Pictures of hyperbolic spaces

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(mpi-mis)

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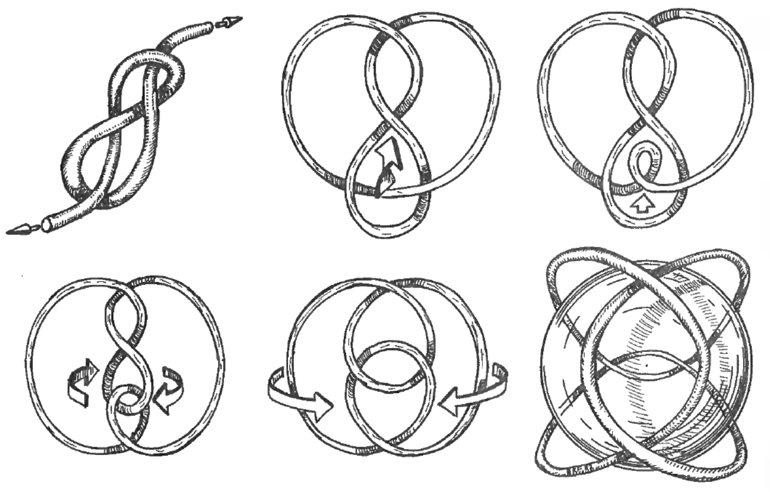
Knot complements

The figure 8 knot $k(5/3)$



[Francis, p.150]

Theorem (Robert Riley (c.1974); William P. Thurston (c.1975))

The complement of the figure 8 knot,

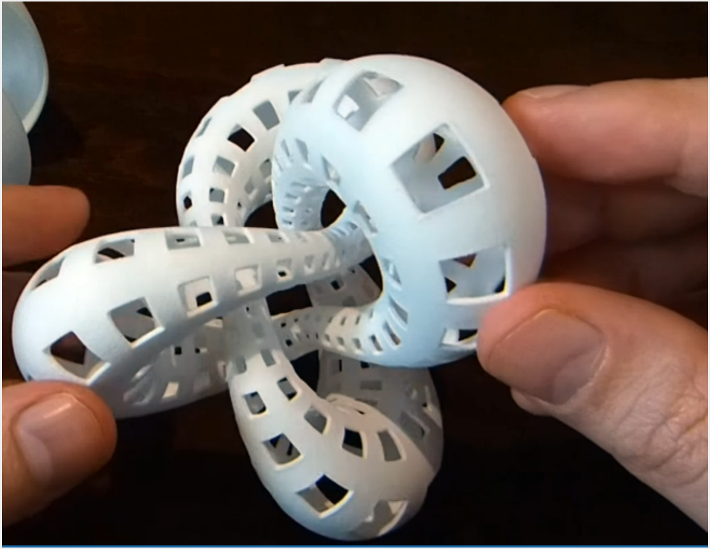
$$S^3 \setminus k(5/3),$$

admits a hyperbolic geometry.

Theorem (Thurston (c.1979))

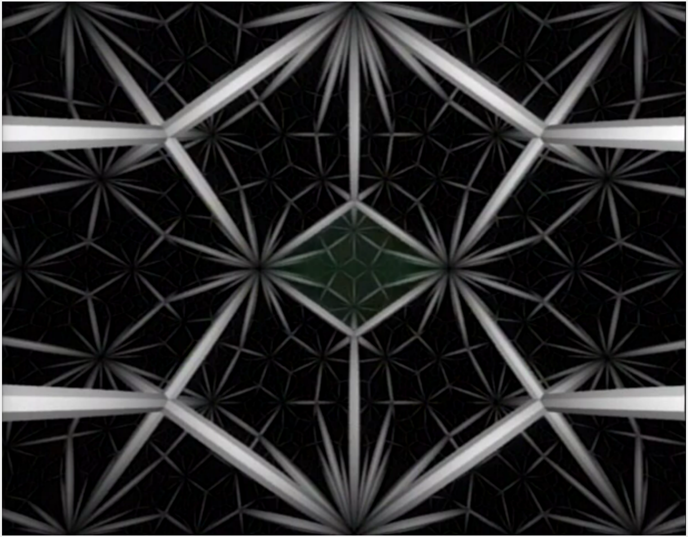
Almost every knot complement[1] admits a hyperbolic geometry.

[1] All but a small family of tabulated exceptions



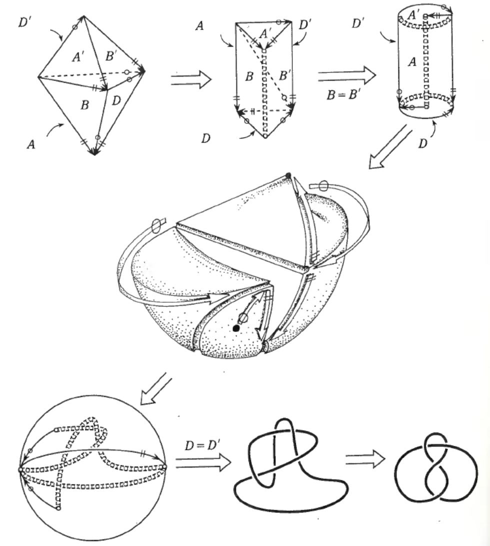
[Guéritaud/Segerman/Schleimer, `https://youtu.be/xGf5jY_v5GE`]

The Borromean rings complement



[Gunn/Maxwell, *Not Knot*]

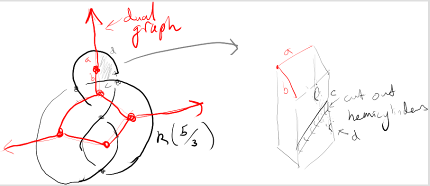
Idea: visualise knot complements by tiling



[Matsuzaki/Taniguchi, p.34]

SnapPea Algorithm (Jeff Weeks, c.1985)

1. Embed the knot in $S^2 \times [-1, 1]$, 'flatly' around $S^2 \times 0$.
2. Cut straight down along the dual graph & the knot graph.



3. Collapse these quadrilateral tubes to tetrahedra (pinch the upper & lower faces to points, and the two hollowed tubes).
4. Glue four cusps onto these vertices to get spherical tetrahedra.
5. Do a bit of fiddling to get the hyperbolic geometry back.

This gives all hyperbolic manifolds

Theorem (Thurston, c.1979)

Every hyperbolic 3-manifold can be obtained by 'Dehn surgery' along some hyperbolic link.

Thus Weeks' algorithm triangulates every hyperbolic 3-manifold.

Limit sets

Theorem

*Let M be a hyperbolic orbifold. Then M is isometric to a orbifold of the form $\mathbb{H}^3/G$ for some discrete group G of hyperbolic isometries (a **Kleinian group**).*

Theorem (Poincaré extension)

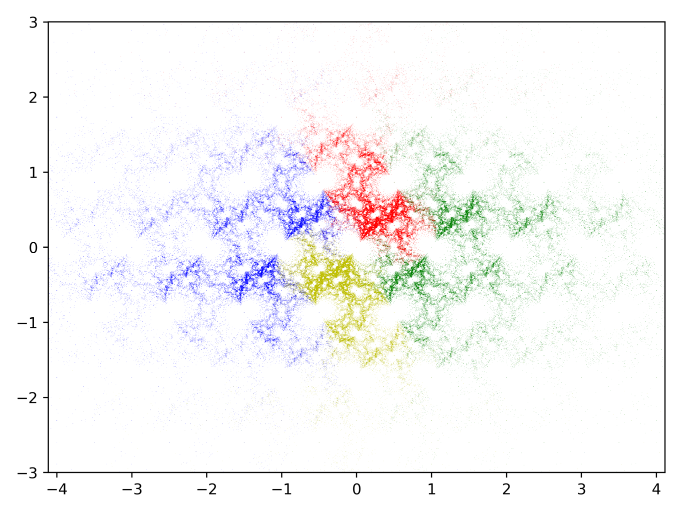
Every isometry of $\mathbb{H}^3$ extends continuously to a unique conformal action on $\hat{\mathbb{C}}$ (the sphere at infinity). Conversely, every conformal map on $\hat{\mathbb{C}}$ extends continuously to a unique hyperbolic isometry on the interior (the ball model of $\mathbb{H}^3$).

The dynamics of the action of a Kleinian group G on $\hat{\mathbb{C}}$ are complicated. There is a partition $\hat{\mathbb{C}} = \Omega(G) \cup \Lambda(G)$ similar to the partition between the Fatou and Julia sets of a holomorphic dynamical system.

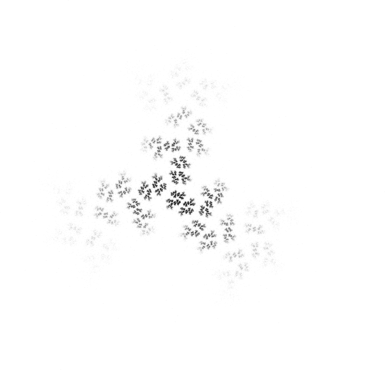
Definition

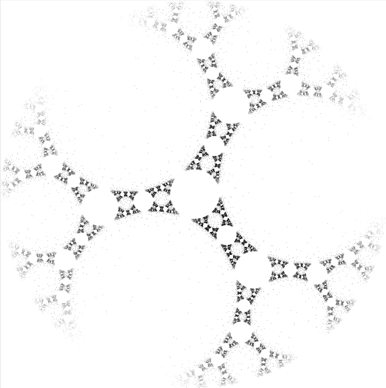
If G is non-elementary, then the **limit set** of G is the closure of the set of fixed points of elements of G.

Figure 8 knot group



Examples





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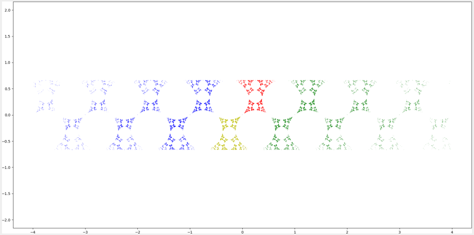
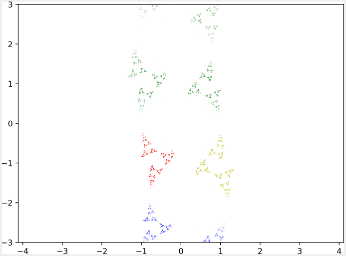




Theorem (Thurston (c.1979); The ending lamination theorem (Epstein/Marden/Minsky))

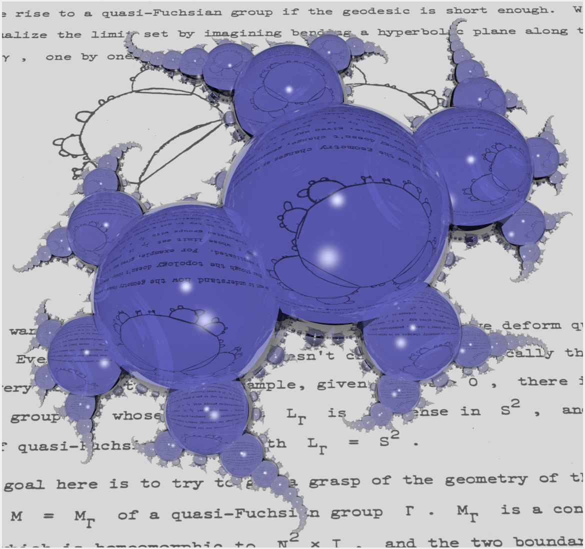
If G is non-degenerate[2] *then there is a strong deformation retract*

$$\frac{\mathbb{H}^3 \cup \Omega(G)}{G} \twoheadrightarrow \frac{\text{h.conv}\,\Lambda(G)}{G}$$

and the 'folding structure' on the convex hull determines the hyperbolic geometry entirely.

[2] non-Fuchsian and non-elementary

Bug on notes of Thurston



[Jeffrey Brock and David Dumas, https://www.dumas.io/poster/]

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- So the problem is reduced to (1) enumerating 'lots of words', and (2) doing matrix products quickly.
- These are standard problems in computational combinatorial group theory.

Bedtime reading

- George K. Francis, *A topological picturebook* (Springer, 1987)
- William P. Thurston, *Geometry and topology of 3-manifolds* (unpublished lecture notes, c.1979)
- William P. Thurston, *Three-dimensional geometry and topology, Vol. 1* (Princeton, 1997)
- Jeff Weeks, "Computation of hyperbolic structures in knot theory". In: *Handb. of Knot Theory* (Elsevier, 2005)
- Jeff Weeks, *The shape of space* (Marcel Dekker, 2001)
- David Mumford, Caroline Series, David Wright, *Indra's pearls* (Cambridge, 2002)
- Jessica Purcell, *Hyperbolic knot theory* (AMS, 2021)
- Title picture: Belmont Trig looking towards Porirua Harbour (13/03/2022).