Discontinuous subgroups of $\mathrm{Aut}(\mathbb{S}^2)$

come in real-algebraic families with stable combinatorics

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František Kupka, *Čáry, plochy, hloubka III* [Lines, surface, depth III] (1913–1923)
National Gallery Prague

Ends of the Cayley graph

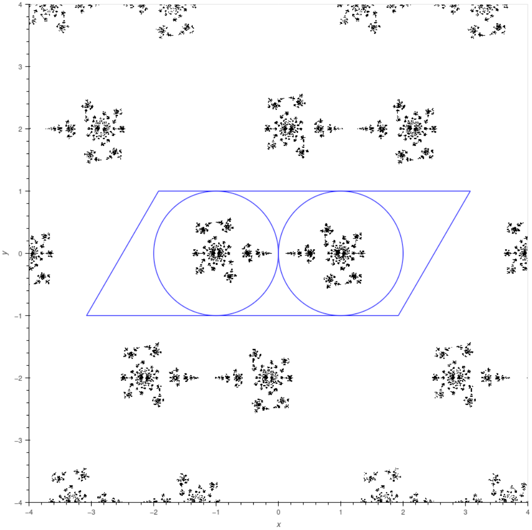
Let G be a group, usually infinite.

representations $G \to \mathrm{PSL}(2,\mathbb{C})$ $\Longleftrightarrow$ conformal actions $G \curvearrowright \mathbb{P}^1\mathbb{C} = \mathbb{S}^2$

Example

For $G = (\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$, a family of representations is

$$G(\alpha,\beta,\lambda) = \left\langle \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{bmatrix} \right\rangle$$



α, β fixed, λ varying

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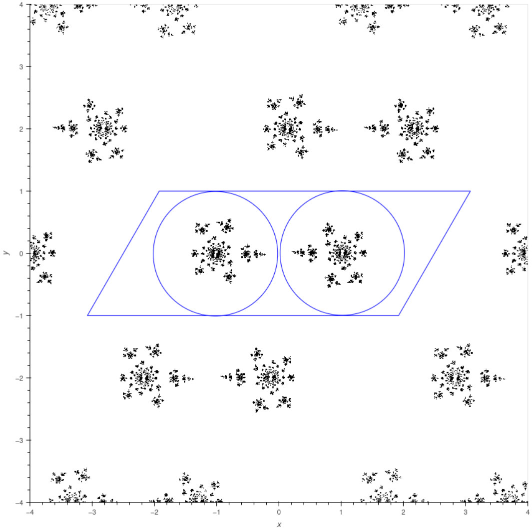

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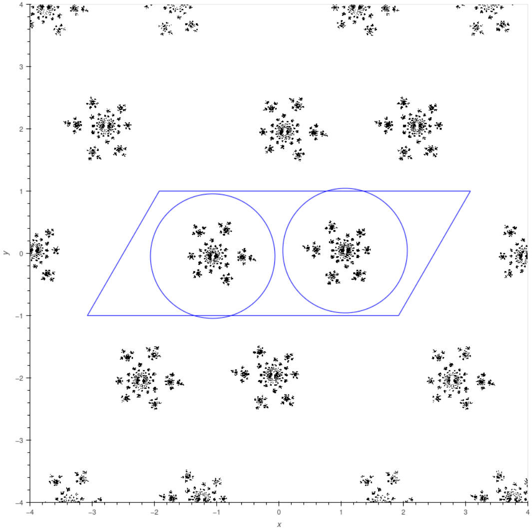

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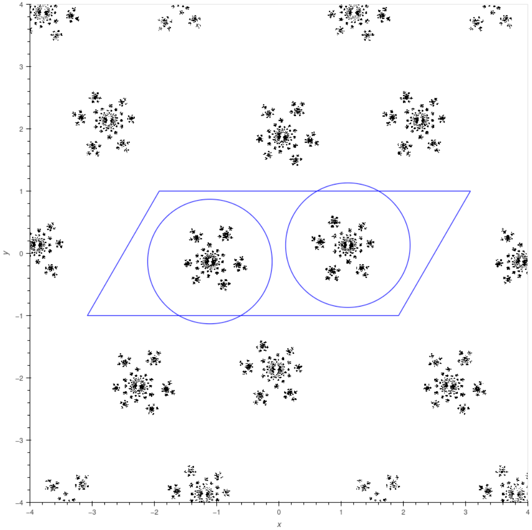

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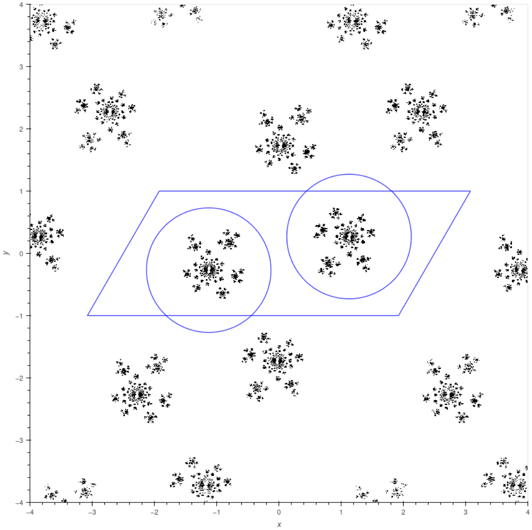

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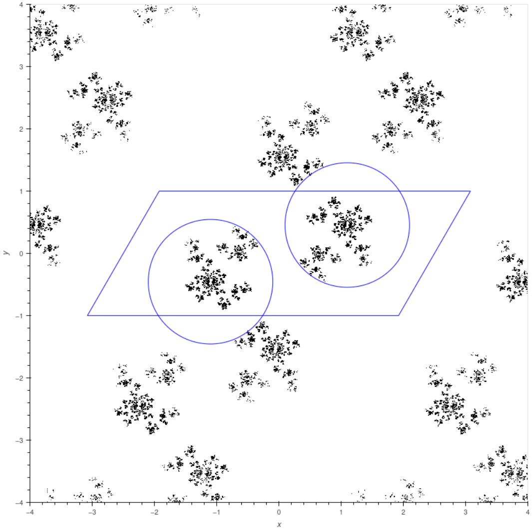

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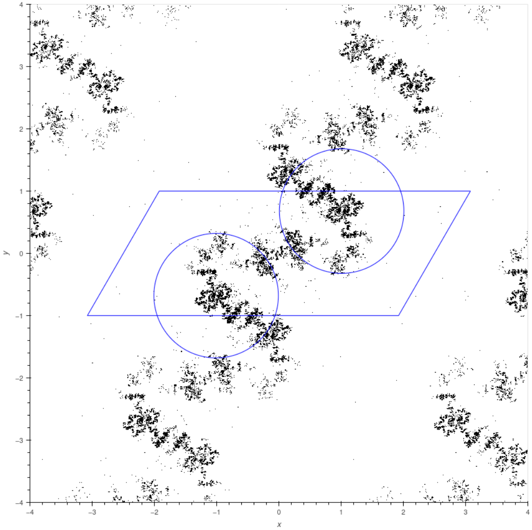

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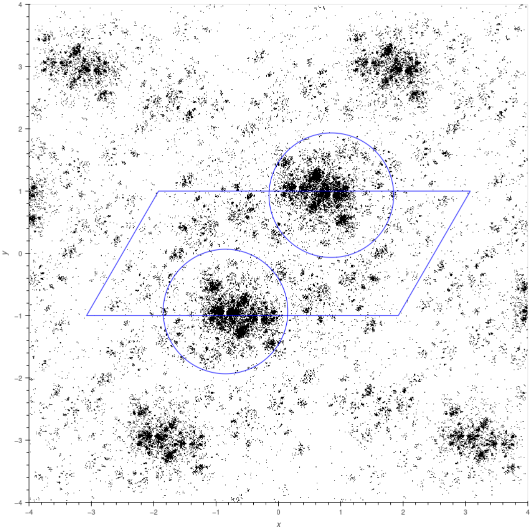

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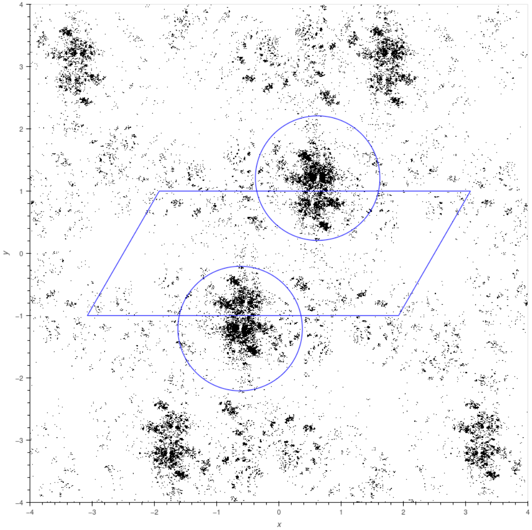

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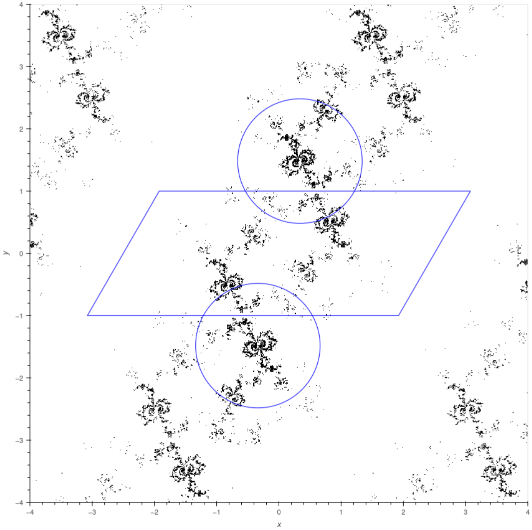

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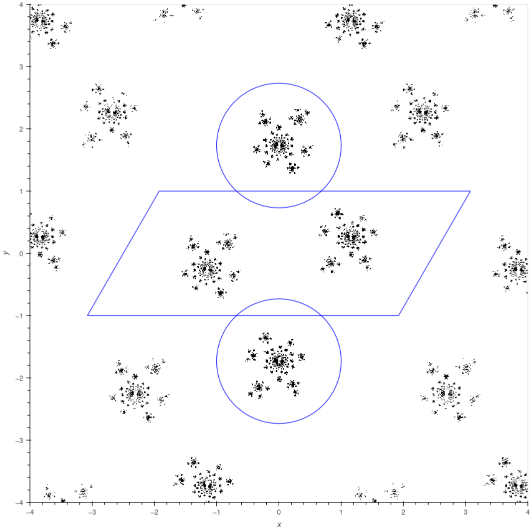

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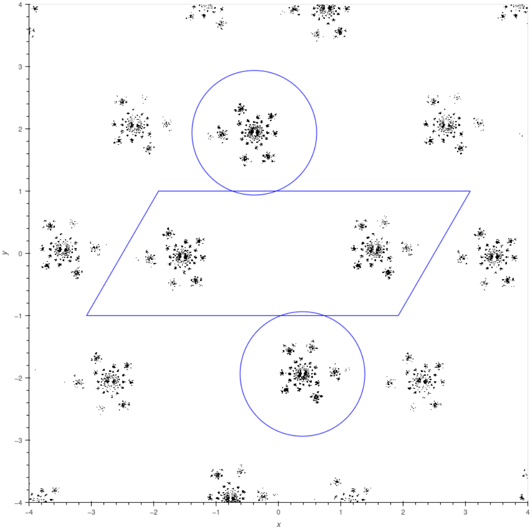

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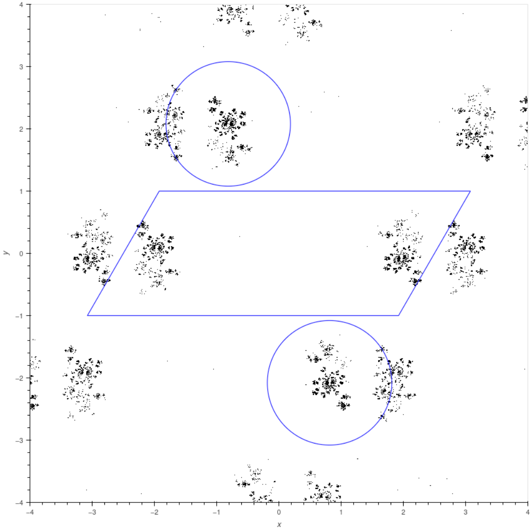

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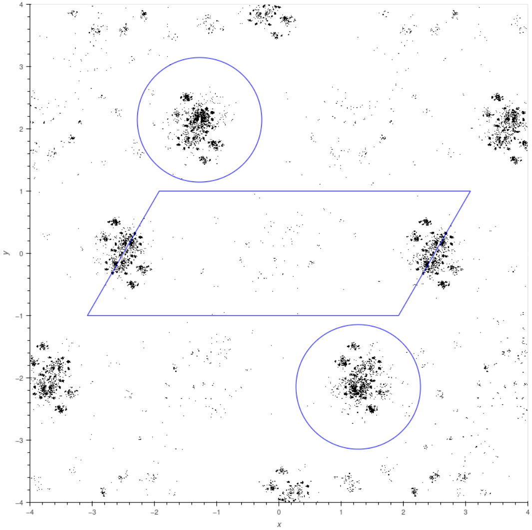

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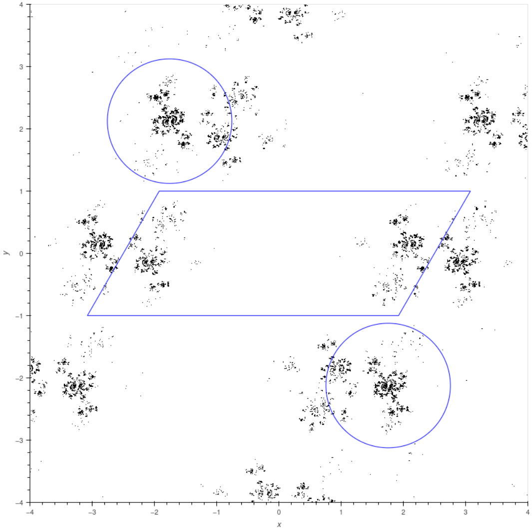

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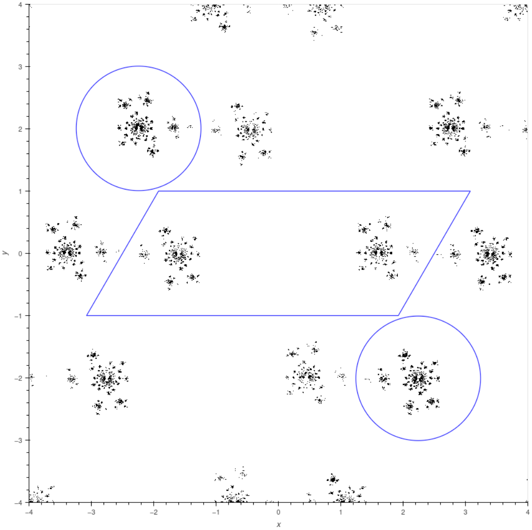

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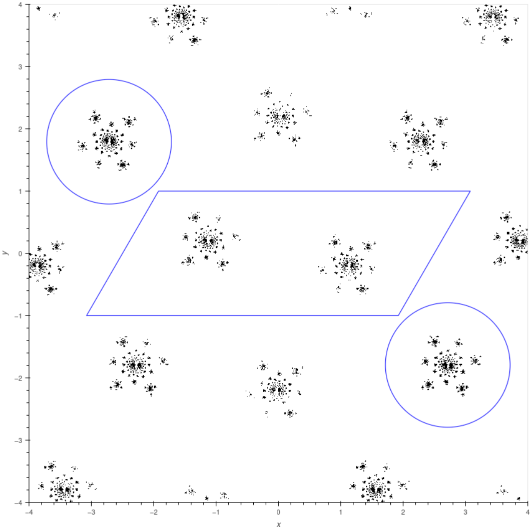

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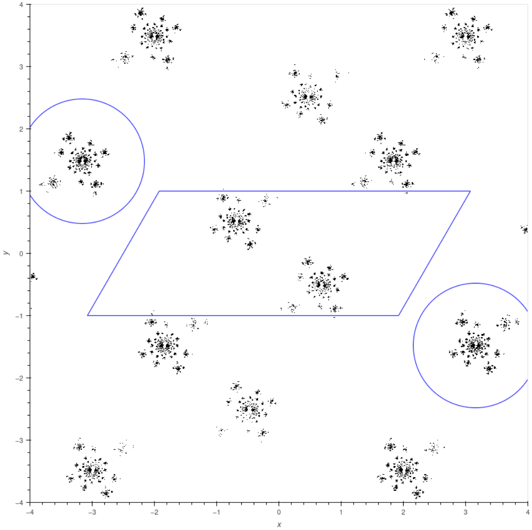

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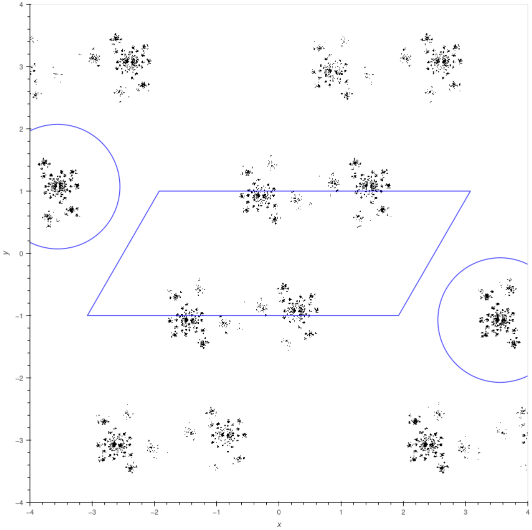

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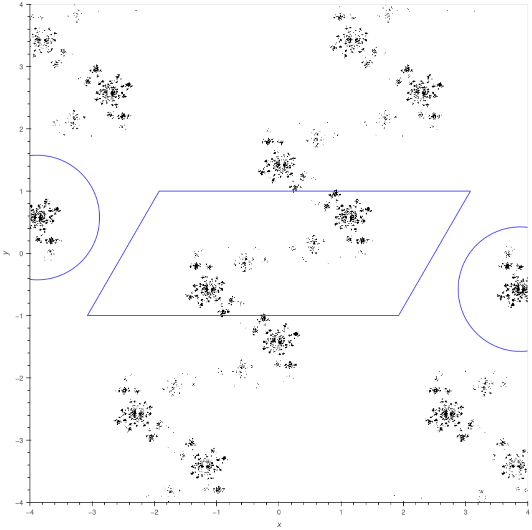

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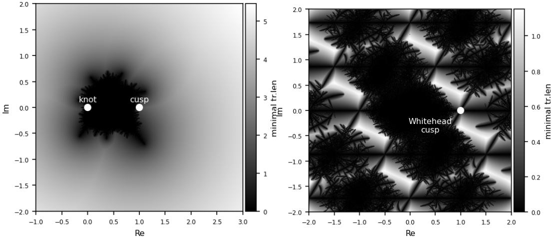
Small deformations of the rep. $G \to \mathrm{PSL}(2,\mathbb{C})$ are sometimes stable[a] (small deformations = small change in ‘global behaviour’).

Given a representation, how do you:

- check whether it is stable under small deformations?
- compute the global geometry (e.g. isomorphism class, quasi-isometry class)?
- compute the extent of the stable locus it lies in, if any?

[a] equiv. act discontinuously on $\mathbb{P}^1\mathbb{C}$

Two slices through $\mathrm{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \mathrm{PSL}(2,\mathbb{C}))$.
White = island of stability



E., “From disc patterns in the plane to character varieties of knot groups” `arXiv:2503.13829 [math.GT]`

Theorem (Ahlfors–Bers–Maskit theorems (c.1970) + Marden isomorphism theorem (1974) + λ-lemma (early 90s) + Ending lamination theorem (conj. Thurston 1982, proved Brock, Canary, Minsky & others c.2004))

For G a finitely generated group:

1. *All reps. in an island of stability in* $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$ *are topologically conjugate.*
2. *Deforming a rep. in an island of stability changes the complex structure on the quotient Riemann surfaces and does nothing else.*

More precisely:

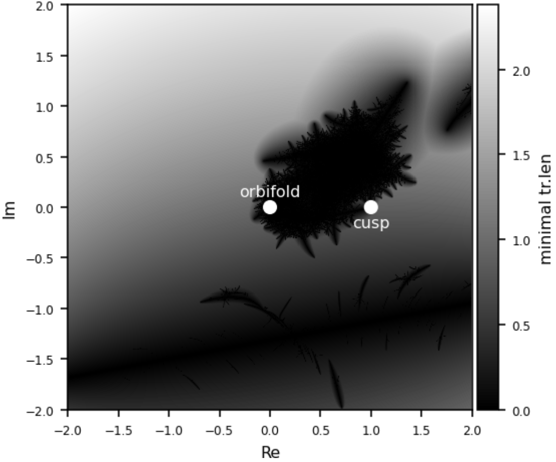
1. *The set of islands is in bijection with:*
   - *quasi-isometry classes of the representations;*
   - *equivalence classes mod conjugacy by quasiconformal maps;*
   - *maximal open sets of reps where the limit points have not congealed into rigid circle packings, possibly with bits filled in (Ahlfors measure* 0 *conj./theorem)*
   - *deformation spaces of hyperbolic metrics on fixed topological* 3*-manifolds;*
2. *Each island is a quotient of a product of Teichmüller spaces by a discrete group.*

Problem

These high-powered theorems are far from effective for doing calculations in real examples.

It's known that the islands of discreteness are embedded very wildly (e.g. not locally connected, see Canary, *Introductory bumponomics*, arXiv 2010); compare with the $\mathrm{PSL}(n, \mathbb{R})$ theory, where components are fairly well understood from real algebraic point of view (e.g. theory of Hitchin).

A slice through $\mathrm{Hom}(\mathbb{Z} * \mathbb{Z}, \mathrm{PSL}(2, \mathbb{C}))$. White = island of stability



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Theorem (E., “Peripheral subgroups of Kleinian groups”, arXiv 2025)

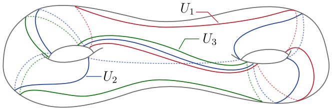
There exists a computable exhaustion of any stable region in any algebraic parameterisation of $X(G)$ by semi-algebraic sets.

Strategy of proof.

1. Find a dense set of semi-algebraic subsets of the desired stable region. These are *pleating varieties* and correspond to groups with very nice coarse geometry.
2. Thicken each semi-algebraic subset to a countable set of full-dimensional open semi-algebraic subsets.
3. Observe that the union of all these semi-algebraic sets is a decomposition of the entire stable region.

□

vertex of
curve complex



curve coding

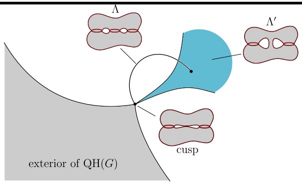
fundamental domain for action on graph of F-peripheral subgroups (*circle chain*)

$$\begin{array}{ccc}
\langle X, YX^{-1}Y^{-1}\rangle & & \\
\mid \langle X \rangle & & \\
\langle X^{-1}, Y^{-1}XY\rangle & \xrightarrow{\langle X^{-1}Y^{-1}XY\rangle} & \langle X^{-1}Y^{-1}X, Y\rangle \\
& & \mid \langle Y \rangle \\
& & \langle Y^{-1}, XYX^{-1}\rangle.
\end{array}$$

more combinatorial

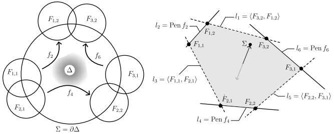
more algebro-geometric

local Ford domains (rational functions of parameters)



semi-algebraic set
in the character variety

system of quadratic inequalities



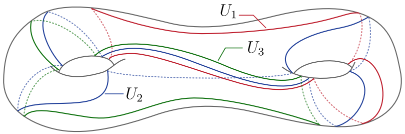
incidence structure in projective space

An F-peripheral subgroup of a rep. $\rho : G \to \mathrm{PSL}(2,\mathbb{C})$ is a subgroup $\Pi \subset \rho(G)$ such that

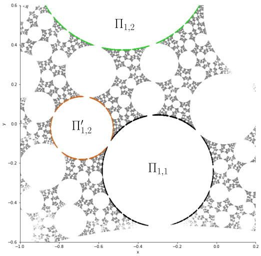
1. Π is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$
2. Π is the stabiliser of a disc $\Delta \subset \Omega(G)$ so that $\Delta/G = \Delta/\Pi$.

Maximal system of peripheral subgroups is called *circle chain*. The set of reps with a circle chain compatible with a fixed system of curves is called a*pleating variety*.

Pleating varieties are semi-algebraic and locally closed in the stability region.



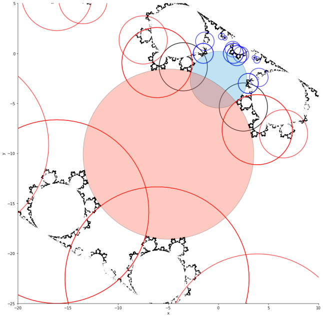
system of curves $\implies$
words in π_1(surface) $\implies$
subgroups gen by those words

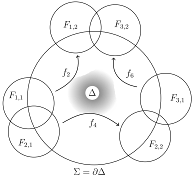


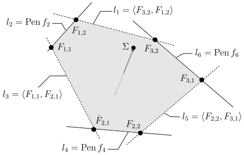
Thickening pleating varieties

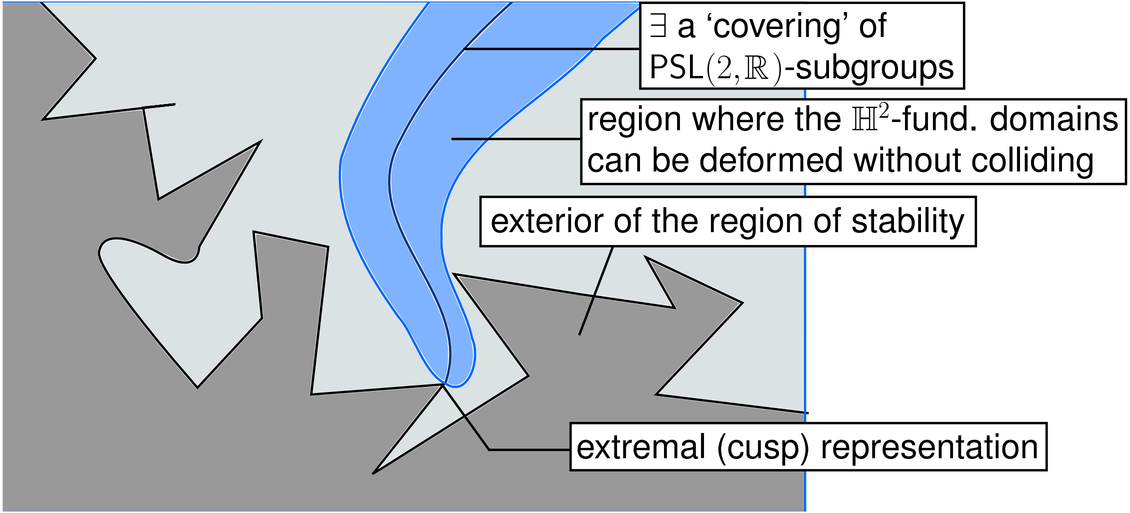
- If ρ is on a pleating variety, then the F-peripheral subgroups induce a 'canonical' fundamental domain for $\rho(G)$.
- This domain is stable under small perturbations of the generators of $\rho(G)$. we can write down semi-algebraic conditions on the pertubations that guarantee stability.
- Method: convert the 'canonical' fundamental domains into incidence structures in $\mathbb{P}^3\mathbb{R}$. Rewrite the action on $\mathbb{P}^1\mathbb{C}$ into one on $\mathbb{P}^3\mathbb{R}$, then work with geometric inequalities.

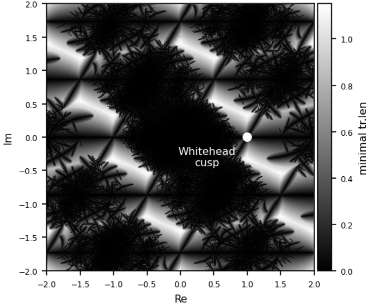
In reality, these conditions are hard to compute even though the proof is fully constructive.








∃ a 'covering' of
PSL(2,ℝ)-subgroups
region where the ℍ²-fund. domains
can be deformed without colliding
exterior of the region of stability
extremal (cusp) representation



- E., *From disc patterns in the plane to character varieties of knot groups.* `arXiv:2503.13829 [math.GT]`
- E., *Peripheral subgroups of Kleinian groups.* `arXiv:2508.00297 [math.GT]`
- Katsuhiko Matsuzaki and Masahiko Taniguchi, *Hyperbolic manifolds and Kleinian groups.* Oxford, 1998.