HOMEWORK 2

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MAT 327

Munkres p.92 qn. 4

Suppose W is open in $X \times Y$. A basis for the topology on $X \times Y$ is $\mathcal{B} = \{U \times V : U \text{ open in } X, V \text{ open in } Y\}$. Thus there is some family $\{(U \times V)_\alpha\}_{\alpha \in I}$ of sets in $\mathcal{B}$ such that $W = \cup(U \times V)_\alpha$.

Claim. $\pi_1(\cup(U \times V)_\alpha) = \cup\pi_1((U \times V)_\alpha)$. Proof:

$$\begin{aligned} x \in \pi_1(\cup(U \times V)_\alpha) &\iff x \in \pi_1((U \times V)_\alpha) \text{ for some } \alpha \\ &\iff x \in \cup\pi_1((U \times V)_\alpha). \end{aligned}$$

But, for each α, $\pi_1((U \times V)_\alpha)$ is open by the definition of $\mathcal{B}$. Hence $\pi_1(\cup(U \times V)_\alpha)$ is open in X, because it is the union of such open sets.

The same argument works to show that $\pi_2(U)$ is open for all sets U open in $X \times Y$: by proving that $\pi_2(\cup(U \times V)_\alpha) = \cup\pi_2((U \times V)_\alpha)$ holds (the same proof as above works), and then arguing that, as a consequence of this, $\pi_2(\cup(U \times V)_\alpha)$ is the union of open sets.

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(i). We need to show that if W is open in $X \times Y$ then it is open in $X' \times Y'$. Suppose $W = \cup(U \times V)_\alpha$ for some families of open sets $U_\alpha \subseteq \mathcal{T}$ and $V_\alpha \subseteq \mathcal{U}$ (we can do this by the definition of the product topology). Then, by hypothesis, $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$; hence each $U_\alpha \in \mathcal{T}'$ and each $V_\alpha \in \mathcal{U}'$. Thus $\cup(U \times V)_\alpha$ is open in $X' \times Y'$, since each $U_\alpha \times V_\alpha$ is a basic open set. Hence $X' \times Y' \supseteq X \times V$, and the former is finer than the latter.

(ii). Yes, the converse holds. Suppose every open set $U \times V$ in $X \times Y$ is open in $X' \times Y'$. Then every basis of $X \times Y$ is a subset of some basis of $X' \times Y'$. In particular, if $U \times V$ is an element of the usual basis of $X \times Y$, i.e. U is open in X and V is open in Y, then $U \times V$ is an element of some basis in $X' \times Y'$ and is therefore open. By (4) above, then, $U = \pi_1(U \times V)$ and $V = \pi_2(U \times V)$ are open in X' and Y' respectively. Hence if $X' \times Y'$ is finer than $X \times Y$, the topology on X' is finer than that on X and the topology on Y' is finer than that on Y.

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Suppose L is a line as a subspace of $X = \mathbb{R}_\ell \times \mathbb{R}$. We can parameterise L by $L(t) = (x_0 + tx_1, y_0 + ty_1)$ $(t \in \mathbb{R})$ for suitable points (x_0, y_0) and (x_1, y_1). Let $U \times V$ be a basic open set on X. Then $U = [a, b)$ and $V = (c, d)$ for suitable real numbers a, b, c, d. Intersecting $U \times V$ with L, and assuming that L is neither horizontal nor

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vertical (if L is horizontal then the topology is inherited only from $\mathbb{R}_\ell$ and if it is vertical then the topology is inherited from $\mathbb{R}$) we obtain the set

$$U = \{(x_0 + tx_1, y_0 + ty_1) : a \leq x_0 + tx_1 < b, c < y_0 + ty_1 < d\}$$

or equivalently, if both x_1 and y_1 are positive, then

$$U = \{L(t) : (a - x_0)/x_1 \leq t < (b - x_0)/x_1, (c - y_0)/y_1 < t < (d - y_0)/y_1\}$$

so, in particular, if $(a - x_0)/x_1 > (c - y_0)/y_1$ then L has the lower limit topology; otherwise, it has the standard order topology. If precisely one of x_1 or y_1 is negative, then the set U will have the upper limit topology instead of the lower imit topology in the first case, as the order relations $>$ will swap to $<$ and vice versa upon division. (Note, order here is defined by $L(t) > L(s) \iff t > s$.)

Similarly, suppose L is a subspace of $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$. Parameterising it in the same way, we have that a basic set of the topology on L is

$$U = \{L(t) : (a - x_0)/x_1 \leq t < (b - x_0)/x_1, (c - y_0)/y_1 \leq t < (d - y_0)/y_1\}$$

and thus the topology on L is always the lower or upper limit topology if x_1 and y_1 are either both positive or both negative. However, if one of them is positive and the other is negative then we obtain the discrete topology because it is possible for the line to intersect an open set in X at the 'closed corner', creating a single-point open set in L.

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Suppose U is open in X and A is closed in X.

Claim 1: $U - A$ is open in X. Indeed, $U - A = (X - A) \cap U$; but $X - A$ is open by the definition of closedness, so $U - A$ is the union of two open sets and is therefore open.

Claim 2: $A - U$ is closed in X. For consider $X - (A - U) = (X - A) \cup U$, which is the union of two open sets and is hence open. Thus the complement of $A - U$ with respect to X is open, and hence $A - U$ is closed.

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Every neighbourhood of x intersects $\cup A_\alpha$; hence, every neighbourhood of x intersects some particular A_α. However, it does not follow that every neighbourhood of x intersects the *same* A_α.

Consider $A_n = \{[1/n, 1) : n \in \mathbb{N}\}$. Then $0 \in \overline{\cup A_n}$ but $0 \notin \cup \overline{A_n}$ since 0 is not in the closure of any of the A_n — there is no A_n such that every neighbourhood of zero intersects it (because given any A_n, the neighbourhood $(-1/n, 1/n)$ of zero is disjoint from it).

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Suppose X is Hausdorff. Claim: $\Delta = \{x \times x : x \in X\}$ is closed; equivalently, $X - \Delta = \{x \times y : x, y \in X, x \neq y\}$ is open.

For every pair $x \times y \in X - \Delta$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. In particular, then, $x \times x \notin U \times V$ (since $x \notin V$) and $y \times y \notin U \times V$. Consider the family of sets

$$\mathcal{A} = \left\{ (U \times V)_{x \times y} : U, V \text{ disjoint and open in } X, x \in U, y \in V \right\}.$$

Then there does not exist $(x \times x) \in X \times X$ such that $x \times x \in A$, so $X - \Delta \supseteq \cup\mathcal{A}$. On the other hand, since all $x \times y$, $x \neq y$, are in $\cup\mathcal{A}$, $\cup\mathcal{A} \supseteq X - \Delta$. Hence $\cup\mathcal{A} = X - \Delta$, and so the latter (being a union of open sets) is open.

Conversely, suppose $\Delta = \{x \times x : x \in X\}$ is closed in $X \times X$. Then $X - \Delta = \{x \times y : x, y \in X, x \neq y\}$ is open. Suppose $x \neq y$. Then $x \times y \in X - \Delta$. Let $U \times V$ be some basic open set in $X \times X$ containing $x \times y$ and contained in $X - \Delta$. Hence $x \times x \notin U \times V$, and $y \times y \notin U \times V$. By (4) above, $\pi_1(U \times V) = U$ and $\pi_2(U \times V) = V$ are open. Hence U is an open set containing x, and V is an open set containing y. Note also, $y \notin U$ (because otherwise $y \times y \in U \times V$); similarly, $x \notin V$. Thus U and V are disjoint open sets containing x and y respectively, and X is Hausdorff.

Challenge Problem

For reference, the following properties will be assumed (where X, Y, and Z are topological spaces):

(1) the projection functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous

(2) if $f : Z \to X$ and $g : Z \to Y$ are continuous, then $f \times g : Z \to X \times Y$ is continuous

(3) if $Y \subseteq X$, then the inclusion function $\iota_Y : Y \to X$ is continuous

(4) if $Y \subseteq X$, and $f : Z \to Y \subseteq X$ is continuous with respect to the codomain X (i.e. $\iota_Y \circ f : Z \to X$ is continuous), then $f : Z \to Y$ is continuous with respect to Y.

Goal: to show that if $A \subseteq X$ and $B \subseteq Y$ are subspaces, then the topology on $A \times B$ as a subspace of $X \times Y$ (call it $\mathcal{T}_1$) is the same as the topology on $A \times B$ as a product of A and B (call it $\mathcal{T}_2$).

Part I ($\mathcal{T}_1 \to \mathcal{T}_2$). The inclusion map from $(A \times B, \mathcal{T}_1)$ into $X \times Y$ is continuous by (3). The projections from $X \times Y$ into X and Y are continuous by (1). Hence the functions

$$f : (A \times B, \mathcal{T}_1) \to (X \times Y) \to X$$
$$g : (A \times B, \mathcal{T}_1) \to (X \times Y) \to Y$$

obtained by composing inclusion and projections are continuous.

Note that the respective ranges of f and g are A and B as subspaces of X and Y. Hence by (4), the functions f and g are continous onto A and B as subsets of X and Y. Thus

$$f' : (A \times B, \mathcal{T}_1) \to (X \times Y) \to A$$
$$g' : (A \times B, \mathcal{T}_1) \to (X \times Y) \to B$$

are continuous.

Note that the product of f' and g' gives us the identity function from $(A \times B, \mathcal{T}_1)$ onto $(A \times B, \mathcal{T}_2)$; by property (2), since f' and g' are continuous their product is continuous and therefore the identity function in this direction is continuous.

Part II ($\mathcal{T}_2 \to \mathcal{T}_1$). Conversely, the projections from $(A \times B, \mathcal{T}_2)$ into A and B are continuous by (1) and the respective inclusion functions from A into X and B into Y are continuous by (3). Hence the functions

$$f : (A \times B, \mathcal{T}_2) \to A \to X$$
$$g : (A \times B, \mathcal{T}_2) \to B \to Y$$

obtained by composing projections and inclusions are continuous. By (2), this implies that $f \times g$ is continuous.

On the other hand, we can get from $(A \times B, \mathcal{T}_1)$ to $X \times Y$ by composing the identity function $\mathrm{Id} : (A \times B, \mathcal{T}_1) \to (A \times B, \mathcal{T}_1)$ and the inclusion function $\iota : (A \times B, \mathcal{T}_1) \hookrightarrow (X \times Y)$. Since the function $h = \iota \circ \mathrm{Id}$ obtained in this way agrees with $f \times g$ on every point of $A \times B$, we have that $h = f \times g$ and therefore h is continuous. But by (4), this implies that Id is continuous.

Thus the identity function from $(A \times B, \mathcal{T}_1)$ to $(A \times B, \mathcal{T}_1)$ is continuous; by part I above, its inverse is continuous; and therefore the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are identical.