Limit sets of cone manifolds

Geometric group theory of indiscrete groups of Möbius transforms

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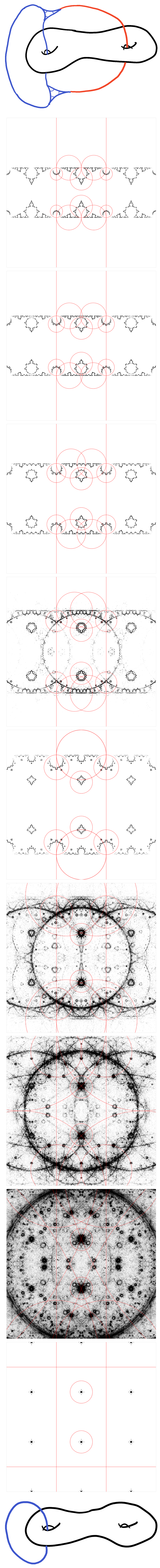

In the strip of images on the right side of the page, we show a deformation from one discrete group of conformal maps of the plane to another, through a family of indiscrete groups that uniformise cone surfaces. The black dots are the images of a single point under the group action.

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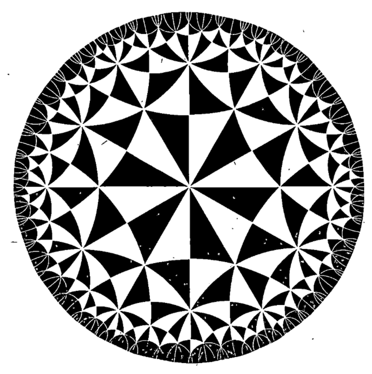
The hyperbolic plane

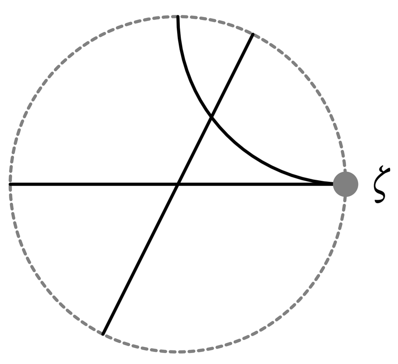
Euclid, c. 300BC, gave five postulates for geometry. His first four now form the basis of modern metric geometry: they govern existence of geodesics, existence of circles, and well-definedness of angles. His fifth is harder to summarise:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles [5].

If you're confused, you're not alone: it feels like it should somehow be a theorem, rather than a postulate. In fact, mathematicians spent over a thousand years trying to deduce it from his other four axioms. They failed for one simple reason: there exist metric geometries in which lines can converge without meeting.



Coxeter's tiling of of $\mathbb{H}^2$ [3] inspired many of Escher's most famous works.

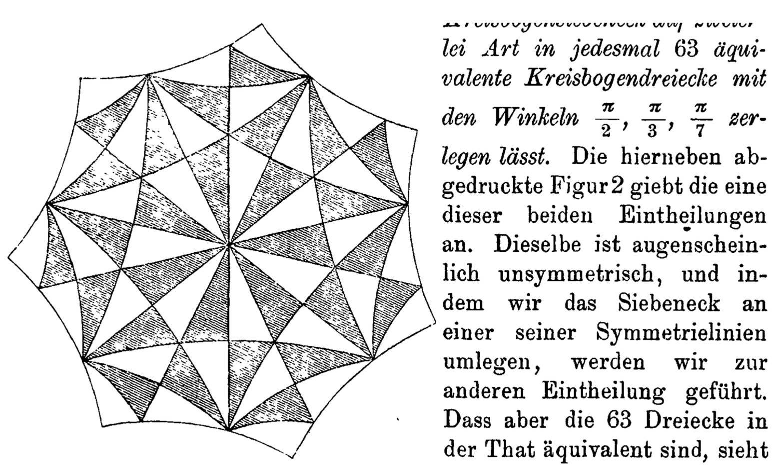


The two geodesics converging at ζ get arbitrarily close but do not meet in $\mathbb{H}^2$.

Hyperbolic geometry, $\mathbb{H}^2$, is a metric geometry on the disc. The boundary of the disc is a 'horizon' which is infinitely far away from every point in $\mathbb{H}^2$, angle measure is the usual angle measure that you get with a protractor, and geodesics are circular arcs that hit the horizon at a right angle.

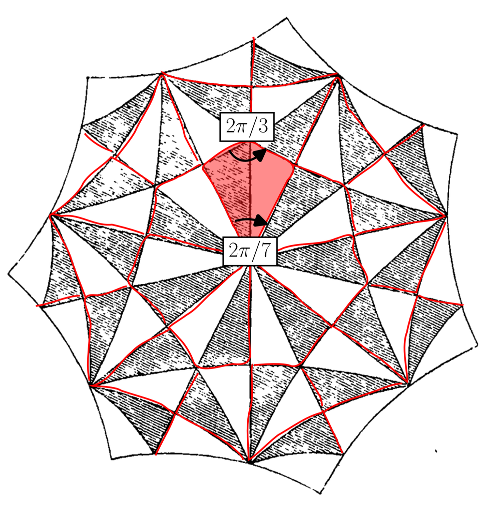
Hyperbolic triangle groups

When p, q, and r are integers satisfying $p^{-1} + q^{-1} + r^{-1} < 1$ then there exists a hyperbolic triangle with angles π/p, π/q, π/r that tiles the hyperbolic plane.



lei Art in jedesmal 63 äquivalente Kreisbogendreiecke mit den Winkeln $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$ zerlegen lässt. Die hierneben abgedruckte Figur 2 giebt die eine dieser beiden Eintheilungen an. Dieselbe ist augenscheinlich unsymmetrisch, und indem wir das Siebeneck an einer seiner Symmetrielinien umlegen, werden wir zur anderen Eintheilung geführt. Dass aber die 63 Dreiecke in der That äquivalent sind, sieht

Fricke drew the triangles with angles $\pi/2$, $\pi/3$, $\pi/7$ in 1893 [6].

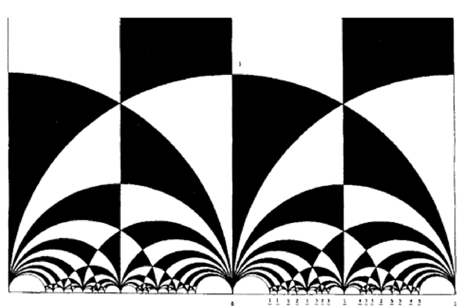


Two rotations generate the kite tiling.

The symmetry group of a triangle tiling is generated by reflections in the sides of one triangle, and it isn't orientation-preserving. Pair the triangles up to form kites: the new symmetry group is generated by two rotations, and it has presentation

$$\langle x, y : x^p = y^q = (xy)^r = 1 \rangle. \qquad (\blacktriangle)$$

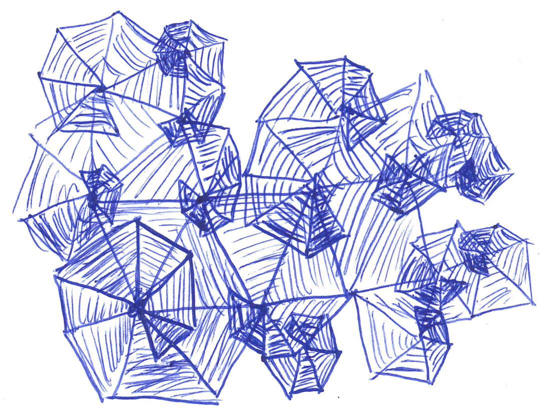
These so-called *triangle groups* are actively studied: M. Conder and D. Young present papers in this meeting on their abstract group theory [2, 7].



The disc is now the upper half-plane so the $(2, 3, \infty)$-triangles stretch infinitely upwards. This tiling comes from $\mathrm{PSL}(2, \mathbb{Z})$ and is of fundamental importance in number theory [1].

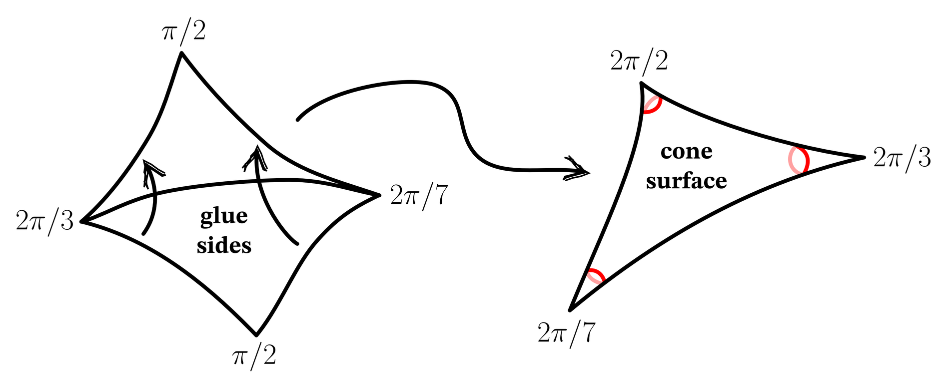
Indiscrete groups and cone manifolds

If we draw a triangle $\triangle$ in $\mathbb{H}^2$ with arbitrary angles, we can still take the group of reflections in its sides. But this group is no longer discrete and the triangle $\triangle$ no longer tiles $\mathbb{H}^2$ since if you keep adding copies around a single vertex, it won't 'close up' perfectly.



The metric space $\mathcal{X}_\triangle$ obtained by gluing copies of $\triangle$ together edge-to-edge is negatively curved and infinitely branched.

This infinitely branched space is a metric space where the group has a nicer action than $\mathbb{H}^2$. The reflection group acts to permute the copies of $\triangle$, and the orientation-preserving half G acts as rotations around the branch points.

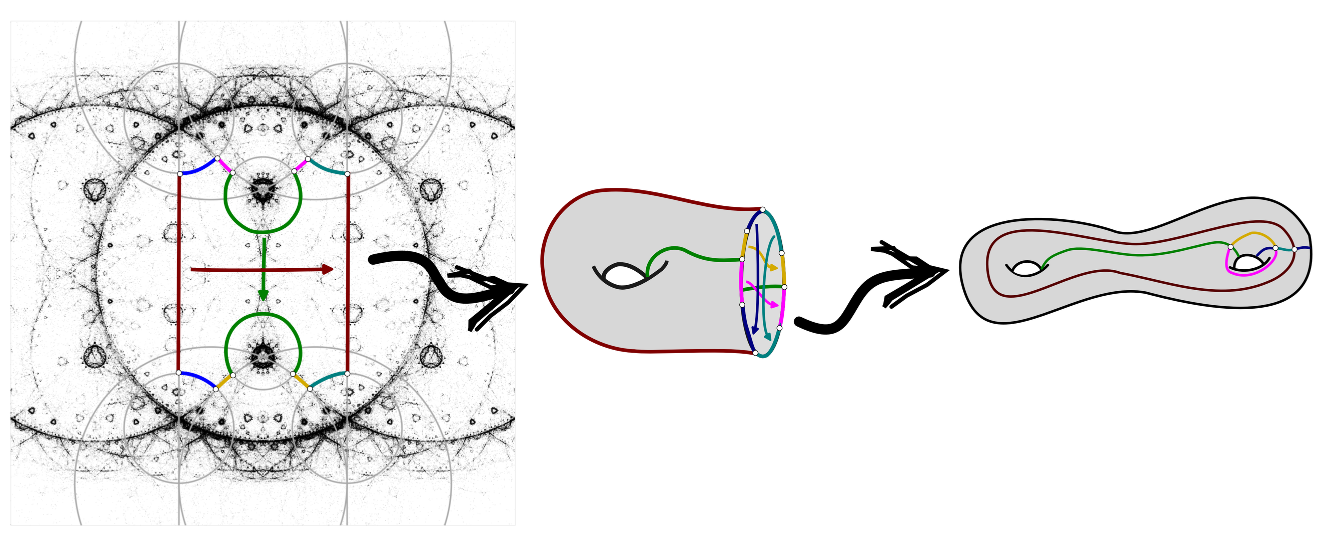


The quotient $\mathcal{X}_\triangle/G$ is a cone surface: it's a 2-sphere that's smooth everywhere except three singularities from the vertices of $\triangle$.

Instead of triangles we can take arbitrary polygons with side-pairings, we can use standard geometric group theory constructions like amalgamated products and HNN-extensions, and package them all up in complexes of indiscrete groups that give us more complicated cone surfaces. The picture is similar one dimension higher, in 3D, except now we glue up polyhedra instead of polygons.

Genus two deformation

At the right running down the page, we show a family of indiscrete groups which were constructed in [4]. At the top and the bottom, the groups are discrete and are the holonomy groups of 3-manifolds. The intermediate groups correspond to cone manifolds where the red cusp arc in the top-most manifold becomes a singular arc of increasing cone angle until at the bottom the angle around it becomes 2π and the singularity is gone.



The groups all have a fundamental domain with twelve sides that glues up to a genus 2 surface.

These groups all come from taking two triangle groups and gluing them together. Can you see the two hyperbolic triangles in the top few pictures on the right?

[1] Lieven le Bruyn. *The Dedekind tessellation.* 2007. URL: `http://www.neverendingbooks.org/the-dedekind-tessellation/` (visited on 11/21/2024).

[2] Marston Conder and Darius Young. *Soluble quotients of triangle groups.* 2024. arXiv: `2410.06571 [math.GR]`.

[3] H.S.M. Coxeter. "Crystal symmetry and its generalizations". In: *Proc. Trans. Roy. Soc. Canada* 51 (1957), pp. 1–13.

[4] Alex Elzenaar. *A case study in deforming 3-manifolds through cone manifolds.* To appear. 2025.

[5] Euclid. "The elements". In: *Greek mathematical works I: Thales to Euclid.* Trans. by Ivor Thomas. Loeb Classical Library 335. Harvard University Press, 1991, pp. 436–479.

[6] Robert Fricke. "Ueber den arithmetischen Charakter der zu den Verzweigungen (2, 3, 7) und (2, 4, 7) gehörenden Dreiecksfunctionen". In: *Math. Ann.* 41 (1893), pp. 443–468.

[7] Darius Young. *Quotient order density of ordinary triangle groups.* 2024. arXiv: `2408.02264 [math.GR]`.