THE COMBINATORICS OF FAREY WORDS AND THEIR TRACES

ALEX ELZENAAR, GAVEN MARTIN, AND JEROEN SCHILLEWAERT

Abstract. The set of Kleinian groups which are free on two parabolic generators is parameterised by the closed Riley slice of Schottky space. A 'Farey word' is a word in such a group which represents a non-boundary-parallel geodesic that can be pinched down to a puncture; in the interior of the Riley slice such a word is loxodromic, and the pinching process corresponds to deforming the word to be parabolic. Keen and Series showed that the geometry of the Riley slice is detected by the real loci of the trace polynomials of these words. We study these trace polynomials from a combinatorial viewpoint, and give a recursion formula for them which enables efficient calculation of the polynomials without performing matrix multiplication; we also present some intriguing examples and conjectures which we would like to bring to the attention of researchers interested in algebraic combinatorics and hypergeometric functions.

The results in this paper are a practical requirement for an upcoming work classifing all arithmetic subgroups of $\mathrm{PSL}(2,\mathbb{C})$ generated by two elements, since they make it possible to carry out a practical computation of Farey words (which provide certificates that certain groups are non-free).

1. Introduction

The work in this paper is important for upcoming work on the enumeration of arithmetic groups of $\mathrm{PSL}(2,\mathbb{C})$ generated by two elements. An arithmetic group is, roughly speaking, a matrix group all of whose entries are algebraic integers, and it is known that there are only finitely many such subgroups of $\mathrm{PSL}(2,\mathbb{C})$ generated by two parabolic or elliptic generators [MM99]. The groups generated by two parabolic elements are known, and there are only four of them [GMM98]. All of these groups are link groups (that is, holonomy groups of knots or links) and lie in the exterior of the set known as the 'parabolic Riley slice'. The enumeration of groups with elliptic generators is the subject of an in-preparation paper by the authors of the current paper, and this enumeration depends on having a practical method of computing traces of long words of specific combinatorial types ('Farey words') that represent geometrically significant curves on the surfaces; it is this practical computation which is made possible by the results of this paper. To be more precise, in order to classify the arithmetic groups it is necessary to have a practical algorithm to decide whether a given group on two elliptic generators is free. The algorithm is constructed by writing down two algorithms:

(AE) Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig
(GM) Institute for Advanced Study, Massey University, Auckland
(JS) Department of Mathematics, The University of Auckland
E-mail addresses: `elzenaar@mis.mpg.de, g.j.martin@massey.ac.nz, j.schillewaert@auckland.ac.nz`.

2020 *Mathematics Subject Classification.* 11B37, 11B39, 11B57, 20F36, 57K20 (primary), 32G15, 30F40, 33C45, 41A10 (secondary).

Key words and phrases. Kleinian groups, Schottky groups, two-bridge links, mapping class groups, trace identities, Stern-Brocot tree, Farey sequences, recurrence relations.

(1) An algorithm which terminates iff the input group $G \leq \mathrm{PSL}(2,\mathbb{C})$ is non-free. This can be implemented naïvely by enumerating words in G in order of length, computing the corresponding matrices, and then terminating if a matrix is seen more than once in the enumeration.
(2) An algorithm which terminates iff the input group $G \leq \mathrm{PSL}(2,\mathbb{C})$ is free. The naïve algorithm here is harder, and relies on some deep analysis of the geometry of G. More precisely, detailed analysis of the work of Keen and Series [KS94] carried out by the authors of the current work in both the parabolic case [EMS23a] and the elliptic case (work in preparation) gives a sequence of subsets covering the set of free subgroups of $\mathrm{PSL}(2,\mathbb{C})$ together with an explicit algebraic description of these subsets as real semialgebraic sets; in order to check if a group is free, one enumerates all of these sets in order and checks if the group is in each one. If the group is free, then the procedure terminates. These semialgebraic sets are cut out by equations corresponding to the traces of the Farey words, and so a computationally efficient way of computing these traces is necessary.

An algorithm which always terminates with a definite output deciding whether a group $G \leq \mathrm{PSL}(2,\mathbb{C})$ is free or not is then the 'interleaving' algorithm 'do step n of algorithm (1); do step n of algorithm (2); if either algorithm terminated then terminate with the right output according to which algorithm terminated; otherwise increment n and keep going'.

We will now explain the mathematical background behind this paper in more detail. A **Kleinian group** is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$; these groups have been intensively studied for a long time in association with hyperbolic geometry and conformal geometry [Bea83; Mar16; Mas87]. The **closed Riley slice** is the deformation space of Kleinian groups which are free on two parabolic generators. After normalisation, every group on two parabolic generators may be written as

$$\Gamma = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \right\rangle ;$$

the closed Riley slice, which we denote by $\overline{\mathcal{R}}$, is therefore naturally identified with the set of $z \in \mathbb{C}$ such that Γ is discrete and free; in this guise, it forms the exterior of the set approximated by the points of Figure 1 (along with the boundary of that set). The interior of $\overline{\mathcal{R}}$ is the **Riley slice**, which can also be characterised as the set of $z \in \mathbb{C}$ such that Γ is free, discrete, and the Riemann surface $\Omega(\Gamma)/\Gamma$ ($\Omega(\Gamma)$ being the domain of discontinuity of Γ, as defined in [Mas87, E.2]) has homeomorphism type a 4-times punctured sphere.

A detailed historical account of the Riley slice, together with background information for the non-expert and motivating applications (including a brief outline of earlier work on the classification of two-generated arithmetic groups), can be found in our proceedings paper [EMS23b]. Though it was first defined in the mid-20th century, recent work on the Riley slice follows on from a 1994 paper of Linda Keen and Caroline Series [KS94] (with some corrections by Yohei Komori and Caroline Series [KS98]). In this paper, they constructed a foliation of the Riley slice via a two-step process:

(1) Firstly, a lamination of $\mathcal{R}$ is defined, with leaves indexed by $\mathbb{Q}/2\mathbb{Z}$. These leaves consist of certain branches of preimages of $(-\infty,-2)$ under a family of polynomials $\Phi_{p/q}$ ($p/q \in \mathbb{Q}$) and are called the **rational pleating rays**.
(2) Then, by a completion construction similar to the completion of $\mathbb{Q}$ via Dedekind cuts, the lamination is extended to a foliation whose leaves are indexed by $\mathbb{R}/2\mathbb{Z}$. The leaves adjoined at this step are called the **irrational pleating rays**.

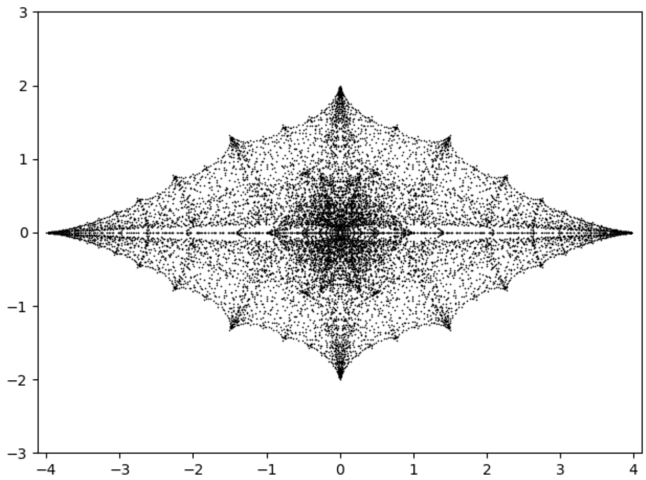
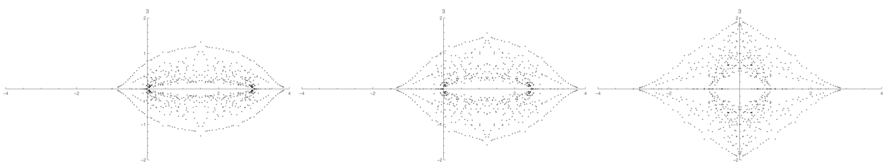


FIGURE 1. An approximation to the exterior of the Riley slice.



(A) $2\pi/3$, $2\pi/3$ (B) $2\pi/3$, $2\pi/4$ (C) $2\pi/3$, $2\pi/\infty$

FIGURE 2. Approximations to the exteriors of elliptic Riley slices with the indicated cone angles. See also Figure 8 of [EMS22].

The polynomials $\Phi_{p/q}$ are constructed as trace polynomials of certain words in the group, which we call **Farey words**; these words enumerate all but one of the non-boundary-parallel closed curves on the sphere, and the distinguished branches on the preimages of $(-\infty, -2)$ correspond to curves in the Riley slice along which the lengths of these curves change in a particularly natural way; in particular, the geometric limit of $\Omega(\Gamma)/\Gamma$ as z travels down the (p/q)-pleating ray towards the preimage of -2 is precisely a pair of three-times punctured spheres, where the additional pair of punctures appears as the loxodromic word $W_{p/q}$ is pinched to a parabolic word with trace -2. The group corresponding to the geometric limit is known as a **cusp group**, and the set of points corresponding to cusp groups are dense in the boundary of the slice by a result of Curtis McMullen [McM91]. The cusp points were studied in detail by David Wright [Wri05].

Punctured spheres can be viewed as a limiting case of cone-pointed spheres as the cone angles tend to zero. It is therefore natural to study the case of spheres with four cone points, which have corresponding groups (after normalisation)

$$\Gamma = \left\langle X = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{bmatrix}, Y_z = \begin{bmatrix} \beta & 0 \\ z & \beta^{-1} \end{bmatrix} \right\rangle;$$

where α and β are roots of unity, and Γ is discrete, free, and isomorphic to $\mathbb{Z}_a * \mathbb{Z}_b$ (where a and b are the respective orders of X and Y_z[1]). The deformation space

[1]There is a technical detail we sweep under the rug here with our notation: since the ambient group is $\mathrm{PSL}(2, \mathbb{C})$ and so $I = -I$, we need to take for example $\alpha = \exp(\pi i/a)$ rather than $\alpha = \exp(2\pi i/a)$.

of these groups is called the (a,b)**-Riley slice**, and we denote it by $\mathscr{R}^{a,b}$; in our paper [EMS22], we show that these spaces admit foliations which are defined in a completely analogous way to the Keen-Series foliation of the parabolic Riley slice. In fact, the polynomials which appear in the construction of this foliation are the trace polynomials of the same Farey words as the parabolic case (i.e. the only changes are in the coefficients of the matrices, not in the construction of the word whose trace is taken).

The goal of this paper is to study the combinatorial properties of these trace polynomials, which we call **Farey polynomials**, in both the parabolic and elliptic cases. Our main result is Theorem 5.7, which gives a recursion formula for the Farey polynomials that is independent of the geometric interpretation of the matrices (it works even if α and β are not roots of unity, in which case the groups are not discrete); in particular, it provides further evidence that the Riley slice exteriors can be studied in a completely dynamical way. This formula may be used to generate pictures of the Riley slice much more quickly than working directly from the definition of the Farey polynomials, as it removes the need for performing slow matrix multiplications; a working software implementation of this in `Python` may be found online [Elz21]. Similar recursions for the commutators $[X^n, Y]-2$ and the traces $\operatorname{tr}^2(X^n)-4$ were found by Alaqad, Gong, and Martin in [AGM21], and for the analogues of the Farey polynomials for the Maskit slice in [MSW02, pp. 283–285]. Links between these Farey words and other combinatorial words which occur in Teichmüller theory have been surveyed by Gilman [Gil06].

Notation. Throughout, $\Gamma = \langle X, Y \rangle$ is discrete and free with

$$X = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{bmatrix} \quad \text{and} \quad Y_z = \begin{bmatrix} \beta & 0 \\ z & \beta^{-1} \end{bmatrix}$$

where $z \in \mathbb{C} \setminus \{0\}$ and α, β are of norm 1. When discussing words we usually do not care about the precise value of z, and to simplify notation we will just write Y for Y_z. We use the convention $x := X^{-1}$, $y := Y^{-1}$ (and we use similar conventions throughout without comment).

1.1. **Structure of the paper.** There are seven sections to this paper beyond this introduction, of varying lengths. Section 2 gives a very quick introduction to the relationship between Farey words and the studies of (i) 2-bridge knots and links, (ii) mapping class groups, and (iii) continued fractions. Readers interested only in the combinatorics and number theory which is contained within the family of polynomials itself can skip to Section 3, which gives a combinatorial definition of the Farey words using cutting sequences; we take care to avoid requiring any knowledge of geodesic coding. In Section 4 we list some definitions and results about Farey sequences from classical number theory. Our main result (Theorem 5.7) is proved in Section 5; in the following section, Section 6, we use our results to compute the commutators of the Farey words in a couple of different ways. The final part of the paper consists of two sections; in Section 7, we apply the work of Chesebro and his collaborators [Che+20; Che20] to show a formal analogy between the Farey polynomials and the Chebyshev polynomials. Then in Section 8 we discuss some applications to the approximation of irrational pleating rays and cusps, and give some computational results which show that there are interesting connections with dynamical systems and number theory for future work to explore. All of the tables have been placed at the end of the paper to avoid breaking the flow of the text, as many are quite large.

Remark. A `MathRepo` page for this paper may be found at `https://mathrepo.mis.mpg.de/farey/index.html`.

1.2. **Acknowledgements.** Much of this paper appeared as part of the first author's MSc thesis [Elz22]. We would like to thank the organisers of the Groups and Geometry workshop on Waiheke Island in December, 2021 for the opportunity to speak, and the attendees of that conference for their interesting questions and helpful comments. The first author also thanks Thu Hien Nguyen for hosting him at the Mathematisches Forschungsinstitut Oberwolfach during June 2022 and sitting through many imperfect explanations of the Birman–Keen–Series theory before the exposition given here finally converged.

2. A panorama of mapping classes and knots

Let S be a 2-sphere with four marked points, two (resp. X and x) labelled with an integer a and the other two (resp. Y and y) labelled with an integer b (here, $0 < a, b \leq \infty$). We view S as the Riemann surface at infinity of some hyperbolic 3-orbifold O homeomorphic to an open 3-ball and with two singular arcs, one of order a joining X to x and one of order b joining Y to y. That is, there are a pair of homotopically distinct and nontrivial loops in the 3-orbifold—represented by elliptic (or parabolic) elements γ_1, γ_2 of respective orders a and b of the holonomy group of O (for the definition see e.g. [Thu97, §3.4])—which each bound singular arcs in O of respective orders a and b whose four endpoints are the marked points of S. The remainder of this section will describe some models for the moduli space of hyperbolic metrics which are induced on O by different arrangements of the arcs in 3-space, and the moduli space of complex structures which are induced on S when it is viewed as the horizon of O.

The key ideas about the interplay between Dehn twists and mapping classes on a punctured sphere and knots and braids in the interior go back a long way, to people like Joan Birman [Bir74] and Caroline Series [Ser85a; Ser85b]. Modern textbook references include [FM12] (for mapping classes) and [Pur20] (for knots). The specific example of the moduli space associated to the four-times punctured sphere is called the **Riley slice**, after Robert Riley who was the first to study it in relation to knot groups [Ril13; Ril72; Ril75a; Ril75b; Ril79; Ril92]; the relation to mapping class groups (the passage from hyperbolic space to the horizon) is essentially due to Linda Keen and Caroline Series [KS94] (with some corrections also due to Yohei Komori and Caroline Series [KS98]), and the case that the marked points are distinguishable in pairs has been studied by the authors of the current paper in a sequence of recent articles [EMS22; EMS23a], and the Master of Science thesis of the first author [Elz22]. A detailed historical account and literature survey may be found in our expository article [EMS23b].

In the following few paragraphs we take notation and definitions from [FM12], with minor modifications to allow us to distinguish points on the surface in pairs.

2.1. **The mapping class group.** Given our surface S with four marked points,[2] we write $\mathrm{Homeo}^+(S)$ for the group of orientation-preserving homeomorphisms of S which preserves both the set $\mathrm{Sing}(S) = \{X, x, Y, y\}$ and its complement and which acts on this 4-set in such a way as to preserve the marking integers—that is, a homeomorphism f is an element of $\mathrm{Homeo}^+(S)$ only if the integral label of z matches that of $f(z)$ for all $z \in \mathrm{Sing}(S)$. (We allow the two integers a and b to be equal, in

[2] We will often need to use standard algebro-topological tools on orbifolds rather than 'standard' topological spaces. Roughly speaking there is no difficulty: simply take the usual definitions and stipulate that two curves are homotopic in the orbifold iff they are homotopic in the underlying topological space such that if the homotopy requires moving across a marked point of order n it must move n pieces of the curve over the marked point. For formal definitions and justifications for why the standard theorems still pass to the orbifold case with only minimal and obvious changes, see Chapter III.$\mathcal{G}$ of [BH99].

which case every orientation-preserving homeomorphism which preserves $\mathrm{Sing}(S)$ is allowed even if it permutes the X's with the Y's). The (marked) **mapping class group** of S is the group $\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\sim$, where $f \sim g$ whenever f and g are isotopic via an isotopy which also preserves $\mathrm{Sing}(S)$ and its complement while respecting integral labels [FM12, §2.1].

2.2. **The braid group.** Again let S be the sphere with four marked points labelled in pairs. The labelling structure may be precisely modelled in the following way: let $C^{\mathrm{ord}}(S, 2, 2)$ be the set of 4 distinct ordered points on the sphere S (that is, $C^{\mathrm{ord}}(S, 2, 2)$ is the set $S^{\times 2} \times S^{\times 2} \setminus \mathrm{BigDiag}(S^{\times 4})$ where $\mathrm{BigDiag}(S^{\times 4})$ is the 'big diagonal' of 4-tuples of points where at least two of the points are repeated); there is a natural action of a subgroup $\mathcal{S}$ of $\mathrm{Sym}(4)$ on $C^{\mathrm{ord}}(S, 2, 2)$ which depends on whether $a = b$: if $a \neq b$ then $\mathcal{S}$ is the Klein 4-group $\mathrm{Sym}(2) \times \mathrm{Sym}(2)$ permuting the coordinates of each factor separately, and if $a = b$ then $\mathcal{S}$ is the whole $\mathrm{Sym}(4)$ and is allowed to permute all four points. The **pairwise configuration space** of four points on S is the quotient

$$C(S, 2, 2) = C^{\mathrm{ord}}(S, 2, 2)/\mathcal{S}.$$

The geometric interpretation is supposed to be clear: the first $S^{\times 2}$ component keeps track of the order a points and the second component keeps track of the order b points.

The homotopy group $\pi_1(C(S, 2, 2))$ (that is, the possible paths up to homotopy traced out by the four points without colliding, where paths must connect points with the same label) is called the **spherical braid group on 4 strands** [FM12, §9.1].

2.3. **The Birman exact sequence.** The main point which forms the basis of the Birman–Keen–Series theory of the mapping class groups of four-times marked spheres is the following. Let S be a sphere with four marked points which bounds a 3-orbifold O in such a way that the marked points are identified in pairs, as above; assume that $a \neq b$, so the pairs are distinguishable from each other. Then there is an exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \hookrightarrow \pi_1(C(S, 2, 2)) \xrightarrow{\mathcal{P}ush} \mathrm{Mod}(S) \longrightarrow 1 \tag{2.1}$$

which generalises the Birman exact sequence [FM12, §4.6; Bir74, §4.1], where

$$\mathrm{Mod}(S) = \mathrm{PSL}(2, \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{PSL}(2, \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \mathrm{Mod}(S_{0,4}).$$

Here, we use $S_{0,4}$ to denote the sphere with four indistinguishable marked points; the map $\mathrm{Mod}(S) \to \mathrm{Mod}(S_{0,4})$ is the evident inclusion map (every mapping class which respects the marking structure in pairs is also a mapping class when that pair-structure is forgotten), the map $\mathcal{P}ush$ is the map which sends a braid $\beta : [0, 1] \to C(S, 2, 2)$ to the induced homeomorphism $\beta(0) \to \beta(1)$, and where $\ker \mathcal{P}ush \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by a homeomorphism Θ which corresponds to a 2π rotation of the four marked points (equivalently, a single 2π twist added to the end of the braid); the image of this in the mapping class group is trivial (the twist can be undone by rotating the 'back' of the sphere via an isotopy without moving the points) and it is an involution in the braid group by the belt trick [Kau87, §VI.1]. It remains to check the two claimed mapping class group equalities:

Lemma 2.2. $\mathrm{Mod}(S_{0,4}) = \mathrm{PSL}(2, \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

Proof. This is a standard argument using the hyperelliptic involution and the standard computation of $\mathrm{Mod}(T^2)$, see e.g. [FM12, §2.2.5]. The point is that $\mathrm{Mod}(T^2) \simeq \mathrm{PSL}(2, \mathbb{Z})$ (proof: write T^2 as the quotient of $\mathbb{C}$ by some lattice Λ,

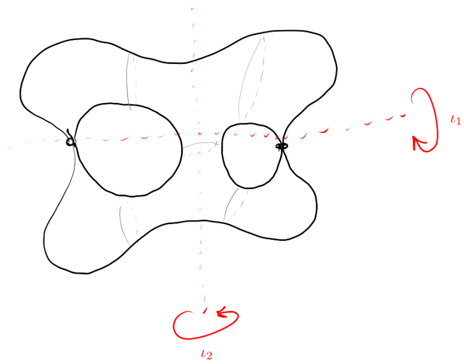


FIGURE 3. The two hyperelliptic involutions on the 4-punctured sphere.

and then show that $\mathrm{PSL}(2,\mathbb{Z})$ is the maximal group which permutes all of the different lattices which produce the same complex structure upon quotienting—see [IT87, §1.2] for some nice pictures); since both $S_{0,4}$ and T^2 are produced by taking quotients of a quadrilateral in $\mathbb{C}$ there is an induced surjective homomorphism $\mathrm{Mod}(S_{0,4}) \to \mathrm{PSL}(2,\mathbb{Z})$ given by topological lifting, and one can show that the kernel of this homomorphism is generated by the two half-rotations ι_1 and ι_2 of Figure 3. □

Via this geometric description of $\mathrm{Mod}(S_{0,4})$ it is easy to see that $\mathrm{Mod}(S)$ is exactly the same, but without the involution ι_2 which swaps the pairs of points with different markings. (If we assume the points in S have the same marking, i.e. $a = b$ and they are all indistinguishable, then $\mathrm{Mod}(S) = \mathrm{Mod}(S_{0,4})$ of course.)

2.4. **Four-plats; or, Viergeflechte; or, rational tangles.** Let S be the 2-sphere with four distinguished points in pairs as above, and let α_1 and α_2 be two disjoint paths on the sphere which join the points in pairs, preserving the integral labelling. If α_1 and α_2 are pushed slightly into the interior of the sphere without passing through each other, the resulting arrangement is called a **rational tangle**. Every two-bridge knot comes from taking such a rational tangle and pairing the four marked points on the sphere with two disjoint paths outside the sphere (which may not preserve the integer labels) which contract onto the sphere via an isotopy such that the images do not cross (this gluing is the so-called **numerator closure**).

There is a natural way to enumerate the rational tangles, essentially due to Schubert [Sch56] and described in [BZ03, §12.B] or [Pur20, Chapter 10]. First, take a sequence $a_0, \dots, a_m$ of integers. Every rational tangle is obtained by laying out four parallel strands (two labelled with the integer a and two with the integer b) and then alternatingly braiding the two leftmost strands and then the two middle strands with plaits of $a_0, a_1, \dots$ crossings (where the sign of each a_i denotes the direction of twisting to produce each braiding cluster); this produces a plait made up of four strands, which are joined like-labelling-to-like at one end to form two plaited cords, one labelled with a and one with b; the numerator closure is then the capping of the four remaining ends.

We now recall a standard fact from classical number theory. If $(a_0, a_1, \dots, a_k)$ is a finite sequence of integers, we define the **simple continued fraction**

$$[a_0; a_1, \dots, a_k] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots + \frac{1}{a_k}}}}}}.$$

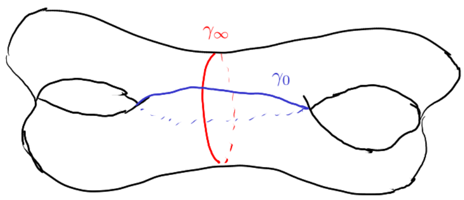


Figure 4. The first homology group of the 4-punctured sphere is isomorphic to $\mathbb{Z}^2$; one possible basis is formed by the two cycles γ_0 and γ_∞ depicted.

Proposition 2.3 ([HW60, Theorem 162]). *Every rational number can be expressed as a finite simple continued fraction in exactly two ways, one with an even and one with an odd number of convergents (number of sequence elements a_n). These are of the form*

$$[a_0; a_1, \dots, a_{N-1}, a_N, 1] \text{ and } [a_0; a_1, \dots, a_{N-1}, a_N + 1]$$

respectively, for some N. □

Remark. The simple continued fraction expression for a given rational number can be computed efficiently by repeated application of the Euclidean algorithm; see, for example, §10.9 of [HW60].

In any case, this relationship gives a bijection between the space of 2-bridge links and the set of rational numbers; the rational number associated to a given 2-bridge link is called the **rational form** or **Schubert normal form** for the link.

2.5. **Homology and 2-bridge links.** By the theory above, for every 2-bridge knot we obtain an element of the braid group $\pi_1(C(S,2,2))$, and hence an element of the mapping class group. (In fact, for each tangle we obtain an element of the braid group, but the image of a tangle under Θ gives the same knot up to isotopy, and so we only really get a well-defined element of the mapping class group.) Note that the involutions ι_1 and ι_2 both preserve the knot structure (whether the two strands/four marked points are indistinguishable or not), and so we in fact have an injection $\phi : \{\text{2-bridge knots}\} \to \mathrm{PSL}(2,\mathbb{Z})$. Observe next that the twisting sequence of a 2-bridge link k is in fact coding a sequence of Dehn twists needed to twist the unbraid into the rational tangle whose closure is k: if γ_0 and γ_∞ are the two curves marked in Figure 4 (which together form a basis for $H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^2$) and τ_0 and τ_∞ are the respective Dehn twists, then a rational tangle whose closure is k is represented by the element

$$\tau_0^{a_0} \tau_\infty^{a_1} \tau_0^{a_2} \tau_\infty^{a_3} \cdots$$

of the mapping class group. The action of the $\mathrm{PSL}(2,\mathbb{Z})$ semidirect multiplicand of $\mathrm{Mod}(S)$ on S can be identified with the usual matrix group action of $\mathrm{PSL}(2,\mathbb{Z})$ on $H_1(S,\mathbb{Z})$ after we choose this basis; take $\gamma_0 = (0,1)^t$ and $\gamma_\infty = (1,0)^t$, and write the standard generating set

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\langle R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \leq \mathrm{PSL}(2,\mathbb{C})$$

(c.f. the case of the punctured torus in [Ser85a]). If A is an element of $\mathrm{PSL}(2,\mathbb{Z})$, say

$$A = \begin{bmatrix} p & r \\ q & s \end{bmatrix},$$

then A sends $\gamma_\infty \mapsto p\gamma_\infty + q\gamma_0$ and $\gamma_0 \mapsto r\gamma_\infty + s\gamma_0$. Observe that this gives a map from $\mathrm{PSL}(2,\mathbb{Z})$ to the space of ordered singular $\mathbb{Z}$-homology bases of S (where the ordering is given by $p\gamma_\infty + q\gamma_0 \preceq r\gamma_\infty + s\gamma_0$ if $p/q \leq r/s$).

Now let $h : \pi_1(S) \to H_1(S)$ be the usual Abelianisation projection. Define the **geometric intersection** of a pair of homology classes $\alpha, \beta \in H_1(S)$ in the usual way, namely $i(\alpha,\beta)$ is the infimum, over all of the choices of σ and γ in the free homotopy classes of all curves in $h^{-1}(\alpha)$ and $h^{-1}(\beta)$ respectively, of $|\sigma \cap \gamma|$.

The following result is standard:

Proposition 2.4. *Suppose $\alpha = p\gamma_\infty + q\gamma_0$ and $\beta = r\gamma_\infty + s\gamma_0$ are arbitrary homology classes.*

(1) $i(\alpha,\beta) = \left|\det \begin{bmatrix} p & r \\ q & s \end{bmatrix}\right|$.
(2) *If $\gcd(p,q) \neq 1$, then $i(\alpha,\alpha) > 0$ (i.e. there is no simple closed curve on S in the homology class of α).*
(3) *If $\gcd(p,q) = 1$, then $i(\alpha,\alpha) = 0$; and further, there is exactly one non-freely-homotopic* geodesic *(with respect to any chosen hyperbolic metric on S) simple closed curve on S which projects to α under h. In this case, both α and $-\alpha$ correspond to the two orientations on this geodesic, and so the geodesic can be identified by the rational number $p/q = (-p)/(-q) \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$; we write $\gamma_{p/q}$ for this geodesic.*

Proof. For the homology class results one can adapt the proof of the analogous 'folk theorem' for compact surfaces, proved in [Mey76; Sch76]; they follow immediately from the Birman exact sequence Equation (2.1). A friendly and concrete sketch of the proof of the intersection number results in the case of punctures may be found as [Mar16, Exercise 2-6] (that reference is phrased in terms of tori but the changes to be made are obvious). The more general (torsion) situation is not too much harder but requires care in studying the lifting. The basic idea is to choose a topological covering $\rho : \mathbb{R}^2 \to S$ where the possible orbifold structure is ignored (more precisely, choose a cover $\mathbb{R}^2 \setminus \mathbb{Z}^2 \to S \setminus \operatorname{Sing} S$ as in the proof of Lemma 2.2), and then observe that in the homotopy class of the simple closed curve $\gamma_{p/q}$ there exists a representative which lifts to a line of slope p/q in $\mathbb{R}^2 \setminus \mathbb{Z}^2$, which has no self-intersection. Suppose we choose a representative line $L_{p/q}$; the lifts of the marked points above and below $L_{p/q}$ alternate in label between X and Y as one moves horizontally, and so $L_{p/q}$ cannot be moved across them via a homotopy. This reduces the situation to that studied in [Mar16] and [KS94; KS98]. $\square$

Now it is an easy exercise to check that

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle R, L\rangle \rtimes \langle Q\rangle \tag{2.5}$$

where L is the matrix

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(the key observation is that Q conjugates L to R^{-1}). Let $\Gamma_1 := \langle L, R\rangle$ be the orientation-preserving (with respect to the action on the upper half-plane) part of $\mathrm{PSL}(2,\mathbb{Z})$; then Γ_1 is in bijection with the space of unordered bases of $H_1(S,\mathbb{Z})$, that is pairs of homotopically distinct and homologically nontrivial curves on S. By direct computation we see now that the action of Γ_1 as a subset of the mapping class group is

$$\begin{aligned} R.\gamma_\infty &= \gamma_\infty, \quad & R.\gamma_0 &= \gamma_{1/1} \\ L.\gamma_\infty &= \gamma_{1/1}, \quad & L.\gamma_0 &= \gamma_0; \end{aligned}$$

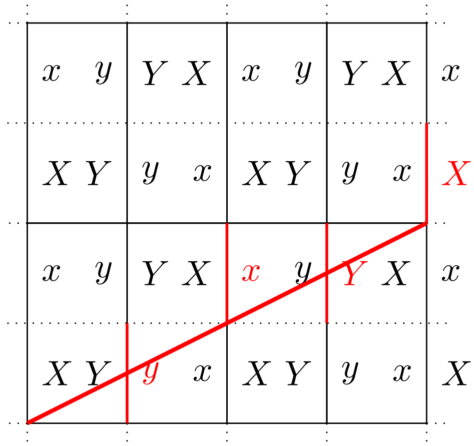


FIGURE 5. The cutting sequence of the 1/2 Farey word.

that is, R acts as τ_∞ and L acts as τ_0. In particular, we have a bijection between the space of two-bridge knots and the group Γ_1 such that the knot with rational form $p/q = [a_0; a_1, \dots, a_N]$ is represented by the matrix

$$L^{a_0} R^{a_1} L^{a_2} \cdots .$$

3. Cutting sequences and Farey words

We define the Farey word[3] of slope p/q via cutting sequences: this is essentially an interpretation of Dehn's classical algorithm for curves on surfaces [Sti83, Chapter 6] in the language of symbolic dynamics and mapping class groups by Birman and Series [BS87], and for the concrete case of interest to us we follow [Ser85a], [Ser91], and [KS94].

Consider the marked tiling of $\mathbb{R}^2$ shown in Figure 5, and let $L_{p/q}$ be the line through $(0,0)$ of slope p/q; now define $S_{p/q} = L_{p/q} \cap 2\mathbb{Z}^2$. Then the **Farey word of slope** p/q, $W_{p/q}$, is the word of length $2q$ such that the ith letter is the label on the right-hand side of the ith vertical line segment crossed by $L_{p/q}$ (i.e. the label to the right of the point $(p/q)i$); if $(p/q)i$ is a lattice point then this definition is ambiguous and by convention we take the label on the north-east side. In other words, the ith letter of $W_{p/q}$ is determined by the parity of $\operatorname{ceil}(p/q)i$ with the convention that $\operatorname{ceil} n = n+1$ for integral n. Observe that $L_{p/q}$ is the lift of a curve $\gamma_{p/q}$ on the surface S of the previous section to the cover $\mathbb{R}^2 \setminus \mathbb{Z}^2$ with the property that $\gamma_{p/q}$ represents the homology class $p\gamma_\infty + q\gamma_0$. (The word $W_{p/q}$, though, has no *a priori* relationship to the word in R and L which represents $\tau_{p/q}$ in the mapping class group—$W_{p/q}$ alternates in X and Y with powers ± 1, while the word in the mapping class group could have arbitrary powers.)

[3]We name these words and the related polynomials after John Farey Sr. as they are closely related to the so-called Farey sequences of rational numbers which we will discuss briefly later in this paper; with regard to this attribution, we quote from the historical notes to Chapter III of Hardy and Wright [HW60, pp. 36–37]: "The history of 'Farey series' is very curious... [their properties] seem to have been stated and proved first by Haros in 1802... Farey did not publish anything on the subject until 1816. [...] Mathematicians generally have followed Cauchy's example in attributing the results to Farey, and the series will no doubt continue to bear his name. Farey has a notice of twenty lines in the *Dictionary of national biography* where he is described as a geologist. As a geologist he is forgotten, and his biographer does not mention the one thing in his life which survives."

Example 3.1. As an example of the construction process, from Figure 5 we can read off that $W_{1/2} = yxYX$; we include a list of the Farey words with $q \leq 12$ as Table 1.

There are various symmetries visible in the Farey words; for instance, they are alternating products of $X^{\pm 1}$ and $Y^{\pm 1}$ which always end in X (this is obvious from the definition). The following slightly less trivial symmetry will be useful in the sequel:

Lemma 3.2. *Let $W_{p/q}$ be a Farey word; then the word consisting of the first $2q-1$ letters of $W_{p/q}$ is conjugate to X or Y according to whether the qth letter of $W_{p/q}$ is $X^{\pm 1}$ or $Y^{\pm 1}$ (i.e. according to whether q is even or odd respectively).*

Proof. This identity comes from considering the rotational symmetry of the line of slope p/q about the point (q,p); it is clear from the symmetry of the picture that the first $p-1$ letters of $W_{p/q}$ are obtained from the $(p+1)$th to $(2p-1)$th letters by reversing the order and swapping the case (imagine rotating the line by 180 degrees onto itself and observe the motion of the labelling). $\square$

Remark. For a more precise statement see e.g. the first lemma of Section 2 of our sister paper [EMS23a].

We do not use the following lemma in this paper, but we state it for interests' sake while we are studying the symmetries of the Farey words.

Lemma 3.3. *Let p/q be a fraction in least terms; then $W_{-p/q}$ is obtained from $W_{p/q}$ by the following process: swap the sign of the exponent of every letter except for the qth and $2q$th letters.*

Proof. For all i such that p/qi is non-integral, we have $\operatorname{ceil} -\frac{p}{q}i = -\operatorname{floor}\frac{p}{q}i = 1 - \operatorname{ceil}\frac{p}{q}i$. In particular, the parity of the ceiling is swapped and so the sign of the exponent is swapped. For i such that p/q is integral (i.e. i is q or $2q$), due to our convention on the ceiling function, we have $\operatorname{ceil} -\frac{p}{q}i = 1 - \frac{p}{q}i$ and $\operatorname{ceil}\frac{p}{q}i = \frac{p}{q}i + 1$, so the parities of $\operatorname{ceil} -\frac{p}{q}i$ and $\operatorname{ceil}\frac{p}{q}i$ are equal and the exponent of the corresponding letter is unchanged. $\square$

The **Farey polynomial of slope** p/q is defined by $\Phi_{p/q} := \operatorname{tr} W_{p/q}$; this is a polynomial in z of degree q, with coefficients rational functions of α and β. If we wish to emphasise the dependence on a and b, we write $\Phi^{a,b}_{p/q}$.

Remark. The Farey polynomials are not to be confused with the so-called **Riley polynomials** $\Lambda_{p/q}$ defined by Riley [Ril72] and studied e.g. by Chesebro [Che20]; the Riley polynomials are also related to knot theory, in particular to the Alexander polynomials, see Chapter 11 of [Kau87] (in particular the final exercise). See also Remark 5.3.13 of [Aki+07].

Example 3.4. We list the Farey polynomials $\Phi^{a,b}_{p/q}$ with $q \leq 4$ in Table 2. This illustrates some of the difficulty in studying these polynomials: they are clearly very symmetric, but quickly become too unwieldy to write explicitly and so actually guessing what the symmetries *are* in general is hard. We also list the first few 'Fibonacci' Farey polynomials $\Phi^{\infty,\infty}_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}$ (as usual, $\mathsf{fib}(1) = 1$, $\mathsf{fib}(2) = 1$, and $\mathsf{fib}(n) := \mathsf{fib}(n-1) + \mathsf{fib}(n-2)$) in Table 3.

4. Farey theory

Our recursion formula, like that for the Maskit slice in [MSW02, pp. 283–285], is a recursion down the **Farey diagram**. We therefore take a quick break from the

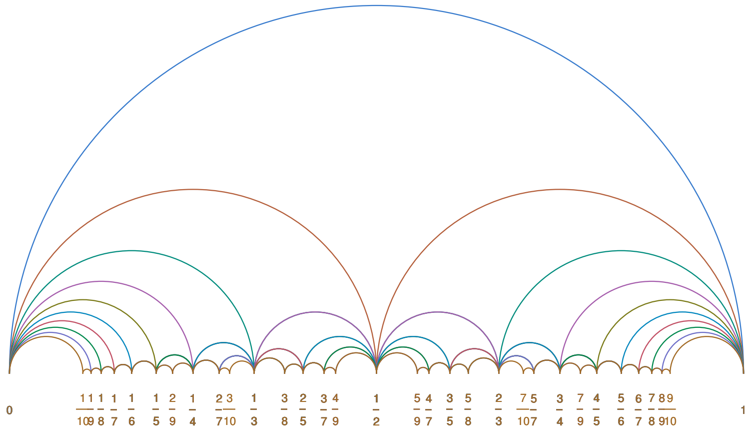


FIGURE 6. The subset of the Farey triangulation of $\mathbb{H}^2$ with vertices $p/q \in [0,1] \cap \mathbb{Q}$ such that $q \leq 10$.

Keen–Series theory to recall some of the notation and ideas, which can be found for instance in [GKP94, §4.5] or [HW60, Chapter III].

We observed in Section 2.5 that there is a subgroup Γ_1 of $\mathrm{PSL}(2,\mathbb{Z})$ generated by the two matrices

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which acts via componentwise fractional linear transformations on the set of ordered pairs $(p/q, r/s)$ with $p/q < r/s$ and $|ps - rq| = 1$ (the action being essentially the faithful action of Γ_1 on itself by left multiplication).

Definition 4.1. If $p/q, r/s \in \hat{\mathbb{Q}}$ satisfy the determinant condition $|ps - rq| = 1$ then we say that p/q and r/s are **Farey neighbours**.

The primary elementary lemma on Farey neighbourliness is the following, which appears in [HW60, §3.3]:

Lemma 4.2. *If p/q and r/s are Farey neighbours with $p/q < r/s$, then $p/q < (p+r)/(q+s) < r/s$, and $(p+r)/(q+s)$ is the unique fraction of minimal denominator between p/q and r/s. More precisely, let u/v be any fraction in $(p/q, r/s)$; then there exist two positive integers λ, μ such that*

$$u = \lambda p + \mu r \text{ and } v = \lambda r + \mu s$$

(so of course the minimal denominator is obtained when $\lambda = \mu = 1$). □

The operation

$$(p/q, r/s) \mapsto (p+r)/(q+s)$$

will be fundamental to our later study; it is called the **mediant** or **Farey addition** operation. We write $(p/q) \oplus (r/s)$ for the Farey addition of p/q to r/s. We will be careful to only combine p/q and r/s in this way if they are Farey neighbours.

We now check that the usual action of Γ_1 on $\hat{\mathbb{Q}}$ as a group of fractional linear transformations agrees with the action of Γ_1 on the space of homology bases, in the following way. Suppose that r_1/s_1 and r_2/s_2 are Farey neighbours with $p/q = r_1/s_1 \oplus r_2/s_2$, and let $A \in \Gamma_1$ be represented by the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix};$$

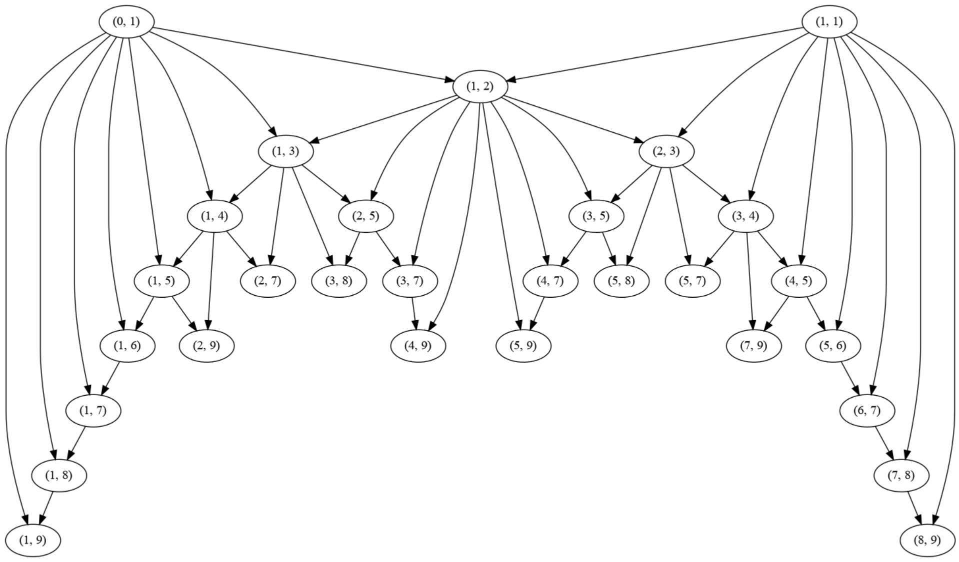


FIGURE 7. The Farey addition graph.

A acts on the pair $\{r_1/s_1, r_2/s_2\}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left\{ \frac{r_1}{s_1}, \frac{r_2}{s_2} \right\} = \left\{ \frac{ar_1 + bs_1}{cr_1 + ds_1}, \frac{ar_2 + bs_2}{cr_2 + ds_2} \right\},$$

a simple computation shows that the Farey sum of the images is exactly

$$\frac{a(r_1 + r_2) + b(s_1 + s_2)}{c(r_1 + r_2) + d(s_1 + s_2)} = \frac{ap + bq}{cp + dq};$$

i.e. the image of p/q under A with respect to the usual action of $\mathrm{PSL}(2, \mathbb{R})$ as a fractional linear transformation.

The ideal triangle Δ spanned by

$$(1/0, 1/1, 0/1) = (\infty, 1, 0)$$

does not bound a fundamental domain for this action, since $X^{-1}(D) = Y^{-1}(D)$; however, the images of Δ under $\mathrm{PSL}(2, \mathbb{Z})$ do tessellate $\mathbb{H}^2$ in the sense that if $W\Delta \cap W'\Delta \neq \emptyset$ then either the intersection is a shared edge or the two images are equal. The simplicial complex thus formed is called the **Farey triangulation** of $\mathbb{H}^2$; we denote it by $\mathcal{D}$. The portion of $\mathcal{D}$ 'above' the segment $[0, 1]$ is depicted in Figure 6. The vertices of $\mathcal{D}$ are exactly the points of $\hat{\mathbb{Q}}$, and the 2-faces are triangles with vertex triples of the form $\{p/q, (p+r)/(q+s), r/s\}$ where $ps - qr = \pm 1$. This second assertion follows immediately from the definition of $\mathcal{D}$ as a tessellation: if $\{a/b, c/d, e/f\} \in \mathcal{D}(2)$, then there exists some matrix

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z})$$

which acts on the original triangle as

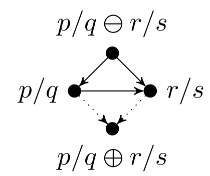
$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix},$$

giving immediately that $(a/b, c/d, e/f) = (p/q, (p+r)/(q+s), r/s)$; i.e. the triangles of $\mathcal{D}$ are exactly the triples $(\alpha, \alpha \oplus \beta, \beta)$ for Farey neighbours $\alpha, \beta \in \hat{\mathbb{Q}}$. The **Farey graph** is the digraph with vertices $\hat{\mathbb{Q}}$ and directed edges from p/q and r/s to $p/q \oplus r/s$ (Figure 7). This is just the 1-skeleton of $\mathcal{D}$ with an added orientation.

It will be convenient, finally, to have the notation $p/q \ominus r/s$ for the fraction $(p-r)/(q-s)$; we shall only use this when it is known that $(p-r)/(q-s)$ and r/s are Farey neighbours (which is implied by neighbourliness of p/q and r/s). If $(p/q, r/s)$ is an edge in the Farey triangulation, then it forms the boundary between the two triangles $(p/q, p/q \oplus r/s, r/s)$ and $(p/q, p/q \ominus r/s, r/s)$.

5. A recursion formula to generate Farey polynomials

In this section, we will give a recursion formula for the Farey polynomials. This recursion will be a recursion 'down the Farey graph', in the sense that its input will be the Farey polynomials at the vertices of a triangle $(p/q, p/q \ominus r/s, r/s)$ and its output will be the Farey polynomial at $p/q \oplus r/s$:



We begin by finding a similar recurrence for the Farey words; we will then produce a recurrence for the polynomials using standard trace identities and elbow grease. One might guess, for instance by analogy with the Maskit slice [MSW02, p. 277], that $W_{p/q}W_{r/s} = W_{p/q\oplus r/s}$. It is easy to check whether or not this is true:

Example 5.1. We use the convention $x := X^{-1}$, $y := Y^{-1}$:

- $W_{1/2} = yxYX$, $W_{1/1} = YX$,
$$W_{1/2}W_{1/1} = yxYXYX, \text{ and}$$
$$W_{1/2\oplus 1/1} = W_{2/3} = yxyXYX.$$
- $W_{1/3} = yXYxYX$, $W_{2/5} = yXYxyXyxYX$,
$$W_{1/3}W_{2/5} = yXYxYXyXYxyXyxYX, \text{ and}$$
$$W_{1/3\oplus 2/5} = W_{3/8} = yXYxYXyxYxyXyxYX.$$

Example 5.1 shows that our guess is almost correct; the corrected statement is:

Lemma 5.2. *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then $W_{p/q\oplus r/s}$ is equal to:*

(1) *$W_{p/q}W_{r/s}$ with the sign of the $(q+s)$th exponent swapped;*
(2) *$W_{r/s}W_{p/q}$ with the sign of the $(2s)$th exponent swapped.*

Proof. We prove (1); the proof of (2) is done in the same way, and we briefly indicate the difference at the end. The situation is diagrammed in Figure 8 for convenience. To simplify notation, in this proof we write $h(i)$ for the height $(\frac{p}{q} \oplus \frac{r}{s})i$. Observe that $h(i)$ is integral only at $i = 0$ and $i = 2q + 2s$ (at both positions trivially the letters in $W_{(p/q)\oplus(r/s)}$ and $W_{p/q}W_{r/s}$ are identical) and at $i = q + s$. The lemma will follow once we check that that at the positions $i \notin \{0, 2q+2s\}$,

$$\text{if } 0 < i \leq 2q \text{ then } (p/q)i < h(i) < \text{ceil}(p/q)i \tag{5.3}$$

and

$$\text{if } 0 < i < 2s \text{ then } (r/s)i + 2q < h(i + 2p) < \text{ceil}[(r/s)i + 2p] : \tag{5.4}$$

indeed, these inequalities show that at every integral horizontal distance the height of the line corresponding to $W_{p/q\oplus r/s}$ is meeting the same vertical line segment as the line corresponding to $W_{p/q}$ or $W_{r/s}$, and so the letter chosen is the same except at $i = q + s$ since at this position the height of the line of slope $(p/q) \oplus (r/s)$,

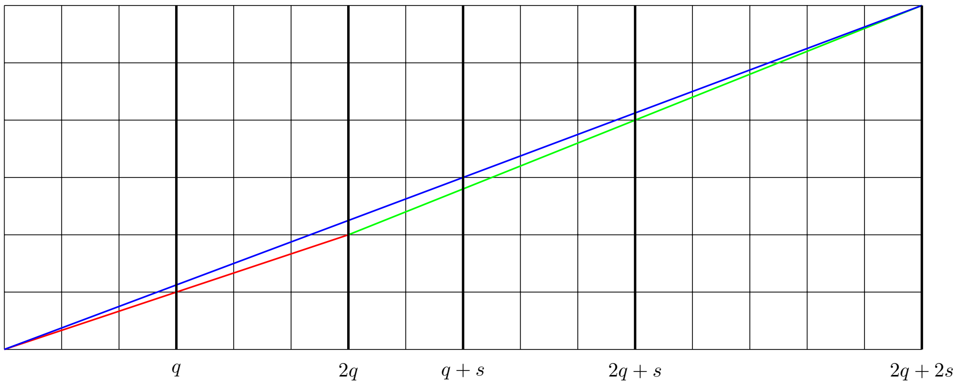


FIGURE 8. Farey addition versus word multiplication for $W_{p/q}$ (red) and $W_{r/s}$ (green).

being integral, is rounded up to $h(i)+1$ while the height of the line of slope r/s is non-integral so is rounded up to the integer $h(i)$.

Observe now that the inequalities Equations (5.3) and (5.4) are equivalent to the following: there is no integer between $(p/q)i$ and $(p/q \oplus r/s)i$ (exclusive) if $0 < i \le 2q$, and there is no integer between $(r/s)i + 2q$ and $h(i+2p)$ if $0 < i < 2s$. But these follow from Lemma 4.2. Indeed, the lemma shows that no integer lies between p/q and $(p/q) \oplus (r/s)$; suppose a/b is a rational between $(p/q)i$ and $h(i)$, then $a = i(\lambda p + \mu(p+r))$ and $b = (\lambda q + \mu(q+s))$ for some positive λ, μ; suppose $a/b \in \mathbb{Z}$, so $\lambda q + \mu(q+s)$ divides $i(\lambda p + \mu(p+r))$. By the case $i = 1$, $\lambda p + \mu(p+r)$ and $\lambda q + \mu(q+s)$ are coprime, so $\lambda q + \mu(q+s)$ divides i; but $\lambda q + \mu(q+s) \ge 2q$. The case of $(r/s)i + 2q$ and $h(i+2p)$ is proved in a similar way. (The proof of part (2) differs only in the relative position of the 'kink' between the concatenated p/q and r/s segments and the $(p+r)/(q+s)$ segment: the kink is at position $2s$ horizontally and lies above the $(p+r)/(q+s)$ line rather than below, so it is at the $(2s)$th position that the 'modified ceiling' makes a nontrivial appearance.) $\square$

Lemma 5.2 is not new; for instance, part (1) appears as Propostion 4.2.4 of [Zha10], and part (2) was known to Keen and Series at least (though as far as we know they did not publish it, see [KS94, Remark at end of §4]). However, the following consequences for the Farey polynomials are comparatively nontrivial to prove and have not appeared before in the literature.

Lemma 5.5 (Product lemma)**.** *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identity holds:*

$$\operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{p/q \oplus r/s} = \begin{cases} 2 + \alpha^2 + \frac{1}{\alpha^2} & \text{if } q+s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q+s \text{ is odd.} \end{cases}$$

Proof. Trace is invariant under cyclic permutations, thus (applying Lemma 5.2) we can write

$$\operatorname{tr} W_{p/q} W_{r/s} = \operatorname{tr} AB \text{ and } \operatorname{tr} W_{p/q \oplus r/s} = \operatorname{tr} AB^{-1},$$

where B is the $(q+s)$th letter of $W_{p/q} W_{r/s}$ and A is the remainder of the word but with the final letters cycled to the front. Now we know that $\operatorname{tr} AB = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1}$ (see the useful list of trace identities found in Section 3.4 of [MR03]), so it suffices to check that $\operatorname{tr} A \operatorname{tr} B = 2 + \alpha^2 + \frac{1}{\alpha^2}$ if $q+s$ is even and $\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$ otherwise.

Case I. $q+s$ is even. Observe now that B is $X^{\pm 1}$ if $q+s$ is even; then $\operatorname{tr} B = 2\Re\alpha$. The identity to show is therefore $\operatorname{tr} A = (2+\alpha^2+\frac{1}{\alpha^2})/(2\Re\alpha)$; recalling that $|\alpha|=1$ and using the double angle formulae we have

$$\frac{2}{2\Re\alpha} + \frac{\alpha^2}{2\Re\alpha} + \frac{\alpha^{-2}}{2\Re\alpha} = \frac{1}{\cos\theta} + \left(\alpha - \frac{1}{2\cos\theta}\right) + \left(\overline{\alpha} - \frac{1}{2\cos\theta}\right) = 2\cos\theta$$

where $\theta = \operatorname{Arg}\alpha$. Thus we actually just need to show $\operatorname{tr} A = 2\cos\theta$. As an aside, this shows that A is parabolic if X is, and is elliptic if X is.

Case II. $q+s$ is odd. In this case, B is $Y^{\pm 1}$ and so $\operatorname{tr} B = 2\Re\beta$; we therefore wish to show that $\operatorname{tr} A = (\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta})/(2\Re\beta)$; again using trigonometry we may simplify the right side,

$$\frac{\alpha\beta}{2\Re\alpha} + \frac{\alpha/\beta}{2\Re\alpha} + \frac{\beta/\alpha}{2\Re\alpha} + \frac{1/(\alpha\beta)}{2\Re\beta} = 2\cos\theta$$

and so again we need only show that $\operatorname{tr} A = 2\cos\theta$ where $\theta = \operatorname{Arg}\alpha$.

Both cases then reduce to the identity $\operatorname{tr} A = \operatorname{tr} X$. It will be enough to show that A is conjugate to X; by construction of A, this is equivalent to showing that in $W_{p/q\oplus r/s}$ the $(q+s+1)$th to $(2q+2s-1)$th letters are obtained from the first $q+s-1$ letters by reversing the order and swapping the case. But this is just Lemma 3.2. □

In the case that X and Y are parabolics and $\alpha=\beta=1$, the two formulae unify to become:

$$\operatorname{tr} W_{p/q} W_{r/s} = 4 - \operatorname{tr} W_{p/q\oplus r/s}.$$

We may similarly prove the following 'quotient lemma':

Lemma 5.6 (Quotient lemma)**.** *Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identity holds:*

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q\ominus r/s} = \begin{cases} 2+\beta^2+\frac{1}{\beta^2} & \text{if } q-s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q-s \text{ is odd.} \end{cases}$$

Proof. We begin by setting up notation. By Lemma 3.2 we may write $W_{p/q} = UAuX$ with $A = X^{\pm 1}$ if q is even and $A = Y^{\pm 1}$ if q is odd; similarly, write $W_{r/s} = VBvX$ with B one of $X^{\pm 1}$ or $Y^{\pm 1}$. Then

$$W_{p/q} W_{r/s}^{-1} = UAuXxVbv = UAuVbv;$$

by Lemma 5.2, we have also that $W_{r/s} W_{p/q\ominus r/s}$ is $W_{p/q}$ with the sign of the exponent of the qth letter swapped; explicitly,

$$W_{p/q\ominus r/s} VBvX = UauX \implies W_{p/q\ominus r/s} = UauXxVbv = UauVbv.$$

Our goal is therefore to compute $\operatorname{tr} UAuVbv + \operatorname{tr} UauVbv$; performing a cyclic permutation again, this is equivalent to $\operatorname{tr} A(uVbvU) + \operatorname{tr} a(uVbvU)$. In this form, this becomes

$$\operatorname{tr} A(uVbvU) + \operatorname{tr} a(uVbvU) = \operatorname{tr} A \operatorname{tr} uVbvU = \operatorname{tr} A \operatorname{tr} b.$$

Consider now the cases for the product $\operatorname{tr} A \operatorname{tr} b$:

	q odd	q even
s odd	$\operatorname{tr}^2 Y$	$\operatorname{tr} X \operatorname{tr} Y$
s even	$\operatorname{tr} X \operatorname{tr} Y$	$\operatorname{tr}^2 X$.

If $p/q, r/s$ are Farey neighbours then it is not possible for both q and s to be even since $ps - rq \equiv 1 \pmod 2$. Further, $q-s$ is odd iff exactly one of p and q is odd, otherwise $q-s$ is even. Thus we see that if $q-s$ is even then

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q\ominus r/s} = \operatorname{tr}^2 Y = (\beta + 1/\beta)^2$$

and if $q - s$ is odd then

$$\operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q \ominus r/s} = \operatorname{tr} X \operatorname{tr} Y = (\alpha + 1/\alpha)(\beta + 1/\beta)$$

which are the claimed formulae. □

Using the product and quotient lemmata, we can prove the desired recursion formula for the trace polynomials.

Theorem 5.7 (Recursion formulae)**.** *Let p/q and r/s be Farey neighbours. If $q+s$ is even, then*

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q\oplus r/s} + \Phi_{p/q\ominus r/s} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2. \tag{5.8}$$

Otherwise if $q + s$ is odd, then

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q\oplus r/s} + \Phi_{p/q\ominus r/s} = 2\left(\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right). \tag{5.9}$$

Proof. Suppose $q + s$ is even; then $q - s$ is also even, so

$$\begin{aligned}
&\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} \\
&\qquad = \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&\qquad = \operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&\qquad = 2 + \alpha^2 + \frac{1}{\alpha^2} + 2 + \beta^2 + \frac{1}{\beta^2}
\end{aligned}$$

where in the final step we used Lemma 5.5 and Lemma 5.6. Similarly, when $q - s$ is odd then $q + s$ is also odd and

$$\begin{aligned}
&\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} \\
&\qquad = \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&\qquad = \operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&\qquad = \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}
\end{aligned}$$

as desired. □

Again in the parabolic case the two formulae unify and the recursion identity becomes

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} = 8; \tag{5.10}$$

in fact, such a unification occurs whenever $\alpha = \beta$.

As an aside, Figure 9 shows the picture we get when we just draw the edges of the Farey graph corresponding to Farey neighbours which appear as products in some iterate of the recursion: it is a nice colouring of the Stern-Brocot tree.[4]

Observe that $0/1$ and $1/0$ are Farey neighbours in $\widehat{\mathbb{Q}}$. Thus, applying Equation (5.8) *formally* to the diamond

[4] There is a standard bijection in computer science between finite binary strings and vertices of the infinite binary tree; the Stern-Brocot tree is the tree structure on $I = (0, 1] \cap \mathbb{Q}$ obtained by writing every $x \in I$ as the image of $1/1$ under a sequence of R's and L's (here R and L are the generators of Γ_1, the orientation-preserving part of $\mathrm{PSL}(2, \mathbb{Z})$, see Section 2.5). For a more explicit discussion, see [GKP94, Section 6.7].

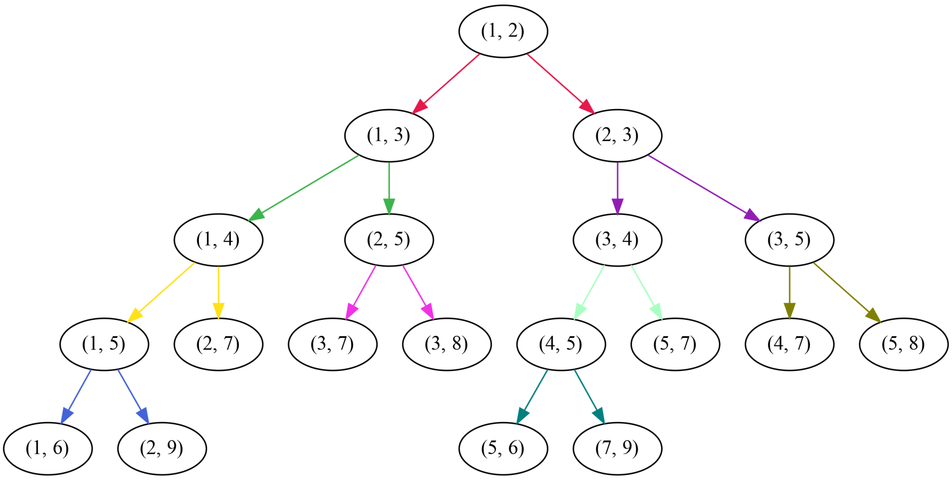
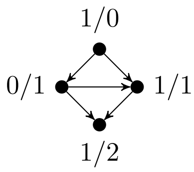


FIGURE 9. The induced colouring of the Stern-Brocot tree.



we obtain

$$\Phi_{0/1}\Phi_{1/1} + \Phi_{1/0} + \Phi_{1/2} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2;$$

substituting for $\Phi_{1/1}$, $\Phi_{1/2}$, and $\Phi_{0/1}$ from Table 2 we get the following formal expression for $\Phi_{1/0}$:

$$\begin{aligned}\Phi_{1/0} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z\right)\left(\alpha\beta + \frac{1}{\alpha\beta} + z\right) - 2 \\ - \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z - z^2 \\ = 2.\end{aligned}$$

Observe that $\Phi_{1/0}^{-1}((-\infty, -2]) = \emptyset$, so this is compatible with the Keen–Series theory; it is also a polynomial of degree q (here, $q = 0$) with constant term 2, which all agrees with the properties of the higher-degree polynomials. On the other hand, it is not monic!

6. Commutators of Farey words

Recall **Fricke's identity** for $A, B \in \mathrm{PSL}(2, \mathbb{C})$ [MR03, (3.15)]:

$$\operatorname{tr}[A, B] = \operatorname{tr}^2 A + \operatorname{tr}^2 B + \operatorname{tr}^2 AB - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB - 2.$$

As a direct consequence of Lemma 5.5, we obtain the following special case for Farey words:

Lemma 6.1. *If p/q and r/s are Farey neighbours with $p/q < r/s$, then*

$$\begin{aligned}\operatorname{tr}\left[W_{p/q}, W_{r/s}\right] = \Phi_{p/q}^2 + \Phi_{r/s}^2 + \Phi_{(p/q)\oplus(r/s)}^2 + \Phi_{p/q}\Phi_{r/s}\Phi_{(p/q)\oplus(r/s)} \\ + \kappa(p/q, r/s)^2 - \kappa(p/q, r/s)\left(\Phi_{p/q}\Phi_{r/s} + 2\Phi_{(p/q)\oplus(r/s)}\right) - 2;\end{aligned}$$

where

$$\kappa(p/q, r/s) := \begin{cases} 2 + \alpha^2 + \frac{1}{\alpha^2} & \text{if } q+s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q+s \text{ is odd.} \end{cases}$$

In particular, if $\alpha = \beta = 1$, then

$$\begin{aligned}(6.2)\quad \operatorname{tr}\left[W_{p/q}, W_{r/s}\right] = \Phi^2_{p/q} + \Phi^2_{r/s} + \Phi^2_{(p/q)\oplus(r/s)} + \Phi_{p/q}\Phi_{r/s}\Phi_{(p/q)\oplus(r/s)} \\ - 4\left(\Phi_{p/q}\Phi_{r/s} + 2\Phi_{(p/q)\oplus(r/s)}\right) + 14.\end{aligned}$$

In the case of the Maskit slice, the commutator of the equivalent of neighbouring Farey words ('slope words') is always -2, since in that case the commutator represents a closed curve about the puncture on the torus. Combining the Fricke identity with the equivalent to Lemma 5.5 in this case gives that Farey triplets (f, g, h) of the trace polynomials of the slope words on the punctured torus satisfy the **Markoff equation**

$$f^2 + g^2 + h^2 = 3fgh$$

(for various perspectives on this see [Aki+07, Section 2.3], [Bow98], [Ser85a], [MSW02], and [CF89, Chapter 7]). A little experimentation shows that no such nice identity holds for the commutators of Farey words in a Riley group—nor should we expect any, since the product of two Farey words is usually strictly loxodromic for any fixed ρ; there is a Markoff-like theory for the 4-times punctured sphere, but it depends on picking hyperbolic representatives for the simple closed curves [BLS86].

We can, however, generalise the process used to compute the commutator in the Maskit case to obtain a second expression for the commutator in the Farey case by using both parts of Lemma 5.2 together. By that lemma,

$$[W_{p/q}, W_{r/s}] = W_{p/q}W_{r/s}(W_{r/s}W_{p/q})^{-1} = \widehat{W}_{(p/q)\oplus(r/s)} \cdot \tilde{W}^{-1}_{(p/q)\oplus(r/s)}$$

where the hat here indicates that the $(q+s)$th exponent is flipped in sign and the tilde indicates a flip in the $(2s)$th exponent. We therefore have two cases: either $q+s < 2s$, or $2s < q+s$. Without loss of generality, assume that $q+s < 2s$ and write $W_{p/q\oplus r/s} = u_1 t u_2 t' u_3$ where u_1 is the word made up of the first $(q+s-1)$ letters, t is the $(q+s)$th letter, u_2 is the $(q+s+1)$th to $(2s-1)$th letters, t' is the $(2s)$th letter, and u_3 is the remainder of the word; then

$$\begin{aligned}\operatorname{tr}\left[W_{p/q}, W_{r/s}\right] &= \operatorname{tr} u_1 T u_2 t' u_3 (u_1 t u_2 T' u_3)^{-1} \\ &= \operatorname{tr} u_1 T u_2 t' u_3 U_3 t' U_2 T U_1 = \operatorname{tr} u_1 T u_2 (t')^2 U_2 T U_1.\end{aligned}$$

By rotational symmetry, this last trace is equal to $\operatorname{tr} u_2 (t')^2 U_2 T^2$. By assumption, t and t' are parabolic so there exists some transformation $M \in \mathrm{PSL}(2, \mathbb{C})$ (perhaps the identity) such that $mt'M = t$. Writing $u = u_2 m$, the trace becomes $\operatorname{tr} ut^2UT^2 = \operatorname{tr}[u, t^2]$. We now use the Fricke identity again, to see that

$$\begin{aligned}\operatorname{tr}[u, t^2] &= \operatorname{tr}^2 u + \operatorname{tr}^2 t^2 + \operatorname{tr}^2 ut^2 - \operatorname{tr} u \operatorname{tr} t^2 \operatorname{tr} ut^2 - 2 \\ &= \operatorname{tr}^2 u + \operatorname{tr}^2 ut^2 - 2\operatorname{tr} u \operatorname{tr} ut^2 + 2 \\ &= (\operatorname{tr} u - \operatorname{tr} ut^2)^2 + 2.\end{aligned}$$

This proves the following result:

Lemma 6.3. *If p/q and r/s are Farey neighbours, then $\operatorname{tr}\left[W_{p/q}(z), W_{r/s}(z)\right] - 2$ is a square in $\mathbb{Z}[z]$.* □

As an amusing corollary, we see that the complicated expression to the right of Equation (6.2) is a positive function.

7. Closed-form solutions for some cases

Recall that the **Chebyshev polynomials** (of the first kind) are the family of polynomials T_n defined via the recurrence relation

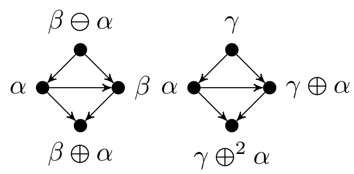
$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned}$$

It is well-known that these polynomials satisfy the product relation

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$$

for $m, n \in \mathbb{Z}_{\geq 0}$. Compare this relation with the relation Equation (5.10) developed above for the parabolic Farey polynomials (but note that the Chebyshev product rule holds for all m, n and the identities for the Farey polynomials hold only for Farey neighbours).

We may apply the theory of 'Farey recursive functions' [Che+20; Che20] in order to explain this analogy. The following diagram may be useful for translating the notation of that paper (right) into the notation we use here (left):



Definition 7.1 (Definition 3.1 of [Che+20])**.** Let R be a (commutative) ring, and suppose $d_1, d_2 : \hat{\mathbb{Q}} \to R$. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is a (d_1, d_2)**-Farey recursive function** if, whenever $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours,

$$\mathcal{F}(\beta \oplus \alpha) = -d_1(\alpha)\mathcal{F}(\beta \ominus \alpha) + d_2(\alpha)\mathcal{F}(\beta). \tag{7.2}$$

Observe that the relation Equation (5.10) looks essentially of this form; to make this clearer, we rewrite it slightly as

$$\Phi(\beta \oplus \alpha) = 8 - \Phi(\beta \ominus \alpha) - \Phi(\alpha)\Phi(\beta) \tag{7.3}$$

(where we set $\beta = p/q$, $\alpha = r/s$, and changed from subscript notation to functional notation). In our case, then, $d_1(\alpha)$ is constantly 1 and $d_2 = -\Phi$. Note also that our relation is not homogeneous. We therefore adapt the definition of [Che+20] to the following:

Definition 7.4. Let R be a (commutative) ring, and suppose $d_1, d_2, d_3 : \hat{\mathbb{Q}} \to R$. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is a (d_1, d_2)**-Farey recursive function** if, whenever $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours,

$$\mathcal{F}(\beta \oplus \alpha) = -d_1(\alpha)\mathcal{F}(\beta \ominus \alpha) + d_2(\alpha)\mathcal{F}(\beta) + d_3(\alpha). \tag{7.5}$$

If d_3 is the zero function, we say that the relation of Equation (7.5) is **homogeneous**, otherwise it is **non-homogeneous**.

The relevant generalisations of the existence-uniqueness results of [Che+20, Section 4] follow easily (the same proofs work, with the usual property that the space of non-homogenous solutions is the sum of a particular solution and the space of homogeneous solutions).

In our case, there is an obvious explicit solution to the non-homogeneous Farey polynomial recursion, Equation (7.3): namely, the map Φ which sends every $\alpha \in \mathbb{Q}$ to the constant polynomial $2 \in \mathbb{Z}[z]$. It therefore remains to solve the corresponding homogeneous equation.

7.1. **A Fibonacci-like subsequence of the homogeneous Farey polynomials.**

Notation. In this section, we work exclusively with the parabolic Farey polynomials, $\Phi^{\infty,\infty}_{p/q}$. We will reuse the symbols α and β for rational numbers (since the roots of unity used to define X and Y become 1 in this situation and so there is no need for special notation for them).

In Table 4, we list the first few polynomials Φ^h which solve the homogeneous recursion relation

$$\Phi^h(\beta \oplus \alpha) = -\Phi^h(\beta \ominus \alpha) - \Phi^h(\alpha)\Phi^h(\beta) \tag{7.6}$$

with the initial values $\Phi^h(0/1) = 2 - z$, $\Phi^h(1/0) = 2$, and $\Phi^h(1/1) = 2 + z$.

The polynomials with numerator 1 listed in the table have very nice properties: immediately one sees that the constant terms alternate in sign and increase in magnitude by 4 each time; also, we have that $\Phi^h_{1/q}(1)$ cycles through the values $3, -5, 2$, $\Phi^h_{1/q}(2)$ cycles through the values $4, -2, -4, 2$; and when we evaluate at 3 and 4 we get a 6-cycle and an arithmetic sequence of step 4 respectively. When we consider $\Phi^h(1/q)(5)$, though, we obtain more interesting behaviour: this is OEIS sequence A100545[5] and satisfies the Fibonacci-type relation

$$\Phi^h_{1/q}(5) = 3\Phi^h_{1/(q-1)}(5) - \Phi^h_{1/(q-2)}(5) \quad \text{with } \Phi^h_{1/1}(5) = 7, \Phi^h_{1/2}(5) = 19.$$

Of course, from the way that we defined the Φ^h such types of relations ought to be expected. In this section, we use the standard diagonalisation technique to explain the behaviour of the sequence $a_q := \Phi^h(1/q)(z)$ for fixed $z \in \mathbb{C}$. From Equation (7.6), we have that

$$a_q = -(2 - z)a_{q-1} - a_{q-2}. \tag{7.7}$$

We may rewrite this equation in matrix form as the following:

$$\begin{bmatrix} 0 & 1 \\ -1 & z-2 \end{bmatrix} \begin{bmatrix} a_{q-2} \\ a_{q-1} \end{bmatrix} = \begin{bmatrix} a_{q-1} \\ a_q \end{bmatrix}. \tag{7.8}$$

One easily computes that the eigenvalues of the transition matrix are

$$\lambda^{\pm} = \frac{z - 2 \pm \alpha}{2}$$

(where $\alpha = \sqrt{z^2 - 4z}$) with respective eigenvectors

$$v^{\pm} = \begin{bmatrix} z - 2 \mp \alpha \\ 2 \end{bmatrix}$$

(note the alternated sign). Thus the transition matrix may be diagonalised as

$$\frac{-1}{2\alpha} \begin{bmatrix} z-2-\alpha & z-2+\alpha \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{z-2+\alpha}{2} & 0 \\ 0 & \frac{z-2-\alpha}{2} \end{bmatrix} \begin{bmatrix} 2 & 2-z-\alpha \\ -2 & z-2-\alpha \end{bmatrix} \tag{7.9}$$

and so a_q is the first coordinate of

$$\frac{-1}{2\alpha} \begin{bmatrix} z-2-\alpha & z-2+\alpha \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \left(\frac{z-2+\alpha}{2}\right)^q & 0 \\ 0 & \left(\frac{z-2-\alpha}{2}\right)^q \end{bmatrix} \begin{bmatrix} 2 & 2-z-\alpha \\ -2 & z-2-\alpha \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix};$$

expanding out, we get

$$\begin{aligned} a_q = 2^{-1-q}\alpha^{-1}\Big(&(a_0(z-2+\alpha) - 2a_1)(z-2-\alpha)^q \\ &+ (a_0(2-z+\alpha) + 2a_1)(z-2+\alpha)^q \Big). \end{aligned} \tag{7.10}$$

[5] http://oeis.org/A100545

We may also characterise the z for which $\Phi^h_{1/q}(z)$ is cyclic: this occurs precisely when the diagonal matrix of Equation (7.9) is of finite order, i.e. whenever both $(z-2\pm\alpha)/2$ are roots of unity.

As an application of the theory above, we have seen that the Chebyshev polynomials also satisfy a second-order recurrence relation with transition matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix}.$$

If we set $z = 2x+2$, then we get back our transition matrix from Equation (7.8). Thus our sequence $\Phi^h_{1/q}(z)$ for fixed z is of the form $W_q(\frac{z-2}{2})$ where W_q is the qth Chebyshev polynomial in the sequence beginning with $W_0 = 2x$ and $W_1 = 2x+4$.

Finally, we consider the solution of the non-homogeneous equation for $\Phi_{1/q}$. Above, we observed that there is a constant solution to the global recursion relation on the entire Stern-Brocot tree; we therefore guess that there is a similar solution to this recursion. Such a solution f will satisfy

$$8 = f(z) + (2-z)f(z) + f(z)$$

and arithmetic gives $f(z) = 8/(4-z)$. Combining this with Equation (7.10) above gives us the following general solution to the non-homogeneous relation:

$$\begin{aligned} a_q = \frac{8}{4-z} + 2^{-1-q}\alpha^{-1}\Big(& (\lambda(z-2+\alpha)-2\mu)(z-2-\alpha)^q \\ & + (\lambda(2-z+\alpha)+2\mu)(z-2+\alpha)^q\Big). \end{aligned}$$

In our case, we have $a_0 = \Phi_{1/0}(z) = 2$ and $a_1 = \Phi_{1/1}(z) = 2+z$. Solving the resulting system of equations gives

$$(\lambda,\mu) = \left(\frac{2z}{z-4}, \frac{2z-z^2}{z-4}\right)$$

and hence

$$a_q = \frac{8}{4-z} + \frac{2^{-q}z}{z-4}\left((-2+z-\sqrt{z^2-4z})^q + (-2+z+\sqrt{z^2-4z})^q\right)$$

7.2. **Solving the homogeneous recursion relation in general.** In the previous section, we computed a closed form formula for $\Phi^h_{1/q}(n)$ using standard techniques from the theory of second-order linear recurrences. We now tackle the general problem of finding a closed-form formula for $\Phi^h_{p/q}(n)$; in order to do this, we use the theory of Section 6 of [Che+20] but with a slight modification: in that paper, the authors define a special case of Farey recursive function, a **Farey recursive function of determinant** d (where $d : \widehat{\mathbb{Q}} \to R$), to be a Farey recursive function $\mathcal{F}$ with $d_1 = d$ and $d_2 = \mathcal{F}$. That is, they replace the recurrence of Equation (7.2) with

$$\mathcal{F}(\beta\oplus\alpha) = -d(\alpha)\mathcal{F}(\beta\ominus\alpha) + \mathcal{F}(\alpha)\mathcal{F}(\beta).$$

This is very similar to our situation, except that instead of $d_2 = \mathcal{F}$ we have $d_2 = -\mathcal{F}$. To reflect this, we shall call a Farey recursive function satisfying a relation of the form

$$\mathcal{F}(\beta\oplus\alpha) = -d(\alpha)\mathcal{F}(\beta\ominus\alpha) - \mathcal{F}(\alpha)\mathcal{F}(\beta). \tag{7.11}$$

a Farey recursive function of ***anti*-determinant** d. We shall work for the time being in this setting (i.e. we shall work with the general function $\mathcal{F}$ rather than the particular example Φ) in order to restate in sufficient generality the theorem which we need (Theorem 6.1 of [Che+20]).

Let $\alpha \in \mathbb{Q}$. The **boundary sequence** $\partial(\alpha)$ is defined inductively by the process of 'continuing to expand down the Farey graph by constant steps'. More precisely, let $\beta \oplus^k \gamma$ denote $\underbrace{((\beta \oplus \gamma) \oplus \cdots) \oplus \gamma}_{k \text{ iterates}}$ for $\beta, \gamma \in \hat{\mathbb{Q}}$ and let γ_L, γ_R be the unique Farey neighbours such that $\alpha = \gamma_L \oplus \gamma_R$; then we set

$$\partial(\alpha) := \{\gamma_L \oplus^k \alpha : k \in \mathbb{Z}_{\geq 0}\} \cup \{\gamma_R \oplus^k \alpha : k \in \mathbb{Z}_{\geq 0}\}.$$

If we allow the Farey graph to embed in the Euclidean upper halfplane by sending $\mathbb{Q} \ni p/q \mapsto (p/q, 1/q) \in \mathbb{H}^2$, then except for the exceptional cases $\alpha = 1/0$ and $\alpha = n/1$ for $n \in \mathbb{Z}$ the subgraph spanned by $\partial(\alpha)$ corresponds to a Euclidean triangle containing α in its interior, see Figure 3 of [Che+20]; for example, the triangle spanned by $\partial(1/n)$ is the triangle with vertices $0, (1/2, 1/2), 1$.

It will be useful to have specific names for the terms in each of the two subsequences and so we set, for $k \in \mathbb{Z}$,

$$\beta_k := \begin{cases} \gamma_L \oplus^{-k-1} \alpha & \text{if } k < -1 \\ \gamma_L & \text{if } k = -1 \\ \gamma_R & \text{if } k = 0 \\ \gamma_R \oplus^k \alpha & \text{if } k > 0. \end{cases} \tag{7.12}$$

For every $\alpha \in \mathbb{Q}$, define

$$M_\alpha = \begin{bmatrix} 0 & 1 \\ -d(\alpha) & \mathcal{F}(\alpha) \end{bmatrix}. \tag{7.13}$$

Given any Farey neighbour β of α, we have

$$M_\alpha^n \begin{bmatrix} \mathcal{F}(\gamma \oplus^0 \alpha) \\ \mathcal{F}(\gamma \oplus^1 \alpha) \end{bmatrix} = \begin{bmatrix} \mathcal{F}(\gamma \oplus^n \alpha) \\ \mathcal{F}(\gamma \oplus^{n+1} \alpha) \end{bmatrix}$$

and so the recursion Equation (7.11) is equivalent to a family of second-order linear recurrences, one down $\partial(\alpha)$ for each α.

We may now state the following theorem:

Theorem 7.14 (Adaptation of Theorem 6.1 of [Che+20])**.** *Let $d : \hat{\mathbb{Q}} \to R$ be a multiplicative function (in the sense that $d(\gamma \oplus \beta) = d(\gamma)d(\beta)$ for all pairs of Farey neighbours $\beta, \gamma \in \mathbb{Q}$) to a commutative ring R, such that $d(\hat{\mathbb{Q}})$ contains no zero divisors. Suppose that $\mathcal{F}$ is a Farey recursive function with anti-determinant d. Given $\alpha \in \mathbb{Q}$, define M_α as in Equation* (7.13) *and $(\beta_k)_{k \in \mathbb{Z}}$ as in Equation* (7.12)*.*

Then, for all $n \in \mathbb{Z}$,

$$M_\alpha^n \begin{bmatrix} \mathcal{F}(\beta_0) \\ \mathcal{F}(\beta_1) \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathcal{F}(\beta_n) \\ \mathcal{F}(\beta_{n+1}) \end{bmatrix} & n \geq 0, \\ \begin{bmatrix} \frac{1}{d(\beta_{-1})} \mathcal{F}(\beta_{-1}) \\ \mathcal{F}(\beta_0) \end{bmatrix} & n = -1, \text{ and} \\ \begin{bmatrix} \frac{1}{d(\beta_{-1}) d_\alpha^{-n-1}} \mathcal{F}_{\beta_n} \\ \frac{1}{d(\beta_{-1}) d_\alpha^{-n-2}} \mathcal{F}_{\beta_{n+1}} \end{bmatrix} & n < -1. \end{cases}$$

We proceed to prove Theorem 7.14 by exactly the same argument as given in [Che+20]. The key point is the following lemma, which is the analogue of the discussion directly preceeding the statement of Theorem 6.1 in that paper.

Lemma 7.15. *With the setup of Theorem 7.14, we have*

$$M_\alpha^{-1}\begin{bmatrix}\mathcal{F}(\beta_0)\\ \mathcal{F}(\beta_1)\end{bmatrix} = \begin{bmatrix}\frac{d(\beta_0)}{d(\alpha)}\mathcal{F}(\beta_{-1})\\ \mathcal{F}(\beta_0)\end{bmatrix}$$

$$M_\alpha^{-2}\begin{bmatrix}\mathcal{F}(\beta_0)\\ \mathcal{F}(\beta_1)\end{bmatrix} = \begin{bmatrix}\frac{1}{d(\alpha)d(\beta_{-1})}\mathcal{F}(\beta_{-2})\\ \frac{1}{d(\beta_{-1})}\mathcal{F}(\beta_{-1})\end{bmatrix}$$

Proof. The formula involving M_α^{-1} comes directly from computing the product on the left via the definition and simplifying with the formula

$$\mathcal{F}(\beta_1) = -d(\beta_0)\mathcal{F}(\beta_{-1}) - \mathcal{F}(\alpha)\mathcal{F}(\beta_0)$$

which is almost exactly the same as Equation (8) of [Che+20]—the single sign change cancels exactly with the sign change between the 'determinant' and 'anti-determinant' recurrences so we get the same overall formula for the M_α^{-1} product as they do in Equation (11) of their paper.

The formula for M_α^{-2} comes from applying the analogues of Equations (9) and (10) of their paper,

$$\mathcal{F}(\beta_{-2}) = -d(\beta_{-1})\mathcal{F}(\beta_0) - \mathcal{F}(\alpha)\mathcal{F}(\beta_{-1})$$
$$d(\alpha) = d(\beta_{-1})d(\beta_0)$$

and simplifying; again the minus signs cancel and we get the same formula. □

Proof of Theorem 7.14. The formula for $n \geq 0$ holds for all Farey recursive formulae as noted above; the formulae for $n = -1$ and $n = -2$ are just the formulae of Lemma 7.15; and we proceed to prove the formula for $n < -2$ by induction. Assume that the formula holds for some fixed $n \leq -2$; then from the definitions we have

$$\mathcal{F}(\beta_{n-1}) = -\mathcal{F}(\alpha)\mathcal{F}(\beta_n) - d(\alpha)\mathcal{F}(\beta_{n+1})$$

and so we can compute

$$\begin{aligned}
M_\alpha^{n-1}\begin{bmatrix}F(\beta_0)\\ F(\beta_1)\end{bmatrix} &= M_\alpha^{-1}M_\alpha^{n}\begin{bmatrix}F(\beta_0)\\ F(\beta_1)\end{bmatrix}\\
&= \frac{1}{d(\alpha)}\begin{bmatrix}-\mathcal{F}(\alpha) & -1\\ d(\alpha) & -0\end{bmatrix}\begin{bmatrix}\frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}F(\beta_n)\\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-2}}F(\beta_{n+1})\end{bmatrix}\\
&= \frac{1}{d(\alpha)}\begin{bmatrix}-\frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}\left(\mathcal{F}(\alpha)\mathcal{F}(\beta_n) + d(\alpha)\mathcal{F}(\beta_{n+1})\right)\\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-2}}\mathcal{F}(\beta_n)\end{bmatrix}\\
&= \begin{bmatrix}-\frac{1}{d(\beta_{-1})d(\alpha)^{-n}}\mathcal{F}(\beta_{n-1})\\ \frac{1}{d(\beta_{-1})d(\alpha)^{-n-1}}\mathcal{F}(\beta_n)\end{bmatrix}
\end{aligned}$$

which is the desired result. □

Corollary 7.16 (Adaptation of Corollary 6.2 of [Che+20])**.** *Let Φ^h be a family of homogeneous Farey polynomials (i.e. a family solving Equation (7.6) for some starting values). Then, for some $\alpha \in \mathbb{Z}$, if M_α is the matrix*

$$\begin{bmatrix}0 & 1\\ -1 & \Phi^h(\alpha)\end{bmatrix}$$

and if $(\beta_k)_{k\in\mathbb{Z}}$ are the boundary values about α as in Equation (7.12), then for all $n \in \mathbb{Z}$ we have

$$M_\alpha^n\begin{bmatrix}\Phi^h(\beta_0)\\ \Phi^h(\beta_1)\end{bmatrix} = \begin{bmatrix}\Phi^h(\beta_n)\\ \Phi^h(\beta_{n+1})\end{bmatrix}.$$

Proof. This follows directly from Theorem 7.14 with the observation that the anti-determinant of Φ^h is the constant function $d(\gamma) = 1$ for all $\gamma \in \mathbb{Q}$. □

Thus to determine Φ^h_α for all $\alpha \in \mathbb{Q}$ it suffices to compute and diagonalise the M_α matrices, using the techniques of Section 7.1. (Of course, we need to diagonalise in the ring of rational functions over $\mathbb{Q}$ rather than the ring of polynomials over $\mathbb{Z}$.) More precisely, we need to compute M_{α_i} for some family (α_i) of rationals with the property that the boundary sets $\partial(\alpha_i)$ cover $\mathbb{Q}$. (In Section 7.1, we did this computation for $\partial(0/1)$.)

In any case, from Corollary 7.16 we immediately have a qualitative result:

Theorem 7.17. *For any $\gamma \in \mathbb{Q}$, there exists a sequence $\dots, \gamma_{-1}, \gamma_0 = \gamma, \gamma_1, \gamma_2, \dots$ of rational numbers such that $\Phi^h_{\gamma_n}(z)$ is a sequence of Chebyshev polynomials $W_n(\Phi^h_\gamma(z)/2)$. (Namely, let γ_{-1} be a neighbour in the Stern-Brocot tree of γ and take the sequence (γ_k) to be precisely the sequence (β_k) of Equation* (7.12) *with $\alpha := \gamma \ominus \gamma_{-1}$.)* □

Remark. Of course, the boundary sequence (γ_k) constructed here is just a geodesic line Λ in the Stern-Brocot tree rooted at γ, defined by choosing one vertical half-ray in the tree starting from γ (where 'vertical' refers to the embedding of Figure 9) and then extending that in the Farey graph in the obvious way by repeated Farey arithmetic with the same difference. There are clearly two such natural choices for Λ given a fixed γ (γ has three neighbours, but two correspond to the same geodesic), and a single natural choice is obtained by taking the unique neighbour of γ which lies above.

We easily compute that the eigenvalues of M_α are

$$\lambda^\pm = \frac{1}{2}\left(\Phi^h_\alpha \pm \sqrt{(\Phi^h_\alpha)^2 - 4}\right).$$

Let $x = \Phi^h_\alpha$ and $\kappa = \sqrt{x^2 - 4}$ (this is the analogue of the constant α from Section 7.1); then the respective eigenvectors are

$$v^\pm = \begin{bmatrix} x \mp \kappa \\ 2 \end{bmatrix}.$$

We therefore may diagonalise M_α as

$$M_\alpha = -\frac{1}{4\kappa}\begin{bmatrix} x-\kappa & x+\kappa \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}(x+\kappa) & 0 \\ 0 & \frac{1}{2}(x-\kappa) \end{bmatrix}\begin{bmatrix} 2 & -x-\kappa \\ -2 & x-\kappa \end{bmatrix};$$

in particular, $\Phi^h(\beta_n)$ is the first component of

$$\begin{aligned} M^n_\alpha &\begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix} \\ &= -\frac{1}{4\kappa}\begin{bmatrix} x-\kappa & x+\kappa \\ 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2^n}(x+\kappa)^n & 0 \\ 0 & \frac{1}{2^n}(x-\kappa)^n \end{bmatrix}\begin{bmatrix} 2 & -x-\kappa \\ -2 & x-\kappa \end{bmatrix}\begin{bmatrix} \Phi^h(\beta_0) \\ \Phi^h(\beta_1) \end{bmatrix} \end{aligned}$$

computing this, we have

$$\Phi^h(\beta_n) = \frac{\begin{array}{c}(\Phi^h(\beta_0)(x+\kappa) - 2\Phi^h(\beta_1))(x-\kappa)^n \\ + (\Phi^h(\beta_0)(\kappa - x) + 2\Phi^h(\beta_1))(x+\kappa)^n\end{array}}{2^{1+n}\kappa}.$$

In particular, we have proved the following quantitative improvement of Theorem 7.17:

Theorem 7.18. *Let β_0 and β_1 be Farey neighbours, and let $\alpha = \beta_1 \ominus \beta_0$. Then we have a closed form formula for $\Phi^h(\beta_n)$ ($n \in \mathbb{Z}$), namely*

$$\Phi^h_{\beta_n} = \frac{\left(\Phi^h_{\beta_0}\left(\Phi^h_\alpha + \kappa\right) - 2\Phi^h_{\beta_1}\right)\left(\Phi^h_\alpha - \kappa\right)^n + \left(\Phi^h_{\beta_0}\left(\kappa - \Phi^h_\alpha\right) + 2\Phi^h_{\beta_1}\right)\left(\Phi^h_\alpha + \kappa\right)^n}{2^{1+n}\kappa}.$$

where $\kappa = \sqrt{\left(\Phi^h_\alpha\right)^2 - 4}$. □

This gives a ‘local’ closed form solution for the recursion around any $\alpha \in \mathbb{Q}$; a ‘global’ solution corresponds to a collection of these solutions, each local to a particular geodesic in the graph and which are compatible on intersections. Unfortunately, our original recurrence relied on knowing only three initial values globally in the graph; while this local formula relies on knowing three initial values which are local on the particular geodesic.

8. Approximating irrational pleating rays

As we mentioned in the introduction to [EMS23a], a version of this theory can be used to give approximations to irrational pleating rays. In order to do this, we must deal with the theory of infinite continued fractions.

We have already used that every rational number can be expressed as a finite simple continued fraction (we stated an exact result as Proposition 2.3). It is also a well-known result in classical number theory that every *irrational* $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ has a unique simple continued fraction approximation of the form

$$\lambda = [a_0, a_1, \dots, a_n, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \frac{1}{\ddots}}}}$$

which can be computed efficiently by repeated application of the Euclidean algorithm (see, for example, §10.9 of [HW60]). We now show that this exhibits λ as a limit of a sequence of rationals ‘down the Farey tree’.

Proposition 8.1. *Suppose that $p/q = [a_1, \dots, a_{N-1}, a_N, 1]$; define*

$$\frac{r_1}{s_1} = [a_1, \dots, a_{N-1}, a_N] \text{ and } \frac{r_2}{s_2} = [a_1, \dots, a_{N-1}].$$

Then r_1/s_1 and r_2/s_2 are Farey neighbours and $p/q = (r_1/s_1) \oplus (r_2/s_2)$.

Proof. That the Farey sum is as claimed follows from Theorem 149 of [HW60]: that is, if $p_n/q_n = [a_1, \dots, a_n]$ then

$$p_n = a_n p_{n-1} + p_{n-2} \text{ and } q_n = a_n q_{n-1} + q_{n-2}.$$

Indeed, take $p/q = p_{N+1}/q_{N+1} = [a_1, \dots, a_{N-1}, a_N, 1]$, then $a_{N+1} = 1$ so $p_{N+1} = 1p_N + p_{N-1}$ and $q_{N+1} = q_N + q_{N-1}$.

That the two are Farey neighbours is exactly Theorem 150 of [HW60], which actually gives slightly more information:

$$p_N q_{N-1} - p_{N-1} q_N = (-1)^{N-1}. \qquad \square$$

In the previous section, we indicated how to compute in closed form the sequence of Farey polynomials corresponding to the Farey fractions

$$\frac{p_1}{q_1}, \frac{p_2}{q_2} = \frac{p_1}{q_1} \oplus \left(\frac{p_2}{q_2} \ominus \frac{p_1}{q_1}\right), \dots, \frac{p_n}{q_n} = \frac{p_1}{q_1} \oplus^{n-1} \left(\frac{p_2}{q_2} \ominus \frac{p_1}{q_1}\right)$$

where p_1/q_1 and p_2/q_2 are Farey neighbours. That is, we gave a way to compute the Farey polynomials down a branch of the Farey tree with constant difference (for instance, we gave the example of $\Phi_{1/q}$, where $\frac{1}{q} = \frac{1}{0} \oplus^q \frac{0}{1}$). The study of partial fraction decompositions here gives, in general, different sequences: the constant addition sequence rooted at an element ξ in the tree is the sequence which constantly chooses the *left* branch when moving down from ξ (with respect to the embedding of Figure 9), while the sequence corresponding to continually adding the previous two items in the tree (and therefore building a continued fraction decomposition) corresponds to the sequence which is eventually constantly moving *rightwards*. For this reason, we might (rather foolishly, as it turns out) expect there to also exist a nice way to compute closed-form expressions for sequences of Farey polynomials corresponding to finite convergents of infinite continued fraction decompositions, and therefore for there to be a reasonable way to approximate irrational pleating rays and attempt to compute expressions for the analytic functions of which they are subsets of zero-sets.

The recursion relation of interest comes, as always, from applying the Farey polynomial operator to the recursion relation $a_n = a_{n-1} \oplus a_{n-2}$; doing this we obtain

$$\Phi_{a_n} = \Phi_{a_{n-1} \oplus a_{n-2}} = 8 - \Phi_{a_{n-1} \ominus a_{n-2}} - \Phi_{a_{n-1}} \Phi_{a_{n-2}}.$$

Observe that $a_{n-1} \ominus a_{n-2}$ is just a_{n-3} and replace the cumbersome notation Φ_{a_k} with x_k to get the relation

$$x_n = 8 - x_{n-3} - x_{n-2} x_{n-1}. \tag{8.2}$$

Unfortunately, this is a non-linear recurrence relation—this means that it is harder to study than the linear left-recursion studied in the previous section, and we are unable to find a closed-form solution. (We believe that if one exists then it will be in terms of combinatorial objects, e.g. Stirling numbers.) However, it does at least give a computationally feasible method for approximating irrational cusp points.

Example 8.3 (The Fibonacci polynomials)**.** Let us compute an approximation to the pleating ray with asymptotic angle π/ϕ, where ϕ is the golden ratio $(\sqrt{5}+1)/2$. It is well-known that

$$\phi = [1, 1, ...] = 1 + \cfrac{1}{1 + \frac{1}{1+\frac{1}{1+\ddots}}}$$

and so by Proposition 8.1 we get that $1/\phi$ is approximated by

$$[0, 1] = 1, [0, 1, 1] = 1/2, [1, 1, 1] = 2/3, \dots ;$$

in other words, by the Fibonacci fractions $\mathsf{fib}(q-1)/\mathsf{fib}(q)$. Being efficiency-minded, we have already computed the corresponding polynomials, which are listed in Table 3; the inverse images of -2 under the first 16 such polynomials are shown in Figure 10, where the $1/\pi$-cusp is approximated by the 'corner' point, at the top-left of each picture. Quite a good approximation seems to be given after only a few pictures, which is good since after about 15 terms the polynomial coefficients and degrees become too big for both `Mathematica` and `MATLAB` to study without hassle. We also note that eventually the cusp point occurs at quite a 'sharp' point in the set; perhaps it would be interesting to use this to distinguish the cusp computationally from the other points in the inverse image.

Example 8.4 ($\sqrt{2}$)**.** For our next trick, we consider another irrational number with a nice continued fraction decomposition:

$$\sqrt{2} = [1, 2, 2, \dots, 2, ...]$$

so we may approximate $1/\sqrt{2}$ by the Farey sequence beginning

$$1, 2/3, 5/7, 7/10, 12/17, \dots.$$

The corresponding Farey polynomials are higher degree and so longer to write down than the Fibonacci polynomials, but it is easy enough for the computer to plot the preimages of -2 for the first 10 or so polynomials; we give the preimage for $\Phi_{19/27}$ in Figure 11.

Having seen some examples, we consider the dynamical system a little more closely. Define the transition map $f : k^3 \to k^3$ by

$$f(x^1, x^2, x^3) = (x^2, x^3, 8 - x^1 - x^2 x^3).$$

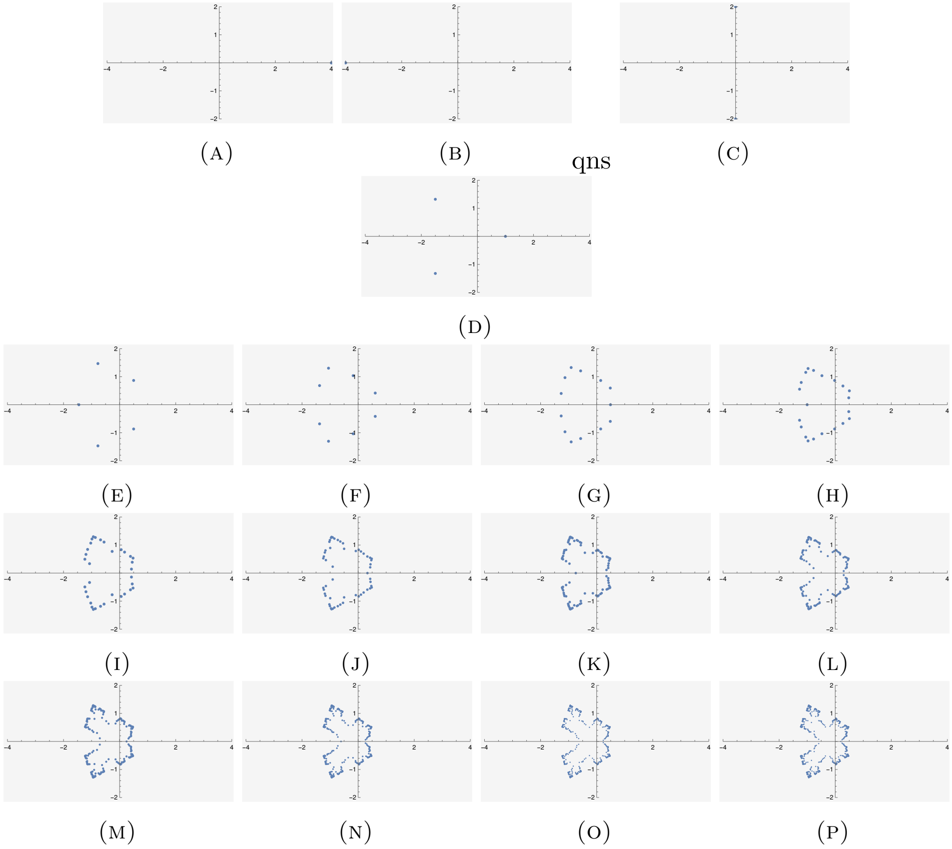


FIGURE 10. The zeros of $\Phi_{\mathsf{fib}(p-1)/\mathsf{fib}(p)} + 2$ for $p \in \{1, \dots, 16\}$.

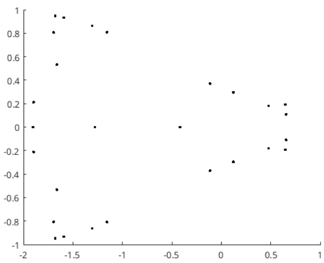


FIGURE 11. $\left(\Phi^{\infty,\infty}_{19/27}\right)^{-1}(-2)$, approximating the $1/\sqrt{2}$ cusp point.

It is easy to check that f has exactly two fixed points, $(-4,-4,-4)$ and $(2,2,2)$, and that its differential is given by

$$D_{x^1,x^2,x^3} f(\xi^1,\xi^2,\xi^3) = f(x^1,x^2,x^3) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -x^2 & -x^3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{bmatrix};$$

at the two fixed points the Jacobian has eigenproperties and determinants as in Table 5. We also give three slices through the phase diagram for this system in Figure 12.

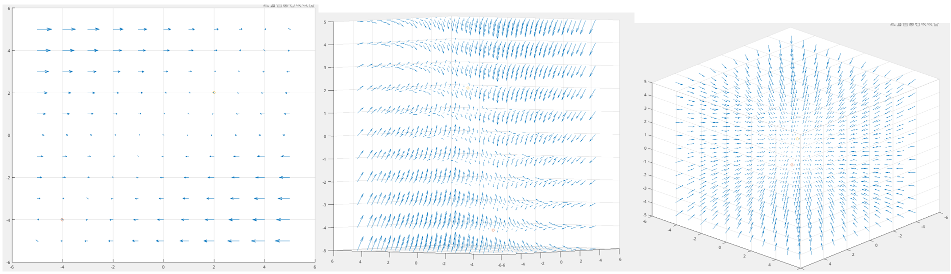


FIGURE 12. The real dynamics of the 'right-recursion' system.

By replacing each x_i with $y_i := x_i - 2$ (since 2 is a fixed point), we may replace Equation (8.2) with the homogeneous relation

$$y_i = -y_{i-3} - y_{i-2}y_{i-1} - 2(y_{i-2} + y_{i-1}); \tag{8.5}$$

with the initial conditions $(y_1, y_2, y_3) = (-z, z, z^2)$, the result is a sequence of **reduced Farey polynomials** $\phi_{p/q} := \Phi^{\infty,\infty}_{p/q} - 2$ which we studied in [EMS23a] (in that paper, these polynomials were called $Q_{p/q}$). We were surprised to observe, through computing a few examples, that the alternating sum of the coefficients of these polynomials seem to be squares (Table 6)!

More precisely, we make the following conjecture.

Conjecture 8.6. *The reduced Farey polynomial $\phi_\alpha(z)$ is always of the form*

$$(-1)^{\epsilon(\alpha)} z^{k(\alpha)} R(\alpha)^2,$$

where $\epsilon : \hat{\mathbb{Q}} \to \{-1, +1\}$, $k : \hat{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$, and $R : \hat{\mathbb{Q}} \to \mathbb{Z}[z]$ are functions which satisfy Farey recursion properties: if $\alpha, \beta \in \hat{\mathbb{Q}}$ are Farey neighbours, then

(1) *$\epsilon(\alpha \oplus \beta) = \epsilon(\alpha)\epsilon(\beta)$;*
(2) *$k(\alpha \oplus \beta) = k(\alpha) + k(\beta)$; and*
(3) *either $R(\alpha \oplus \beta) = R(\alpha)R(\beta) + R(\alpha \ominus \beta)$ or $R(\alpha \oplus \beta) = R(\alpha)R(\beta) - R(\alpha \ominus \beta)$ depending on the position of $\alpha \oplus \beta$ in $\hat{\mathbb{Q}}$. In the case of the arboreal geodesic $\mathsf{fib}(p)/\mathsf{fib}(p+1)$, the rule with a minus sign occurs iff $\mathsf{fib}(p)$ is even; the 'bad points' in $\hat{\mathbb{Q}}$ where this minus sign rule occurs are shown in red in Figure 13).*

The functions ϵ, k, and R satisfy the initial conditions in the following table:

α	$\epsilon(\alpha)$	$k(\alpha)$	$R(\alpha)$
$0/1$	-1	$+1$	1
$1/1$	$+1$	$+1$	1
$1/2$	$+1$	0	z

This conjecture is a more precise version of Conjecture 1 of [EMS23a], and (if the rule for R could be guessed) can probably be proved by an application of Farey induction applied to Equation (8.5) (compare with Lemma 5.3.12 and Remark 5.3.13 of [Aki+07]), but really the interest to us is the following observation: our initial recurrence relation is uniform on $\hat{\mathbb{Q}}$ (that is, the recurrence rule is the same no matter the position in the Farey graph), but from it there naturally arises a recurrence rule which depends on some *parity* structure in the graph. This is the *second* such recurrence relation which has appeared in this paper (the elliptic Farey polynomial recurrence also had parity), and roughly speaking in the geometric context of the Riley slices this parity is coming from the categorisation of two-bridge knots as either starting with an east-west twist or a north-south twist.

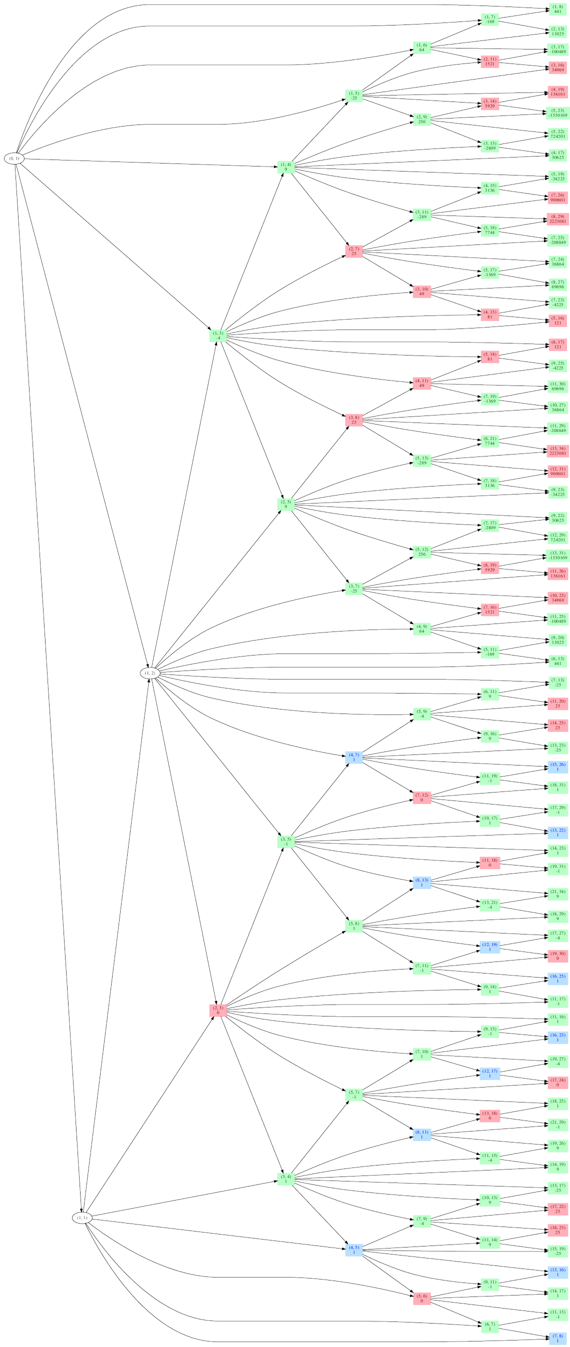


FIGURE 13. Points up to height 8 in the Farey graph coloured according to which recursion rule for the function R in Conjecture 8.6 is correct at that fraction: in green are points $\alpha \oplus \beta$ such that $R(\alpha \oplus \beta) = R(\alpha)R(\beta) + R(\alpha \ominus \beta)$, in red are points such that the rule $R(\alpha \oplus \beta) = R(\alpha)R(\beta) - R(\alpha \ominus \beta)$ holds, and in blue are the points such that $0 = R(\alpha \ominus \beta)$ (so both rules hold). The full picture may be viewed at the URL
`https://aelzenaar.github.io/farey/badpoints.ps`.

We end this paper with a few open questions which we hope might be answered by researchers in other fields.

(1) Do other parity-dependent recurrence relations on the Farey graph occur in nature?
(2) Is there an analogue of the theory of Farey recursive functions with constant difference that is applicable to the elliptic Farey polynomial recurrence?
(3) Is there a geometric proof for Theorem 5.7 (for instance, via some study of the lengths of the represented geodesics)?
(4) As a stepping stone to (3), is there a more combinatorial reason (for instance, in terms of geodesics on the marked sphere) for parity to matter for the elliptic Farey polynomial recurrence? That is, take two adjacent triangles in the Farey graph; what is it about the arrangements of the corresponding geodesics which causes their lengths to depend on each other by either the odd relation or the even relation?
(5) Is there a way of determining computationally which roots of $\Phi^{a,b}_{p/q} + 2$ correspond to the cusp points? Computation shows that the shape of the sets $\left(\Phi^{\infty,\infty}_{p/q}\right)^{-1}(-2)$ as $q \to \infty$ seems to become quite regularly shaped (possibly with polygonal convex hull), with two extremal points corresponding to the cusps (see Figure 10 and Figure 11). If this can be made precise and shown to hold in general, then it would provide such a method for finding cusps.
(6) Can one give closed forms for the factors of $\phi^{\infty,\infty}_{p/q} = \Phi^{a,b}_{p/q} - 2$?
(7) How quickly does the sequence of Farey polynomials coming from the continued fraction decomposition approximate the irrational pleating ray?

List of Tables

TABLE 1. Farey words $W_{p/q}$ for small q.

p/q	$W_{p/q}$
$0/1$	yX
$1/1$	YX
$1/2$	$yxYX$
$1/3$	$yXYxYX$
$2/3$	$yxyXYX$
$1/4$	$yXyxYxYX$
$3/4$	$yxyxYXYX$
$1/5$	$yXyXYxYxYX$
$2/5$	$yXYxyXyxYX$
$3/5$	$yxYXYxyXYX$
$4/5$	$yxyxyXYXYX$
$1/6$	$yXyXyxYxYxYX$
$5/6$	$yxyxyxYXYXYX$
$1/7$	$yXyXyXYxYxYxYX$
$2/7$	$yXyxYxyXyXYxYX$
$3/7$	$yXYxyXYxYXyxYX$
$4/7$	$yxYXyxyXYxyXYX$
$5/7$	$yxyXYXYxyxYXYX$
$6/7$	$yxyxyxyXYXYXYX$
$1/8$	$yXyXyXyxYxYxYxYX$
$3/8$	$yXYxYXyxYxyXyxYX$
$5/8$	$yxYXYxyxYXyxyXYX$
$7/8$	$yxyxyxyxYXYXYXYX$
$1/9$	$yXyXyXyXYxYxYxYxYX$
$2/9$	$yXyXYxYxyXyXyxYxYX$
$4/9$	$yXYxyXYxyXyxYXyxYX$
$5/9$	$yxYXyxYXYxyXYxyXYX$
$7/9$	$yxyxYXYXYxyxyXYXYX$
$8/9$	$yxyxyxyxyXYXYXYXYX$
$1/10$	$yXyXyXyXyxYxYxYxYxYX$
$3/10$	$yXyxYxyXyxYxYXyXYxYX$
$7/10$	$yxyXYXyxyxYXYxyxYXYX$
$9/10$	$yxyxyxyxyxYXYXYXYXYX$
$1/11$	$yXyXyXyXyXYxYxYxYxYxYX$
$2/11$	$yXyXyxYxYxyXyXyXYxYxYX$
$3/11$	$yXyxYxYXyXYxYxyXyXYxYX$
$4/11$	$yXYxYXyXYxyXyxYxyXyxYX$
$5/11$	$yXYxyXYxyXYxYXyxYXyxYX$
$6/11$	$yxYXyxYXyxyXYxyXYxyXYX$
$7/11$	$yxYXYxyxYXYxyXYXyxyXYX$
$8/11$	$yxyXYXYxyxyXYXyxyxYXYX$
$9/11$	$yxyxyXYXYXYxyxyxYXYXYX$
$10/11$	$yxyxyxyxyxyXYXYXYXYXYX$
$1/12$	$yXyXyXyXyXyxYxYxYxYxYxYX$
$5/12$	$yXYxyXyxYXyxYxyXYxYXyxYX$
$7/12$	$yxYXyxyXYxyxYXyxYXYxyXYX$
$11/12$	$yxyxyxyxyxyxYXYXYXYXYXYX$

TABLE 2. Farey polynomials $\Phi^{a,b}_{p/q}(z)$ for small q.

p/q	
$0/1$	$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z$
$1/1$	$\alpha\beta + \frac{1}{\alpha\beta} + z$
$1/2$	$2 + \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z + z^2$
$1/3$	$\frac{1}{\alpha\beta} + \alpha\beta + \left(3 - \frac{1}{\alpha^2} - \alpha^2 - \frac{1}{\beta^2} - \beta^2 + \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2}\right) z$
	$+ \left(\alpha\beta - 2\frac{\alpha}{\beta} - 2\frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z^2 + z^3$
$2/3$	$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \left(-3 + \alpha^2 + \frac{1}{\alpha^2} - \frac{1}{\alpha^2\beta^2} - \alpha^2\beta^2 + \beta^2 + \frac{1}{\beta^2}\right) z$
	$+ \left(-2\alpha\beta - \frac{2}{\alpha\beta} + \frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) z^2 - z^3$
$1/4$	$2 + \left(\frac{\alpha}{\beta^3} - \frac{\alpha^3}{\beta^3} + \frac{2}{\alpha\beta} - 3\frac{\alpha}{\beta} + \frac{\alpha^3}{\beta} + \frac{\beta}{\alpha^3} - 3\frac{\beta}{\alpha} + 2\alpha\beta - \frac{\beta^3}{\alpha^3} + \frac{\beta^3}{\alpha}\right) z$
	$+ \left(6 - \frac{2}{\alpha^2} - 2\alpha^2 - \frac{2}{\beta^2} + 3\frac{\alpha^2}{\beta^2} - 2\beta^2 + 3\frac{\beta^2}{\alpha^2}\right) z^2$
	$+ \left(\frac{1}{\alpha\beta} - 3\frac{\alpha}{\beta} - 3\frac{\beta}{\alpha} + \alpha\beta\right) z^3 + z^4$
$3/4$	$2 + \left(\frac{1}{\alpha^3\beta^3} - \frac{1}{\alpha\beta^3} - \frac{1}{\alpha^3\beta} + \frac{3}{\alpha\beta} - 2\frac{\alpha}{\beta} - 2\frac{\beta}{\alpha} + 3\alpha\beta - \alpha^3\beta - \alpha\beta^3 + \alpha^3\beta^3\right) z$
	$+ \left(6 - \frac{2}{\alpha^2} - 2\alpha^2 - \frac{2}{\beta^2} + \frac{3}{\alpha^2\beta^2} - 2\beta^2 + 3\alpha^2\beta^2\right) z^2$
	$+ \left(\frac{3}{\alpha\beta} - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + 3\alpha\beta\right) z^3 + z^4$

TABLE 3. Farey polynomials of the form $\Phi^{\infty,\infty}_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}(z)$ for small q.

$\frac{\mathsf{fib}(q-1)}{\mathsf{fib}(q)}$	
$0/1$	$2 - z$
$1/1$	$2 + z$
$1/2$	$2 + z^2$
$2/3$	$2 - z - 2z^2 - z^3$
$3/5$	$2 + z + 2z^2 + 3z^3 + 2z^4 + z^5$
$5/8$	$2 + 4z^4 + 8z^5 + 8z^6 + 4z^7 + z^8$
$8/13$	$2 - z - 2z^2 - 5z^3 - 12z^4 - 22z^5 - 32z^6 - 44z^7 - 54z^8$
	$- 53z^9 - 38z^{10} - 19z^{11} - 6z^{12} - z^{13}$
$13/21$	$2 + z + 2z^2 + 7z^3 + 14z^4 + 31z^5 + 64z^6 + 124z^7 + 214z^8$
	$+ 339z^9 + 498z^{10} + 699z^{11} + 936z^{12} + 1148z^{13} + 1216z^{14}$
	$+ 1064z^{15} + 746z^{16} + 409z^{17} + 170z^{18} + 51z^{19} + 10z^{20} + z^{21}$
$21/34$	$2 + z^2 + 8z^4 + 24z^5 + 68z^6 + 192z^7 + 516z^8 + 1256z^9 + 2834z^{10}$
	$+ 5912z^{11} + 11460z^{12} + 20816z^{13} + 35598z^{14} + 57248z^{15}$
	$+ 86446z^{16} + 122560z^{17} + 163199z^{18} + 203952z^{19} + 238564z^{20}$
	$+ 259704z^{21} + 260686z^{22} + 238320z^{23} + 195694z^{24} + 142328z^{25}$
	$+ 90451z^{26} + 49552z^{27} + 23058z^{28} + 8952z^{29} + 2831z^{30} + 704z^{31}$
	$+ 130z^{32} + 16z^{33} + z^{34}$
$34/55$	$2 - z - 4z^2 - 10z^3 - 34z^4 - 103z^5 - 286z^6 - 791z^7$
	$- 2078z^8 - 5221z^9 - 12680z^{10} - 29824z^{11} - 67872z^{12}$
	$- 149896z^{13} - 321800z^{14} - 671896z^{15} - 1364228z^{16}$
	$- 2692102z^{17} - 5158232z^{18} - 9587668z^{19} - 17273444z^{20}$
	$- 30141702z^{21} - 50903644z^{22} - 83138942z^{23} - 131230688z^{24}$
	$- 200056876z^{25} - 294348624z^{26} - 417663240z^{27} - 571010576z^{28}$
	$- 751328456z^{29} - 950188464z^{30} - 1153232920z^{31} - 1340813030z^{32}$
	$- 1490107333z^{33} - 1578696308z^{34} - 1589182962z^{35} - 1513960786z^{36}$
	$- 1358696535z^{37} - 1142850158z^{38} - 896137319z^{39} - 651440922z^{40}$
	$- 436582355z^{41} - 268228504z^{42} - 150207744z^{43} - 76207672z^{44}$
	$- 34797892z^{45} - 14193584z^{46} - 5125756z^{47} - 1621110z^{48}$
	$- 442809z^{49} - 102556z^{50} - 19630z^{51} - 2990z^{52}$
	$- 341z^{53} - 26z^{54} - z^{55}$

TABLE 4. Selected polynomials Φ^h satisfying the homogeneous recuresion relation for small q, with the initial values as given.

p	q	$\Phi^h_{p/q}$
1	0	2
0	1	$-z+2$
1	1	$z+2$
1	2	z^2-6
1	3	$z^3-2z^2-7z+10$
2	3	$-z^3-2z^2+7z+10$
1	4	$z^4-4z^3-4z^2+24z-14$
3	4	$z^4+4z^3-4z^2-24z-14$
1	5	$z^5-6z^4+3z^3+34z^2-55z+18$
2	5	$-z^5+2z^4+13z^3-22z^2-41z+58$
3	5	$z^5+2z^4-13z^3-22z^2+41z+58$
4	5	$-z^5-6z^4-3z^3+34z^2+55z+18$
1	6	$z^6-8z^5+14z^4+32z^3-119z^2+104z-22$
1	7	$z^7-10z^6+29z^5+10z^4-186z^3+308z^2-175z+26$
1	8	$z^8-12z^7+48z^6-40z^5-220z^4+648z^3-672z^2+272z-30$
1	9	$z^9-14z^8+71z^7-126z^6-169z^5+1078z^4-1782z^3+1308z^2-399z+34$
1	10	$z^{10}-16z^9+98z^8-256z^7+35z^6+1456z^5-3718z^4+4224z^3-2343z^2+560z-38$

TABLE 5. Eigenproperties of the right-recurrence relation.

fixed point	eigenvalue	eigenvector
-4	$\frac{5+\sqrt{21}}{2}$	$\left(-\frac{-5+\sqrt{21}}{5+\sqrt{21}}, \frac{2}{5+\sqrt{21}}, 1\right)^t$
	-1	$(1,-1,1)^t$
	$\frac{5-\sqrt{21}}{2}$	$\left(-\frac{5+\sqrt{21}}{-5+\sqrt{21}}, -\frac{2}{-5+\sqrt{21}}, 1\right)^t$
	determinant:	-1
2	$\frac{-1+i\sqrt{3}}{2}$	$\left(\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}, 1\right)^t$
	-1	$(1,-1,1)^t$
	$\frac{-1-i\sqrt{3}}{2}$	$\left(\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}, 1\right)^t$
	determinant:	-1

TABLE 6. $\sqrt{\left|\phi_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}(-1)\right|}$ for small q.

q	$\sqrt{\left\|\phi_{\mathsf{fib}(q-1)/\mathsf{fib}(q)}(-1)\right\|}$
1	1
2	1
3	1
4	0
5	1
6	1
7	1
8	2
9	3
10	5
11	17
12	88
13	1491
14	131225
15	195656563
16	25675032478184
17	5023488609594854052817
18	128978233205135262131911855731900891
19	647920685391665774371139137077100918906183250131690881763
20	835676652588773441434735633165055341787601058478245210146 00110051618636610092033658769403650

References

[AGM21] Hala Alaqad, Jianhua Gong, and Gaven J. Martin. *Chebyshev poolynomials and inequalities for Kleinian groups*. 2021. arXiv: 2103.05696 [math.CV].

[Aki+07] Hirotaka Akiyoshi, Makoto Sakuma, Masaaki Wada, and Yasushi Yamashita. *Punctured torus groups and 2-bridge knot groups I*. Lecture Notes in Mathematics 1909. Springer, 2007. ISBN: 9783540718079. DOI: 10.1007/978-3-540-71807-9.

[Bea83] Alan F. Beardon. *The geometry of discrete groups*. Graduate Texts in Mathematics 91. Springer-Verlag, 1983. ISBN: 0387907882. DOI: 10.1007/978-1-4612-1146-4.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Grundlehren der mathematischen Wissenshaften 319. Springer, 1999. ISBN: 9783540643241. DOI: 10.1007/978-3-662-12494-9.

[Bir74] Joan S. Birman. *Braid, links, and mapping class groups*. Annals of Mathematics Studies 82. Princeton University Press, 1974. ISBN: 0691081492.

[BLS86] A. F. Beardon, J. Lehner, and M. Sheingorn. "Closed geodesics on a Riemann surface with application to the Markov spectrum". In: *Transactions of the American Mathematical Society* 295.2 (1986), pp. 635–647.

[Bow98] Brian H. Bowditch. "Markoff triples and quasifuchsian groups". In: *Proceedings of the London Mathematical Society* 77.3 (1998), pp. 697–736. DOI: 10.1112/s0024611598000604.

[BS87] Joan S. Birman and Caroline Series. "Dehn's algorithm revisited, with applications to simple curves on surfaces". In: *Combinatorial group theory and topology*. Ed. by Stephen M. Gersten and John R. Stallings. Annals of Mathematics Studies 111. Princeton University Press, 1987, pp. 451–478. ISBN: 0691084092.

[BZ03] Gerhard Burde and Heiner Zieschang. *Knots*. 2nd ed. Studies in Mathematics 5. de Gruyer, 2003.

[CF89] Thomas W. Cusick and Mary E. Flahive. *The Markoff and Lagrange spectra*. Vol. 30. Mathematical Surveys and Monographs. American Mathematical Society, 1989. ISBN: 978-0-8218-1531-1. DOI: 10.1090/surv/030.

[Che+20] Eric Chesebro, Cory Emlen, Kenton Ke, Denise Lafontaine, Kelly McKinnie, and Catherine Rigby. "Farey recursive functions". In: *Involve* 41 (2020), pp. 439–461. DOI: 10.2140/involve.2021.14.439. arXiv: 2008.13303.

[Che20] Eric Chesebro. "Farey recursion and the character varieties for 2-bridge knots". In: *Characters in low-dimensional topology*. Ed. by Olivier Collin, Stefan Friedl, Cameron Gordon, Stephan Tillmann, and Liam Watson. Contemporary Mathematics 760. American Mathematical Society, 2020, pp. 9–34. ISBN: 9781470452094. arXiv: 1902.01968 [math.GT].

[Elz21] Alex Elzenaar. *Riley slice computational package*. GitHub repository. 2021. URL: https://github.com/aelzenaar/riley.

[Elz22] Alex Elzenaar. "Deformation spaces of Kleinian groups". Master thesis. The University of Auckland, 2022. URL: https://aelzenaar.github.io/msc_thesis.pdf.

[EMS22] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *The elliptic cousins of the Riley slice*. In preparation. 2022.

[EMS23a] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *Approximations of the Riley slice*. in press. 2023. DOI: 10.1016/j.exmath.2022.12.002. arXiv: 2111.03230 [math.GT].

[EMS23b] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021-22 MATRIX annals*. Springer, 2023. arXiv: 2204.11422 [math.GT]. To appear.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Princeton Mathematical Series. Princeton University Press, 2012. ISBN: 9780691147949.

[Gil06] Jane Gilman. "Informative words and discreteness". In: *Combinatorial group theory, discrete groups, and number theory*. Ed. by Benjamin Fine, Anthony M. Gaglione, and Dennis Spellman. Vol. 421. American Mathematical Society, 2006, pp. 147–155. ISBN: 978-0-8218-3985-0. DOI: 10.1090/conm/421/08034.

[GKP94] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics. A foundation for computer science*. 2nd ed. Addison-Wesley, 1994. ISBN: 0201558025.

[GMM98] Fred W. Gehring, Colin Maclachlan, and Gaven J. Martin. "Two-generator arithmetic Kleinian groups II". In: *Bulletin of the London Mathematical Society* 30.3 (1998), pp. 258–266. DOI: 10.1112/S0024609397004359.

[HW60] Godfrey H. Hardy and Edward M. Wright. *An introduction to the theory of numbers*. 4th ed. Oxford University Press, 1960.

[IT87] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer, 1987. ISBN: 978-4-431-68174-8. DOI: 10.1007/978-4-431-68174-8.

[Kau87] Louis H. Kauffman. *On knots*. Annals of Mathematics Studies 115. Princeton University Press, 1987. ISBN: 0691084343.

[KS94] Linda Keen and Caroline Series. "The Riley slice of Schottky space". In: *Proceedings of the London Mathematics Society* 3.1 (69 1994), pp. 72–90. DOI: 10.1112/plms/s3-69.1.72.

[KS98] Yohei Komori and Caroline Series. "The Riley slice revisited". In: *The Epstein birthday schrift*. Ed. by Igor Rivin, Colin Rourke, and Caroline Series. Geometry and Topology Monographs 1. Mathematical Sciences Publishers, 1998, pp. 303–316. DOI: 10.2140/gtm.1998.1.303. arXiv: math/9810194.

[Mar16] Albert Marden. *Hyperbolic manifolds. An introduction in 2 and 3 dimensions*. 2nd ed. First edition was published under the title "Outer circles". Cambridge University Press, 2016. ISBN: 9781107116740.

[Mas87] Bernard Maskit. *Kleinian groups*. Grundlehren der mathematischen Wissenshaften 287. Springer-Verlag, 1987. ISBN: 978-3-642-61590-0. DOI: 10.1007/978-3-642-61590-0.

[McM91] Curt McMullen. "Cusps are dense". In: *Annals of Mathematics* 133 (1991), pp. 217–247. DOI: 10.2307/2944328.

[Mey76] Mark D. Meyerson. "Representing homology classes of closed orientable surfaces". In: *Proceedings of the American Mathematical Society* 61.1 (1976), pp. 181–182. DOI: 10.2307/2041691.

[MM99] Colin Maclachlan and Gaven J. Martin. "2-generator arithmetic Kleinian groups". In: *Journal für die Reine und Angewandte Mathematik* 511 (1999), pp. 95–117. DOI: 10.1515/crll.1999.511.95/html.

[MR03] Colin Maclachlan and Alan W. Reid. *The arithmetic of hyperbolic 3-manifolds*. Graduate Texts in Mathematics 219. Springer-Verlag, 2003. ISBN: 978-1-4757-6720-9. DOI: 10.1007/978-1-4757-6720-9.

[MSW02] David Mumford, Caroline Series, and David Wright. *Indra's pearls. The vision of Felix Klein*. Cambridge University Press, 2002. ISBN: 0521352533.

[Pur20] Jessica S. Purcell. *Hyperbolic knot theory*. Graduate Studies in Mathematics 209. American Mathematical Society, 2020. ISBN: 9781470454999.

[Ril13] Robert Riley. "A personal account of the discovery of hyperbolic structures on some knot complements". In: *Expositiones Mathematicae* 31.2 (2013), pp. 104–115. DOI: 10.1016/j.exmath.2013.01.003. arXiv: 1301.4601.

[Ril72] Robert Riley. "Parabolic representations of knot groups, I". In: *Proceedings of the London Mathematics Society*. 3rd ser. 24 (1972), pp. 217–242. DOI: 10.1112/plms/s3-24.2.217.

[Ril75a] Robert Riley. "A quadratic parabolic group". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 77 (1975), pp. 281–288. DOI: 10.1017/s0305004100051094.

[Ril75b] Robert Riley. "Parabolic representations of knot groups, II". In: *Proceedings of the London Mathematics Society*. 3rd ser. 31 (4 1975), pp. 495–512. DOI: 10.1112/plms/s3-31.4.495.

[Ril79] Robert Riley. "An elliptical path from parabolic representations to hyperbolic structures". In: *Topology of low-dimensional manifolds—Proceedings, Sussex 1977*. Ed. by Roger Fenn. Lecture Notes in Mathematics 722. Springer-Verlag, 1979, pp. 99–133. ISBN: 3540095063.

[Ril92] Robert Riley. "Algebra for Heckoid groups". In: *Transactions of the American Mathematical Society* 334.1 (1992), pp. 389–409. DOI: 10.1090/s0002-9947-1992-1107029-9.

[Sch56] Horst Schubert. "Knoten mit zwei Brücken". In: *Mathematische Zeitschrift* 65 (1956), pp. 133–170. DOI: 10.1007/bf01473875.

[Sch76] James A. Schafer. "Representing homology classes on surfaces". In: *Canadian Mathematical Bulletin* 19.3 (1976), pp. 373–374. DOI: 10.4153/CMB-1976-058-4.

[Ser85a] Caroline Series. "The geometry of Markoff numbers". In: *The Mathematical Intelligencer* 7.3 (1985), pp. 20–29. DOI: 10.1007/bf03025802.

[Ser85b] Caroline Series. "The modular surface and continued fractions". In: *Journal of the London Mathematical Society*. 2nd ser. 31 (1985), pp. 69–80. DOI: 10.1112/jlms/s2-31.1.69.

[Ser91] Caroline Series. "Geometric methods of symbolic coding". In: *Ergodic theory, symbolic dynamics and hyperbolic spaces*. Ed. by Tim Bedford, Michael Keane, and Caroline Series. Oxford University Press, 1991, pp. 125–151. ISBN: 0198596855.

[Sti83] John Stillwell. *Classical topology and combinatorial group theory*. 2nd ed. Graduate Texts in Mathematics 72. Springer, 1983. ISBN: 978-1-4612-4372-4. DOI: 10.1007/978-1-4612-4372-4.

[Thu97] William P. Thurston. *Three-dimensional geometry and topology*. Ed. by Silvio Levy. Vol. 1. Princeton University Press, 1997. ISBN: 0691083045.

[Wri05] David Wright. "Searching for the cusp". In: *Spaces of Kleinian groups*. Ed. by Yair N. Minsky, Makoto Sakuma, and Caroline Series. LMS Lecture Notes 329. Cambridge University Press, 2005, pp. 301–336. DOI: 10.1017/cbo9781139106993.016.

[Zha10] Qingxiang Zhang. "Two elliptic generator Kleinian groups". PhD thesis. Massey University, 2010. URL: http://hdl.handle.net/10179/2044.