Riley slice $\mathcal{R}$ is the moduli space parameterising the Kleinian groups, generated by two parabolic elements, whose domains of discontinuity have quotient surface a four-times punctured sphere S^2_4. More precisely, define a family $(\Gamma_\mu)_{\mu\in\mathbb{C}\setminus\{0\}}$ of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ by

$$\Gamma_\mu := \left\langle X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, Y_\mu = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \right\rangle;$$

The assumption $\mu \neq 0$ implies that Γ_μ is not abelian and has free subgroups of all ranks. The group Γ_μ acts on the Riemann sphere $\hat{\mathbb{C}}$ and, and there is a maximal open set $\Omega(\Gamma_\mu) \subset \mathbb{C}$ (possibly empty) on which this group acts discontinuously (the *ordinary set*); the complement of this set in $\hat{\mathbb{C}}$ is the *limit set* $\Lambda(\Gamma_\mu)$ and is the closure of the set of fixed points of elements of

Γ_μ (so, since ∞ is fixed by f, $\infty \in \Lambda(\Gamma_\mu)$).[1] The quotient $\Omega(\Gamma_\mu)/\Gamma_\mu$ is a Riemann surface. When Γ_μ is free and discrete, the Riemann surfaces so obtained are supported on one of three homeomorphism classes of topological space: the empty set; a disjoint union of two three-times punctured spheres; and a four-times punctured sphere [42]. It happens that the first two types of space may be viewed as geometric deformations of four-times punctured spheres, and so it is natural to consider the set of all μ such that Γ_μ is free and discrete and such that this quotient is a four-times punctured sphere; the boundary of this set is then the set of μ such that the quotient is one of the other two types of surface.

Thus, the Riley slice is defined by

$$\mathcal{R} = \{\mu \in \mathbb{C} : \Omega(\Gamma_\mu)/\Gamma_\mu \text{ is topologically a four-times punctured sphere}\}.$$

Denote the Möbius transformations of $\hat{\mathbb{C}}$ representing X and Y by f and g respectively, so

$$f(z) = z + 1 \text{ and } g(z) = \frac{z}{\mu z + 1}$$

for $z \in \mathbb{C}$. We will abuse notation and write Γ_μ for $\langle f, g \rangle$ as well as for the matrix group.

Notation. Often we will be working with a fixed Γ_μ; in this situation we will often write Γ and Y for Γ_μ and Y_μ without comment.

One can also view the Riley slice as the quotient of the Teichmüller space $\mathcal{T}_{0,4}$ of genus 0 surfaces with 4 punctures by a group generated by a Dehn twist τ about a simple closed curve which separates one pair of punctures from another in S^2_4. This is a special case of a more general theorem: the Riley slice can be equivalently defined as the space of quasiconformal conjugates of any group which it parameterises [33], and given any finitely generated non-elementary Kleinian group G the quasiconformal deformation space is the quotient of the Teichmüller space of $S = \Omega(G)/G$ by the subgroup of the mapping class group of S generated by Dehn twists about the simple closed curves which bound compression discs in the underlying 3-manifold; this result is attributed by Bers [7, §2.4] to Bers and Greenberg [8] and Marden [36], and modern expositions can be found in the textbook of Matsuzaki and Taniguchu [43] and in Chapter 5 of the textbook by Marden [37]. In the Riley slice case, $\mathcal{T}_{0,4} = \mathbb{H}^2$ and the relevant subgroup of the

[1] One may also define the ordinary set in the following way, if Γ_μ is discrete and non-elementary (true for every group in $\mathcal{R}$): it is the largest domain in $\mathbb{C}$ on which the transformations of Γ_μ are equicontinuous. In this way the ordinary set is analogous to the *Fatou set* of a dynamical system, as in Section 3.2 below.

mapping class group is generated by a single parabolic isometry τ, so the quotient $\mathbb{H}^2/\langle\tau\rangle$ is a disc with a puncture and admits an intrinsic hyperbolic metric induced by the Teichmüller metric.

The theory of Keen and Series [27] endows the Riley slice with a foliation structure. This structure consists of a set of curves parameterised by $\mathbb{Q}$ which radiate out from the boundary of the slice and which are dense in the slice (the so-called *rational pleating rays*) together with a natural completion (in the sense that we may add curves parameterised by $\mathbb{R}\setminus\mathbb{Q}$ in order to fill out the entire slice). In Figure 1 we illustrate the Riley slice together with a selection of rational pleating rays.

The exterior of the Riley slice (the bounded region of Figure 1) is also of interest: it includes all the groups Γ_μ which are discrete but not free, among them are, for instance, all hyperbolic two bridge knot groups. These lie along or at the endpoint of a rational pleating ray. Recently [2, 3] gave a complete description of all these discrete groups outside the Riley slice as Heckoid groups and their near relatives. For each such group there are at most two Nielsen classes of parabolic generating pairs. The boundary of the Riley slice is a Jordan curve [48], and it is believed to have outward directed cusps (this was established for the related Earle and Maskit slices by Miyachi [45]). The non-discrete groups are generically free, but every neighbourhood of a non-discrete group contains a supergroup containing any given two parabolic groups—discrete or otherwise—and a group with any prescribed number of distinct Nielsen classes, [39].

Our motivation for the study of the Riley slice here is the continuation of a longstanding programme to identify all the finitely many generalised arithmetic triangle groups in $\mathrm{PSL}(2,\mathbb{C})$ [12, 21, 23, 34, 40]. For this programme, one needs quite refined computational descriptions of other one complex dimensional moduli spaces such as the moduli space of $S^2_{p,q}$, the 2-sphere with four cone points (two of order p and two of order q). Arithmetic criteria developed and described in [22] identify those algebraic integers in $\mathbb{C}$ which give rise to discrete subgroups of arithmetic lattices. Obtaining degree bounds, and numerically identifying these points, is a challenging task, see [19, 34, 40]. Once an algebraic integer is identified there are further problems. *A priori*, the relevant group is discrete, but we need to know if it is in fact free on the generators (that is, we need to characterise the analogue of the Riley slice) and if it is not, identify the abstract group and hyperbolic 3-manifold quotient. We illustrate some relevant data in Figure 2.

In order to be able to resolve these issues we need to be able to *provably* decide if a point in $\mathbb{C}$ actually lies in the Riley slice or its analogue. Solving this problem also has computational applications, including identifying all

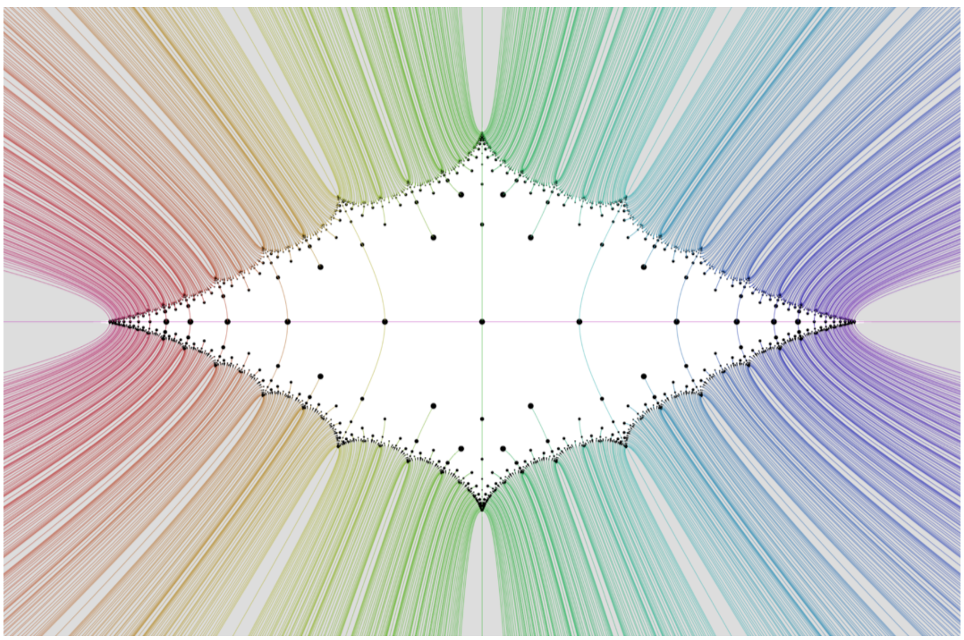

Figure 1: The Riley slice is the unbounded region, foliated by 'rational pleating rays' (the coloured curves). The symmetries of this space include complex conjugation and $\mu \leftrightarrow -\mu$ interchanging g with g^{-1}. Picture courtesy of Y. Yamashita.

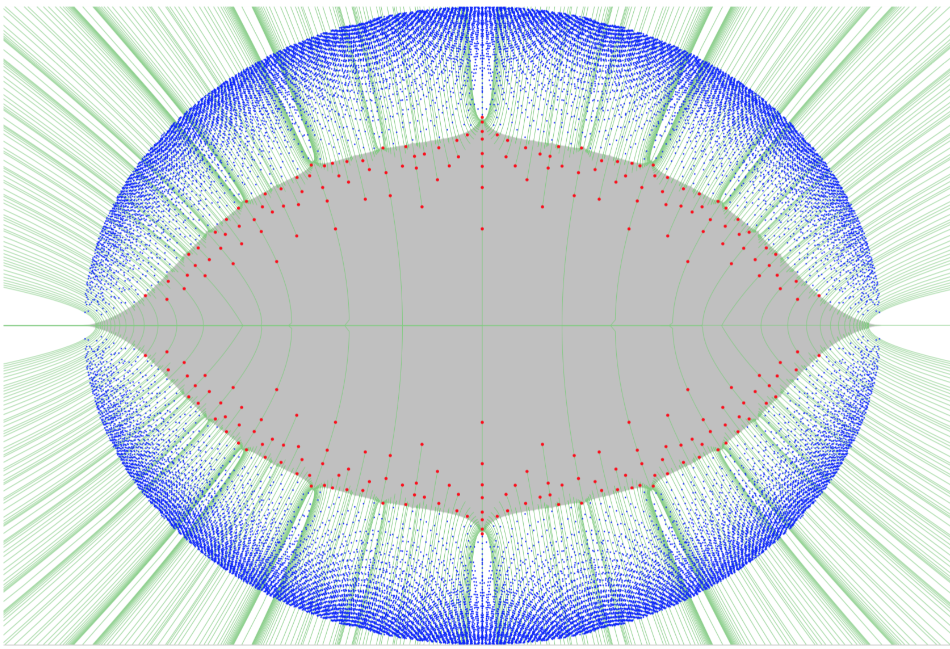

Figure 2: $\mathcal{R}_{3,3}$. The analogue of the Riley slice is the exterior of the grey region and foliated by rational pleating rays. The $15,909$ algebraic integers satisfying the arithmetic criteria described found by Flammang and Rhin [19] above are blue (if, by visual inspection, are outside the grey region) and red (if, by visual inspection, are inside the grey region). These red points yield either lattices or rigid groups with circle packing limit sets in $\mathrm{PSL}(2,\mathbb{C})$ that are generated by two elliptic elements of order three. Picture in collaboration with Y. Yamashita.

the finitely many generalised arithmetic triangle groups in $\mathrm{PSL}(2,\mathbb{C})$ [12, 21, 23, 34].

Main results. Our first main result, Theorem 3.5 (p.16), sets up a dynamical system whose stable region contains $\mathcal{R}$; this gives a system of polynomials which we call $Q_{p/q}$ whose filled Julia sets lie in the exterior of the Riley slice. As an incidental consequence of the theory used to derive this result, we characterise the Farey word traces of the discrete groups which lie on pleating ray extensions (Theorem 3.4).

With the technology of Keen and Series, we may identify whether a point lies on a rational pleating ray, but the union of these rays has measure 0 so it is not so useful to check whether a point lies in the Riley slice. In this paper, we show that a well defined open neighbourhood of each rational pleating ray lies in the Riley slice (or its torsion generated analogue) so that we can 'capture' points. This is the content of our second main theorem, Theorem 4.1 (p.23), which extends the theory of Keen and Series to give such neighbourhoods.

Structure of the paper. In section 2 we introduce the Farey words, which represent simple closed curves on the four-times punctured sphere which are not boundary-parallel; the basis of the theory is the relationship between the combinatorics and algebra of these words and the deformations of the curves that they represent. In Section 3 we define the Farey polynomials $P_{p/q}(\mu)$ and give our first main result, Theorem 3.5, together with some conjectures on the structure of the Farey polynomials and the dynamical system that they generate. In Section 4 we motivate our second main result, Theorem 4.1, and place it in context with prior work by Lyndon and Ullman. We conclude the paper in Section 5 with the proof of Theorem 4.1 together with some related estimates.

Our related work. In an upcoming paper [18] we extend the theory of neighbourhoods of cusp points to the elliptic setting (an example of an elliptic Riley slice is given in Figure 3); this is more than just a slight modification of the argument for the parabolic case and it relies on more accurate estimates like those hinted at in Figure 11 on page 24 below.

We have also recently posted a preprint [17] which gives various combinatorial identities involving the Farey words and their trace polynomials, including a much more efficient method for determining the Farey polynomials computationally via a recurrence formula, along with some applications to

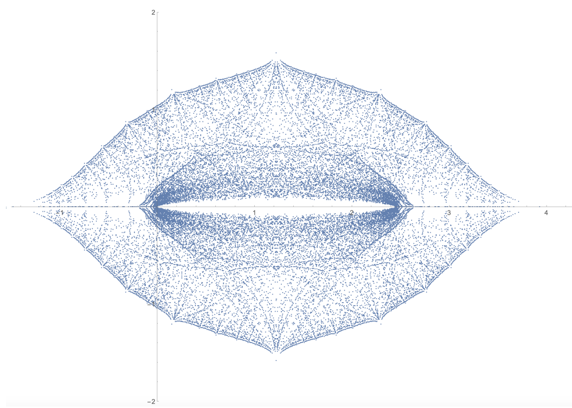

Figure 3: Points approximating the slice of Schottky space corresponding to a sphere with paired cone points, of respective cone angles $2\pi/6$ and $2\pi/8$.

the geometry of the Riley slice boundary. Closed forms for certain sequences of Farey polynomials can also be computed, allowing the approximation of the Riley slice near the cusp point at $+4$ by a sequence of well-behaved neighbourhoods of the form described in Section 4 of the current paper. In Figure 4 we include a picture produced from the first 100 polynomials of this approximating sequence; it took just under two minutes to compute the roots symbolically in `Mathematica` from scratch.

2 Farey words and their traces

A *Farey word* is a word in x and y representing a simple closed curve on the four-times punctured sphere which is not homotopic to a cusp (Figure 5). The definition of these words in terms of rational slopes p/q is explained in [27, §2.3] with some corrections in [29]. The exact details are not useful to us here; however, it will be useful to know the broad structure of the Farey words. The main structural result is the following, which is essentially immediate from the combinatorial definition given in [27] and is well-known to experts in any case, see for example [30, Lemma 3.1].

Lemma 2.1. *Let p/q be a rational slope and $W_{p/q}$ a Farey word. Then $W_{p/q}$ has word length $2q$. Further:*

1. *If q is even, then there are $u, v \in \langle x, y\rangle$ such that (up to cyclic permutation, which changes neither the trace nor the represented geodesic)*

$$W_{p/q} = xux^{-1}u^{-1} = vyv^{-1}y^{-1}.$$

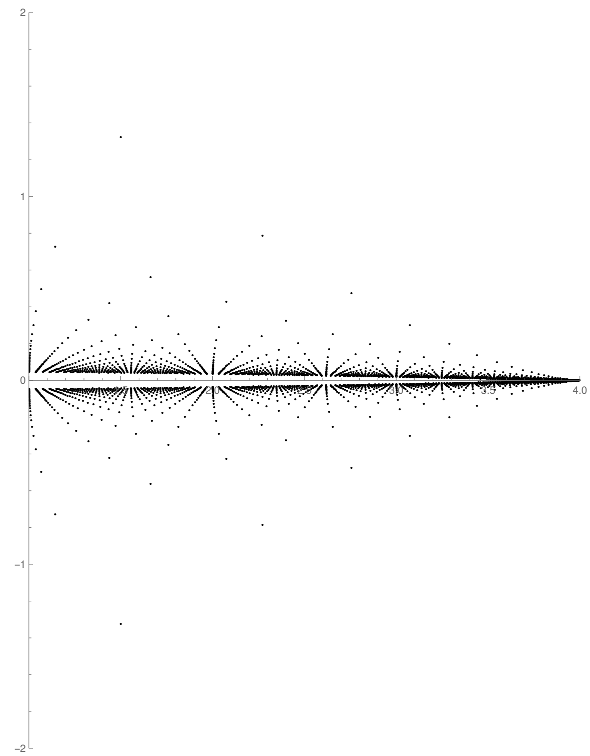

Figure 4: Inverse images of -2 under a sequence of Farey polynomials approximating the $+4$-cusp.

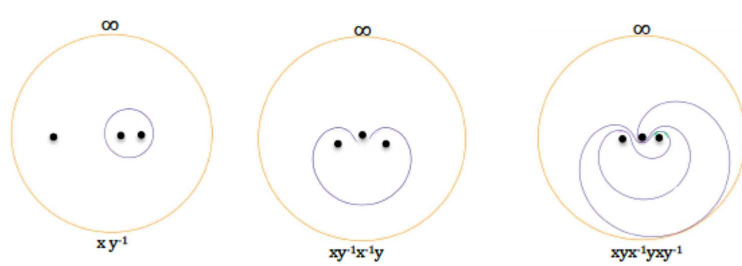

Figure 5: Simple closed curves on a 4-times punctured sphere, from left: $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$. One puncture is at ∞, disks bound neighbourhoods of the other punctures.

Table 1: Farey words and their corresponding cusp points.

p	q	Farey word $W_{p/q}$	$Q_{p/q}(\mu) = P_{p/q}(\mu) - 2$	Approx. cusp point
1	2	$xyx^{-1}y^{-1}$	μ^2	$2i$
4	7	$xyx^{-1}y^{-1}xyx^{-1}yxy^{-1}x^{-1}yxy^{-1}$	$\mu\left(-1+2\mu-\mu^2+\mu^3\right)^2$	$0.427505 + 1.57557i$
3	5	$xyx^{-1}y^{-1}xy^{-1}x^{-1}yxy^{-1}$	$-\mu(1-\mu+\mu^2)^2$	$0.773301 + 1.46771i$
5	8	$xyx^{-1}y^{-1}xy^{-1}x^{-1}yx^{-1}y^{-1}xyx^{-1}yxy^{-1}$	$\mu^4(2-2\mu+\mu^2)^2$	$1.05642 + 1.30324i$
2	3	$xyx^{-1}yxy^{-1}$	$z(z-1)^2$	$1.5 + (\sqrt{7}/2)i$
5	7	$xyx^{-1}yxy^{-1}xy^{-1}x^{-1}yx^{-1}y^{-1}xy^{-1}$	$-\mu(1+2\mu-3\mu^2+\mu^3)^2$	$1.85181 + 0.911292i$
3	4	$xyx^{-1}yx^{-1}y^{-1}xy^{-1}$	$\mu^2(\mu-2)^2$	$2.27202 + 0.786151i$
4	5	$xyx^{-1}yx^{-1}yxy^{-1}xy^{-1}$	$\mu(1-3\mu+\mu^2)^2$	$2.75577 + 0.474477i$
1	1	xy^{-1}	$-\mu$	4

2. *If q is odd, then there are $u, v \in \langle x, y\rangle$ such that (again up to cyclic permutation)*

$$W_{p/q} = xuy^{(-1)^p}u^{-1} = vx^{(-1)^{p+1}}v^{-1}y^{-1}$$

In particular if q is even, then $W_{p/q}$ is a commutator in two different ways. The word length of $W_{p/q}$ is $2q$. $\square$

We can view $W_{p/q}$ as a word $W_{p/q}(\mu)$ in Γ_μ by performing the substitution $x \mapsto X$, $y \mapsto Y_\mu$:

$$W_{p/q}(\mu) = \begin{pmatrix} a_{p/q}(\mu) & b_{p/q}(\mu) \\ c_{p/q}(\mu) & d_{p/q}(\mu) \end{pmatrix} \qquad a_{p/q}d_{p/q} - b_{p/q}c_{p/q} = 1. \tag{1}$$

The entries of $W_{p/q}(\mu)$ are polynomials of degree q in the symbol μ. In particular, the trace $\operatorname{tr} W_{p/q}(\mu)$ is a polynomial of degree q in μ; we call this polynomial the *Farey polynomial* of slope p/q and denote it by $P_{p/q}(\mu)$. The polynomial $Q_{p/q} := P_{p/q} - 2$ also turns out to be very useful in the sequel. In Table 1, we list examples of Farey words with small-denominator slopes, together with their corresponding polynomials.

Notation. Just as we write f and g for the Möbius transformations associated to X and Y, we write $h_{p/q}(\mu)$ for the Möbius transformation associated to $W_{p/q}(\mu)$.

Our computational exploration of the matrices suggested the following result.

Theorem 2.2. *With the notation of (1),*

$$Q_{p/q}(\mu) = \operatorname{tr} W_{p/q}(\mu) - 2 = a_{p/q}(\mu) + d_{p/q}(\mu) - 2 = c_{p/q}(\mu).$$

Proof. Using Lemma 2.1 we will show this reduces to the well known Fricke identity in $\mathrm{PSL}(2,\mathbb{C})$ (see Formula (3.15) of [35]),

$$\operatorname{tr}[A,B] = \operatorname{tr}^2 A + \operatorname{tr}^2 B + \operatorname{tr}^2 AB - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB - 2.$$

We put $A = X^{-1}$ and $B = W_{p/q}$; in the following we suppress μ since it is fixed. Note that, by Lemma 2.1 and the conjugacy invariance of trace, $\operatorname{tr} X^{-1}W_{p/q}$ is either $\operatorname{tr}(X)$ or $\operatorname{tr}(Y)$ depending on whether q is even or odd. In our situation both of these traces are 2. Thus, supposing q is odd (the result if q is even follows with a similar calculation),

$$\begin{aligned} c_{p/q}^2 &= \operatorname{tr}[X, W_{p/q}] - 2 = \operatorname{tr}[X^{-1}, W_{p/q}] - 2 \\ &= \operatorname{tr}^2(X^{-1}) + \operatorname{tr}^2(W_{p/q}) + \operatorname{tr}^2(Y) - \operatorname{tr}(X^{-1}) \operatorname{tr}(W_{p/q}) \operatorname{tr}(Y) - 4 \\ &= 4 + (a_{p/q} + d_{p/q})^2 - 4 \operatorname{tr}(a_{p/q} + d_{p/q}) \\ &= (a_{p/q} + d_{p/q} - 2)^2 \end{aligned}$$

Thus $c_{p/q} = \pm(a_{p/q} + d_{p/q} - 2)$. When $\mu = 1$, the positive square root occurs. Since the identity is continuous in μ, it follows that the positive square root is the correct choice for all μ. $\square$

Remark. We use the Fricke identity above as this identity is quite central in our explorations of Farey polynomials. However, as pointed out by the referee there is an easier proof, a quick sketch of which goes as follows. The $(2,1)$ component of $W_{p/q}$ is invariant under conjugations which preserve X. By Lemma 2.1 $W_{p/q}$ is the product of the non-commuting parabolic elements X and $X^{-1}W_{p/q}$. Moreover $(X, X^{-1}W_{p/q}$ is conjugate to (X, Y'_{μ}) for some $\mu' \in \mathbb{C} \setminus 0$. By the above $c = \mu'$ and thus $c = \operatorname{tr}(W_{p/q}) - 2 = Q_{p/q}$.

The importance of Farey words in this setting is that in order for an isomorphic family of discrete groups to approach the boundary of a moduli space, a simple closed curve has to shrink to a cusp. That is, a word in the reference group has to become parabolic. The limit of a sequence of finitely generated Kleinian groups (where the number of generators is fixed) with generators converging is again a Kleinian group by Jørgensen's algebraic convergence theorem [25]. Thus we have the following result:

Lemma 2.3. *All the points in the Riley slice boundary represent discrete groups.* $\square$

The groups on the boundary of the Riley slice for which $\Omega(\Gamma_\mu)/\Gamma_\mu$ is a disjoint union of triply punctured spheres (the surface that is naturally obtained by shrinking a simple closed curve on S_4^2) are called *cusp groups*.

A point in $\partial\mathcal{R}$ which is not a cusp group has empty ordinary set (since the quotient cannot support moduli) and is degenerate [6].

Parabolic Möbius transformations are easily identified by the trace condition,

$$\beta(f) = \operatorname{tr}^2(f) - 4 = 0 \text{ if and only if } f \text{ is parabolic.}$$

Here, and in what follows, we have abused notation and written $\operatorname{tr}(f)$ for the trace of the matrix representative of f in $\mathrm{SL}(2,\mathbb{C})$. Keen and Series [27] study the boundary of the Riley slice by considering what happens for a fixed slope p/q as $\operatorname{tr}(f_{p/q}) \to -2$, $\operatorname{tr}(f_{p/q}) \in \mathbb{R}$. In fact Keen and Series show that the Farey polynomial $P_{p/q}$ has a branch so that the pleating ray

$$\mathcal{P}_{p/q} = P_{p/q}^{-1}\big((-\infty, -2]\big)$$

lies entirely in the closure of the Riley slice and meets the boundary at a point μ corresponding to a cusp group where $P_{p/q}(\mu) = -2$. These cusp groups have a limit set consisting of a circle packing; two examples are depicted in Figure 6 and the approximate positions of low-order cusp points are given in Table 1. A result of McMullen [44] shows these limits to be dense in the boundary of the Riley slice.

3 The dynamics of the Farey polynomials

In this section, we prove various algebraic and dynamical results of the Farey polynomials.

3.1 Discrete groups which lie on pleating ray extensions

We begin with three elementary lemmata.

Lemma 3.1. *Let $\langle A, B\rangle$ be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with $\operatorname{tr}^2 A = 4$, $\operatorname{tr}^2 B = 4$, $\operatorname{tr}(AB) - 2 = \nu \neq 0$, and such that neither A nor B is the identity. Then $\langle A, B\rangle$ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to the group Γ_ν.*

Proof. Let f and g be the Möbius transformations of $\hat{\mathbb{C}}$ representing A and B. Then f and g are parabolic with fixed points z_f and z_g. Since $\mu \neq 0$, the mappings f and g do not commute and $z_f \neq z_g$. Choose a Möbius transformation h so that $h(z_f) = \infty$, $h(z_g) = 0$, and $hfh^{-1}(0) = 1$. Then $hfh^{-1}(z) = z + 1$ and $hgh^{-1}(z) = z/(\alpha z + 1)$ for some $\alpha \in \mathbb{C}$. We compute that

$$2 + \alpha = \operatorname{tr} hfh^{-1}hgh^{-1} = \operatorname{tr} fg = \operatorname{tr}(AB) = 2 + \nu$$

and the result follows. $\square$

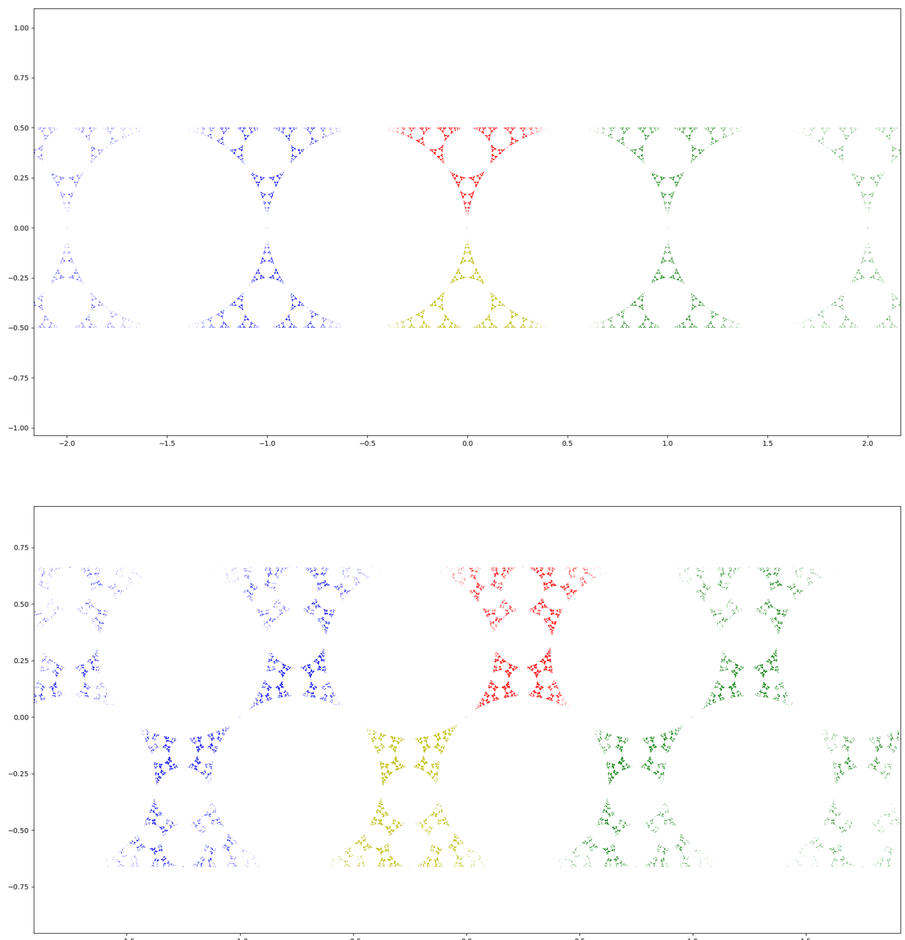

Figure 6: Circle packing limit sets of a cusp group. Top 1/2, bottom 2/3.

Lemma 3.2. *Let Γ_μ be discrete and $\mu \neq 0$. Then for all rational slopes p/q,*

$$|Q_{p/q}(\mu)| \geq 1$$

unless $P_{p/q}(\mu) = 2$. This estimate is sharp.

Remark. In fact, the union of all of the inverse images of the unit disc under the polynomials $Q_{p/q}$ fills the Riley slice complement [33, Lemma 3].

Proof of Lemma 3.2. Write $W_{p/q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose first that $c \neq 0$. Then the Shimitzu–Leutbecher inequality [31] applied to the discrete group $\langle f, h_{p/q} \rangle$ gives

$$1 \leq \operatorname{tr}[f, h_{p/q}] - 2 = |c|^2 = |a + d - 2|^2$$

which is the desired result by Theorem 2.2. If $c = 0$, then $h_{p/q}$ is parabolic and also fixes ∞.

The figure-eight knot-complement group is Γ_{μ_0} with $\mu_0 = \frac{1}{2}(1 + i\sqrt{3})$. With $p/q = 1/1$ this shows the inequality to be sharp, while the relator in this group is the 3/5-Farey word, and $h_{3/5} = \mathrm{Id}$, and

$$P_{3/5}(\mu) - 2 = -\mu(1 - \mu + \mu^2)^2$$

so $P_{3/5}(\mu_0) = 2$. $\square$

Every two-bridge link is obtained by taking the denominator closure of some rational tangle [49, Chapter 10]. In this way we obtain a classification of two-bridge links by rational numbers; this invariant was defined by Schubert [52] in the language of knots and Conway [13] in the language of tangles, and is called the *Schubert normal form* (q, p) or *rational form* q/p of the knot (in either case, q and p are integers with $\gcd(q, p) = 1$). The rational form of the figure eight knot complement is 5/3. More generally, the relator in the two bridge knot or link complement with rational form q/p is the p/q-Farey word.

Next we recall the following elementary result [27, Lemma 3.2].

Lemma 3.3. *Let Γ_μ be discrete. If $\operatorname{tr} h_{p/q}$ is real, then $\langle f, h_{p/q} \rangle$ is Fuchsian.*

Proof. The group $\langle f, f^{-1}h_{p/q} \rangle$ is generated by two parabolics whose product is hyperbolic. $\square$

As a consequence of these results, we can characterise the traces of the Farey words for discrete groups on the pleating ray extensions.

Theorem 3.4. *Let Γ_μ be discrete and $\mu \neq 0$. Let p/q be a rational slope and $P_{p/q}(\mu) \in \mathbb{R}$. Then either*

1. $P_{p/q}(\mu) \geq 6$ *or* $P_{p/q}(\mu) = 2 + 4\cos^2\left(\frac{\pi}{r}\right)$, *where* $r \geq 3$, *or*

2. $P_{p/q}(\mu) \leq -2$ *or* $P_{p/q}(\mu) = 2 - 4\cos^2\left(\frac{\pi}{r}\right) = -2\cos\left(\frac{2\pi}{r}\right)$, *where* $r \geq 3$.

In particular, on the extension of the pleating ray $\mathcal{P}_{p/q}$, that is $P_{p/q}^{-1}((-\infty, 0])$, the only allowable values for a discrete group are

$$P_{p/q}(\mu) = -2\cos(\frac{2\pi}{r}), \quad r \geq 3$$

Each of these values occurs.

Proof. The Möbius transformation $f^{-1}h_{p/q}$ is parabolic. There is an involution Φ conjugating f^{-1} to $f^{-1}h_{p/q}$. The group $\langle f, h_{p/q}\rangle$ is at most index two in $\langle \Phi, f\rangle$ and hence the latter is discrete. If $P_{p/q}(\mu) - 2$ is real, then $\langle f, h_{p/q}\rangle$ is Fuchsian by Lemma 3.3. We have (in the notation of [20])

$$\begin{aligned}\gamma(f, \Phi) &= \operatorname{tr}[f, \Phi] - 2 = \operatorname{tr}(f\Phi f^{-1}\Phi^{-1}) - 2 = \operatorname{tr}(f\Phi f^{-1}\Phi^{-1}) - 2\\ &= \operatorname{tr}(ff^{-1}h_{p/q}) - 2 = \operatorname{tr} h_{p/q} - 2 = P_{p/q}(\mu) - 2.\end{aligned}$$

and also

$$\beta(f) = \operatorname{tr}^2(f) - 4 = 0, \quad \beta(\Phi) = -4.$$

Then with the assumption that $\gamma(f, \Phi) \neq 0$ the discussion following (4.9) of [20, Theorem 4.5] tells us that $\langle f, \Phi\rangle$ is discrete only if either

- $\gamma(f, \Phi) \geq 4$, or $\gamma(f, \Phi) = 4\cos^2\frac{\pi}{r}$, $r \geq 3$, or
- $\gamma(f, \Phi) \leq -4$, or $\gamma(f, \Phi) = -4\cos^2\frac{\pi}{r}$, $r \geq 3$.

The converse follows from the classification of discrete groups in the Riley slice complement given by the papers [3, §2] and [2]. □

Remark. The special case of Theorem 3.4 for Fuchsian groups (that is, groups lying on the 0/1 and 1/1 pleating rays) was first proved by Knapp [28].

3.2 Dynamical properties

We first give a short overview of the dynamical systems terminology which we shall use (following for example [46]). A *dynamical system* is a set S (the *stable region*) together with a set Φ of functions $\phi : S \to S$ closed under iteration; in our case we will actually have an entire semigroup $\mathcal{Q}$ of such functions. If S is a metric space, the *Fatou set* of a dynamical system is the maximal open set of S on which the functions of Φ are equicontinuous; the *Julia set* is the complement of the Fatou set.[2] The Julia set is often 'thin' and so we want to 'thicken' it by 'filling in the interior'. This motivates the definition of the *filled Julia set* which has the Julia set as boundary: namely, if S happens to be a field k, with complete metric coming from some absolute value $|\cdot|_v$, we define the filled Julia set of Φ to be

$$\mathcal{K}(\Phi) := \{x \in k : \sup_{\phi\in\Phi} |\phi(x)| < \infty\}.$$

(Of course in this paper we are working only over $\mathbb{C}$, so all of this makes sense.) The complement of $\mathcal{K}(\Phi)$ is the *attracting basin of* ∞. Finally, suppose ϕ is a rational function over $\mathbb{C}$ and let x be a fixed point of α. Then we variously say that x is

- *superattracting* if $\phi'(x) = 0$,
- *attracting* if $|\phi'(x)| < 1$,
- *neutral* if $|\phi'(x)| = 1$, and
- *repelling* if $|\phi'(x)| > 1$.

With all this in mind, our first main theorem of the paper is the following result.

Theorem 3.5. *For each rational slope p/q we have $Q_{p/q}(\mathcal{R}) \subset \mathcal{R}$.*

Remark. This theorem appears implicitly in Lemma 2 of [33], and we give a proof along very similar lines but in more modern language.

Proof of Theorem 3.5. Let $\Gamma_\mu = \langle f, g\rangle$ and let $h_{p/q}$ be the transformation corresponding to the Farey word $W_{p/q}(\mu)$. Consider the group

$$\tilde{\Gamma} = \langle f, h_{p/q}\rangle = \langle f, f^{-1}h_{p/q}\rangle$$

[2] The Fatou set is analogous to the ordinary set of a Kleinian group, while the Julia set is analogous to the limit set.

This group is generated by two parabolics by Lemma 2.1. Thus $\tilde{\Gamma}$ is a conjugate of Γ_ν where

$$\nu = \operatorname{tr}(ff^{-1}h_{p/q}) - 2 = P_{p/q}(\mu) - 2$$

by Lemma 3.1. As a conjugate of a subgroup of $\langle f, g\rangle$ the group Γ_ν is discrete. It is also free with nonempty ordinary set, moreover $\Lambda(\tilde{\Gamma})$ is a subset of $\Lambda(\Gamma_\mu)$ and the Möbius image of $\Lambda(\Gamma_\nu)$. Since Γ_ν is discrete and free, $\nu \in \overline{\mathcal{R}}$. By the limit set remarks, ν does not lie on the boundary of $\mathcal{R}$; so $\nu \in \mathcal{R}$. $\square$

Corollary 3.6. *For each rational slope p/q we have that the algebraic set*

$$\mathbf{Z}(Q_{p/q}) = \{z \in \mathbb{C} : Q_{p/q}(z) = 0\}$$

is contained within the exterior of the closure of the Riley slice, i.e. within $\mathbb{C} \setminus \overline{\mathcal{R}}$.

Of course more is true here.

Corollary 3.7. *Let $z_0 \notin \overline{\mathcal{R}}$. For each rational slope p/q we have*

$$\{z \in \mathbb{C} : Q_{p/q}(z) = z_0\} \subset \mathbb{C} \setminus \overline{\mathcal{R}}.$$

The semigroup generated by the polynomials

$$\mathcal{Q} = \langle Q_{p/q} : \ p/q \text{ is a rational slope}\rangle$$

with the operation of functional composition now sets up a dynamical system on $\hat{\mathbb{C}}$ for which $\mathcal{R}$ lies in the stable region. No filled Julia set for any polynomial $Q_{p/q}$ can meet $\mathcal{R}$; for three examples, see Figure 7. Equivalently, $\mathcal{R}$ lies in the superattracting basin of ∞ for every polynomial. Every polynomial $Q_{p/q}$ has zero as a fixed point, $Q_{p/q}(0) = 0$.

Our computational evidence suggests the following conjecture:

Conjecture 3.8. *If p/q is a rational slope, then $Q_{p/q}$ factors as*

- $Q_{p/q}(z) = \pm z u(z)^2$, $u(0) = 1$, *when q is odd.*
- $Q_{p/q}(z) = z^{2[(n+1)/2]} v(z)^2$, *where $q = 2^n r$, r odd.*

Remark. An anonymous referee kindly provided us with a proof of a version of this conjecture using the machinery of Markoff triples, which we include below as Proposition 3.13.

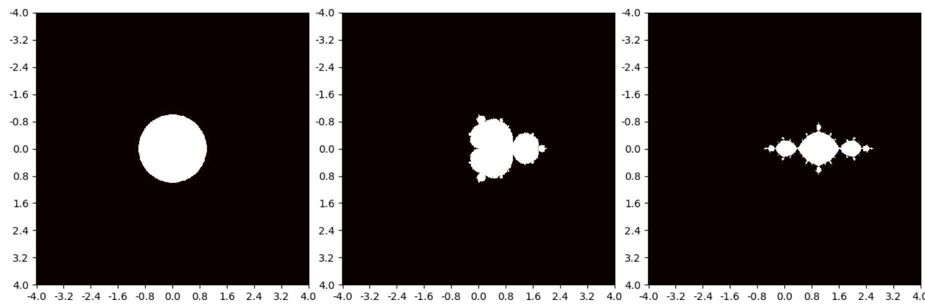

Figure 7: Filled Julia sets for $Q_{1/2}$ (left), $Q_{1/3}$ (middle), and $Q_{1/4}$ (right).

We wish to explore this a little further. The following lemma is a simple consequence of the form of a Farey word.

Lemma 3.9. *As* $\mu \to 0$ *we have*

- $W_{p/q}(\mu) \to X$ *if* q *is odd.*
- $W_{p/q}(\mu) \to \mathrm{Id}$ *if* q *is even.* □

From this lemma, we easily classify the fixed point type of 0.

Theorem 3.10. *If* q *is even, then* 0 *is a superattracting fixed point for* $Q_{p/q}$. *If* q *is odd, then* 0 *is not superattracting.*

Proof. By Theorem 2.2 we have $a+d-2 = c = Q_{p/q}$. Substituting $ad-bc = 1$ we obtain $c(a-b) = (a-1)^2$. If q is even, then $a(\mu) \to 1$ as $\mu \to 0$ so the right hand side of this equality is a polynomial with a double root at 0. On the left, $a(\mu) - b(\mu) \to 1$ as $\mu \to 0$; in particular, $(a-b)(0) \neq 0$. Thus both roots at 0 must come from factors of $c(\mu)$, i.e. $\mu^2 \mid c(\mu)$ and so $c'(0) = 0$.

For q odd, we have again $c(a-b) = (a-1)^2$; again the right side has a double root at 0, but on the left we have a factor $(a-b)$ which becomes 0 at 0; since c also has a root at 0, it follows that both $(a-b)$ and c have single roots at 0. □

Remark. Of course it would follow easily from Conjecture 3.8 that 0 is a *neutral* fixed point.

In [39] it is shown that the semigroup generated by all word polynomials has the complement of the Riley slice as its Julia set. As such, the roots of the word polynomials are dense in $\mathbb{C} \setminus \mathcal{R}$ as backward orbits are dense in the Julia set. In the context of Farey words this suggests that the roots of all

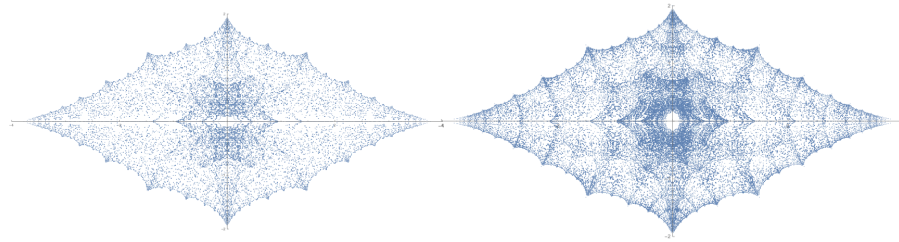

Figure 8: For all rational slopes with $q \leq 377$, Left: Root set of the polynomials $Q_{p/q}$. This set lies in $\mathbb{C}\backslash\overline{\mathcal{R}}$. Right: Root set of the equation $Q_{p/q} = -4$. This latter set lies in $\mathbb{C} \setminus \mathcal{R}$ and contains all the cusp groups. Notice the appearance of clustering around the pleating rays.

compositions of Farey words are dense. However somewhat more appears true—see Figure 8.

These pictures are quite quickly generated and give a good approximation to the Riley slice even for much smaller bounds on the denominators q. Notice that in view of Lemma 3.2 there is an open pre-image of the unit disk about each point also lying in $\mathbb{C} \setminus \overline{\mathcal{R}}$.

3.3 Proof of a version of Conjecture 3.8

Remark. The arguments of this section were generously provided to us by one of the anonymous referees.

It is well-known that the Farey polynomials (and the theories of geodesic laminations on surfaces lying on the boundary of genus 2 Schottky space) are heavily related to the theory of the Markoff Diophantine equation $X^2+Y^2+Z^2 = 3XYZ$; some entry points into the relevant literature include a paper by Bowditch [9] (which originated the detailed study of this relationship) and a pair of papers by Series [53, 54] (which fully work out the theory for the punctured torus). Essentially the link comes from the Fricke identity which appeared above in our proof of Theorem 2.2, and we pause to give the relevant definitions.

Definition 3.11 (Group representations)**.** Let M be a hyperbolic manifold, with $G = \pi_1(M)$. Of course G does not detect all the hyperbolic structure, which comes from an action of G on $\mathbb{H}^3$ as a discrete group of isomorphisms;

equivalently, the hyperbolic structure is given by a discrete and faithful group representation $\varsigma : G \to \mathrm{PSL}(2, \mathbb{C})$, where $\varsigma(G)$ is the holonomy group of M. It is often useful to look at non-faithful representations, but the full space $\mathrm{Hom}_{\mathrm{discr.}}(G, \mathrm{PSL}(2, \mathbb{C}))$ is usually too big, in the sense that it includes representations which kill 'large parts' of the manifold—more precisely, send loxodromic elements to parabolic elements and produce a geometric 'pinch'. This leads to the definition of a *type-preserving representation*: a discrete representation $\varsigma : G \to \mathrm{PSL}(2, \mathbb{C})$ such that $\varsigma(g)$ is parabolic only if g is parabolic, and such that $\mathrm{tr}^2 \varsigma(g) = \mathrm{tr}^2 g$ if g is elliptic.

Definition 3.12 (Farey sequences)**.** There is a natural ideal[3] triangulation $\mathcal{D}$ of $\mathbb{H}^2$, called the *Farey triangulation*, constructed iteratively via the following process: (1) the triangle spanned by $(0/1, 1/1, 1/0)$ lies in $\mathcal{D}$; (2) for every triangle $(p/q, r/s, t/u)$ in the triangulation— where all the fractions are written in simplest form and where $p/q < r/s < t/u$ with the obvious interpretation when one is ∞—the triangles

$$\left(\frac{p}{q}, \frac{p+r}{q+s}, \frac{r}{s}\right), \qquad \left(\frac{r}{s}, \frac{r+t}{s+u}, \frac{t}{u}\right), \qquad \text{and } \left(\frac{r}{s}, \frac{r-p}{s-q}, \frac{p}{q}\right),$$

all lie in $\mathcal{D}$ (secretly this definition comes from writing $\mathrm{PSL}(2, \mathbb{Z}) = \langle R, L\rangle \rtimes \langle Q\rangle$ where $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and then allowing this group to act on $\mathbb{H}^2$ as a group of hyperbolic isometries). Two rational numbers α, β are said to be *Farey neighbours* if they are joined by an edge of $\mathcal{D}$.

The Farey triangulation is very important in number theory, in relation to both classical parts of the subject (e.g. continued fraction representations) and modern parts (e.g. modular forms). We will not need much about it, we just remark that the triangles cover $\mathbb{H}^2$ without overlap and the set of vertices is precisely $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. In addition, if $(p/q, r/s, t/u) \in \mathcal{D}$ (where we take ordered triples as in the definition, where we gave each triangule in $\mathcal{D}$ a canonical orientation) then (i) $p/q < r/s < t/u$ and (ii) $r = p + t$, $s = q + u$. (It is always guranteed that new triangles, as produced via the processes of the definition, already are labeled in simplest form and the triplets are in the right order.) Conversely, any triplet $(p/q, r/s, t/u)$ of elements of $\hat{\mathbb{Q}}$ satisfying (i) and (ii) is a triangle in $\mathcal{D}$. These results can all be found in Chapter III of Hardy and Wright [24].

[3] *Ideal*: all the vertices lie on the boundary, $\hat{\mathbb{R}}$.

Let T be a punctured torus. The complex structures on T are parameterised by the *Maskit slice* [26, 47], the set of all $\mu \in \mathbb{C}$ such that

$$\left\langle \begin{bmatrix} -i\mu & -i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\rangle$$

is a discrete group; there is a construction of distinguished words and trace polynomials parameterised by rational numbers for the punctured torus which is completely analogous to the Farey polynomial construction, essentially because both families of groups are constructed by taking geometric limits of genus two Schottky groups; we write $\beta_{p/q}$ for the word in $\pi_1(T)$ indexed by $p/q \in \hat{\mathbb{Q}}$ (for the detailed construction see e.g. [47, Chapter 9]). The words and the trace polynomials $\phi(p/q) = \operatorname{tr} \beta_{p/q}(\mu)$ satisfy particularly nice relations, namely whenever p/q and r/s are Farey neighbours we have the product identity

$$\phi(\frac{p+r}{q+s}) + \phi(\frac{p-r}{q-s}) = \phi(p/q)\phi(r/s) \tag{2}$$

and the Markoff identity

$$\phi(p/q)^2 + \phi(r/s)^2 + \phi(\frac{p+r}{q+s})^2 = 3\phi(p/q)\phi(r/s)\phi(\frac{p+r}{q+s})$$

If $\tilde{\varsigma} : \pi_1(T) \to \mathrm{PSL}(2,\mathbb{C})$ is a type-preserving representation, then the map $\hat{\mathbb{Q}} \ni p/q \mapsto \operatorname{tr} \tilde{\varsigma}(\beta_{p/q}) \in \mathbb{C}$ is called the *Markoff map* associated to $\tilde{\varsigma}$; and the analogous identities continue to hold even after passing through a representation. Observe that, given a function ϕ which satisfies the product identity then it is defined by its values on the triangle $(0/1, 1/1, 1/0)$; further, if a function $\phi : \hat{\mathbb{Q}} \to \mathbb{C}$ satisfies both the product and the Markoff identity then it in fact comes from a the trace polynomial of a representation [4, Lemma 2.3.7], and so we may safely use the term *Markoff map* to refer to any such function.

Remark. The identity (2) does have an analogous version for the Farey polynomials, but it is more complicated [17] and the Markoff theory is not worked out fully in this case; the argument we give below will involve a reduction from the spherical case to the toric case and then applying the Markoff theory.

We now state and prove a version of Conjecture 3.8.

Proposition 3.13. *Fix some $\rho \in \mathbb{C}$, and let x be a square root of $-\rho$. Define a Markoff map $\phi : \hat{\mathbb{Q}} \to \mathbb{C}$ by $(\phi(0), \phi(1), \phi(\infty)) = (x, ix, 0)$. Then:*

1. $Q_{p/q}(\rho) = -\phi(p/q)^2$*;*

2. there is a family of integer polynomials $G_{p/q}(\rho)$ *such that*

$$Q_{p/q}(\rho) = \begin{cases} (-1)^p \rho (G_{p/q}(\rho))^2 & \text{if } q \text{ is odd;} \\ (-1)^{p-1} \rho^2 (G_{p/q}(\rho))^2 & \text{if } q \text{ is even.} \end{cases}$$

Part (2) may be found as the combination of Lemma 5.3.12 and Remark 5.3.13 in [4]. We therefore prove only part (1) here.

Proof of part (1) of the proposition. There is a natural covering diagram

$$\begin{tikzcd}
& \mathbb{R}^2 \setminus \mathbb{Z}^2 \arrow[dl, "\mathbb{Z}^2"'] \arrow[dd, "\tilde{G}"] \arrow[dr, "G"] & \\
T \arrow[dr, "\mathbb{Z}/2\mathbb{Z}"'] & & S \arrow[dl, "\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}"] \\
& \mathcal{O} &
\end{tikzcd}$$

called the *Fricke diagram* [4, §2.1], where $\mathcal{O}$ is an orbifold surface with three π cone points and one puncture and G and $\tilde{G}$ are generated by rotations about points in $\mathbb{Z}^2$ and $(\frac{1}{2}\mathbb{Z})^2$ respectively.

By usual covering theory, $\pi_1(S)$ is a normal subgroup of $\pi_1(\mathcal{O})$ of index 4; for each rational number p/q the element $\alpha_{p/q} \in \pi_1(S)$ represented by the simple loop of slope p/q has a "square root" in $\pi_1(\mathcal{O})$, i.e. there is an element $\beta_{p/q} \in \pi_1(\mathcal{O})$ such that $\alpha_{p/q} = \beta_{p/q}^2$. Further, the element $\beta_{p/q}$ is contained in $\pi_1(T)$ as a subgroup of $\pi_1(\mathcal{O})$.

Let $\varsigma_T : \pi_1(T) \to \mathrm{PSL}(2, \mathbb{C})$ be the representation which induces the Markoff map ϕ, so $\phi(p/q) = \operatorname{tr} \varsigma_T \beta_{p/q}$. This representation extends to a representation $\pi_1(\mathcal{O}) \to \mathrm{PSL}(2, \mathbb{C})$ and then restricts to a representation $\varsigma_S : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ [4, Lemma 2.3.7] (there is a dependence here on the explicit construction of the extension, which is not canonical). We have already seen in the introduction that if S is a four-times punctured sphere which appears as the boundary Riemann surface for a quotient $\mathbb{H}^3/\Gamma_\rho$ then $\Gamma_\rho \simeq \pi_1(S)/\langle \gamma_\infty \rangle$; in terms of representation theory, the quotient group $\pi_1(S)/\langle \gamma_\infty \rangle$ is equal to a free group $\langle K_0, K_1 \rangle$ and the representation ς_S decends to a representation $\varsigma_S^* : \langle K_0, K_1 \rangle \to \mathrm{PSL}(2, \mathbb{C})$ such that $(\varsigma(K_0), \varsigma(K_1)) = (X, Y_\rho)$. All these representations are defined in terms of explicit matrix representatives, giving respective lifts to $\mathrm{SL}(2, \mathbb{C})$ which we notate by adding a $\tilde{}$, allowing us to take traces (not just tr^2). With the particular matrix representatives chosen, $W_{p/q}(\rho) = -\widetilde{\varsigma_S}(\alpha_{p/q})$.

We may now construct the chain of equalities

$$\begin{aligned} Q_{p/q}(\mu) + 2 = \operatorname{tr} W_{p/q}(\mu) &= -\operatorname{tr} \widetilde{\varsigma_S}(\alpha_{p/q}) = -\operatorname{tr} \widetilde{\varsigma_T}(\beta_{p/q}^2) \\ &= -(\operatorname{tr}^2 \widetilde{\varsigma_T}(\beta_{p/q}) - 2) = -\phi(p/q)^2 + 2 \end{aligned}$$

(the fourth equality coming from the identity $\operatorname{tr} AB + \operatorname{tr} AB^{-1} = \operatorname{tr} A \operatorname{tr} B$ with $A = B$); this completes the proof of (1). $\square$

If both p and q are odd, the polynomial $G_{p/q}(\rho)$ is equal to $\Lambda_{q,p}(-\rho)$ where $\Lambda_{q,p}$ is the so-called Riley polynomial introduced by Riley in [50]. It is immediate from the explicit description of Λ given in Equation 3.10 of that paper (p. 227) that $\Lambda_{q,p}(0) = 1$, and so we obtain the statement $u(0) = 1$ from Conjecture 3.8 as long as p is odd.

4 Neighbourhoods of rational pleating rays

In this section, we give some motivation and intuition for our second main result. Our first main result gave a method of approximating the Riley slice exterior using the Farey polynomials and some related Julia sets; our second result is an approximation of the *interior* using the Farey polynomials. Here is the precise statement:

Theorem 4.1 (Existence of open neighbourhoods). *Let $P_{p/q}$ be a Farey polynomial. Then there is a branch $P_{p/q}^{-1}$ of the inverse of $P_{p/q}$ such that*

$$P_{p/q}^{-1}(\mathcal{H}), \text{ where } \mathcal{H} = \{\operatorname{Re} z < -2\},$$

is an open subset of $\mathcal{R}^{\infty,\infty}$.

The bounds given in the theorem are illustrated in Figure 9.

4.1 Motivating remarks

The idea behind Theorem 4.1 is very simple: the Keen–Series theory [27] depends on the existence of round discs (which they call *F-peripheral discs*) in the domains of discontinuity of the groups Γ_ρ which glue up along their edges to form the quotient surface S_4^2; the pleating rays are arcs in the Riley slice such that deforming the groups along these arcs preserves the 'roundness' of a given set of these peripheral discs (this is [27, Proposition

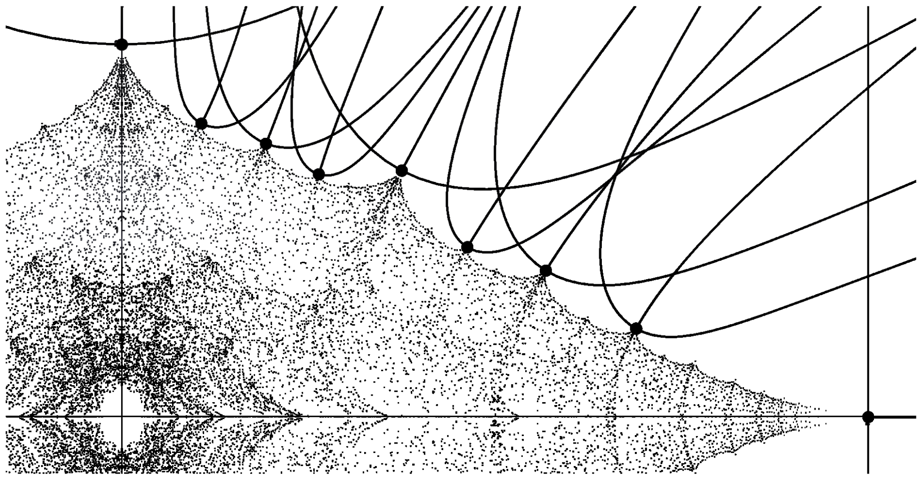

Figure 9: The Riley slice with neighbourhoods for our pleating ray values illustrated.

3.1]). In order to find neighbourhoods of these rays, we simply allow deformations in both dimensions, rather than simply in the direction of the pleating ray. Of course, deformations off the pleating ray do not preserve the roundness of the F-peripheral discs; but we claim that the quasidiscs (that is, quasiconformal images of discs) obtained still 'glue up' correctly, and limits of them continue to be quasidiscs (rather than the boundary becoming space-filling)—this last property (Lemma 5.11) allows us to prove an analogue of the closedness part of [27, Theorem 3.7], which is important because our proof follows a similar thread to the subsequent arguments of that paper: we define a certain subset of $\mathbb{C}$, namely the set $\mathcal{N}_{p/q}$ of $\rho \in \mathbb{C}$ which admit 'canonical peripheral quasidiscs of slope p/q' in analogy to the non-conjugate pairs of F-peripheral discs; we then prove that any group in some $\mathcal{N}_{p/q}$ lies in the Riley slice (this is Lemma 5.10 below, the analogue of [27, Lemma 3.5]); and then, via an open-closed argument like [27, Theorem 3.7], we see that $\mathcal{N}_{p/q}$ is precisely the set of Theorem 4.1.

In order to carry out this procedure, we need some information about the precise nature of the action of the quasiconformal deformations on the discs: more precisely, we will need to know that the combinatorial properties of the round peripheral discs are preserved even when we deform off the pleating ray and they turn into quasidiscs. Recall that the Riley slice $\mathcal{R}$ is topologically a punctured disc in the plane, and as such admits a hyperbolic metric which we denote by $\mathrm{dist}_{\mathcal{R}} : \mathcal{R} \times \mathcal{R} \to [0, \infty)$.

Theorem 4.2. *Let α be a curve in $\mathcal{R}$ which lies a bounded hyperbolic distance from a pleating ray (that is, there exists an $M < \infty$ such that for each $\rho \in \alpha$ there is some $\nu \in \mathcal{P}_{p/q}$ with $\mathrm{dist}_{\mathcal{R}}(\rho, \nu) \leq M$). Then the quasiconformal map conjugating Γ_ρ to Γ_ν has distortion no more than e^M.*

Proof. Let $\rho \in \alpha$ and $\nu \in \mathcal{P}_{p/q}$ and let $M := \mathrm{dist}_{\mathcal{R}}(\rho, \nu)$. Let $\pi : \mathbb{D}^2 \to \mathcal{R}$ be the hyperbolic universal covering map with $\pi(0) = \rho$ and $\pi(\tanh(M/2)) = \nu$. The holomorphically parameterised family of discrete groups $\{\Gamma_{\Phi(z)} : z \in \mathbb{D}^2\}$ induces an equivariant ambient isotopy of $\hat{\mathbb{C}}$ by the equivariant extended λ-lemma [15, 55, 56] (the argument is a standard one, see for example the exposition in [5, §12.2.1]). If we move ρ in $\mathcal{R}$ then the motion of the fixed point set extends to a holomorphically parameterised quasiconformal ambient isotopy, equivariant with respect to the groups Γ_ρ, of the whole Riemann sphere. By the fourth assertion of the extended λ-lemma as stated as [5, Theorem 12.3.2] the distortion of this ambient isotopy is exactly the exponential of the hyperbolic distance between the start (at 0) and finish (at $\tanh(M/2)$), that is e^M. $\square$

Consider deforming a point $\rho \in \mathcal{R}$ towards the Riley slice boundary along a curve α which lies a bounded distance away from a pleating ray $\mathcal{P}_{p/q}$. Theorem 4.2 shows that if $\nu \in \mathcal{P}_{p/q}$ is the hyperbolic projection of ρ onto the pleating ray then the combinatorial properties of circles in the limit set of Γ_ν transfer directly to combinatorial properties of quasicircles in the limit set of Γ_ρ, since there is a uniformly bounded distortion mapping one to the other. These quasicircles bound what we will call the *F-peripheral quasidiscs* of the group Γ_ν.

Most of the information that the Keen–Series theory provides is topological and their arguments could be used almost directly if we knew these uniform bounds. However, there is no way that we can compute or even estimate the hyperbolic metric of the Riley slice near the boundary to identify a curve such as α for every rational pleating ray. What we do is guess (motivated by examining a lot of examples on the computer) that such a curve is $\alpha = (\Phi^{\infty,\infty}_{p/q})^{-1}(\{z = -2 + it : t > 0\})$, where we take the branch of the inverse of $\Phi^{\infty,\infty}_{p/q}$ with the correct asymptotic behaviour.

An important point that we need to take into account when we modify the proof of [27, Theorems 3.7 and 4.1] and the corrected version [29, Theorem 2.4] is that, as mentioned above, the peripheral quasicircles could become quite entangled and eventually become space filling curves. We avoid this situation by modifying the peripheral quasidiscs as we move, so they have large scale "bounded geometry" (though the small scale geometry is uncontrolled). An important observation is that along the rational pleating ray the isometric circles of the Farey word $W_{p/q}$ are disjoint. We deform in such a way that this property is preserved, and it is for this reason that we choose the set $\mathcal{H}$ in the theorem statement: we will prove that $W_{p/q}$ has disjoint isometric discs when its trace lies in this region (Lemma 5.2), though we believe that this region can even be enlarged (see Section 4.2). Further, if we do not move too far away from the pleating ray these isometric circles do not start spinning around one another. This information allows us to construct a "nice" precisely invariant set stabilised by X and $W_{p/q}$—this turns out to be one of the peripheral quasidiscs which *does have* bounded geometry. Existence of this peripheral quasidisc (which we call *canonical* peripheral quasidisc) guarantees we have the correct quotient from the action of Γ_ρ on the ordinary set; and then the open-closed argument carries through.

4.2 Lyndon and Ullman's results

Here, we recall the main result of a paper of Lyndon and Ullman [32] and examine it in the context of our pleating neighbourhoods. In the process we will make some conjectures about improvements we think are possible to make to Theorem 4.1. In particular, we believe that the set $\mathcal{H}$ in the theorem can be enlarged to a cone with angle $\pi/3$.

Theorem 4.3 (Theorem 3, [32])**.** *Let K denote the Euclidean convex hull of the set $\mathbb{D}(2) \cup \{\pm 4\}$ ($\mathbb{D}(2)$ is the disc of radius 2 about 0). Then $\mathbb{C} \setminus \mathcal{R} \subseteq K$.* □

See Figure 10 for a depiction of this bound; as a consequence, the Riley slice $\mathcal{R}$ is contained within a conic region with apex at -4, bounded by two rays tangent to the disc $\mathbb{D}(2)$. Since the two lines are orthogonal to the radii of the circle, a simple trigonometric calculation shows that the cone angle is $\pi/3$, and so the interior of the cone is the set

$$\mathcal{W} = \{z \in \mathbb{C} : \frac{-\pi}{6} < \arg(z+4) < \frac{\pi}{6}\}.$$

Let φ be the branch of $z \mapsto -(-z-4)^{3/5} - 4$ conformally mapping $\mathbb{C} \setminus \overline{\mathcal{W}}$ to the half-space $H = \{z : \operatorname{Re} z < -4\}$ (here, $3/5 = \pi/(2\pi - \pi/3)$). Then $H \subseteq \mathbb{C} \setminus \overline{\mathcal{W}}$, and $\varphi(H)$ is the sector

$$\{z \in \mathbb{C} : \frac{7\pi}{10} < \arg(z+4) < \frac{13\pi}{10}\}.$$

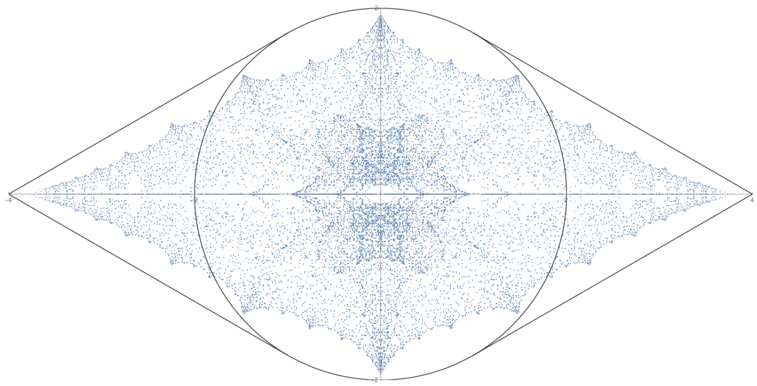

Figure 10: The convex hull of $\mathbb{D}(2) \cup \{\pm 4\}$ contains $\mathbb{C} \setminus \mathcal{R}$.

Because φ is conformal it is now straightforward to see that the distance in the hyperbolic metric of $\mathbb{C} \setminus \overline{\mathcal{W}}$ between the line $\ell_1 = -4 + i\mathbb{R}$ and the rational pleating ray $\ell_2 = (-\infty, -4]$ (which are parallel hyperbolic geodesics since they meet at the same point at infinity, hence lie a constant distance apart) is

$$\mathrm{dist}_{\mathbb{C}\setminus\overline{\mathcal{W}}}(\ell_1, \ell_2) = \int_{\frac{7\pi}{10}}^{\pi} \frac{d\theta}{|\cos\theta|} = \int_0^{3\pi/10} \frac{d\theta}{\cos(\theta)} = \frac{1}{2}\ln\left[5 + 2\sqrt{5}\right] \approx 1.1241.$$

From Theorem 4.2 we now have the following corollary.

Corollary 4.4. *Let $\nu \in -4 + i\mathbb{R}$. Then there is $\rho \in (-\infty, -4]$, the rational pleating ray $\mathcal{P}_{1/1}$, so that Γ_ν and Γ_ρ are K-quasiconformally conjugate for some deformation K satisfying*

$$K \le \sqrt{5 + 2\sqrt{5}} \approx 3.077\ldots.$$

Proof. The only thing left to observe is that the contraction principle for the hyperbolic metric shows that

$$\mathrm{dist}_{\mathcal{R}}(\rho, \nu) \le \mathrm{dist}_{\mathbb{C}\setminus\bar{\mathcal{W}}}(\rho, \nu) = \frac{1}{2}\ln\left[5 + 2\sqrt{5}\right]$$

for the point ρ closest to ν, and hence by Theorem 4.2

$$K \le e^{\mathrm{dist}_{\mathcal{R}}(\rho,\nu)} \le \sqrt{5 + 2\sqrt{5}}.$$

This proves the corollary. $\square$

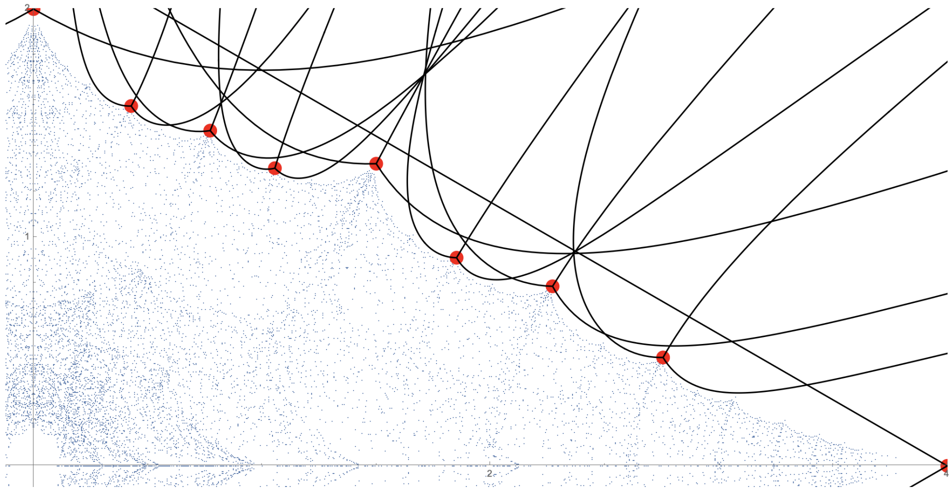

Figure 11: Preimages of the rays $\{\arg(z+4) = -\frac{\pi}{6}\}$ and $\{\arg(z-4) = \frac{\pi}{6}\}$.

We believe these estimates for larger neighbourhoods of the $(-\infty, -4]$ pleating ray persist in general in the parabolic case (namely, take preimages of this cone rather than of $\mathcal{H}$), but proving this adds additional complications in the construction we give as the isometric circles of $W_{p/q}$ may no longer be disjoint. We offer Figure 11, which is a slight modification of Figure 9, as computational support for this conjecture. Instead of looking at the branch of the inverse of $P_{p/q}$ defined on $\{\operatorname{Re} z < -2\}$, to produce this image we compute the preimages of the conic region of opening $\frac{\pi}{3}$ given by Theorem 4.3.

In the elliptic case, additional difficulty arises in finding an analogue for Theorem 4.3 in order to even 'guess' the right cone to pull back to a neighbourhood. We will discuss this further in our upcoming joint paper [18].

5 Proof of Theorem 4.1

In this section, we carry out the proof sketch that we gave in Section 4.1. It may be useful to have a reference to a specific example. Figure 12 shows pictures of the geometric objects we will be interested in for two specific cusp groups.

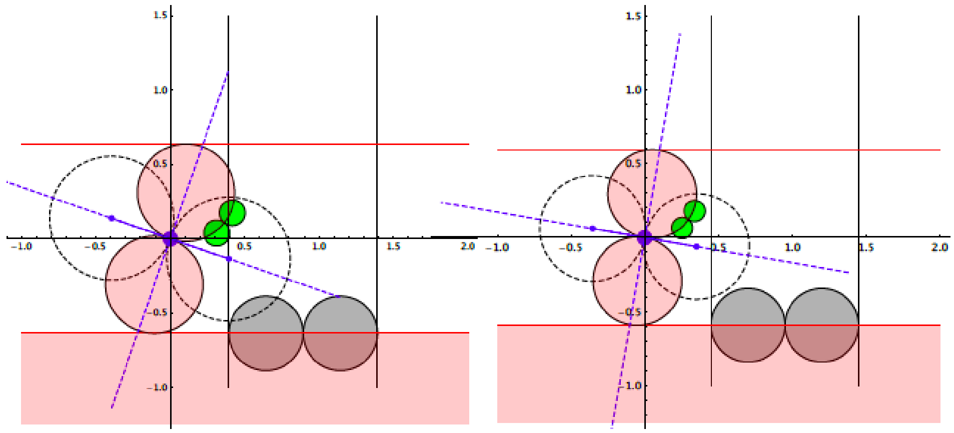

Figure 12: Geometric objects in the limit sets of the 3/4-cusp group (left) and the 4/5-cusp group (right). The isometric circles of $W_{p/q}$ are shaded grey; their images under the involution $\Phi : z \mapsto 1/(\rho z)$ are shaded green. This involution defines the non-conjugate peripheral disc $\langle Y, \Phi W_{p/q} \Phi^{-1} \rangle$. The non-conjugate peripheral discs are shaded in red (one is a lower half plane). Fixed points of the involution and its action are also illustrated.

5.1 Products of parabolics

As noted earlier (Lemma 2.1), an important property of a Farey word $W_{p/q}$ is that it can be written as a product of parabolic elements in two essentially different ways. For $\rho \in \mathcal{R}$ there are only two conjugacy classes of parabolics, those represented by X and Y [41, VI.A]; this is just a reflection of the fact that the deletion of a non-boundary-parallel curve on the 4-punctured sphere leaves two doubly punctured discs. To find these parabolics we just look for a couple of conjugates of X and Y whose product is $W_{p/q}$. Keen and Series studied the set of all such pairs (this is the data encoded in the circle chain sets $\mathcal{U}_{p/q}$ of [27]); in our analysis, we will only look closely at the pair $\{X, X^{-1}W_{p/q}\}$. The group $\langle X, W_{p/q} \rangle = \langle X, X^{-1}W_{p/q} \rangle$ is generated by two parabolics, and so can therefore only be discrete and free on its generators if

$$\operatorname{tr}(XX^{-1}W_{p/q}) - 2 = \operatorname{tr}(W_{p/q}) - 2 \in \overline{\mathcal{R}}$$

(since a group generated by two parabolics is discrete and free if and only if it is conjugate to a group in the Riley slice or its boundary; the Riley parameter ρ of a group generated by two parabolics A and B is just $\operatorname{tr} AB - 2$, since $\rho + 2 = \operatorname{tr} XY_\rho$.). If $\operatorname{tr}(W_{p/q}) \in \mathbb{R}$, then the traces of X, $X^{-1}W_{p/q}$, and

$W_{p/q}$ are real (the first two are ± 2) and so $\langle X, W_{p/q}\rangle$ is Fuchsian (this is a straightforward but tedious argument, see e.g. Project 6.6 of [47]). It is groups of this form (and their conjugates) which produce the *round* F-peripheral circles of [27].

That it suffices to only look at $\{X, X^{-1}W_{p/q}\}$ is a consequence of the following general result:

Lemma 5.1. *Suppose that $u_1, u_2, v_1, v_2 \in \mathrm{PSL}(2,\mathbb{C})$ are parabolics such that $\operatorname{tr} u_1u_2 = \operatorname{tr} v_1v_2$. Then the two groups $\langle u_1, u_2\rangle$ and $\langle v_1, v_2\rangle$ are conjugate in $\mathrm{PSL}(2,\mathbb{C})$.*

The upshot of this lemma is that if we were to pick a different pair whose product was $W_{p/q}$ then we get exactly the same geometry, up to a well-defined conjugation in $\hat{\mathbb{C}}$. We give a proof that provides slightly more information; an elementary proof that proves exactly the statement given is easy to write down (the groups can be conjugated to Γ_μ and $\Gamma_{\mu'}$, then the indicated traces are $2+\mu$ and $2+\mu'$ respectively so the groups themselves are conjugate to each other).

Proof of Lemma 5.1. There exist involutions $\phi_u, \phi_v \in \mathrm{PSL}(2,\mathbb{C})$ such that $\phi_u u_1 \phi_u^{-1} = u_2$ and $\phi_v v_1 \phi_v^{-1} = v_2$.

$$\operatorname{tr}^2 \phi_u = \operatorname{tr}^2 \phi_v = 0 \text{ and } \operatorname{tr}^2 u_1 = \operatorname{tr}^2 v_1 = 4.$$

Also,

$$\begin{aligned}\operatorname{tr}[u_1,\phi_u] &= \operatorname{tr} u_1\phi_u u_1^{-1}\phi_u^{-1} = \operatorname{tr} u_1u_2^{-1} = \operatorname{tr} u_1 \operatorname{tr} u_2 - \operatorname{tr} u_1u_2 \text{ and}\\ \operatorname{tr}[v_1,\phi_v] &= \operatorname{tr} v_1\phi_v v_1^{-1}\phi_v^{-1} = \operatorname{tr} v_1v_2^{-1} = \operatorname{tr} v_1 \operatorname{tr} v_2 - \operatorname{tr} v_1v_2;\end{aligned}$$

the two right-hand sides are equal (since $\operatorname{tr} u_i = \operatorname{tr} v_j$ for all i, j, and the product traces are equal by assumption) so $\operatorname{tr}[u_1,\phi_u] = \operatorname{tr}[v_1,\phi_v]$. In [23] it is shown that any pair of two-generator groups with the same trace square of the generators and the same trace of the commutators are conjugate in $\mathrm{PSL}(2,\mathbb{C})$. Thus $\langle v_1, \phi_v\rangle$ and $\langle u_1, \phi_u\rangle$ are conjugate and so are their subgroups $\langle u_1, u_2\rangle$ and $\langle v_1, v_2\rangle$. □

5.2 Rotation angles and isometric discs

Let $f \in \mathrm{PSL}(2,\mathbb{C})$ with $\operatorname{tr} f = -2 + ti$ ($t \in \mathbb{R}$). Then f has complex translation length $\tau_f + i\theta_f$, where τ_f and θ_f are respectively the real translation length and the rotation angle f given by the formulae

$$\frac{\tau_f}{2} = \mathrm{Re}\left[\sinh^{-1}\left(\frac{i}{2}\sqrt{t(4i+t)}\right)\right] \text{ and } \frac{\theta_f}{2} = \mathrm{Im}\left[\sinh^{-1}\left(\frac{i}{2}\sqrt{t(4i+t)}\right)\right]$$

We also have the following asymptotics.

$$\text{As } t \to 0,\ \begin{cases} \frac{\tau_f}{\sqrt{2t}} \to 1, \\ \frac{\theta_f}{\sqrt{2t}} \to -1. \end{cases} \qquad \text{For } 0 < t < 1,\ \begin{cases} 1 \leq \frac{\tau_f}{\sqrt{2t}} \leq 1.03642..., \\ -1 \leq \frac{\theta_f}{\sqrt{2t}} \leq -0.954\ldots. \end{cases}$$

In addition, $\theta_f \to -\pi$ as $t \to \infty$.

Suppose that some (now arbitrary) $f \in \mathrm{PSL}(2, \mathbb{C})$ is represented by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the isometric discs of f are the two discs D_1 and D_2 given by

$$D_1 = \left\{ z \in \mathbb{C} : |z - \frac{a}{c}| \leq \frac{1}{|c|} \right\}, \quad D_2 = \left\{ z \in \mathbb{C} : |z + \frac{d}{c}| \leq \frac{1}{|c|} \right\}$$

The isometric circles are the boundaries of these two discs. We say that f has *disjoint isometric discs* if these discs have disjoint interior. This is clearly equivalent to the condition $|a + d| \geq 2$.

The mapping f pairs these discs in the sense that

$$f(D_1) = \hat{\mathbb{C}} \setminus \overline{D_2}.$$

Thus $\hat{\mathbb{C}} \setminus \overline{D_1 \cup D_2}$ is a fundamental domain for the action of f on $\hat{\mathbb{C}}$. Notice that when $c \neq 0$, $f(\infty) = \frac{a}{c}$ and that $f^{-1}(\infty) = -\frac{d}{c}$ are the centers of the isometric discs.

We now specialise to the case that f is the Farey word $W_{p/q}$. Label the entries of the matrix representing $W_{p/q}(\rho)$ as follows:

$$W_{p/q}(\rho) = \begin{pmatrix} a_{p/q}(\rho) & b_{p/q}(\rho) \\ c_{p/q}(\rho) & d_{p/q}(\rho) \end{pmatrix} \qquad a_{p/q}d_{p/q} - b_{p/q}c_{p/q} = 1.$$

The isometric discs of $W_{p/q}$ are the two discs

$$D_1 = B\left(\frac{a_{p/q}(\rho)}{c_{p/q}(\rho)}, \frac{1}{|c_{p/q}(\rho)|} \right) \text{ and } D_2 = B\left(\frac{-d_{p/q}(\rho)}{c_{p/q}(\rho)}, \frac{1}{|c_{p/q}(\rho)|} \right), \qquad (3)$$

Lemma 5.2. *Let* $\mathrm{tr}(W_{p/q}) = x + it$ *with* $x \leq -2$*. Then the Farey word* $W_{p/q}(\rho)$ *has disjoint isometric discs.*

Proof. This is a simple computation: $|a_{p/q} + d_{p/q}| = \sqrt{x^2 + t^2} \geq \sqrt{x^2} = |x| \geq 2$. $\square$

We now compute with the identity of Theorem 2.2 that

$$\frac{a_{p/q}(\rho)}{c_{p/q}(\rho)} + \frac{d_{p/q}(\rho)}{c_{p/q}(\rho)} = \frac{2 + c_{p/q}(\rho)}{c_{p/q}(\rho)} = 1 + \frac{2}{c_{p/q}(\rho)}. \tag{4}$$

Theorem 2.2 and (4) together have the following consequence.

Corollary 5.3. *Let* $\operatorname{tr}(W_{p/q}) = -x + it$ *for* $x \geq 2$ *and* $t \in \mathbb{R}$. *Then the group* $\langle X, W_{p/q} \rangle$ *is discrete and free on the indicated generators.*

Proof. For clarity we drop the subscript p/q as it is fixed. Let S be the vertical strip of width one given by

$$\left\{ z \in \mathbb{C} : \frac{1}{2}\left(\operatorname{Re}\left(\frac{a(\rho) - d(\rho)}{c(\rho)}\right) - 1\right) < \operatorname{Re}(z) < \frac{1}{2}\left(\operatorname{Re}\left(\frac{a(\rho) - d(\rho)}{c(\rho)}\right) + 1\right)\right\}.$$

Using the notation of Lemma 5.2 for the isometric discs of $W_{p/q}$, set $\widetilde{D_1} = D_1 - 1$ and $\widetilde{D_2} = D_2 + 1$ so each $\widetilde{D}_i$ is a translate of the respective D_i (to the left and right respectively; see Figure 13). Essentially following the construction on pp.1392–1393 of [32], define $\widetilde{S}$ by

$$\widetilde{S} = (S \cup D_1 \cup D_2) \setminus \left(\widetilde{D}_1 \cup \widetilde{D_2}\right). \tag{5}$$

Theorem 2.2 implies that the discs $\widetilde{D}_1$ and D_2 are tangent. Two things now follow. Firstly, the translates of $\widetilde{S}$ by $n \in \mathbb{Z}$ fill the plane. Secondly, $\widetilde{S}$ contains the isometric circles of $W_{p/q}$. The Klein combination theorem [41, Theorem VII.A.13] now implies the result since $\hat{\mathbb{C}} \setminus (D_1 \cup D_2)$ is a fundamental domain for the action of $W_{p/q}$. $\square$

There is one further piece of information we would like out of Corollary 5.3: that the point of tangency of the isometric discs $\tilde{D}_1$ and D_2 and their translates is a parabolic fixed point. To save space, write $h = h_{p/q}$ for the function on $\hat{\mathbb{C}}$ corresponding to the action of $W_{p/q}$. The point of tangency can be calculated to be

$$z_\infty := \frac{a - d}{2c} - \frac{1}{2} = \frac{a - d - c}{2c} = \frac{1 - d}{c}.$$

(where we continue to use a, b, c, d for the entries of a matrix representing $W_{p/q}$). Then

$$h(z_\infty) = \frac{a z_\infty + b}{c z_\infty + d} = \frac{a \frac{1-d}{c} + b}{c \frac{1-d}{c} + d} = \frac{1 + c - d}{c} = z_\infty + 1$$

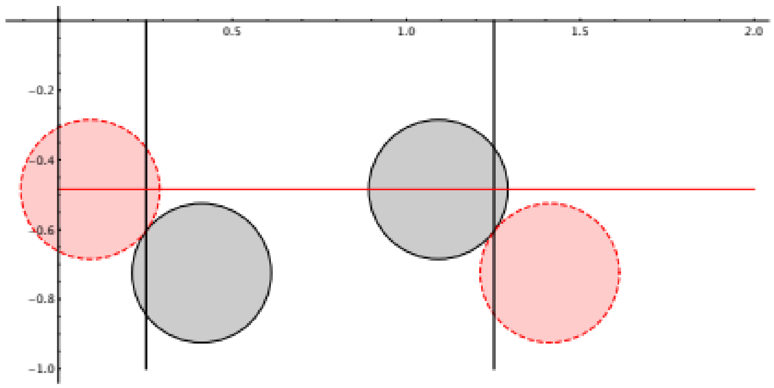

Figure 13: The isometric circles of $W_{3/4}$ when $\operatorname{tr} W_{3/4} = -2 + i$ (in grey) and their translates (in red).

and so with f representing X we have shown $f^{-1}h(z_\infty) = z_\infty$ so z_∞ is a fixed point of $f^{-1}h$. Since $X^{-1}W_{p/q}$ is parabolic as previously observed we have proved the following lemma.

Lemma 5.4. *The point $z_\infty = \frac{1-d}{c} \in \partial\widetilde{S}$ (with $\widetilde{S}$ defined by (5)), a point of tangency of the isometric discs of $W_{p/q}$ and their unit translates, is a parabolic fixed point.* □

5.3 Canonical peripheral quasidisks

In this section we show that the geometry of the peripheral quasicircles is controlled by the pairing of the isometric circles of $W_{p/q}$. We begin by studying the geometry for $\rho \in \mathcal{P}_{p/q}$, and then we allow ρ to move holomorphically off the pleating ray, inducing a quasiconformal deformation of the limit set and hence the peripheral discs, with a quasiconformality constant that we can explicitly bound.

Fix $\rho \in \mathcal{P}_{p/q}$, and as above we write $h_{p/q}(\rho)$ for the Möbius transformation represented by $W_{p/q}(\rho)$ which maps ∂D_2 onto ∂D_1 (c.f. (3)). Let $z_0 \in \partial D_2$ be the unique closest point of ∂D_2 to ∂D_1. Since $h_{p/q}(\rho)$ is hyperbolic (its trace is real, since ρ is on the p/q-pleating ray) it maps z_0 to the point of ∂D_1 which is closest to ∂D_2. Let $L(\rho)$ be the Euclidean line segment joining z_0 to $h_{p/q}(\rho)(z_0)$.

We now allow ρ to move off the pleating ray; more precisely, we choose a holomorphic motion $\Phi : \mathbb{D} \times A \to A$ such that A is a sufficiently small neighbourhood of the pleating ray (really the point is that we can choose A to be the neighbourhood which we claim the existence of in Theorem 4.1

without moving out of the Riley slice) and then take $\tilde{\rho} = \Phi(\lambda, \rho)$ for some $\lambda \in \mathbb{D}$. After this deformation, $L(\tilde{\rho})$ is still a line segment joining two points on the boundaries of $\partial D_1(\tilde{\rho})$ and $\partial D_2(\tilde{\rho})$ such that the endpoint $z_0(\tilde{\rho})$ on $\partial D_2(\tilde{\rho})$ is mapped by $h_{p/q}(\tilde{\rho})$ onto the other endpoint of $L(\tilde{\rho})$; but now the line segment itself does not form the projection of the axis of $h_{p/q}(\tilde{\rho})$ (though the projection of the orbit is symmetric with respect to $L(\tilde{\rho})$) and $z_0(\tilde{\rho})$ and $h_{p/q}(\tilde{\rho})(z_0(\tilde{\rho}))$ are not the closest points of the two circles.

The line segment $L_{p/q}(\tilde{\rho})$ will lie entirely in $\widetilde{S}$ provided that $\tilde{\rho}$ is close enough to ρ that the isometric discs of $W_{p/q}$ have not twisted too far around. In particular, it is enough if the absolute value of the difference between the real parts of the centers of the isometric discs exceeds twice the radius of the isometric discs. That is, if

$$\Big| \operatorname{Re} \frac{a_{p/q}(\tilde{\rho}) + d_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})} \Big| \geq \frac{2}{|c_{p/q}(\tilde{\rho})|}.$$

Using Theorem 2.2, we calculate that

$$\begin{aligned} \operatorname{Re} \frac{a_{p/q}(\tilde{\rho}) + d_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})} &= \operatorname{Re} \frac{c_{p/q}(\tilde{\rho}) + 2}{c_{p/q}(\tilde{\rho})} = 1 + \operatorname{Re} \frac{2}{c_{p/q}(\tilde{\rho})} \\ &= 1 + \frac{2}{|c_{p/q}(\tilde{\rho})|^2} \operatorname{Re} c_{p/q}(\tilde{\rho}); \end{aligned}$$

which is implied by

$$\big||c_{p/q}(\tilde{\rho})|^2 + 2 \operatorname{Re} c_{p/q}(\tilde{\rho})\big| \geq 2|c_{p/q}(\tilde{\rho})|.$$

This is true if $\operatorname{Re} c_{p/q}(\tilde{\rho}) \leq -4$—and in particular if $\operatorname{Re} \operatorname{tr} W_{p/q} < -2$, by Theorem 2.2—so under these conditions the line segment $L_{p/q}(\tilde{\rho})$ has the property that it lies entirely in $\widetilde{S}$ with its endpoints on $\partial\widetilde{S}$; and as we mentioned above the endpoints of $L_{p/q}(\tilde{\rho})$ are identified by $h_{p/q}(\tilde{\rho})$.

For convenience, introduce now the notation $\Gamma_{p/q}(\tilde{\rho}) = \langle f, h_{p/q}(\tilde{\rho})\rangle$ where f and $h_{p/q}(\tilde{\rho})$ are the Möbius transformations with respective matrices X and $W_{p/q}(\tilde{\rho})$. We have identified a fundamental domain $\widetilde{S}$ for the action of $\Gamma_{p/q}(\tilde{\rho})$ on $\Omega(\Gamma_{p/q}(\tilde{\rho}))$. The quotient

$$\Omega(\Gamma_{p/q}(\tilde{\rho}))/\Gamma_{p/q}(\tilde{\rho})$$

is the four-times punctured sphere $S_{0,4}$, since

$$\Gamma_{p/q}(\tilde{\rho}) = \big\langle f, h_{p/q}(\tilde{\rho})\big\rangle = \big\langle f, f^{-1}h_{p/q}(\tilde{\rho})\big\rangle$$

is a circle-pairing group generated by two parabolics. The line segment $L_{p/q}(\tilde{\rho})$ projects to a simple closed curve (though not a geodesic in general) in the homotopy class of $h_{p/q}(\tilde{\rho})$ and separates one pair of punctures from another. We remark that the projection of $L_{p/q}(\tilde{\rho})$ is smooth away from one corner (namely, the point of projection of the segment endpoints) and the angle at that corner tends to π as $\operatorname{Im}\tilde{\rho} \to 0$.

By construction the Schottky lift of the projection of $L_{p/q}(\tilde{\rho})$ into $\mathbb{S}^2_4$ is a Jordan curve through ∞. We set

$$\mathcal{L}_{p/q}(\tilde{\rho}) = \bigcup_{g \in \langle f, h_{p/q} \rangle} g(L_{p/q}(\tilde{\rho})). \tag{6}$$

(Formally, we take the *closure* of this union to be $\mathcal{L}_{p/q}$ since the $L_{p/q}$ are open segments.) This curve consists of (the closure of) the translates of $L_{p/q}$ by f^n, $n \in \mathbb{Z}$, together with images which lie in the union of the two isometric disks of $h_{p/q}$ and their integer translates.

In fact the curve $\mathcal{L}_{p/q}(\tilde{\rho})$ is a quasiline—the image of a line under an entire quasiconformal mapping. Unfortunately we have no control on the distortion here even though we expect that we are a bounded hyperbolic distance from a Fuchsian group on the rational pleating ray $\mathcal{P}_{p/q}$, so the easiest way to see that $\mathcal{L}_{p/q}$ is indeed a quasiline is to use Ahlfors' "three point criterion" [1]. Since $\mathcal{L}_{p/q}(\tilde{\rho})$ is invariant under a translation it will be a quasiline provided the Ahlfors criterion holds at smaller scales—say after only a few translates. At smaller scales the criterion holds simply by compactness, $\mathcal{L}_{p/q}(\tilde{\rho})$ is a piecewise smooth Jordan arc and at the connecting points the angle is not zero so the curve has no cusps.

We note that

$$h_{p/q}(\tilde{\rho})(\infty) = \frac{a_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})} \text{ and } h_{p/q}(\tilde{\rho})^{-1}(\infty) = -\frac{d_{p/q}(\tilde{\rho})}{c_{p/q}(\tilde{\rho})}$$

and these are parabolic fixed points on $\mathcal{L}_{p/q}(\tilde{\rho})$ (conjugates of the fixed points of f) as well as being the centers of the isometric circles. The parabolic fixed point we earlier identified in Lemma 5.4, $z_\infty = \frac{1-d}{c}$, also lies in $\mathcal{L}_{p/q}(\tilde{\rho})$ and is not the parabolic fixed point of a conjugate of f (since it is not conjugate in the abstract group $\langle X, Y \rangle$ from which the rational words come, a consequence of the fact that they represent simple closed curves on the four-times punctured sphere.) The translates of the endpoints of $L_{p/q}(\tilde{\rho})$ under $\langle h_{p/q}(\tilde{\rho}) \rangle$ lie on a log-spiral connecting the fixed points of $h_{p/q}(\tilde{\rho})$. This is illustrated in the examples of Figure 14.

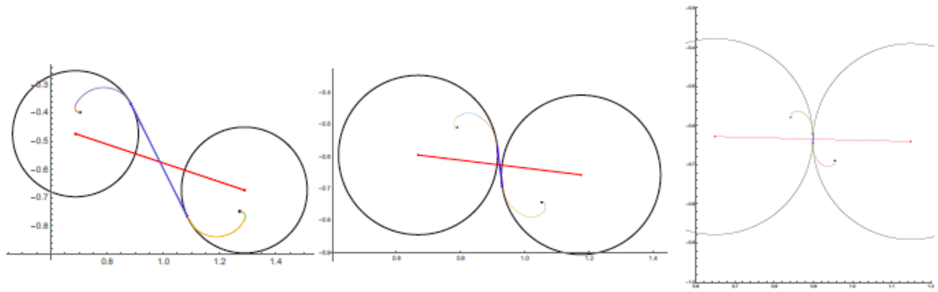

Figure 14: A log-spiral connecting the fixed points of $h_{3/4}(\tilde{\rho})$, the Möbius transformation representing $W_{3/4}(\tilde{\rho})$. Red lines connect the isometric circle centers, and the spirals connect the fixed points of $h_{3/4}(\tilde{\rho})$. Here, $\tilde{\rho}$ is chosen such that $\mathrm{tr}(W_{3/4}) = -2 + it$ where the values of t shown from left to right are $t = 2$, $t = 0.5$, and $t = 0.1$.

If we denote by $H^{\pm}_{p/q}(\tilde{\rho})$ the components of $\mathbb{C} \setminus \mathcal{L}_{p/q}(\tilde{\rho})$, then

$$H^{\pm}_{p/q}(\tilde{\rho})/\Gamma_{p/q}(\tilde{\rho})$$

is a twice punctured disc with boundary given by a projection of $L_{p/q}(\tilde{\rho})$.

We can give some bounds on the position of the invariant quasiline; in particular, this shows that it has bounded large-scale geometry (as we discussed in Section 4.1).

Lemma 5.5. *The invariant quasiline $\mathcal{L}_{p/q}$ lies in the strip*

$$\left\{ z \in \mathbb{C} : \mathrm{Im}\left(\frac{a_{p/q}}{c_{p/q}}\right) + \frac{1}{|c_{p/q}|} \leq \mathrm{Im}\, z \leq \mathrm{Im}\left(-\frac{d_{p/q}}{c_{p/q}}\right) - \frac{1}{|c_{p/q}|} \right\}$$

(where all objects are taken with respect to $\Gamma_{\tilde{\rho}}$).

Proof. By construction $\ell_{p/q}$ lies in, and separates $\widetilde{S}$. Its translates together with the translates of the isometric discs of $W_{p/q}$ separate both the ordinary set of $\langle f, h_{p/q}\rangle$ and the plane into two parts. The strip is the smallest horizontal strip containing the isometric circles of $W_{p/q}$. Note that $\mathrm{Im}\,\frac{a_{p/q}}{c_{p/q}} \geq \mathrm{Im}\,-\frac{d_{p/q}}{c_{p/q}}$ and that both are negative. This particular fact holds if we choose, as we may, ρ to be in the positive quadrant of $\mathbb{C}$. $\square$

Our computational investigations suggest that in fact the width of this strip can be improved to where the spiral "turns over". This appears proportional to the difference of the imaginary parts of the fixed points. A consequence would be that as $\operatorname{Im}\tilde{\rho} \to 0$ the strip turns into a line and the quasilines $\mathcal{L}_{p/q}(\tilde{\rho})$ converge to the line through the fixed points of $h_{p/q}(\tilde{\rho})$, which is a line in the limit set of the cusp group.

Definition 5.6. By analogy with Keen and Series, we call the component $H_{p/q}(\tilde{\rho})$ of $\mathbb{C} \setminus \mathcal{L}_{p/q}(\tilde{\rho})$ which does not contain 0 a *canonical peripheral quasidisc* if

1. $\Lambda(\Gamma_{p/q}(\tilde{\rho})) = \overline{H_{p/q}(\tilde{\rho})} \cap \Lambda(\Gamma_{\tilde{\rho}})$, and
2. $\operatorname{tr}(W_{p/q}((\tilde{\rho}))) \in \{z = x + iy \in \mathbb{C} : x < -2\}$.

Notice that if $\tilde{\rho} \in \mathcal{R}$, then there is *some* slope p/q such that $\Gamma_{\tilde{\rho}}$ admits the canonical peripheral quasidisc $H_{p/q}(\tilde{\rho})$, since each such group is quasiconformally conjugate to one on a pleating ray where there is such a peripheral circle. There seems to be no way of guaranteeing that the large scale geometry of the boundary quasiline is bounded, but we do know that the geometry is bounded for the special case of $\mathcal{L}_{p/q}(\tilde{\rho})$.

5.4 Completing the proof

We now give a series of lemmata imitating the proofs given for the case of a pleating ray by Keen and Series [27]. Set $S^*_{p/q} = \widetilde{S} \cap H_{p/q}$; this is a fundamental domain of $\Gamma_{p/q}$ defined by the isometric circles of $h_{p/q}$ and the line segment $\ell_{p/q}$. Recall the parabolic cusp point given by 5.4 in $\partial H_{p/q}$ (and also in $S^*_{p/q}$). The following lemma is immediately clear from construction.

Lemma 5.7. *An F-peripheral disc in the sense of Keen and Series is a canonical peripheral quasidisc.* □

In fact, in this case $h_{p/q}$ is hyperbolic, with disjoint isometric discs and $\ell_{p/q}$ is a segment of the line through its fixed points (and also through isometric circles) and orthogonal to them.

Recall that in the Keen–Series theory it was important that the F-peripheral discs moved continuously with ρ; since the defining points of $L_{p/q}$ move continuously with ρ, the analogous result is true:

Lemma 5.8. *Fix a rational slope p/q. The quasiline $L_{p/q}$ moves continuously with ρ and the data $a_{p/q}, b_{p/q}, c_{p/q}$ and $d_{p/q}$, as does the associated fundamental domain $S^*_{p/q}$.* □

Remark. In fact, the defining points (vertices of $S^*_{p/q}$) move *holomorphically*, but as a set $S^*_{p/q}$ does not.

Next the analogue of [27, Proposition 3.1].

Lemma 5.9. *Fix a rational slope p/q. The set*

$$\{\rho : \Gamma_\rho \text{ admits the canonical peripheral quasidisc } H_{p/q}\}$$

is open.

Proof. By definition $\mathrm{tr}(W_{p/q}) \in \{\mathrm{Re}\, z < -2\}$. Choose a small neighbourhood of ρ so that this remains true. That is, $\mathrm{tr}(W_{p/q}(\rho')) \in \{\mathrm{Re}\, z < -2\}$ for ρ' close to ρ. Each Γ_ρ is geometrically finite [42], and therefore each parabolic fixed point is doubly cusped (see e.g. [41] and [38]). Let U be a horodisc neighbourhood of the parabolic fixed point in $\partial S^*_{p/q}$ (not ∞). As $\ell_{p/q} \in \partial H_{p/q}$ is in the ordinary set for $\Gamma_{p/q}$ it is in the ordinary set of Γ_ρ and projects to a loop bounding a doubly punctured disc in S^2_4. It follows that $S^*_{p/q} \setminus U$ is compactly supported away from $\Lambda(\Gamma_\rho)$. This limit set moves holomorphically and so for small time t the varying $(S^*_{p/q})_t \setminus U_t$ lie in the ordinary set of Γ_{ρ_t}. The images of $(S^*_{p/q})_t \setminus U_t$ under $(\Gamma_{p/q})_t$ tessellate $(H_{p/q})_t$, apart from the deleted cusp neighbourhoods which we now put back to find a canonical peripheral quasidisc $(H_{p/q})_t$. $\square$

For $p/q \in \mathbb{Q}$, let $\mathcal{N}_{p/q}$ be the set defined by

$$\mathcal{N}_{p/q} := \{\rho \in \mathbb{C} : \Gamma_\rho \text{ admits a canonical peripheral quasidisc } H_{p/q}\}.$$

We prove a version of [27, Lemma 3.5], for $\mathcal{N}_{p/q}$ rather than the pleating ray $\mathcal{P}_{p/q}$.

Lemma 5.10. *Fix a rational slope p/q. If $\rho \in \mathcal{N}_{p/q}$, then $\rho \in \mathcal{R}$.*

Proof. We have $\Gamma_{p/q} = \langle f, h_{p/q}\rangle = \langle f, f^{-1}h_{p/q}\rangle$. As described earlier there is another group $\Gamma'_{p/q}$ generated by two parabolics in Γ_ρ whose product is also $h_{p/q}$. These groups are not conjugate in Γ_ρ but are conjugate when the $\mathbb{Z}_2$ symmetry that conjugates X to Y is added. This symmetry leaves the limit set set-wise invariant. Hence both groups are quasi-Fuchsian with canonical peripheral quasidiscs. The remainder of the argument is as in [27, Lemma 3.5]. Briefly, both of the sets

$$H_{p/q}/\Gamma_{p/q} = H_{p/q}/\Gamma_\rho \text{ and } H'_{p/q}/\Gamma'_{p/q} = H'_{p/q}/\Gamma_\rho$$

are two different twice punctured discs in the quotient glued along a common boundary (a translation arc of $h_{p/q}$ which lies in $H_{p/q} \cap H'_{p/q}$). Then the quotient is S^2_4 and hence $\rho \in \mathcal{R}$ by definition. $\square$

It is in the proof of the next lemma (the analogue of the closedness half of [27, Theorem 3.7]) where we use the fact that the quasidiscs $H_{p/q}$ have bounded geometry. Without this, the invariant quasicircles for the peripheral discs could either become space-filling curves or collapse entirely. This indeed happens in general with the formation of B-groups, or the geometrically infinite groups on the boundary of $\mathcal{R}$.

Lemma 5.11. *Fix a rational slope p/q. Suppose that Γ_{ρ_j} admits canonical peripheral quasidiscs $H^j_{p/q}$, and that $\operatorname{tr}(W^j_{p/q}) \to z_0$ with $\operatorname{Re}(z_0) < -2$. Then there is a subsequence ρ_{j_k} which converges to some $\rho \in \mathcal{R}$ such that Γ_ρ admits a canonical peripheral quasidisc of the same slope.*

Proof. That $\operatorname{tr}(W^j_{p/q}) \to z_0$ where $\operatorname{Re}(z_0) < -2$ means that $a_{p/q}, b_{p/q}, c_{p/q}$ and $d_{p,q}$ all have finite limits and that $c_{p/q} \not\to 0$ by Lemma 3.2 and Theorem 2.2; therefore we can apply Lemma 5.5 to conclude that the invariant lines bounding $L_{p/q}$ also have a limiting height above and below. It follows that there is a non-empty open set U such that, for j sufficiently large, $U \subset H^j_{p/q}$. Each of the groups Γ_{ρ_j} is discrete (and free) and, after passing to a subsequence if necessary, the limit group Γ_ρ is also discrete (and free). Thus the ordinary set of Γ_ρ must contain U. By Lemma 5.10 we have $\rho_j \in \mathcal{R}$ and hence $\rho \in \overline{\mathcal{R}}$. If $\rho \in \mathcal{R}$ we are done. Otherwise $\rho \in \partial\mathcal{R}$, and Γ_ρ has nonempty ordinary set $\Omega_\rho = \hat{\mathbb{C}} \setminus \Lambda(\Gamma_\rho)$. Since ρ lies in the boundary of $\mathcal{R}$ the quotient surface Ω/Γ_ρ can support no moduli. The group Γ_ρ is torsion-free with non-empty ordinary set (it contains U), so the quotient is a union of triply punctured spheres and the point ρ must be a cusp group (these results are all found in the paper [42]). Notice that $h_{p/q}$ will have its fixed points in the boundary of a component of the ordinary set, which are now round circles. Thus $\Gamma_{p/q}$ is Fuchsian (since it is *a priori* quasi-Fuchsian and has limit set dense in a round circle), $\operatorname{tr}(h_{p/q})$ is real and therefore $\operatorname{tr}(h_{p/q}) \in (-\infty, -2)$. But these groups lie on the pleating ray in $\mathcal{R}$ and so have F-peripheral discs. This completes the proof. □

We now complete the proof of Theorem 4.1. Consider the set $\mathcal{Z}_{p/q}$ defined by

$$\mathcal{Z}_{p/q} = \{\rho \in \mathcal{R} : \operatorname{Re} P_{p/q}(\rho) < -2\}.$$

We show that $\mathcal{N}_{p/q}$ is a connected component of $\mathcal{Z}_{p/q}$, by showing (as in the proof of [27, Theorems 3.7] and [29, Theorem 2.4]) that $\mathcal{N}_{p/q}$ is a non-empty clopen subset of $\mathcal{Z}_{p/q}$.

By Lemma 5.10, $\mathcal{N}_{p/q} \subseteq \mathcal{R}$.

We make four observations.

1. $\mathcal{N}_{p/q} \subseteq \mathcal{Z}_{p/q}$ since, by Definition 5.6, $\operatorname{Re}\operatorname{tr} W_{p/q}(\rho) < -2$ for $\rho \in \mathcal{N}_{p/q}$;
2. Note that $\mathcal{N}_{p/q}$ is closed in $\mathcal{Z}_{p/q}$ by Lemma 5.11.
3. By definition, $\mathcal{Z}_{p/q}$ is open in $\mathbb{C}$ (it is the inverse image of an open set); since $\mathcal{N}_{p/q}$ is also open in $\mathbb{C}$ (Lemma 5.9) it is open in $\mathcal{Z}_{p/q}$.
4. Finally, $\mathcal{N}_{p/q} \neq \emptyset$ since (by Lemma 5.7) it contains the (non-empty) p/q pleating ray.

Thus $\mathcal{N}_{p/q}$ is a union of non-empty connected components of $\mathcal{Z}_{p/q}$ contained in $\mathcal{R}$. By the Keen–Series theory, there are at most two such connected components, namely the components corresponding to the pleating rays of asymptotic slopes $\pm\pi p/q$ ([27, Theorem 4.1] and [29, Theorem 2.4]); and clearly we hit both of these components. In any case, picking a branch of the inverse of $P_{p/q}$ corresponding to these arguments will give a connected component of $\mathcal{N}_{p/q}$, and such a component is the desired neighbourhood of the cusp lying inside the Riley slice.

References

[1] Lars Ahlfors. "Quasiconformal reflections". In: *Acta Mathematica* 109 (1963), pp. 291–301. DOI: 10.1007/BF02391816.

[2] Shunsuke Aimi, Donghi Lee, Shunsuke Sakai, and Makoto Sakuma. "Classification of parabolic generating pairs of Kleinian groups with two parabolic generators". In: (2020). arXiv: 2001.11662 [math.GT].

[3] Hirotaka Akiyoshi, Ken'ichi Ohshika, John Parker, Makoto Sakuma, and Han Yoshida. "Classification of non-free Kleinian groups generated by two parabolic transformations". In: *Transactions of the American Mathematical Society* 374 (3 2021), pp. 1765–1814. DOI: 10.1090/tran/8246. arXiv: 2001.09564 [math.GT].

[4] Hirotaka Akiyoshi, Makoto Sakuma, Masaaki Wada, and Yasushi Yamashita. *Punctured torus groups and 2-bridge knot groups I*. Lecture Notes in Mathematics 1909. Springer, 2007. ISBN: 978-3-540-71807-9. DOI: 10.1007/978-3-540-71807-9.

[5] Kari Astala, Tadeusz Iwaniec, and Gaven J. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series 48. Princeton University Press, 2009. ISBN: 978-0-691-13777-3.

[6] Lipman Bers. “On boundaries of Teichmüller spaces and on Kleinian groups I”. In: *Annals of Mathematics* 91 (3 1970), pp. 570–600. DOI: 10.2307/1970638.

[7] Lipman Bers. “Uniformization, moduli, and Kleinian groups”. In: *Bulletin of the London Mathematical Society* 4 (1972), pp. 257–300. DOI: 10.1112/blms/4.3.257.

[8] Lipman Bers and L. Greenberg. “Isomorphisms between Teichmüller spaces”. In: *Advances in the theory of Riemann Surfaces – proceedings of the 1969 Stony Brook conference*. Ed. by Lars Ahlfors and Lipman Bers. Annals of Mathematics Studies 66. Princeton University Press, 1971, pp. 53–79. ISBN: 0-691-08081-X.

[9] Brian H. Bowditch. “Markoff triples and quasifuchsian groups”. In: *Proceedings of the London Mathematical Society* 77.3 (1998), pp. 697–736. DOI: 10.1112/s0024611598000604.

[10] Joël Lee Brenner. “Quelques groupes libres de matrices”. French. In: *Comptes Rendus Mathématique, Académie des Sciences, Paris* (241 1955), pp. 1689–1691. URL: https://gallica.bnf.fr/ark:/12148/bpt6k31937/f1695.item.

[11] Bomshik Chang, S. A. Jenning, and Rimhak Ree. “On certain pairs of matrices which generate free groups”. In: *Canadian Journal of Mathematics* 10 (1958), pp. 279–284. DOI: 10.4153/CJM-1958-029-2.

[12] Marston D. E. Conder, Colin Maclachlan, Gaven J. Martin, and Eamonn A. O’Brien. “2-generator arithmetic Kleinian groups III”. In: *Mathematica Scandinavica* 90 (2 2002), pp. 161–179. DOI: 10.7146/math.scand.a-14368.

[13] J.H. Conway. “An enumeration of knots and links, and some of their algebraic properties”. In: *Computational Problems in Abstract Algebra*. Ed. by John Leech. Pergamon Press, 1970, pp. 329–358. ISBN: 978-0-08-012975-4.

[14] Marc Culler. “Lifting representations to covering groups”. In: *Advances in Mathematics* 59.1 (1986), pp. 64–70. DOI: 10.1016/0001-8708(86)90037-X.

[15] Clifford J. Earle, Irwin Kra, and Samuel L. Krushkal’. “Holomorphic motions and Teichmüller spaces”. In: *Transactions of the American Mathematical Society* 343 (2 1994), pp. 927–948. DOI: 10.2307/2154750.

[16] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021-22 MATRIX annals*. Springer, 2022. arXiv: `2204.11422 [math.GT]`. To appear.

[17] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *The combinatorics of Farey words and their traces*. 2022. arXiv: `2204.08076 [math.GT]`.

[18] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *The elliptic cousins of the Riley slice*. In preparation. 2022.

[19] Valérie Flammang and Georges Rhin. "Algebraic integers whose conjugates all lie in an ellipse". In: *Mathematics of Computation* 74 (252 2005), pp. 2007–2015. DOI: `10.1090/s0025-5718-05-01735-7`.

[20] Frederick W. Gehring, Jane Gilman, and Gaven J. Martin. "Kleinian groups with real parameters". In: *Communications in Contemporary Mathematics* 3 (2001), pp. 163–186. DOI: `10.1142/S0219199701000342`.

[21] Frederick W. Gehring, Colin Maclachlan, and Gaven J. Martin. "Two-generator arithmetic Kleinian groups II". In: *Bulletin of the London Mathematical Society* 30 (3 1998), pp. 258–266. DOI: `10.1112/S0024609397004359`.

[22] Frederick W. Gehring, Colin Maclachlan, Gaven J. Martin, and Alan W. Reid. "Arithmeticity, discreteness and volume". In: *Transactions of the American Mathematical Society* 347 (1997), pp. 3611–3643. DOI: `10.1090/S0002-9947-97-01989-2`.

[23] Frederick W. Gehring and Gaven J. Martin. "Commutators, collars and the geometry of Möbius groups". In: *Journal d'Analyse Mathématique* 63 (1994), pp. 175–219. DOI: `10.1007/BF03008423`.

[24] Godfrey H. Hardy and Edward M. Wright. *An introduction to the theory of numbers*. 4th ed. Oxford University Press, 1960.

[25] Troels Jørgensen. "On discrete groups of Möbius transformations". In: *American Journal of Mathematics* 98 (1976), pp. 739–749. DOI: `10.2307/2373814`.

[26] Linda Keen and Caroline Series. "Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori". In: *Topology* 32.4 (1993), pp. 719–749. DOI: `10.1016/0040-9383(93)90048-z`.

[27] Linda Keen and Caroline Series. "The Riley slice of Schottky space". In: *Proceedings of the London Mathematics Society* 3.1 (69 1994), pp. 72–90. DOI: `10.1112/plms/s3-69.1.72`.

[28] Anthony W. Knapp. “Doubly generated Fuchsian groups”. In: *Michigan Mathematical Journal* 15.3 (1968), pp. 289–304. DOI: `10.1307/mmj/1029000032`.

[29] Yohei Komori and Caroline Series. “The Riley slice revisited”. In: *The Epstein birthday schrift*. Ed. by Igor Rivin, Colin Rourke, and Caroline Series. Geometry and Topology Monographs 1. Mathematical Sciences Publishers, 1998, pp. 303–316. DOI: `10.2140/gtm.1998.1.303`. arXiv: `math/9810194`.

[30] Donghi Lee and Makoto Sakuma. “Epimorphisms between 2-bridge link groups: homotopically trivial simple loops on 2-bridge spheres”. In: *Proceedings of the London Mathematical Society*. 3rd ser. 104.2 (2012), pp. 359–386. DOI: `10.1112/plms/pdr036`.

[31] Armin Leutbecher. “Über die Heckeschen Gruppen $G(\lambda)$”. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 81 (1967), pp. 199–205. DOI: `10.1007/BF02992399`.

[32] Roger C. Lyndon and Joseph L. Ullman. “Groups generated by two parabolic linear fractional transformations”. In: *Canadian Journal of Mathematics* 21 (1969), pp. 1388–1403. DOI: `10.4153/cjm-1969-153-1`.

[33] Mikhail Yu. Lyubich and V. V. Suvorov. “Free subgroups of $\mathrm{SL}_2(\mathbb{C})$ with two parabolic generators”. In: *Journal of Soviet Mathematics* 41.2 (Apr. 1988), pp. 976–979. DOI: `10.1007/BF01247092`.

[34] Colin Maclachlan and Gaven J. Martin. “The $(6, p)$-arithmetic hyperbolic lattices in dimension 3”. In: *Pure and Applied Mathematics Quarterly* 7 (2). Special Issue: In honor of Frederick W. Gehring, pp. 365–382. DOI: `10.4310/PAMQ.2011.v7.n2.a5`.

[35] Colin Maclachlan and Alan W. Reid. *The arithmetic of hyperbolic 3-manifolds*. Graduate Texts in Mathematics 219. Springer-Verlag, 2003. ISBN: 978-1-4757-6720-9. DOI: `10.1007/978-1-4757-6720-9`.

[36] Albert Marden. “An inequality for Kleinian groups”. In: *Advances in the theory of Riemann Surfaces – proceedings of the 1969 Stony Brook conference*. Ed. by Lars Ahlfors and Lipman Bers. Annals of Mathematics Studies 66. Princeton University Press, 1971, pp. 295–296. ISBN: 0-691-08081-X.

[37] Albert Marden. *Hyperbolic manifolds. An introduction in 2 and 3 dimensions*. 2nd ed. First edition was published under the title “Outer circles”. Cambridge University Press, 2016. ISBN: 978-1-107-11674-0.

[38] Albert Marden. “The geometry of finitely generated Kleinian groups”. In: *Annals of Mathematics* 99 (1974), pp. 383–462. DOI: 10.2307/1971059.

[39] Gaven J. Martin. “Nondiscrete parabolic characters of the free group F_2: supergroup density and Nielsen classes in the complement of the Riley slice”. In: *Journal of the London Mathematical Society* 103.4 (2020), pp. 1402–1414. DOI: 10.1112/jlms.12412. arXiv: 2001.10077.

[40] Gaven J. Martin, K. Selahi, and Y. Tamashita. *The $(4, p)$-arithmetic hyperbolic lattices in dimension* 3. To appear.

[41] Bernard Maskit. *Kleinian groups*. Grundlehren der mathematischen Wissenshaften 287. Springer-Verlag, 1987. ISBN: 978-3-642-61590-0. DOI: 10.1007/978-3-642-61590-0.

[42] Bernard Maskit and Gadde Swarup. “Two parabolic generator Kleinian groups”. In: *Israel Journal of Mathematics* 64.3 (1989), pp. 257–266. DOI: 10.1007/bf02882422.

[43] Katsuhiko Matsuzaki and Masahiko Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford University Press, 1998. ISBN: 0-19-850062-9.

[44] Curt McMullen. “Cusps are dense”. In: *Annals of Mathematics* 133 (1991), pp. 217–247. DOI: 10.2307/2944328.

[45] Hideki Miyachi. “Cusps in complex boundaries of one-dimensional Teichmüller space”. In: *Conformal Geometry and Dynamics* 7 (2003), pp. 103–151. DOI: 10.1090/S1088-4173-03-00065-1.

[46] Shunsuke Morosawa, Yasuichiro Nishimura, Masahiko Taniguchi, and Tetsuo Ueda. *Holomorphic dynamics*. Cambridge Studies in Advanced Mathematics 66. Cambridge University Press, 2000. ISBN: 0-521-66258-3.

[47] David Mumford, Caroline Series, and David Wright. *Indra’s pearls. The vision of Felix Klein*. Cambridge University Press, 2002. ISBN: 0-521-35253-3.

[48] Ken’ichi Ohshika and Hideki Miyachi. “Uniform models for the closure of the Riley slice”. In: *Contemporary Mathematics* 510 (2010), pp. 249–306. DOI: 10.1090/conm/510/10029.

[49] Jessica S. Purcell. *Hyperbolic knot theory*. Graduate Studies in Mathematics 209. American Mathematical Society, 2020. ISBN: 978-1-4704-5499-9.

[50] Robert Riley. “Parabolic representations of knot groups, I”. In: *Proceedings of the London Mathematics Society*. 3rd ser. 24 (1972), pp. 217–242. DOI: 10.1112/plms/s3-24.2.217.

[51] I. N. Sanov. “A property of a representation of a free group”. Russian. In: *Doklady Akademiya Nauk SSSR* 57.7 (1947), pp. 657–659.

[52] Horst Schubert. “Knoten mit zwei Brücken”. In: *Mathematische Zeitschrift* 65 (1956), pp. 133–170. DOI: 10.1007/bf01473875.

[53] Caroline Series. “The geometry of Markoff numbers”. In: *The Mathematical Intelligencer* 7.3 (1985), pp. 20–29. DOI: 10.1007/bf03025802.

[54] Caroline Series. “The modular surface and continued fractions”. In: *Journal of the London Mathematical Society*. 2nd ser. 31 (1985), pp. 69–80. DOI: 10.1112/jlms/s2-31.1.69.

[55] Zbigniew Słodkowski. “Holomorphic motions and polynomial hulls”. In: *Proceedings of the American Mathematical Society* 111 (1991), pp. 347–355. DOI: 10.1090/s0002-9939-1991-1037218-8.

[56] Zbigniew Słodkowski. “Natural extensions of holomorphic motions”. In: *Journal of Geometric Analysis* 7 (4 1997), pp. 637–651. DOI: 10.1007/bf02921638.