Cone manifolds and combination theorems

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Topology Seminar
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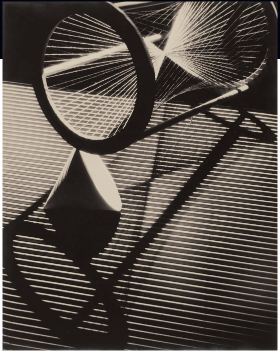


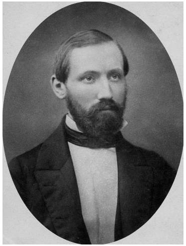
György Kepes
Untitled (String Structure, Lines, Circles) (1940)

Multivalued functions are common in complex analysis:-

- $z \mapsto \sqrt{z}$ is 2-valued.
- $z \mapsto \log z$ is infinitely valued.

If you walk continuously in a circle around 0, the value changes based on how many times you walk around.

Bernhard Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (1851): **multi-valued functions are projections of single-valued functions on a surface**.



Bernhard Riemann, 1850

Riemannstr., Leipzig Zentrum-Süd, own photo 2022

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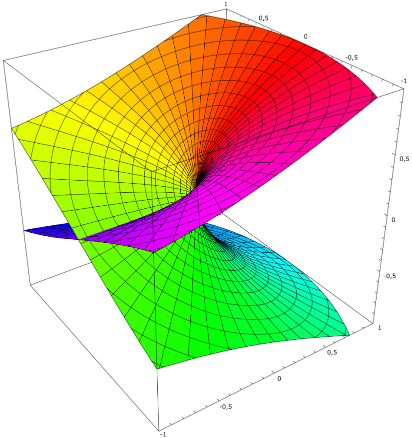

surface of $\sqrt{z}$

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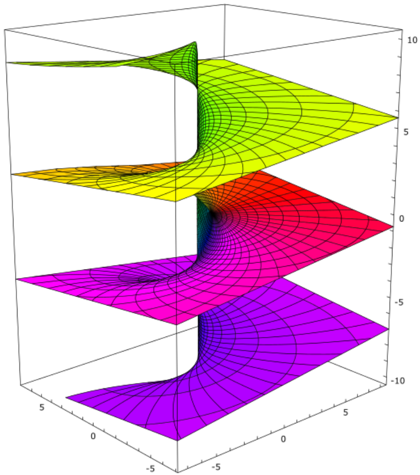


surface of $\log z$

Problem

For any Riemann surface S, construct an analytic *single-valued* surjection $\mathbb{C} \to S$.

Theorem (Koebe–Poincare uniformisation)

If S is a surface with $\chi(S) < 0$, then there exists a holomorphic function from the unit disc Δ onto S.

These functions have a lot of symmetry. If $f : \Delta \to S$ is one of these functions, then its symmetry group (deck transformation group) is called a **Fuchsian group**.

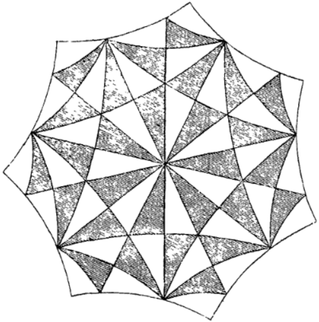
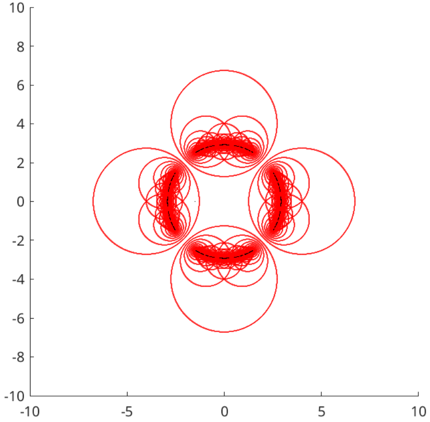
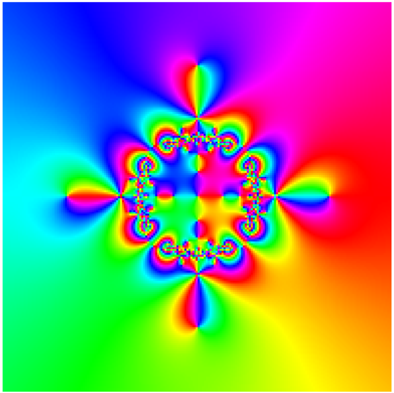


Fig. 2.

F. Klein, "Ueber den arithmetischen Charakter der zu den Verzweigungen $(2, 3, 7)$ und $(2, 4, 7)$ gehörenden Dreiecksfunctionen", *Math. Ann.* 41 (1892). $(2, 3, 7)$-triangle group tiling of the unit disc.

Schottky genus two group: R = 2.728427e+00





- $\mathbb{H}^2 = \mathbb{B}^2$:-
  - Conformal automorphism group: $\mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{SO}(2, 1)$
  - A discrete group $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ is called **Fuchsian**.
  - Any hyperbolic surface is of the form $\mathbb{H}^2/\Gamma$ for some Fuchsian group.
  - $\partial\mathbb{H}^2 = \mathbb{S}^1$ is not very interesting.
- $\mathbb{H}^3 = \mathbb{B}^3$:-
  - Conformal automorphism group: $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3, 1)$
  - A discrete group $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is called **Kleinian**.
  - Any hyperbolic 3-fold is of the form $\mathbb{H}^2/\Gamma$ for some Kleinian group.
  - $\partial\mathbb{H}^3 = \mathbb{S}^2$ is the Riemann sphere. Γ acts here conformally.

Theorem (Bers, Maskit)

Given any countable collection $\{S_i\}$ of Riemann surfaces with $\chi(S_i) < 0$, you can find a single Kleinian group Γ that uniformises all of those surfaces at once: there is a collection of disjoint open sets $\{U_i \subset \mathbb{C}\}$ such that $U_i/\Gamma = S_i$ for all i.

Intermezzo

Henri Poincaré
1887

Felix Klein
in Leipzig, ca.1880–1886

H. Poincaré (in Paris), letter to F. Klein. 8 Dec. 1881

…I am infinitely grateful for the obliging offer you make me and am fully prepared to avail myself of it. I shall shortly send you the letter you request; I would ask you, however, how much room you are prepared to devote to it in the *Annales* [Klein's journal]. I know that your journal's clientele is numerous and the space you can allow each article is necessarily limited and I would not wish to abuse your benevolence…

(trans. H.P. de Saint-Gervais, p. 399)

F. Klein (in Leipzig), letter to H. Poincaré. 13 Jan. 1882

...I have not yet thanked you personally for sending your article...it will go to press in a few days...

...Would you, in particular, examine the short commentary that I have...appended to your article, and in which I protest, as strongly as I can, against the two names *Fuchsian* and *Kleinian*, citing Schottky with respect to the latter and, incidentally, pointing to Riemann as the one who initiated all these investigations?...

...As far as the *proofs* are concerned, they are difficult. I always operate with Riemann's ideas respecting "geometria situs". This is very difficult to get clear...

(trans. H.P. de Saint-Gervais, p. 401)

H. Poincaré (in Paris), letter to F. Klein. 30 Mar. 1882

...Recently you were good enough to have pulished in the *Mathematische Annalen* an article of mine on single valued functions that replicate themselves under linear substitutions, with a note appended in which you explain why you find the names I have given to these transcendental functions unsuitable....

...If I believed I should bestow on these new functions the name of Mr Fuchs, it was not out of disregard for the value of your work...on the contrary I would be the first to appreciate its great importance. However it was impossible for me to ignore the remarkable discoveries published by the Heidelberg professor in *Crelle's Journal* ...

...As far as Kleinian functions are concerned...it was Mr. Schottky who discovered the figure you discuss in your letter, but it was you who *outlined their fundamental importance*...

(trans. H.P. de Saint-Gervais, p. 403)

F. Klein (in Leipzig), letter to H. Poincaré. 3 Apr. 1882

…If I had to say two words concerning [Fuchs' note], they would be the effect that I judge it to be completely beside the point. I claim only that Fuchs has never published anything on "Fuchsian functions"…[his article] may be considered to be concerned with "Fuchsian functions" insofar that it deals with modular functions, but, lacking geometric intuition, Fuchs has not correctly recognised the proper character of the latter…

…The situation is precisely the reverse of what Fuchs claims. It was not that I took his ideas, but rather that I showed that his topic should be treated using my ideas…

…I should add that on my part I have no intention of prolonging our *terminological* disagreement…Let us rather compete to see which of us is best equipped to advance the theory in question! …

(trans. H.P. de Saint-Gervais, p. 405)

H. Poincaré (in Paris), letter to F. Klein. 4 Apr. 1882

...I have just received your letter...You say you wish to cease our sterile debate ...I can only congratulate you on your resolution...

...If I named Kleinian functions after you, it was for the reasons I gave and not, as you insinuate, *by way of compensation*, since there is nothing to compensate you for; I will only recognize a property right prior to mine when you can show me that someone had earlier investigated the discontinuity of the groups and the single-valuedness of the functions in even just a slightly general case...

...In any case it would be ridiculous for us to continue falling out over a name. *Name ist Schall und Rauch*...

(trans. H.P. de Saint-Gervais, p. 406)

Koebe and Maskit's function groups

Paul Koebe
in Jena or Weimar, 1920 or 1921.
The Pólya picture album, ed. G.L. Alexanderson (1987), p.53.

Bernard Maskit and Lipman Bers
"Bernard Maskit Memorial Tribute", *Not. AMS*, August 2025

Koebe was so candid that he never concealed that he was a famous man. "It is the talk of all Europe: Koebe mailing reprints." In hotels he never registered as Koebe. He travelled incognito because he could not stand waiters and chambermaids asking him whether he was a relative of the famous function theoretician. Among colleagues he was given the nickname of "the greatest Luckenwalde function theoretician." Younger people, when introduced to Koebe, almost automatically reacted: "Ah, the great function theoretician." This earned one of my friends an assistantship with Koebe.

H. Freudenthal, "A Bit of Gossip: Koebe", Math. Intell. 6 (1984)

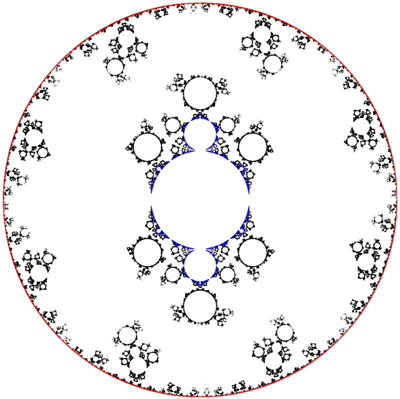
Definition

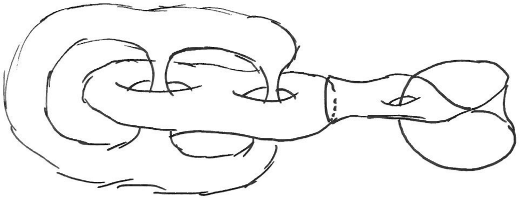
A **function group** is a Kleinian group G such that there exists a non-empty, open, connected set U on the Riemann sphere with the properties

1. U is G-invariant: $G(U) \subset U$.
2. G acts discontinuously on U: If $x \in U$ then there exists a compact $K \subset U$ with $x \in \operatorname{Int} K$ such that $g(K) \cap K$ is empty for all but finitely many $g \in G$.

Such groups were first constructed by Koebe (*Annalen*, 1912). The name comes from the idea that they provide an alternate kind of uniformisation of Riemann surfaces, i.e. they define 'functions' from a connected planar set U onto a Riemann surface U/G. They were fully classified by Maskit (*Acta*, 1977).

An unnervingly complicated example





$$\omega = \sqrt{-4i\sqrt{2\sqrt{2}-2} + 2\sqrt{2} - 6}$$

$$M = \begin{bmatrix} 1+\sqrt{2} & \omega \\ \omega^* & 1+\sqrt{2} \end{bmatrix} \quad R = \begin{bmatrix} \exp(\pi i/8) & 0 \\ 0 & \exp(-\pi i/8) \end{bmatrix}$$

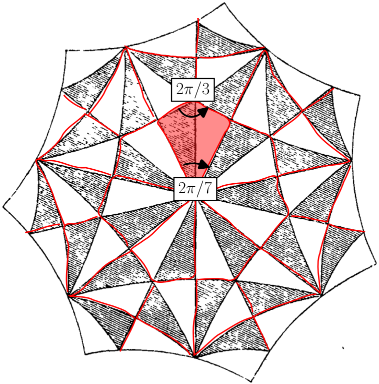
$$N_1 = \begin{bmatrix} 1+i & 1/4 \\ 4 & 1-i \end{bmatrix} \quad N_2 = \begin{bmatrix} 1-i & 1/4 \\ 4 & 1+i \end{bmatrix} \quad T = \begin{bmatrix} 2 & -3i/4 \\ 4i & 2 \end{bmatrix}$$

$$G = \langle {\color{red}M, RMR^{-1}, R^2MR^{-2}, R^3MR^{-3}}, {\color{blue}N_1, N_2, T} \rangle$$

Definition

Let T be a triangle in $\mathbb{H}^2$ with angles π/p, π/q, π/r. Embed $\mathbb{H}^2$ as the upper halfspace of the Riemann sphere, and let G be the orientation-preserving half of the group generated by the reflections in the sides of T.

Then G is called the (p,q,r)-**triangle group.**



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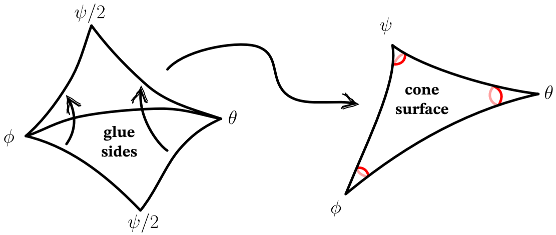
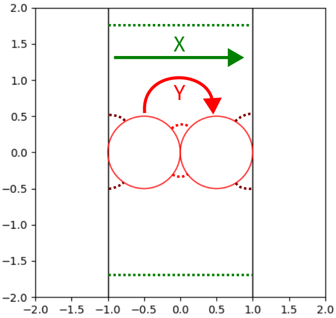


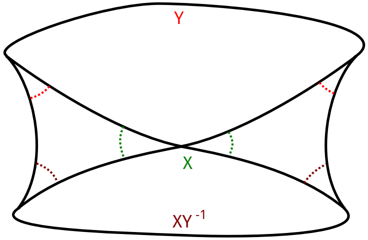
If p,q,r are integers, then G is discrete and $\mathbb{H}^2/G$ is an orbifold surface.

(∞, ∞, ∞)-TRIANGLE GROUPS

The traditional matrix representation for the (∞, ∞, ∞)-triangle group is

$$G = \left\langle X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle.$$





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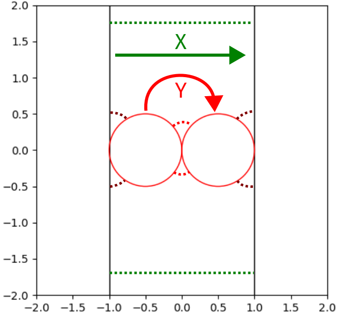
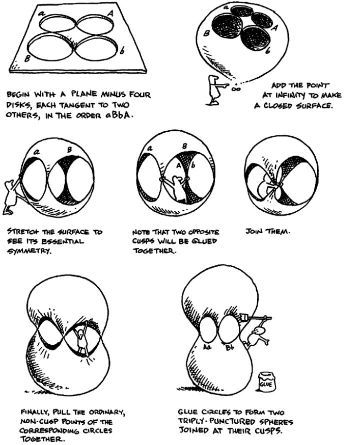


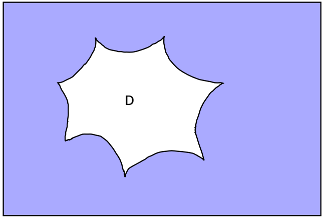


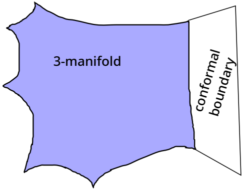
Cartoon by Larry Gonick, included in Mumford, Series, Wright. *Indra's Pearls.* p.216

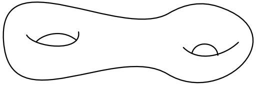
A four-punctured sphere group

Theorem (Klein, *Annalen* 1882)

If G and G' are Kleinian groups with (open) fundamental domains D and D' such that $\hat{\mathbb{C}} \setminus D \subset D'$ and $\hat{\mathbb{C}} \setminus D' \subset D$, then $G * G'$ is Kleinian with fundamental domain $D \cap D'$.







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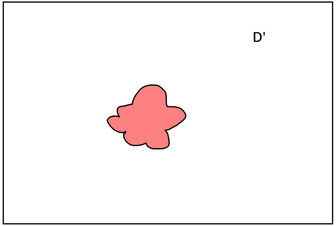

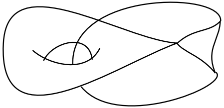

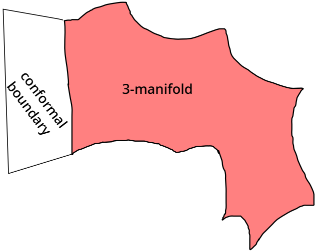

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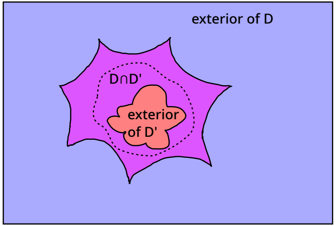
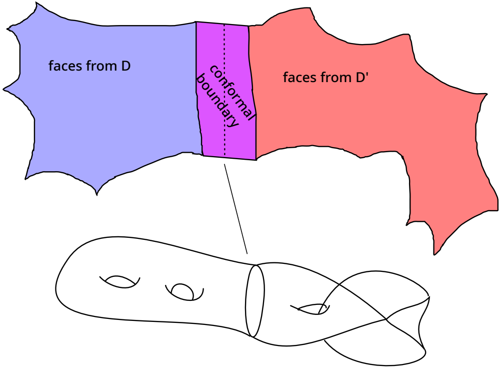


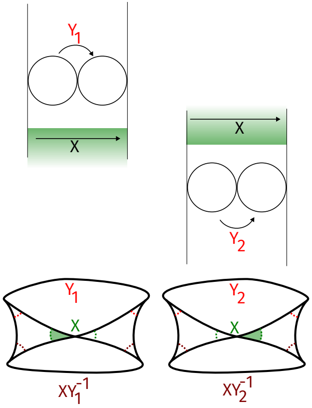


A FOUR-PUNCTURED SPHERE GROUP

Theorem (Maskit, *TAMS* 1965 & 1968)

- *If $G > H < G'$ are Kleinian,*
- *D & D' are fundamental polyhedra for G & G', and*
- *regions of D & D' with faces paired only by H can be sliced off so that the remaining pieces match up perfectly,*
- *then $\langle G, G' \rangle = G *_H G'$ is Kleinian with fundamental domain $D \cap D'$.*

(Maskit's result is more general.)



$\Omega(G)/G$ $\qquad$ $\Omega(G')/G'$

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 1+2i & 2 \\ 2 & 1-2i \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 1-2i & 2 \\ 2 & 1+2i \end{bmatrix}$$

$$G = \langle X, Y_1 \rangle$$
$$G' = \langle X, Y_2 \rangle$$
$$H = \langle X \rangle$$

A FOUR-PUNCTURED SPHERE GROUP

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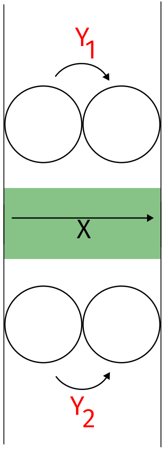
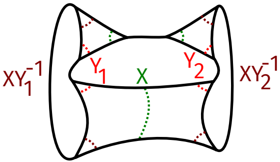


$\Omega(G *_H G')/(G *_H G')$



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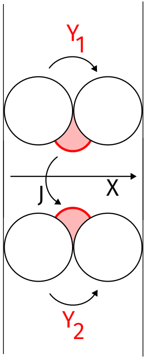
$G *_H G' = \langle X, Y_1, Y_2 \rangle$

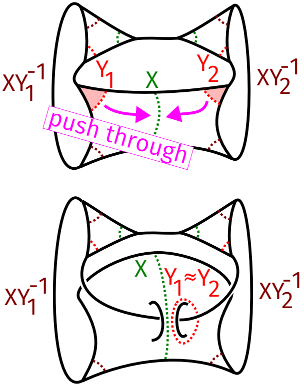
Theorem (Maskit, *TAMS* 1965 & 1968)

If G is Kleinian with fundamental polyhedron D, and if there are two hyperplane slices through D cutting off pieces that do not have faces paired to other places in D, leaving some remaining piece $D' \subset D$, and if $J \in \mathrm{PSL}(2, \mathbb{C})$ sends one of these faces to the other with $J(D') \cap D' = \varnothing$, then $\langle G, J \rangle = G *_J$ is Kleinian with fundamental domain D'.

(Maskit's result is more general.)



$$J = \begin{bmatrix} 3 & -2i \\ 4i & 3 \end{bmatrix}$$



$$(G *_H G')*_J = \langle X, Y_1, Y_2, J \rangle$$

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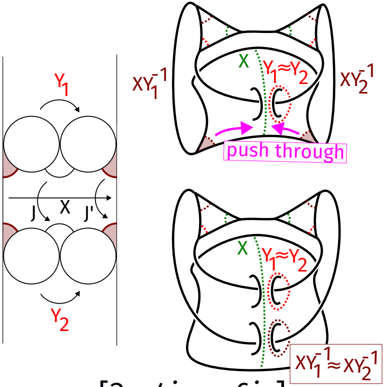


$$J' = \begin{bmatrix} 3+4i & -6i \\ 4i & 3-4i \end{bmatrix}$$

$$((G *_H G')*_J)*_{J'} = \langle X, Y_1, Y_2, J, J' \rangle$$

Why I do not try to give formal statements of the Maskit combination theorems…

E.5. Theorem. *Let J_1 and J_2 be geometrically finite subgroups of the discrete group G_0, and let $G_1 = \langle f \rangle$ be infinite cyclic. Assume that B_1 and B_2 are jointly f-blocked closed topological discs, and that $A_0 \neq \varnothing$. Let D_0 be a maximal fundamental set for G_0. Set $G = \langle G_0, f \rangle$, and set $D = D_0 \cap (A \cup W_1)$, where $W_m = \partial B_m$. Then the following statements hold.*

(i) $G = G_0 *_f$.

(ii) *G is discrete.*

(iii) *If (B_1, B_2) is precisely invariant under (J_1, J_2) in G_0, then every non-loxodromic element of G is conjugate to an element of G_0.*

(iv) *W_1 is a precisely embedded (J_1, G)-block; if B_1 and B_2 are both strong G_0-blocks, then W_1 is a strong G-block.*

(v) *If $\{W_k'\}$ is a sequence of distinct G-translates of W_1, then* $\mathrm{dia}(W_k') \to 0$.

(vi) *There is a sequence of distinct translates of W_1 nesting about the point x if and only if x is a limit point of G, and x is not a translate of a limit point of G_0.*

(vii) *If B_1 and B_2 are both strong, and x is a limit point of G which is not G-equivalent to a limit point of G_0, then x is a point of approximation.*

(viii) *D is a fundamental set for G. If D_0 is constrained, W_1 and W_2 intersect ∂D_0 in a finite set of points, and if there is a constrained fundamental set E_m for J_m so that, except perhaps for some excluded cusps, D_0 and E_m agree near W_m, then D is constrained.*

(ix) *A_0 is precisely invariant under G_0 in G. Let $Q = \bar{A}_0 \cap \Omega(G_0)$; then $\Omega(G)/G = Q/G_0$, where the two possibly disconnected and possibly empty boundaries, $(W_1 \cap \Omega(G_0))/J_1 = (W_1 \cap \Omega(J_1))/J_1$ and $(W_2 \cap \Omega(G_0))/J_2 = (W_2 \cap \Omega(J_2))/J_2$ are identified; the identification is that given by f (that is, if $x \in W_1$, then p(x) is identified with $p \circ f(x)$).*

(x) *G is geometrically finite if and only if G_0 is geometrically finite.*

(xi) *Assume that W_1 and W_2 are both strong. Let C_m be a spanning disc for W_m where $C_2 = f(C_1)$. Let B_m^3 be the topological half-space cut out of $\mathbb{H}^3$ by C_m, where B_m^3 spans B_m. Then $\mathbb{H}^3/G$ can be realized as $\mathbb{H}^3/G_0$, where the images of B_1^3/J_1 and B_2^3/J_2 have been deleted, and the two resulting boundaries, C_1/J_1 and C_2/J_2, are identified; the identification being given by f.*

This is just the statement of the second combination theorem!

I cut out the technical machinery like the definition of "jointly f-blocked pair" which in turn depends on knowing what a "(J_m, G_0)-block" is.

B. Maskit. *Kleinian groups.* pp. 161–162.

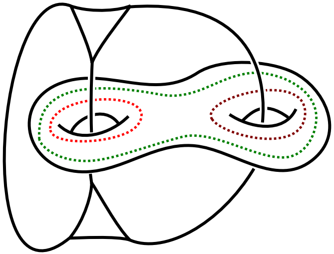
We have followed Koebe's 1912 algorithm to prove:

Lemma (E., Lemma 1 of 2411.17940)

The group

$$B = \left\langle X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, Y_1 = \begin{bmatrix} 1+2i & 2 \\ 2 & 1-2i \end{bmatrix}, J = \begin{bmatrix} 3 & -2i \\ 4i & 3 \end{bmatrix}, J' = \begin{bmatrix} 3+4i & -6i \\ 4i & 3-4i \end{bmatrix} \right\rangle$$

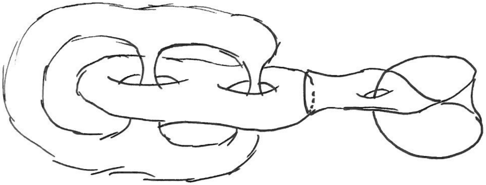
is a geometrically finite function group such that $M = \mathbb{H}^3/B$ is homeomorphic to $S_2 \times (-1,1)$, where one end of M is a compact surface of genus 2 and where the other end of M is a union of two thrice-punctured spheres.



Theorem (Maskit, *Acta* 1975 & 1977)

Function groups are equivalent to hyperbolic structures on compression bodies. Every function group is obtained from quasi-Fuchsian groups, elementary groups, and singly degenerate groups by a tower of amalgamated products and HNN-extensions, and this tower determines the group up to q.c. conjugacy.

It is not true that the distinguished component of the group projects down to the entire compression end:



The big technical problem is defining the necessary combinatorial structure to keep track of all parabolics and elliptics (the decomposition is *not* just cutting along compression discs since this keeps track only of topological information).

Deforming function groups

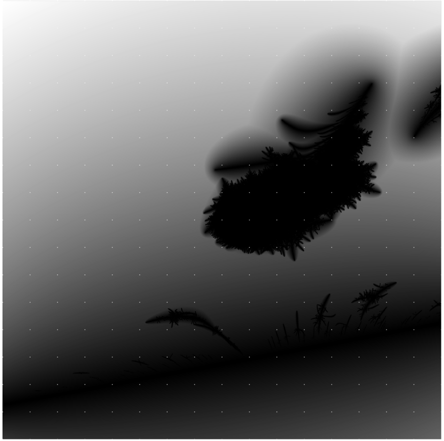
Deformation space

Given G define a deformation space

$$X(G) = \mathrm{Hom}(G, \mathrm{PSL}(2,\mathbb{C})) \mathbin{//} \mathrm{PSL}(2,\mathbb{C}).$$

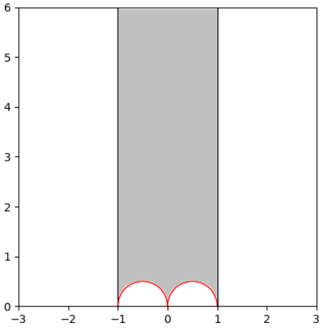
- Subsets of $X(G)$ where the reps are discrete are quotients of Teichmüller spaces & are embedded wildly.
- Outside them, the geometry of the reps changes chaotically.
- Working out how the Teichmüller spaces relate to each other is an important open question.

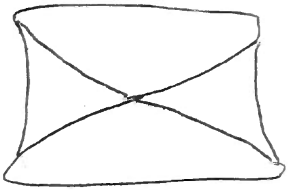
Compare with $\mathrm{PSL}(n,\mathbb{R})$ character varieties which have a rich theory of components studied by Hitchin, Labourie, Bonahon, Wienhard…



E., "From disc patterns in the plane to character varieties of knot groups" `arXiv:2503.13829 [math.GT]` & submitted

(∞, ∞, ∞)-triangle groups are rigid: there are no deformations of the conformal structure that they uniformise. But they do have nontrivial algebraic deformations by varying the angle of one cusp from $0 = 2\pi/\infty$ through $2\pi/n$ to $2\pi/1 = 2\pi$.





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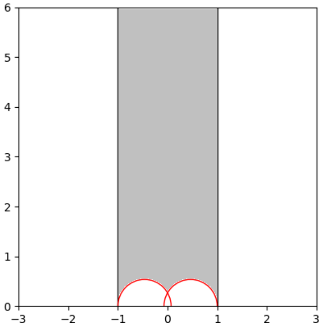

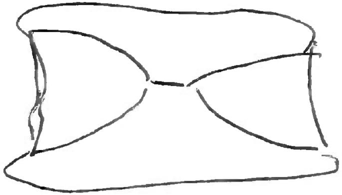

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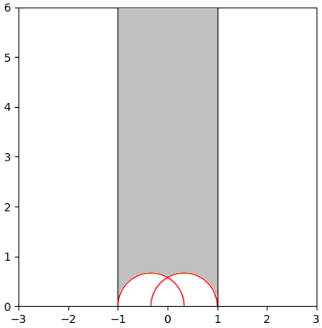

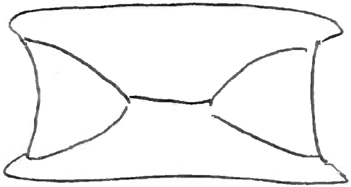

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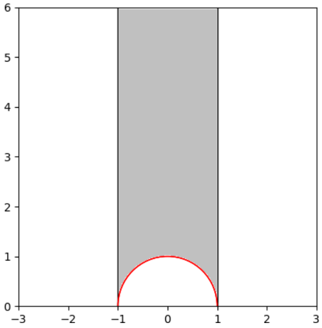

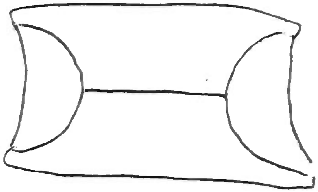

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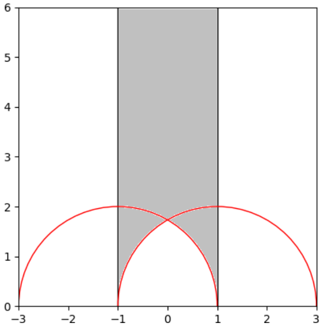

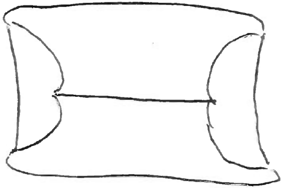

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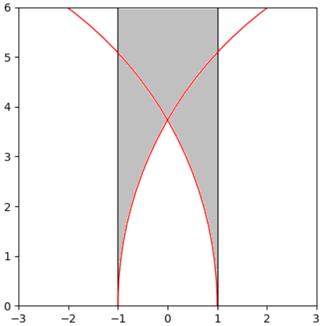

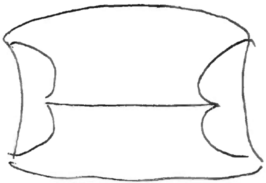

Cone-deforming triangle groups

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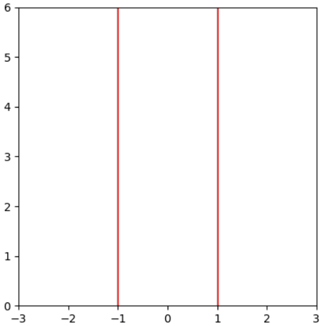
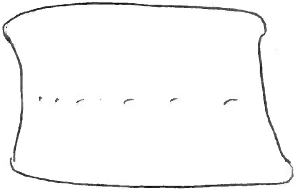
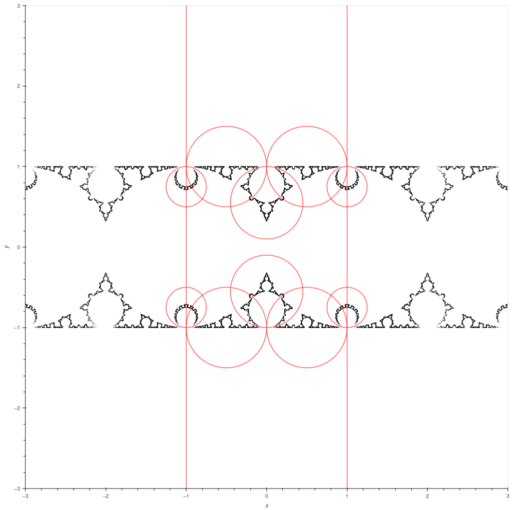
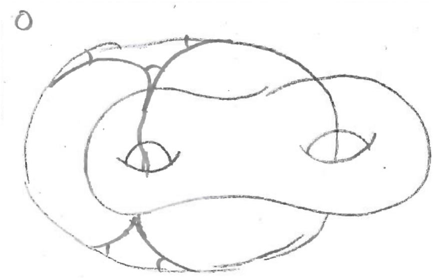




Cone-deforming function groups





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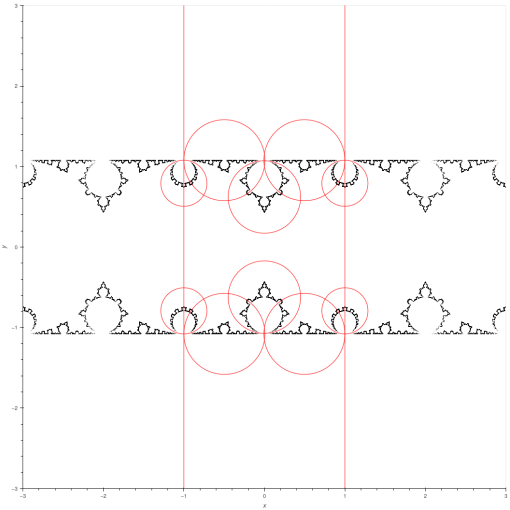

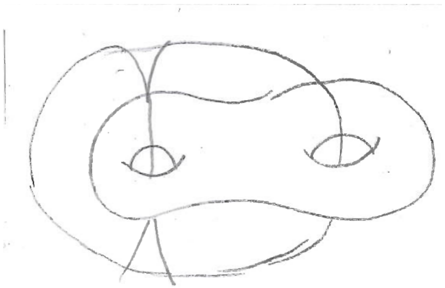

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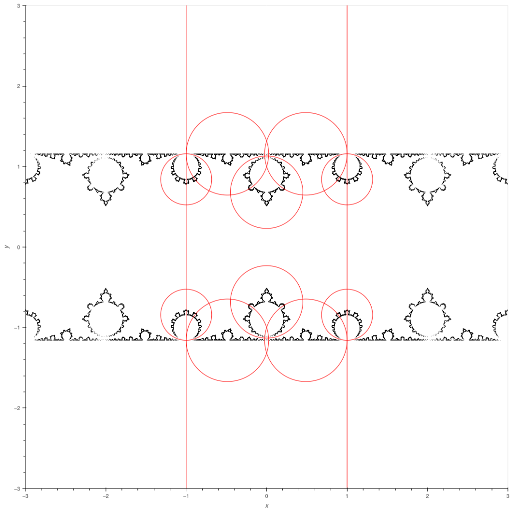

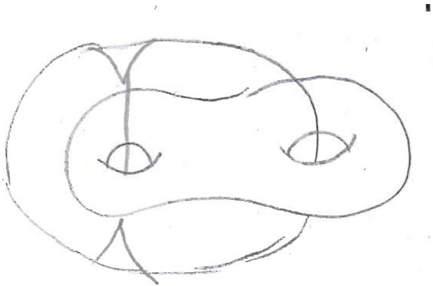

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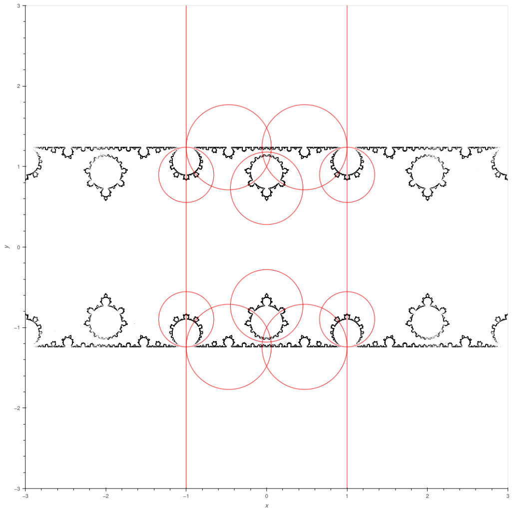

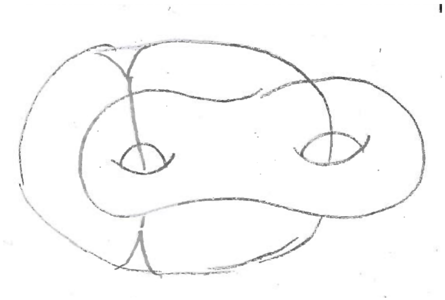

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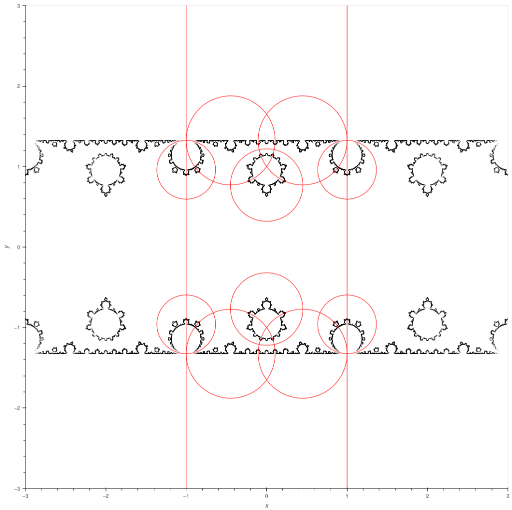

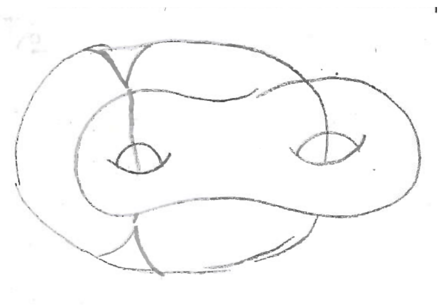

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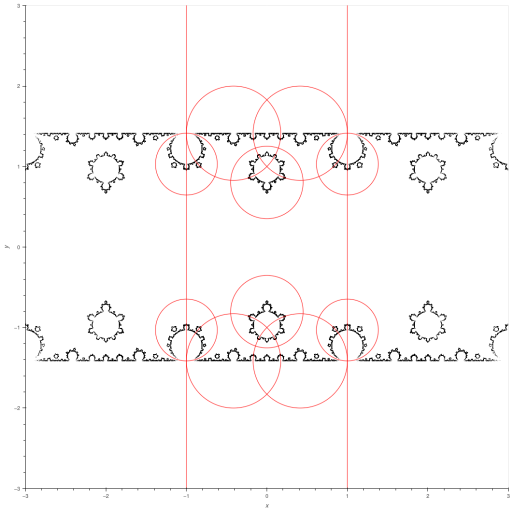

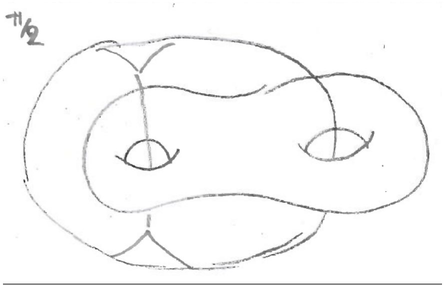

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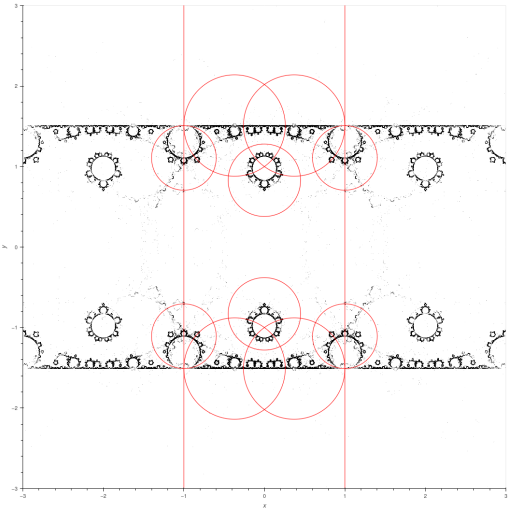

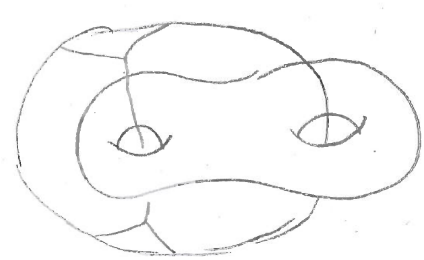

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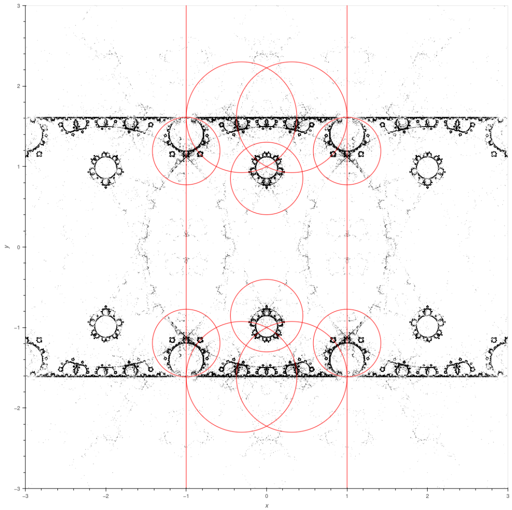

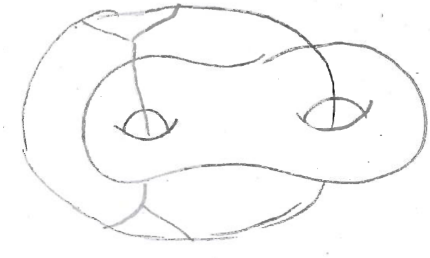

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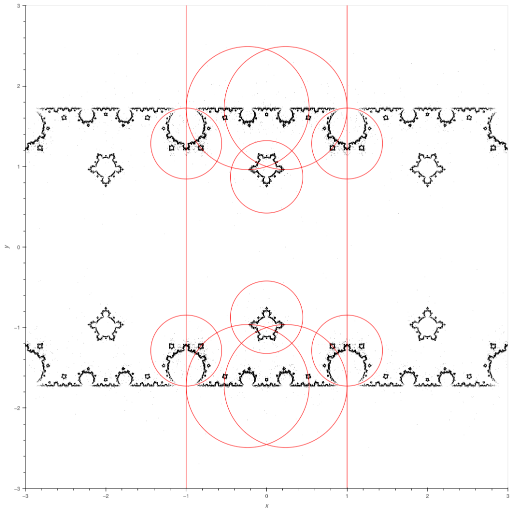

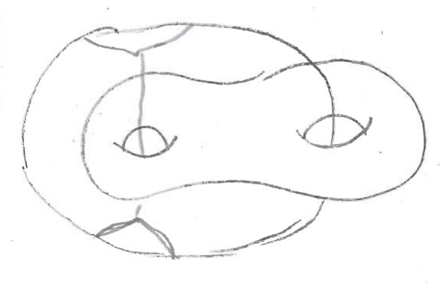

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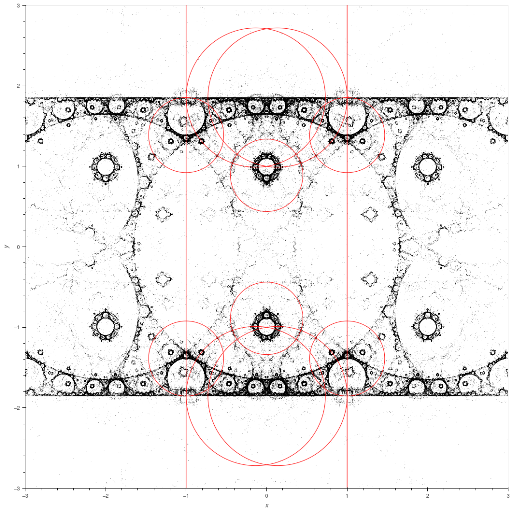

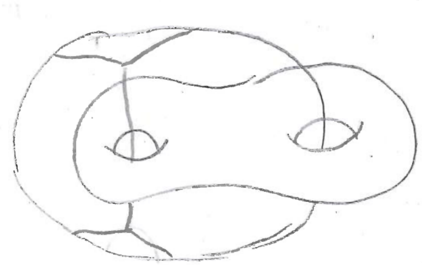

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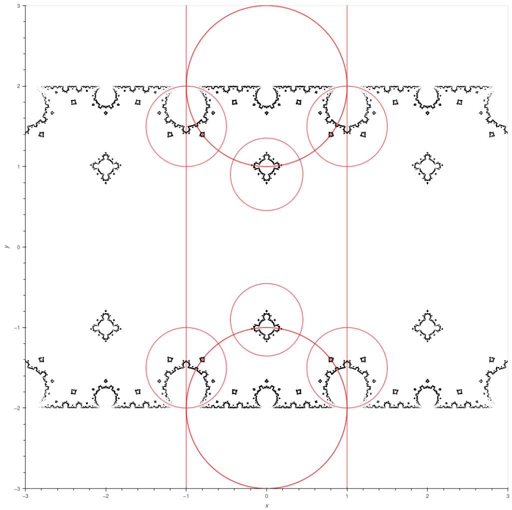

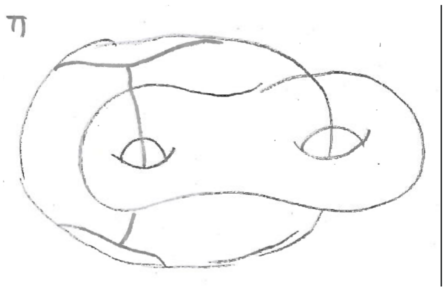

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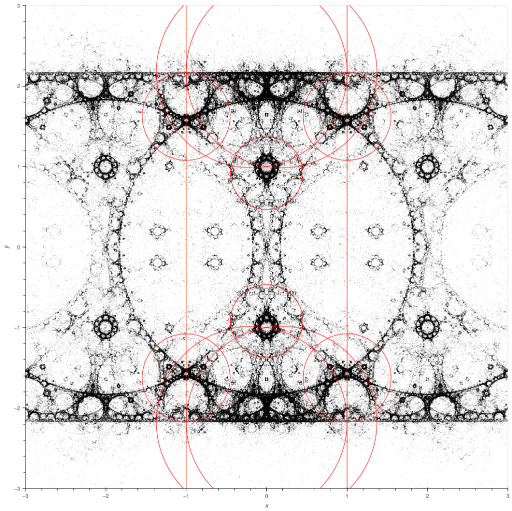

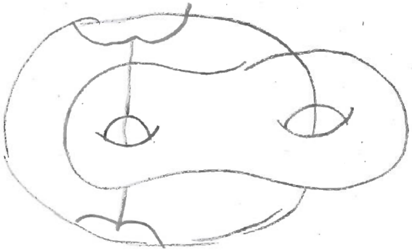

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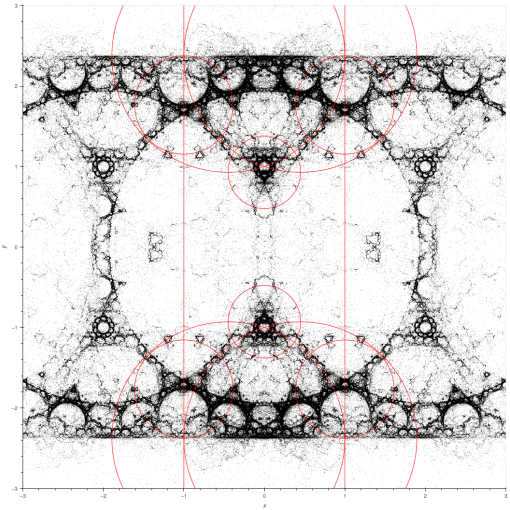

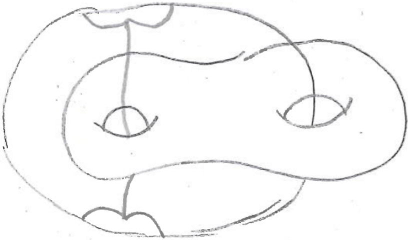

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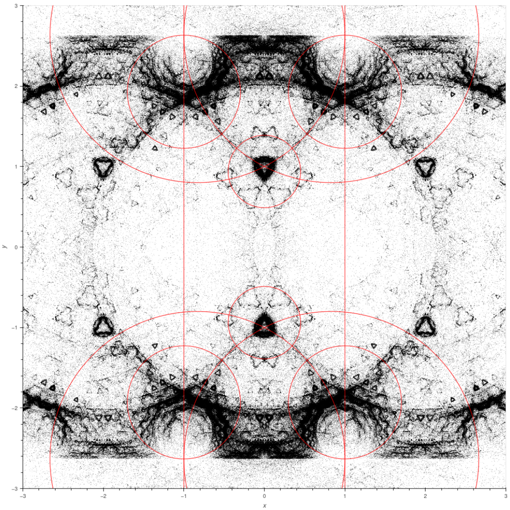

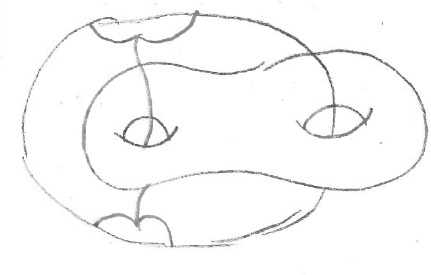

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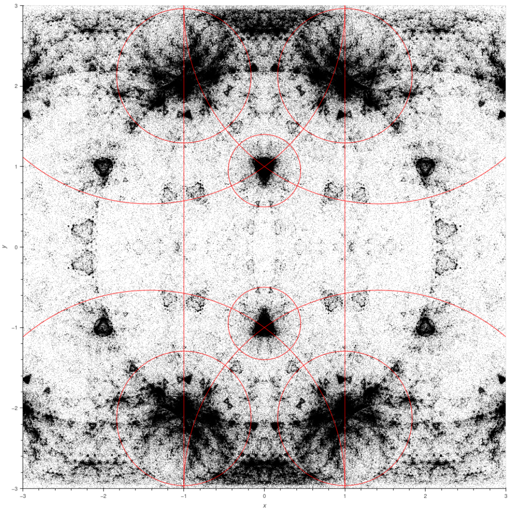

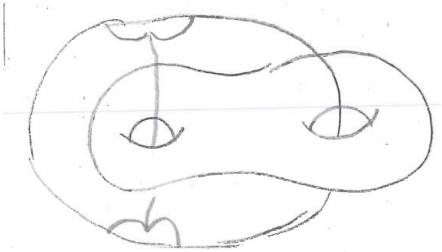

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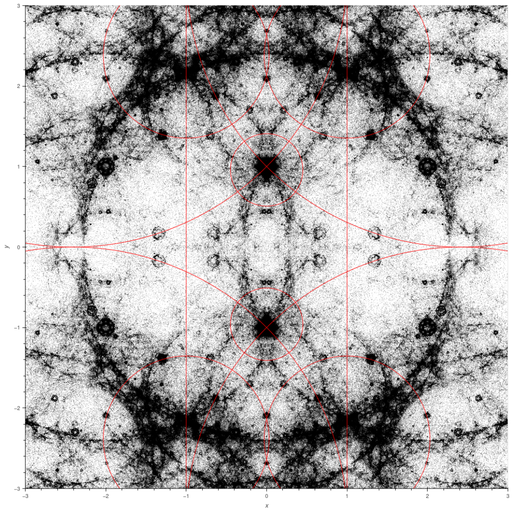

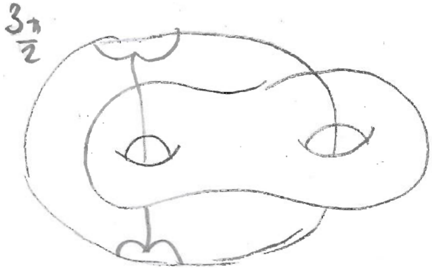

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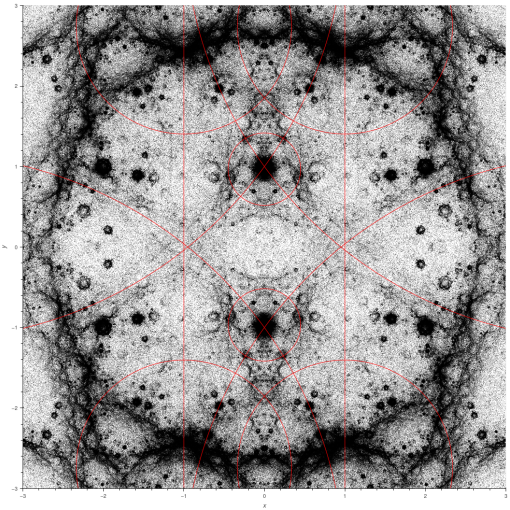

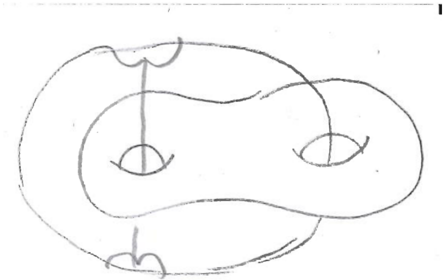

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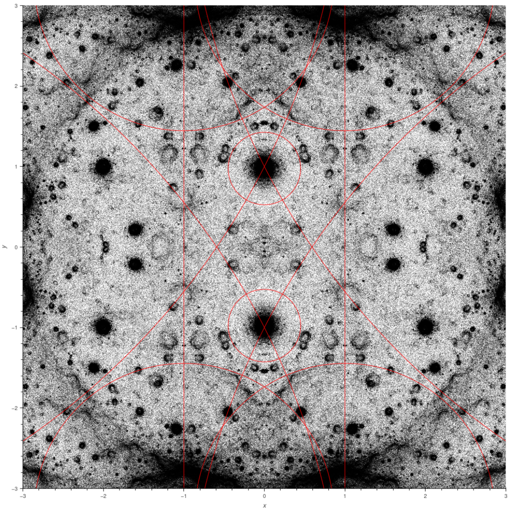

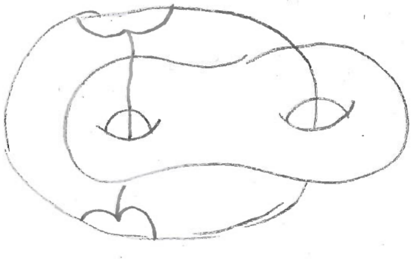

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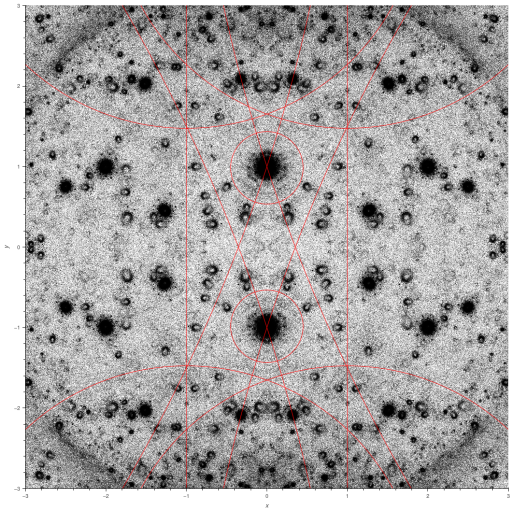

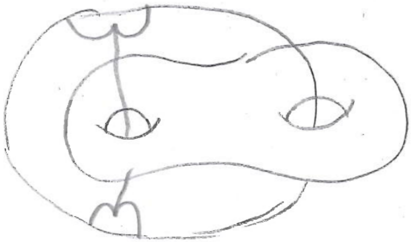

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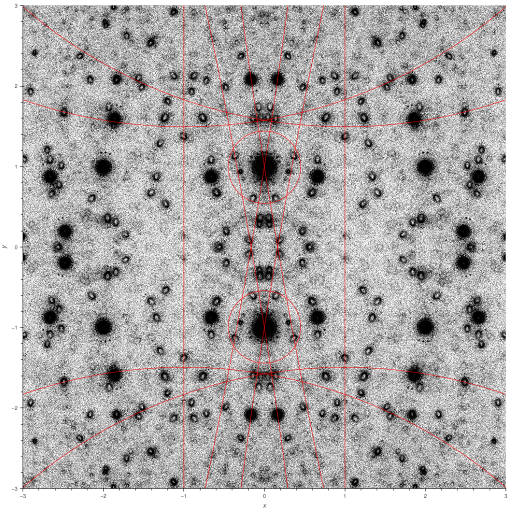
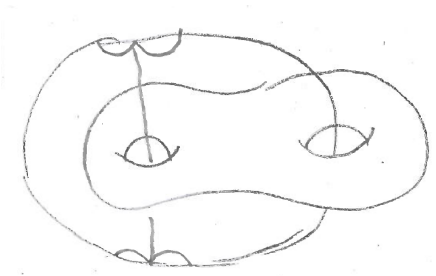
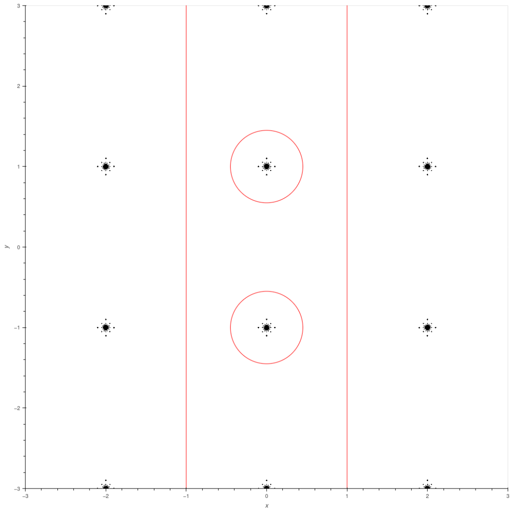
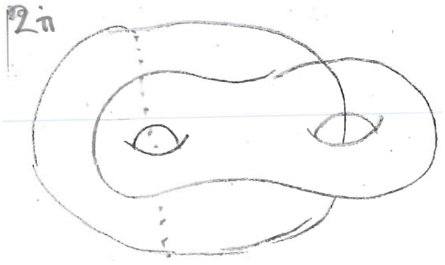


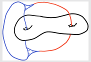


Cone-deforming function groups

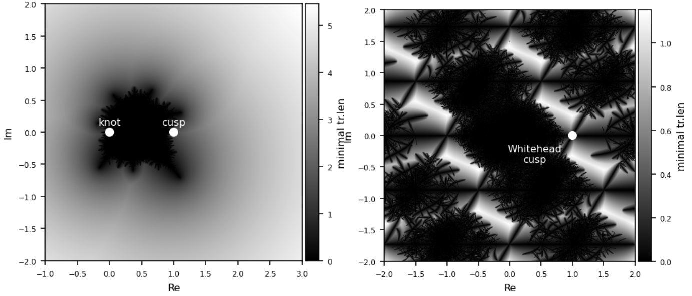




Theorem (E., 2411.17940)

There is a smooth path of cone manifolds inside the character variety $X(\pi_1(S_{2,0}))$ that joins the deformation space of hyperbolic metrics on  *to the deformation space of hyperbolic metrics on* .

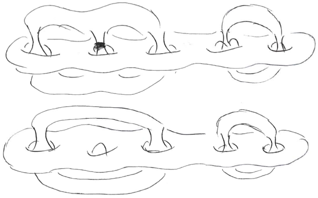
Two slices through $\mathrm{Hom}((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}, \mathrm{PSL}(2, \mathbb{C}))$. The 'cusp' point is the same group in each picture.



E., "From disc patterns in the plane to character varieties of knot groups" `arXiv:2503.13829 [math.GT]` & submitted

Theorem (E., 2411.17940)

For any pair of (topological) compression bodies with the same compression end that differ by exactly one handle, the procedure of obtaining from the other by gluing in a 2-handle can be realised by a continuous family of cone manifolds: there is a smooth path of cone manifold groups in the character variety that joins one embedded Teichmüller space to the other.



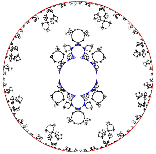
Proof.

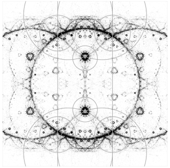
Realise the non-compression end by a chain of (∞, ∞, ∞)-triangle groups. The procedure from earlier (slicing off ends of polyhedra and regluing) is totally local & it does not matter how much flora exists away from the distingushed cone arc. 𓅓𓈙

Moral: there are still nontrivial results to be obtained from the planar theory of Kleinian groups, using only slight generalisations of the tools employed by Poincaré and Klein in the 1880s.

> *[In Maskit's book,] the approach is largely oriented toward developing the planar theory of Kleinian groups by exploring their action on the Riemann sphere … this approach may make it appear that Maskit's work is somewhat removed from the exciting work on geometrization and hyperbolic 3-manifolds that has occurred over the last 40 years, but this is mostly an issue of point of view, since these areas are either inextriably intertwined or in some cases identical.*
>
> *Anderson, Basmajian, Hidalgo, Susskind, and Taylor, "Bernard Maskit memorial tribute", Not. AMS, Aug. 2025.*

Bedtime reading





- E., *Changing topological type of compression bodies through cone manifolds.* `arXiv:2411.17940 [math.GT]`
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- Henri Poincaré, *Papers on Fuchsian functions* (trans. J. Stillwell). Springer, 1985.
- Albert Marden, “Kleinian groups and 3-dimensional topology”. *A crash course in Kleinian groups*, Springer LNM 400, 1974.
- Bernhard Maskit, *Kleinian groups.* Springer, 1987.
- Henri Paul de Saint-Gervais, *Uniformization of Riemann surfaces: Revisiting a hundred-year-old theorem.* Euro. Math. Soc., 2016.