Fix Σ compact with $\chi(\Sigma) < 0$, and choose a maximal system of disjoint curves $\gamma_1, \ldots, \gamma_n \subset \Sigma$ (i.e. a simplex in the curve complex for Σ). Consider the subvariety of

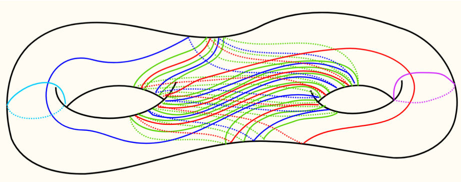
$$X = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbb{C})) /\!\!/ \mathrm{PSL}(2, \mathbb{C})$$

given by

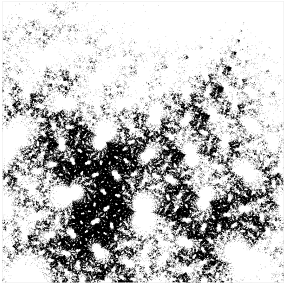
$$Y = \{\rho \in X : \forall_i \ \mathrm{tr}^2 \rho(\gamma_i) = 4\}.$$

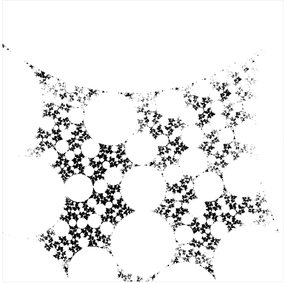
Abstract problem. What are the discrete non-elementary (not nec. faithful) representations in Y?

Simple case. genus 2, with 2 curves fixed passing through the handles (studied e.g. by Riley):

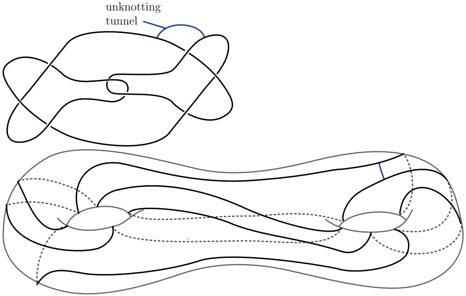


tr^2 blue $= 4$ & tr^2 green $= 4$ & tr^2 pink $= 4$

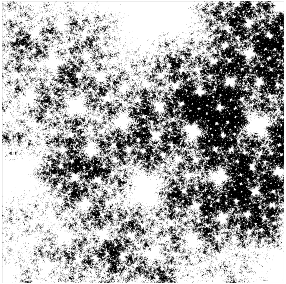




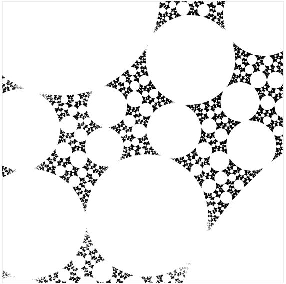
Heuristic in genus 2. Three general curves are dual to a θ-graph. Viewing Σ as the Heegaard surface of the standard Heegard splitting $\mathcal{H}^+ \cup_\Sigma \mathcal{H}^- = \mathbb{S}^3$, we have a knotted θ-graph in $\mathbb{S}^3$. It 'should be' that forcing the three general curves to $\mathrm{tr}^2 = 4$ will either give a maximal cusp (all go parabolic) or a tunnel number 1 link (one general curve is sent to the identity by the rep $\iff$ a 2-handle is glued onto $\mathcal{H}^-$ along that curve). But this is true only if the combinatorics are nice: the three curves need to bound embedded discs in $\mathcal{H}^+$ (up to $\mathrm{Mod}(\Sigma)$) for this to work. When the combinatorics are not nice, more complicated things can happen.



`arXiv:2503.13829`, Ex. 3.7



cofinite orbifold group (with an order 3 elliptic)



cusp group