Apocrypha and ephemera on the boundaries of moduli space

or, What I did on my holiday.

Alex Elzenaar

January 10, 2023

Henry Moore: *Bronze Form* (1988). In situ, Wellington Botanic Garden ki Paekākā. Photo by author (2022)

Introduction and context

When Thurston reinvented hyperbolic geometry in the '70s and '80s, he brought numerous combinatorial ideas with him from his earlier work on foliations. We will survey some of these combinatorial ideas and their relation to work I have been doing for the past couple of years, including some recent joint work with Sam Fairchild and Ángel Ríos Ortiz. One can view all of hyperbolic geometry post-Thurston as being essentially complicated combinatorics and geometry of circles on the sphere (the geometry being encoded by so-called *Kleinian groups*, which arose much earlier in conformal geometry).

Remark. These notes are essentially a compressed version of what was supposed to be the second third of my graduate seminar at the University of Auckland in 2021, which was sadly interrupted by the August lockdown. (The three thirds were 'classical', i.e. pre-Thurston conformal geometry and complex dynamics; 'Thurston', i.e. hyperbolic geometry; and 'arithmetic', following [36]. Thus in some sense we *de jure* assume an awful lot of classical Kleinian group theory, from e.g. [5, 38]; but we *de facto* do not need it, since everything is highly intuitive.)

Contents

List of Figures

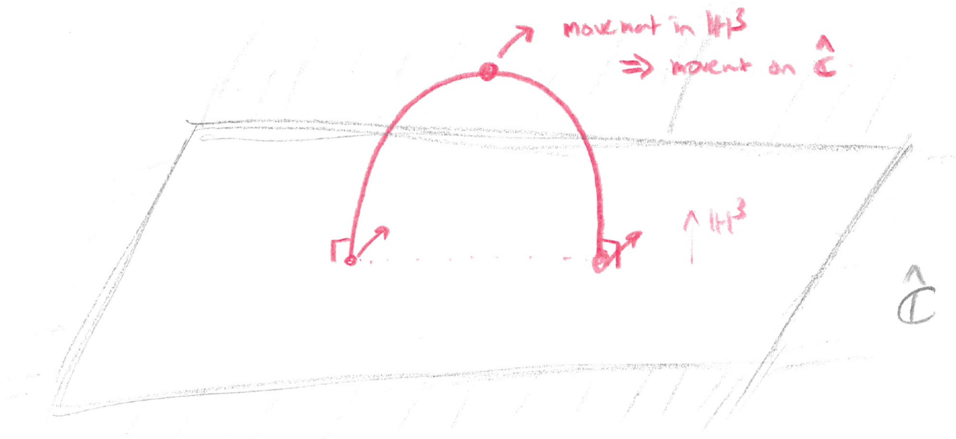


Figure 1: A geodesic in $\mathbb{H}^3$ lying on the visual boundary $\hat{\mathbb{C}}$.

§1. A crash course on Kleinian groups

As Kra [34] and Series [53] (among many others) have done before us, we will first have a crash course on Kleinian groups. But unlike Kra and Series, we will take a post-Thurston view immediately. For the basic material on hyperbolic metric spaces, we follow [10]. I will try to be consistent in notation with the lecture notes from my 2021 graduate course [18] which in turn are consistent with the books of Maskit [38] and Beardon [5], and the notes of Thurston [58]. A much nicer overview may be found in Thurston's famous survey paper [59].

We recall that **hyperbolic 3-space** is the unique Riemann manifold of constant curvature -1. We will primarily use the Poincaré model, supported on the open smooth manifold

$$\mathbb{H}^3 := \{x = (z, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$$

with Riemann metric given by

$$ds = \frac{|dz + dt|}{t}.$$

Geodesic lines in $\mathbb{H}^3$ are exactly the (Euclidean) semicircles and half-lines orthogonal to $\mathbb{C}$. There is a natural compactification of $\mathbb{H}^3$ given by adjoining the sphere at infinity, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This is exactly the visual boundary of $\mathbb{H}^3$ as a CAT(0) space, and as such isometries of $\mathbb{H}^3$ extend to to conformal maps on the boundary (Fig. 1). In fact since the visual boundary is defined to be the space of geodesic half-rays modulo the relation of being eventually parallel, and because this relation is trivial in $\mathbb{H}^3$ by simple Euclidean geometry of circles, we get an isomorphism of topological groups $\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{Conf}(\hat{\mathbb{C}})$. By standard results from undergraduate analysis we have further isomorphisms between $\mathrm{Conf}(\hat{\mathbb{C}})$ and the groups of Möbius transformations $\mathbb{M}$ and of fractional linear transformations $\mathrm{PSL}(2, \mathbb{C})$. As always a discrete subgroup of $\mathbb{M}$ (and by canonical identification, any of these groups) is called **Kleinian**.

We will now move to manifolds modelled on hyperbolic space, but we might as well be slightly more general and use Thurston's language of pseudogroups (essentially a version of sheaf theory). A **pseudogroup** on a topological space X is a set $\mathcal{G}$ of homeomorphisms defined on open subsets of X such that:

1. (Group-type axioms)
    a) If $f, g \in \mathcal{G}$ then $f \circ g \in \mathcal{G}$;
    b) If $f \in \mathcal{G}$ then $f^{-1} \in \mathcal{G}$;
2. (Sheaf-type axioms)
    a) The set $\{\operatorname{dom} f : f \in \mathcal{G}\}$ is an open cover of X;
    b) If $f \in \mathcal{G}$ and $V \subseteq \operatorname{dom} f$ then $f|_V \in \mathcal{G}$;
    c) Suppose $f \in \mathrm{Homeo}(X)$ and $(V_\alpha)_{\alpha\in A}$ is an open cover of $\operatorname{dom} f$. If $f|_{V_\alpha} \in \mathcal{G}$ for all $\alpha \in A$, then $f \in \mathcal{G}$.

Now a $\mathcal{G}$**-manifold**, for a pseudogroup $\mathcal{G}$ on some subset $X \subseteq \mathbb{R}^n$, is an n-manifold M such that the charts of M land in X and the transition maps of M lie in $\mathcal{G}$. For instance, a Riemann surface is a $\mathcal{C}$-manifold, where $\mathcal{C}$ is the pseudogroup of biholomorphic maps on subsets of $\mathbb{C} \simeq \mathbb{R}^2$.

1.1 Definition. An $\mathbb{H}^3$**-manifold** (or just **manifold** for us) is a $\mathcal{H}^3$-manifold, where $\mathcal{H}^3$ is the pseudogroup of hyperbolic isometries on $\mathbb{H}^3$.

Of course, one can replace $\mathbb{H}^3$ with one's favourite Thurston geometry (which is Sol).

It is natural to extend the definition of a manifold to allow for singularities. Let $\mathcal{G}$ be a pseudogroup on $X \subseteq \mathbb{R}^n$ and let O be a Hausdorff topological space. A $\mathcal{G}$**-orbifold structure** on O is given by the following data:

1. An open cover $\{V_i\}_{i\in I}$ of O;
2. For each i, a finite subpseudogroup $\Gamma_i \leq \mathcal{G}$ and a simply connected open set $X_i \subseteq X$ such that $\operatorname{dom} f \supseteq X_i$ and $f(X_i) \subseteq X_i$ for all $f \in \Gamma_i$ together with a continuous map $q_i : X_i \to V_i$ which descends to a homeomorphism $X_i/\Gamma_i \to V_i$;
3. For all $x_i \in X_i$ and $x_j \in X_j$ with $q_i(x_i) = q_j(x_j)$, a diffeomorphism h from an open connected neighbourhood W of x_i to a neighbourhood of x_j such that $q_j h = q_i|_W$.

Given a hyperbolic orbifold O, we can define a **developing map** via analytic continuation, following a version of the traditional construction of a universal cover in this way. The precise construction can be found in [47, §13.3], but the point is that the lifting of paths has to be compatible with the local group actions (Fig. 2). We say that O is **complete** if the development fills up the entirety of $\mathbb{H}^3$. Compare this with the following 2D example.

1.2 Example. Let G be the elementary group generated by $z \mapsto 2z$. A fundamental domain for this group is the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. The quotient orbifold is not complete since the tessellation by the fundamental domain misses 0 and ∞. It is an affine torus, by the way (Fig. 3) and the incompleteness is seen via the accumulation of the red curve in the quotient; see Exercise 3.1.10 of [60] for this example, and later on Example 3.3.4 for other affine manifolds including a second structure on the torus.

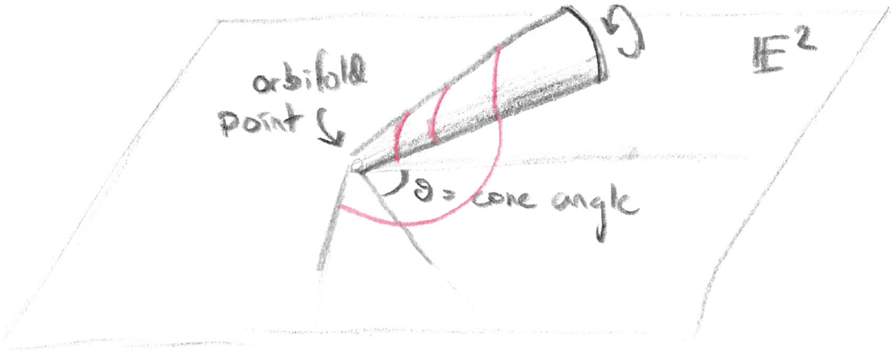


Figure 2: Unrolling a cone onto $\mathbb{E}^2$. The apex is a cone point: it is the centre of a rotation in the holonomy group (the symmetry group of the unrolling). See how the red path lifts, just like in the construction of a universal cover; but since we need to preserve geometry, some points might be missing.

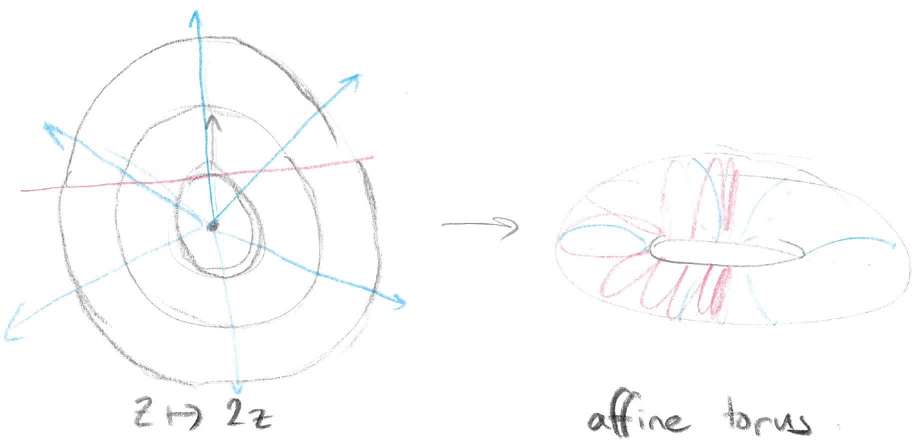


Figure 3: The affine torus produced as a quotient of $\mathbb{C} \setminus \{0\}$ by a dilation.

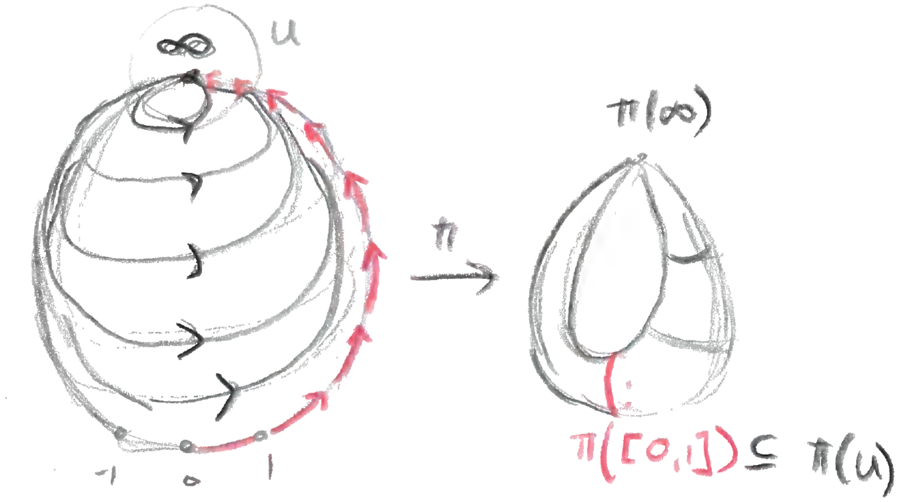


Figure 4: Attracting or repelling fixed points lead to non-Hausdorff quotients. Here, every neighbourhood of the projection of ∞ pulls back to an open set containing ∞ on the Riemann sphere; but every such neighbourhood contains infinitely many translates of the red segment $[0, 1]$ under the group action, and so every point of $\pi([0, 1])$ lies in every neighbourhood of $\pi(\infty)$. In fact, the only neighbourhood of $\pi(\infty)$ is the entire quotient!

Note, this definition of completeness coincides with the usual definition of completeness for a metric space (in the torus example, take any sequence in $\mathbb{C}$ converging to 0 which projects to a sequence of distinct points, then this is a non-converging Cauchy sequence).

1.3 Theorem. *If O is a connected complete $\mathbb{H}^3$-orbifold, then there is a Kleinian group Γ such that $O \simeq \mathbb{H}^3/O$. Further, Γ is isomorphic to the orbifold fundamental group $\pi_1^\circ(O)$. Conversely, if Γ is Kleinian then $\mathbb{H}^3/O$ is an $\mathbb{H}^3$-orbifold, in fact a hyperbolic $K(\Gamma, 1)$.*

The group Γ is called the **holonomy group** of O. Often rather than speaking about a group Γ we speak of a faithful discrete representation $\rho : \pi_1^\circ(O) \to \mathrm{PSL}(2, \mathbb{C})$, it amounts to the same thing.

Now a Kleinian group G does not just act on $\mathbb{H}^3$, it also acts on the sphere $\hat{\mathbb{C}} = \partial\mathbb{H}^3$. However, discreteness of G does *not* imply that the quotient $\hat{\mathbb{C}}/G$ is well behaved. For instance, take the group G of Example 1.2: the quotient $\hat{\mathbb{C}}/G$ is not Hausdorff at the projection of ∞—take the sequence 2^{-n}, all of these are identified by the group to a point ξ and every neighbourhood of 0 contains an element of the sequence so in the quotient every neighbourhood of 0 contains $\xi \neq 0$. For another example, take the group generated by the single translation $z \mapsto z + 1$ and look at the image of ∞ under the canonical projection map π (Fig. 4).

It turns out that all the bad points arise in this form (as fixed points and limits of fixed points), so we define the **limit set** of G to be $\Lambda(G) = \overline{\mathrm{Fix}(G)}$ where $\mathrm{Fix}(G)$ is the set of fixed points of non-torsion elements of G. The complement $\Omega(G) := \hat{\mathbb{C}} \setminus \Lambda(G)$ has Hausdorff quotient under G, in fact $\Omega(G)/G$ is a Riemann surface (with marked orbifold points). This gives us two geometric objects associated to G:

- the **complete hyperbolic manifold without boundary** $N_G = \mathbb{H}^3/G$
- the **Riemann surface at infinity** $S_G = \Omega(G)/G$

Now one can play with some examples and see that the set $\Omega(G)$ is identified the set of lifts of points on the visual boundary of N_G: if g is a geodesic ray in $\mathbb{H}^3$ ending at a limit point, then it projects to a closed geodesic in the quotient. Hence we can identify S_G with the visual boundary of N_G, and the result is $O_G = (\mathbb{H}^3 \cup \Omega(G))/G$, the **Kleinian manifold** (although in general it is orbi not mani).

The main point of all of this discussion is that it can be reversed. This is known as the **Poincaré polyhedron theorem** [47, §13.5]: if you take a hyperbolic polyhedron P together with a set of isometries pairing the sides up, then (subject to some geometric conditions) the group G generated by the side-pairings is discrete and $\mathbb{H}^3/G$ is isometric to the space obtained by taking P and identifying the sides. This works even if $\overline{P} \cap \partial\mathbb{H}^3$ is non-empty, in which case the intersection forms a fundamental domain for the action of G as a surface group and the quotient surface is obtained by pairing arcs of circles.

Now for some examples. I want to talk about moduli, and there are two 'easy' moduli spaces (arguably), called the **Riley slice** and the **Maskit embedding**. So we will give some relevant groups.

1.4 Example (Figure 8 knot group). Let $k \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the figure 8 knot (the best knot!!!). It is a result of Riley [49, Theorem 1] that $\pi_1(k)$ (recall, the fundamental group of a knot is the fundamental group of its complement in S^3) is isomorphic to the Kleinian group

$$\Gamma_{-\omega} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix} \right\rangle$$

where $\omega = \exp(2\pi i/3)$; the limit set is shown as Fig. 5. For a survey of the history see [20]. In any case, since $\Gamma_{-\omega}$ is non-elementary the quotient manifold $\mathbb{H}^3/\Gamma_{-\omega}$ is a hyperbolic manifold with fundamental group $\Gamma_{-\omega}$, and in fact it is homeomorphic to $S^3 \setminus k$. Nice pictures can be found in [41, p. 34] and in [25, p. 152]. This was in fact the example that led Thurston to the geometrisation conjecture (for various accounts of this history see [59, 48, 12] and the historical notes to Section 10.3 on p.504 of [47]).

1.5 Example (A Riley group). Define the group

$$\Gamma_{3i} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3i & 1 \end{bmatrix} \right\rangle.$$

See the limit set of Fig. 6. This group uniformises a manifold homeomorphic to the 3-ball minus two arcs, and hyperbolic metric coming from a braiding of the arcs of slope 2/1; see Fig. 7.

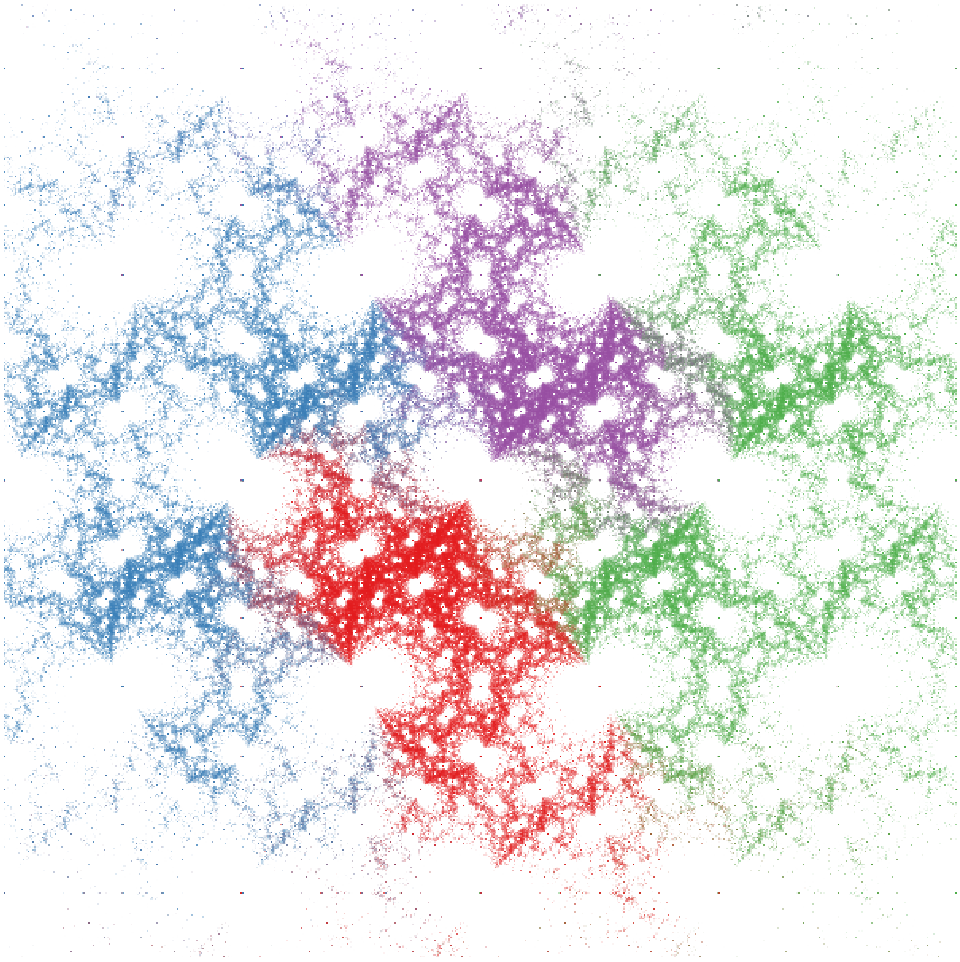


Figure 5: Limit set of the figure 8 knot group.

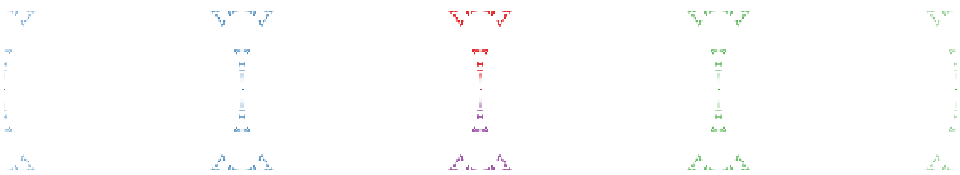


Figure 6: Limit set of the $\rho = 3i$ Riley group.

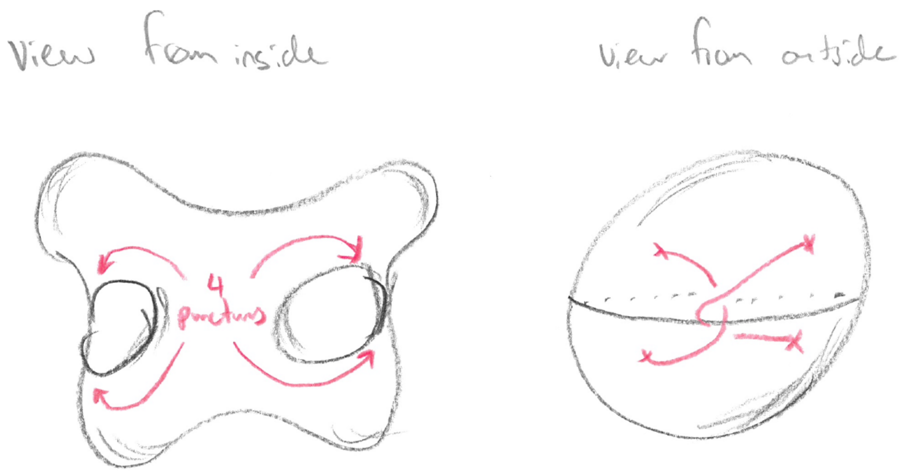


Figure 7: Two views of the quotient structures which arise from Γ_{3i} (Example 1.5). On the left, we see the hyperbolic structure on the Riemann surface at infinity (the quotient $\Omega(\Gamma_{3i})/\Gamma(3_i)$); observe that the punctures can be compactified by adding points 'infinitely far away' in the hyperbolic surface metric. On the right, we see the conformal structure at infinity from outside the three-manifold; the manifold is a ball with two arcs drilled out, and the arcs are twisted. We can untwist the arcs by isotopy in the 3-manifold, but tracking the movement of the 'fluid' we see that this will twist up the complex structure on the boundary.

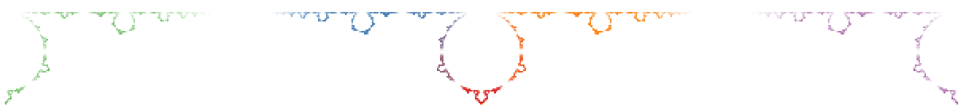


Figure 8: Limit set of the $\mu = 3i$ Maskit group.

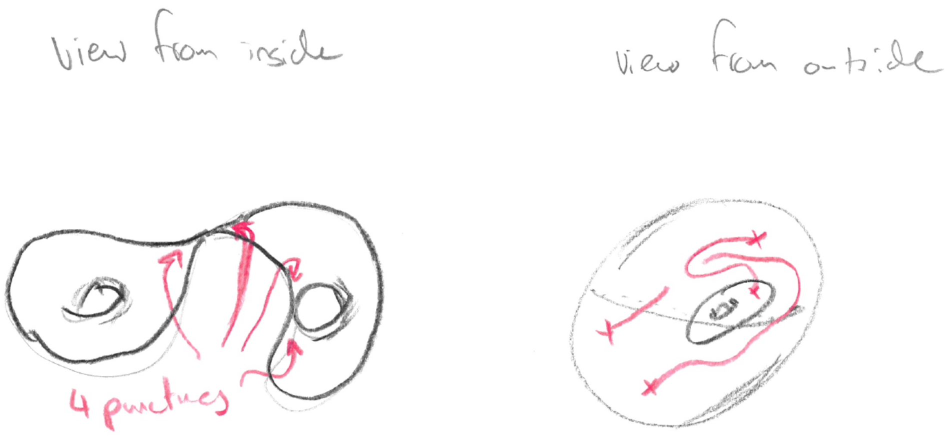


Figure 9: Two views of the quotient structures which arise from G_{3i} (Example 1.6). On the left, we see the hyperbolic structure on the Riemann surface at infinity (the quotient $\Omega(G_{3i})/G(3_i)$), and observe there is a similar kind of compactification as described in the caption of Fig. 7. On the right, we see the conformal structure at infinity from outside the three-manifold.

1.6 Example (A Maskit group)**.** Define the group

$$G_{3i} = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -i(3i) & i \\ i & 0 \end{bmatrix} \right\rangle.$$

See the limit set of Fig. 8. This group uniformises a manifold homeomorphic to a 3-ball with a deleted torus and two deleted arcs, see Fig. 9.

§2. Sociology

We would like topological methods for detecting ghosts of finite structures on the boundary at infinity; the main result which we want to state is the so-called *ending lamination theorem*, Theorem 2.2. In order to understand the part of this theorem which deals with Riemann surfaces on the boundary of the 3-manifold, we need to understand how the geometry of the boundary is reflected in the core of the manifold.

Let $\mathrm{h.conv}\,\Lambda(G)$ be the hyperbolic convex hull of $\Lambda(G)$, which looks something like Fig. 10, and set $M_G \coloneqq (\mathrm{h.conv}\,\Lambda(G))/G$. This manifold is the **convex core** of O_G; if G is non-elementary and non-Fuchsian, then M_G is a deformation retract of O_G. The boundary ∂M_G is a **pleated surface**; that is, there is a hyperbolic surface S together with an isometry $f : S \to \partial M_G$ (with respect to the intrinsic metric of ∂M_G) such that every point $s \in S$ lies in the interior of a geodesic in S which is mapped by f to a geodesic arc in ∂M_G, and such that f is homotopically incompressible (i.e. $f_* : \pi_1(S) \to \pi_1(M_G)$ has trivial kernel). The **pleating locus** of ∂M_G is the set of points through which there is exactly one geodesic in S which is mapped to a geodesic in M_G. The pleating locus is the canonical example of a measured geodesic lamination.

2.1 Definition. A **lamination** L on a surface M is a closed subset $A \subseteq M$ (the **support** of L) together with a local product structure on A: that is, there is a family of open sets

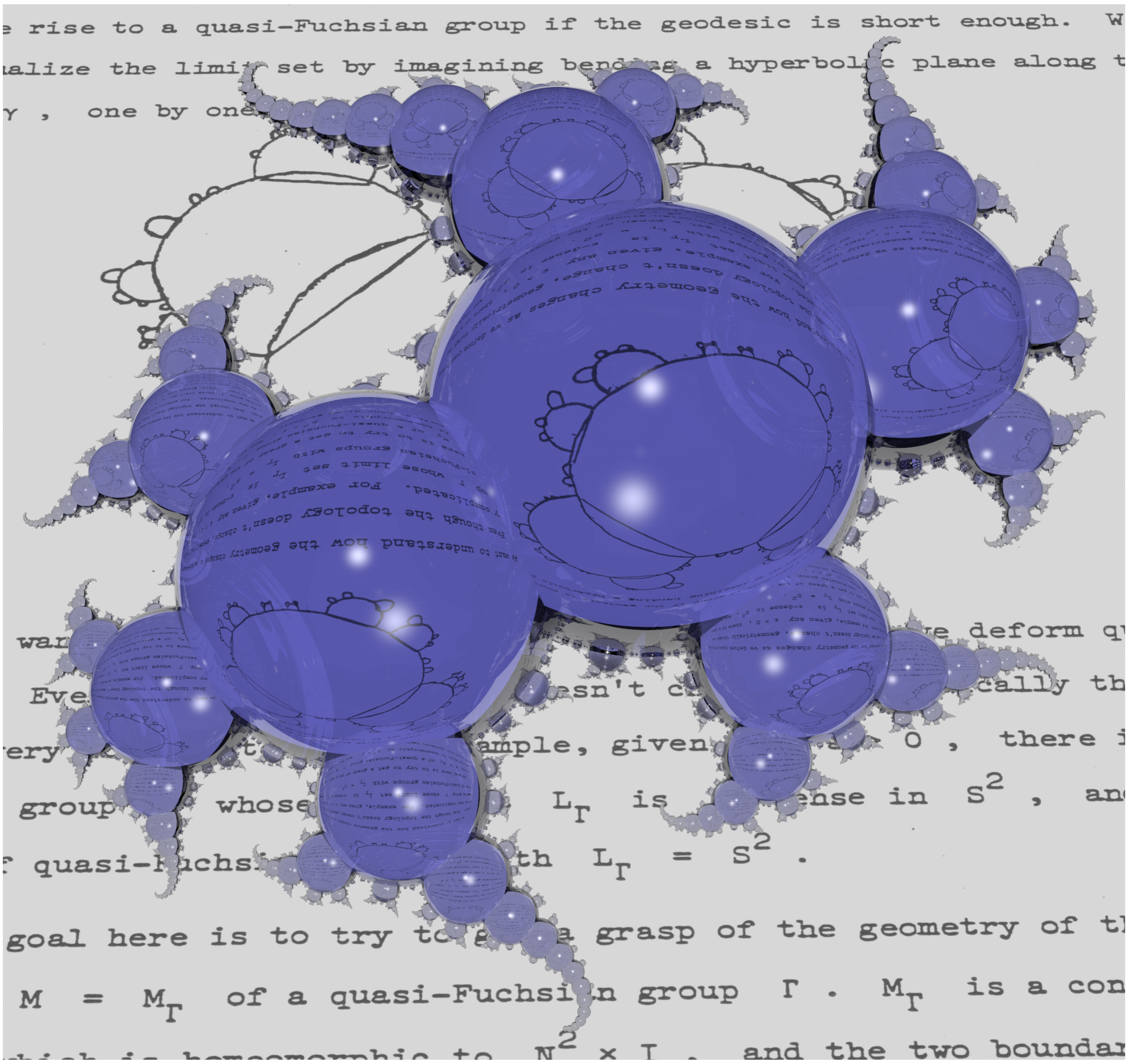


Figure 10: Jeffrey Brock and David Dumas, *Bug on notes of Thurston*. https://www.dumas.io/poster/.

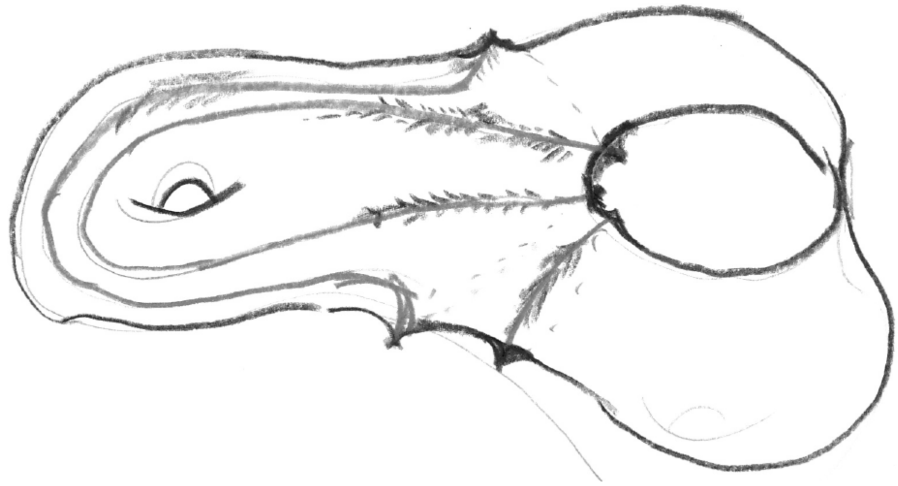


Figure 11: A geodesic measured lamination is a union of disjoint geodesics, where each geodesic (leaf of the lamination) is labelled with a weight that can be thought of as the 'angle of bending' as you walk across the leaf. The boundary of the convex core of a 3-manifold, $\partial\,\mathrm{h.conv}\,\Lambda(G)/G$, is really obtained as a physical pleated surface in an ambient space, and so in this case the angle is a real-life angle that you can measure with a protractor.

$\{U_i\}$ of M which cover A, together with charts $\phi_i : U_i \to \mathbb{R}\times\mathbb{R}$ for each i, such that $\phi_i(A\cap U_i) = \mathbb{R}\times B$ ($B\subseteq\mathbb{R}$) for each i, and such that the transition maps are of the form

$$\phi_i\phi_j^{-1}(x,y) = (f_{i,j}(x,y), g_{i,j}(y))$$

for all i, j and for all $x\in\mathbb{R}, y\in B$. If the leaves are all geodesics, then L is a **geodesic lamination**.

A **transverse measure** on L is a regular measure defined on the set of embedded intervals in M which are transverse to every leaf that they meet. A lamination with a transverse measure is called a **measured lamination**.

If S is a pleated surface in a manifold M, then S has a natural measure, namely the bending measure. See Fig. 11

For the following, see the exposition of Minsky [43]. I believe it was first conjectured by Thurston, for instance it appears in his 1982 list of problems [59].

2.2 Theorem (Ending lamination theorem). *If M is a hyperbolic 3-manifold with finitely generated fundamental group, then M is determined up to isometry by its homeomorphism class and its end invariants.*

For completeness, we should say what we mean formally by 'end'. Let X be a topological space, and suppose that there is an ascending sequence

$$K_1 \subset K_2 \subset \cdots$$

Figure 12: Ends of a hyperbolic 3-manifold.

of compact subsets such that $\bigcup_{i\in\mathbb{N}} \int X_i = X$. Then an **end** of X is a *descending* sequence of open sets

$$U_1 \supset U_2 \supset \cdots$$

such that each U_i is a connected component of the respective complement $X \setminus K_i$.

We can now say what we mean by 'end invariants'. There are three kinds, shown in Fig. 12.

1. If M is finite volume, then the ends of M are empty, or cusps. In this case, the end invariants are trivial.

2. If M is infinite volume but $\pi_1(M)$ is geometrically finite, then the end invariants are the complex structures on the Riemann surfaces ∂M.

3. If M is infinite volume and $\pi_1(M)$ is geometrically infinite, then the end invariants are certain laminations. What is going on here is that the group was obtained by taking a finite end and a dense lamination on that end, and pinching the lamination down to zero (i.e. the convex core bending measure becomes zero); the finite component of the end vanishes, but the ghostly lamination is left behind.

We will briefly give definitions and pictures for the quasi-sociology of Kleinian groups. We will begin with sociology, though: that is, the representation variety. Our primary reference is Kapovich [29, Chapter 8], for a change.

Let $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of faithful discrete representations of a finitely generated group[1] G into $\mathbb{M} \simeq \mathrm{PSL}(2,\mathbb{C})$. We say that the sequence (ρ_j) **converges algebraically** to some $\rho \in \mathrm{Hom}(G,\mathbb{M})$ if we have $\rho_j(g) \to \rho(g)$ for all $g \in G$, with respect to the standard topology on $\mathbb{M}$. Let $D(G,\mathbb{M})$ be the space of discrete, faithful representations of G; then $D(G,\mathbb{M})$ is closed in the representation variety $R(G,\mathbb{M}) = \mathrm{Hom}(G,\mathbb{M})/\mathbb{M}$ (this is Chuckrow's theorem [29, Theorem 8.4]).

[1]We recall, finitely generated does NOT imply geometrically finite (we shall see some examples, but the first were due to Greenberg [26]), but it DOES imply tameness on the level of 3-manifolds (this is Marden's tameness theorem) and it DOES imply analytic finiteness of the Riemann surface (this is Ahlfors' finiteness theoem).

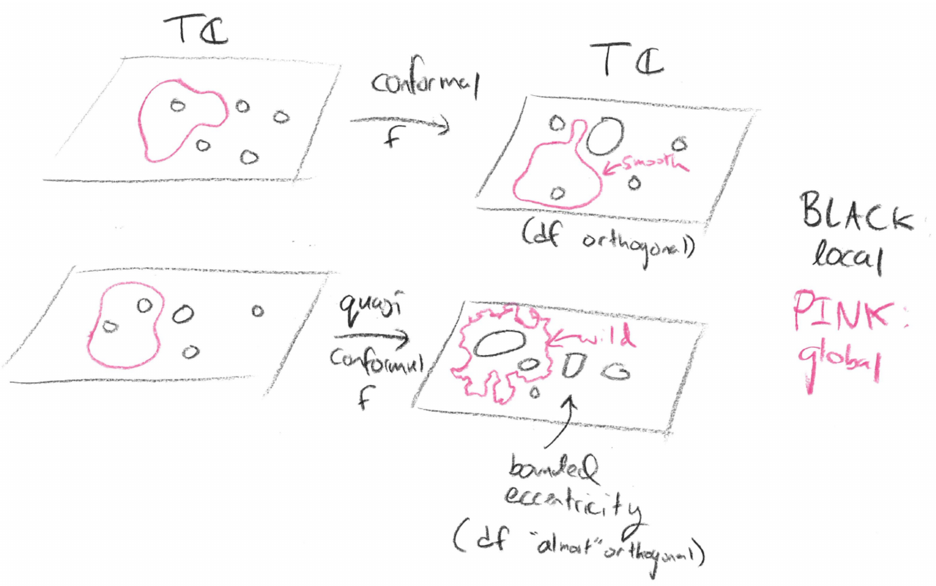


Figure 13: Local and global pictures of a quasiconformal map.

A second notion of convergence does not coincide with convergence of representations. Suppose that $G_j \leq \mathbb{M}$ is a sequence of subgroups ($j \in \mathbb{N}$). The **geometric limit** of the sequence is a group G_∞ such that

1. for each convergent sequence $(g_{j_i}) \leq G_j$, the limit $\lim_{i\to\infty} g_{j_i}$ lies in G_∞
2. for eacg $g \in G_\infty$ there is a sequence $g_j \in G_j$ such that $g_j \to g$.

The point is that geometric convergence of groups corresponds to quasi-isometric convergence of the 3-manifolds. The geometric limit might be strictly larger than the algebraic limit. If the Hausdorff limit of the sequence $\Lambda(G_j)$ is equal to the limit set of the geometric limit, then the algebraic and geometric limits coincide.

Recall before we continue that a quasiconformal map is a homeomorphism $f : U \to \hat{\mathbb{C}}$ such that the local dilation $\mu = (\partial f/\partial \bar{z}) \Big/ (\partial f/\partial z)$ is globally bounded (μ is a bounded complex function on U, called the **Beltrami coefficient**). Locally on $T\mathbb{C}$ a quasiconformal map sends infinitesimal circles to infinitesimal ellipses of bounded eccentricity, but globally on $\mathbb{C}$ it can have fairly wild behaviour—see Fig. 13.

Fix a basepoint $\rho \in D(G, \mathbb{M})$. By the λ-lemma of Mañé, Sad, and Sullivan with extensions by Słodkowski and Earle, Kra, and Krushkal' (for which see [3] and [20]) small holomorphic deformations of the coefficients of ρ to new representations $\tilde{\rho}$ induce quasiconformal maps $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\rho}(G) = \phi\rho(G)\phi^{-1}$. Conversely, if ϕ is a quasiconformal map with Beltrami coefficient μ satisfying $g^*\mu = \mu$ for all $g \in G$ then $\phi\rho\phi^{-1}$ is a discrete faithful representation. We define the **Teichmüller space** of G to be the open subset of $D(G, \mathbb{M})$

$$T(G) = \frac{\{\rho \in \mathrm{Hom}(G, \mathbb{M}) : \exists \rho\text{-equivariant q.c. homeo. } \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}{\mathbb{M}}.$$

If $\Omega(G)/G$ is the disjoint union of the connected Riemann surfaces $\Sigma_1, ..., \Sigma_n$, then $T(G)$ is a quotient space of the space $T(\Sigma_1) \times \cdots T(\Sigma_n)$ by a subgroup $\widehat{\mathrm{Mod}}(G)$ of the mapping class group $\mathrm{Mod}(G)$; if G is geometrically finite, then $\widehat{\mathrm{Mod}}(G)$ is the group generated by Dehn twists along curves bounding compression discs in O_G.

We now do some quick examples.

2.3 Example (The Maskit slice, $\mathcal{M}$)**.** The Teichmüller space of the group G_{3i} from Example 1.6 is naturally identified with (a natural quotient of) the set of $\mu \in \mathbb{C}$ such that

$$G_\mu = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -i\mu & i \\ i & 0 \end{bmatrix} \right\rangle.$$

is discrete and has quotient surface homeomorphic to that of G_{3i}. See [44, 32]

A **Fuchsian group** is a pair (G, Δ) where Δ is a round open disc and G is a Kleinian group which preserves Δ and $\hat{\mathbb{C}} \setminus \Delta$. As an exercise one can show that $\Lambda(G) \subseteq \partial\Delta$. See Fig. 15 for a picture of the geometry. A **quasi-Fuchsian group** is a Kleinian group of the form $\tilde{G} = \phi G \phi^{-1}$ where G is Fuchsian and ϕ is a quasiconformal homeomorphism. In this case $\tilde{G}$ preserves the topological discs $\phi\Delta$ and $\phi(\hat{\mathbb{C}} \setminus \Delta)$ and $\Lambda(\tilde{G})$ is a subset of the topological circle bounding these quasidiscs. We say that $\tilde{G}$ is **of the first kind** if $\Lambda(\tilde{G}) = \partial\phi\Delta$. In this case, $\Omega(\tilde{G})/\tilde{G}$ is a disjoint union of two hyperbolic Riemann surfaces. A **Bers slice** is a subset of the moduli space of $\tilde{G}$ such that the complex structure of one component is held fixed. This gives an embedding of the Teichmüller space of the non-fixed component into the Teichmüller space of the group. A **Maskit slice** is a Bers slice such that the fixed component is a thrice-punctured sphere.

Here is an example for quasi-Fuchsian groups which are of the second kind (i.e. the limit set does not fill the quasicircle).

2.4 Example (The Riley slice, $\mathcal{R}$)**.** The Teichmüller space of the group Γ_{3i} from Example 1.5 is naturally identified with the set of $\rho \in \mathbb{C}$ such that

$$G_\mu = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ 2 & 1 \end{bmatrix} \right\rangle.$$

is discrete and has quotient surface homeomorphic to that of Γ_{3i}. See [33, 20, 19, 22]. See Fig. 16, Fig. 17, and Fig. 18.

Both Example 2.3 and Example 2.4 are slices through the boundary of genus 2 Schottky space; we can even pass naturally from one to the other via blowing up cusps (more precisely, we use the Maskit combination theorems of [38, Chapter VII], see the lecture notes of Series [55] for a very comprehensive and explicit description of the combinatorial group theory).

Let $\mathcal{PML}(\Sigma)$ be the projective space of measured laminations on the surface Σ. Then $\mathcal{PML}(\Sigma) \cup T(\Sigma)$ is the **Thurston compactification** of the Teichmüller space. It descends to the Teichmüller space of a group G since a Dehn twist along a compression disc can be undone by isotopy in the 3-manifold. (The point here is that the twisting is done through

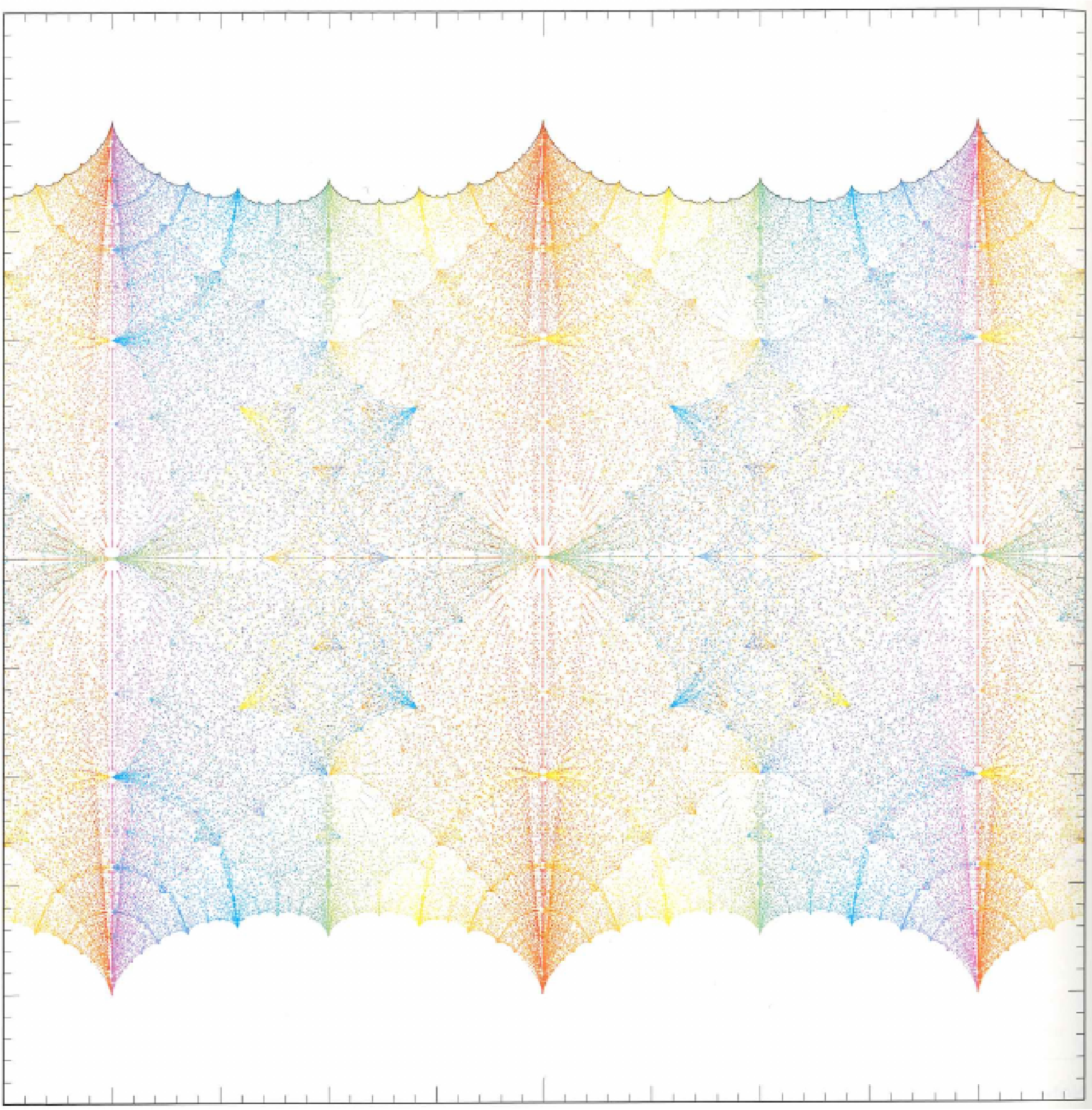


Figure 14: Maskit slice from [44, p. 288].

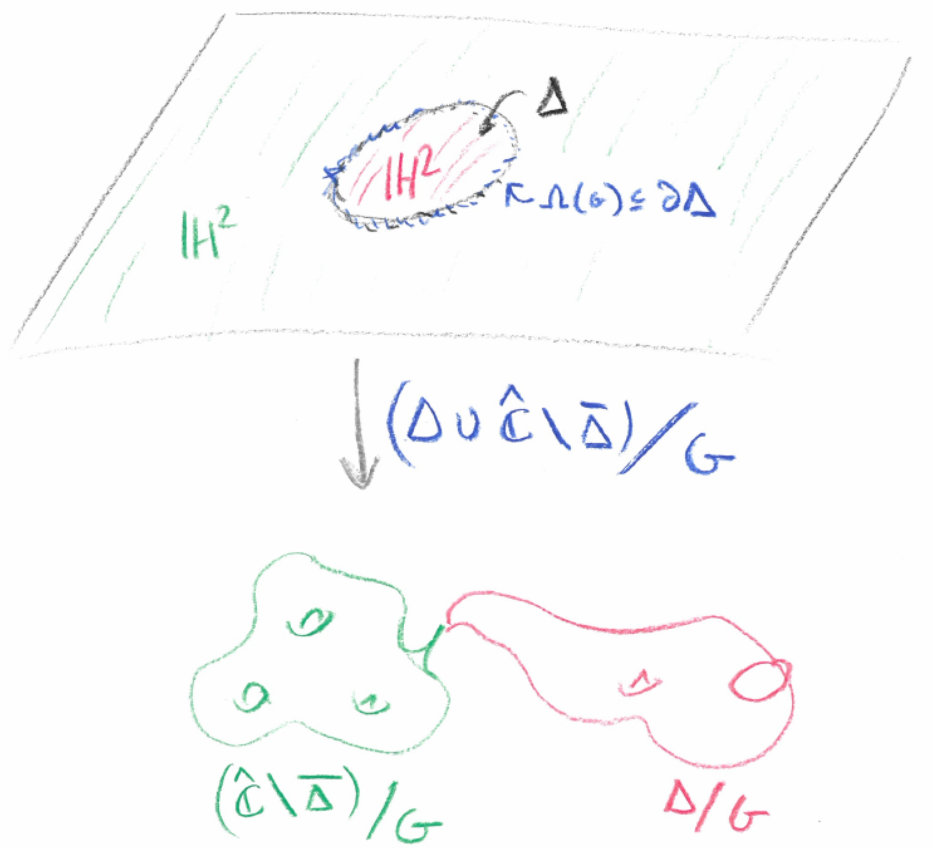


Figure 15: The action of a Fuchsian group produces two Riemann surfaces as quotients of two copies of $\mathbb{H}^2$; if the group has limit set dense in the circle then the two surfaces are disjoint—possibly joined by a shared cusp, as in this picture—or are glued together along deleted discs (this second situation is not pictured here, but easy examples are given by the Riley groups with real parameter).

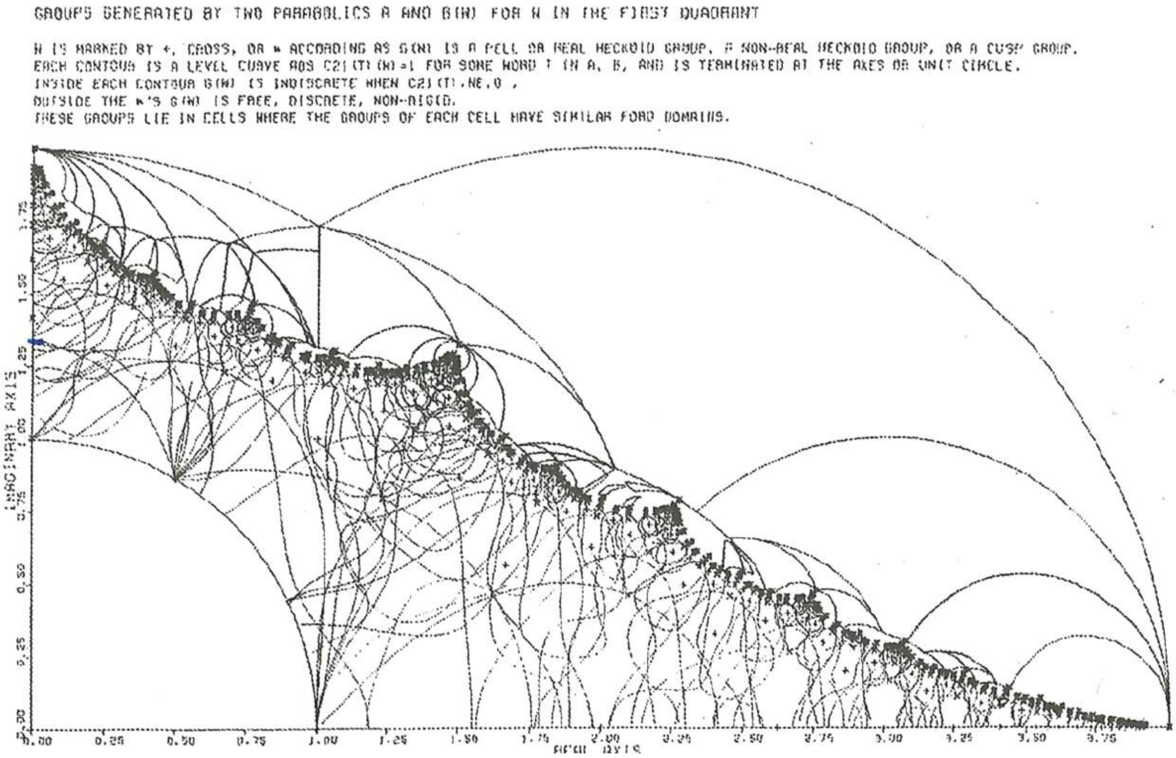


Figure 16: Riley's plot of two-bridge link groups in the $(+,+)$-quadrant of $\mathbb{C}$, taken from [1, Figure 0.2a].

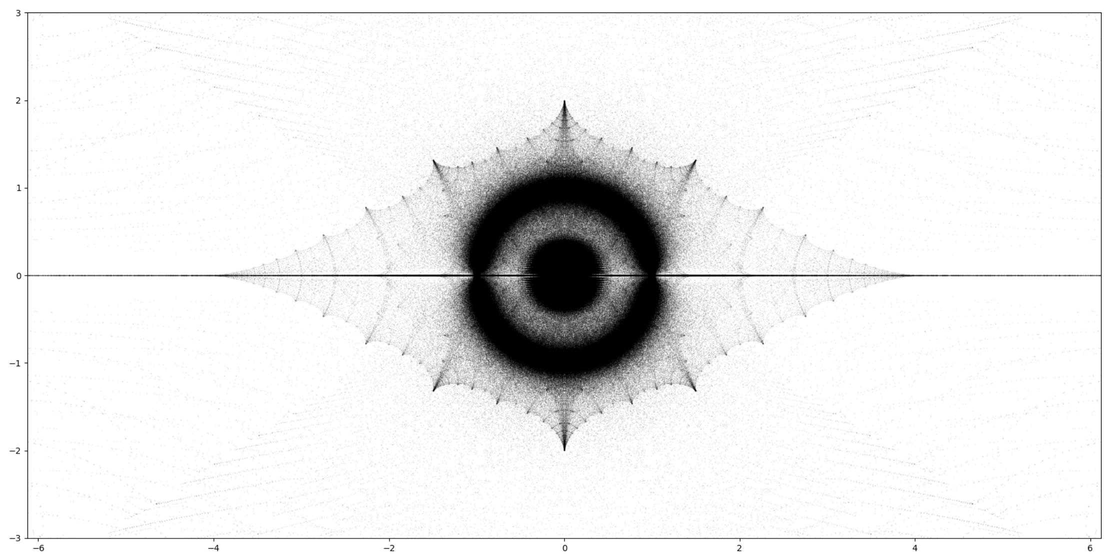


Figure 17: A higher resolution Riley slice.

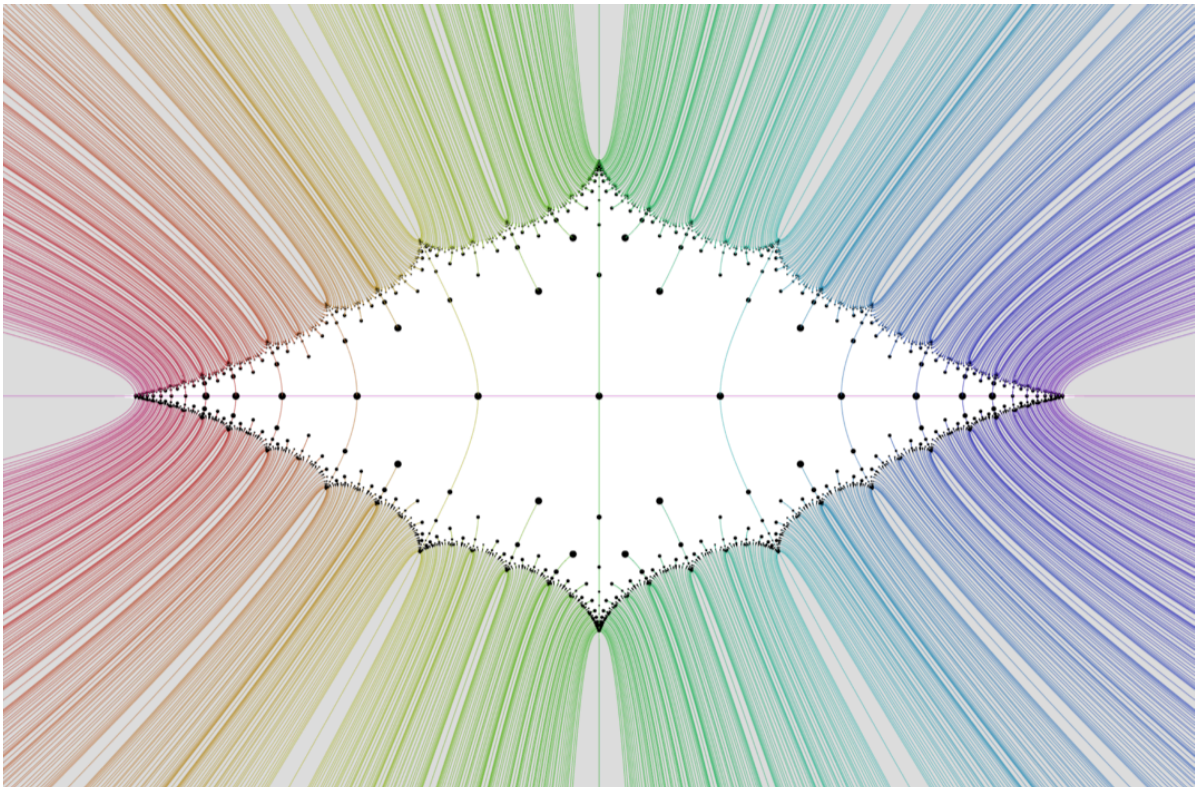


Figure 18: Riley slice with pleating rays, courtesy Y. Yamashita.

the 3-manifold, not intrinsically on the surface, and so it is necessary for there to be an ambient manifold.) Even better, if G is quasi-Fuchsian then there is a natural coordinate system on the Bers slice of G given by measured laminations [51]; and it is natural to conjecture that this construction extends to all geometrically finite groups—for instance the Riley slice theory verifies this for a class of quasi-Fuchsian groups of the second kind. (The general argument should be a straightforward extension of the argument for a single component. We will discuss this in [16].) In any case, the point is to assign to a group G the bending lamination of $\partial\, \mathrm{h.conv}\, \Lambda(G)/G$.

Remark. The main theme of the remainder of this note is the interplay between the different ways of keeping track of the canonical measured lamination.

Giving data of a geodesic lamination is equivalent to giving the data of **train tracks** on the manifold. Train tracks were introduced by Thurston (Fig. 19); the point is that one can conglomerate together parallel geodesics. Let us give a formal definition following Kapovich [29] and Penner [45]. See also the classic notes inspired by Thurston's surface lectures [23].

2.5 Definition. Let S be a hyperbolic surface. A set of **train tracks** on S is given by a trivalent graph G, together with an embedding $G \to S$ such that every edge is C^1 and such that at every vertex there is a distinguished tangent line. We call the edges **branches** and the vertices **junctions**[2] Isomorphism of train tracks is given by C^1 isotopy.[3]

A **transverse measure** on a set of train tracks τ is an assignment of a non-negative real number to each branch of τ, such that if v is a vertex then the 'incoming' measure is the sum of the two 'outgoing' measures. The support of the measured train track is the subgraph of branches with positive measure.

A train track is produced by 'compressing together' parallel leaves of a lamination, as in Fig. 20, while keeping track of the measure additively.

§3. B-groups and other degeneracies

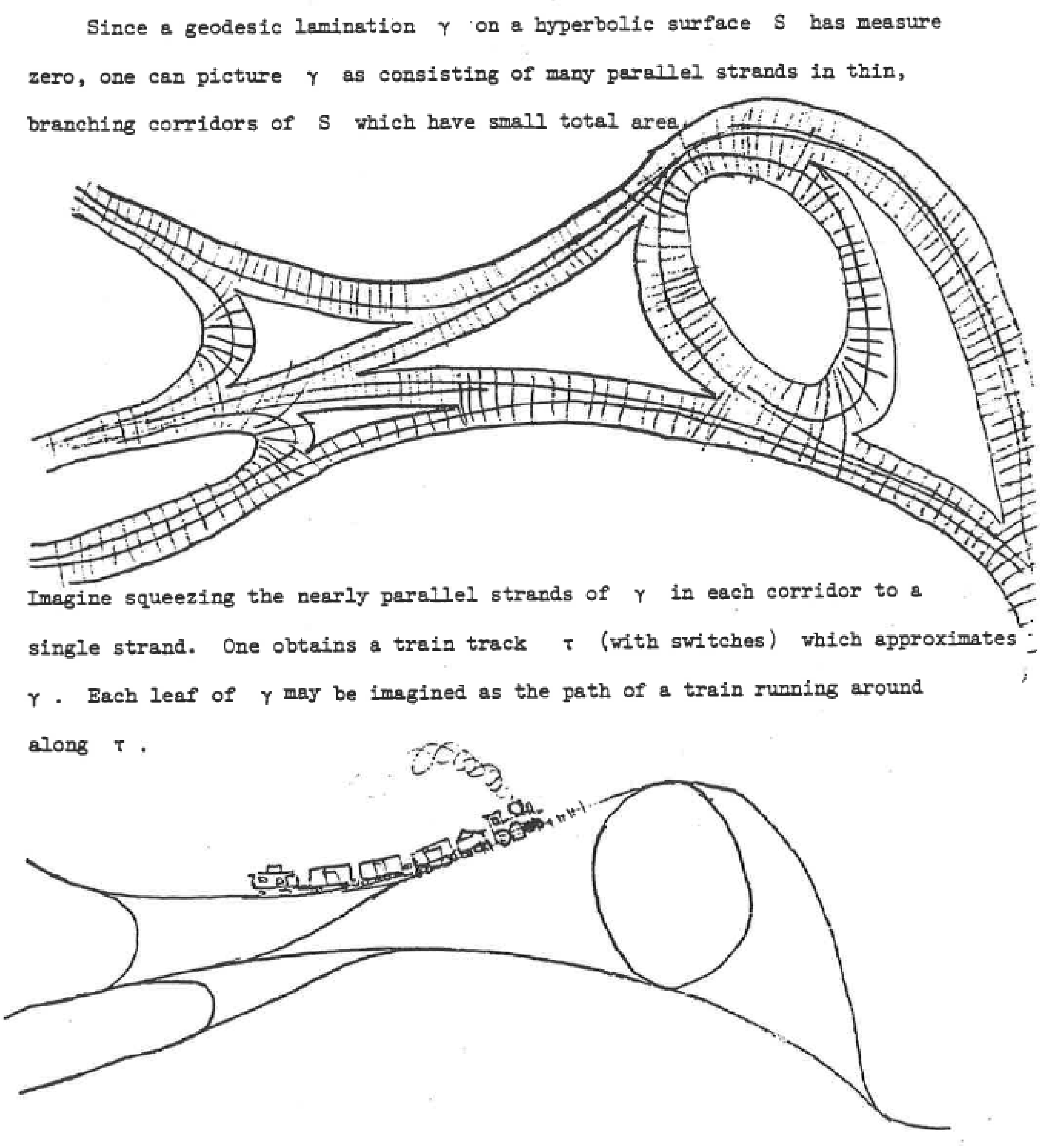
We will talk now about the kinds of groups which lie on the boundary. The original work on this is due to Bers [7] and Maskit [39]. A nice but slightly outdated survey was written by Canary [14].

Fix a geometrically finite Fuchsian group G, and let $\Gamma \in \partial T(G)$ (where the boundary is taken with respect to the algebraic topology). Automatically, Γ is group-isomorphic to G. There are several cases; let $\rho : G \to \Gamma \leq \mathbb{M}$ be a corresponding representation in $D(G, \mathbb{M})$.

1. If there exists some $\eta \in G$ such that η is loxodromic but $\rho\eta$ is parabolic, then Γ is called a **cusp group**. It is a famous result of McMullen [42] that such groups

[2]We now come to a terminological issue. In English, the apparatus by which a train is deflected from one track to another is called a *set of points*, while the point at which this deflection occurs is called a *junction*; in America, the apparatus is called a **switch** (and a junction is still a junction). Thus, in translating Thurston's work to English, one is tempted to replace 'switches' with 'points'; unfortunately, using the word 'point' would be ambiguous since we are doing geometry, and so we must settle for the slightly less correct translation of 'junction'.

[3]Actually slightly weaker isomorphism is needed for the equivalence with laminations, but this is not important.



Since a geodesic lamination γ on a hyperbolic surface S has measure zero, one can picture γ as consisting of many parallel strands in thin, branching corridors of S which have small total area.

Imagine squeezing the nearly parallel strands of γ in each corridor to a single strand. One obtains a train track τ (with switches) which approximates γ. Each leaf of γ may be imagined as the path of a train running around along τ.

Figure 19: Reproduction of page 8.51 of Thurston's notes [58].

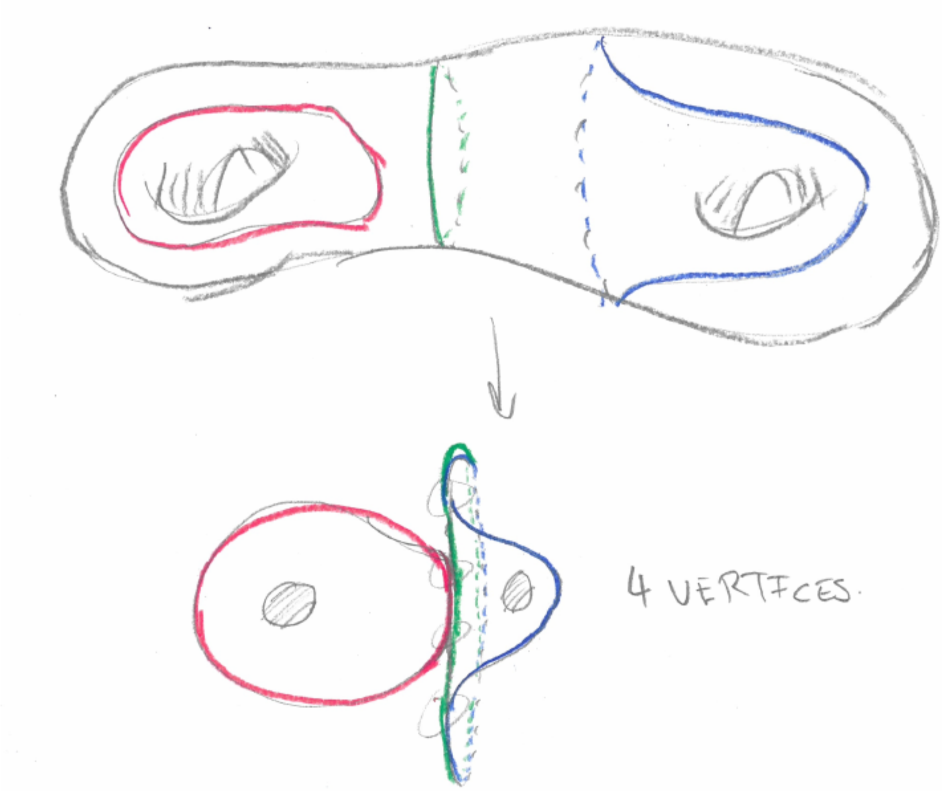


Figure 20: Compressing a (maximal) geodesic lamination to a train track in genus two.

are dense in the boundary. In this case, Γ is geometrically finite and $\Lambda(\Gamma)$ is a nil-measure set. If every component of $\Omega(\Gamma)/\Gamma$ is a thrice-punctured sphere, then Γ is a **maximal cusp**. Such groups are dense in the boundary of every Schottky space (again by McMullen). We have already seen that complex structure is determined by the limit set; cusp groups have circle-packing limit sets, and are geometrically finite [31].

2. If Γ has a simply connected invariant component, then it is called a **B-group**. These were introduced by Bers [7], and by his results every group on the boundary of a quasi-Fuchsian space is a B-group. A B-group is called **degenerate** if it has exactly one component. These exist. In fact, most boundary groups of quasi-Fuchsian spaces of the first kind are degenerate [7, Theorem 14]. One can think of degenerate groups in this case being 'irrational', and cusp groups being 'rational'. All degenerate groups are geometrically infinite, by the previously cited result of Greenberg [26]. It is an open conjecture of Bers that every degenerate group does in fact arise on the boundary of some quasi-Fuchsian space.

3. On the boundary of quasi-Fuchsian spaces of the second kind, the equivalent of a degenerate group is a group with dense limit set in $\hat{\mathbb{C}}$ (rather than dense in a disc). These may be geometrically finite in special cases, but I believe in general they can also be geometrically infinite. (Example?)

The point, compared with the previous section, is the following picture.

3.1 Conjecture. *Let G be a geometrically finite Kleinian group which supports quasiconformal deformations, and let $Q = \mathrm{QH}(G)$ be its deformation space. Then:*

1. *There exists a geometric coordinate system on Q, namely $Q = \prod_{s=1}^{n} \mathcal{ML}(S_i)$ where $S_1, \dots, S_n$ are the components of $\Omega(G)/G$ (c.f. Fig. 22).[4]*

2. *The boundary ∂Q admits a natural partition into cusp groups and non-cusp groups, where the non-cusp groups have limit sets of positive measure. Cusp groups are dense in ∂Q, but most groups (in the sense of category) on the boundary are non-cusp groups. Cusp groups correspond to choosing a lamination on the surface which is not dense and pinching a corresponding loxodromic element (or system of loxodromic elements) to parabolics, while non-cusp groups correspond to picking a dense lamination on a component of the surface and shrinking down the corresponding piece of the convex core to zero volume (so the component vanishes). See Fig. 21 for an explicit picture of this on a higher dimension Teichmüller space.*

3. *Topologically, Q is homeomorphic to a sphere with deleted balls. (Can we guess a more precise result?)*

[4] This is not quite true, and we actually need to take a quotient here—for instance, in the Riley slice, we have to quotient out by the geodesic γ_∞. But it is true enough.

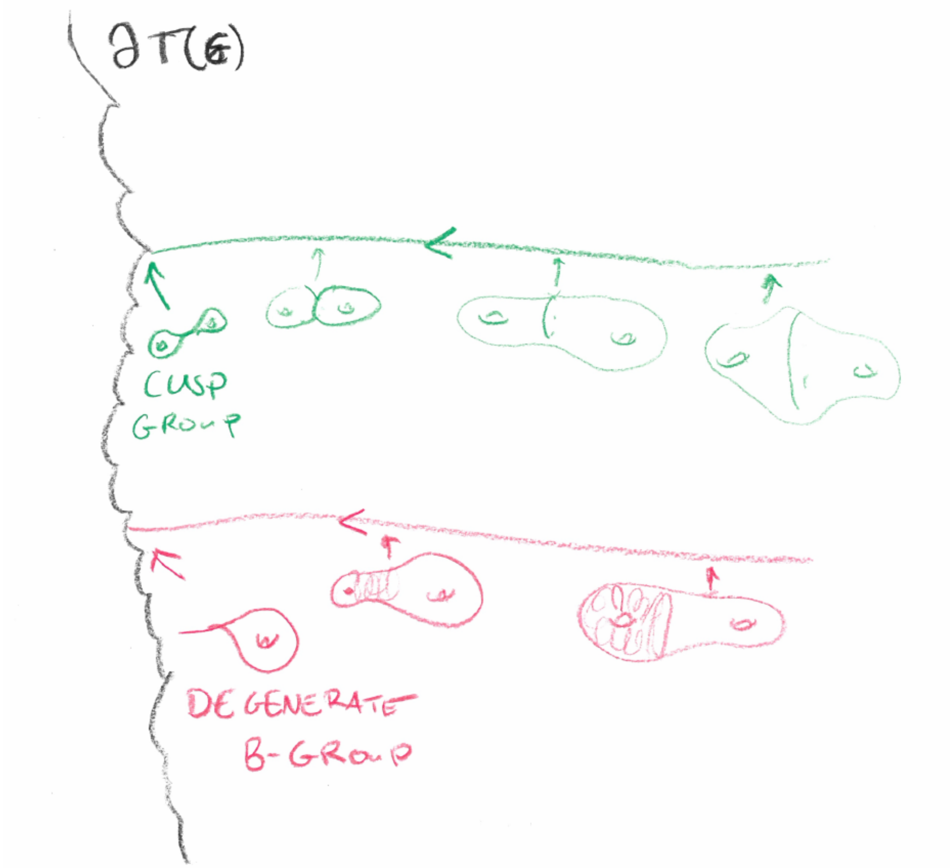


Figure 21: Here we see two deformations to the boundary of the Teichmüller space of groups uniformising genus two compact surfaces. The green degeneration is a pinching along a lamination with measure zero, which produces a new pair of cusps and no virtual ends, while the red end pinches down a geodesic with nonzero area and produces an unpaired cusp and the ghost of a virtual end.

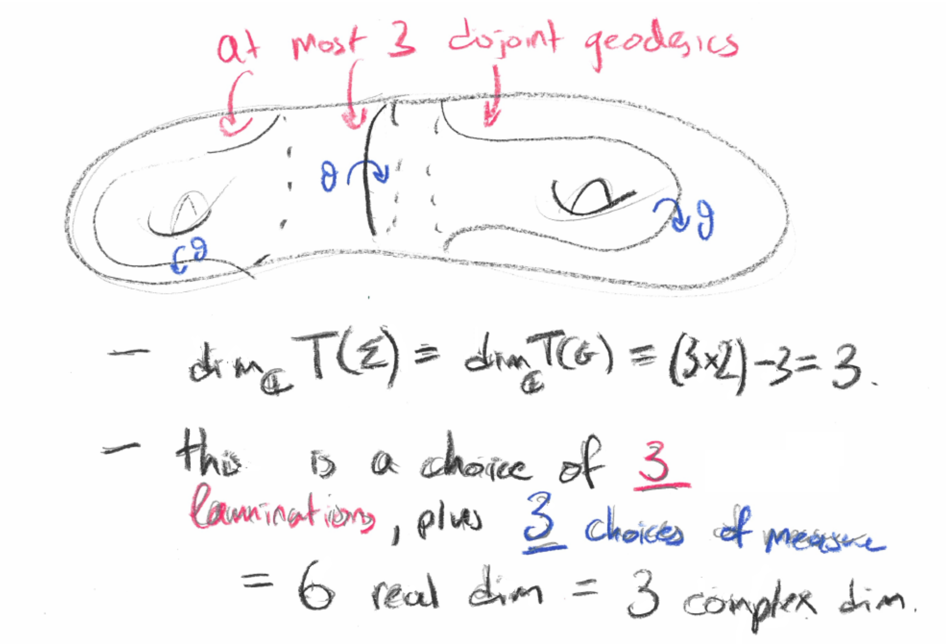


Figure 22: We see that the dimension of $T(G)$ comes from a choice of a maximal system of geodesics and a measure on each.

We can give some rough pictures based on our extensions to the theory of Keen and Series which appear in [21]: see Fig. 23. This gives a geometrically finite (by results of Maskit and Swarup [40]—though this is not typical) boundary group where the Riemann surface vanishes. (The corresponding 3-manifold will be related to some wild knot, which is not usually treated in textbooks; we recommend [30][5] and [50] which do deal with wild objects.)

Let us restrict ourselves to cusp groups; in fact, maximally cusped groups for the time being. If Γ is geometrically finite, then its limit points are of three kinds (Fig. 24).

1. rank 1 cusps (fixed points of cyclic maximal parabolic subgroups);

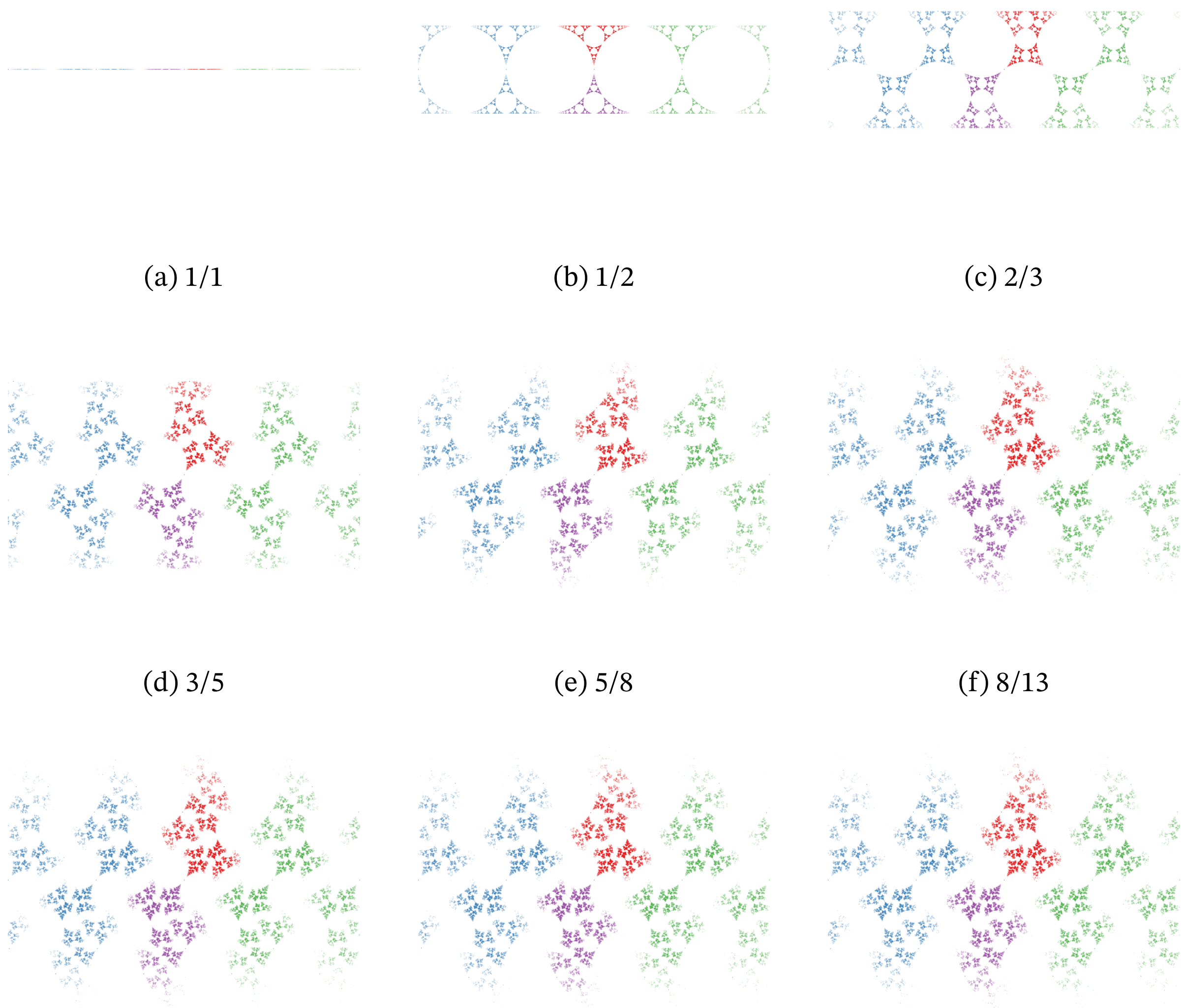
2. rank 2 cusps (fixed points of rank 2 maximal parabolic subgroups); and

3. points of approximation (accumulation points of isometric circles).

In addition, all rank 1 cusps are doubly cusped. The corresponding cusp structures in the 3-manifold are Margulis cusps (rank 1) and Margulis tubes (rank 2).

The proof of the following is just by looking at the picture.

3.2 Lemma. *Consider a Riemann surface consisting of a disjoint union of an even number of thrice-punctured spheres, $R = R_1 \cup \cdots \cup R_k$, and suppose that we additionally specify a matching structure on the punctures (i.e. for every puncture p let p' be a puncture distinct from p, such that $(p')' = p$). Then there exists a 3-manifold realising this matching structure at infinity.*



(a) 1/1 (b) 1/2 (c) 2/3

(d) 3/5 (e) 5/8 (f) 8/13

(g) 13/21 (h) 21/34 (i) 34/55

Figure 23: Approximations to the $1/\phi = 2/(1+\sqrt{5})$ boundary group along cusp groups. Observe the apparent breakdown of technology! For computational details see [21, 15].

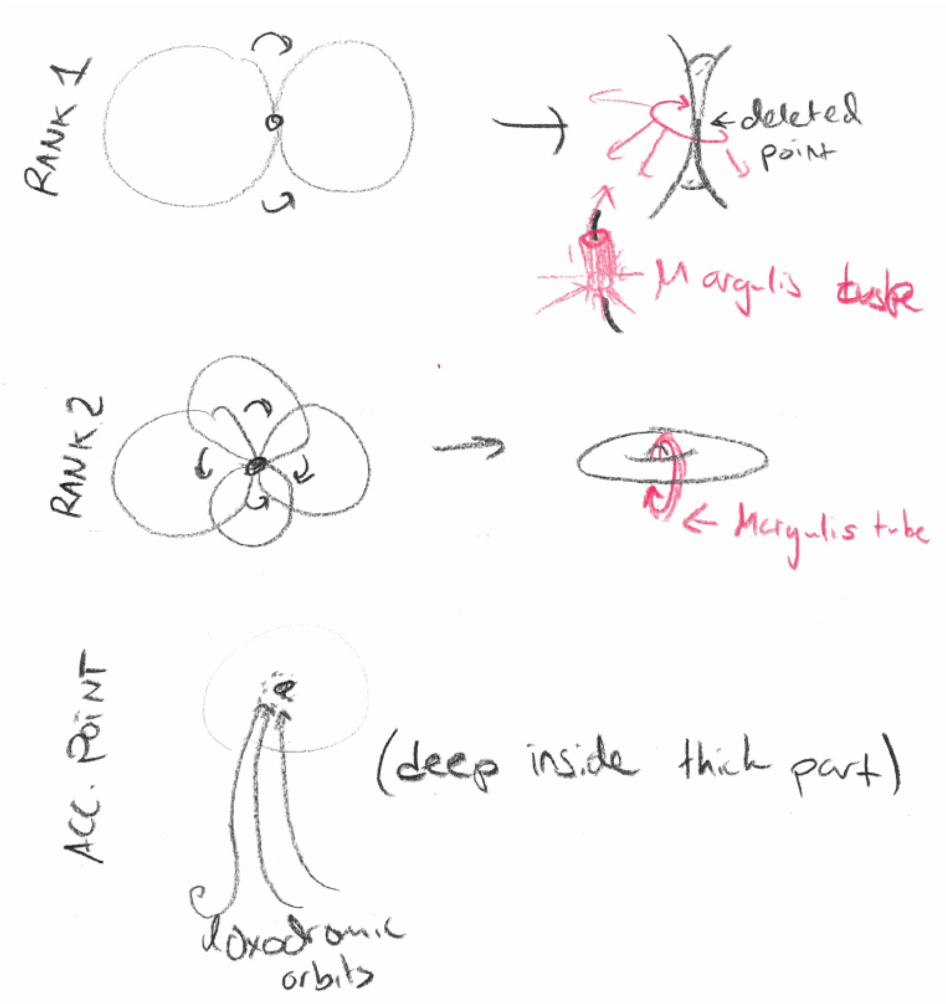


Figure 24: The three kinds of limit point.

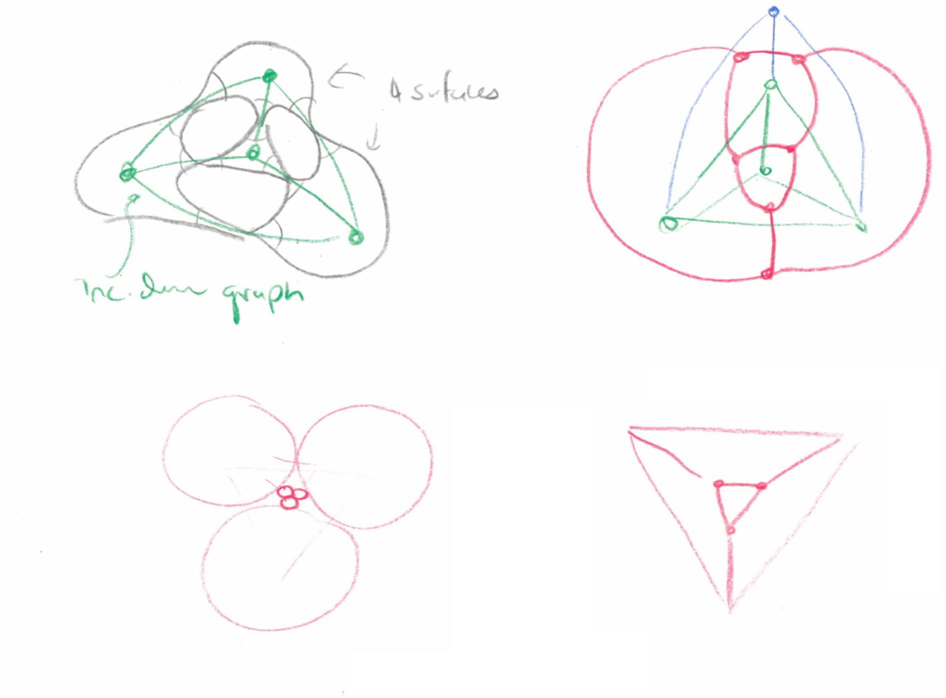


Figure 25: The construction of a cusp group realising a graph curve. Clockwise from top left: (i) the incidence data for the curve, and the surface as a union of thrice-punctured spheres; (ii) the construction of the doubled dual $\hat{G}$; (iii) the tangency data for circles; (iv) a sufficiently symmetric choice of tangent circles realising this incidence.

Note, giving a matching structure in this way is equivalent to giving a trivalent graph, where *we allow loops*. (Note, $2|e| = 3|v|$ so $|v|$ is even as expected.) If G is a trivalent graph, then we can construct explicitly a Schottky-type group which realises the arrangement G. We will deal with the planar case in this section. (Note, $K_{3,3}$ is the simplest example of a non-planar trivalent graph, it is a very instructive example to see what kinds of problems arise.) We define a 'doubled dual' graph $\hat{G}$: glue two copies of G to each other along the bounding edges, and take the dual graph of the result, keeping track of the quotient $q : \hat{G} \to G$. Now construct an arrangement of tangent circles with the following properties.

1. the tangency relation of the circles is given by $\hat{G}$;

2. if $q(C) = q(C')$, $q(D) = q(D')$, and $C \sim C'$ (hence $D \sim D'$) then the line segment l between the centres of C and C' is equal in length to the line segment m between the centres of D and D', and there are circles K through C and D and K' through C' and D' such that both l and m are orthogonal to K and K'.

3.3 Lemma. *If $\hat{G}$ is constructed from a planar trivalent graph G as described, then there exists an arrangement of tangent circles satisfying (1) and (2).*

A **graph curve** is a connected projective algebraic curve which is a union of projective lines, each meeting exactly three others and with all intersections transverse [4]. There is a bijection between graph curves and trivalent graphs without loops. Topologically, a graph curve is a union of 2-spheres, such that each sphere is tangent to exactly three others (c.f. [11, pp. 142–144].) There is a natural construction called **plumbing** [35] (see also [37, Exercise 4.28]) which allows us to replace paired cusps with algebraic singularities on the level of abstract surfaces without losing any information.

Putting all this together, we have a construction which allows us to realise any planar trivalent graph as a maximally cusped Riemann surface via a Kleinian group, and since thrice-punctured spheres have trivial Teichmüller space the complex structure is exactly that of the corresponding graph curve. We can now ask two natural questions:

1. What about non-planar arrangements?

2. Can we classify *all* of the maximally cusped groups which realise a particular graph curve?

§4. Braids, links, and mapping class groups: living in a post-Birman world

We have already seen the mapping class group arise in the study of moduli, but we only mentioned it as an aside. In this section, we will use it more extensively. If S is a Riemann surface, then the **mapping class group** or **modular group** of S is the group[6]

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

where $\mathrm{Homeo}^+(S, \partial S)$ is the group of orientation-preserving self-homeomorphisms of S which preserve ∂S pointwise.

[5] This book is worth a look at, just for the rather whimsical pictures.

[6] Alternatively one sees $\mathrm{MCG}(S)$ or, in an instance of terrible compromise, $\mathrm{Mcg}(S)$...

4.1 Theorem. *The mapping class group is generated by Dehn twists.*

Suppose that S is genus g with n punctures, and $\overline{S}$ is the canonical compactification. One of the most important results for us is the **Birman exact sequence** [8],

$$1 \longrightarrow K \hookrightarrow \pi_1(C(S,n)) \xrightarrow{\mathcal{Push}} \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(\overline{S}) \to 0.$$

We can even write down K explicitly in most cases; for instance in the case of the four-punctured sphere, we explain the whole thing in the expository section of [21]. This allows us to give combinatorial paths through the moduli space: we walk from one projective lamination fibre to another by mapping class actions, and then the only thing to do is choose the measure (which, as we have seen, is just a choice of arbitrary real numbers on the train track). Again in special cases this is worked out very explicitly [9, 54, 32, 33, 21]—in addition I highly recommend the exposition of Series in [56, 57]—and I have a paper in preparation for some more complicated cases [17].

§4.1. Example: Riley groups, again; adapted from [21]

Let S be a 2-sphere with four marked points, two (resp. X and x) labelled with an integer a and the other two (resp. Y and y) labelled with an integer b (here, $0 < a, b \leq \infty$). We view S as the Riemann surface at infinity of some hyperbolic 3-orbifold O homeomorphic to an open 3-ball and with two singular arcs, one of order a joining X to x and one of order b joining Y to y. That is, there are a pair of homotopically distinct and nontrivial loops in the 3-orbifold—represented by elliptic (or parabolic) elements γ_1, γ_2 of respective orders a and b of the holonomy group of O (for the definition see e.g. [60, §3.4])—which each bound singular arcs in O of respective orders a and b whose four endpoints are the marked points of S. The remainder of this section will describe some models for the moduli space of hyperbolic metrics which are induced on O by different arrangements of the arcs in 3-space, and the moduli space of complex structures which are induced on S when it is viewed as the horizon of O.

Given our surface S with four marked points, we write $\mathrm{Homeo}^+(S)$ for the group of orientation-preserving homeomorphisms of S which preserves both the set $\mathrm{Sing}(S) = \{X, x, Y, y\}$ and its complement and which acts on this 4-set in such a way as to preserve the marking integers—that is, a homeomorphism f is an element of $\mathrm{Homeo}^+(S)$ only if the integral label of z matches that of $f(z)$ for all $z \in \mathrm{Sing}(S)$. (We allow the two integers a and b to be equal, in which case every orientation-preserving homeomorphism which preserves $\mathrm{Sing}(S)$ is allowed even if it permutes the X's with the Y's). The (marked) **mapping class group** of S is the group $\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\sim$, where $f \sim g$ whenever f and g are isotopic via an isotopy which also preserves $\mathrm{Sing}(S)$ and its complement while respecting integral labels [24, §2.1].

The labelling structure may be precisely modelled in the following way: let $C^{\mathrm{ord}}(S,2,2)$ be the set of 4 distinct ordered points on the sphere S (that is, $C^{\mathrm{ord}}(S,2,2)$ is the set $S^{\times 2} \times S^{\times 2} \setminus \mathrm{BigDiag}(S^{\times 4})$ where $\mathrm{BigDiag}(S^{\times 4})$ is the 'big diagonal' of 4-tuples of points where at least two of the points are repeated); there is a natural action of a subgroup $\mathcal{S}$ of $\mathrm{Syn}(4)$ on $C^{\mathrm{ord}}(S,2,2)$ which depends on whether $a = b$: if $a \neq b$ then $\mathcal{S}$ is the Klein 4-group

$\mathrm{Syn}(2)\times\mathrm{Syn}(2)$ permuting the coordinates of each factor separately, and if $a = b$ then $\mathcal{S}$ is the whole $\mathrm{Syn}(4)$ and is allowed to permute all four points. The **pairwise configuration space** of four points on S is the quotient

$$C(S,2,2) = C^{\mathrm{ord}}(S,2,2)/\mathcal{S}.$$

The geometric interpretation is supposed to be clear: the first $S^{\times 2}$ component keeps track of the order a points and the second component keeps track of the order b points.

The homotopy group $\pi_1(C(S,2,2))$ (that is, the possible paths up to homotopy traced out by the four points without colliding, where paths must connect points with the same label) is called the **spherical braid group on 4 strands** [24, §9.1].

The main point which forms the basis of the Birman–Keen–Series theory of the mapping class groups of four-times marked spheres is the following. Let S be a sphere with four marked points which bounds a 3-orbifold O in such a way that the marked points are identified in pairs, as above; assume that $a \neq b$, so the pairs are distinguishable from each other. Then the Birman exact sequence becomes

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \hookrightarrow \pi_1(C(S,2,2)) \xrightarrow{\mathcal{Push}} \mathrm{Mod}(S) \longrightarrow 1$$

where

$$\mathrm{Mod}(S) = \mathrm{PSL}(2,\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{PSL}(2,\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}) = \mathrm{Mod}(S_{0,4}).$$

Here, we use $S_{0,4}$ to denote the sphere with four indistinguishable marked points; the map $\mathrm{Mod}(S) \to \mathrm{Mod}(S_{0,4})$ is the evident inclusion map (every mapping class which respects the marking structure in pairs is also a mapping class when that pair-structure is forgotten), the map $\mathcal{Push}$ is the map which sends a braid $\beta : [0,1] \to C(S,2,2)$ to the induced homeomorphism $\beta(0) \to \beta(1)$, and where $\ker \mathcal{Push} \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by a homeomorphism Θ which corresponds to a 2π rotation of the four marked points (equivalently, a single 2π twist added to the end of the braid); the image of this in the mapping class group is trivial (the twist can be undone by rotating the 'back' of the sphere via an isotopy without moving the points) and it is an involution in the braid group by the belt trick Fig. 26.

Let α_1 and α_2 be two disjoint paths on the sphere which join the points in pairs, preserving the integral labelling. If α_1 and α_2 are pushed slightly into the interior of the sphere without passing through each other, the resulting arrangement is called a **rational tangle**. Every two-bridge knot comes from taking such a rational tangle and pairing the four marked points on the sphere with two disjoint paths outside the sphere (which may not preserve the integer labels) which contract onto the sphere via an isotopy such that the images do not cross (this gluing is the so-called **numerator closure**).

There is a natural way to enumerate the rational tangles, essentially due to Schubert [52] and described in [13, §12.B] or [46, Chapter 10]. First, take a sequence $a_0, \dots, a_m$ of integers. Every rational tangle is obtained by laying out four parallel strands (two labelled with the integer a and two with the integer b) and then alternatingly braiding the two leftmost strands and then the two middle strands with plaits of $a_0, a_1, \dots$ crossings (where

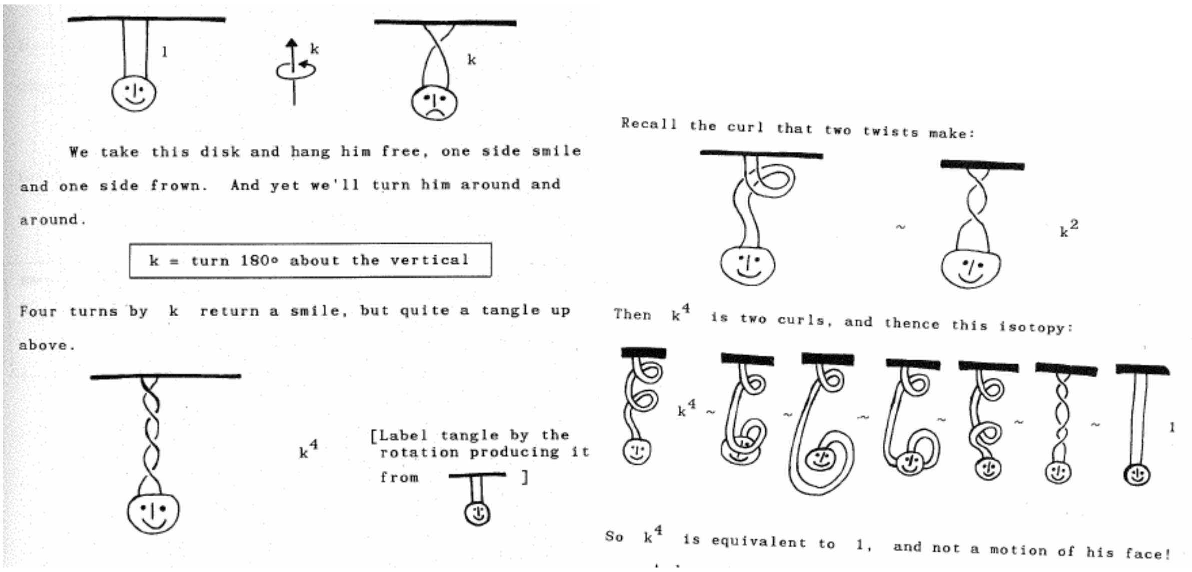


Figure 26: The belt trick [30, §VI.1].

the sign of each a_i denotes the direction of twisting to produce each braiding cluster); this produces a plait made up of four strands, which are joined like-labelling-to-like at one end to form two plaited cords, one labelled with a and one with b; the numerator closure is then the capping of the four remaining ends.

We now recall a standard fact from classical number theory. If $(a_0, a_1, \ldots, a_k)$ is a finite sequence of integers, we define the **simple continued fraction**

$$[a_0; a_1, \ldots, a_k] \coloneqq a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}}}.$$

Every rational number can be expressed as a finite simple continued fraction in exactly two ways, one with an even and one with an odd number of convergents (number of sequence elements a_n) [27, Theorem 162]. These can be computed efficiently [27, §10.9]. This relationship gives a bijection between the space of 2-bridge links and the set of rational numbers; the rational number associated to a given 2-bridge link is called the **rational form** or **Schubert normal form** for the link.

By the theory above, for every 2-bridge knot we obtain an element of the braid group $\pi_1(C(S, 2, 2))$, and hence an element of the mapping class group. (In fact, for each tangle we obtain an element of the braid group, but the image of a tangle under Θ gives the same knot up to isotopy, and so we only really get a well-defined element of the mapping class group.) Note that the involutions ι_1 and ι_2 both preserve the knot structure (whether the two strands/four marked points are indistinguishable or not), and so we in fact have an injection $\phi : \{\text{2-bridge knots}\} \to \mathrm{PSL}(2, \mathbb{Z})$. Observe next that the twisting sequence of a 2-bridge link k is in fact coding a sequence of Dehn twists needed to twist the unbraid into the rational tangle whose closure is k: if γ_0 and γ_∞ are the two curves marked in Fig. 27

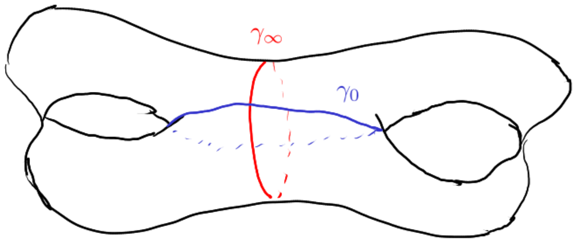


Figure 27: The first homology group of the 4-punctured sphere is isomorphic to $\mathbb{Z}^2$; one possible basis is formed by the two cycles γ_0 and γ_∞ depicted.

(which together form a basis for $H_1(S, \mathbb{Z}) \simeq \mathbb{Z}^2$) and τ_0 and τ_∞ are the respective Dehn twists, then a rational tangle whose closure is k is represented by the element

$$\tau_0^{a_0} \tau_\infty^{a_1} \tau_0^{a_2} \tau_\infty^{a_3} \cdots$$

of the mapping class group. The action of the $\mathrm{PSL}(2, \mathbb{Z})$ semidirect multiplicand of $\mathrm{Mod}(S)$ on S can be identified with the usual matrix group action of $\mathrm{PSL}(2, \mathbb{Z})$ on $H_1(S, \mathbb{Z})$ after we choose this basis; take $\gamma_0 = (0, 1)^t$ and $\gamma_\infty = (1, 0)^t$, and write the standard generating set

$$\mathrm{PSL}(2, \mathbb{Z}) = \left\langle R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \leq \mathrm{PSL}(2, \mathbb{C})$$

(c.f. the case of the punctured torus in [56]). If A is an element of $\mathrm{PSL}(2, \mathbb{Z})$, say

$$A = \begin{bmatrix} p & r \\ q & s \end{bmatrix},$$

then A sends $\gamma_\infty \mapsto p\gamma_\infty + q\gamma_0$ and $\gamma_0 \mapsto r\gamma_\infty + s\gamma_0$. Observe that this gives a map from $\mathrm{PSL}(2, \mathbb{Z})$ to the space of ordered singular $\mathbb{Z}$-homology bases of S (where the ordering is given by $p\gamma_\infty + q\gamma_0 \leq r\gamma_\infty + s\gamma_0$ if $p/q \leq r/s$).

Now let $h : \pi_1(S) \to H_1(S)$ be the usual Abelianisation projection. Define the **geometric intersection** of a pair of homology classes $\alpha, \beta \in H_1(S)$ in the usual way, namely $i(\alpha, \beta)$ is the infimum, over all of the choices of σ and γ in the free homotopy classes of all curves in $h^{-1}(\alpha)$ and $h^{-1}(\beta)$ respectively, of $|\sigma \cap \gamma|$.

The following result is standard:

4.2 Proposition. *Suppose $\alpha = p\gamma_\infty + q\gamma_0$ and $\beta = r\gamma_\infty + s\gamma_0$ are arbitrary homology classes.*

1. $i(\alpha, \beta) = \left| \det \begin{bmatrix} p & r \\ q & s \end{bmatrix} \right|$.

2. *If* $\gcd(p, q) \neq 1$, *then* $i(\alpha, \alpha) > 0$ *(i.e. there is no simple closed curve on S in the homology class of α).*

3. *If* $\gcd(p, q) = 1$, *then* $i(\alpha, \alpha) = 0$*; and further, there is exactly one non-freely-homotopic* geodesic *(with respect to any chosen hyperbolic metric on S) simple closed curve on S*

which projects to α under h. In this case, both α and $-\alpha$ correspond to the two orientations on this geodesic, and so the geodesic can be identified by the rational number $p/q = (-p)/(-q) \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$; we write $\gamma_{p/q}$ for this geodesic.

Now it is an easy exercise to check that

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle R, L \rangle \rtimes \langle Q \rangle \tag{4.3}$$

where L is the matrix

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(the key observation is that Q conjugates L to R^{-1}). Let $\Gamma_1 := \langle L, R \rangle$ be the orientation-preserving (with respect to the action on the upper half-plane) part of $\mathrm{PSL}(2, \mathbb{Z})$; then Γ_1 is in bijection with the space of unordered bases of $H_1(S, \mathbb{Z})$, that is pairs of homotopically distinct and homologically nontrivial curves on S. By direct computation we see now that the action of Γ_1 as a subset of the mapping class group is

$$\begin{aligned} R.\gamma_\infty &= \gamma_\infty, & R.\gamma_0 &= \gamma_{1/1} \\ L.\gamma_\infty &= \gamma_{1/1}, & L.\gamma_0 &= \gamma_0; \end{aligned}$$

that is, R acts as τ_∞ and L acts as τ_0. In particular, we have a bijection between the space of two-bridge knots and the group Γ_1 such that the knot with rational form $p/q = [a_0; a_1, \ldots, a_N]$ is represented by the matrix

$$L^{a_0} R^{a_1} L^{a_2} \cdots.$$

§4.2. Changing the other manifold invariant: the 3-topology

Mapping classes are the objects which move from one lamination (i.e. complex structure at infinity) to another, without changing the 3-manifold homeomorphism type. In order to change the 3-manifold homeomorphism type but not the conformal structure at the visual boundary, we deal with Dehn surgeries; instead of cutting out an annulus and then regluing with a twist, we drill and fill tori.[7] Recall that simple closed curves (i.e. results of taking a meridian and twisting) on a torus are parameterised by $\hat{\mathbb{Q}}$; hence Dehn surgeries are parameterised by a choice of loop and a rational number. See the book by the two Italians [6] and the book by Purcell [46]. Just like every complex structure can be obtained by mapping class actions, we can get every 3-manifold structure WITHOUT BOUNDARY(!) via (non-intersecting) Dehn surgeries:

4.4 Theorem (Lickorish-Wallace). *Every closed, orientable, connected 3-manifold is obtained by performing Dehn surgery on a link in the 3-sphere.*

We guess that in the case of nontrivial but finite boundary then there is a similar result. (Is this known?)

Using all of this heavy machinery, we should be able to deal with the non-planar case.

4.5 Conjecture. *Consider a Riemann surface consisting of a disjoint union of an even number of thrice-punctured spheres, $R = R_1 \cup \cdots \cup R_k$, with a matching structure on the punctures as above. Then there exists an enumeration of the 3-manifolds realising this matching structure at infinity.*

[7]The terminology is explained by the observation that Thurston's parents were dentists. (Reference??)

§5. Combinatorialisations of Keen–Series theory

As well as studying laminations, one can do very combinatorial analysis of circle packings in the limit sets. This follows on from the work of Jørgensen [28], and the point is that one can work directly with chains and complexes of generators in a single group; for instance this point of view has been taken quite far by the Japanese school of knot theorists in order to give an exhaustive study of the geometry of the Riley slice, which includes the study of the knot groups which are at the kernel of the moduli space [1, 2].

§6. Questions and Problems

The guiding objects in the study of $\partial\,\mathrm{QH}(G)$ are, following Thurston, Bers, Marden, Maskit, Ahlfors, Kapovich, etc.:

- Laminations on surfaces and their maximal degenerations (these are tropical objects)
- Circle packings on the 2-sphere (these have dual graphs, which are tropical objects)

Question. Can the tropical point of view be made explicit, even though the natural completion of these spaces is larger than the corresponding algebraic completion?

1. What algebraic structure can be placed onto an algebraic curve in order to lift a complex structure to the structure of a visual boundary of a 3-manifold? (Answer: a quadratic differential...?)
2. What is the algebraic analogue of the procedure 'measured lamination → train track'?
3. Give an explicit computable example of a degenerate B-group, not in terms of a limiting process. (This then gives an explicit example of a geometrically infinite group which is finitely generated.) Compare the non-constructive result, [26].
4. Prove or disprove: if $\Gamma \in \partial\,\mathrm{QH}(G)$ where G supports deformation, then Γ is a cusp iff its matrix entries are algebraic over $\mathbb{C}$. A similar result for groups in the interior lying on rational lamination locii.
5. Write a completion of the space of graph curves which is in natural bijection with the (projection of the) Thurston or Bers boundary of the relevant Teichmüller space. (That is, Schottky groups are to smooth tropical varieties as B-groups are to...?)
6. There should be a duality between some objects from tropical geometry and:-
   a) braid groups on n braids (c.f. the book of the four Japanese mathematicians)
   b) points in $\mathbb{Z}^n$ visible from the origin (?)
   c) (measured) laminations on a $n/2$-punctured sphere
7. Determine necessary and sufficient conditions for a circle packing on the sphere to be the limit set of a maximal cusp.
8. Give similar results about circle packing limit sets for higher dimensional groups.

9. Make explicit the action of the mapping class group on the combinatorics of train tracks and laminations for arbitrary geometrically finite groups, giving a description similar to the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathcal{ML}(S_{0,4})$ and $\mathcal{ML}(S_{1,1})$.

10. Understand [1].

References

[1] Hirotaka Akiyoshi, Makoto Sakuma, Masaaki Wada, and Yasushi Yamashita. *Punctured torus groups and 2-bridge knot groups I*. Lecture Notes in Mathematics 1909. Springer, 2007. ISBN: 978-3-540-71807-9. DOI: `10.1007/978-3-540-71807-9` (cit. on pp. 19, 36, 37).

[2] Hirotaka Akiyoshi et al. "Classification of non-free Kleinian groups generated by two parabolic transformations". In: *Transactions of the American Mathematical Society* 374 (3 2021), pp. 1765–1814. DOI: `10.1090/tran/8246`. arXiv: `2001.09564 [math.GT]` (cit. on p. 36).

[3] Kari Astala, Tadeusz Iwaniec, and Gaven J. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series 48. Princeton University Press, 2009. ISBN: 978-0-691-13777-3 (cit. on p. 15).

[4] Dave Bayer and David Eisenbud. "Graph curves". In: *Advances in Mathematics* 86 (1991), pp. 1–40. DOI: `10.1016/0001-8708(91)90034-5` (cit. on p. 30).

[5] Alan F. Beardon. *The geometry of discrete groups*. Graduate Texts in Mathematics 91. Springer-Verlag, 1983. ISBN: 0-387-90788-2. DOI: `10.1007/978-1-4612-1146-4` (cit. on pp. 2, 4).

[6] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, 1992. ISBN: 978-0-387-55534-8. DOI: `10.1007/978-3-642-58158-8` (cit. on p. 35).

[7] Lipman Bers. "On boundaries of Teichmüller spaces and on Kleinian groups I". In: *Annals of Mathematics* 91 (3 1970), pp. 570–600. DOI: `10.2307/1970638` (cit. on pp. 21, 24).

[8] Joan S. Birman. *Braid, links, and mapping class groups*. Annals of Mathematics Studies 82. Princeton University Press, 1974. ISBN: 0-691-08149-2 (cit. on p. 31).

[9] Joan S. Birman and Caroline Series. "Dehn's algorithm revisited, with applications to simple curves on surfaces". In: *Combinatorial group theory and topology*. Ed. by Stephen M. Gersten and John R. Stallings. Annals of Mathematics Studies 111. Princeton University Press, 1987, pp. 451–478. ISBN: 0-691-08409-2 (cit. on p. 31).

[10] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Grundlehren der mathematischen Wissenshaften 319. Springer, 1999. ISBN: 978-3-540-64324-1. DOI: `10.1007/978-3-662-12494-9` (cit. on p. 4).

[11] Egbert Brieskorn and Horst Knörrer. *Plane algebraic curves*. Trans. by John Stillwell. Modern Birkhaüser Classics. Birkhaüser, 1986. ISBN: 978-3-0348-0492-9 (cit. on p. 30).

[12] Matthew G. Brin, Gareth A. Jones, and David Singerman. "Commentary on Robert Riley's article "A personal account of the discovery of hyperbolic structures on some knot complements"". In: *Expositiones Mathematicae* 31.2 (2013), pp. 99–103. DOI: 10.1016/j.exmath.2013.01.002 (cit. on p. 8).

[13] Gerhard Burde and Heiner Zieschang. *Knots*. 2nd ed. Studies in Mathematics 5. de Gruyer, 2003 (cit. on p. 32).

[14] Richard D. Canary. "Pushing the boundary". In: *In the tradition of Ahlfors and Bers, III*. Ed. by William Abikoff and Andrew Haas. Contemporary Mathematics 355. American Mathematical Society, 2004, pp. 109–121. DOI: 10.1090/conm/355/06448 (cit. on p. 21).

[15] Alex Elzenaar. "Deformation spaces of Kleinian groups". Master thesis. The University of Auckland, 2022. URL: https://aelzenaar.github.io/msc_thesis.pdf (cit. on p. 27).

[16] Alex Elzenaar. *Embedding deformation spaces of Kleinian groups into $\mathbb{C}^n$*. In preparation (cit. on p. 21).

[17] Alex Elzenaar. *Functions on homology bases*. In preparation (cit. on p. 31).

[18] Alex Elzenaar. *The dynamics and geometry of Kleinian groups*. Lecture notes. 2021. URL: https://aelzenaar.github.io/tchng/kleiniangroups2021.pdf (cit. on p. 4).

[19] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *Approximations of the Riley slice*. 2021. arXiv: 2111.03230 [math.GT] (cit. on p. 16).

[20] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021-22 MATRIX annals*. Springer, 2023. arXiv: 2204.11422 [math.GT]. To appear (cit. on pp. 8, 15, 16).

[21] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *The combinatorics of Farey words and their traces*. 2022. arXiv: 2204.08076 [math.GT] (cit. on pp. 26, 27, 31).

[22] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *The elliptic cousins of the Riley slice*. In preparation. 2022 (cit. on p. 16).

[23] A. Fahti, P. Laudenbach, and V. Poenaru. *Thurston's work on surfaces*. Trans. by Djun Kim and Dan Margalit. Mathematical Notes 48. Princeton University Press, 2012. ISBN: 978-0-691-14735-2 (cit. on p. 21).

[24] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Princeton Mathematical Series. Princeton University Press, 2012. ISBN: 978-0-691-14794-9 (cit. on pp. 31, 32).

[25] George K. Francis. *A topological picturebook*. Springer, 1987 (cit. on p. 8).

[26] L. Greenberg. "Fundamental polyhedra for Kleinian groups". In: *Annals of Mathematics* 84.3 (1996), pp. 433–441. DOI: 10.2307/1970456 (cit. on pp. 14, 24, 36).

[27] Godfrey H. Hardy and Edward M. Wright. *An introduction to the theory of numbers*. 4th ed. Oxford University Press, 1960 (cit. on p. 33).

[28] Troels Jørgensen. “On pairs of once-punctured tori”. In: *Kleinian groups and hyperbolic 3-manifolds*. LMS Lecture Note Series 299. Cambridge University Press, 2003, pp. 183–297 (cit. on p. 36).

[29] Michael Kapovich. *Hyperbolic manifolds and discrete groups*. Progress in Mathematics 183. Birkhaüser, 2001. ISBN: 978-0-8176-4913-5. DOI: 10.1007/978-0-8176-4913-5 (cit. on pp. 14, 21).

[30] Louis H. Kauffman. *On knots*. Annals of Mathematics Studies 115. Princeton University Press, 1987. ISBN: 0-691-08434-3 (cit. on pp. 26, 33).

[31] Linda Keen, Bernard Maskit, and Caroline Series. *Geometric finiteness and uniqueness for Kleinian groups with circle packing limit sets*. Stony Brook IMS Preprint #1991/23. Dec. 1991. arXiv: math/9201299 [math.DG]. URL: https://www.math.stonybrook.edu/preprints/ims91-23.pdf (cit. on p. 24).

[32] Linda Keen and Caroline Series. “Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori”. In: *Topology* 32.4 (1993), pp. 719–749. DOI: 10.1016/0040-9383(93)90048-z (cit. on pp. 16, 31).

[33] Linda Keen and Caroline Series. “The Riley slice of Schottky space”. In: *Proceedings of the London Mathematics Society* 3.1 (69 1994), pp. 72–90. DOI: 10.1112/plms/s3-69.1.72 (cit. on pp. 16, 31).

[34] Irwin Kra. “Deformation Spaces”. In: *A crash course on Kleinian groups: Lectures given at a special session at the January 1974 meeting of the American Mathematical Society at San Francisco*. Ed. by Lipman Bers and Irwin Kra. Lecture Notes in Mathematics 400. Springer-Verlag, 1974, pp. 48–70. DOI: 10.1007/bfb0065675 (cit. on p. 4).

[35] Irwin Kra. “Horocyclic coordinates for Riemann surfaces and moduli spaces, I. Teichmüller and Riemann spaces of Kleinian groups”. In: *Journal of the American Mathematical Society* 3.3 (1990), pp. 499–578. DOI: 10.2307/1990927 (cit. on p. 30).

[36] Colin Maclachlan and Alan W. Reid. *The arithmetic of hyperbolic 3-manifolds*. Graduate Texts in Mathematics 219. Springer-Verlag, 2003. ISBN: 978-1-4757-6720-9. DOI: 10.1007/978-1-4757-6720-9 (cit. on p. 2).

[37] Albert Marden. *Hyperbolic manifolds. An introduction in 2 and 3 dimensions*. 2nd ed. First edition was published under the title “Outer circles”. Cambridge University Press, 2016. ISBN: 978-1-107-11674-0 (cit. on p. 30).

[38] Bernard Maskit. *Kleinian groups*. Grundlehren der mathematischen Wissenshaften 287. Springer-Verlag, 1987. ISBN: 978-3-642-61590-0. DOI: 10.1007/978-3-642-61590-0 (cit. on pp. 2, 4, 16).

[39] Bernard Maskit. “On boundaries of Teichmüller spaces and on Kleinian groups II”. In: *Annals of Mathematics* 91 (3 1970), pp. 607–639. DOI: 10.2307/1970640 (cit. on p. 21).

[40] Bernard Maskit and Gadde Swarup. "Two parabolic generator Kleinian groups". In: *Israel Journal of Mathematics* 64.3 (1989), pp. 257–266. DOI: 10.1007/bf02882422 (cit. on p. 26).

[41] Katsuhiko Matsuzaki and Masahiko Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford University Press, 1998. ISBN: 0-19-850062-9 (cit. on p. 8).

[42] Curt McMullen. "Cusps are dense". In: *Annals of Mathematics* 133 (1991), pp. 217–247. DOI: 10.2307/2944328 (cit. on p. 21).

[43] Yair N. Minsky. "The classification of Kleinian surface groups, I: Models and bounds". In: *Annals of Mathematics* 171.1 (2010), pp. 1–107. DOI: 10.4007/annals.2010.171.1. arXiv: math/0302208 [math.GT] (cit. on p. 13).

[44] David Mumford, Caroline Series, and David Wright. *Indra's pearls. The vision of Felix Klein*. Cambridge University Press, 2002. ISBN: 0-521-35253-3 (cit. on pp. 16, 17).

[45] R. C. Penner and J. L. Harer. *Combinatorics of train tracks*. Annals of Mathematics Studies 125. Princeton University Press, 1992. ISBN: 0-691-02531-2 (cit. on p. 21).

[46] Jessica S. Purcell. *Hyperbolic knot theory*. Graduate Studies in Mathematics 209. American Mathematical Society, 2020. ISBN: 978-1-4704-5499-9 (cit. on pp. 32, 35).

[47] John G. Ratcliffe. *Foundations of hyperbolic manifolds*. Graduate Texts in Mathematics 149. Springer, 1994. ISBN: 978-3-030-31597-9. DOI: 10.1007/978-3-030-31597-9 (cit. on pp. 5, 8).

[48] Robert Riley. "A personal account of the discovery of hyperbolic structures on some knot complements". In: *Expositiones Mathematicae* 31.2 (2013), pp. 104–115. DOI: 10.1016/j.exmath.2013.01.003. arXiv: 1301.4601 (cit. on p. 8).

[49] Robert Riley. "A quadratic parabolic group". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 77 (1975), pp. 281–288. DOI: 10.1017/s0305004100051094 (cit. on p. 8).

[50] Dale Rolfsen. *Knots and links*. Corrected edition. AMS Chelsea, 2003. ISBN: 0-8218-3436-3 (cit. on p. 26).

[51] Kevin P. Scannell and Michael Wolf. "The grafting map of Teichmüller space". In: *Journal of the American Mathematical Society* 15.2 (2002), pp. 893–927. DOI: 10.1090/S0894-0347-02-00395-8 (cit. on p. 21).

[52] Horst Schubert. "Knoten mit zwei Brücken". In: *Mathematische Zeitschrift* 65 (1956), pp. 133–170. DOI: 10.1007/bf01473875 (cit. on p. 32).

[53] Caroline Series. "A crash course on Kleinian groups". In: *Rendiconti dell'Istituto di Matematica dell'Università di Trieste* 37 (), pp. 1–38. URL: https://rendiconti.dmi.units.it/volumi/37/01.pdf (cit. on p. 4).

[54] Caroline Series. "Geometric methods of symbolic coding". In: *Ergodic theory, symbolic dynamics and hyperbolic spaces*. Ed. by Tim Bedford, Michael Keane, and Caroline Series. Oxford University Press, 1991, pp. 125–151. ISBN: 0-19-859685-5 (cit. on p. 31).

[55] Caroline Series. “Lectures on pleating coordinates for once punctured tori”. In: *Departmental Bulletin Paper, Kyoto University: Hyperbolic Spaces and Related Topics* 1104 (1999), pp. 30–90. URL: `http://hdl.handle.net/2433/63221` (cit. on p. 16).

[56] Caroline Series. “The geometry of Markoff numbers”. In: *The Mathematical Intelligencer* 7.3 (1985), pp. 20–29. DOI: `10.1007/bf03025802` (cit. on pp. 31, 34).

[57] Caroline Series. “The modular surface and continued fractions”. In: *Journal of the London Mathematical Society*. 2nd ser. 31 (1985), pp. 69–80. DOI: `10.1112/jlms/s2-31.1.69` (cit. on p. 31).

[58] William P. Thurston. *The geometry and topology of three-manifolds*. Unpublished notes. 1979. URL: `http://library.msri.org/books/gt3m/` (cit. on pp. 4, 22).

[59] William P. Thurston. “Three dimensional manifolds, Kleinian groups, and hyperbolic geometry”. In: *Bulletin (new series) of the American Mathematical Society* 6.3 (1982), pp. 357–381. DOI: `10.1090/s0273-0979-1982-15003-0` (cit. on pp. 4, 8, 13).

[60] William P. Thurston. *Three-dimensional geometry and topology*. Ed. by Silvio Levy. Vol. 1. Princeton University Press, 1997. ISBN: 0-691-08304-5 (cit. on pp. 5, 31).

Index