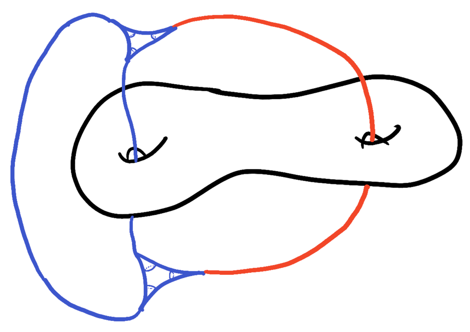
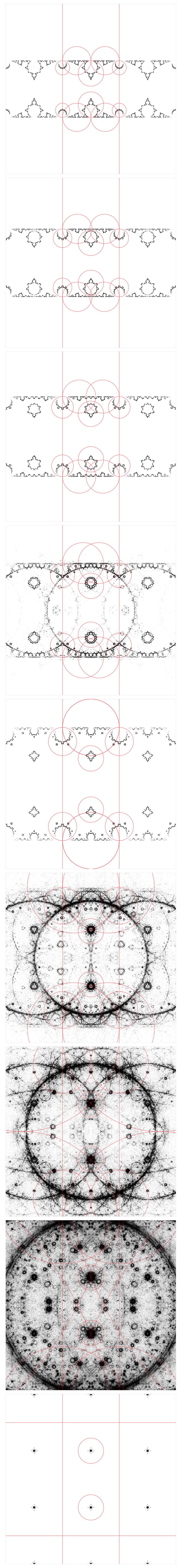
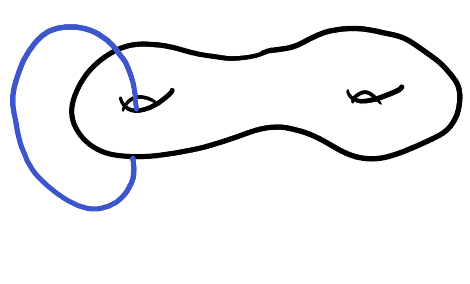
Limit sets of cone manifolds

Geometric group theory of indiscrete groups of Möbius transforms

Alexander Elzenaar
School of Mathematics, Monash University
https://aelzenaar.github.io/ — alexander.elzenaar@monash.edu







In the strip of images on the right side of the page, we show a deformation from one discrete group of conformal maps of the plane to another, through a family of indiscrete groups that uniformise cone surfaces. The black dots are the images of a single point under the group action.

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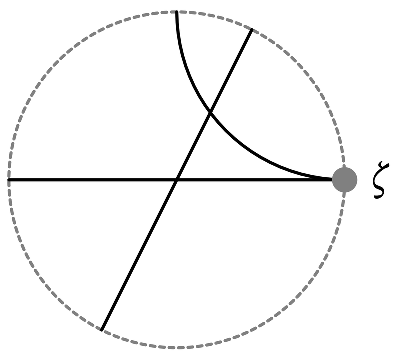
The hyperbolic plane

Euclid, c. 300BC, gave five postulates for geometry. His first four now form the basis of modern metric geometry: they govern existence of geodesics, existence of circles, and well-definedness of angles. His fifth axiom is harder to summarise:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles [5].

If you're confused, you're not alone: it seems like it should be a theorem rather than a postulate. In fact, mathematicians spent over a thousand years trying to deduce it from his other four axioms. They failed for one simple reason: there exist metric geometries in which lines can converge without meeting.

Hyperbolic geometry, $\mathbb{H}^2$, is a metric geometry on the disc $\{z \in \mathbb{C} : |z| < 1\}$. Writing down the explicit metric is possible but unenlightening [1, §7.2]. All we need is that that geodesics in this geometry are circular arcs orthogonal to the boundary circle, that this circle is a 'horizon' which is infinitely far away from every point in $\mathbb{H}^2$, and that angle measure is the usual angle measure that you get with a protractor. Here is a picture of three geodesics in $\mathbb{H}^2$ forming a counterexample to Euclid's postulate (the point ζ is on the horizon, so the two geodesics converging there get arbitrarily close but do not meet in $\mathbb{H}^2$):



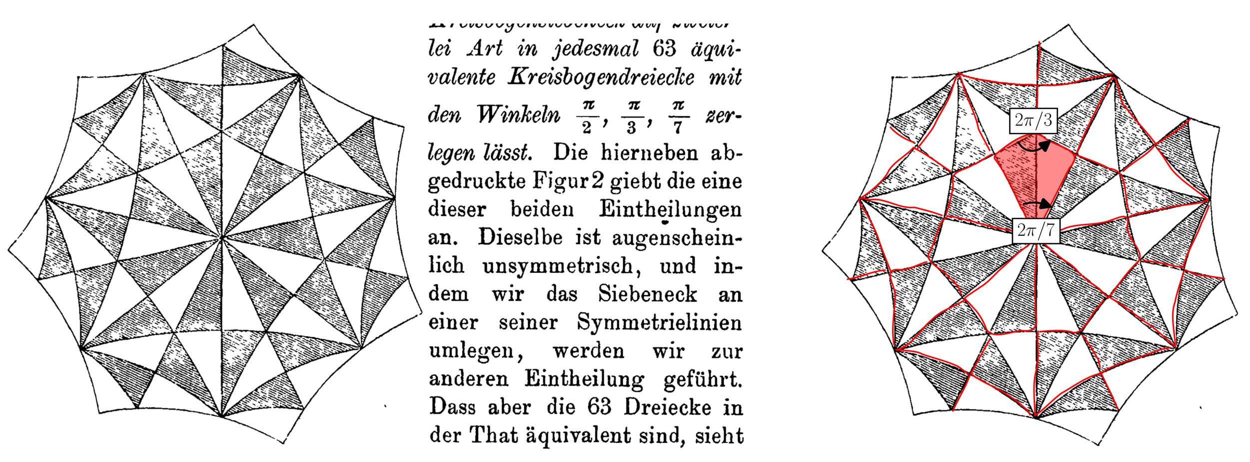
Hyperbolic triangle groups

Euclid studied lines and circles drawn in the sand and the ways you can construct complicated patterns from simple ones. One of the drivers of mathematics is the desire to understand such construction problems: how can massive systems of complexity arise from simple pieces? One model for this kind of problem is group theory, and to start with we consider groups with presentation

$$\langle x, y : x^p = y^q = (xy)^r = 1 \rangle. \qquad (\blacktriangle)$$

These so-called *triangle groups* are actively studied: M. Conder and D. Young present papers in this very meeting on their abstract group theory [3, 7].

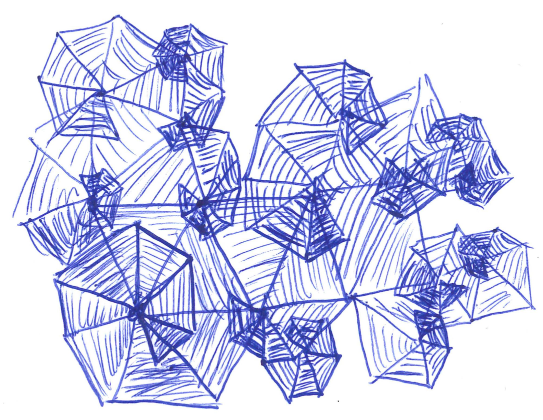
When p, q, and r are integers satisfying $p^{-1} + q^{-1} + r^{-1} < 1$ then there exists a hyperbolic triangle with angles π/p, π/q, π/r that tiles the hyperbolic plane. We take the example on the left, $(p, q, r) = (2, 3, 7)$, from Fricke's 1893 paper on triangle groups [6, Fig. 2]:


lei Art in jedesmal 63 äquivalente Kreisbogendreiecke mit den Winkeln $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{7}$ zerlegen lässt. Die hierneben abgedruckte Figur 2 giebt die eine dieser beiden Eintheilungen an. Dieselbe ist augenscheinlich unsymmetrisch, und indem wir das Siebeneck an einer seiner Symmetrielinien umlegen, werden wir zur anderen Eintheilung geführt. Dass aber die 63 Dreiecke in der That äquivalent sind, sieht

The symmetry group of the tiling is the group generated by reflections in the sides of one triangle. Since it contains reflections, it isn't orientation-preserving. Instead of individual triangles, look at the tiling on the right by quadrilaterals: it has symmetry group generated by the two indicated rotations, and has presentation ($\blacktriangle$) with $p = 3$ and $q = 7$. All tessellations of the hyperbolic plane by (p, q, r)-triangles have symmetry groups with index preserving half of the form ($\blacktriangle$).

Indiscrete groups and cone manifolds

If we draw a (geodesic-sided) triangle $\triangle$ in $\mathbb{H}^2$ with arbitrary angles, we can still form the group of reflections in its sides and take the orientation-preserving half G. However, this group G is no longer discrete, and the triangle $\triangle$ no longer tiles $\mathbb{H}^2$: if you keep adding copies of the triangle around a single vertex, it will no longer 'close up' exactly. Every point of $\mathbb{H}^2$ will have infinitely many copies of the triangle on top of it, all in slightly different positions. However, we can produce a covering space of $\mathbb{H}^2$ where the group acts nicely. By standard results in geometric group theory [2, Chapter II.11], the metric space $\mathcal{X}_\triangle$ obtained by gluing copies of $\triangle$ together edge-by-edge (with singularities at the vertices) is still negatively curved: it is an infinitely branched non-simply-connected cover of $\mathbb{H}^2$ on which the group G acts to permute the copies of $\triangle$.



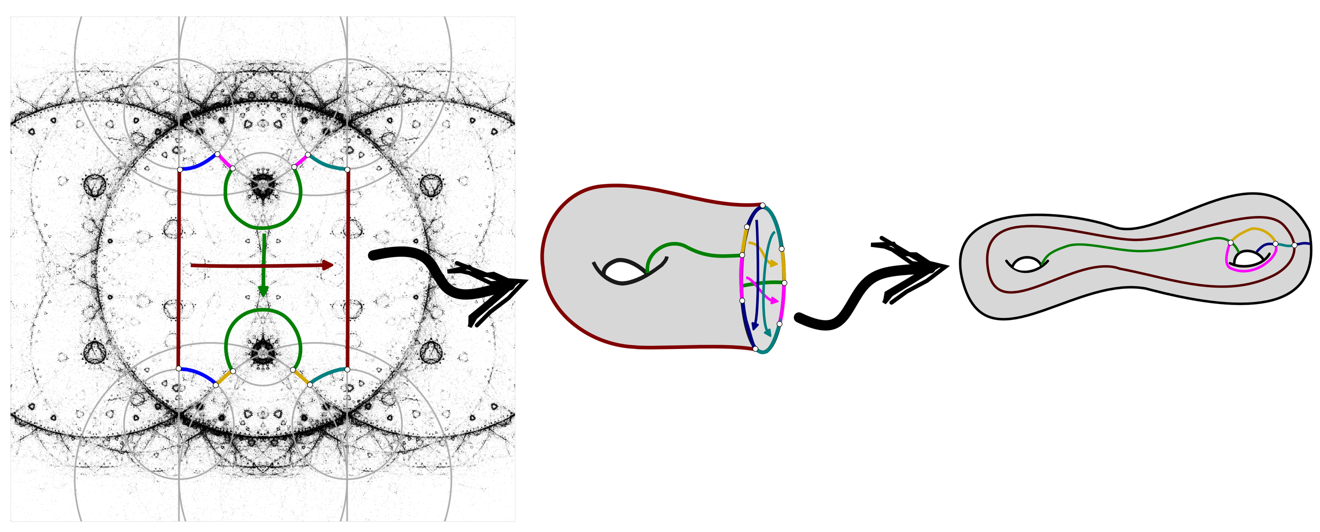
The quotient $\mathcal{X}_\triangle/G$ is a cone surface: it is a topological manifold that is homeomorphic to a 2-sphere, and is smooth everywhere except three singularities at the projections of the vertices of $\triangle$. This point of view can be extended: instead of triangles we can take arbitrary polygons with side-pairings, and we can use constructions like amalgamated products, HNN-extensions, and package them all up in complexes of indiscrete groups that give the cone manifolds structure.

The situation one dimension higher, in 3D, is very interesting. A discrete subgroup G of $\mathrm{Isom}^+(\mathbb{H}^3)$ is called a *Kleinian group* and $\mathbb{H}^3/G$ is a hyperbolic orbifold with holonomy group G. The group G also acts on $\partial\mathbb{H}^3 = \mathbb{S}^2$ as a group of conformal maps, and the projection of the maximal set where it acts discontinuously on $\mathbb{S}^1$ is the conformal boundary of $\mathbb{H}^3/G$. The general rigidity and deformation theory of Kleinian groups (and hence 3-orbifolds) is highly dependent on the analysis of this action. If M is a cone manifold then we can still construct a holonomy group, but now it's indiscrete: it naturally acts as an isometry group on a branched complex rather than $\mathbb{H}^3$ and it acts conformally on a branched complex over the sphere rather than on $\mathbb{S}^2$ itself.

Genus two deformation

At the right running down the page, we show a family of indiscrete groups which were constructed in [4] (the exact numeric parameters differ slightly from the preprint). At the top, the group is discrete and uniformises the 3-manifold which is the exterior of the genus 2 surface minus two thrice-punctured spheres with the punctures joined in three cusps as shown; at the bottom, the group is discrete and uniformises the exterior of the genus 2 surface minus a single loop. The intermediate groups correspond to cone angles where the deleted arc in red from the top-most manifold becomes a singular arc of increasing cone angle until at the bottom the angle becomes 2π and the singularity is gone.

These groups are all obtained from triangle groups via a sequence of HNN-extensions and amalgamated products. The resulting groups each have an invariant component with a fundamental domain that has twelve sides, and which glues up to the genus two component as shown:



The two triangle group components are off the top and bottom of this diagram, but can you see them in the top few pictures on the right?

References

[1] Alan F. Beardon. *The geometry of discrete groups.* Graduate Texts in Mathematics 91. Springer, 1983.

[2] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Grundlehren der mathematischen Wissenschaften 319. Springer, 1999.

[3] Marston Conder and Darius Young. *Soluble quotients of triangle groups.* 2024. arXiv: `2410.06571 [math.GR]`.

[4] Alex Elzenaar. *A case study in deforming 3-manifolds through cone manifolds.* To appear. 2025.

[5] Euclid. "The elements". In: *Greek mathematical works I: Thales to Euclid.* Trans. by Ivor Thomas. Loeb Classical Library 335. Harvard University Press, 1991, pp. 436–479.

[6] Robert Fricke. "Ueber den arithmetischen Charakter der zu den Verzweigungen (2, 3, 7) und (2, 4, 7) gehörenden Dreiecksfunctionen". In: *Mathematische Annalen* 41 (1893), pp. 443–468.

[7] Darius Young. *Quotient order density of ordinary triangle groups.* 2024. arXiv: `2408.02264 [math.GR]`.