Generalised Ford domains

arising from peripheral geometry

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Jan Zrzavý, *Krajina se stromem* [Landscape with a tree] (1916)
National Gallery Prague

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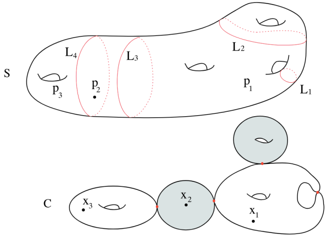
- The theory of uniformisation of Riemann surfaces, where they connect to the birational geometry of algebraic curves;



Arbarello, Cornalba, Griffiths, *Geometry of algebraic curves, II*, p.653.

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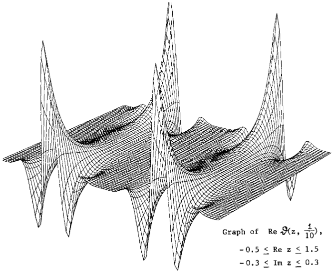

Mumford, *Tata lectures on theta, I*, frontispiece.

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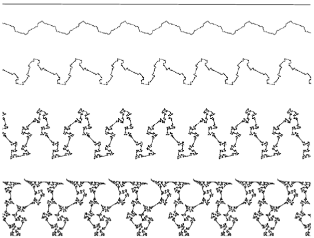
- The theory of uniformisation of Riemann surfaces, where they connect to the birational geometry of algebraic curves;
- Analytic number theory and automorphic forms, c.f. Langlands programme;
- Complex dynamics, via the McMullen–Sullivan dictionary and the theory of holomorphic motions.



Astala, Iwaniec, Martin, *Elliptic PDEs and q.c. mappings in the plane*, p.334.

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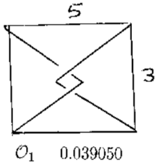
Additionally, since the 1970s: Kleinian groups are the holonomy groups of hyperbolic 3-orbifolds.

Applications go both ways:

- From Kleinian groups to topology: classification of the minimal volume hyperbolic 3-orbifolds by Gehring, Marshall, Martin (*Ann. of Math.* 2009 & 2012)
- From topology to Kleinian groups: the density theorem proved by Namazi & Souto (*Acta*, 2012) and Ohshika (*Geom. Top.*, 2011).



The minimal volume hyperbolic 3-orbifold, proved by a mixture of classical tools (Jørgensen's inequality, trace relations) and hyperbolic geometry (e.g. collaring results).
Image from Zimmermann, Rend. Ist. Mat. Univ. Trieste, 2001.

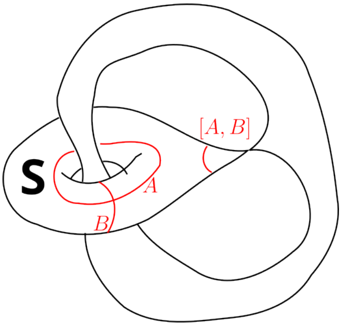
One possible approach to "Kleinian groups" is to ask: "How do they look?" It makes sense when the groups have been associated with natural fundamental polyhedrons.

Troels Jørgensen, "On pairs of once-punctured tori", c.1975 (reprinted 2003)

Important historical example

Let S be a once-punctured torus. The group $\pi_1(S)$ is free on 2 generators. Hyperbolic structures on the thickened torus $S \times [0,1]$ correspond to embeddings $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ such that $[A,B]$ is parabolic.

Problem. Find a fundamental domain for such an embedding.



S is the once-punctured torus. Let $\mathrm{QF}(S)$ be the quasi-Fuchsian space of reps $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$.

- (Bers) Each point in $\mathrm{QF}(S)$ is determined by a pair of points $v^-, v^+ \in \mathrm{Teich}(S) = \mathbb{H}^2$.
- Triples of curves on S with intersection number 1 are identified with vertices of the Farey graph $K(S)$. So 2-cells in the dual $K(S)^*$ correspond to triples in $\pi_1(S)$.

Theorem (Jørgensen, 1975; Akiyoshi, Sakuma, Wada, Yamashita, 2007)

For $G \in \mathrm{QF}(S)$ let $v^-, v^+ \in \mathbb{H}^2$ be the corresponding Teichmüller points. We can embed $K(S)$ into $\mathbb{H}^2$ and find a dual complex $K(S)^* \subset \mathbb{H}^2$ so that a fundamental domain for G is cut out by the isometric circles of elements corresponding to cells of $K(S)^*$ that lie on the path $[v^-, v^+]$.

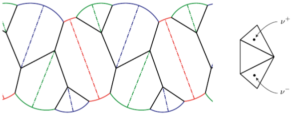


Fig. 0.9. $(0.594262 - 0.143287i, 0.221545 - 0.125063i, 0.184193 + 0.26835i)$

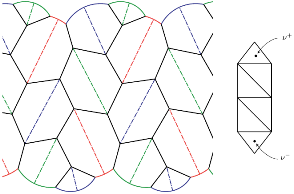
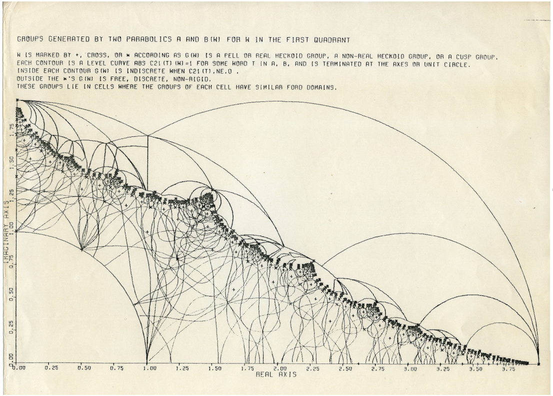


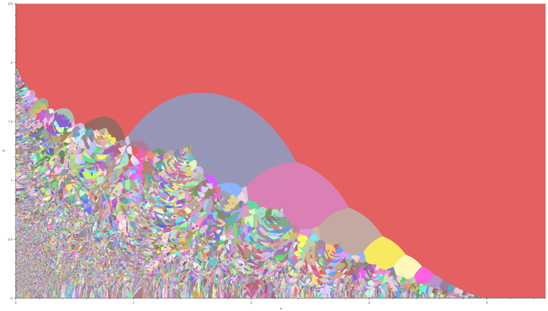
Fig. 0.10. $(0.652971 - 0.175526i, 0.196392 - 0.153203i, 0.150637 + 0.328729i)$

Akiyoshi, Sakuma, Wada, Yamashita, *Punctured Torus Groups and 2-Bridge Knot Groups I*, 2007

A similar picture for the Riley slice.



R. Riley, unpublished.



E., G.J. Martin, J. Schillewaert, to appear.

- These Ford domain constructions rely heavily on quasi-Fuchsian structure.
  - product structure $\implies$ can be built by ‘stacking’ slices of fixed geometry.
  - The surface geometry is simple (basically controlled by the Farey graph) so it is practical to actually make constructions.

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- Downsides:
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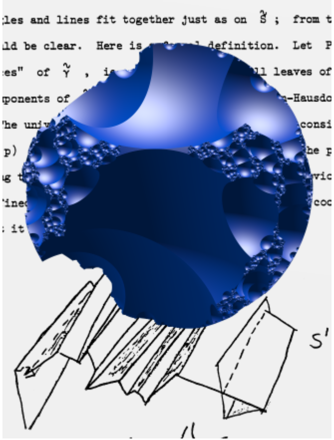
  - Ford domains do not detect pleating structure directly either, meaning the interaction with the ELT in general is murky.
- **Idea:** glue together *local* Ford domains, one for each flat piece of the convex core. Still have problem of understanding the complex of curves (this is unavoidable) but the resulting domains have nice algebraic structures associated to them that come from the pleating locus structure.

- Fix a geometrically finite Kleinian group G so that $\partial CC(\mathbb{H}^3/G)$ has measure 0 pleating locus.



Brock and Dumas, *Thurston's Jewel*, `https://www.dumas.io/convex/`

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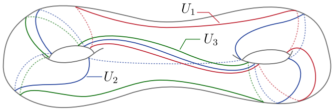
- For each flat piece $D \subset \partial CC(\mathbb{H}^3/G)$, consider the subgroup $\Pi \leq G$ consisting of loops in $\mathbb{H}^3/G$ that are isotopic to loops on D.



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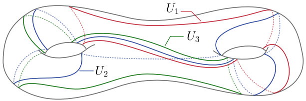

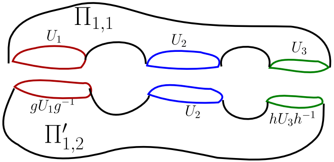

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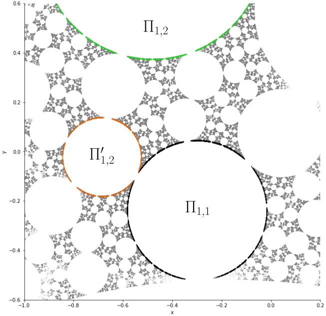
- The group Π is a Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{C})$ that preserves an open disc $\Delta(\Pi) \subset \Omega(G)$ (the disc under the dome of h.conv $\Lambda(G)$ supporting D). It is called a **F-peripheral subgroup** of G.



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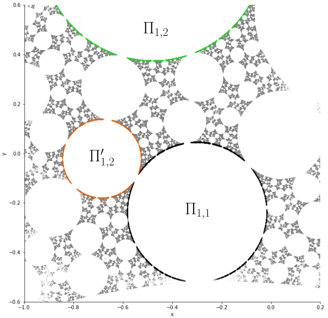
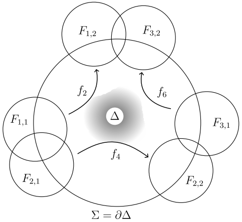
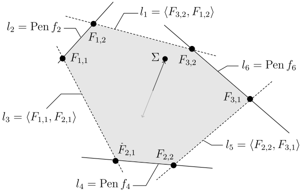
- A maximal set $\{\Pi_1, \ldots, \Pi_n\}$ of non-conjugate F-peripheral subgroups so that all the peripheral discs $\Delta(\Pi)$ for each surface of $\Omega(G)/G$ form a connected set is called a **circle chain**.



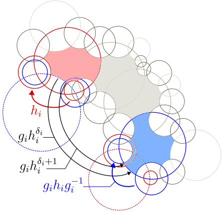
For a maximal lamination, the flat pieces are 3-holed spheres. These groups have nice simple fundamental domains cut out by circles.





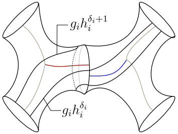
Theorem

If $\tilde{G} \in \mathrm{QH}(G)$ has a circle chain covering the entirety of $\Omega(\tilde{G})/\tilde{G}$, then we can construct a fundamental domain for $\tilde{G}$ by patching fundamental domains of the F-peripheral groups together.



Proof. Start with one F-peripheral group and then start gluing fundamental domains for thrice-punctured sphere groups onto it arbitrarily, following a tree on the surface dual to the lamination.

When you get to the end, the fundamental domains will not match up across the final lamination curve: there is some holonomy which needs to be taken into account, by splitting the 'end' edges up into pieces that get paired by maps differing by a Dehn twist.



Theorem

Suppose $G = G(0)$ is deformed to $G(t)$ by modifying its matrix entries according to some algebraic function in t. For $t \approx 0$, there is a way of modifying the fundamental domain of $G(0)$ into one for $G(t)$ so that the edges of the domain depend algebraically on t.

The proof is a technical geometric construction, and uses the projective structure of the space of circles on the sphere as well as some real algebraic geometry.

Just as with the theory of Jørgensen we get a set of open sets in $\mathrm{QH}(G)$ in which groups support a fundamental domain on $\hat{\mathbb{C}}$ with a fixed combinatorial structure, but now with geometry derived from the ending laminations of G directly.

Discreteness problem. *For a group $G \leq \mathrm{PSL}(2, \mathbb{C})$ given in terms of some data, e.g. generating matrices, determine whether G is discrete.*

In the BSS model of computation (strong enough to do real algebraic geometry), this is undecidable (Kapovich, Int. J. Alg. Comp., 2016). In the case of $\mathrm{PSL}(2, \mathbb{R})$, there are algorithms due to Fine & Rosenberger (LMS Lec. Not. Ser. 211, 1995) and Keen (Mem. AMS., 1995). In the case of $\mathrm{PSL}(2, \mathbb{Q}_p)$, there is a similar algorithm due to Markowitz (J. Alg., 2025).

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Weaker (local) problem. *For a given geometrically finite 3-manifold group $G \leq \mathrm{PSL}(2,\mathbb{C})$, determine whether some representation $\rho : G \to \mathrm{PSL}(2,\mathbb{C})$ lies in the deformation space $\mathrm{QH}(G)$.*

Our results give a countable set of semi-algebraic sets, determined by combinatorial data on $\mathbb{H}^3/G$, which fill the deformation space $\mathrm{QH}(G)$. Thus we obtain a semi-algorithm solving the weaker problem. However, there are major barriers to actually use it in any practical applications e.g. arithmetic group enumeration.

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