Assignment 4

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MATHS 326

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Question One

(a) The symmetry group of the bracelet is C_6.

Cycle type	Count	Fixed colourings
$[1^6]$	1	2^6
$[2^3]$	1	2^3
$[3^2]$	2	2^2
$[6]$	2	2^1
	6	

So the number of orbits of C_6 on the colourings is

$$\frac{1 \cdot 2^6 + 1 \cdot 2^3 + 2 \cdot 2^2 + 2 \cdot 2^1}{6} = 14.$$

There are 14 distinct colourings up to rotation.

(b) If we also include reflections, then the symmetry group of the bracelet is D_6 (of order 12).

Cycle type	Count	Fixed colourings
$[1^6]$	1	2^6
$[2^3]$	1	2^3
$[3^2]$	2	2^2
$[6]$	2	2^1
$[1^2, 2^2]$	3	2^4
$[2^3]$	3	2^3
	12	

So the number of orbits of D_6 on the colourings is

$$\frac{1 \cdot 2^6 + 1 \cdot 2^3 + 2 \cdot 2^2 + 2 \cdot 2^1 + 3 \cdot 2^4 + 3 \cdot 2^3}{12} = 13.$$

There are 13 distinct colourings up to rotation and reflection.

Question Two

The cube's symmetry group has order 24.

Cycle type on edges	Count	Fixed colourings
$[1^{12}]$	1	2^{12}
$[4^3]$	6	2^3
$[2^6]$	3	2^6
$[3^4]$	8	2^4
$[1^2, 2^5]$	6	2^7
	24	

So the number of orbits of the group on the edge colourings is

$$\frac{1 \cdot 2^{12} + 6 \cdot 2^3 + 3 \cdot 2^6 + 8 \cdot 2^4 + 6 \cdot 2^7}{24} = 218.$$

There are 218 distinct colourings up to rotation.

Question Thre

(a) Labelling the vertices with the numbers 1 to 5, and writing the edge between vertex 1 and vertex 2 by 12, we have the following:

$$12 \mapsto 12$$
$$13 \mapsto 24 \mapsto 15 \mapsto 23 \mapsto 14 \mapsto 25 \mapsto 13$$
$$34 \mapsto 45 \mapsto 53.$$

(b) The symmetry group of K_5 is S_5. We have the following:

Cycle type on vertices	Count	Cycle type on edges	Fixed colourings
$[1^5]$	1	$[1^{10}]$	k^{10}
$[1^3, 2]$	10	$[1^4, 2^3]$	k^7
$[1^2, 3]$	20	$[1, 3^3]$	k^4
$[1, 4]$	30	$[2, 4^2]$	k^3
$[1, 2, 2]$	15	$[1^2, 4^2]$	k^4
$[2, 3]$	20	$[1, 3, 6]$	k^3
$[5]$	24	$[5^2]$	k^2
	120		

Hence the cycle index of the symmetry group is

$$\frac{1}{120}(a_1^{10} + 10a_1^2a_2^3 + 20a_1a_3^3 + 30a_2a_4^2 + 15a_1^2a_4^2 + 20a_1a_3a_6 + 24a_5^2).$$

(c) Counting the number of non-isomorphic graphs on five vertices is equivalent to counting the number of 2-colourings of K^5. Hence we need only calculate

$$\frac{1}{120}(2^{10} + 10(2^5) + 20(2^4) + 30(2^3) + 15(2^4) + 20(2^3) + 24(2^2)) = 20.$$

There are 20 distinct graphs on five vertices up to isomorphism.

Question Four

The symmetry group of the pentagonal prism is $C_2 \times C_5$.

Cycle type	Count	Cycle monomial
$[1^7]$	1	a_1^7
$[1^2, 5]$	4	$a_1^2a_5$
$[1^5, 2]$	1	$a_1^5a_2$
$[2, 5]$	4	a_2a_5
	10	

So the cycle index on the face colourings is

$$\frac{1}{10}(a_1^7 + 4a_1^2a_5 + a_1^5a_2 + 4a_2a_5).$$

Let x, y, and z be the number of green, blue, and red edges respectively; so the generating function for the colorings of a particular edge is $f(x, y, z) = x + y + z$. Hence, we want the coefficient of $x^2y^2z^2$ in

$$\frac{1}{10}((x+y+z)^7 + 4(x+y+z)^2(x^5+y^5+z^5) + (x+y+z)^5(x^2+y^2+z^2) \\ + 4(x^2+y^2+z^2)(x^5+y^5+z^5)).$$

This coefficient is

$$\frac{1}{10}\left(\frac{7!}{2!2!2!} + 0 + 3\cdot\frac{5!}{2!2!0!} + 0\right) = 72.$$

There are 72 distinct colourings such that each colour appears precisely twice, up to rotation.

Question Five

The symmetry group of the dodecahedron has order sixty.

Cycle type	Count	Cycle monomial
$[1^{12}]$	1	a_1^{12}
$[1^2, 5^2]$	24	$a_1^2a_5^2$
$[2^6]$	15	a_2^6
$[3^4]$	20	a_3^4
	60	

So the cycle index on the face colourings is

$$\frac{1}{60}(a_1^{12} + 24a_1^2a_5^2 + 15a_2^6 + 20a_3^4).$$

Let x, y, and z be the number of red, green, and white faces respectively; so the generating function for the colorings of a particular face is $f(x, y, z) = x + y + z$. Hence, we want the coefficient of $x^3y^3z^6$ in

$$\frac{1}{60}\left((x+y+z)^{12} + 24(x+y+z)^2(x^5+y^5+z^5)^2 + 15(x^2+y^2+z^2)^6 + 20(x^3+y^3+z^3)^4\right).$$

This coefficient is

$$\frac{1}{60}\left(\frac{12!}{3!3!6!} + 0 + 0 + 20\cdot\binom{4}{1}\cdot\binom{3}{1}\cdot\binom{2}{2}\right) = 312.$$