Spherical designs & numerical cubature

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Joint work with Shayne Waldron

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This talk, August 8, 2025

Many applications (e.g. error correcting codes, quantum tomography, signal analysis) require us to find a spanning set of $\mathbb{R}^n$ which balances maximal redundancy (bigger sets are better) with efficiency (bigger sets require more storage/bandwidth). If we fix the number of vectors in the basis, it's clear that the most best arrangement (where all vectors carry maximum information) is one where the vectors are equally spaced.

Example (SIC (Symmetric Informationally Complete) sets)

A set of d^2 unit vectors $(v_j)_{j=1}^{d^2}$ in $\mathbb{C}^2$ is a **SIC** if the lines $(\mathbb{C}v_j)$ are equiangular. This is equivalent to the condition

$$|\langle v_j, v_k\rangle|^2 = \frac{1}{d+1}, \forall (j \neq k).$$

Zauner's conjecture (1999): there is a SIC for every dimension d.

- If $t \in \mathbb{N}$ then a **spherical half-design** of order t in $\mathbb{R}^d$ is a set $(v_j)_{j=1}^n$ of unit vectors such that

$$\forall_{f \in \mathrm{Hom}_t(\mathbb{R}^d)} \quad \int_{\mathbb{S}^{d-1}} f \, d\omega = \frac{1}{n} \sum_{j=1}^{n} f(x).$$

  i.e. averaging homogeneous polynomials over the design integrates those polynomials on the sphere.
- If the set is a spherical half-design of order s for all $s \leq t$ then it's called a **spherical t-design**.
- **Theorem (Seymour-Zaslavsky, 1984):** there is a spherical t-design in $\mathbb{R}^d$ for all t and d, so long as n is big enough.
- **Game:** find designs with smallest possible n.

t-DESIGNS: EXAMPLES

- The Mercedez-Benz frame: 
- For any t the vertices of the regular $(t+1)$-gon form a t-design in $\mathbb{R}^2$.
- The 12 vertices of the regular icosahedron are a 5-design in $\mathbb{R}^3$.

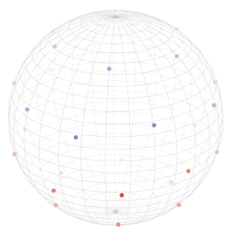
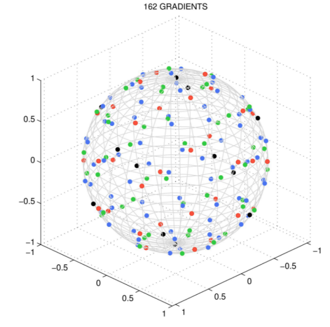


FIG. 1. Visualization of a $t = 7$ spherical t-design. Each blue point has a corresponding antipodal red point.

A.C. Simmonett, B.R. Brooks, T.A. Darden, "Efficient and scalable electrostatics via spherical grids and treecode summation", *J. Chem. Phys.* **162**, 2025. `https://doi.org/10.1063/5.0264934`



Spherical t-designs of orders 5 (black), 7 (red), 9 (green) and 11 (blue), of respective sizes 12, 32, 48 and 70.

D. Gasbarra, S. Pajevic, P.J. Basser, "Eigenvalues of random matrices with isotropic Gaussian noise and the design of Diffusion Tensor Imaging experiments", *SIAM J. Imaging Sci.* **10**, 2017. `https://doi.org/10.1137/16M1098693`. arXiv:1707.06953 [stat.ME]

Theorem (Waldron, 2016)

Let $(v_j)_{j=1}^n$ be a set of distinct vectors (maybe not unit) in $\mathbb{R}^d$. Then

$$\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle v_j \mid v_k\rangle|^{2t} \geq \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+1)\cdots(d+2(t-1))}\left(\sum_{j=1}^{n}\|v_j\|^{2t}\right)^2$$

and equality holds if and only if the vectors give a cubature rule:

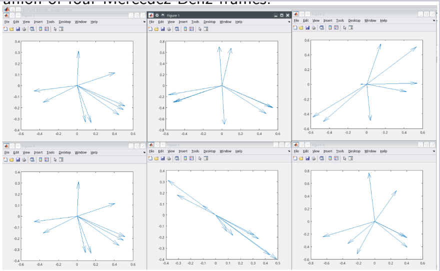
$$\forall_{f\in \mathrm{Hom}_{2t}(\mathbb{R}^d)}, \quad \int_{\mathbb{S}^{d-1}} f\,d\omega = \frac{1}{\sum_{j=1}^{n}\|v_j\|^{2t}}\sum_{j=1}^{n} f(v_j).$$

*If equality holds, the vectors are called a **spherical** (t,t)**-design**.*

Examples due to Bramwell (2011), Hughes–Waldron (2018), Mohammadpour–Waldron (2019):

- A spherical $(2,2)$-design of 16 vectors in $\mathbb{R}^5$: the union of
  - 6 equiangular vectors at an angle of $1/5$, of length $\sqrt[4]{20/21}$ and
  - 10 equiangular vectors at an angle of $1/3$, of length $\sqrt[4]{36/35}$.
- A 19-design of 720 vectors for $\mathbb{R}^4$, found as a union of orbits of a real reflection group (obtained from a 360 vector $(9,9)$-design).

Every equal-norm $(2,2)$-design of 12 vectors in $\mathbb{R}^4$ is constructed as a union of four Mercedez-Benz frames.



(t,t)-DESIGNS AND THE FRAME POTENTIAL

Strategy for finding new examples: take

$$\sum_{j=1}^{n}\sum_{k=1}^{n} |\langle v_j \mid v_k\rangle|^{2t} \geq \frac{1\cdot 3\cdot 5\cdots(2t-1)}{d(d+1)\cdots(d+2(t-1))}\left(\sum_{j=1}^{n}\|v_j\|^{2t}\right)^2$$

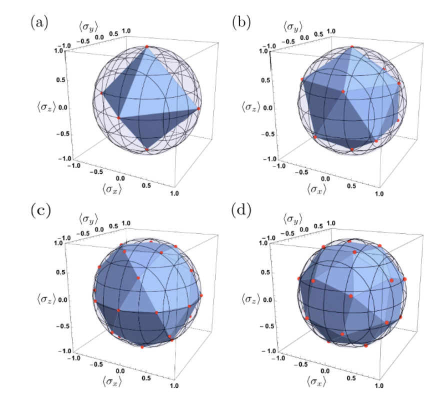
and subtract the right hand side from the left, giving a function $f_{t,d,n} : (\mathbb{R}^d)^n \to \mathbb{R}$ called the **frame potential**. Then minimise this function numerically over manifolds of high dimension.

For the computer people

We used the helpful software Manopt (`https://www.manopt.org/`). The computer code we used is available on the Intenet at `https://github.com/aelzenaar/tightframes`.

State of the art in 2018

t	d	n_w	n_e	comment
1	d	d	d	orthonormal bases
t	2	$t+1$	$t+2$	equiangular lines in $\mathbb{R}^2$
2	3	6	6	equiangular lines in $\mathbb{R}^3$
	4	11	12	wtd is due to Reznick
	5	16	20	
	6	22	24	
	7	28	28	equiangular lines in $\mathbb{R}^4$
	8	45	> 45	
3	3	11	16	wtd is due to Reznick
	4	23	> 23	
	5	41	> 41	
4	3	16	25	
	4	43	> 43	
5	3	24	35	

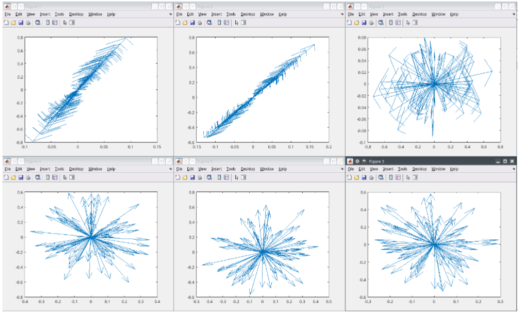


A. Ketterer, N. Wyderka, O. Gühne, "Entanglement characterization using quantum designs", *Quantum* **4** 325 (2020). https://doi.org/10.22331/q-2020-09-16-325. arXiv:2004.08402 [quant-ph]

State of the art as of our work

t	largest d now known	comment
2	18 (was 7)	infinite families for $d \geq 8$
3	9 (was 3)	
4	7 (was 3)	infinite families for $d > 4$ $d = 7$, $n = 504$
5	6 (was 3)	infinite families $d = 6$, $n = 604$
6	5 (all new!)	infinite families
7	4 (all new!)	infinite families
8	4 (all new!)	
9	4 (all new!)	$d = 4$: 360 vectors, unique
10	3 (all new!)	

An equal norm $(3,3)$-design in $\mathbb{R}^6$ of 96 vectors



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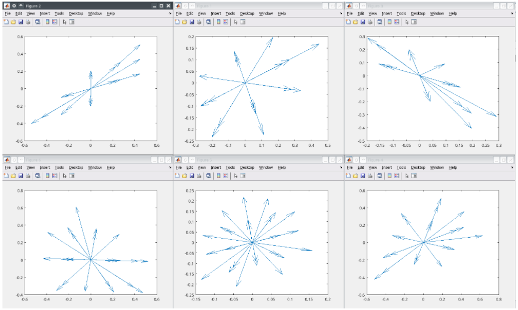
An weighted $(3, 3)$-design in $\mathbb{R}^6$ of 63 vectors



This has been constructed analytically as a union of orbits of two vectors

State of the Art as of Our Work

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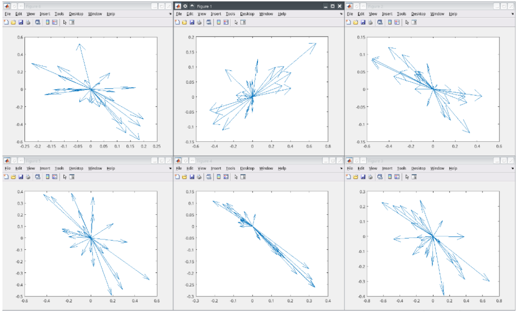
10	3 (all new!)	

An weighted $(5,5)$-design in $\mathbb{R}^8$ of 31 vectors



State of the art as of our work

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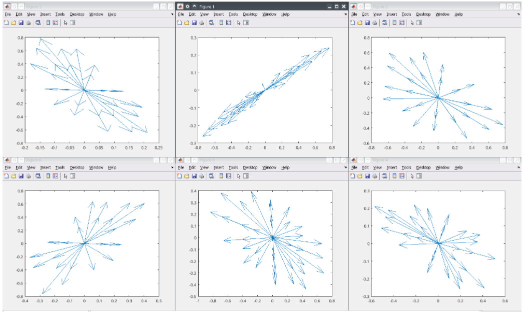
6	5 (all new!)	infinite families
7	4 (all new!)	infinite families
8	4 (all new!)	
9	4 (all new!)	$d = 4$: 360 vectors, unique
10	3 (all new!)	

An equal-norm $(5,5)$-design in $\mathbb{R}^4$ of 60 vectors



From Sheppard-Todd group 30

Bedtime reading

- A. Elzenaar and S. Waldron, “Putatively optimal projective spherical designs with little apparent symmetry”, *J. Combin. Des.* **33**(6), 2025. `https://doi.org/10.1002/jcd.21979`
- A. Elzenaar and S. Waldron, “A repository of spherical (t, t)-designs”. `https://doi.org/10.5281/zenodo.6443356`
- S. Waldron, *An introduction to finite tight frames.* Birkhaüser, 2018.

- J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups* (3e). Grundlehren Math. Wiss. 290. Springer, 1999.
- B. Reznick, *Sums of even powers of real linear forms.* Mem. AMS 463, 1992.

- S. V. Borodachov, P. G. Boyvalenkov, P. D. Dragnev, D. P. Hardin, E. B. Saff & M. M. Stoyanova, “Bounds on discrete potentials of spherical (k, k)-designs”, *Des. Codes Cryptogr.*, 2025. `https://doi.org/10.1007/s10623-025-01659-z`