Assignment 4

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MATHS 326

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Question One

(a) The symmetry group of the bracelet is C_6.

Cycle type	Count	Fixed colourings
$[1^6]$	1	2^6
$[2^3]$	1	2^3
$[3^2]$	2	2^2
$[6]$	2	2^1
	6	

So the number of orbits of C_6 on the colourings is

$$\frac{1 \cdot 2^6 + 1 \cdot 2^3 + 2 \cdot 2^2 + 2 \cdot 2^1}{6} = 14.$$

There are 14 distinct colourings up to rotation.

(b) If we also include reflections, then the symmetry group of the bracelet is D_6 (of order 12).

Cycle type	Count	Fixed colourings
$[1^6]$	1	2^6
$[2^3]$	1	2^3
$[3^2]$	2	2^2
$[6]$	2	2^1
$[1^2, 2^2]$	3	2^4
$[2^3]$	3	2^3
	12	

So the number of orbits of D_6 on the colourings is

$$\frac{1 \cdot 2^6 + 1 \cdot 2^3 + 2 \cdot 2^2 + 2 \cdot 2^1 + 3 \cdot 2^4 + 3 \cdot 2^3}{12} = 13.$$

There are 13 distinct colourings up to rotation and reflection.

Question Two

The cube's symmetry group has order 24.

Cycle type on edges	Count	Fixed colourings
$[1^{12}]$	1	2^{12}
$[4^3]$	6	2^3
$[2^6]$	3	2^6
$[3^4]$	8	2^4
$[1^2, 2^5]$	6	2^7
	24	

So the number of orbits of the group on the edge colourings is

$$\frac{1 \cdot 2^{12} + 6 \cdot 2^3 + 3 \cdot 2^6 + 8 \cdot 2^4 + 6 \cdot 2^7}{24} = 218.$$

There are 218 distinct colourings up to rotation.

Question Thre

(a) Labelling the vertices with the numbers 1 to 5, and writing the edge between vertex 1 and vertex 2 by 12, we have the following:

$$12 \mapsto 12$$
$$13 \mapsto 24 \mapsto 15 \mapsto 23 \mapsto 14 \mapsto 25 \mapsto 13$$
$$34 \mapsto 45 \mapsto 53.$$

(b) The symmetry group of K_5 is S_5. We have the following:

Cycle type on vertices	Count	Cycle type on edges	Fixed colourings
$[1^5]$	1	$[1^{10}]$	k^{10}
$[1^3, 2]$	10	$[1^4, 2^3]$	k^7
$[1^2, 3]$	20	$[1, 3^3]$	k^4
$[1, 4]$	30	$[2, 4^2]$	k^3
$[1, 2, 2]$	15	$[1^2, 4^2]$	k^4
$[2, 3]$	20	$[1, 3, 6]$	k^3
$[5]$	24	$[5^2]$	k^2
	120		

Hence the cycle index of the symmetry group is

$$\frac{1}{120}(a_1^{10} + 10a_1^2a_2^3 + 20a_1a_3^3 + 30a_2a_4^2 + 15a_1^2a_4^2 + 20a_1a_3a_6 + 24a_5^2).$$

(c) Counting the number of non-isomorphic graphs on five vertices is equivalent to counting the number of 2-colourings of K^5. Hence we need only calculate

$$\frac{1}{120}(2^{10} + 10(2^5) + 20(2^4) + 30(2^3) + 15(2^4) + 20(2^3) + 24(2^2)) = 20.$$

There are 20 distinct graphs on five vertices up to isomorphism.

Question Four

The symmetry group of the pentagonal prism is $C_2 \times C_5$.

Cycle type	Count	Cycle monomial
$[1^7]$	1	a_1^7
$[1^2, 5]$	4	$a_1^2a_5$
$[1^5, 2]$	1	$a_1^5a_2$
$[2, 5]$	4	a_2a_5
	10	

So the cycle index on the face colourings is

$$\frac{1}{10}(a_1^7 + 4a_1^2a_5 + a_1^5a_2 + 4a_2a_5).$$

Let x, y, and z be the number of green, blue, and red edges respectively; so the generating function for the colorings of a particular edge is $f(x,y,z) = x + y + z$. Hence, we want the coefficient of $x^2y^2z^2$ in

$$\frac{1}{10}((x+y+z)^7 + 4(x+y+z)^2(x^5+y^5+z^5) + (x+y+z)^5(x^2+y^2+z^2) \\ + 4(x^2+y^2+z^2)(x^5+y^5+z^5)).$$

This coefficient is

$$\frac{1}{10}\left(\frac{7!}{2!2!2!} + 0 + 3\cdot\frac{5!}{2!2!0!} + 0\right) = 72.$$

There are 72 distinct colourings such that each colour appears precisely twice, up to rotation.

Question Five

The symmetry group of the dodecahedron has order sixty.

Cycle type	**Count**	**Cycle monomial**
$[1^{12}]$	1	a_1^{12}
$[1^2, 5^2]$	24	$a_1^2a_5^2$
$[2^6]$	15	a_2^6
$[3^4]$	20	a_3^4
	60	

So the cycle index on the face colourings is

$$\frac{1}{60}(a_1^{12} + 24a_1^2a_5^2 + 15a_2^6 + 20a_3^4).$$

Let x, y, and z be the number of red, green, and white faces respectively; so the generating function for the colorings of a particular face is $f(x,y,z) = x + y + z$. Hence, we want the coefficient of $x^3y^3z^6$ in

$$\frac{1}{60}\left((x+y+z)^{12} + 24(x+y+z)^2(x^5+y^5+z^5)^2 + 15(x^2+y^2+z^2)^6 + 20(x^3+y^3+z^3)^4\right).$$

This coefficient is

$$\frac{1}{60}\left(\frac{12!}{3!3!6!} + 0 + 0 + 20\cdot\binom{4}{1}\cdot\binom{3}{1}\cdot\binom{2}{2}\right) = 312.$$