Two-bridge knots, genus two surfaces, and discrete groups with two generators

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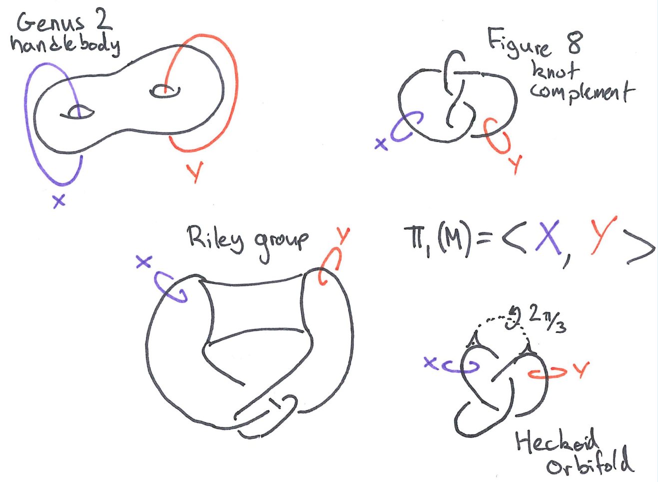
Monash University,
Melbourne

Hodgsonfest, Uni. Melbourne, 2024

What do all these 3-manifolds have in common?



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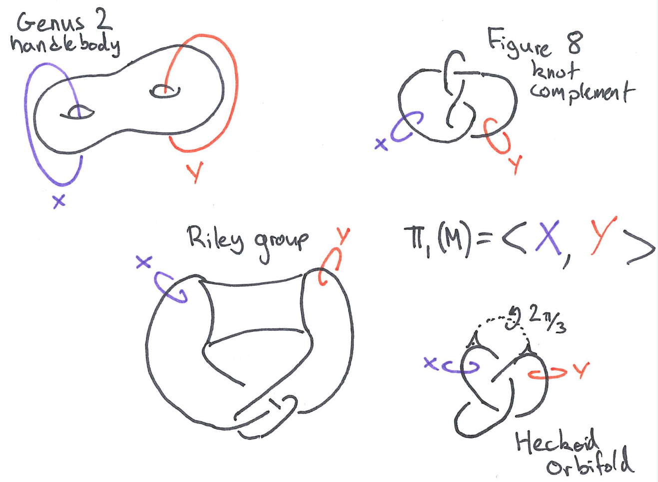
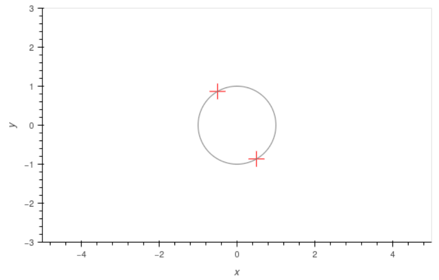
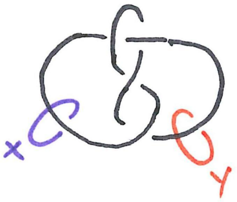


What do the holonomy groups of orbifolds with two parabolic or elliptic generators look like, and where do they live?

Warnings and disclaimers

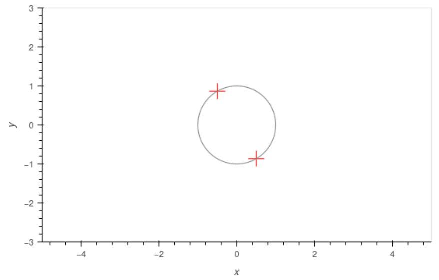
- I will focus here only on visual aspects of the work, so we will hand-wave a lot of technical detail. If you want exact statements of theorems, come to my talk in Auckland in December (or see endnotes for references).
- A lot of this work is joint with Gaven Martin and Jeroen Schillewaert, and was motivated by Martin's 20+ year programme to enumerate all the arithmetic groups on two generators in $\mathrm{PSL}(2, \mathbb{C})$.
- Many of the background or parabolic generator results are due to a large number of authors, too many to list here. The most influential authors for us were Robert Riley, Linda Keen, Caroline Series, R.C. Lyndon, J.L. Ullman, Hirotaka Akiyoshi, Hideki Miyachi, Ken'ichi Ohshika, Makoto Sakuma, Yasushi Yamashita, and Eric Chesebro, but this is very far from an exhaustive list.

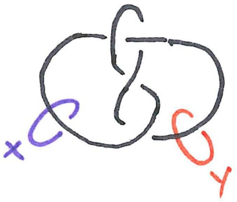




The holonomy group of the figure eight knot is

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle$$





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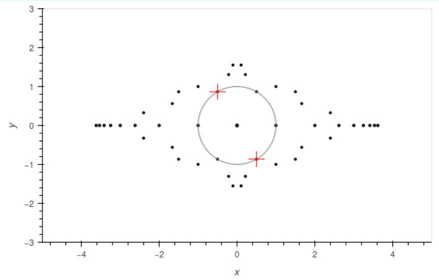
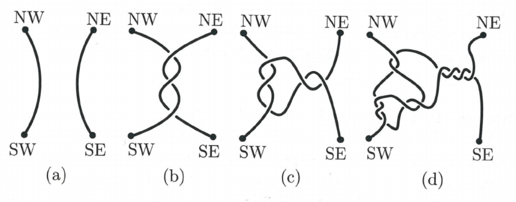
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle$$

$e^{2\pi i/3} \approx -0.5 + 0.866i$ — remember this number!



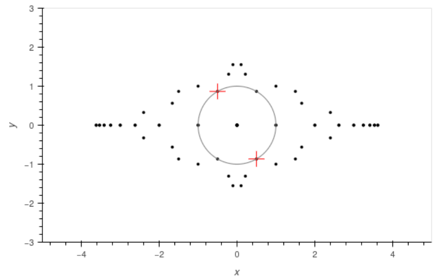


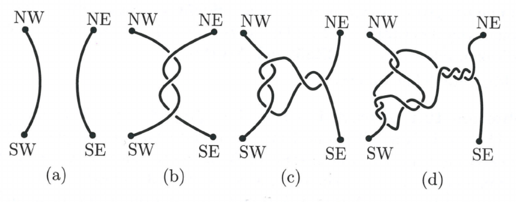
J. Purcell, *Hyperbolic knot theory*, Fig. 10.1.

Every hyperbolic 2-bridge link has a discrete holonomy group of the form

$$\Gamma_\rho = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$

for some $\rho \in \mathbb{C}$.



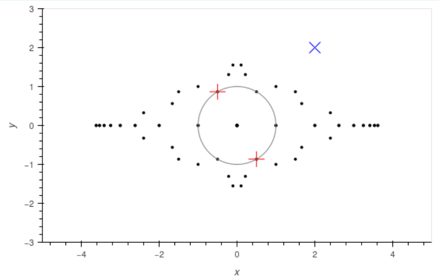


J. Purcell, *Hyperbolic knot theory*, Fig. 10.1.

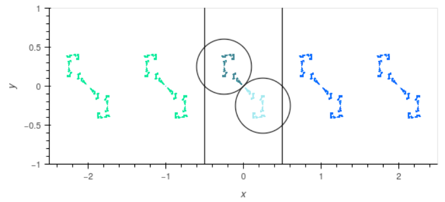
Every **non**-hyperbolic 2-bridge link's π_1 has a **non-faithful** representation of the form

$$\Gamma_\rho = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$

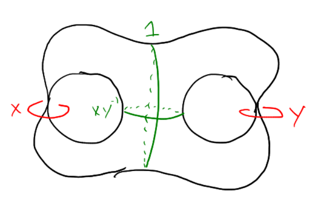
for some $\rho \in \mathbb{R}$.

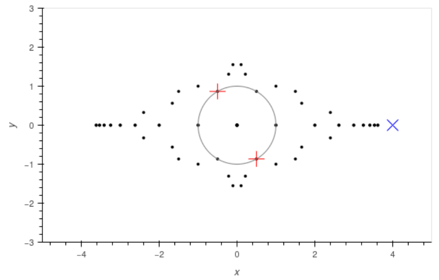


What if we make ρ bigger?

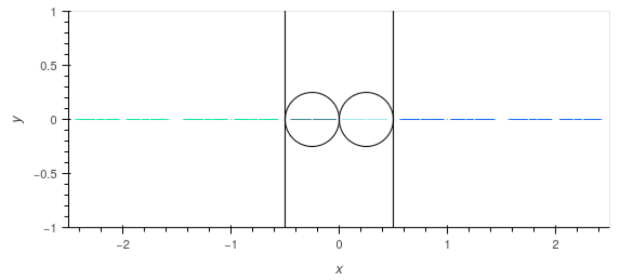


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2+2i & 1 \end{bmatrix} \right\rangle$$



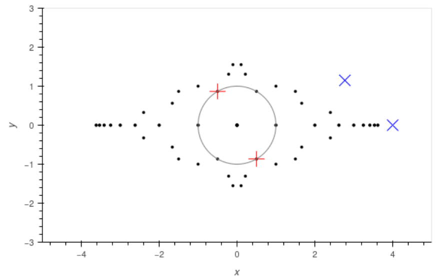


Let's try a bunch of values of ρ.



$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (4.0 + 0.0i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete (and Fuchsian). Quotient manifold is $S_{0,3} \times [0, 1]$.



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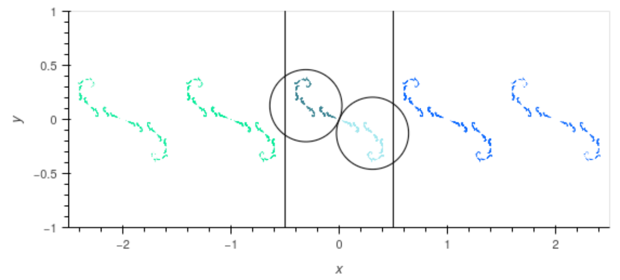
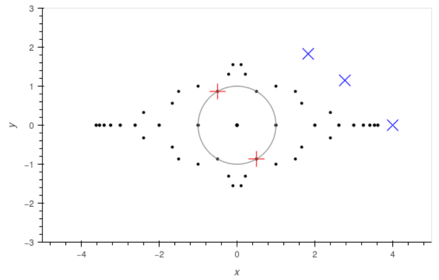


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (2.77164 + 1.14805i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete! Quotient manifold is a 3-ball minus two ideal arcs (boundary is $S_{0,4}$).



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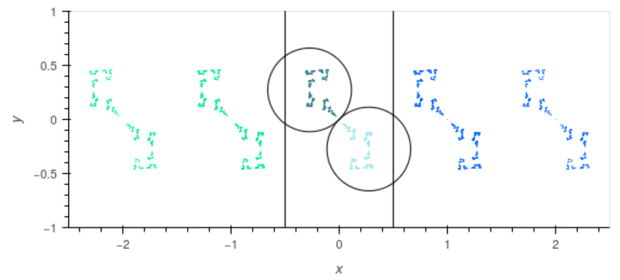
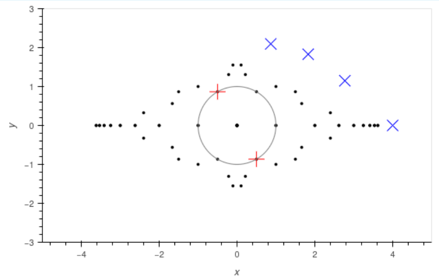


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ {\color{blue}(1.82843 + 1.82843i)} & 1 \end{bmatrix} \right\rangle$$

Definitely discrete! Quotient manifold is a 3-ball minus two ideal arcs (boundary is $S_{0,4}$).



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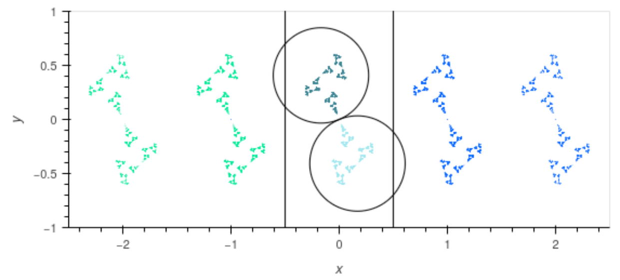
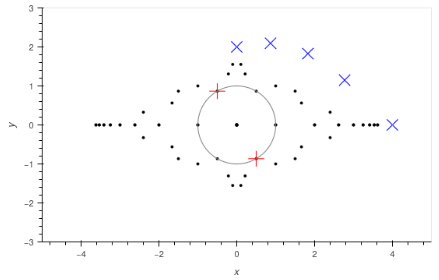


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.867907 + 2.09531i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete! Quotient manifold is a 3-ball minus two ideal arcs (boundary is $S_{0,4}$).



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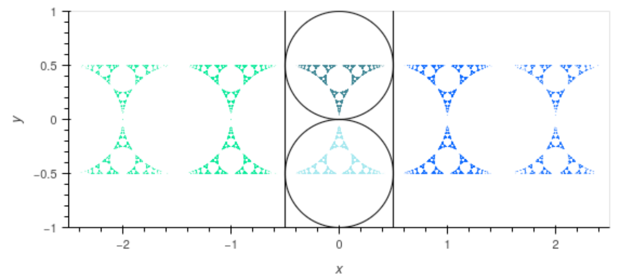
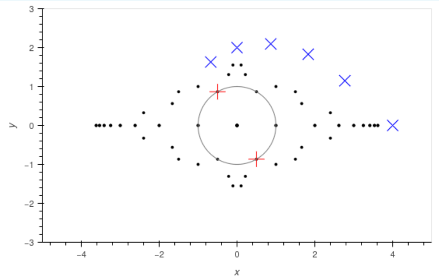


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 + 2.0i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete! Quotient manifold is $S_{0,3} \times [0, 1]$.



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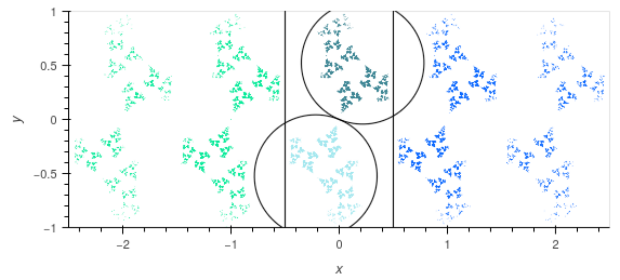
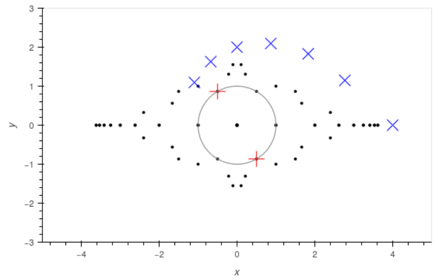


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.675028 + 1.62966i) & 1 \end{bmatrix} \right\rangle$$

Definitely discrete! Quotient manifold is a 3-ball minus two ideal arcs (boundary is $S_{0,4}$).



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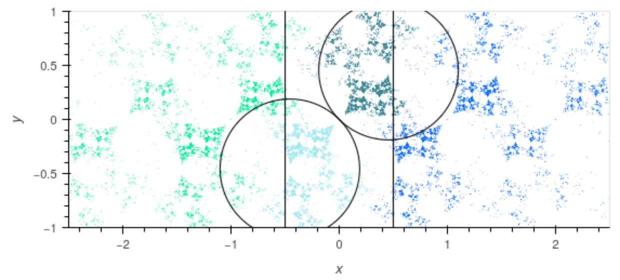
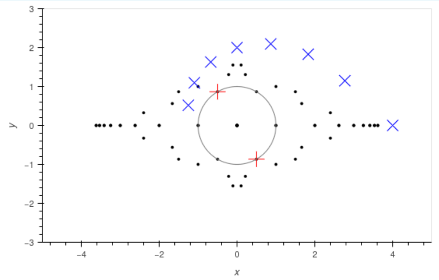


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.09638 + 1.09638i) & 1 \end{bmatrix} \right\rangle$$

Looks indiscrete.



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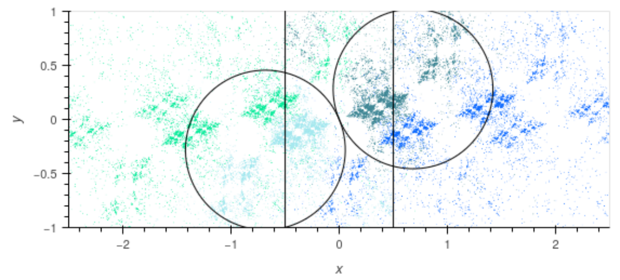
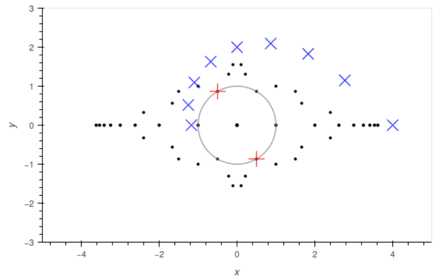


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.25116 + 0.518249i) & 1 \end{bmatrix} \right\rangle$$

Looks indiscrete.



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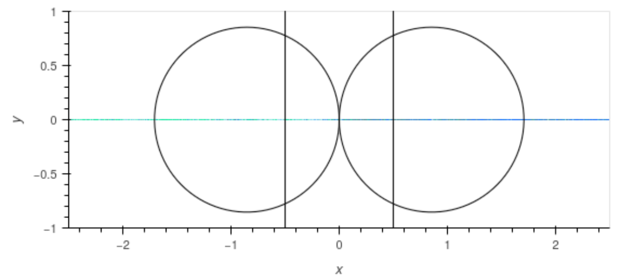
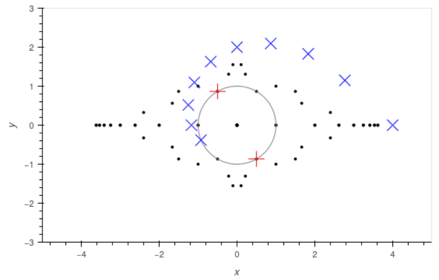


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.17157 + 0.0i) & 1 \end{bmatrix} \right\rangle$$

Almost definitely indiscrete.



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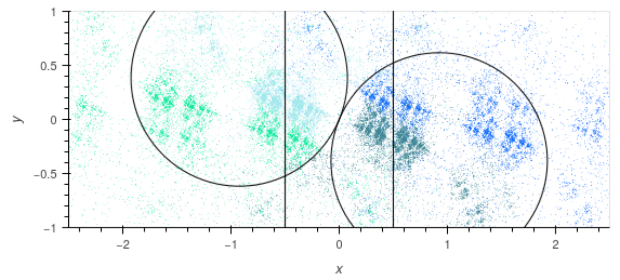
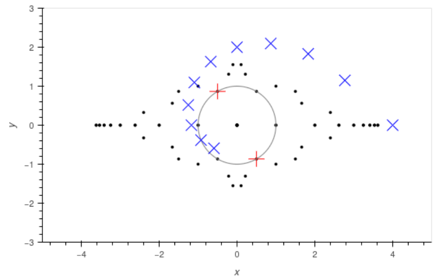


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.92388 - 0.382683i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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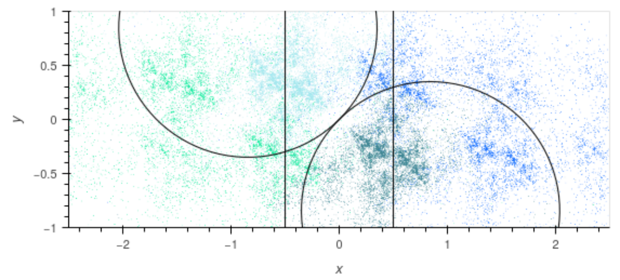
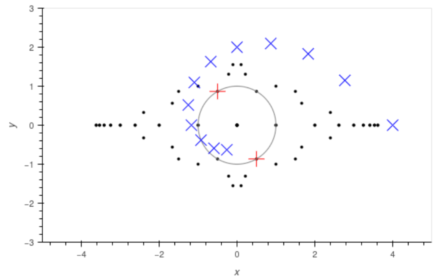


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.592359 - 0.592359i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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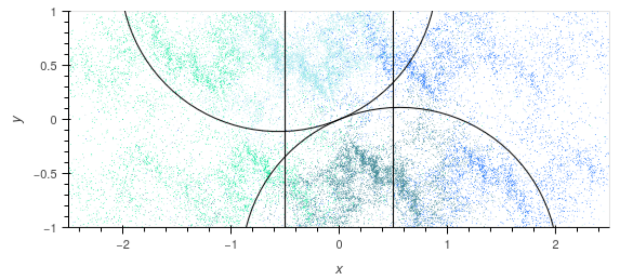
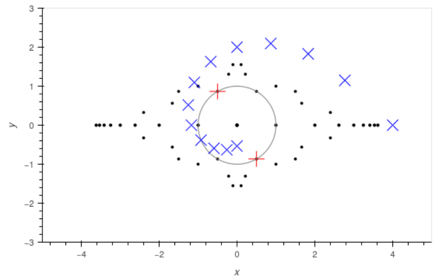


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.261516 - 0.631356i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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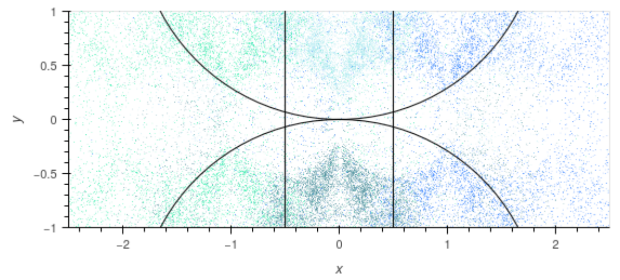
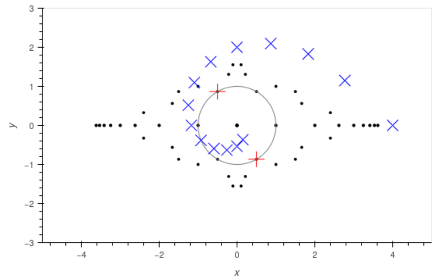


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 - 0.535898i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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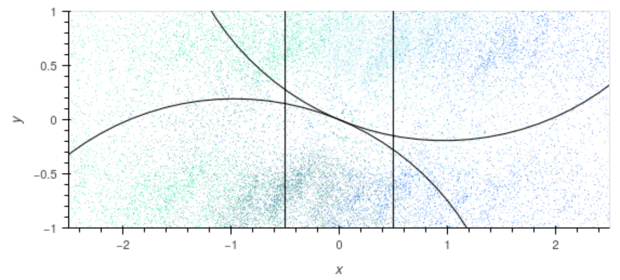
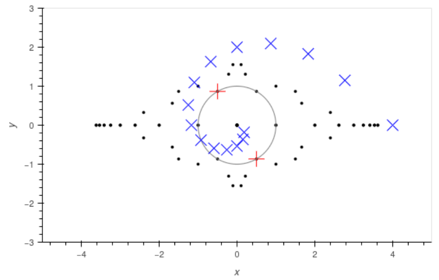


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.150949 - 0.364423i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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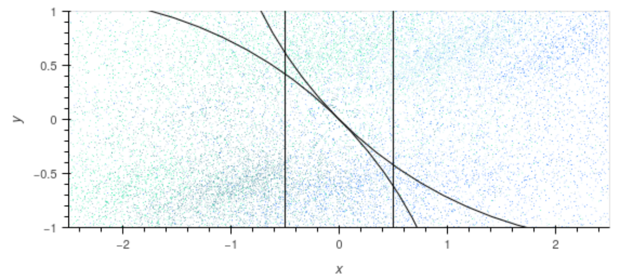
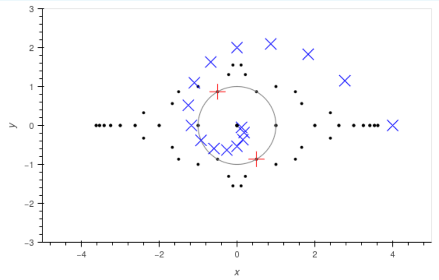


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.182676 - 0.182676i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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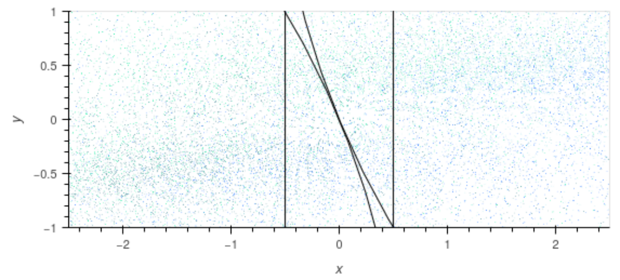
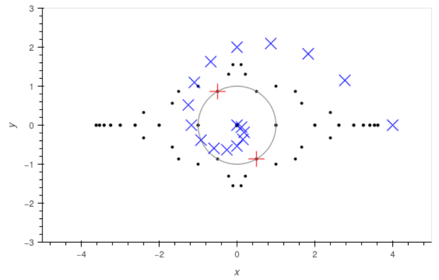


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.117348 - 0.0486072i) & 1 \end{bmatrix} \right\rangle$$

Indiscrete by Shimizu-Leutbecher.



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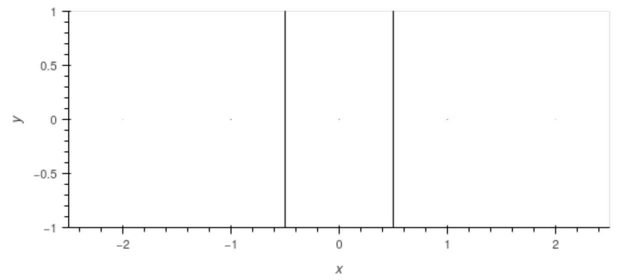
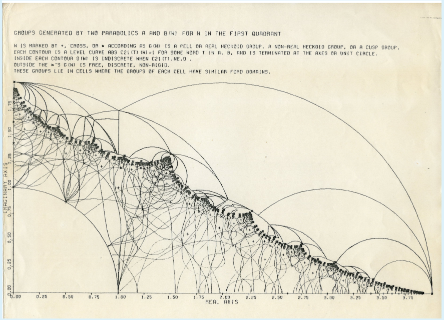


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 + 0.0i) & 1 \end{bmatrix} \right\rangle$$

Elementary group!

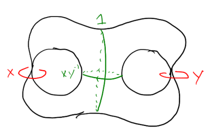
$$\Gamma_\rho = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle$$



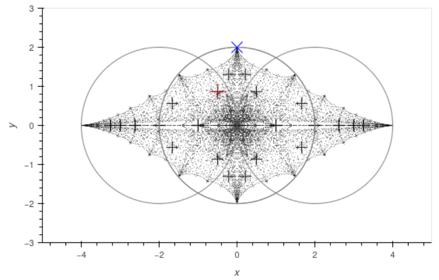
Robert Riley did these experiments in the early 1970s, and produced this plot ((+, +)-quadrant of $\mathbb{C}$).

Definition

The **Riley slice** $\mathcal{R}$ is the set of ρ such that the group Γ_ρ is discrete, and $\mathbb{H}^3/\Gamma_\rho$ is the interior of a ball with two rank one cusps drilled out.



The name is due to Linda Keen and Caroline Series (1994), but we can trace the history of this moduli space back to the 1950s.



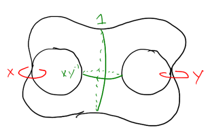
Still $\mathcal{R}$: $p = \infty, q = \infty$

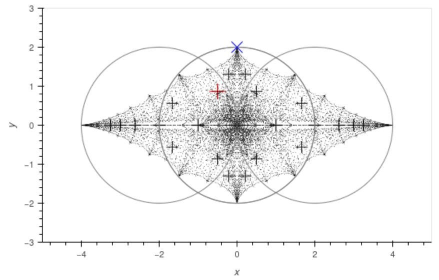
We generalise to the case that the two generators are allowed to be finite orders p and q.

$$\Gamma_\rho = \left\langle \begin{bmatrix} e^{\pi i/p} & 1 \\ 0 & e^{-\pi i/p} \end{bmatrix}, \begin{bmatrix} e^{\pi i/q} & 0 \\ \rho & e^{\pi i/-p} \end{bmatrix} \right\rangle$$

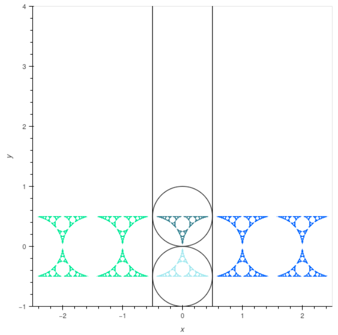
Definition

$\mathcal{R}_{p,q}$ is the set of ρ such that the group Γ_ρ is discrete, and $\mathbb{H}^3/\Gamma_\rho$ is the interior of a ball with one ideal singular arc of order p and one singular arc of order q.

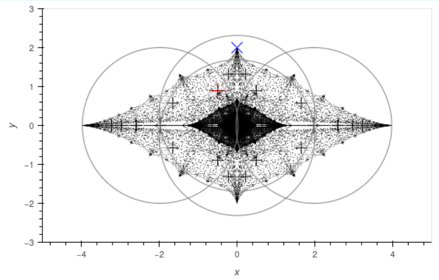




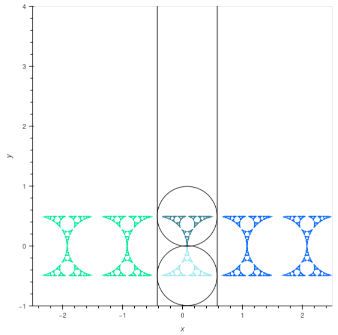
$$p = \infty, q = \infty$$



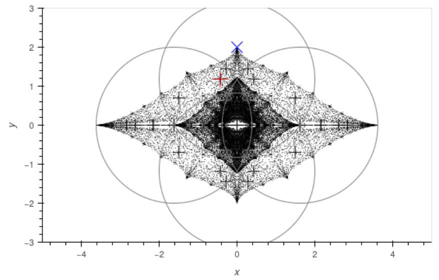
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (0.0 + 2.0i) & 1 \end{bmatrix} \right\rangle$$



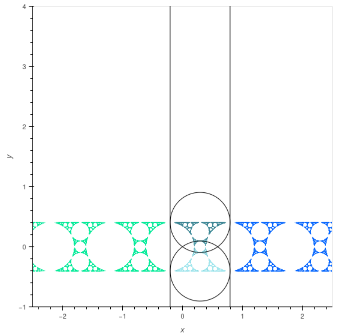
$p = \infty, q = 20$



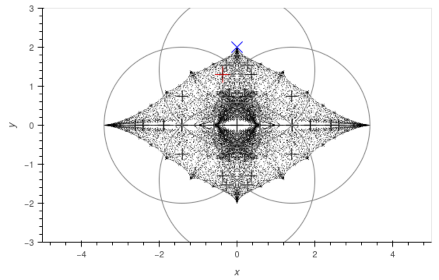
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{\pi i/20} & 0 \\ (0.0 + 2.0i) & e^{-\pi i/20} \end{bmatrix} \right\rangle$$



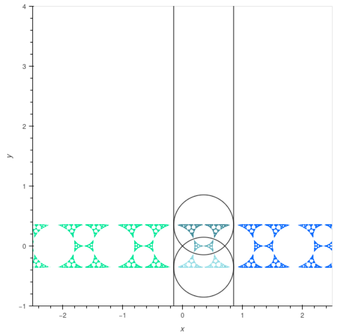
$p = \infty, q = 5$



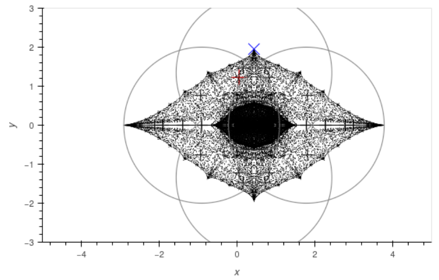
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{\pi i/5} & 0 \\ (0.0 + 2.0i) & e^{-\pi i/5} \end{bmatrix} \right\rangle$$



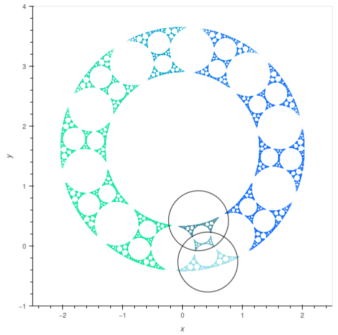
$p = \infty, q = 4$



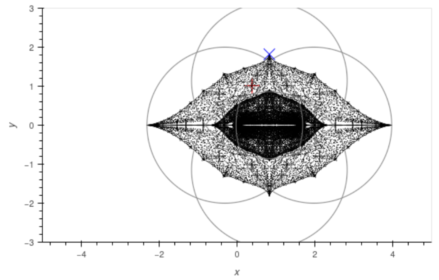
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (0.0 + 2.0i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$



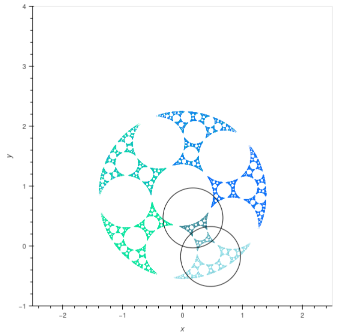
$p = 10, q = 4$



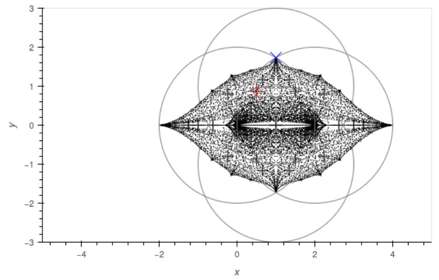
$$\left\langle \begin{bmatrix} e^{\pi i/10} & 1 \\ 0 & e^{-\pi i/10} \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (0.437016 + 1.95167i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$



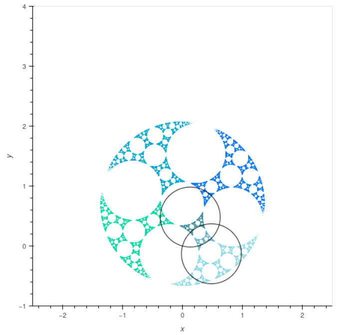
$p = 5, q = 4$



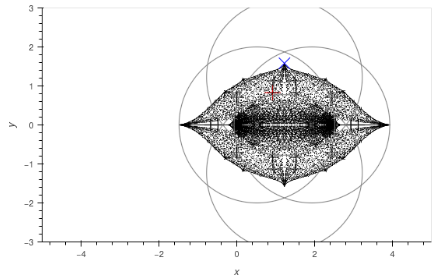
$$\left\langle \begin{bmatrix} e^{\pi i/5} & 1 \\ 0 & e^{-\pi i/5} \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ (0.831254 + 1.81907i) & e^{-\pi i/4} \end{bmatrix} \right\rangle$$



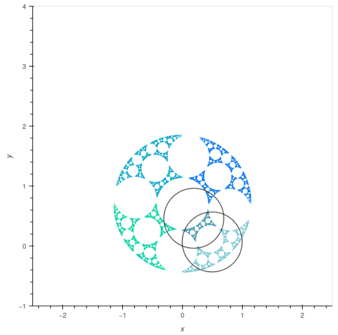
$p = 4, q = 4$



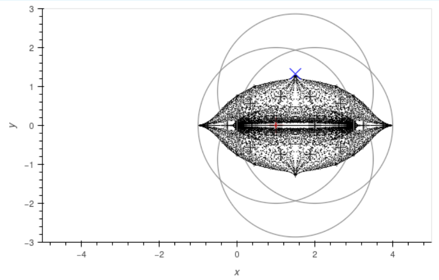
$$\left\langle \begin{bmatrix} e^{\pi i/4} & 1 \\ 0 & e^{-\pi i/4} \end{bmatrix}, \begin{bmatrix} e^{\pi i/4} & 0 \\ {\color{blue}(1.0 + 1.73205i)} & e^{-\pi i/4} \end{bmatrix} \right\rangle$$



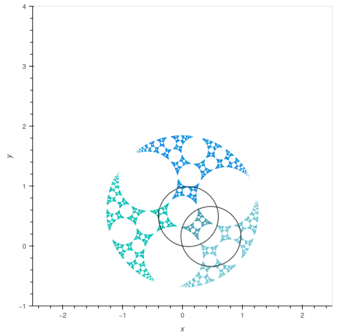
$p = 4, q = 3$



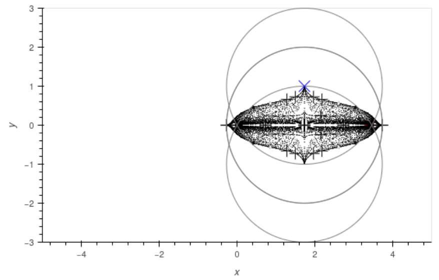
$$\left\langle \begin{bmatrix} e^{\pi i/4} & 1 \\ 0 & e^{-\pi i/4} \end{bmatrix}, \begin{bmatrix} e^{\pi i/3} & 0 \\ (1.22474 + 1.58114i) & e^{-\pi i/3} \end{bmatrix} \right\rangle$$



$p = 3, q = 3$

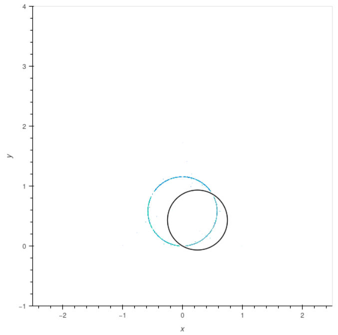


$$\left\langle \begin{bmatrix} e^{\pi i/3} & 1 \\ 0 & e^{-\pi i/3} \end{bmatrix}, \begin{bmatrix} e^{\pi i/3} & 0 \\ (1.5 + 1.32288i) & e^{-\pi i/3} \end{bmatrix} \right\rangle$$

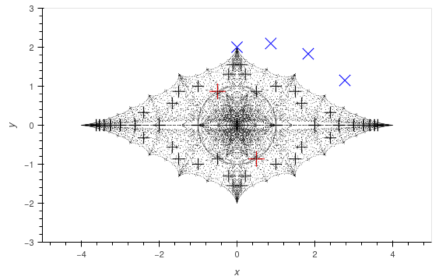


This is (up to finite index) the space of quantisations of $\mathrm{SL}(2,\mathbb{Z}) = \mathcal{B}_3$. We (E., Gong, Martin, Schillewaert, 2024) used this theory to solve a conjecture of Morier-Genoud, Ovsienko, and Veselov about quantum rationals and Burau representations.

$p = 3, q = 2$

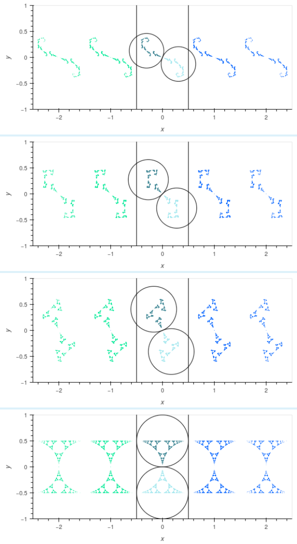


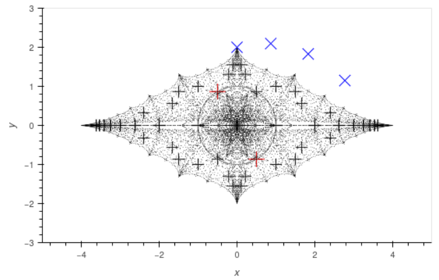
$$\left\langle \begin{bmatrix} e^{\pi i/3} & 1 \\ 0 & e^{-\pi i/3} \end{bmatrix}, \begin{bmatrix} e^{\pi i/2} & 0 \\ (1.5 + 1.32288i) & e^{-\pi i/2} \end{bmatrix} \right\rangle$$



Back to the case $p = q = \infty$ (two parabolic generators) to make things easier to visualise.

We saw earlier that going 'around' the deformation space twists the isometric circles:





Back to the case $p = q = \infty$ (two parabolic generators) to make things easier to visualise.

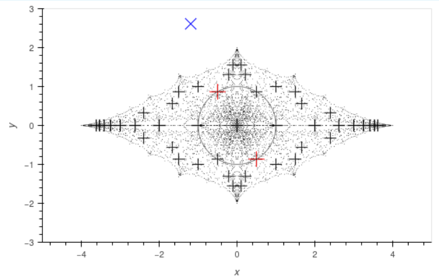
Rough idea

Going radially into the deformation space boundary preserves coarse peripheral structures.

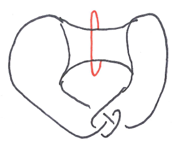
What is a coarse peripheral structure?

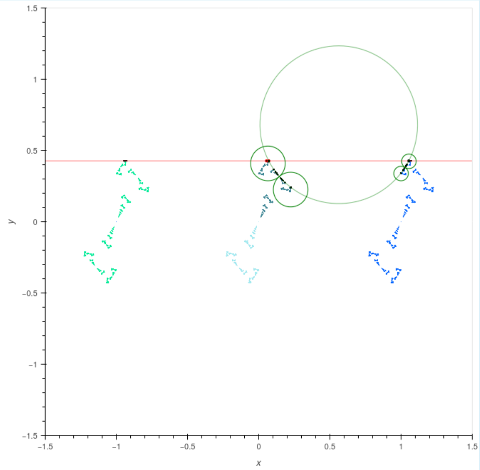
Slightly more precise idea

Teichmüller theory tells us that natural curves to the boundary of moduli space should come from preserving relative lengths of curves on the boundary surfaces of the manifolds.

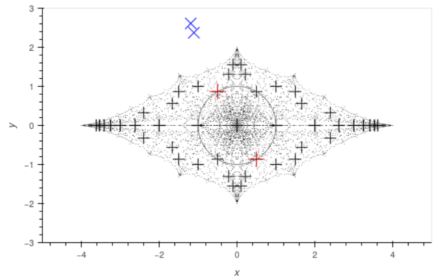


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing length: trlen $W_{3/5} = 9.65185$

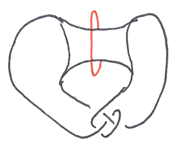


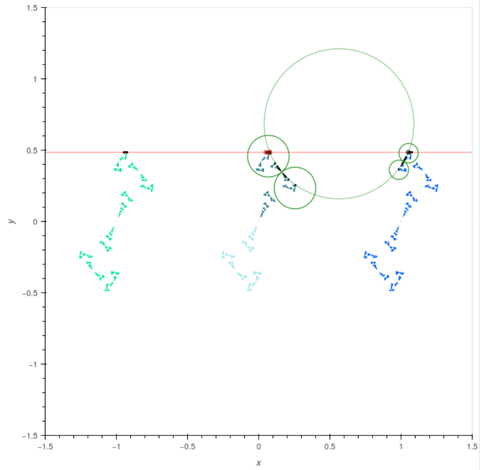


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.18884 + 2.61036i) & 1 \end{bmatrix} \right\rangle$$

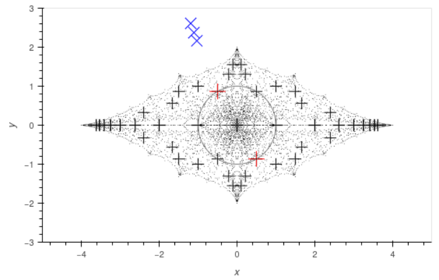


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 8.58525$

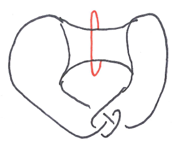


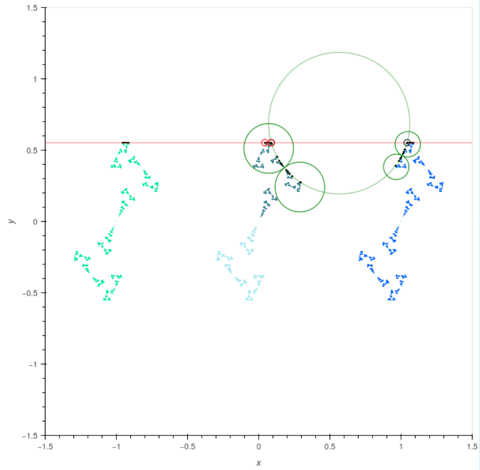


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.10637 + 2.37397i) & 1 \end{bmatrix} \right\rangle$$

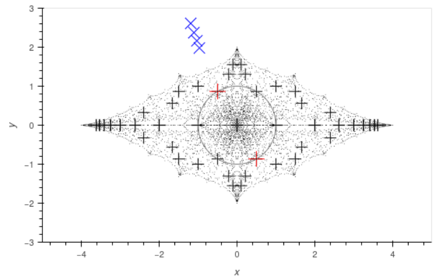


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 7.5011$

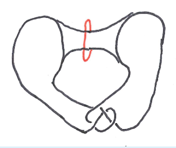


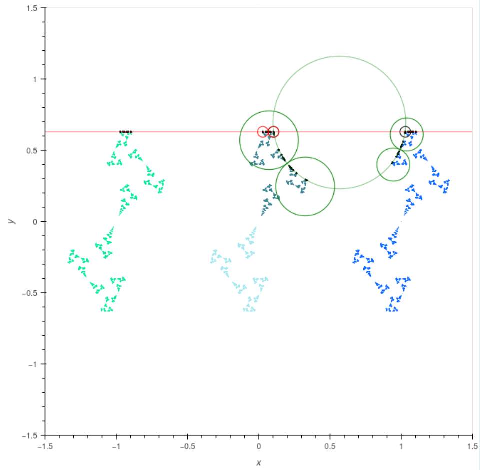


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-1.03184 + 2.16328i) & 1 \end{bmatrix} \right\rangle$$

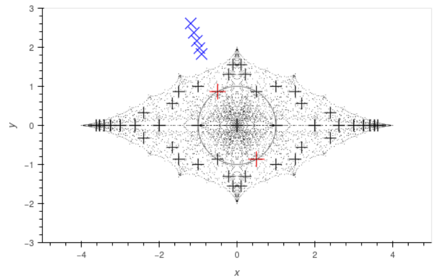


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 6.41179$

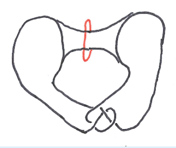


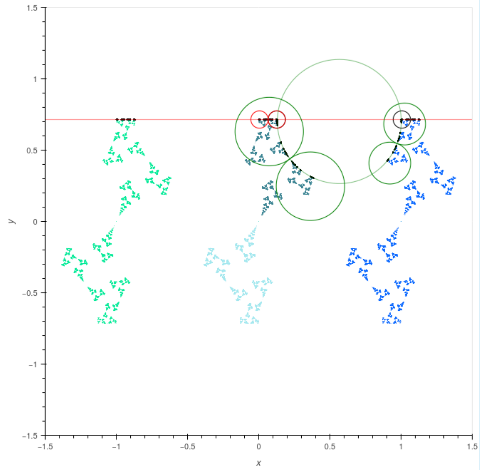


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.965971 + 1.98003i) & 1 \end{bmatrix} \right\rangle$$

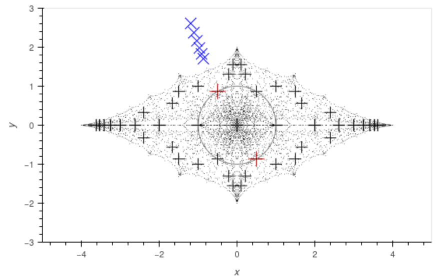


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 5.33801$

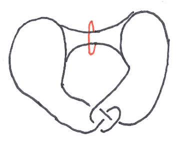


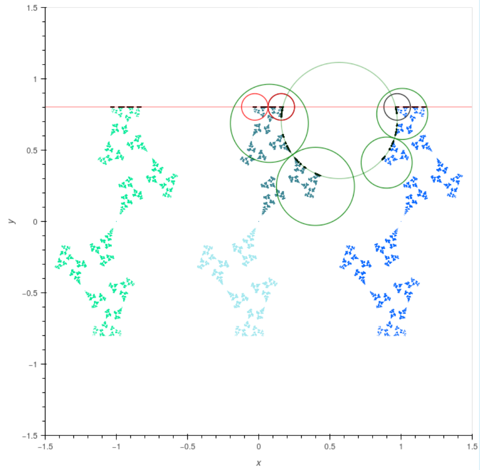


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.909739 + 1.82631i) & 1 \end{bmatrix} \right\rangle$$

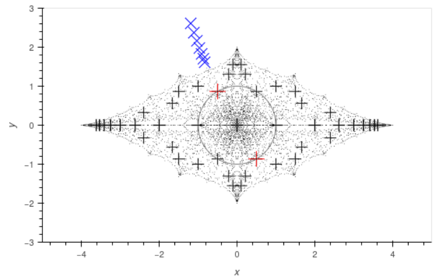


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing length: $\operatorname{trlen} W_{3/5} = 4.31081$

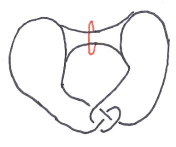


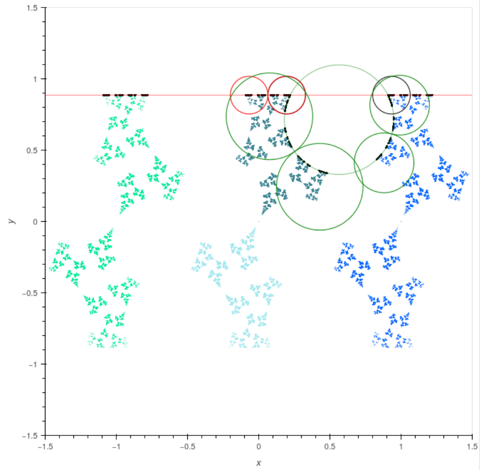


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.864178 + 1.70401i) & 1 \end{bmatrix} \right\rangle$$

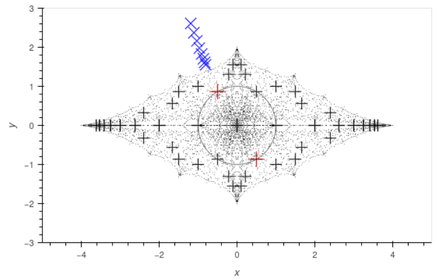


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 3.36929$

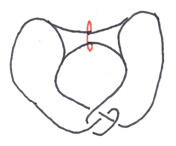


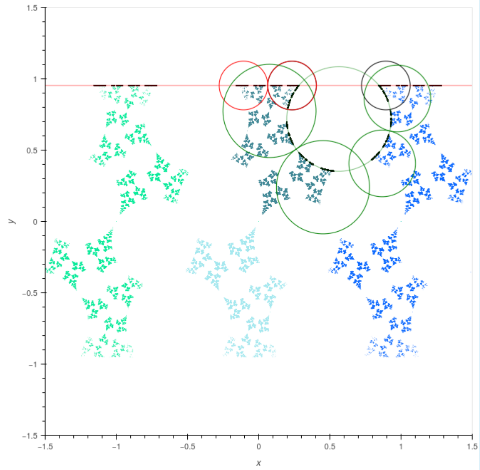


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.829848 + 1.61343i) & 1 \end{bmatrix} \right\rangle$$

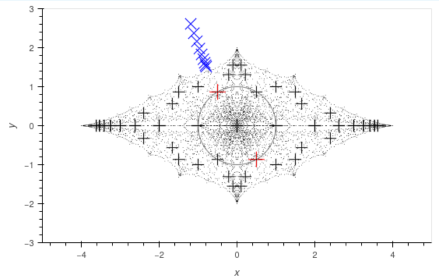


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: $\operatorname{trlen} W_{3/5} = 2.55086$

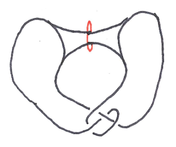


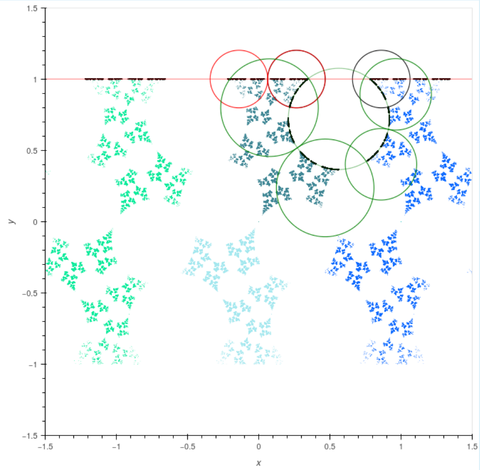


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.806173 + 1.55185i) & 1 \end{bmatrix} \right\rangle$$

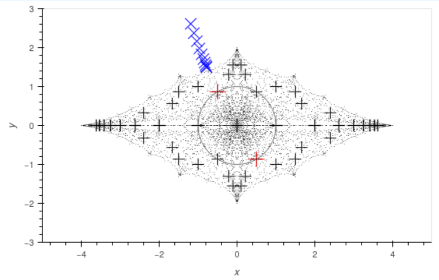


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5}$ = 1.87793

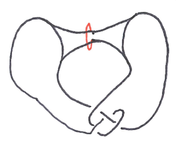


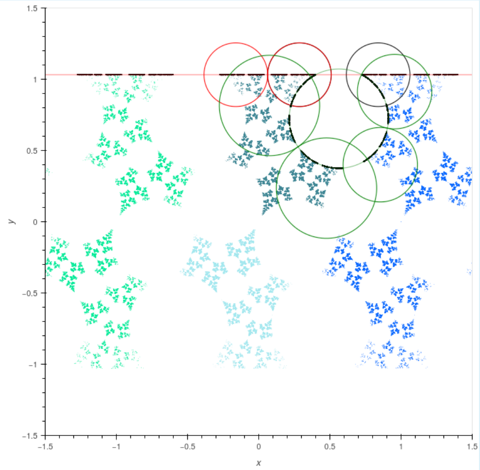


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.791287 + 1.51354i) & 1 \end{bmatrix} \right\rangle$$

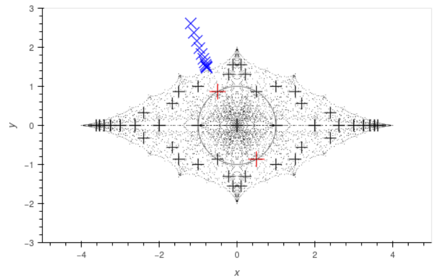


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 1.35137$

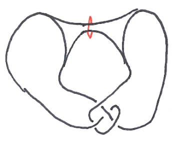


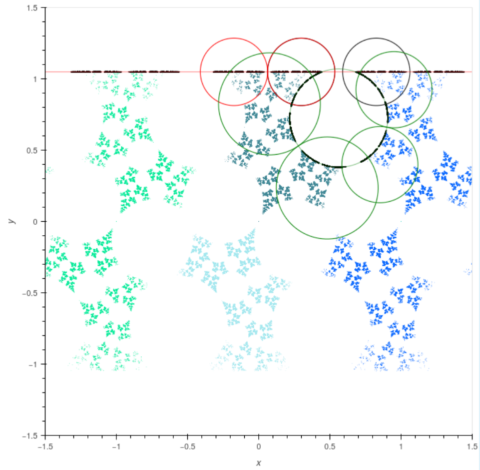


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.782668 + 1.49152i) & 1 \end{bmatrix} \right\rangle$$

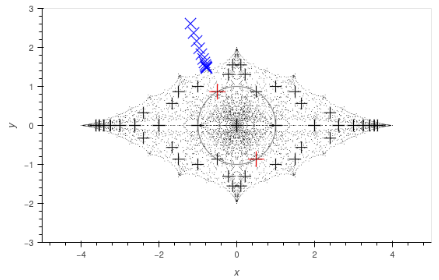


Choose ρ so the geodesic $W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$ has no holonomy and decreasing length: trlen $W_{3/5} = 0.954957$

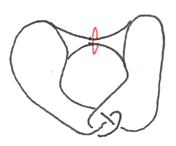


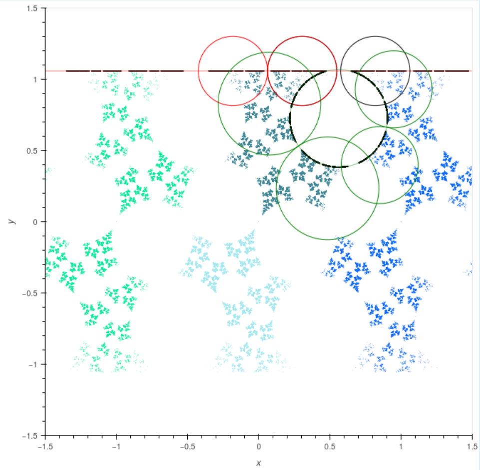


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.777994 + 1.47962i) & 1 \end{bmatrix} \right\rangle$$

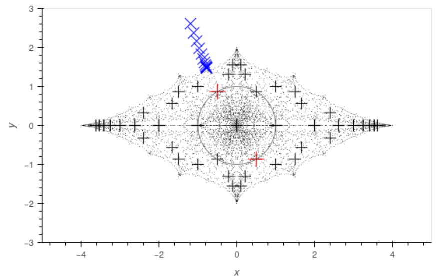


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing length: $\operatorname{trlen} W_{3/5} = 0.664851$

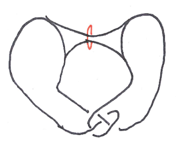


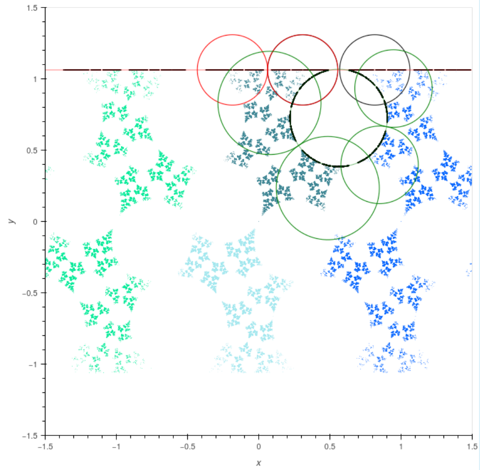


$$\left\langle \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ (-0.775579 + 1.47349i) & 1\end{bmatrix}\right\rangle$$

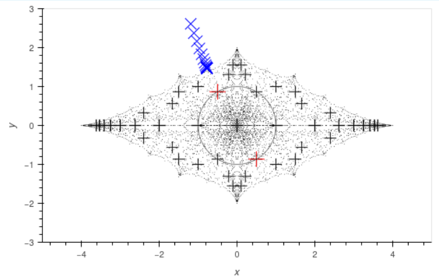


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 0.456906$

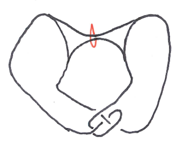


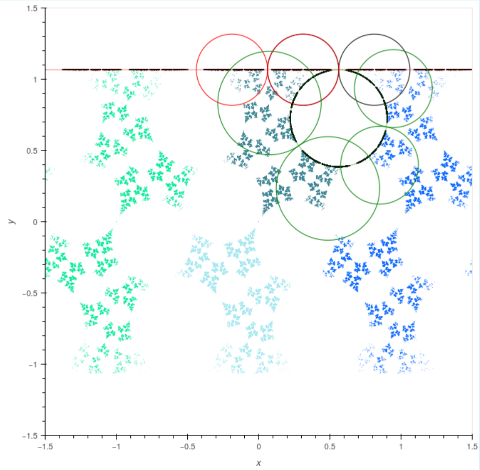


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.774378 + 1.47044i) & 1 \end{bmatrix} \right\rangle$$

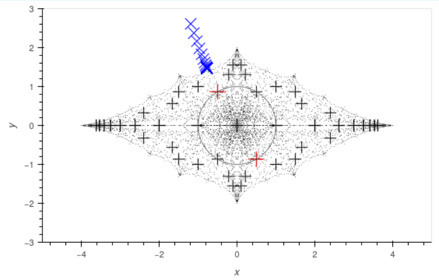


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 0.0587254$

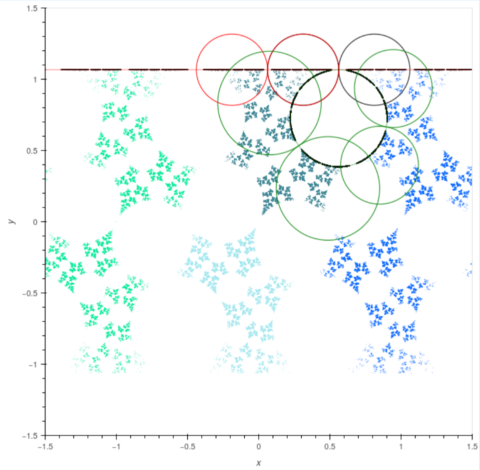


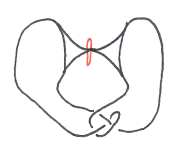


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.773319 + 1.46776i) & 1 \end{bmatrix} \right\rangle$$

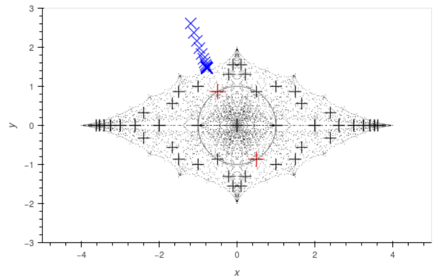


Choose ρ so the geodesic
$W_{3/5} = XY^{-1}X^{-1}YXYX^{-1}Y^{-1}XY$
has no holonomy and decreasing
length: trlen $W_{3/5} = 0.0$



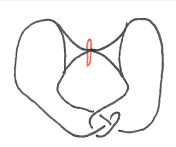


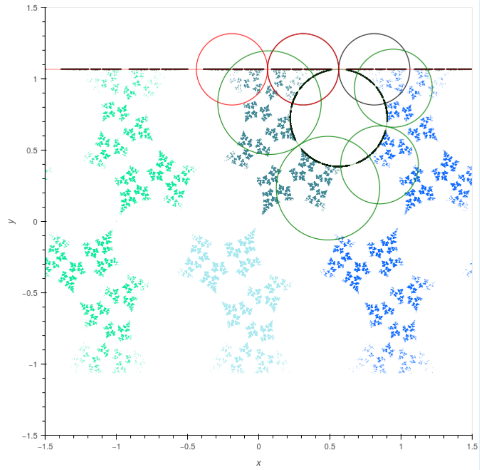
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.773301 + 1.46771i) & 1 \end{bmatrix} \right\rangle$$



The geodesic has turned into a loop around a cusp.

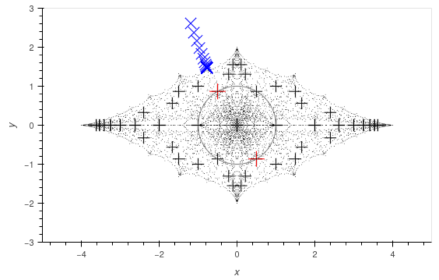
trlen $W_{3/5}$ = 0.0



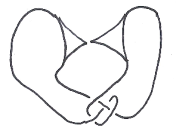


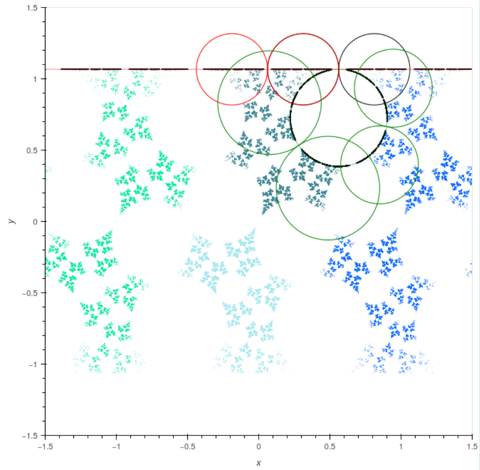
We smoothly decreased tr $W_{3/5}$ from −120 to −2.

But why stop here?

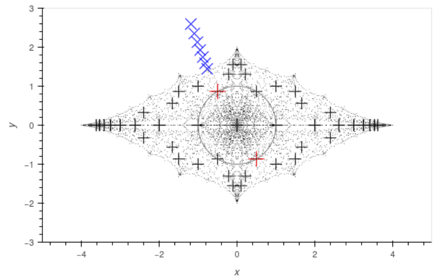


$\operatorname{trlen} W_{3/5} = -6.28i$

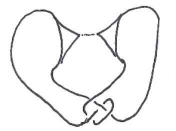


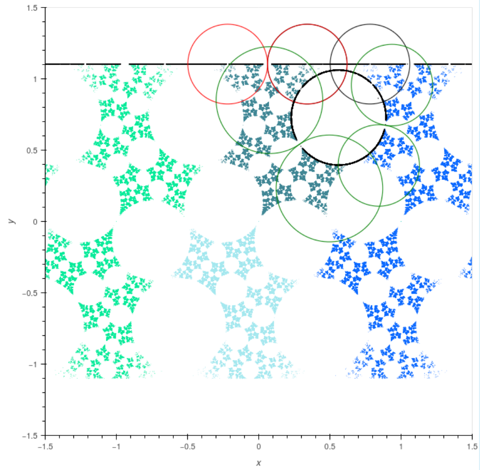


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.773301 + 1.46771i) & 1 \end{bmatrix} \right\rangle$$

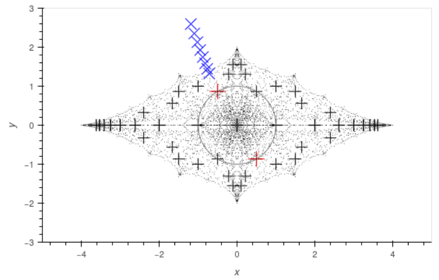


$$\operatorname{trlen} W_{3/5} = -4.95678i$$

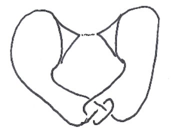


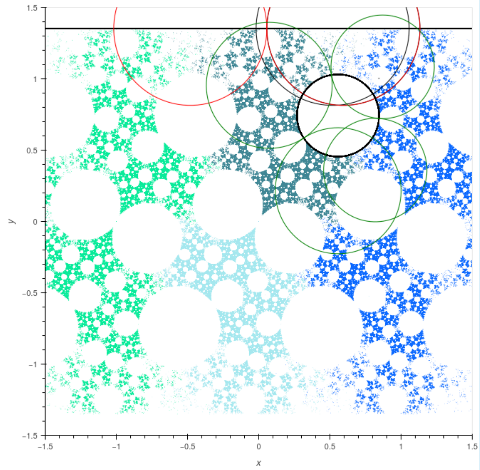


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.76416 + 1.44462i) & 1 \end{bmatrix} \right\rangle$$

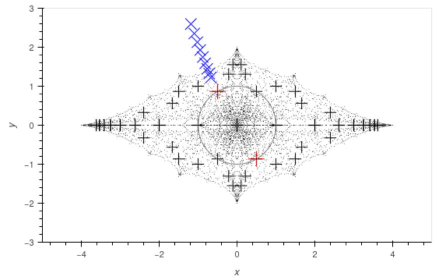


$\text{trlen}\, W_{3/5} = -3.00102i$

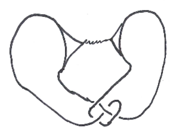


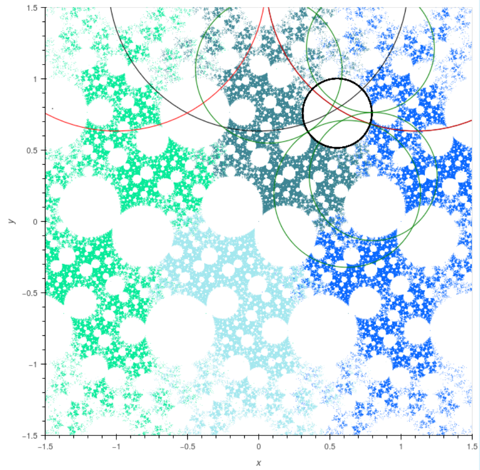


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.715145 + 1.32335i) & 1 \end{bmatrix} \right\rangle$$

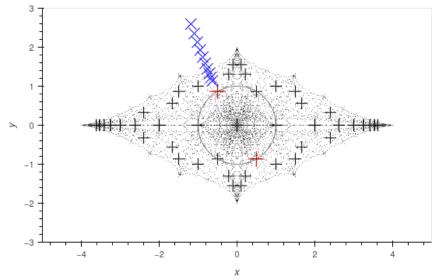


$$\operatorname{trlen} W_{3/5} = -2.03425i$$

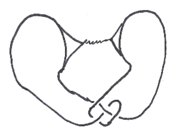


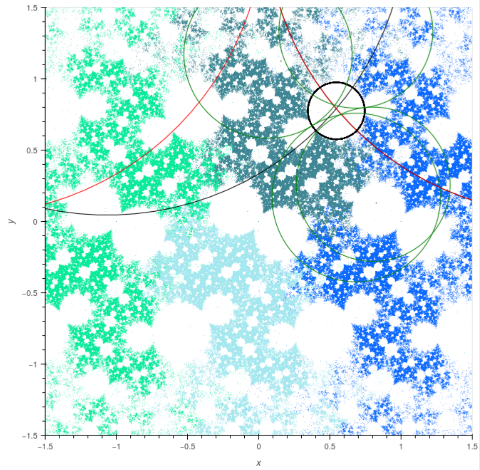


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.671951 + 1.22058i) & 1 \end{bmatrix} \right\rangle$$

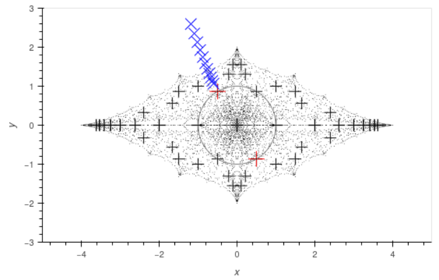


$$\operatorname{trlen} W_{3/5} = -1.40599i$$

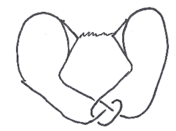


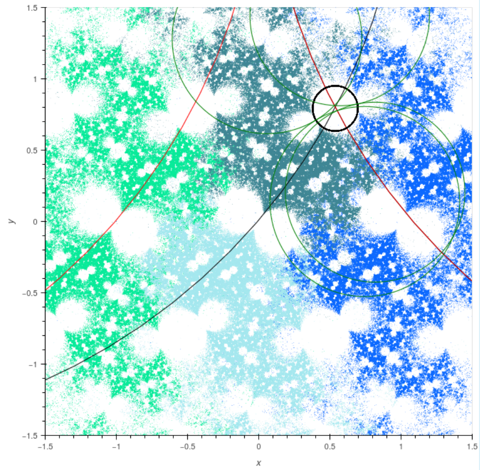


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.634562 + 1.1353i) & 1 \end{bmatrix} \right\rangle$$

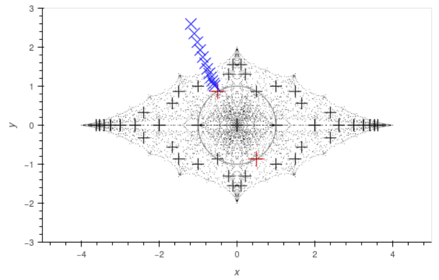


$$\operatorname{trlen} W_{3/5} = -0.973641i$$

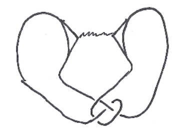


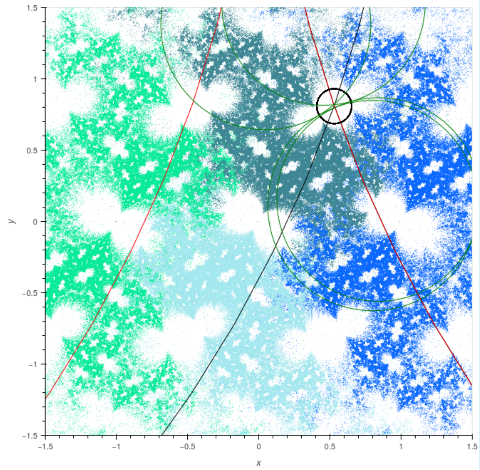


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.602914 + 1.06621i) & 1 \end{bmatrix} \right\rangle$$

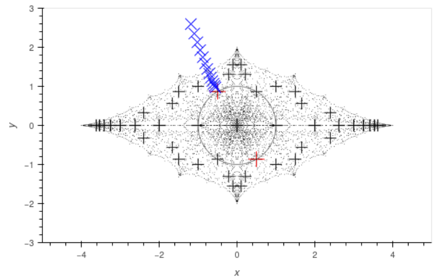


$\operatorname{trlen} W_{3/5} = -0.67106i$

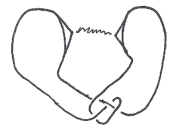


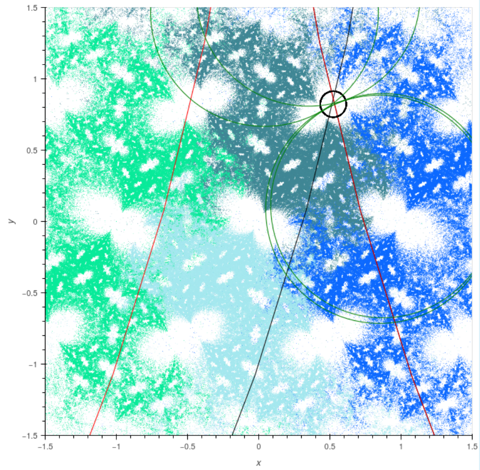


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.57683 + 1.01168i) & 1 \end{bmatrix} \right\rangle$$

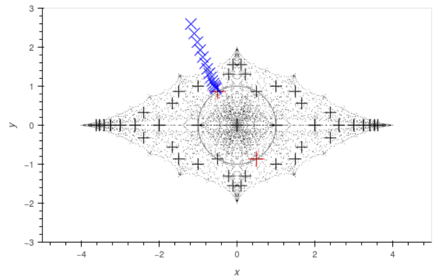


$\operatorname{trlen} W_{3/5} = -0.458907i$

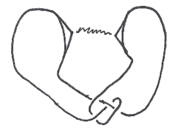


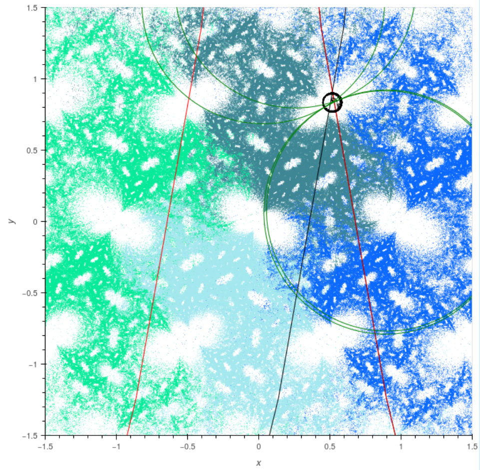


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.55597 + 0.969794i) & 1 \end{bmatrix} \right\rangle$$

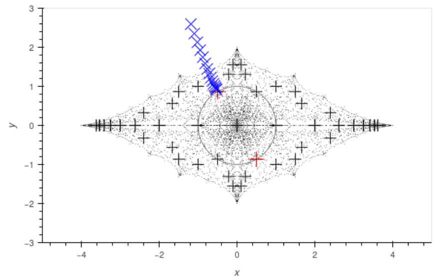


$$\operatorname{trlen} W_{3/5} = -0.310876i$$

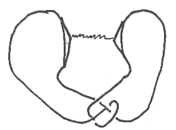


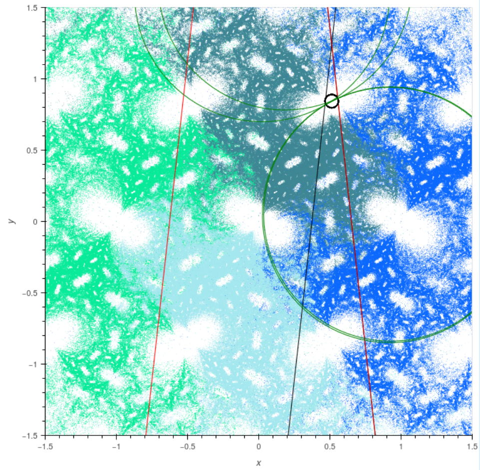


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.53981 + 0.938492i) & 1 \end{bmatrix} \right\rangle$$

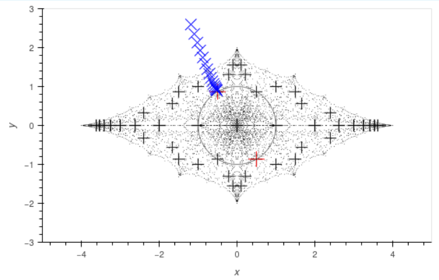


$$\operatorname{trlen} W_{3/5} = -0.208427i$$

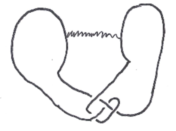


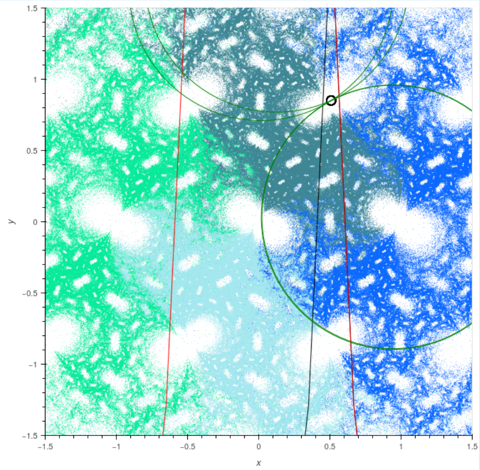


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.527683 + 0.915694i) & 1 \end{bmatrix} \right\rangle$$

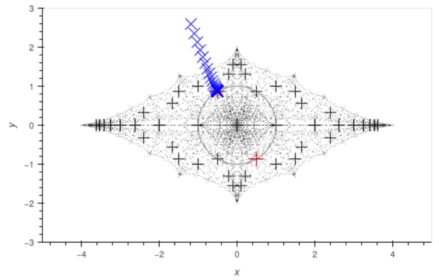


$$\operatorname{trlen} W_{3/5} = -0.138219i$$

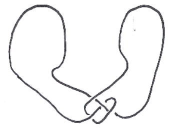




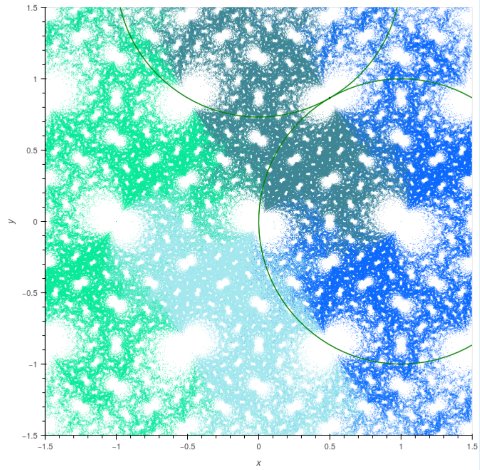
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.518852 + 0.899484i) & 1 \end{bmatrix} \right\rangle$$



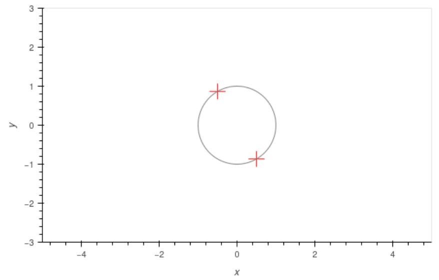
trlen $W_{3/5} = 0.0 + 0.0i$

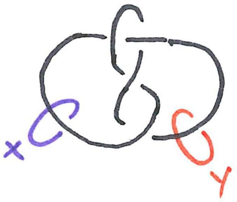


$W_{3/5}$ has become the identity.



$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.5 + 0.86603i) & 1 \end{bmatrix} \right\rangle$$

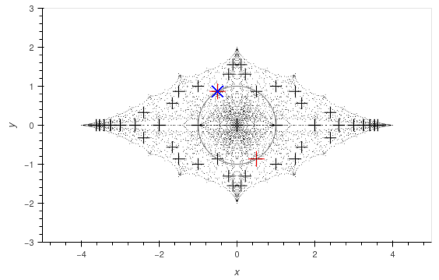


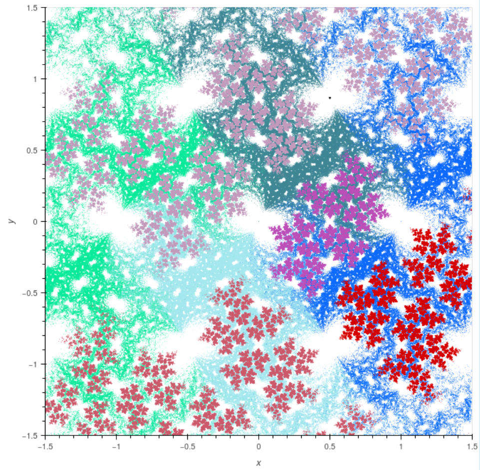


The holonomy group of the figure eight knot is

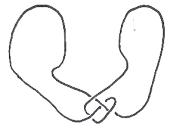
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle$$

$$e^{2\pi i/3} \approx -0.5 + 0.866i$$
$$W_{3/5} = Id$$

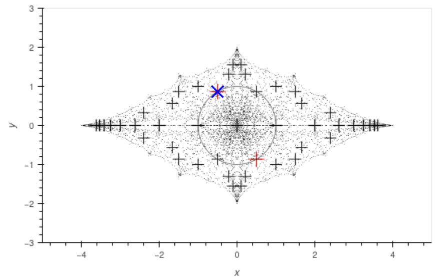




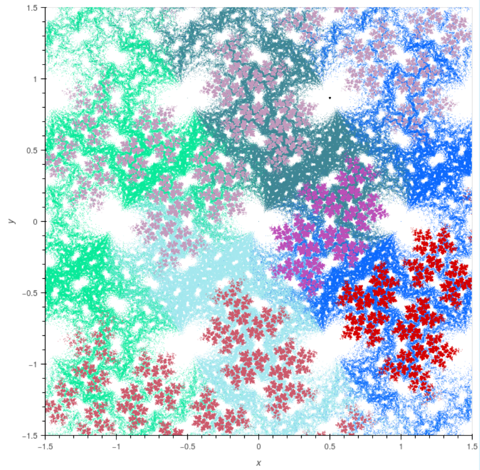
$\mathrm{trlen}\, W_{3/5} = 0.0$

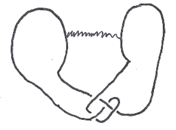


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.5 + 8.6603i) & 1 \end{bmatrix} \right\rangle$$

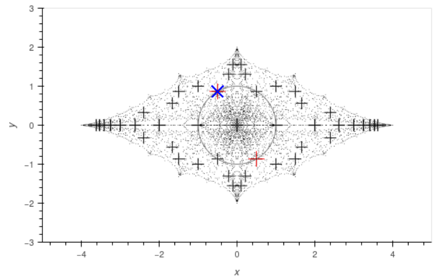


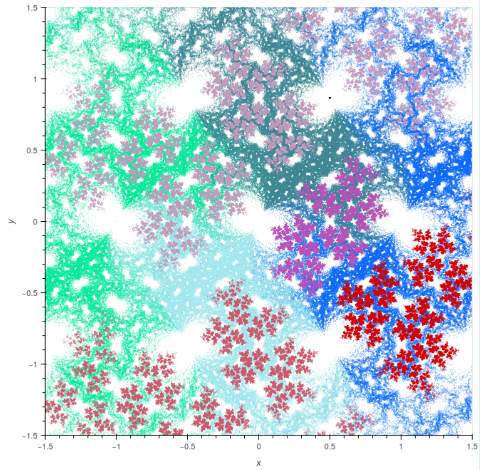
$\text{trlen}\, W_{3/5} = 0.0 - 0.000662171i$



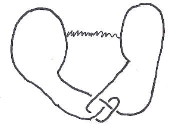


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.500096 + 0.866191i) & 1 \end{bmatrix} \right\rangle$$

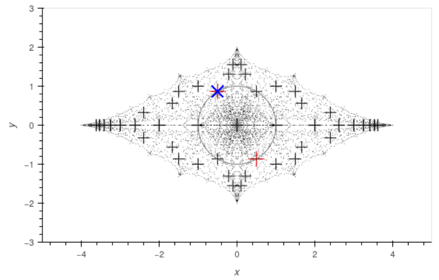


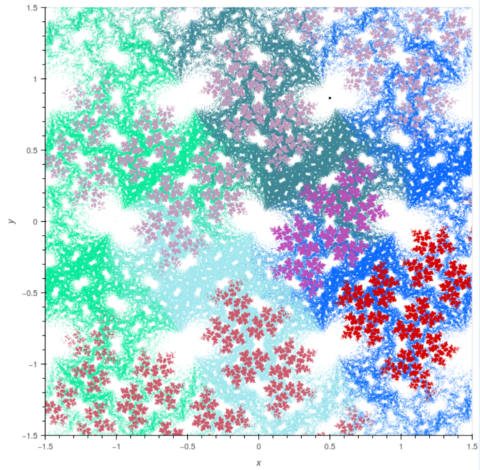


$\operatorname{trlen} W_{3/5} = 0.0 - 0.00115269i$

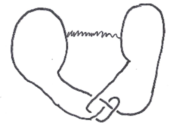


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.500166 + 0.866313i) & 1 \end{bmatrix} \right\rangle$$

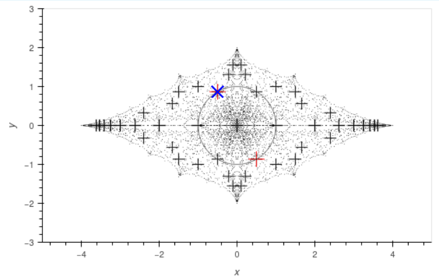


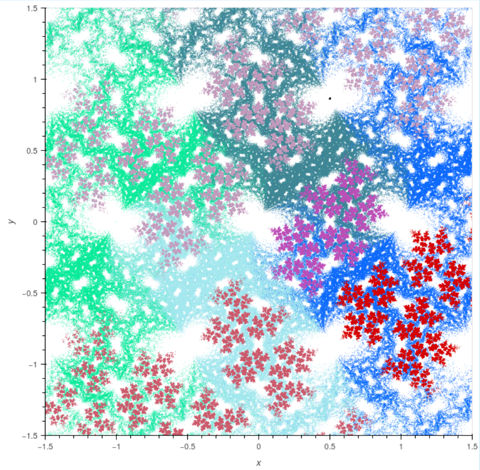


$$\operatorname{trlen} W_{3/5} = 0.0 - 0.00556915i$$

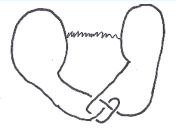


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.500802 + 0.867416i) & 1 \end{bmatrix} \right\rangle$$

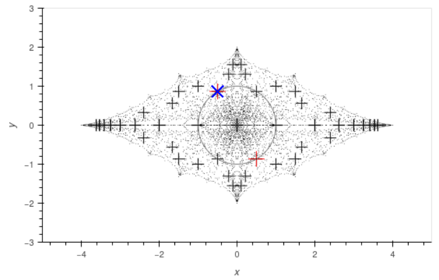




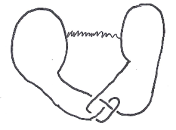
$$\mathrm{trlen}\, W_{3/5} = 0.0 - 0.00915563i$$

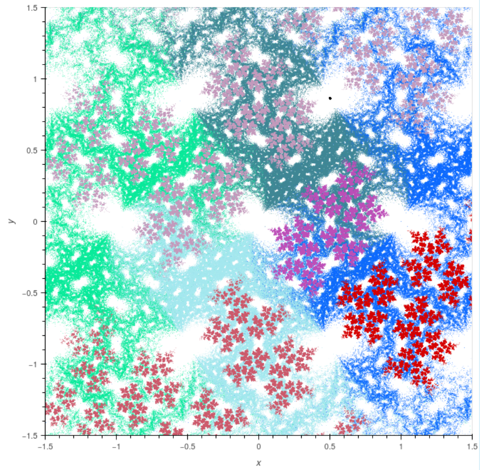


$$\left\langle \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ (-0.501316 + 0.868309i) & 1\end{bmatrix} \right\rangle$$

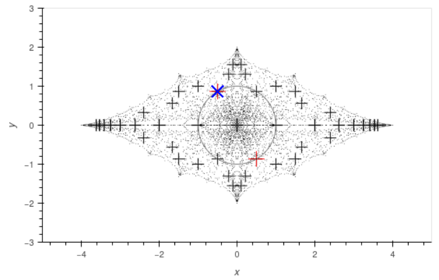


$$\operatorname{trlen} W_{3/5} = 0.0 - 0.0148524i$$

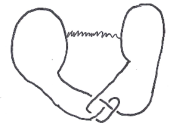


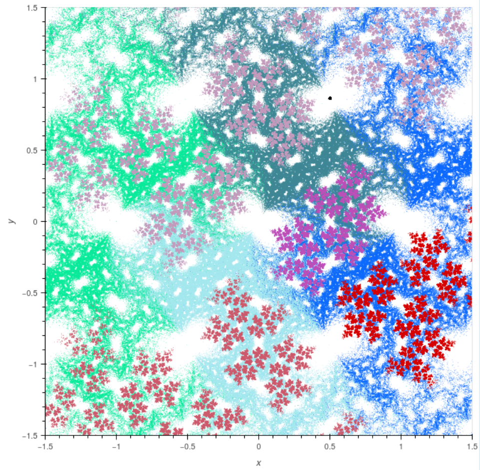


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.50213 + 0.869725i) & 1 \end{bmatrix} \right\rangle$$

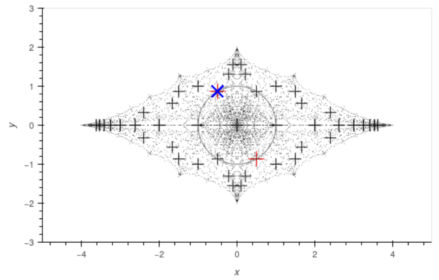


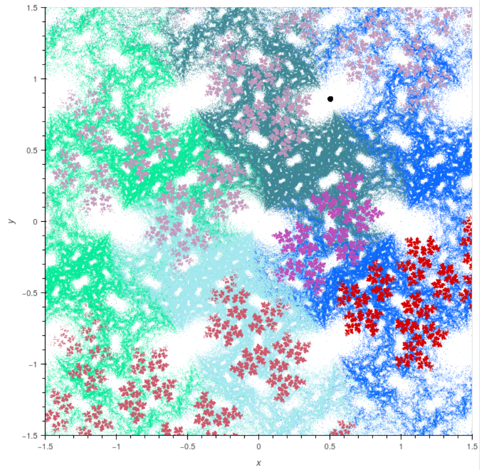
$$\operatorname{trlen} W_{3/5} = 0.0 - 0.023783i$$



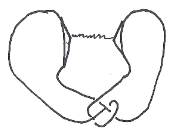


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.503398 + 0.871937i) & 1 \end{bmatrix} \right\rangle$$

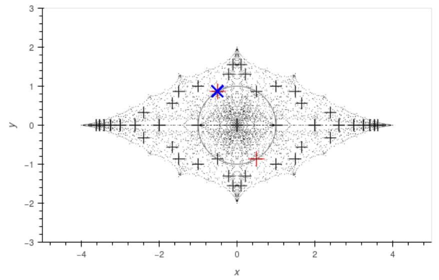




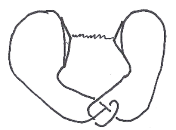
$$\operatorname{trlen} W_{3/5} = 0.0 - 0.0587296i$$

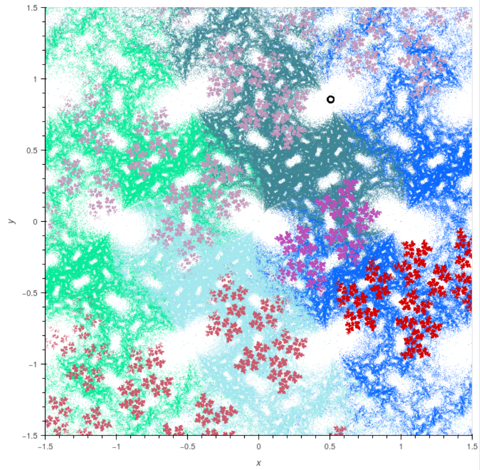


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.508269 + 0.880505i) & 1 \end{bmatrix} \right\rangle$$

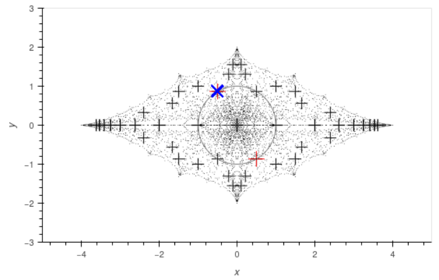


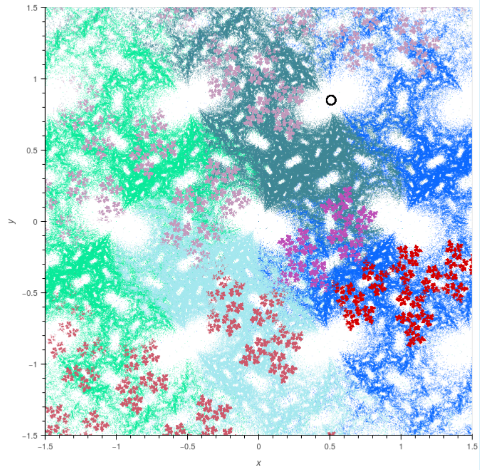
$$\operatorname{trlen} W_{3/5} = 0.0 - 0.0906263i$$



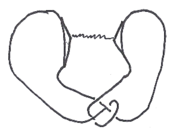


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.512595 + 0.888203i) & 1 \end{bmatrix} \right\rangle$$

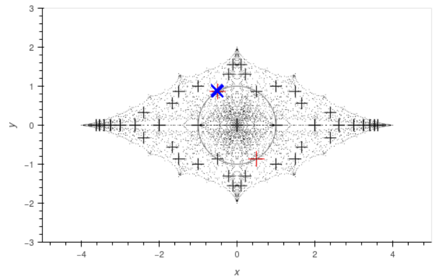


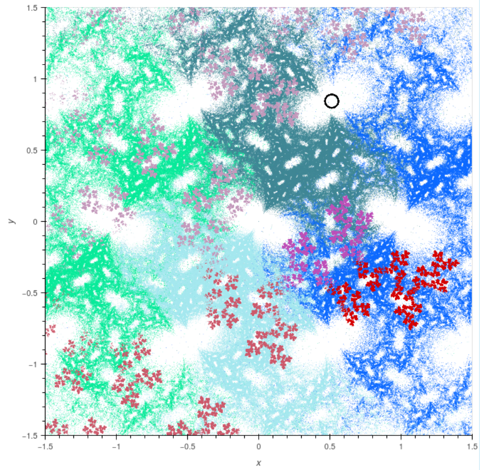


$\operatorname{trlen} W_{3/5} = 0.0 - 0.138219i$

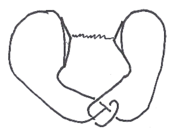


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.518852 + 0.899484i) & 1 \end{bmatrix} \right\rangle$$

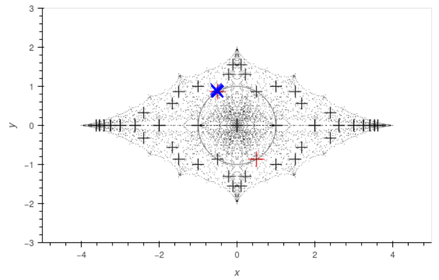




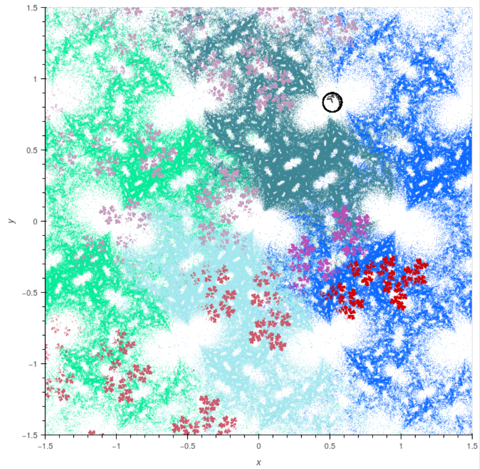
$$\operatorname{trlen} W_{3/5} = 0.0 - 0.208427i$$

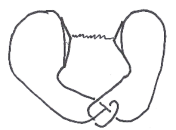


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.527683 + 0.915694i) & 1 \end{bmatrix} \right\rangle$$

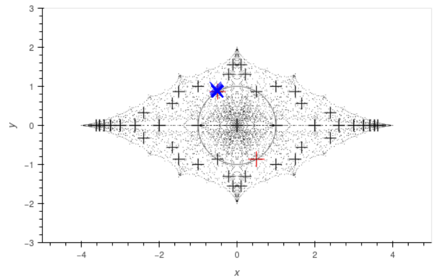


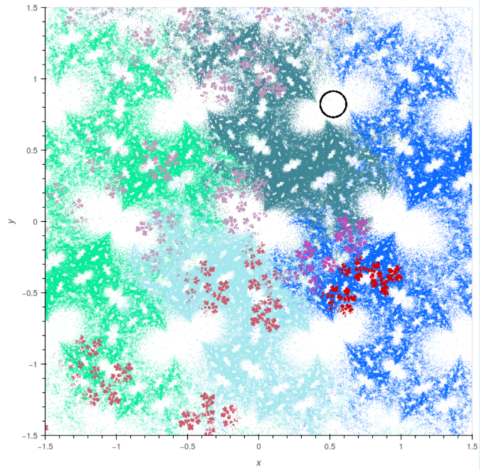
$\mathrm{trlen}\, W_{3/5} = 0.0 - 0.310876i$



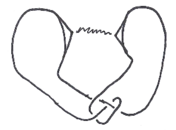


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.53981 + 0.938492i) & 1 \end{bmatrix} \right\rangle$$

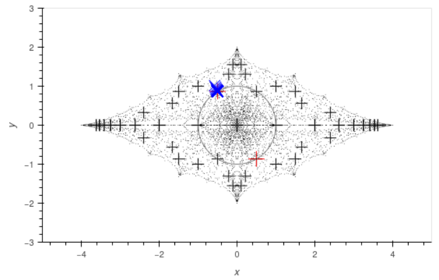




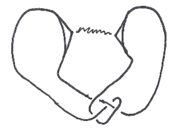
$$\operatorname{trlen} W_{3/5} = 0.0 - 0.458907i$$

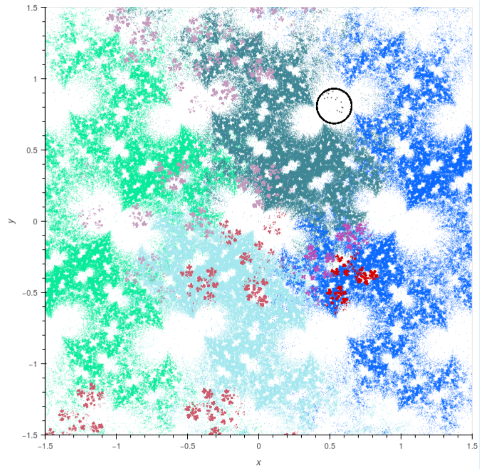


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.55597 + 0.969794i) & 1 \end{bmatrix} \right\rangle$$

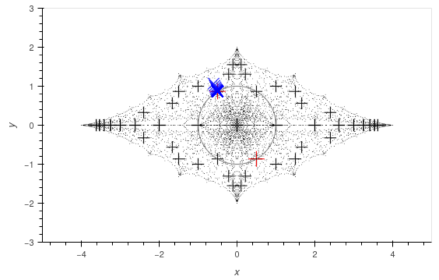


$$\operatorname{trlen} W_{3/5} = 0.0 - 0.67106i$$

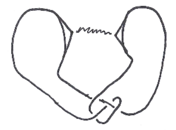


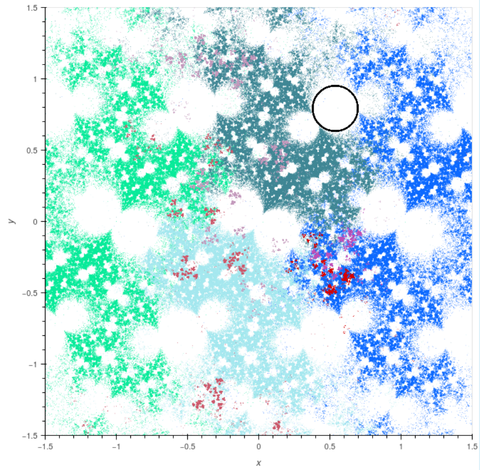


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.57683 + 1.01168i) & 1 \end{bmatrix} \right\rangle$$

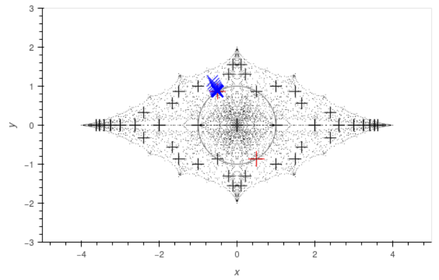


$$\operatorname{trlen} W_{3/5} = 0.0 - 0.973641i$$

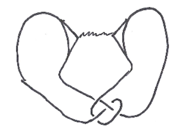


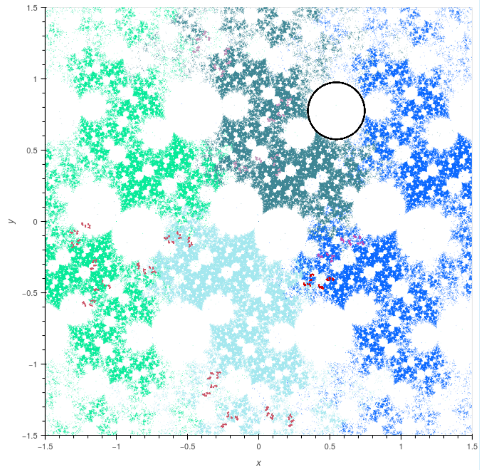


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.602914 + 1.06621i) & 1 \end{bmatrix} \right\rangle$$

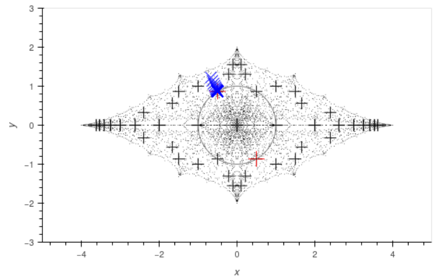


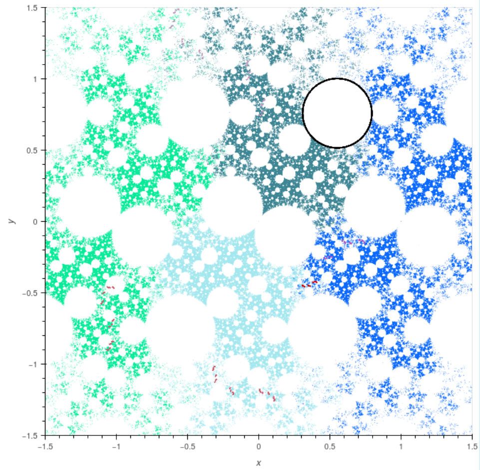
$$\operatorname{trlen} W_{3/5} = 0.0 - 1.40599i$$





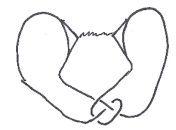
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.634562 + 1.1353i) & 1 \end{bmatrix} \right\rangle$$

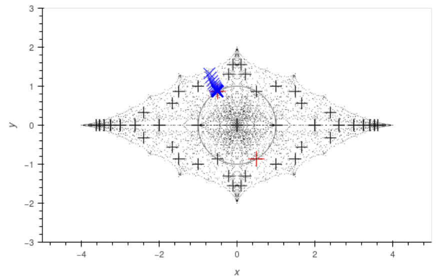




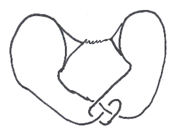
$\operatorname{trlen} W_{3/5} = 0.0 - 2.03425i$

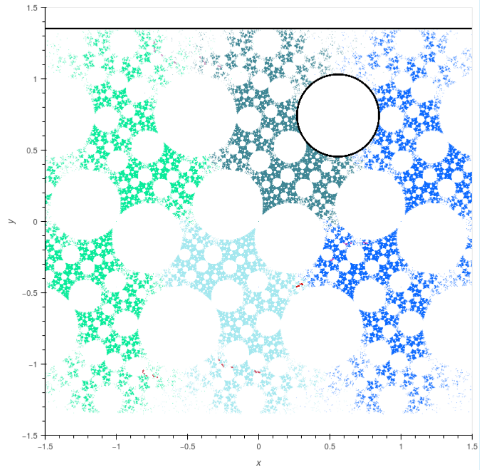
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.671951 + 1.22058i) & 1 \end{bmatrix} \right\rangle$$



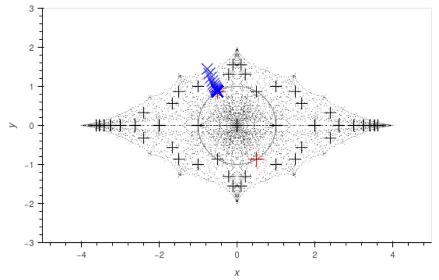


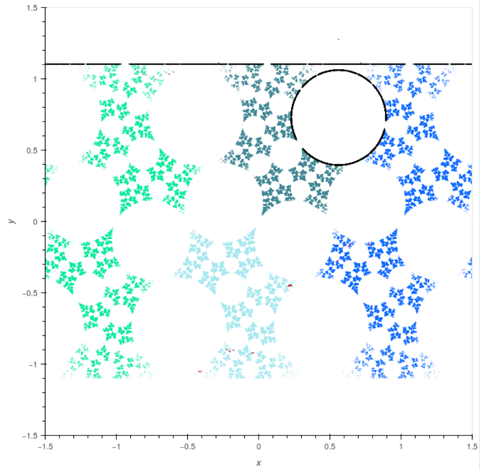
$\operatorname{trlen} W_{3/5} = 0.0 - 3.00102i$



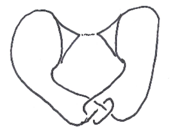


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.715145 + 1.32335i) & 1 \end{bmatrix} \right\rangle$$

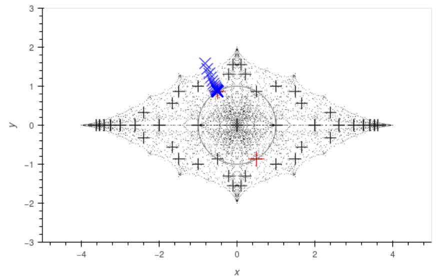




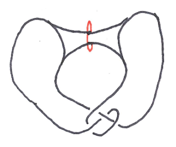
$$\operatorname{trlen} W_{3/5} = 0.0 - 4.95678i$$

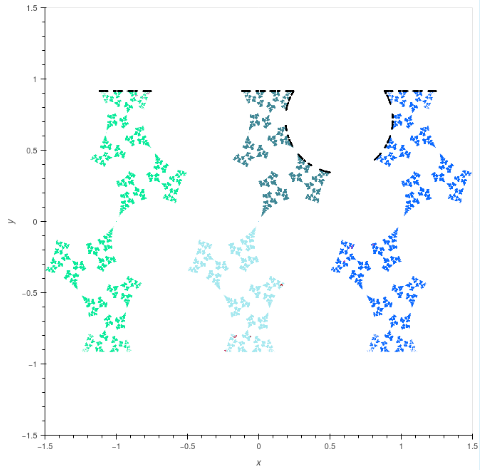


$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.76416 + 1.44462i) & 1 \end{bmatrix} \right\rangle$$

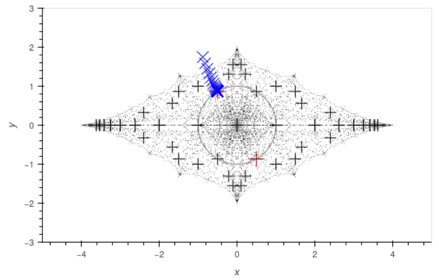


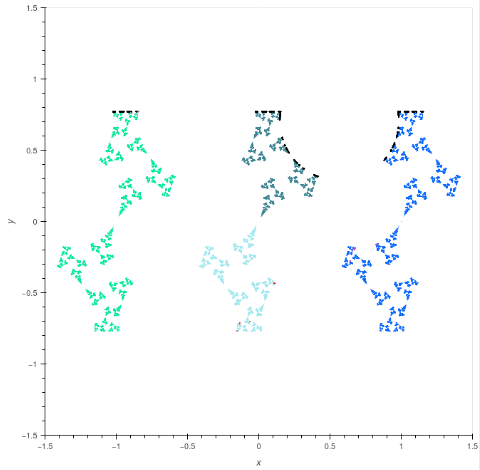
$\operatorname{trlen} W_{3/5} = 3.01983 - 6.28319i$





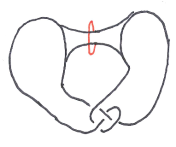
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.819013 + 1.58515i) & 1 \end{bmatrix} \right\rangle$$

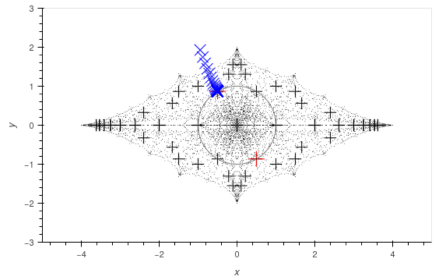




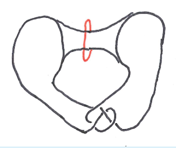
$$\operatorname{trlen} W_{3/5} = 4.68338 - 6.28319i$$

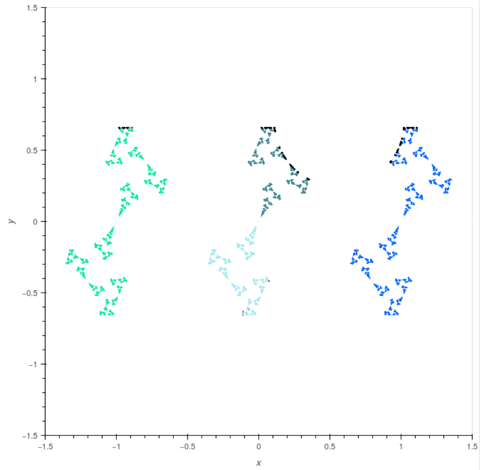
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.879751 + 1.74557i) & 1 \end{bmatrix} \right\rangle$$



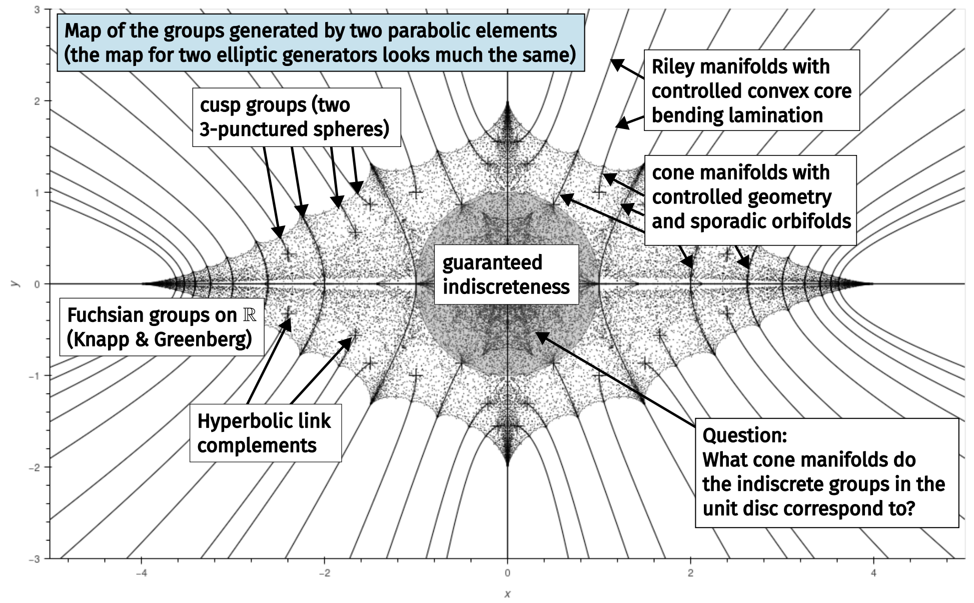


$\mathrm{trlen}\, W_{3/5} = 6.05734 - 6.28319i$

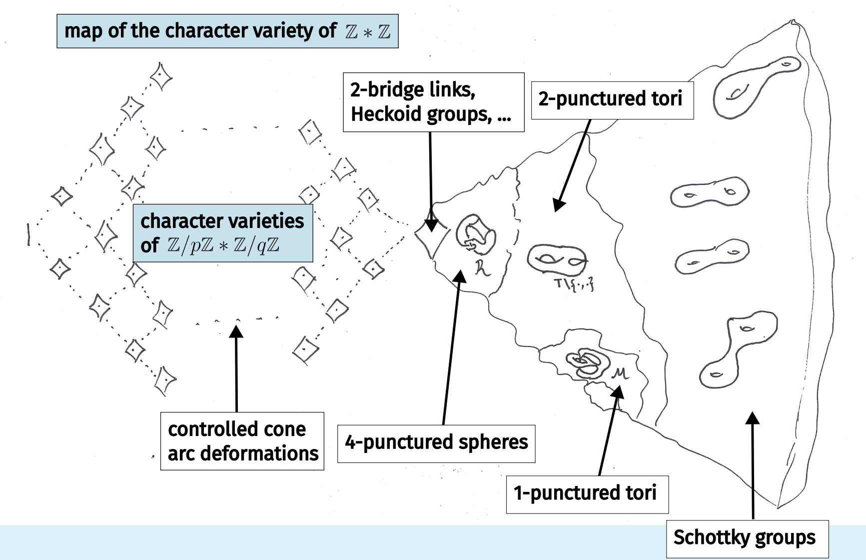




$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ (-0.946453 + 1.92636i) & 1 \end{bmatrix} \right\rangle$$



Map of the groups generated by two parabolic elements
(the map for two elliptic generators looks much the same)
cusp groups (two
3-punctured spheres)
Riley manifolds with
controlled convex core
bending lamination
cone manifolds with
controlled geometry
and sporadic orbifolds
guaranteed
indiscreteness
Fuchsian groups on ℝ
(Knapp & Greenberg)
Hyperbolic link
complements
Question:
What cone manifolds do
the indiscrete groups in the
unit disc correspond to?
y
x
3
2
1
0
−1
−2
−3
−4
−2
0
2
4



Warning: there are lots of weird geometrically infinite things on boundaries! I only show nice groups...

Bedtime reading

- A.J.E., Gaven J. Martin, and Jeroen Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In *2021-2022 MATRIX Annals*.
- A.J.E., Jianhua Gong, G.J.M, and J.S. *Bounding deformation spaces of Kleinian groups with two generators*. arXiv preprint (2024).
- A.J.E., G.J.M., and J.S. *Deformation spaces of Kleinian groups generated by two elements of finite order*. Preprint to appear shortly.
- Eric Chesebro, G.J.M., and J.S. *2-elliptic generated Kleinian groups are Heckoid*. Preprint to appear shortly.
- Linda Keen and Caroline Series. "The Riley slice of Schottky space". Proc. LMS (1994).
- C.S. *The suggestive power of pictures*. `https://youtu.be/YOMr4eviQWs`.
- Sangbum Cho and Darryl McCullough. "The tree of knot tunnels". Geom. Topol. (2009).
- Title picture: Caspar David Friedrich, *Erinnerungen an das Riesengebirge* [Memories of the Riesengebirge], c.1835.

Theorem (E., G.J. Martin, J. Schillewaert (2021+), after Keen and Series (1994))

The quasiconformal deformation space $\mathcal{R}^{p,q}$ of Kleinian groups which split as a free product $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ and which have quotient surface that is a pair of 3-marked spheres admits a natural lamination where leaves are groups with the same projective convex core bending lamination. On each leaf, enumerated by the Farey triangulation, the groups split into peripheral subgroups of given geometry; and extensions of these out of $\mathcal{R}^{p,q}$ further into $X(\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z})$ eventually ends at an orbifold group with singular locus a 2-bridge link. Every leaf also admits a semi-algebraic neighbourhood in $\mathcal{R}^{p,q}$ that gives membership (hence discreteness) certificates for groups in $\mathcal{R}^{p,q}$. All this can be effectively computed.

Keen and Series (1994): case $p = q = \infty$.

Arithmetic group applications

Theorem (E., G.J. Martin, J. Schillewaert (announced 2024, to appear))

1. *There are finitely many arithmetic groups in* $\mathrm{PSL}(2,\mathbb{C})$ *generated by two parabolic or elliptic elements.*
2. *There are finitely many thin groups in* $\mathrm{PSL}(2,\mathbb{C})$ *which do not split as free products.*
3. *The groups in (1) and (2) can be effectively tabulated.*

Proof.

Use algebraic number theory to bound maximal degree of algebraic integers that can arise in generators, and use earlier work (necessary criterion for arithmeticity) of Maclachlan and Martin to come up with a finite list of possible groups. Use joint work with J. Gong (+EMS) and E. Chesebro (+MS) to give semi-algorithm that can check all these finitely many groups to find their surface quotient topology and isomorphism type.

Cone manifold deformations

We follow ideas of Akioshi, “Thin representations for the one-cone torus group”, Topol. Appl. (2019) but applied to $\mathcal{R}$ not $\mathcal{M}$.

- On each pleating ray the groups splits as an amalgamated product of two subgroups generated by primitive p and q order elliptics. The product of the generators is the amalgamation curve γ (of course you also amalgamate the generators).
- From this we obtain a fundamental polyhedron for the larger group. This is standard application of something like Maskit’s first combination theorem, which you can rephrase as a statement about gluing polyhedrons and obtaining isomorphism and side-pairing action by the resulting group generated by side-pairings regardless of discreteness.
- Travelling down the pleating ray and then the extended pleating ray is just a smooth deformation of some angles on this fundamental polyhedron. As you continue outside the deformation space, the polyhedron does not collide with itself, and so the group generated by its side pairings is the holonomy group of the cone manifold where γ has become a cone arc of steadily increasing angle; but this group is just the indiscrete group on the extended pleating ray.

Computation notes

- Figures produced in Python using Bella `https://github.com/aelzenaar/bella` (for mathematical computations) and HoloViews `https://holoviews.org/` (for visualisation)
- Algorithms for limit set calculation can be found in David Mumford, Caroline Series, and David Wright. *Indra's pearls*. Cambridge Uni. Press, 2002.
- Moduli space and pleating ray computations are based on theory in E., M., and S., "The combinatorics of Farey words and their traces". Groups, Geom. and Dyn. (accepted, preprint on the arXiv); basically we walk down a given asymptotic branch of a polynomial inverse using Newton's algorithm.
- See also: David Wright, "Searching for the cusp", in *Spaces of Kleinian groups*, Lond. Math. Soc. Lec. Notes 329, 2011.
- See also: visualisations by Emily Dumas (`https://www.dumas.io/limset/`).