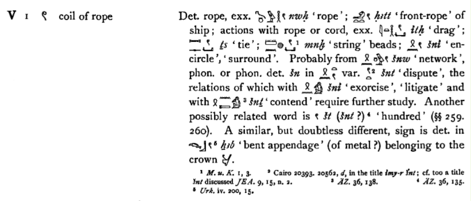
“The dynamic in the static”*

Manifolds, braids, and classical number theory

Alex Elzenaar

Max-Planck-Institut für
Mathematik in den
Naturwissenschaften

Regiomontanus PhD Seminar
Universität Leipzig



V 1 coil of rope

Det. rope, exx. nwḥ ‘rope’; ḥ3tt ‘front-rope’ of ship; actions with rope or cord, exx. ỉtḥ ‘drag’; ṯs ‘tie’; mnḫ[1] ‘string’ beads; šnỉ ‘encircle’, ‘surround’. Probably from šnw ‘network’, phon. or phon. det. šn in var.[2] šnt ‘dispute’, the relations of which with šnỉ ‘exorcise’, ‘litigate’ and with[3] šnṯ ‘contend’ require further study. Another possibly related word is št (šnt ?)[4] ‘hundred’ (§§ 259. 260). A similar, but doubtless different, sign is det. in[5] ḳ3b ‘bent appendage’ (of metal ?) belonging to the crown.

[1] M. u. K. 1, 3. [2] Cairo 20393. 20562, d, in the title ỉmy-r šnt; cf. too a title šnt discussed JEA. 9, 15, n. 2. [3] ÄZ. 36, 138. [4] ÄZ. 36, 135. [5] Urk. iv. 200, 15.

---

*M.C. Escher, letter to his nephew Rudoph Escher, 22 Feb. 1957.

§I. Knots

Gauss, age 17

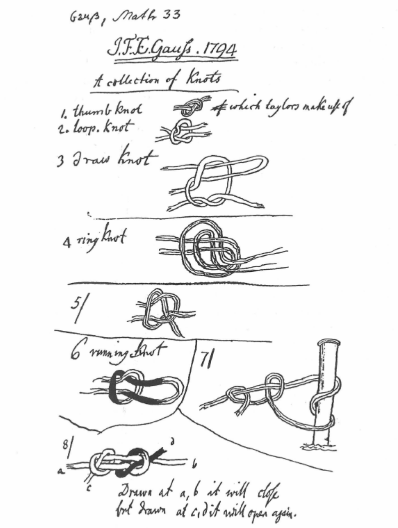
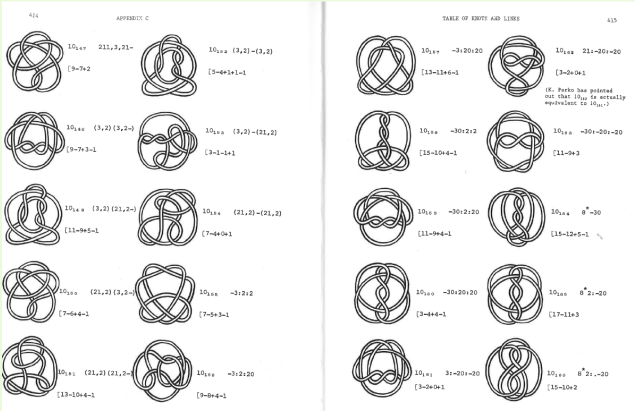


Fig. 2. Copy of a page from the notebooks of C. F. Gauss (1777–1855)

J. C. Thurner and P. v.d.Griend (eds.), *History and science of knots*, p. x.

Definition

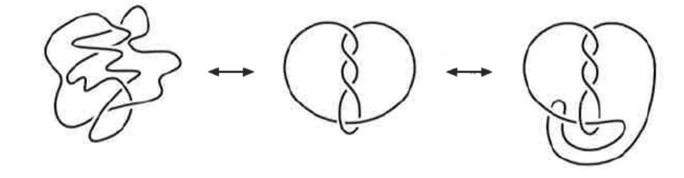
A knot is an embedding $S^1 \to S^3$. A link is an embedding $S^1 \sqcup \cdots \sqcup S^1 \to S^3$.



D. Rolfsen, *Knots and links*, pp. 414–415.

Definition

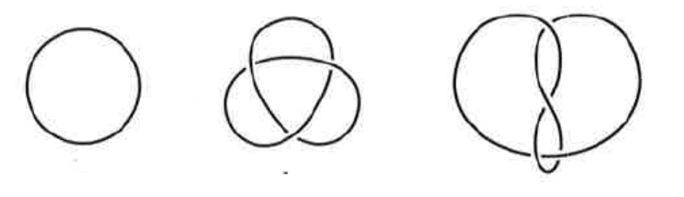
Two knots are equivalent if there is an ambient isotopy of S^3 which transforms one to the other.



C. Adams, *The knot book*, p. 2.

Exercise

How do you know these three knots are different?



C. Adams, *The knot book*, p. 2.

If k is a knot or link, then $S^3 \setminus k$ is a smooth oriented 3-manifold.

Theorem (Gordon–Luecke (1989))

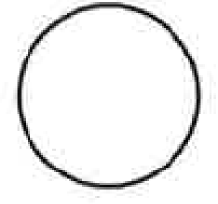
Knots are determined† by their complements.‡*

Hence if k is a knot, $\pi_1(S^3 \setminus k)$ is a knot invariant. A presentation can be computed mechanically from a diagram of the knot.

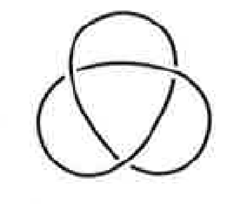
*which are tame

†modulo ambient isotopy

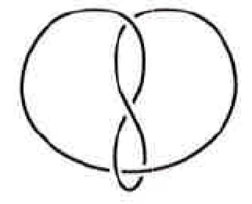
‡in S^3 moduli orientation-preserving homeomorphisms



$\langle x \rangle$



$\langle x, y : xyx = yxy \rangle$



$\langle x, y : x^{-1}yxy^{-1}xy = yx^{-1}yx \rangle$.

It is a nontrivial computational problem to check that these groups are not isomorphic.

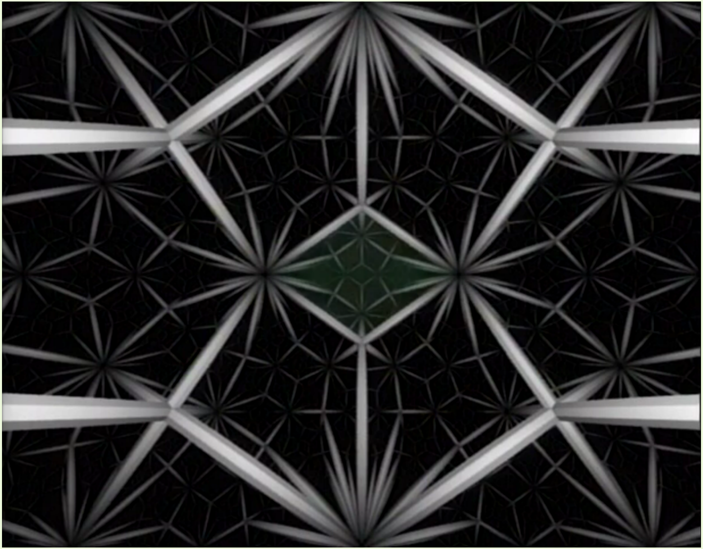
Theorem (William Thurston (1970s))

Most 3-manifolds are hyperbolic. More precisely, they are a quotient $\mathbb{H}^3/G$ where G is a discrete group of hyperbolic isometries.

That is, locally most 3-manifolds look like a polyhedron (maybe not finitely sided) in $\mathbb{H}^3$ with faces glued.

Screenshot from *Hyperbolica* (CodeParade, 2022).

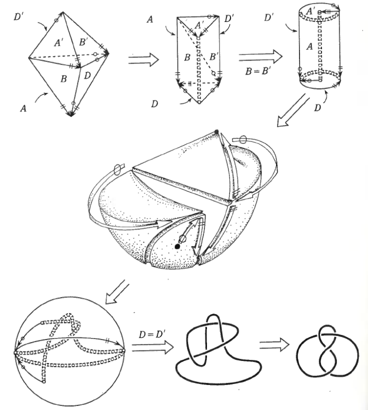
The complement of the Borromean rings



From *Not Knot* (Geometry Center, 1991). https://www.youtube.com/watch?v=4aN6vX7qXPQ

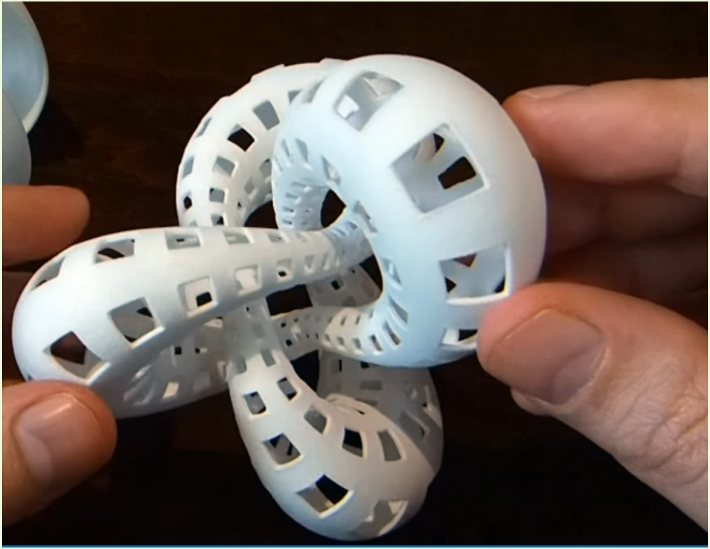
Theorem (Robert Riley (c.1974); William P. Thurston (c.1975))

The figure 8 knot complement admits a hyperbolic geometry.



Matsuzaki and Taniguchi, *Hyperbolic manifolds and Kleinian groups*, p.34.

The hyperbolic structure



[Guéritaud/Segerman/Schleimer, `https://youtu.be/xGf5jY_v5GE`]

Theorem (Gromov–Jørgensen–Thurston)

The set of volumes of hyperbolic manifolds is a well-ordered subset of $\mathbb{R}$. The set of manifolds with any given volume is finite.

Hyperbolic volume of the complement turns out to be a very good link invariant. It can be computed algorithmically.

Example

The volume of the figure eight knot complement is

$$-6\int_0^{\pi/3} \log|2\sin\theta|\,d\theta = 2.02988\ldots$$

§II. Braids

What is...a braid?

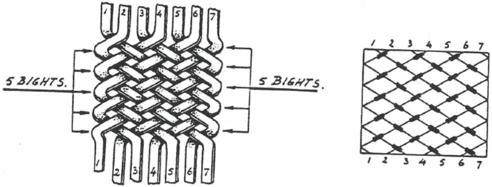


Fig. 6. The (7P, 5B) regular flat braid, with Turk's Head coding

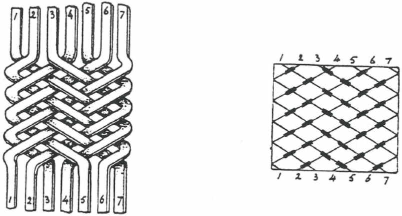
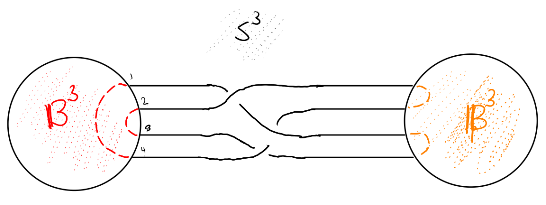


Fig. 7. The (7P, 5B) regular flat braid, with Two Pass Headhunter's coding.
Figs. 6 and 7 demonstrate two different braids with the same whole string run

J. C. Thurner and P. v.d.Griend (eds.), *History and science of knots*, p. 284.

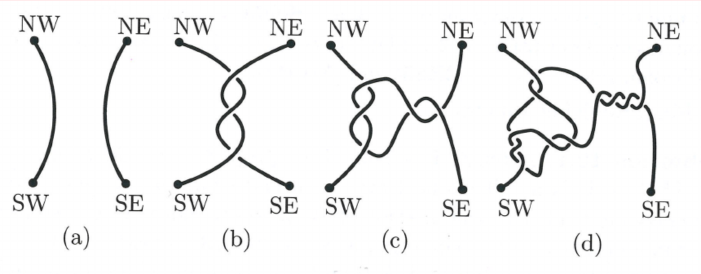
Instead of braiding strings glued onto two planes, braid strings glued on two 2-spheres in S^3 (mod ambient isotopy of S^3).



Theorem

Every braid with ends glued on spheres admits a unique completion to a knot/link which cannot be simplified by 'untwisting'.

We will care only about braids with four strands, completed at one end. We will call these objects **rational tangles**.



J. Purcell, *Hyperbolic knot theory*, p. 208.

Every rational tangle is given by a sequence of integers, this one is [4, −2, −2, 3].

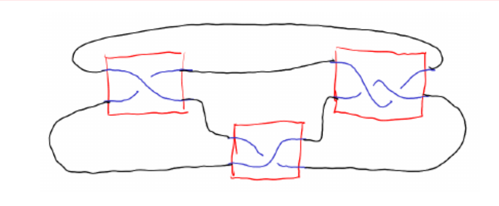
A two-bridge knot (or link) is the knot obtained by completing a rational tangle.

Theorem (Schubert (1956), Conway (1970))

Rational tangles and two-bridge links are indexed by $\mathbb{Q} \cup \{\infty\}$:

$$[a_n, a_{n-1}, \dots, a_1] \leftrightarrow a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}$$

We write $k(p/q)$ for the link indexed by $p/q \in \mathbb{Q}$.



Example

The figure eight knot has rational form $1 + 1/(1 + 1/2) = 5/3$.

Theorem (Riley (1972))

Every two-bridge link $k(p/q)$ has a fundamental group on two generators and one relation

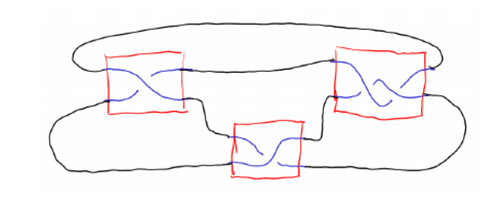
$$\langle X, Y : W_{p/q} X = Y W_{p/q} \rangle$$

where $W_{p/q}$ is some word in X and Y depending only on p/q. This group admits a representation into $\mathrm{PSL}(2, \mathbb{C})$ given by

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \qquad Y_\rho = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix}$$

*where $\rho \in \mathbb{C}$ depends only on p/q.**

*Different authors use p/q or q/p for different corresponding objects.



Example

In this case the Riley representation is faithful and the fundamental group is

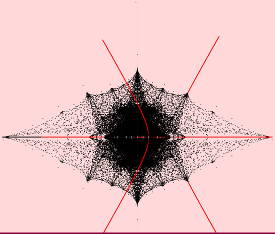
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -e^{2\pi i/3} & 1 \end{bmatrix} \right\rangle.$$

The corresponding word is $W_{5/3} = Y^{-1}X^{-1}YXYX^{-1}Y^{-1}XYX$.

The manifold obtained by taking the complement of the rational tangle t (i.e. $S^3 \setminus B^3 \cup t$) does not have a unique hyperbolic structure. The space of all possible hyperbolic structures is one dimensional over $\mathbb{R}$, and the set of all hyperbolic structures is indexed by the component of the set

$$\{\rho \in \mathbb{C} : \operatorname{tr} W_{p/q}(\rho) \in (-\infty, -2)\}$$

with asymptotic angle $\pi p/q$. The p/q knot complement is somehow the 'limit' of the sequence of geometric structures on complements of p/q tangles.



Theorem (E.–Martin–Schillewart (2022))

If $\begin{vmatrix} p & r \\ q & s \end{vmatrix} = \pm 1$, *then*

$$\operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{(p+r)/(q+s)} + \operatorname{tr} W_{|p-r|/|q-s|} = 8$$

as a polynomial in ρ.

Really, this is a recursion down the tree of continued fractions. Doing a horizontal twist corresponds to 'adding' 0/1, and doing a vertical twist corresponds to 'adding' 1/0.

Example polynomials

0/1	$2-z$
1/1	$2+z$
1/2	$2+z^2$
2/3	$2-z-2z^2-z^3$
3/5	$2+z+2z^2+3z^3+2z^4+z^5$
5/8	$2+4z^4+8z^5+8z^6+4z^7+z^8$
8/13	$2-z-2z^2-5z^3-12z^4-22z^5-32z^6-44z^7-54z^8-53z^9-38z^{10}-19z^{11}-6z^{12}-z^{13}$
13/21	$2+z+2z^2+7z^3+14z^4+31z^5+64z^6+124z^7+214z^8+339z^9+498z^{10}+699z^{11}+936z^{12}$ $+1148z^{13}+1216z^{14}+1064z^{15}+746z^{16}+409z^{17}+170z^{18}+51z^{19}+10z^{20}+z^{21}$
21/34	$2+z^2+8z^4+24z^5+68z^6+192z^7+516z^8+1256z^9+2834z^{10}+5912z^{11}+11460z^{12}$ $+20816z^{13}+35598z^{14}+57248z^{15}+86446z^{16}+122560z^{17}+163199z^{18}$ $+203952z^{19}+238564z^{20}+259704z^{21}+260686z^{22}+238320z^{23}+195694z^{24}$ $+142328z^{25}+90451z^{26}+49552z^{27}+23058z^{28}+8952z^{29}+2831z^{30}+704z^{31}$ $+130z^{32}+16z^{33}+z^{34}$

Advertisement: Minicourse on knot theory and geometry

When?	Two lectures every week of July.
Where?	Dept. of Mathematics, The University of Auckland.
What?	Classical knot theory. Geometric knot theory and hyperbolic invariants. Braids and mapping classes. Knot polynomials (Alexander, Conway, Jones, HOMFLY-PT).
Prereqs?	Basic topology (what is π_1). Passing familiarity with classical hyperbolic geometry in 2 or 3 dimensions.
Email	aelz176@aucklanduni.ac.nz

Bedtime reading

- A.J.E., Gaven Martin, and Jeroen Schillewaert, "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021-22 MATRIX annals*. Springer, to appear.
- —, "The combinatorics of the Farey words and their traces". arXiv:2204.08076 [math.GT], 2022.
- William P. Thurston, "Three dimensional manifolds, Kleinian groups and hyperbolic geometry". In: *Bulletin (NS) of the AMS* **6**(3) pp.357–381, 1982.
- Benson Farb and Dan Margalit, *A primer on mapping class groups*. Princeton, 2012.
- Jessica Purcell, *Hyperbolic knot theory*. AMS, 2021.
- Title picture: A. Gardiner, *Egyptian grammar*. Griffith Institute, 1957.

Proof of the 2022 theorem

Suppose $p/q < r/s$ and $\begin{vmatrix} p & r \\ q & s \end{vmatrix}$.

- (Word products.) By careful consideration of the ergodic behaviour of the lift of the curves represented by $W_{p/q}$, $W_{r/s}$, and $W_{(p+r)/(q+s)}$ to the universal cover $\mathbb{H}^2$ of the four-punctured sphere, we see that $W_{(p+r)/(q+s)} = W_{p/q}W_{r/s}$ with the $(q+s)$th generator in the word inverted.
- (Product and quotient lemmata.) Then by standard trace identities in $\mathrm{PSL}(2,\mathbb{C})$ we see that

$$\operatorname{tr} W_{p/q}W_{r/s} + \operatorname{tr} W_{(p+r)/(q+s)} = \begin{cases} \operatorname{tr}^2 X & \text{if } q+s \text{ is even} \\ \operatorname{tr} X \operatorname{tr} Y & \text{if } q+s \text{ is odd} \end{cases}$$

and

$$\operatorname{tr} W_{p/q}W_{r/s}^{-1} + \operatorname{tr} W_{|q-s|/|q-s|} = \begin{cases} \operatorname{tr}^2 Y & \text{if } q-s \text{ is even} \\ \operatorname{tr} X \operatorname{tr} Y & \text{if } q-s \text{ is odd.} \end{cases}$$

Proof of the 2022 theorem (ctd)

We proved that

$$\operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{(p+r)/(q+s)} = \begin{cases} \operatorname{tr}^2 X & \text{if } q+s \text{ is even} \\ \operatorname{tr} X \operatorname{tr} Y & \text{if } q+s \text{ is odd} \end{cases}$$

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{|q-s|/|q-s|} = \begin{cases} \operatorname{tr}^2 Y & \text{if } q-s \text{ is even} \\ \operatorname{tr} X \operatorname{tr} Y & \text{if } q-s \text{ is odd;} \end{cases}$$

- (Standard identity.) In $\mathrm{PSL}(2, \mathbb{C})$, $\operatorname{tr} A \operatorname{tr} B = \operatorname{tr} AB + \operatorname{tr} AB^{-1}$.
- Adding the displayed equations and applying the standard identity gives the recurrence. (In fact we have proved more, we only claimed the special case $\operatorname{tr} X = \operatorname{tr} Y = 2$ but we have proved it for arbitrary X and Y.)